

THIRTY-SEVENTH ANNUAL REPORT
OF THE
NATIONAL ADVISORY COMMITTEE
FOR AERONAUTICS

1951

INCLUDING TECHNICAL REPORTS
NOS. 1003 to 1058



UNITED STATES
GOVERNMENT PRINTING OFFICE
WASHINGTON : 1952

CONTENTS

	Page
Letter of Transmittal	v
Letter of Submittal	vii
Thirty-seventh Annual Report	1
Part I. Technical Activities	3
Research Organization and Procedures	3
Aerodynamic Research	4
Power Plants for Aircraft	17
Aircraft Construction	26
Operating Problems	34
Part II. Committee Organization	47
Part III. Financial Report	53
Technical Reports	55

TECHNICAL REPORTS

No.	Page	No.	Page
1003. Correlation of Physical Properties with Molecular Structure for Some Dicyclic Hydrocarbons Having High Thermal-Energy Release Per Unit Volume. By P. H. Wise, K. T. Serijan, and I. A. Goodman, NACA.....	55	1020. Measurements of Average Heat-Transfer and Friction Coefficients for Subsonic Flow of Air in Smooth Tubes at High Surface and Fluid Temperatures. By Leroy V. Humble, Warren H. Lowdermilk, and Leland G. Desmon, NACA.....	343
1004. A Lift-Cancellation Technique in Linearized Supersonic-Wing Theory. By Harold Mirels, NACA.....	65	1021. Analysis of Plane-Plastic-Stress Problems With Axial Symmetry in Strain-Hardening Range. By M. H. Lee Wu, NACA.....	359
1005. Analytical Determination of Coupled Bending-Torsion Vibrations of Cantilever Beams by Means of Station Functions. By Alexander Mendelson and Selwyn Gendler, NACA.....	77	1022. Temperature Distribution in Internally Heated Walls of Heat Exchangers Composed of Non-circular Flow Passages. By E. R. G. Eckert and George M. Low, NACA.....	381
1006. Analysis of Thrust Augmentation of Turbojet Engines by Water Injection at Compressor Inlet Including Charts for Calculating Compression Processes with Water Injection. By E. Clinton Wilcox and Arthur M. Trout, NACA.....	97	1023. Diffusion of Chromium in Alpha Cobalt-Chromium Solid Solutions. By John W. Weeton, NACA.....	397
1007. Horizontal Tail Loads in Maneuvering Flight. By Henry A. Pearson, William A. McGowan, and James J. Donegan, NACA.....	117	1024. Calculation of the Lateral Control of Swept and Unswept Flexible Wings of Arbitrary Stiffness. By Franklin W. Diederich, NACA.....	413
1008. A Small-Deflection Theory for Curved Sandwich Plates. By Manuel Stein and J. Mayers, NACA.....	129	1025. Experimental and Theoretical Studies of Area Suction for the Control of the Laminar Boundary Layer on an NACA 64A010 Airfoil. By Albert L. Braslow, Dale L. Burrows, Neal Tetervin, and Fioravante Visconti, NACA.....	433
1009. Investigation of Fretting by Microscopic Observation. By Douglas Godfrey, NACA.....	135	1026. NACA Investigation of Fuel Performance in Piston-Type Engines. By Henry C. Barnett, NACA.....	453
1010. A Recurrence Matrix Solution for the Dynamic Response of Aircraft in Gusts. By John C. Houbolt, NACA.....	145	1027. Buckling of Thin-Walled Cylinder Under Axial Compression and Internal Pressure. By Hsu Lo, Harold Crate, and Edward B. Schwartz, NACA.....	647
1011. Dynamics of a Turbojet Engine Considered as a Quasi-Static System. By Edward W. Otto and Burt L. Taylor, III, NACA.....	177	1028. Effect of Aspect Ratio on the Air Forces and Moments of Harmonically Oscillating Thin Rectangular Wings in Supersonic Potential Flow. By Charles E. Watkins, NACA.....	657
1012. Investigation of the NACA 4-(5)(08)-03 and NACA 4-(10)(08)-03 Two-Blade Propellers at Forward Mach Numbers to 0.725 to Determine the Effects of Camber and Compressibility on Performance. By James B. Delano, NACA.....	189	1029. Compressive Strength of Flanges. By Elbridge Z. Stowell, NACA.....	675
1013. Effects of Wing Flexibility and Variable Air Lift Wing Bending Moments During Landing Impacts of a Small Seaplane. By Kenneth F. Merten and Edgar B. Beck, NACA.....	221	1030. Investigation of Separation of the Turbulent Boundary Layer. By G. B. Schubauer and P. S. Klebanoff, National Bureau of Standards....	689
1014. Study of Effects of Sweep on the Flutter of Cantilever Wings. By J. G. Barmby, H. J. Cunningham, and I. E. Garrick, NACA.....	229	1031. A study of the Use of Experimental Stability Derivatives in the Calculation of the Lateral Disturbed Motions of a Swept-Wing Airplane and Comparison With Flight Results. By John D. Bird and Byron M. Jaquet, NACA.....	709
1015. Analysis of Turbulent Free-Convection Boundary Layer on Flat Plate. By E. R. G. Eckert and Thomas W. Jackson, NACA.....	255	1032. A Comparison of Theory and Experiment for High-Speed Free-Molecule Flow. By Jackson R. Stalder, Glen Goodwin, and Marcus O. Creager, NACA.....	735
1016. Effect of Tunnel Configuration and Testing Technique on Cascade Performance. By John R. Erwin and James C. Emery, NACA.....	263	1033. Comparison Between Theory and Experiment for Wings at Supersonic Speeds. By Walter G. Vincenti, NACA.....	757
1017. Investigation of Frequency-Response Characteristics of Engine Speed for a Typical Turbine-Propeller Engine. By Burt L. Taylor, III, and Frank L. Oppenheimer, NACA.....	279	1034. Investigation of Spoiler Ailerons for Use as Speed Brakes or Glide-Path Controls on Two NACA 65 Series Wings Equipped With Full-Span Slotted Flaps. By Jack Fischel and James M. Watson, NACA.....	769
1018. A Theoretical Analysis of the Effect of Time Lag in an Automatic Stabilization System on the Lateral Oscillatory Stability of an Airplane. By Leonard Sternfield and Ordway B. Gates, Jr., NACA.....	291	1035. Analysis of Means of Improving the Uncontrolled Lateral Motions of Personal Airplanes. By Marion O. McKinney, Jr., NACA.....	795
1019. Relation Between Inflammables and Ignition Sources in Aircraft Environments. By Wilfred E. Scull, NACA.....	303		

No.	Page	No.	Page
1036. Experimental Investigation of the Effects of Viscosity on the Drag and Base Pressure of Bodies of Revolution at a Mach Number of 1.5. By Dean R. Chapman and Edward W. Perkins, NACA.....	805	1048. A Study of Effects of Viscosity on Flow Over Slender Inclined Bodies of Revolution. By H. Julian Allen and Edward W. Perkins, NACA....	1103
1037. General Method and Thermodynamic Tables for Computation of Equilibrium Composition and Temperature of Chemical Reactions. By Vearl N. Huff, Sanford Gordon, and Virginia E. Morrell, NACA.....	829	1049. Experimental Investigation of the Effect of Vertical-Tail Size and Length and of Fuselage Shape and Length on the Static Lateral Stability Characteristics of a Model With 45° Sweptback Wing and Tail Surfaces. By M. J. Queijo and Walter D. Wolhart, NACA.....	1117
1038. Wind-Tunnel Investigation of Air Inlet and Outlet Openings on a Streamline Body. By John V. Becker, NACA.....	887	1050. Formulas for the Supersonic Loading, Lift and Drag of Flat Swept-Back Wings With Leading Edges Behind the Mach Lines. By Doris Cohen, NACA.....	1146
1039. On the Particular Integrals of the Prandtl-Busemann Iteration Equations for the Flow of a Compressible Fluid. By Carl Kaplan, NACA....	909	1051. An Analysis of Base Pressure at Supersonic Velocities and Comparison With Experiment. By Dean R. Chapman, NACA.....	1187
1040. Spectra and Diffusion in a Round Turbulent Jet. By Stanley Corrsin and Mahinder S. Uberoi, Johns Hopkins University.....	915	1052. A Summary of Lateral-Stability Derivatives Calculated for Wing Plan Forms in Supersonic Flow. By Arthur L. Jones and Alberta Alksne, NACA.....	1211
1041. Equations and Charts for the Rapid Estimation of Hinge-Moment and Effectiveness Parameters for Trailing-Edge Controls Having Leading and Trailing Edges Swept Ahead of the Mach Lines. By Kenneth L. Goin, NACA.....	937	1053. Investigation of Turbulent Flow in a Two-Dimensional Channel. By John Laufer, California Institute of Technology.....	1247
1042. Some Effects of Nonlinear Variation in the Directional-Stability and Damping-in-Yawing Derivatives on the Lateral Stability of an Airplane. By Leonard Sternfield, NACA.....	1009	1054. Integrals and Integral Equations in Linearized Wing Theory. By Harvard Lomax, Max A. Heaslet, and Franklyn B. Fuller, NACA.....	1267
1043. A Numerical Method for the Stress Analysis of Stiffened-Shell Structures Under Nonuniform Temperature Distributions. By Richard R. Heldenfels, NACA.....	1019	1055. Comparison of Theoretical and Experimental Heat-Transfer Characteristics of Bodies of Revolution at Supersonic Speeds. By Richard Scherrer, NACA.....	1301
1044. The Method of Characteristics for the Determination of Supersonic Flow Over Bodies of Revolution at Small Angles of Attack. By Antonio Ferri, NACA.....	1039	1056. Theoretical Antisymmetric Span Loading for Wings of Arbitrary Plan Form at Subsonic Speeds. By John DeYoung, NACA.....	1317
1045. Supersonic Flow around Circular Cones at Angles of Attack. By Antonio Ferri, NACA.....	1055	1057. Analysis of the Effects of Boundary-Layer Control on the Take-Off and Power-Off Landing Performance Characteristics of a Liaison Type of Airplane. By Elmer A. Horton, Laurence K. Loftin, Jr., Stanley F. Racisz, and John H. Quinn, Jr., NACA.....	1353
1046. A General Integral Form of the Boundary-Layer Equation for Incompressible Flow With an Application to the Calculation of the Separation Point of Turbulent Boundary Layers. By Neal Tetervin and Chia Chiao Lin, NACA.....	1067	1058. Influence of Chemical Composition on Rupture Properties at 1200° F. of Forged Chromium-Cobalt-Nickel-Iron Base Alloys in Solution-Treated and Aged Condition. By E. E. Reynolds, J. W. Freeman, and A. E. White, NACA..	1385
1047. The Stability of the Compression Cover of Box Beams Stiffened by Posts. By Paul Seide and Paul F. Barrett, NACA.....	1087		

Letter of Transmittal

To the Congress of the United States:

In compliance with the provisions of the act of March 3, 1915, as amended, establishing the National Advisory Committee for Aeronautics, I transmit herewith the Thirty-seventh Annual Report of the Committee covering the fiscal year 1951.

HARRY S. TRUMAN.

THE WHITE HOUSE,
JANUARY 28, 1952.

Letter of Submittal

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

WASHINGTON, D. C., *November 15, 1951.*

DEAR MR. PRESIDENT: In compliance with the act of Congress approved March 3, 1915, as amended (U. S. C. 1946, title 50, sec. 153), I have the honor to submit herewith the Thirty-seventh Annual Report of the National Advisory Committee for Aeronautics covering the fiscal year 1951.

The development of new types of aircraft slowed materially since the end of World War II but, since Korea, progress is now rapidly accelerating. The magnitude of the military effort required to check world-wide aggression makes it evident that America's aircraft must be superior in performance and military effectiveness to those of any other nation.

The Committee seeks, through timely and adequate research, to assure the success of the expanding aircraft program. The Committee believes that in order to support the military effort effectively, its scope of operations should be increased to the extent that availability of scientific manpower will permit.

Respectfully submitted.

JEROME C. HUNSAKER,
Chairman.

THE PRESIDENT,
The White House, Washington, D. C.

VII

Preceding Page Blank

National Advisory Committee for Aeronautics

Headquarters, 1724 F Street NW., Washington 25, D. C.

Created by act of Congress approved March 3, 1915, for the supervision and direction of the scientific study of the problems of flight (U. S. Code, title 50, sec. 151). Its membership was increased from 12 to 15 by act approved March 2, 1929, and to 17 by act approved May 25, 1948. The members are appointed by the President, and serve as such without compensation.

JEROME C. HUNSAKER, Sc. D., Massachusetts Institute of Technology, *Chairman*

ALEXANDER WETMORE, Sc. D., Secretary, Smithsonian Institution, *Vice Chairman*

DETLEV W. BRONK, Ph. D., President, Johns Hopkins University.

JOHN H. CASSADY, Vice Admiral, United States Navy, Deputy Chief of Naval Operations.

EDWARD U. CONDON, Ph. D., Director, National Bureau of Standards.

HON. THOMAS W. S. DAVIS, Assistant Secretary of Commerce.

JAMES H. DOOLITTLE, Sc. D., Vice President, Shell Oil Co.

R. M. HAZEN, B. S., Director of Engineering, Allison Division, General Motors Corp.

WILLIAM LITTLEWOOD, M. E., Vice President, Engineering, American Airlines, Inc.

THEODORE C. LONNQUEST, Rear Admiral, United States Navy, Deputy and Assistant Chief of the Bureau of Aeronautics.

HON. DONALD W. NYROP, Chairman, Civil Aeronautics Board.

DONALD L. PUTT, Major General, United States Air Force, Acting Deputy Chief of Staff (Development).

ARTHUR E. RAYMOND, Sc. D., Vice President, Engineering, Douglas Aircraft Co., Inc.

FRANCIS W. REICHELDERFER, Sc. D., Chief, United States Weather Bureau.

GORDON P. SAVILLE, Major General, United States Air Force, Deputy Chief of Staff (Development).

HON. WALTER G. WHITMAN, Chairman, Research and Development Board, Department of Defense.

THEODORE P. WRIGHT, Sc. D., Vice President for Research, Cornell University.

HUGH L. DRYDEN, Ph. D., *Director*

JOHN F. VICTORY, LL. D., *Executive Secretary*

JOHN W. CROWLEY, JR., B. S., *Associate Director for Research*

E. H. CHAMBERLIN, *Executive Officer*

HENRY J. E. REID, D. ENG., Director, Langley Aeronautical Laboratory, Langley Field, Va.

SMITH J. DEFANCE, B. S., Director, Ames Aeronautical Laboratory, Moffett Field, Calif.

EDWARD R. SHARP, Sc. D., Director, Lewis Flight Propulsion Laboratory, Cleveland Airport, Cleveland, Ohio

TECHNICAL COMMITTEES

AERODYNAMICS

POWER PLANT FOR AIRCRAFT

AIRCRAFT CONSTRUCTION

OPERATING PROBLEMS

INDUSTRY CONSULTING

Coordination of Research Needs of Military and Civil Aviation

Preparation of Research Programs

Allocation of Problems

Prevention of Duplication

Consideration of Inventions

LANGLEY AERONAUTICAL LABORATORY
Langley Field, Va.

AMES AERONAUTICAL LABORATORY
Moffett Field, Calif.

LEWIS FLIGHT PROPULSION LABORATORY
Cleveland Airport, Cleveland, Ohio

Conduct, under unified control, for all agencies, of scientific research on the fundamental problems of flight

OFFICE OF AERONAUTICAL INTELLIGENCE
Washington, D. C.

Collection, classification, compilation, and dissemination of scientific and technical information on aeronautics

THIRTY-SEVENTH ANNUAL REPORT

OF THE

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

WASHINGTON, D. C., *November 15, 1951.*

To the Congress of the United States:

In accordance with the act of Congress, approved March 3, 1915 (U. S. C. title 50, sec. 151), which established the National Advisory Committee for Aeronautics, the Committee submits its thirty-seventh annual report for the fiscal year 1951.

The United States is engaged in expanding military aviation to levels never before reached except in the midst of a major war. In Korea, our military aircraft are engaged in combat with airplanes of an unfriendly nation evidently able to build military aircraft of increasing capabilities. In this environment, the NACA is responsible for conducting an adequate program of scientific research to open the way for the design of aircraft and missiles of superior performance.

Since World War II the pace of technical development has increased. Until then, improvement in aircraft performance as a result of the application of scientific research proceeded at what now seems to be a relatively slow and orderly rate. Modest increases in speed, climb, range, or altitude were set as reasonable goals. Compressibility effects at high speeds were just beginning to be encountered and indicated a formidable barrier near the velocity of sound. This barrier has been found by research and experiment to be less formidable than supposed, and we now see the possibility of radical gains in airplane performance that are of great military significance. Such gains are also attainable by a potential enemy.

The increased complexity of modern high performance aircraft results in greatly increased costs, whether measured in manpower or dollars, and places more at stake in the success or failure of the research upon which superior performance is predicated. In view of the magnitude of the national effort to build up air power adequate to our national security, scientific research should proceed on a corresponding scale.

In the years immediately following World War II in appreciation of the probable significance to national security of applications of new knowledge of supersonic

aerodynamics and jet propulsion, the President and the Congress supported the NACA in an intensive program of fundamental scientific studies in these fields of aeronautical science. There was an urgent need to overcome a serious deficiency of basic information resulting from wartime concentration of research teams on immediate problems associated with the improvement of airplanes in service. Only by concentration on fundamentals may we expect to find leads to radically new developments which should pay off in the design of future aircraft. We do not know how long the present period of tension will last nor do we know what discoveries an enemy may make.

In addition, the Committee must not neglect less spectacular research to provide engineering information essential to the safety and economy of aircraft of all types, both civil and military. In this, the NACA has responsibility for the scientific study and practical solution of current problems, as crystallized in the discussions of our several subcommittees of experts. These subcommittees include designers, users, and research scientists. Research initiated to solve a current problem can be readily translated into practice. Such research, for example, makes possible an increase in thrust of current production engines and improved performance of the next lot of airplanes. The need for extended work in this area has been reflected in our budget estimates.

The military research and development program has been increased threefold, but to date the funds and manpower authorized for NACA have not expanded to support adequately the military effort. Therefore, the Committee urges an expansion in NACA effort to the extent that availability of scientific manpower will permit.

Presently, emphasis is on the military applications of research, but many of the results of NACA research effort are of value to civil aviation, as for example the work on ice prevention, fire prevention, effects of atmospheric turbulence, and work on aircraft stability and control. Much research for the improvement of

military aircraft is extended to obtain data needed by those responsible for the design, operation, or regulation of civil aircraft.

During the current year the Committee completed the installation of a transonic ventilated throat in the 16-foot wind tunnel at the Langley Aeronautical Laboratory. This installation is based on an invention by members of the NACA staff and is considered to be of exceptional importance because it permits model airplane tests at transonic air speeds in wind tunnels, hitherto impossible because of the choking of the conventional wind tunnel as the air speed approaches the velocity of sound.

In the formulation of its research programs, the Committee has been materially assisted by some 400

members of its 26 technical committees. These members include scientists from universities, engineers from industry, and experts from the civil and military agencies of the Government. The Committee continues to enjoy the loyal and effective support of its civil service staff in the execution of its policies and directives.

Parts I, II, and III of this annual report present a résumé of the unclassified scientific activities of the Committee, a description of the Committee's organization and membership, and the financial report for the fiscal year 1951.

Respectfully submitted.

JEROME C. HUNSAKER,
Chairman.

Part I—TECHNICAL ACTIVITIES

RESEARCH ORGANIZATION AND PROCEDURES

COMMITTEE LABORATORIES

Research of the National Advisory Committee for Aeronautics is conducted largely at its three laboratories—Langley Aeronautical Laboratory, Langley Field, Va.; Ames Aeronautical Laboratory, Moffett Field, Calif.; and Lewis Flight Propulsion Laboratory, Cleveland, Ohio. A subsidiary station is located at Wallops Island, Va., as a branch of the Langley Laboratory for conducting research on models in flight in the transonic and supersonic speed ranges. At Edwards, Calif., is located the NACA High-Speed Flight Research Station for research on transonic and supersonic airplanes in flight. The total number of employees, both technical and administrative, at these five stations and headquarters in Washington was 7,705 at the end of the fiscal year 1951.

TECHNICAL COMMITTEES AND RESEARCH COORDINATION

In carrying out its function of coordinating aeronautical research the Committee is assisted by a group of technical committees and subcommittees. The members of these committees are chosen for their particular knowledge in a specific field of aeronautics. They are selected from other Government agencies concerned with aviation including the Department of Defense, from the aircraft and air transport industries, and from scientific and educational institutions. These committees provide for the interchange of ideas and prevention of research duplication except where parallel efforts are desired. There are four technical committees under the National Advisory Committee for Aeronautics:

1. Committee on Aerodynamics.
2. Committee on Power Plants for Aircraft.
3. Committee on Aircraft Construction.
4. Committee on Operating Problems.

Each committee is supported by three to eight technical subcommittees.

In addition to the four technical committees, there is an Industry Consulting Committee established to assist the main Committee in formulation of general

policy. Membership of this Committee, as well as the technical committees and subcommittees, is listed in part II of this report.

Research coordination is further effected through discussions between Committee technical personnel and the research staffs of the aviation industry, educational and scientific organizations, and other aeronautical agencies. The Research Coordination Office is assisted by a west coast representative who maintains close contact with the aeronautical research and engineering staffs of that geographical area.

RESEARCH SPONSORED IN SCIENTIFIC AND RESEARCH INSTITUTIONS

For the 1951 fiscal year the NACA continued to make use of the unique research talents and facilities of universities and other nonprofit scientific institutions to find solutions for aeronautical problems. Both research proposals and the results from investigations have been carefully reviewed to maintain the quality of this part of the NACA program, and reports of the useful results of the sponsored research have been given the same wide distribution as other NACA reports.

The diversity of this sponsored research is indicated by the fact that 19 of the NACA technical subcommittees reviewed proposals for or the results of such investigations during the past year and that 48 reports of sponsored research were released. The technical aspects of the program are included with the descriptions of work under the various technical sections.

During the 1951 fiscal year the following institutions participated in the contract research program:

National Bureau of Standards, Forest Products Laboratory, Armour Research Foundation, Battelle Memorial Institute, Polytechnic Institute of Brooklyn, Brown University, California Institute of Technology, University of California, University of California at Los Angeles, Carnegie Institute of Technology, University of Cincinnati, Cornell University, University of Florida, Georgia Institute of Technology, Harvard University, Illinois Institute of Technology, University of Illinois, Iowa State College, Johns Hopkins University, Massachusetts Institute of Technology, University

of Michigan, Mount Washington Observatory, New York University, University of Notre Dame, Ohio State University, Pennsylvania State College, Princeton University, Purdue University, Stanford University, Stevens Institute of Technology, Syracuse University, Texas Agricultural and Mechanical College, University of Virginia, and University of Washington.

RESEARCH INFORMATION

Research results obtained in the Committee's laboratories and through research contracts are distributed in the form of Committee publications. Formal Reports, printed by the Government Printing Office, contain unclassified information and are available to the general public from the Superintendent of Documents. Technical Notes, released by the Committee in limited numbers, also contain unclassified research results of a more or less interim nature and are distributed to interested organizations throughout the country. Frequently, research results which will ultimately be published in Report form are issued initially as Technical Notes in the interest of speedy dissemination of data to American technical people and organizations. Translations of foreign material are issued in the form of Technical Memorandums.

In addition to unclassified publications the Committee prepares a large number of reports containing classified research information. For reasons of national security, these reports are controlled in their circulation. From time to time the classified reports are examined to determine whether it is in the national interest to declassify them. If it is found desirable to declassify the reports, they may be published as unclassified papers.

Another important means of transmitting quickly and efficiently the latest information in a particular field of research directly to designers and engineers working in that field is the holding of technical conferences from time to time at an appropriate NACA laboratory. Several conferences of this nature were held during the past year.

The Office of Aeronautical Intelligence was established in 1918 as an integral part of the Committee's

activities. Its functions are the collection and classification of technical knowledge on the subject of aeronautics, including the results of research and experimental work conducted in all parts of the world, and its dissemination to the Department of Defense, aircraft manufacturers, educational institutions, and other interested people. American and foreign reports obtained are analyzed, classified, and brought to the attention of the proper persons through the medium of public and confidential bulletins.

AERONAUTICAL INVENTIONS

By act of Congress, approved July 2, 1926 (U. S. C. title 10, sec. 310-r), an Aeronautical Patents and Design Board was established consisting of the Assistant Secretaries for Air of the Departments of War, Navy, and Commerce. In accordance with that act as amended by an act, approved March 3, 1927, the National Advisory Committee for Aeronautics is charged with the function of analyzing and reporting upon the technical merits of aeronautical inventions and designs submitted to any agency of the Government. The Aeronautical Patents and Design Board is authorized, upon the favorable recommendation of the Committee to "determine whether the use of the design by the Government is desirable or necessary and evaluate the design and fix its worth to the United States in an amount not to exceed \$75,000."

Recognizing its obligation to the public in this respect the Committee has continued to accord to all correspondence on such matters full consideration. All proposals received have been carefully analyzed and evaluated and the submitters have been advised concerning the probable merits of their suggestions. Many personal interviews have been granted inventors who visited the Committee's offices, and technical information has been supplied when requested.

* * * * *

The following detailed summary of unclassified research, completed during the fiscal year 1951, is organized to reflect the areas of research over which each technical committee and related subcommittee has cognizance.

AERODYNAMIC RESEARCH

During the past year there has been an increase in research directed specifically at the problems of high-speed airplanes and guided missiles. Aerodynamic research to permit efficient operation of airplanes at transonic speeds has continued to receive special emphasis, and information has been obtained in this speed range from flights of special research airplanes, rocket-powered free-flight models, freely-falling bodies, and

from very small models on wind-tunnel bumps or by the wing-flow method in flight. There has, however, been a serious need for techniques to permit detailed systematic large-scale experiments to be made in wind tunnels through the speed of sound. As a result of extensive research on wind-tunnel design, suitable techniques have been developed, and the Langley 8-foot and 16-foot wind tunnels were modified to permit testing

through the speed of sound. These modified tunnels are contributing to a better understanding of aerodynamic problems in the transonic speed range.

Theoretical and experimental research on high-speed airplane design problems has been directed generally at the establishment of technical information to permit the selection of the most efficient configurations for flights at transonic and supersonic speeds which will also have satisfactory flying characteristics at low speeds for landing and take-off. As the high-speed requirements increased, the landing requirements have become more difficult to meet and it has been necessary to continue research at low speeds on these problems.

In addition to studies of performance problems of guided missiles, considerable attention has been directed during the past year to the problems of stability and control. In this field particular emphasis has been placed on the cross-coupling effects experienced by various missile configurations and on automatic control problems.

The NACA is assisted in guiding the aerodynamic research activities of its laboratories by the Committee on Aerodynamics and its eight technical subcommittees. Results of the research which are most pertinent to the design and development of military airplanes and guided missiles are summarized and presented at technical conferences held during the year with the representatives of the military services and their contractors, in order to assist designers in making early application of these results.

In the sections which follow a description is given of some of the Committee's recent unclassified work in aerodynamics.

FLUID MECHANICS

Investigation of Airfoils at Low Speeds

Low-speed investigations of two-dimensional airfoils have been devoted to a large extent to the study of flow phenomena at high lifts and the development of methods for the improvement of high lift characteristics. Reported in Technical Note 2235 is an investigation of the boundary-layer and stalling characteristics of the NACA 64A-010 airfoil section in the Ames 7- by 10-foot wind tunnels. The results show the small region of separated flow or "bubble" near the leading edge, characteristic of airfoils with small leading-edge radii. At an angle of attack of 9.5° the separated flow failed to reattach to the surface, causing the stall. Since there was no turbulent separation at the trailing edge, the lift-curve peak was sharp and the stall occurred suddenly and with no warning.

An investigation was also conducted in the Ames 7- by 10-foot wind tunnels to determine the possibility of

delaying the flow separation that occurs near the leading edge of the NACA 63₁-012 airfoil section and of improving its stalling characteristics by modifications of the contour near the leading edge. This study is reported in Technical Note 2228. The greatest improvement was obtained with modifications that incorporated an increase in camber and modifications to the leading-edge shape were more effective than drooped-nose flaps.

The results of a comprehensive investigation of the stalling characteristics of airfoil sections at low subsonic speeds in the Ames 7- by 10-foot wind tunnels have been summarized in Technical Note 2502. Several types of stall encountered with airfoils ranging in thickness from that for a slow, light airplane to that for a possible supersonic application are described in detail in conjunction with the characteristic flows in the boundary layer.

The increases in airplane lift-drag ratio associated with increasing aspect ratio are limited by the high profile-drag coefficients associated with the thick wing-root sections which are structurally necessary on wings of high aspect ratio. An investigation has therefore been made of NACA 64-series airfoils of 32- and 40-percent-chord thickness in the Langley low-turbulence tunnels to determine whether the high profile-drag coefficients of thick airfoils can be reduced by boundary-layer control through a single suction slot. The results obtained for the 32- and 40-percent-thick sections together with other available airfoil section data were used to calculate the characteristics of a number of hypothetical wings. These calculations indicate that the maximum attainable lift-drag ratio of structurally feasible wings and of the corresponding airplanes may be increased appreciably by the use of such thick wings with boundary-layer control. The results are presented in Technical Note 2405.

It has been found that in the higher ranges of landing speeds compressibility effects on maximum lift can become appreciable. As part of a program to explore this phenomenon, the effects of Mach number and Reynolds number on the maximum lift coefficient of a wing of NACA 66-series airfoil sections were determined in the Langley 19-foot pressure tunnel and are published in Technical Note 2251. The ranges of Mach number and Reynolds number extended up to 0.34 and 8.0 million, respectively. It was found that for a given value of Mach number, the values of maximum lift coefficient were increased when the Reynolds number was increased. For a given value of Reynolds number, an increase in Mach number at first caused only small reductions in maximum lift coefficient, but the reduction was large after the critical speed was exceeded.

Theoretical Investigations of Two-Dimensional Compressible Flows

Theoretical investigations of two-dimensional compressible flows are being continued with particular reference to the difficult transonic speed range. At the Langley Laboratory the Prandtl-Busemann small-perturbation method has been utilized to obtain the flow of a compressible fluid past a wave-shaped, infinitely long wall. When the essential assumption for transonic flow is introduced—that all Mach numbers in the region of flow are nearly unity—the expression for the velocity potential takes the form of a power series in the transonic similarity parameter. On the basis of this form of the solution, an attempt has been made to solve the nonlinear differential equation for transonic flow past a wavy wall. The results of this analysis exhibit clearly the inherent difficulties encountered in nonlinear flow problems. Nevertheless, the investigation has been carried in a rigorous manner to the point where the question of the existence or nonexistence of a mixed potential flow free of discontinuities can be settled by the behavior of a single power series in the transonic similarity parameter. The calculation of the coefficients of this dominant power series has been reduced to a routine computing problem by means of recursion formulas resulting from the solution of the differential equation and the boundary condition at the wall. One of the results of the analysis is the rigorous statement of a limit of applicability of the transonic similarity concept. The results are presented in TN 2383.

In Technical Note 2253 it is shown that it is possible to solve the Prandtl-Busemann iteration equations by means of the familiar concepts of a distribution of sources and sinks. The method is limited to the treatment of thin, sharp-nose, symmetric, two-dimensional profiles. An explicit expression is derived for the second-order velocity potential and velocity components, and a method for obtaining higher-order terms is indicated. The velocity at the surface of the Kaplan bump is evaluated to illustrate the method.

The relaxation method has been proposed as a means of calculating pressure distributions on two-dimensional bodies at high speeds, and a comparison is shown in Technical Note 2174 between the experimental pressure distribution for an NACA 0012 profile and the theoretical pressure distribution obtained by this method. The comparison showed good agreement at low speeds, but at higher Mach numbers the theoretical calculations indicated higher negative pressure coefficients than were obtained experimentally.

An integral method of calculating two-dimensional compressible flows past thin airfoils, with particular reference to the transonic speed range is presented in

Technical Note 2130. By an appropriate choice of streamline curvature, this method permits the flow pattern about a thin airfoil to be calculated in a comparatively simple manner. The results presented include velocity distributions for various symmetric sections. Comparison with other theories and with available experimental data indicate at least qualitatively good agreement.

The transonic similarity law for two-dimensional flow, derived by von Karman, has been investigated by an iteration procedure and is presented in Technical Note 2191. The boundary condition of zero velocity normal to the body was satisfied at the body rather than near the body, as was done by von Karman. The results showed that the velocity potential can be expressed in terms of a similarity parameter and were in agreement with those of von Karman.

Theoretical investigations of two-dimensional compressible flows by several educational institutions under contract to the NACA are being continued. At the New York University an approximate method of solving the nonlinear differential equation for two-dimensional subsonic compressible flow by means of the variational method is being used. The general problem of steady irrotational flow past an arbitrary body is formulated and two examples have been carried out, namely, the flow past a circular cylinder and the flow past a thin curved surface. The variational method yields results of velocity and pressure distributions which compare well with those found by other methods.

The problem of two-dimensional flow behind a curved stationary shock wave has been considered analytically at the Massachusetts Institute of Technology. The method, which is presented in Technical Note 2364, assumes a given shock-wave shape, which automatically determines certain initial conditions on the flow variables; and the flow pattern, including any body shape, follows from the initial conditions. Approximate analytic expressions are found for the stream function in the subsonic region following the shock and, after the stream function is obtained, the flow density is determined by Bernoulli's equation, which connects the density with the derivatives of the stream function. The final solution can then be determined from the velocity field thus obtained.

Technical Note 2356 presents the results of a study, made at Cornell University, of the two-dimensional transonic flow past airfoils. The problem of constructing solutions for transonic flow over symmetric airfoils is treated and simplification of the mapping of the incompressible flow is emphasized. In the case of a symmetric Joukowski airfoil without circulation, the mapping is relatively simple, but the coefficients in the power series are difficult to evaluate. As a means of

simplification, an approximate flow is used which differs only slightly from the exact incompressible one when the thickness is small. Flow with circulation is also considered by the same method.

At the Johns Hopkins University an investigation has been made of the transient behavior of an airfoil in supersonic flow due to pitching, flapping, and vertical gusts by means of the Fourier integral method. By using a convolution integral, the aerodynamic effect of any arbitrary motion of the airfoil can be determined, if the aerodynamic response of the airfoil to a step-function disturbance is known. As useful examples, the harmonically oscillating airfoil with different modes or degrees of freedom is analyzed for the complete time history, for the case in which the motion starts abruptly from rest. In carrying out the investigation, a new integral has been introduced which is believed to play an equivalent part in supersonic flutter to the Theodorsen function in the flutter problems of incompressible flow. The results are presented in Technical Note 2333.

Investigation of Wings and Bodies at Transonic and Supersonic Speeds

An essential part of the correlation of aerodynamic data is a knowledge of the correspondence between the flow fields about similar bodies at related speed conditions. An investigation was made, therefore, at the Ames Laboratory which extended similarity laws for transonic flow to include finite-span wings. Under the assumptions of transonic small-perturbation potential theory, it was shown that similitude in pressure, lift, pitching moment, and pressure-drag coefficients depend upon the constancy of two parameters. The application of these results led to an essential improvement in the manner of relating experimental wing characteristics at near-sonic speeds. This work has been reported in Technical Note 2273.

Source distribution methods for evaluating the aerodynamics of thin finite wings at supersonic speeds are summarized and extended in Report 951. The fundamental equations were derived in general form for both time-independent and time-dependent wing motions and were applied to solve a large variety of aerodynamic problems. In Technical Note 2303 a method was developed for relating the solution of a wing to the known solution of a simpler wing. This method permits complicated flow fields to be solved in a straightforward manner.

At the Johns Hopkins University, von Karman's Fourier integral method in supersonic wing theory, derived directly from the basic concepts of the harmonic source and doublet, has been applied to investigate the general solution of the wave drag of a tapered swept wing with a symmetrical diamond airfoil profile, cover-

ing various geometrical arrangements of the planform. The theory has also been applied to find the downwash distribution in the plane of the wing on which the pressure distribution is preassigned. A number of examples are given. This work has been done under contract to the NACA and the results are presented in Technical Note 2317.

The flow around cones without axial symmetry and moving at supersonic velocity has been analyzed and the results of this analysis are presented in Technical Note 2236. Singular points were shown to exist in the flow around the cone if no axial symmetry exists. The concept of a vortical layer around the cone at small angles of attack was introduced, and the correct values of the first-order terms of the velocity components were determined. It was shown that good agreement with experimental results can be obtained if the complete equation for the pressure distribution is used.

The transonic similarity law, as applied to bodies of revolution, has been investigated and the results presented in Technical Note 2239. The results obtained were in approximate agreement with those of von Karman in the region of the flow field not too close to the body. In the neighborhood of the body a different similarity law was obtained. In another investigation the integral method used to obtain compressible flow past two-dimensional shapes (Technical Note 2130) was applied to the calculation of the compressible flow past slender bodies of revolution. Good agreement of the resulting velocities on ellipsoids of revolution with those obtained by other methods was found. The investigation is reported in Technical Note 2245.

The pressure acting on the base of a body of revolution flying at supersonic velocity is of considerable importance because the base drag can, in some cases, represent more than half of the total drag. It is essential that the factors contributing to this drag be explored to enable a reasonable drag prediction to be made in the design of missiles and to provide information from which possible base-drag reductions may be realized. One such factor is the presence of tail surfaces near the base of the body. Since the pressures in the vicinity of the trailing edge of the tail surfaces at zero angle of attack are normally less than the free-stream values, the interaction of these pressures with the flow behind the base may produce a reduction in the base pressure and, hence, an increase in the base drag. Accordingly, experimental measurements of the base pressure on an unboattailed body of revolution in combination with rectangular tail surfaces were made at the Ames Laboratory to investigate the effects of tail surfaces on base drag. The results of the investigation showed that the addition of tail surfaces with the trailing edge near the base of the body incurred a significant increase in base

drag. For example, the increase was about 70 percent at a Mach number of 1.5 for a cruciform tail having a 10-percent-thick biconvex airfoil section. The base-drag penalty due to tail surfaces was found to increase with an increase in the airfoil thickness ratio and with the number of tail surfaces and to decrease with an increase in Mach number. It was also found that this base-drag increment could be essentially eliminated by moving the tail so that the trailing edge was about one chord-length ahead of or behind the base of the body. The results are presented in Technical Note 2360.

Research in the Hypersonic Speed Range

In the Thirty-sixth Annual Report several facilities and techniques permitting supersonic research to be extended into the hypersonic speed range were described. During the past year this work has been continued and, in addition, progress has been made in extending theoretical calculations into the hypersonic range.

The accuracy of Tsien's hypersonic similarity rule, and the range of Mach numbers and fineness ratios over which it can be applied, have been investigated theoretically. A first investigation, which ignored rotation effects, showed that the rule applied with good accuracy over a range from moderate supersonic speeds up to hypersonic speeds for a wide range of body fineness ratios. This work has been reported in Technical Note 2250. An investigation of the effects of rotation, presented in Technical Note 2399, showed that the omission of these effects in the previous investigation did not have a significant effect on accuracy within the expected range of validity of hypersonic similarity.

An analysis has been performed to determine the lift and drag characteristics of conical bodies, flat plates, cylinders and spheres in a free-molecule flow field. The calculations were made for a range of Mach numbers from 0 to 24. It was found that the aerodynamic coefficients approached constant values at the very high speeds. It was also found that the lift-drag ratio for bodies in a free-molecule flow field are extremely low. The results are reported in Technical Note 2423.

Theoretical investigations have also been made to determine the similarity law for hypersonic flow about slender three-dimensional shapes in terms of customary aerodynamic parameters. A first investigation, which considered only steady flows, employed the law to determine relation for correlating the pressures acting on, and the aerodynamic coefficients of, related bodies and results are reported in Technical Note 2443. In the special case of inclined bodies of revolution, the expressions for the aerodynamic coefficients were extended to include some significant effects of the viscous cross force.

A theoretical analysis which determines the airfoil profile having minimum pressure drag for a given

structural requirement was reported in Technical Note 2264. The general method was developed and applied to linearized flow theory. The principal results of the investigation were that the optimum airfoil at hypersonic velocities has a trailing-edge thickness slightly less than the maximum airfoil thickness and the optimum airfoil at supersonic velocities generally has a moderately thick trailing edge. Curves were developed which enable optimum airfoil profiles to be determined rapidly for a given Mach number, base pressure, and structural requirement.

Condensation of the components of air has been found to occur in hypersonic wind tunnels. Because of the rapid expansion and cooling of the air in the nozzles of these tunnels, the temperature of the air drops below the liquefaction point. In order to study this condensation process, the Langley Laboratory has made an investigation of optical methods for measuring size and concentration of condensation particles, and it has been found that the polarization and angular distribution of light scattered by condensation particles can be used to measure these quantities. The results of this investigation are presented in Technical Note 2441.

Boundary-Layer Research

The existence of localized regions of laminar-boundary-layer separation on airfoils both behind the position of minimum pressure at the ideal angle of attack and near the leading edge at high angles of attack has been recognized, but the parameters controlling such regions have not been fully understood. Consequently, an experimental investigation was made at the Langley Laboratory to study the localized region of laminar separation behind the point of minimum pressure on an NACA 66₃-018 airfoil section at zero angle of attack and at several Reynolds numbers. The results, presented in Technical Note 2338, confirmed the idea that such separation regions are characterized by a length of laminar boundary layer followed by transition and subsequent reattachment as a turbulent boundary layer and established some of the quantitative characteristics of the phenomenon.

At the Lewis Laboratory an analysis has been made of the stability of the laminar boundary layer between parallel streams of an incompressible fluid. Calculations based on this analysis showed that the flow instability occurs at much lower Reynolds number for the free boundary layer between streams than for the boundary layer on a flat plate with no pressure gradient. The results of this study are presented in Report 979.

An investigation of methods for treating the three-dimensional compressible laminar boundary layer is reported in Technical Note 2279. The equations of mo-

tion are shown therein to be simplified by the introduction of a two-component vector potential and by the use of a transformation which converts the equations into nearly incompressible form. Several examples, including the flow over flat plates with arbitrary leading-edge contours, are discussed.

Numerical solutions of quantities appearing in the von Karman momentum equation for the development of a turbulent boundary layer in plane and in radial compressible flows along thermally insulated surfaces are presented in Technical Note 2337 for a range of Mach number from 0.1 to 10. Through the use of these tables, approximate calculation of boundary-layer growth is reduced to routine arithmetic computation.

Experimental data, from several sources, for skin friction of turbulent boundary layers under adverse pressure gradients are analyzed in Technical Note 2431. Data obtained by momentum-balance methods were compared with data obtained by hot-wire and heat-transfer methods. A new integral energy parameter was introduced and its relation to skin-friction data was demonstrated on the basis of available material.

At the Ames Laboratory an analysis has been made to determine the skin-friction and heat-transfer characteristics of the turbulent boundary layer on a flat plate at supersonic speeds. A review of existing analyses has also been made and a test performed in the Ames heat transfer tunnel to measure the skin friction in the turbulent boundary layer of a flat plate at a Mach number of 2.4. It was found that the analysis which most closely corresponded to the experimental results was an extended version of the Frankl-Voishel analysis, which has been reported in Technical Memorandum 1053. The test data and analysis have been reported in Technical Note 2305.

A series of heat-transfer investigations were conducted on bodies of revolution with laminar boundary layers at Mach numbers from 1.49 to 2.13. In addition, a comparison between theory and experiment has been made for a body of revolution with a laminar boundary layer and a nonuniform surface temperature. The theoretical and experimental heat-transfer results are in good agreement for both heated and cooled bodies with both uniform and nonuniform surface temperatures. The qualitative effects of heat transfer on transition were found to be in agreement with the implications of Lees' boundary-layer-stability theory.

Detailed measurements of shock-wave boundary-layer interaction which have been made in wind tunnels have been limited to small or moderate Reynolds numbers, and previous flight tests at high Reynolds numbers have been limited to pressure measurements. Therefore, tests were made on an airplane wing in flight to

investigate the region of shock-wave interaction with a thick turbulent boundary layer at full-scale flight Reynolds numbers, utilizing both a schlieren apparatus and pressure measurements. Good correlation with theoretical and wind-tunnel investigations of boundary-layer shock-wave interaction was obtained, particularly with respect to the lower Mach numbers at which a forked or bifurcated type of shock wave appeared. The boundary layer did not appear to thicken behind the normal shock wave. Considerable thickening, associated with separation, did occur, however, with increasing Mach number after the formation of the forked shock wave. The density gradient in the boundary layer appeared to increase markedly just behind the shock wave. This stronger gradient, however, appeared to be dissipating at approximately five to six boundary-layer thicknesses behind the shock.

At the Lewis Laboratory supersonic flow against blunt bodies placed in boundary layers or wakes has been investigated and is discussed in Technical Note 2418. It was concluded that wedge-shaped or conical dead-air regions should form ahead of the body if part of the upstream velocity profile is subsonic and if the body is fairly thick relative to the initial boundary layer or wake thickness. A quantitative theoretical analysis of this type of flow was made and the results were compared with experiment.

In a contract investigation carried out at the California Institute of Technology, measurements have been made at Mach numbers from about 1.3 to 1.5 of reflection characteristics and the relative upstream influence of shock waves impinging on a flat surface with both laminar and turbulent boundary layers. The difference between impulse and step waves is discussed and their interaction with the boundary layer were compared in Technical Note 2334. General considerations on the experimental production of shock waves from wedges and cones and examples of reflection of shock waves from these bodies were presented, as were also some examples of reflection of shock waves from supersonic shear layers.

At the National Bureau of Standards an investigation sponsored by the NACA was conducted on a turbulent boundary layer near a smooth surface with pressure gradients sufficient to cause flow separation. The Reynolds number was high, but speeds were entirely within the incompressible flow range. The investigation consisted of measurements of mean flow, three components of turbulence intensity, turbulent shearing stress, and correlations between two fluctuation components at a point between the same component at different points. Results are given in Technical Note 2133 in the form of tables and graphs. The discussion deals

first with separation and then with the more fundamental question of basic concepts of turbulent flow.

Aerodynamic Heating and Heat Transfer

As was pointed out in the Thirty-sixth Annual Report, knowledge of aerodynamic heating and heat transfer phenomena has become of increasing importance as airplane and missile flight speeds have increased into the supersonic region, and during the past year considerable effort has been directed toward the understanding of these phenomena.

Tests have been conducted at the Ames Laboratory to determine the effect of surface heating on the transition Reynolds numbers on a flat plate at a Mach number of 2.4. It was found that surface heating had a strong effect in lowering the minimum Reynolds number for transition. Measured skin-friction coefficients for the laminar portion of the boundary layer were in excellent agreement with other investigations and were 35 percent higher than the values predicted theoretically by Crocco. The data have been reported in Technical Note 2351.

Most aircraft or missiles will not have a constant surface temperature. Consequently, it is necessary to determine the effect of a variable surface temperature on the local heat-transfer rate in order to design skin-cooling systems. An analysis has been made which shows the effect of a variable stream-wise surface temperature on local heat-transfer rates. An interesting side light arising from the investigation shows that the use of plug-type heat meters, in which a small segment of the aircraft skin is heated or cooled to a different temperature than the surrounding skin, may introduce large errors in the determination of the local heat-transfer coefficients. This analysis has been published as Technical Note 2345.

An analytical method is presented in Technical Note 2296 for obtaining turbulent temperature recovery factors for a thermally insulated surface in supersonic flow. The analysis, conducted at the Lewis Laboratory, indicated that the recovery factor decreased with increasing Mach number. For the range of Prandtl number considered (0.65 to 0.75), the recovery factors at a stream Mach number of 10 were, on the average, about 5 percent lower than the limiting values at zero Mach number.

A comparison has been made of free-molecule-flow theory with experimental measurements of drag and temperature-rise characteristics of a transverse circular cylinder. The measured values of the cylinder center-point temperature confirmed the salient point of the heat-transfer analysis, which was the prediction that an insulated cylinder would attain a temperature higher than the stagnation temperature of the stream. Good

agreement was obtained between the theoretical and experimental values for the drag coefficient. The data have been reported in Technical Note 2244.

Experiments have been made at the Ames Laboratory to determine the heat-transfer characteristics of cylinders over a wide range of densities encompassing the free-molecule, slip and continuum flow regimes, and the results are presented in Technical Note 2438. Good correlation of the data was obtained over a range of Reynolds numbers from 0.02 to 100 and a range of Mach numbers from 1.9 to 3.2. The Knudsen number (ratio of mean free molecular path to cylinder diameter) varied from 0.02 to 10. Several important conclusions were drawn from the results of this investigation: It was found that fully developed free-molecule flow occurred when the Knudsen number was 2 or greater. For Knudsen numbers greater than 0.2, the temperature recovery factor exceeded unity even though free-molecular flow was not fully developed.

In an investigation at the Lewis Laboratory, analyses were made for fully developed laminar and turbulent flow in smooth circular tubes of fluids having a Prandtl number of 1.0. The analyses took into consideration the variation of fluid properties. Velocity and temperature profiles, together with local heat-transfer and friction coefficients, were predicted. In order to check the analysis for turbulent flow, velocity and temperature profiles and corresponding local heat-transfer and friction coefficients were also experimentally determined for air in smooth tubes at high heat-transfer rates. The analytical and experimental results were found to be in good agreement. Both indicated that for the turbulent flow of gases the radial velocity and temperature profiles tended to become more uniform as the surface-to-fluid temperature ratio increased. At constant Reynolds number, the local heat-transfer and friction coefficients decreased with an increase in surface-to-fluid temperature ratio. The effects of temperature ratio could be eliminated when the fluid properties, including density in the Reynolds number, were evaluated at a temperature close to the average of the fluid and surface temperatures. The results of these analyses are presented in Technical Notes 2242 and 2410.

Average heat-transfer and friction coefficients have also been measured for the turbulent flow of air in smooth tubes. This investigation covered an over-all range of Reynolds numbers up to 500,000, tube exit Mach numbers up to 1.0, inlet-air temperatures from 535° to 3050° R., average surface temperatures from 535° to 1500° R., length-diameter ratios from 30 to 120, and heat fluxes to 150,000 BTU per hour per square foot. Three tube entrance configurations were studied. Most of the data are for heat addition to the air; some results are for heat extraction from the air. Data have

also been obtained for tubes of noncircular cross section. An effect of surface-to-fluid temperature ratio was found to exist when the average coefficients were correlated with other pertinent variables by conventional methods. The effect was eliminated in the same manner as for the local coefficients. Correlation equations for heat transfer and friction were obtained which adequately represent all the experimental data for smooth tubes.

An analytical investigation of the compressible flow processes occurring in the passages of high heat-flux exchangers is continuing at the Lewis Laboratory. Charts are presented in Technical Notes 2186 and 2328 for convenient determination of the compressible-flow pressure drop sustained by monatomic and diatomic gases (assuming constant heat capacities during the process) flowing at high subsonic speeds in constant-area passages under the simultaneous influence of friction and heat addition.

STABILITY AND CONTROL

Static Stability Investigations

The use of end plates has been suggested as a possible means of relieving some of the adverse lateral stability and control problems experienced with swept wings. An investigation was therefore conducted in the Langley 300-mph 7- by 10-foot tunnel to determine the effects of end plates of various sizes and shapes on the stability and control characteristics of several swept wings. The results of this investigation, presented in Technical Note 2229, indicate that at low lift coefficients the addition of end plates increased the slope of the lift curve, reduced maximum lift-drag ratio, generally decreased maximum lift coefficient, and increased longitudinal stability. It was also found that the variation of effective dihedral with lift coefficient for the wing-end plate combinations investigated was reduced by an increase in the size of the end plate.

A knowledge of the character of the downwash fields behind wings is required for the rational design of horizontal tail surfaces as well as for the analysis of the longitudinal stability characteristics of an airplane. One theoretical study of this problem has resulted in the development of a method for calculating the downwash field behind lifting surfaces at subsonic and supersonic speeds. A description of the method, together with an illustrative example of its use, is contained in Technical Note 2344. In connection with another study, a series of charts and tables was prepared from which the downwash behind horseshoe vortices in incompressible flow can be obtained. The use of these charts and tables in connection with the calculation of the downwash behind wings of arbitrary planform is

described in Technical Note 2353, which also contains illustrative examples. Report 983 describes the use of line vortex theory for the calculation of supersonic downwash and Technical Note 2141 contains charts for the estimation of downwash behind rectangular, trapezoidal, and triangular wings which were prepared on the basis of line vortex theory.

Studies of Dynamic Stability

At the Ames Laboratory the dynamic lateral stability characteristics of a dive-bomber type of airplane were investigated in flight to determine the cause of a small amplitude continuous motion referred to as "snaking." The results of this investigation, reported in Technical Note 2195, indicate that the floating characteristics of the rudder were affected by Mach number and had an appreciable effect on the snaking of the airplane. Rudder modifications resulting in an increase in the restoring tendency of the rudder proved successful in eliminating this undesirable lateral oscillation.

The effects of control centering springs on the apparent spiral stability characteristics of a typical high-wing personal-owner airplane have been investigated in flight. Control centering was provided for both the ailerons and rudder by means of preloaded springs. Results of the investigation, reported in Technical Note 2413, show that the airplane appeared to be spirally unstable with controls free when the centering springs were disengaged because of moments resulting from out-of-trim control position. With centering springs engaged, however, the airplane quickly returned to straight and level flight with the controls freed following an abrupt disturbance. The report also includes information concerning the flying qualities of the airplane in rough air.

At the Langley Laboratory free-flight-tunnel studies were made to determine the effects of mass distribution on the low-speed dynamic lateral stability and control characteristics of a free-flying model. In this investigation the longitudinal and lateral mass distributions of the model were varied, both independently and simultaneously, while the relative density factor was held constant. The results of the investigation, presented in Technical Note 2313, show that increases in the rolling and yawing moments of inertia reduced the controllability of the model by increasing the time required to reach a given angle of bank and caused the flying qualities to become less desirable. The study also showed that, as moment-of-inertia increased, the oscillatory stability generally decreased and flight behavior became progressively less satisfactory.

A study has been made of methods for the evaluation of dynamic stability parameters from flight-test data. The ability to evaluate these parameters by this tech-

nique will make it possible to obtain information on the variation of these factors in the critical transonic range. The results of this study are reported in Technical Note 2340 and mathematical aids for the analysis presented in this paper are reported in Technical Note 2341.

A summary of available methods for estimating subsonic dynamic lateral stability and airplane response characteristics and for estimating the aerodynamic stability derivatives required in these calculations has been made. The results of this summary are presented in Technical Note 2409. This paper presents methods for obtaining time histories of lateral motions, period and damping characteristics, and the lateral stability boundaries, and also provides references to related experimental data. A brief discussion of the evaluation of transonic and supersonic stability parameters is included.

A matrix method has been derived to determine the longitudinal stability coefficients and frequency response characteristics of aircraft from arbitrary maneuvers. This method is presented in Technical Note 2370 and can be applied to time history measurements of such quantities as angle of attack, pitching velocity, load factor, elevator angle, and hinge moment to obtain over-all coefficients. With simple additional computations, it can also be used to determine frequency response characteristics and may be applied to other problems expressed by linear differential equations.

A theoretical investigation was made using LaPlace transformations to determine the effect of nonlinear stability derivatives on airplane lateral stability characteristics, with particular emphasis placed on snaking. The nonlinearities assumed corresponded to the condition where values of the directional stability and damping-in-yaw derivatives are zero for small angles of sideslip. Results of this study are presented in Technical Note 2233 and indicate that under certain conditions the assumed nonlinearities caused a motion which had different rates of damping for large and small amplitudes and little damping at the small amplitudes.

Another theoretical study was made to determine the effects of fuel motion on airplane dynamics. Results of this study are presented in Technical Note 2280. In this report, the general equations of motion for an airplane with spherical fuel tanks are presented. The motion of the fuel is approximated by the motion of solid pendulums. The analysis applies to fuel tanks of any shape if the fuel motion can be represented in terms of undamped harmonic oscillators. The study shows that fuel motion may have an appreciable effect on the dynamic behavior of an airplane.

Analytical and experimental investigations have been made to determine the effects of wing interference on

vertical tail effectiveness at low speeds. These studies have been reported in Technical Notes 2332 and 2175. Comparisons of the estimated and experimental results indicate that the inclusion of a factor to account for wing interference effects provides agreement between the experimental and estimated values of the tail contribution to the rolling derivatives.

The lift and pitching moment produced by angle of attack, steady state pitching, and constant vertical acceleration for a series of thin sweptback streamwise-tip wings of arbitrary sweep and taper have been determined through the use of linearized supersonic flow theory. The method of analysis and the results from the angle of attack and steady state pitching studies are presented in Technical Note 2294 for a range of supersonic Mach numbers that allow the wing leading edge to be subsonic and the trailing edge to be either supersonic or subsonic. The method of analysis and studies of the lift and pitching moment due to constant vertical acceleration are reported in Technical Note 2315 for a range of supersonic Mach numbers for which the wing leading edge is subsonic and the trailing edge supersonic. Design charts are presented in both of these reports that permit rapid estimates of wing lift and pitching moment for given values of aspect ratio, taper ratio, Mach number, and leading-edge sweep.

Automatic Stability Studies

The application of automatic controls to the operation of aircraft complicates the analysis of aircraft dynamics. In order to gain a more thorough knowledge of the factors involved in the automatic control and stabilization of airplanes and missiles, research has been intensified on means of solving current and anticipated problems in this field.

A survey has been made of various techniques used in analysis of the stability and performance of automatically controlled aircraft. This survey deals with methods commonly applied to the linear, continuous control type of system ordinarily used in aircraft, and has been published as Technical Note 2275.

An investigation was carried out to evaluate the suitability of proposed methods for analyzing servomechanism systems incorporating feedback. This study is described in Technical Note 2373 and deals with an extension of frequency response techniques for the analysis of an autopilot-aircraft combination. This paper contains comparisons between experimental data and results obtained by the use of the practical methods of calculation which were developed.

An experimental investigation was conducted to determine the response characteristics of four airplane configurations employing an autopilot sensitive to yawing acceleration. The results of this study are pre

sented in Technical Note 2395 and show that predictions based on the assumed autopilot characteristics did not agree with experimental data because the assumptions did not satisfactorily approximate the frequency response characteristics of the autopilot.

A theoretical method was derived for determining the control gearings and time lags necessary for a specified damping of the lateral motions of an aircraft equipped with an autopilot. The method was applied to a typical airplane equipped with an autopilot which deflects the rudder in proportion to yawing angular acceleration. The results of this study, presented in Technical Note 2307, indicate that the types of motion predicted for this airplane-autopilot system by the derived method are in good agreement with those obtained by a step-by-step method.

Research on Controls

Theoretical and experimental research on the effects of numerous control design variables on the characteristics of controls has been continued during the past year.

In one theoretical study, an analysis was made of the effects of sweep, aspect ratio, taper ratio, and Mach number on the characteristics of inboard trailing-edge flaps at supersonic speeds. This study considers cases where the Mach lines lie behind the leading and trailing edges of the flap. Design charts developed from the analysis and presented in Technical Note 2205 enable a rapid estimation to be made of the characteristics due to deflection for control surfaces for which the Mach lines from the flap tips do not intersect on the control surface.

In another theoretical analysis, supersonic control characteristics were determined through the use of existing conical-flow theory. This analysis applies to a broad range of trailing-edge control configurations having supersonic edges and covers such variables as wing aspect ratio, taper ratio, sweep and control location. The results of this study, presented in Technical Note 2221, are in the form of equations and charts from which lift, pitching-moment, rolling-moment, and hinge-moment coefficients may be determined.

The stability and control characteristics of wings of trapezoidal planform were investigated by means of linearized supersonic theory to determine the characteristics of various wing-flap combinations. Expressions for the stability characteristics were derived by treating the configuration as a complete wing, and the control characteristics were determined by treating it as a rectangular wing with half-delta tip flaps. The results, reported in Technical Note 2336, were compared with corresponding results for several other wing-flap combinations and it was found that, of the wings con-

sidered, a triangular wing with either half-delta tip flaps or trailing-edge controls exhibited the most favorable characteristics.

As part of the general NACA program to investigate the applicability of various types of lateral control devices to wings suitable for high-speed flight, an experimental investigation was conducted in the Langley 300-mph 7- by 10-foot tunnel to determine the effect of aspect ratio on the lateral control characteristics of a series of untapered, unswept, low-aspect-ratio wings equipped with continuous type retractable spoiler ailerons. One swept wing was included in the study for purposes of comparison, and, in addition to being equipped with the continuous type spoiler, was also investigated when equipped with a stepped or segmented spoiler aileron. The results of the study, reported in Technical Note 2347, indicate that for the unswept wings continuous spoiler ailerons became progressively more effective in producing roll as wing aspect ratio was increased. The continuous spoiler aileron was less effective on the swept wing than on a straight wing of comparable aspect ratio. On the swept wing it was found that the effectiveness of the continuous spoiler generally increased as the spoiler was moved progressively inboard, while the opposite was true for the stepped spoiler.

An experimental investigation was conducted in the Langley 300-mph 7- by 10-foot tunnel to determine the lateral control characteristics of an untapered semispan wing having either 0° or 45° of sweep and equipped with a plain unsealed aileron. The span and spanwise location of the aileron were varied in order to determine the optimum configuration for rolling effectiveness. The results of this investigation, reported in Technical Notes 2199 and 2316, indicate that for either the swept or unswept wing, a given partial span aileron was most effective in producing roll and in maintaining effectiveness over a large angle of attack range when located on the outboard portion of the wing. It was also found that, although the rate of change of aileron hinge-moment coefficient with angle of attack could not always be satisfactorily predicted for the swept wing, existing empirical methods for predicting the rate of change of rolling moment coefficient and aileron hinge-moment coefficient with aileron deflection provided satisfactory agreement with the experimental results for both the straight and swept wings.

In order to determine the effect of aspect ratio on the low-speed lateral control characteristics of low-aspect-ratio wings, a series of four unswept, untapered wings with aspect ratios from 1 to 6 was investigated in the Langley 300-mph 7- by 10-foot tunnel. The wings were equipped with plain sealed ailerons of various spans located at several spanwise stations. The results of this study are presented in Technical Note 2348 and

show that the variation of aileron effectiveness with aspect ratio could not be accurately predicted for all aileron spans by any of the theoretical methods utilized in this investigation.

Studies of Damping Derivatives

Theoretical and experimental studies have been extended in the past year to develop methods for the reliable evaluation of damping derivatives in all speed ranges.

One theoretical method, described in Technical Note 2197, was developed for calculating the pressure distribution and damping in pitch at supersonic Mach numbers for thin swept wings having subsonic edges. The method consists of the calculation of a basic pressure distribution and the correction of this distribution to account for the effects of the subsonic trailing edges and tips. These data are then used to determine the damping-in-pitch characteristics.

Technical Note 2285 presents an evaluation of the supersonic damping in roll of cruciform delta wings by means of linearized theory. Both subsonic and supersonic leading edges were considered. The effect of increasing the number of wing panels from four to an arbitrary number under the restriction of low-aspect ratio was determined, and the damping for an infinite number of wing panels was evaluated without restriction as to aspect ratio or Mach number. A similar theoretical study is reported in Technical Note 2270 in which the theory of slender wings was used to determine the characteristics in roll of slender cruciform wings. This analysis shows that, for the cruciform wing, the damping-in-roll is considerably greater than that of a plane wing having the same aspect ratio and that the rolling effectiveness is less than that of a plane wing.

Experimental studies of damping derivatives have been carried out in the rolling and curved flow facilities of the Langley stability tunnel. One investigation was conducted to determine the effects of horizontal tail location on the low-speed static longitudinal stability and damping in pitch of a model with 45° swept wings and tail. The results, published in Technical Note 2381, indicate that at high angles of attack, lowering the horizontal tail increased the static longitudinal stability and decreased the damping in pitch. A related investigation is reported in Technical Note 2382. In this study, horizontal tail size and tail length are variables. The study shows that the contribution of the horizontal tail to static longitudinal stability was directly related to tail size and length, while damping in pitch was related to tail size and the square of the tail length.

Another investigation was conducted in the Langley stability tunnel to determine the effect of vertical tail

area and tail length on the yawing stability of a model with a swept wing. Technical Note 2358 contains the results of this study, which indicate that the effects of wing-fuselage interference for the midwing configuration studied were small over most of the angle-of-attack range. Large interference effects on the vertical tail effectiveness were apparently produced by both the wing and fuselage at moderate and high angles of attack, but because these effects tended to cancel each other, the gross effects were small.

Investigation of Flying Qualities

The increasing operational speeds of current and proposed aircraft have aggravated the problems involved in defining and achieving flying qualities compatible with safety, performance, and piloting requirements. Research to isolate the effects of numerous design parameters on flying qualities has continued to receive attention.

Two design factors which can significantly affect the flying qualities of an airplane are the wing airfoil section and the type of lateral-control device employed. To determine the effects of one of these design variables on the flying qualities of an airplane model, an investigation was conducted in the Langley free-flight tunnel to determine the effects of round and sharp leading-edge airfoil sections on the dynamic lateral stability and control characteristics. Two models were used in the study, one representative of a current fighter airplane with respect to inertia characteristics and the other representative of a possible future design in which mass is concentrated along the fuselage. The results of the investigation, reported in Technical Note 2219, indicate that airfoil section has no apparent effect on the flying qualities of the model with high fuselage inertia. The normal inertia model when equipped with the wing having a round-nose airfoil section exhibited adverse yawing during aileron rolls which increased with decreases in directional stability. This same model, however, exhibited no adverse yawing when fitted with the wing having a sharp-nose airfoil, even at low values of directional stability. This difference in flying characteristics is attributed to the adverse yawing moment due to rolling of the round-nose wing, which on the other model is apparently masked by the high fuselage inertia.

Another investigation was conducted in the Langley free-flight tunnel to compare the dynamic lateral-control characteristics of stepped plug ailerons with those of conventional plain flap ailerons. The model employed in this study had a low-aspect-ratio swept wing equipped with full span flaps. The results of the investigation, published in Technical Note 2247, show that when the lateral stability characteristics were satisfactory the controllability of the model was better with

plug ailerons alone than with conventional ailerons alone because the loss of rolling effectiveness due to the adverse yaw associated with the flap aileron was more objectionable than the lag associated with the plug ailerons. For conditions of low dynamic lateral stability, the conventional ailerons alone provided better controllability than either plug ailerons alone or a combination of flap ailerons and rudder. Although the time lag of the plug ailerons was excessive according to existing flying qualities requirements, the pilot considered the controllability of the model to be satisfactory.

The improvement of passenger comfort in airplanes, through a reduction of accelerations caused by rough air, has become of greater interest as airplane operational speeds have increased. One method which has been suggested for providing increased comfort in high-speed airplanes is to have the wing flaps operated by an angle-of-attack or acceleration-sensing device in such a manner as to reduce accelerations due to gusts. A theoretical study of such a system has been made and is reported in Technical Note 2416. The results of the study show that flaps with characteristics similar to those of conventional landing flaps are unsuitable for acceleration alleviation because of airplane pitching motions resulting from flap deflection. The analysis indicates that the flaps should produce zero pitching moment about the wing aerodynamic center and downwash at the tail in the opposite direction from that normally expected in order to be satisfactory for this application. Various means of achieving these characteristics are suggested, and it is shown that flaps possessing these desirable characteristics would be effective in reducing accelerations in rough air when combined with an angle-of-attack or acceleration-sensing device.

In the previous study no attempt was made to consider the practical problems involved in the design of a mechanism for acceleration alleviation. To help evaluate one of these problems, an experimental investigation was made to determine the ability of a vane mounted ahead of the nose of an airplane to give an indication of the average angle of attack over the entire wing span during flight through rough air. The results of this investigation are presented in Technical Note 2415 and show that the device gives a sufficiently accurate indication of average angle of attack over the wing span to allow its use in an acceleration alleviation system.

Spinning Investigations

The spin and recovery characteristics of airplanes are dependent on many aerodynamic and structural design parameters. In order to provide the designer with information on which of these parameters may be of primary importance to the recovery characteristics of

any particular aircraft configuration, research on the problems of spinning has continued in the Langley 20-foot free-spinning tunnel.

One investigation was conducted to determine the effects of mass and dimensional variations on the spin and recovery characteristics of a model representative of current four-place light airplanes. The results, which are reported in Technical Note 2352, indicate that satisfactory recovery characteristics could be obtained for all of the mass distributions studied provided that the proper sequence of control movements for recovery was followed. It was also found that unless the rudder can be made to float against the spin, recovery by releasing controls may be difficult to achieve unless the elevator floats below neutral. Other requirements for spin recovery set forth in current regulations could probably be met for the model arrangements investigated by restricting center-of-gravity location and providing a high value of tail damping.

Because a previous study had indicated that a horn-balanced rudder might possess the desirable characteristic of floating against the spin, an experimental investigation was undertaken to determine the floating characteristics of full-length plain and horn-balanced rudders during rotary tests at spinning attitudes of a model of a typical low-wing light airplane. The effects of the horizontal tail and wing on the rudder floating characteristics were also determined. The results of this investigation, reported in Technical Note 2359, indicate that for this configuration the rudder was in the wake of the stalled wing and oscillated violently for high spinning angles of attack. At lower angles of attack, or for the case in which the tail was outside the wake of the stalled wing, the horn-balanced rudder had more desirable floating characteristics than a plain rudder, although neither would fulfill the floating deflection requirements for control-free spin recovery.

AIRCRAFT PROPELLERS

In the field of propeller vibration, Technical Note 2308 has been published in continuation of the investigation of first-order propeller vibration on a twin-engine, straight-wing airplane in the Ames 40- by 80-foot tunnel. A procedure has been developed for computing the flow field at the propeller plane to a sufficient degree of accuracy to permit satisfactory calculation of the propeller vibratory stresses. In the procedure, account is taken of the upwash contributions of the wing, fuselage, and nacelles and their mutual interference effects. The analysis shows that the nacelle caused an unexpectedly large effect which, if ignored, could readily cause disagreement between the magnitude of computed and measured vibratory stresses.

In another propeller vibration investigation, the existing analysis applicable to the prediction of first-order exciting forces for the straight-wing propeller case has been extended to include the swept-wing propeller combination. With the aid of the straight-wing analysis, a typical swept-wing propeller-driven airplane design has been studied to examine the magnitude of the once per revolution exciting forces which could exist. It has been found that in addition to the usual exciting forces, a new and relatively important one is added where a swept wing is used; this results from the relative fore and aft movement of the propeller blade with respect to the leading edge of the wing which is swept with respect to the propeller plane of rotation. It was found further that the effects of Mach number are such as to reduce the magnitude of the once-per-revolution exciting forces.

SEAPLANES

Hydrodynamic studies have continued in the Langley tanks to provide basic and design data for the development of water-based airplanes.

One study reported in Technical Note 2297 investigated the use of high angles of dead rise on high-length-beam-ratio flying-boat hulls as a means for reducing water loads encountered during rough-water operation. An increase in angle of dead rise from 20° to 40° increased the take-off stability and substantially improved the spray characteristics of a high-length-beam ratio hull. An expected decrease in take-off performance was evidenced by increases in take-off time and distance of 25 and 30 percent, respectively. The over-all rough-water landing behavior was improved; the maximum vertical and angular accelerations were reduced approximately 55 and 30 percent, respectively. The reduction in vertical acceleration was in good agreement with that predicted by impact theory.

ROTARY WING AIRCRAFT

Based on the results of flight investigations of several single-rotor helicopters, preliminary qualitative requirements for satisfactory flying and handling qualities of helicopters have been established. Progress in designing helicopters to meet these requirements, however, has been handicapped by the need for a method permitting sufficiently accurate prediction of the flying qualities of a helicopter at the design stage. To help fill this need, existing rotor theory, accurate for the calculation of rotor performance and blade motion, has been extended (Technical Note 2309) to permit the prediction of those rotor characteristics that influence the flying qualities. Variation of the longitudinal derivatives of rotor resultant force, rotor pitching mo-

ment, and rotor torque with operating parameters such as rotor angle of attack, collective pitch, forward speed, and rotational speed may be determined. The usual simplifying assumption that the rotor resultant force vector is perpendicular to the rotor tip path plane is shown by the results of this theory to lead in many cases to grossly incorrect longitudinal stability derivatives. The theory also indicates that the increase in rotor load factor with an incremental increase in angle of attack is approximately linear with increasing forward speed. This is in contrast to the airplane where the increase is as the square of the forward velocity.

Increases in the forward speed of helicopters are expected to require increases in rotor tip speeds in order to avoid excessive tip stalling on the retreating blade. At tip speeds within the transonic range the rotor will suffer some performance loss due to compressibility. In order to gain an insight into the magnitude of this compressibility-induced performance loss and to furnish a check on theoretical methods of estimating the loss, two conventional full-scale rotors, one having linear twist and the other untwisted, have been tested to tip speeds up to 770 feet per second on the Langley helicopter test tower. The results of this study, reported in Technical Note 2277, show that both rotors suffered increasing compressibility losses as the tip speed increased to the maximum speed studied. Linear twist delayed the onset of compressibility losses. For the blades investigated good agreement was obtained between the measured and predicted drag-divergence Mach number.

As part of a general investigation of the aerodynamic characteristics of various multirotor configurations, an investigation to determine the static-thrust performance of two full-scale coaxial helicopter rotors has been conducted in the Langley full-scale tunnel. One coaxial rotor was equipped with blades tapered in both planform and thickness and the other with blades tapered in thickness only. The results, presented in Technical Note 2318, show the hovering performance of each rotor in the coaxial configuration and with the upper rotor removed. The effect of application of yaw control on the hovering performance of the coaxial configurations is also presented. A comparison of measured and predicted hovering performance is included.

As a part of the investigation of multirotor configurations a study was made of the air-flow patterns through small scale single, coaxial, and tandem rotor models. The balsa-dust technique of air-flow visualization was employed. The photographic results, presented in Technical Note 2220, provide a qualitative interpretation of the transient and steady-state flow through the rotors.

A theoretical study of the rigid-body oscillations in hovering of helicopter rotor blades has been made by

the Polytechnic Institute of Brooklyn under NACA sponsorship. The study, presented in Technical Note 2226, includes a determination of the rigid-body frequency and damping characteristics of the coupled flapping and lagging oscillations of helicopter blades on which the lagging hinge axis is offset from the flapping hinge axis and both hinges are inclined. The effect of offset of the flapping hinge axis from the axis of rotation of the rotor is also considered.

The analysis and numerical examples indicate that significant increases in the damping of the lagging motions, which ordinarily border on instability, can be obtained by suitable inclinations of the hinge axis, particularly the lagging axis. Offsetting the flapping and lagging hinge axis also increases the natural lagging frequency.

UPPER ATMOSPHERE RESEARCH

The study of the compatibility of the tentative standard atmospheric temperature distribution as set forth

in Technical Note 1200 with the observed free periods of oscillation of the atmosphere has continued. Technical Notes 2209 and 2314 present the results of a continued study of atmospheric tides by the Institute for Advanced Study, under NACA sponsorship. The investigations indicate that certain simplifying assumptions made in the development of the basic ocean tide equations invalidates these relations for application to the study of atmospheric tides. A method for including the previously neglected terms in the tidal equations has been outlined and two atmospheres of different temperature distribution have been studied.

A review by the Subcommittee on the Upper Atmosphere of experimental information on the characteristics of the atmosphere from sea level to a height of 32 kilometers has indicated that the standard atmosphere tentatively proposed in Technical Note 1200 represents to a satisfactory degree of accuracy the characteristics of the atmosphere within this height range.

POWER PLANTS FOR AIRCRAFT

With the ever-increasing trend for aircraft operation at higher altitudes and higher Mach numbers there has arisen a multitude of complex power plant problems. Efficient diffusers at high Mach number, greater air flow handling ability of compressors, increased combustion efficiencies at high altitudes without blowout or instability, increased turbine inlet gas temperatures, reduction of strategic material content, increased thrust augmentation with afterburners, and the over-all engine control and component matching require extensive power plant research. For the purpose of obtaining the most efficient operation of each part of the power plant, these problems have been approached through theoretical and experimental investigations. Power plants, such as the turbojet, the turbopropeller, the ram jet, the rocket, and combinations of these engines utilizing chemical and nuclear fuels are currently under investigation. As a result of this research improved subsonic and supersonic operation of the interceptor, the long-range bomber, and the guided missile can be expected.

NACA efforts in the aircraft-propulsion field have been assisted by the Committee on Power Plants for Aircraft and its seven subcommittees. The following discussion is limited to unclassified research.

AIRCRAFT FUELS RESEARCH

Intensive effort has been applied in the past year to the field of aircraft-fuel research with consideration being given to current aircraft fuels as well as fuels of the future. As might be expected, investigations conducted since the advent of jet propulsion have gradually led toward the development of special fuels for future

power plants. Concurrently with research on future fuels it has been necessary to maintain a high degree of emphasis on current aircraft fuels in order to assure the availability of satisfactory fuels in the event of an emergency. The greatest problem in the current aircraft-fuel program is the formalization of a sound fuel specification that will assure maximum availability and superior performance.

Synthesis and Analysis

Because many of the aircraft considered for the future are volume-limited with respect to fuel storage, hydrocarbon synthesis has been directed primarily toward fuels that will release high energy per unit volume and consequently permit the attainment of desired flight range. Fuels of this type are not readily available even in small quantities required for research studies. For this reason synthesis facilities of the NACA are in continuous operation isolating pure high-density hydrocarbons to be used in engine-performance studies.

During the past year the synthesis and purification of some of the high-density hydrocarbons have been described in Technical Notes 2230 and 2260. Nineteen compounds were isolated and those having satisfactory physical and chemical properties will ultimately be evaluated in typical turbojet-combustor performance investigations. The synthesis projects are planned in an orderly manner to permit the analysis of correlations between molecular structure and properties. By this process considerable time and money is saved in that an examination of the correlation will indicate

which molecular structures have obvious disadvantages, thus obviating costly and tedious synthesis.

In addition to high-density hydrocarbons, other compounds have been synthesized for investigation as possible fuel components to provide greater combustion stability over wide operating ranges and better ignition characteristics. Hydrocarbons such as cyclopropane derivatives are being prepared for study of the effect of molecular structure on rates of flame propagation. Results of the synthesis of ten cyclopropane hydrocarbons have been reported in Technical Notes 2258, 2259, and 2398. The ten hydrocarbons were obtained in high purity for the first time and physical constants and infrared spectra are given in the cited references. Synthesis procedures were developed, and physical properties and heats of combustion were determined for four alkylsilanes. This was the first experimental determination of heats of combustion for these compounds.

In the course of conducting synthesis projects, occasional results of general interest with regard to experimental technique are obtained. When this occurs, the data are published for the information and use of various synthesis laboratories. A study of this nature has recently been reported in Technical Note 2342. In this publication an evaluation of packed distillation columns used by the NACA is described.

Aside from the determination of physical properties of synthesized materials, the NACA analytical laboratories participate in studies to improve methods of analysis of fuels. Particular attention has been given to the analytical procedure for determination of aromatic and olefinic hydrocarbons in wide-boiling petroleum fractions. These hydrocarbon classes are known to be significant in relation to engine performance and therefore the quantities present in a given fuel stock must be accurately known. A procedure for the determination of aromatics and olefins has been developed which is applicable to the determination of aromatics and olefins in petroleum stocks with final boiling points below 600° F. Accuracies of 1 percent are attained with analysis times of less than 8 hours.

Fuels Performance Evaluation

The problem of relating easily measured hydrocarbon fuel properties to engine performance has received considerable attention since the advent of jet propulsion. The object of this research is to permit the prediction of engine performance of fuels from fuel properties that may be determined with relative ease. Combustion properties of interest are spontaneous ignition temperature, flame velocities, and inflammability limits.

Spontaneous ignition temperatures for 109 hydrocarbons and commercial fuels have been determined by a crucible method. From these data trends in the varia-

tion of ignition temperature with molecular structure have been established. The performance of ten of these fuels has been studied in a single tubular combustor.

During the past year flame-velocity studies were made of 37 pure hydrocarbons. The classes of compounds examined were alkanes, alkenes, alkynes, benzene, and cyclohexane. The results of this study indicated that the flame velocities of the normal alkanes were the same except for methane which had a velocity about 16 percent lower than the alkanes of higher molecular weight. The alkenes and the alkynes had velocities somewhat higher than those of the alkanes, particularly in the low molecular weight range. The results of this investigation have been published in the *Journal of the American Chemical Society* (vol. 73, No. 1, January 1951).

The flame velocity investigation was later extended to include the alkadienes. Ten compounds in this class were studied and it was found that the flame velocities were higher than those of the alkanes and approximately equal to the flame velocities of the alkynes investigated. Performance of certain fuels in these classes of compounds are being studied in a single tubular combustor.

The flame-velocity data accumulated in the foregoing investigations have been used as the basis for correlations between molecular structure and flame velocity. One such correlation indicated that the maximum flame velocity is a function of the concentrations of the various types of carbon-hydrogen bonds in the inflammable mixture. From this correlation the maximum flame velocities were calculated for 34 hydrocarbons. The average difference between the calculated and observed flame velocity is less than 2 percent. The results of this analysis were published in the *Journal of the American Chemical Society* (vol. 73, No. 4, April 1951).

As another part of this research an investigation is being conducted to determine the inflammability limits of pure hydrocarbon-air mixtures at different pressures. The limits were determined for 17 pure normal paraffins, branched paraffins, and mono-olefins. It was found that the lean inflammability limits were about the same for all of the fuels, however, the rich limits increased markedly with increased molecular weight.

The tendency of certain fuels to deposit carbon in combustors thereby lowering performance and causing mechanical difficulties continues as one of the more important problems for research. Brief investigations have been made of several fuels which contain components allowable in the jet fuel specification but which appear marginal from a carbon deposition standpoint. A study of these fuels with several current turbojet engines is intended to show how fuel specification and combustor design may be compromised for best performance and maximum fuel availability.

COMBUSTION RESEARCH

Because aircraft engines must produce enormous power from a small package, it is axiomatic that the combustion problem is to release tremendous quantities of heat energy from the fuel in a small volume and in a short space of time. NACA research on combustion is directed at obtaining an understanding of the basic physics and chemistry behind the phenomena involved in combustion and demonstrating how these phenomena can be put to use to meet all the requirements of flight. Basic studies are made on detailed problems of combustion such as evaporation, ignition, flame propagation limits, flame velocity, flame quenching, with variables such as composition, pressure, temperature, velocity, and turbulence. Studies on applying these data involve (a) systematic studies of the effect of operating variable and design variables on engine performance such as combustion efficiency, altitude operational limits, smoke and carbon formation, pressure drop and temperature profile, and (b) attempts to correlate basic combustion laws with engine performance.

Fundamentals of Combustion

To gain an insight into the problem of combustion stability at high altitudes, a study has been made of the chemical and physical factors affecting the inflammability limits at different pressures. The results of the investigation of chemical factors were discussed under the fuels research program. An investigation was also made of the effects of walls on inflammability limits. The results indicated that the minimum pressure for flame propagation increased with diminishing tube diameter (greater relative wall area). In connection with the inflammability limits of fuels, an investigation was conducted to determine the combustion efficiencies of hydrocarbon-air systems at reduced pressures. With quiescent fuel-air mixtures and with small diffusion flames, combustion efficiencies close to 100 percent were obtained at pressures much lower than those found in turbojet combustors at high altitude; in general, efficiencies were high at pressures approaching the limiting values for inflammation.

Several projects are in progress to ascertain the basic mechanism of flame propagation in order to gain insight into the actual combustion process that occurs in an engine. These studies supplement the studies of the effect of hydrocarbon structure on flame velocity reported in the fuels program. In one phase of this work, the effect of initial mixture temperature on flame velocity was evaluated. The first of these studies (Technical Note 2170) evaluated the flame velocities and blow-off limits of propane-air flames. These studies

were later extended to include methane-air and ethylene-air flames (Technical Note 2374). As a result of these projects, equations were developed to permit the estimation of flame velocity at various initial mixture temperatures in the range of 34° to 344° C.

In order to achieve a better understanding of the cause of flame propagation, several studies have been made of the effect of free radicals on flame propagation. Inasmuch as hydrogen diffuses more readily than other radicals in a flame, it has been suggested by past investigators that the rate of flame propagation is related to the hydrogen-atom concentration in the flame zone. An evaluation of this theory was made by adding light water and heavy water to mixtures of carbon monoxide and air and measuring the flame velocities. It was reasoned that hydrogen from heavy water would diffuse less rapidly than that from light water; consequently, a lower flame velocity would result. Measurements of the flame velocities substantiated this reasoning.

In an analysis of flame-velocity data for 35 pure hydrocarbons, it was concluded that the maximum flame velocities were consistent with the active radical theory of flame propagation. Only one pure hydrocarbon, ethylene, was found to be inconsistent with the theory. This analysis was published in the Journal of the American Chemical Society (vol. 73, No. 1, January 1951).

In connection with flame-velocity investigations, a method for determining the distribution of luminous emitters in a Bunsen flame was found and is described in Technical Note 2246. As an example of the application of this method, an intensity distribution in a Bunsen cone image resulting from the free radical C_2 radiation was analyzed.

In order to correlate the flame-propagation velocities of a burner with the local turbulence intensity in the flow, quantitative turbulence data are needed. Such data have been obtained from an investigation, the results of which are presented in Technical Note 2361. Data were obtained for turbulent-velocity-fluctuation components and mean-velocity distributions in a subsonic jet issuing from a pipe in which fully developed turbulent flow was established. It was found that at the jet exit the axial-fluctuation-velocity component was about 2.5 times as great as the radial component at the pipe center line. Near the pipe wall, the axial-fluctuation-velocity component was about three or four times as great as the radial component. The axial components of the turbulent fluctuation velocities diminished rapidly downstream of the pipe wall, whereas the radial components were practically constant so that the axial and radial components approached a condition of equal magnitude with increasing distance from the pipe exit

LUBRICATION AND WEAR

Fundamentals of Friction and Wear

The current trend in the selection of aircraft-power-plant lubricants is to materials of low viscosity. The primary reason for this is to meet low temperature operating and starting requirements. It may be expected that such usage will lead to difficulties because of the low load capacities obtained with the light oils. One means of compensating for the reduced load capacity is by the use of EP (extreme pressure) additives. One of the objectives of lubrication and wear research is to determine for surfaces lubricated with EP additives whether a change in chemical reactivity could appreciably affect the rate of production of a protective film and thus appreciably affect the critical sliding velocity at which surface welding begins. Among the additives under study are benzyl chloride $C_6H_5CH_2Cl$, p-dichlorobenzene $C_6H_4Cl_2$, free sulfur S, benzyl disulfide $(C_6H_5CH_2)_2S_2$, and phenyl monosulfide $(C_6H_5)_2S$, at sliding velocities from 75 to 7,000 feet per minute and initial Hertz surface stresses up to 194,000 p. s. i. (Technical Note 2144). Results indicate that higher critical sliding velocities may be obtained with additives whose active atoms have the greatest chemical reactivity. The study showed that the factor of activity of the individual active atoms was more important than the number of such atoms available for reaction.

A further study of the effect of additives on critical sliding velocities involved the use of fatty acids (Technical Note 2366). This investigation was conducted because it has been found that with an EP additive lubrication may be ineffective at high sliding velocities because a chemical reaction between the surfaces and the additive to provide a lubricating film may not have time to occur. Further, studies at low temperatures have established the point that with the fatty-acid additive, high contact temperatures or pressures are unnecessary for the formation of an effective lubricating film. Studies have been completed of stearic acid as an additive in cetane at sliding velocities up to 7,000 feet per minute on clean steel surfaces and surfaces coated with ferric oxide Fe_2O_3 (Technical Note 2366). The experiments indicate that the type of surface oxide and the thickness of the oxide film are important in determining the effectiveness of stearic acid as an additive.

Most of the bearings employed in the turbine-type aircraft power plants are rolling-contact bearings. Early service experience indicated that one of the principal sources of failure in bearings has been lubrication failures at the cage locating surfaces (retainer or separator). These failures are due in main to the difficulty in obtaining proper lubrication because of design con-

figurations and to "high-temperature soaking" (above 500° F.) of the turbine bearing after engine shut-down. One means of reducing the severity of the problem is to obtain a cage material having a lesser tendency toward metallic adhesion to steel under marginal conditions of lubrication than the currently used materials. Sliding friction experiments have been made to obtain fundamental comparative information on friction and wear properties which are a general measure of adhesion (Technical Note 2384). The materials studied were brass, bronze, beryllium copper, Monel, Nichrome V, 24S-T aluminum, nodular iron, and gray cast iron at sliding velocities up to 18,000 feet per minute. On the basis of wear and resistance to welding only, the cast irons were the most promising materials investigated in that they showed the least wear and the least tendency to surface failure when run dry. On the basis of mechanical properties, the nodular iron is superior to gray cast iron. Results obtained with brass, beryllium copper, and aluminum indicate that these materials are not particularly suited for cages.

It has been observed that the running of piston rings, particularly nitrided cylinder barrels, induces the formation on the ring of a surface coating which had different and, in some respects, superior properties compared with those of the bulk ring material. An investigation has been completed, whose objective was to determine to what extent these rubbing conditions could have produced transfer of metal from the barrel to the ring (Technical Note 2271). The study concerned itself with metal transfer between nitrided steel of several hardnesses and also between chromium plate and nitrided steel. The technique employed consisted of making one of two rubbing surfaces radioactive, carrying out a friction test and then examining the other surface for signs of radioactivity. The effects on material transfer of, load, speed, distance of travel, repeated travel over the same path, hardness of the moving surface, and the type of chromium plate were investigated. In the case of all materials studied, there was an observable amount of transfer. The amount of transfer was roughly proportional to the distance traveled and was independent of whether this travel was repeated a number of times over the same track or was continuously over fresh surface. In addition, the amount of transfer for a given distance of travel was constant over a range of low speeds, but started to decrease at higher speeds. This indicates that a possible pretreatment to obtain a desirable surface layer might consist of running rings in a special cylinder having walls of selected composition and controlled hardness to give surface coatings highly improved characteristics in a minimum length of time.

Fretting

Fretting is defined as the surface failure that may occur when closely fitting metal surfaces undergo slight relative motion. Recent research conducted by the NACA indicates that fretting is the result of localized or very concentrated friction phenomena. Adhesion is believed to be the primary friction phenomenon involved in that it causes the removal of finely divided oxidizable metal. Other phenomena, such as the rubbing off of oxide films, welding from frictional heat, and abrasion, undoubtedly contribute to surface destruction. The effectiveness of the intermetallic compound molybdenum disulfide (MoS_2) as a fretting inhibitor was studied, utilizing a steel ball, vibrated in contact with glass flats at 120 cycles per second, an amplitude of 0.001 inch, and a normal load of 0.2 pound (Technical Note 2180). A coating of dry molybdenum disulfide bonded to steel by rubbing a mixture of molybdenum disulfide and syrup in intimate contact with clean steel at elevated temperatures proved the most effective. This coating delayed the onset of fretting to 28,000,000 cycles in contrast to less than 30 cycles for a clean uncoated steel.

Bearing Research

A dependable bearing to carry a radial load at extreme speeds and at high ambient temperatures is desired for use as the turbine-support bearing in aircraft gas turbines where the gravity loads are usually under 1,000 pounds and the DN (diameter in millimeters multiplied by revolutions per minute) values are as high as 1×10^6 . To develop such a bearing it is necessary to know the operating characteristics and limitations of conventional rolling-contact bearings at high speeds and how these characteristics and limitations may be improved and extended by such means as improved lubrication methods and design modifications. A study has been completed utilizing three types of bearings to determine experimentally the operating characteristics of conventional cylindrical roller bearings at high speeds at DN values from 0.3×10^6 to 1.65×10^6 and static radial loads from 7 to 1,613 pounds with circulatory oil feed (Technical Note 2128). The operating temperatures of the bearings were found to differ most appreciably in the low-load high-speed range where the roller-riding cage-type bearing exhibited significantly lower operating temperatures than the one- and two-piece inner-race-riding cage-type bearings. However, the operation of the roller-riding bearings was considerably rougher than that of the inner-race-riding bearings and they showed prohibitive cage and roller wear.

A continuation of the study outlined above was undertaken to compare and to evaluate the effects of oil-

inlet location, jet angle, and jet velocity on outer- and inner-race bearing temperatures for single-jet lubrication over a wide range of variables (Technical Note 2216). The effect of oil flows of 0.6 to 12.9 pounds per minute and oil inlet velocities of 13 to 200 feet per second were studied utilizing a one-piece inner-race-riding brass cage bearing over a load range of 113 to 368 pounds and DN values of 0.3×10^6 to 1.43×10^6 . For the conditions studied, the inner- and outer-race temperatures were found to be at a minimum when the oil was directed at the cage-locating surface perpendicular to the bearing face. A mathematical correlation based on heat transfer considerations was obtained such that for a given operating condition a representative single straight line resulted regardless of whether the bearing speed, oil flow, or jet diameter was varied.

COMPRESSOR AND TURBINE RESEARCH

Compressor Research

Both high-speed and long-range aircraft utilizing gas-turbine power plants require light compact compressors with large air-flow capacities and high pressure ratios and efficiencies. Compressor research, therefore, is being directed toward solving aerodynamic problems of compressing air efficiently in a compressor of minimum frontal area, minimum length, and minimum complexity, so that the unit will be mechanically sturdy and easily manufactured. A reliable engine having comparatively low initial cost can thus be obtained. Compressor research consists of theoretical work on compressor aerodynamics, guide vane studies, cascade investigations, and single-stage and multistage compressor investigations.

The theoretical approach to the aerodynamics of compressors has been concentrated on obtaining a working knowledge of the flow phenomena associated with the compressor and its operating conditions. As an approach to the solution of the three-dimensional flow through compressors, a general through-flow theory corresponding to axially symmetric flow has been developed (Technical Note 2302). The theory is applicable to both *direct* and *inverse* problems and is derived primarily for use in machines having thin blades of high solidity.

In order to approximate the circumferential variation of the flow conditions, which is neglected in the through-flow solution, several different approaches have been used. When the radial component of flow is negligible, the flow follows along cylindrical surfaces and may be analyzed in terms of the equivalent two-dimensional cascade. The method developed for designing cascade blades (Technical Note 2281) with prescribed velocity distribution for subsonic compressible flow based on a linear pressure-volume relation has

been extended, and expressions for estimating the accuracy of solutions have been developed.

For cases where the radial component of velocity is not negligible, a better approximation to the actual flow is obtained by assuming the flow to take place along the surfaces of revolution, as obtained from the axially symmetric solution, and then determining the flow between the blades along these surfaces of revolution. Both the direct and inverse problem for subsonic flow have been analyzed (Technical Note 2407).

A rapid but approximate blade-element-design method (Technical Note 2408) for compressible or incompressible nonviscous flow in high-solidity stators or rotors of axial-, radial-, or mixed-flow compressors has been developed. The method is based on channel-type flow between blade elements on a specified surface of revolution that lies between the hub and shroud and is concentric with the axis of the compressor. The blade element is designed for prescribed velocities along the blade-element profile.

The characteristic equations for the axially symmetric flow in supersonic impellers have been developed and used to investigate flows in several configurations in order to ascertain the effect of variations of the boundary conditions on the internal flow and work input (Technical Note 2388).

A theoretical discussion of the application of blade boundary-layer control to increase the efficiency and the stage pressure ratio and to improve off-design performance of turbomachines is given in Technical Note 2371. Also presented is a method based on potential flow of a compressible fluid for designing suction, or ejection slotted, blades having a prescribed velocity distribution along the blade and in the slot.

The boundary-layer profiles on the casing of an axial-flow compressor were measured behind the guide vanes and behind the rotor. Using these measurements and three-dimensional boundary-layer momentum-integral equations which were developed (Technical Note 2310), a qualitative consideration of boundary-layer behavior on the walls of an axial-flow compressor was made. This consideration shows that the important parameters concerning the secondary flows in the boundary layers are the turning of the flow and the product of the curvature of the streamline outside the boundary layer by the boundary-layer thickness.

Turbine Research

The problems associated with the turbine component of the turbojet engine are similar to those of the compressor when good performance is required for application to transonic and supersonic aircraft. The problems in obtaining light, efficient, and dependable turbine components for driving compressors are in general

the result of compromising the turbine aerodynamics with mechanical considerations to obtain reasonable stresses. The turbine research is therefore being aimed at improving the turbine performance. Turbine cooling is also commanding considerable attention as a means of improving the compromise between the requirements of stress and aerodynamics. Turbine research consists of theoretical turbine aerodynamics in which the theories and tools developed for compressors are readily applicable, cascade investigations, and single-stage and multistage turbine research.

In order to facilitate the determination of the best design compromises in the design of turbines, a method was developed for the computation and graphical presentation of a series of possible turbine designs for any specific application. Design charts are made to aid in the study of the effects of turbine radius ratio and diameter on important design parameters such as Mach number, turning angle, and blade root stress. The method for constructing these charts is presented in Technical Note 2402.

Non-twisted-type rotor blades are desirable where internal passages in the blades are required for turbine cooling. An analytical evaluation has been made of the aerodynamic characteristics of turbines with non-twisted rotor blades by comparison with the corresponding characteristics of free-vortex turbines. The results of this investigation, including working charts for aid in designing non-twisted-rotor-blade turbines, are presented in Technical Note 2365.

Turbine Cooling

The research program for the application of cooling to gas turbines has two objectives. The first is the elimination of critical metals from the turbine rotor of gas-turbine engines so that alloy steels of higher strength and lower critical material content can be used while extending the useful life of turbine blades. The second objective is to provide the means of operating gas turbines at higher gas temperatures to achieve the large potential gains in thrust of the turbojet engine, and the increase in power and the lower specific fuel consumption inherent in the turbine-propeller engine. Past analyses and cascade investigations show that the blade-temperature reduction required to permit substitution of noncritical metals can be achieved with acceptable coolant flow.

Solutions for a system of generalized equations for the laminar boundary layer with heat transfer have been developed for the determination of local heat-transfer coefficients. Numerical solutions have been obtained for low Mach numbers over a wide range of coolant flow, temperature ratio, and pressure gradient with allowance for fluid property variations. Solutions to the equa-

tions for the high Mach number range are now being effected through the use of automatic computing machines (Technical Note 2207).

Analytical methods for computing one-dimensional spanwise or chordwise temperature distributions in liquid-cooled turbine blades or in simplified shapes used to approximate sections of liquid-cooled turbine blades have been summarized (Technical Note 2321).

Blade Vibration and Flutter

An experimental spin rig was used to determine the effect of root design on the damping of loosely mounted blades. The conventional ball-and-dovetail roots were shown to have the same characteristics in that tightening occurred at relatively low values of centrifugal force, thus eliminating the advantages of the loose fit. The fir-tree and the hinge designs, however, had different characteristics, the fir-tree tightening at the relatively high centrifugal loadings and the hinge root remaining loose throughout the speed range investigated.

Methods have been developed for computing the natural modes and frequencies of arbitrarily shaped twisted cantilever beams. Use has been made of the concept of station functions. An analysis is being made by this method to compute the natural modes and frequencies of a group of different compressor and turbine blades. The computed values will be compared with experimental values in order to evaluate the analytical method (Technical Note 2300).

ENGINE PERFORMANCE AND OPERATION

Performance and Operating Characteristics

The performance of turbojet, turbopropeller, and ram-jet engines has been investigated over a range of altitudes and flight Mach numbers. In addition to obtaining altitude performance characteristics, the effects of Reynolds number, burner blow-out limits, altitude starting characteristics, and component performance were studied. As a result of these investigations basic design changes were recommended which result in improved power plant performance for future engines. Various principles of thrust augmentation have been evaluated experimentally under a variety of engine operating conditions.

An analysis was made to obtain an expression for the optimum jet-pressure ratio for any turbine-propeller engine. The results of this analysis, reported in Technical Note 2178, are presented in the form of charts from which the jet-pressure ratio for the division of power giving maximum thrust may be obtained for engines either with or without intercooling, reheat, regeneration, or any combination of these modifications.

Analytical evaluations of different methods of power extraction from an axial-flow-type turbojet engine have

been completed. These results have been presented in terms of generalized parameters that facilitate their application to different engines. The data cover the range of power extractions available from the engines on a range considered ample for all auxiliary-power requirements. The power-extraction methods are compressor-outlet air bleed, shaft-power extraction, turbine-inlet or tail-pipe hot-gas bleed, and the use of a tail-pipe heat exchanger on a turbojet engine as an auxiliary power (or energy) source. The first three parts of this investigation are reported in Technical Notes 2166, 2202, and 2304.

Engine Controls

A variety of engine control loops have been evaluated experimentally to determine effects of interaction and stability on engine operation. Several theoretical studies were conducted to determine optimum control arrangements for various engine types.

An analysis was made of the effect of independent variations in the component efficiency characteristics, flight conditions, and engine size on the time constant and the turbine-inlet-temperature overshoot of a turbojet engine with a centrifugal compressor and a turbine with choked stator. The dynamic factors were calculated from the thermodynamic equations of engine-component performance. This investigation is reported in Technical Note 2182.

The general form of transfer functions for a turbojet engine with tailpipe burning was developed and the relations among the variables in these functions were found from the transfer functions and from engine thermodynamics (Technical Note 2183). By means of these relations, the dynamic characteristics of the engine can be found from steady-state data and one transient relation.

As reported in Technical Note 2378, a rational analytic method for the design of automatic control systems was developed that starts from certain arbitrary criteria on the transient behavior of the system and derives those physically realizable characteristics of the controllers that satisfy the original criteria.

Accurate knowledge of the dynamic response characteristics of turbine-propeller engines and the factors that affect these characteristics are of great importance in the design of quick-acting, stable controls for this type engine. An investigation of the dynamics of a turbine-propeller engine was made at sea level (Technical Note 2184) and in the altitude wind tunnel employing the frequency-response technique for a range of altitudes.

Directing research effort toward the improvement of the turbine-propeller engine requires an appreciation of the relative importance of the various engine-component characteristics. An analysis is being made to

determine the influence of interdependent characteristics, such as component size, efficiency, and flow capacity, on the performance of the turbine-propeller engine. Results are evaluated on an airplane-performance basis so as to reveal the ultimate effects of the variables under investigation.

Thrust Augmentation

A theoretical analysis of thrust augmentation of turbine-propeller engines is in progress. Methods of augmentation considered are tail-pipe burning, compressor-inlet water injection, and a combination of water-injection and tail-pipe burning. Augmented performance with tail-pipe burning has been determined for high subsonic and for supersonic speeds. Calculations are in progress on the augmented performance with water injection over a range of flight conditions from take-off to transonic flight at altitude, and on the performance at high speeds with a combination of water injection and tail-pipe burning.

An analysis is being conducted for the prediction of the jet flows, pumping characteristics, and jet thrust obtainable from conical ejectors in order to determine the rules for the design of such ejectors for maximum effectiveness. A one-dimensional analysis of conical-air-ejector performance was made using an assumed pressure gradient along the conical secondary shroud.

ENGINE MATERIALS RESEARCH

High Temperature Materials

As a part of the search for materials having physical properties suitable for use at higher and higher operating temperatures, considerable effort is being expended in trying to improve the physical properties of the current materials. Factors which influence the high temperature strength of alloys are melting practice, forging, or casting procedures, and the heat treatments selected. To gain a better understanding how heat treatment affects the structures of alloys and the resulting physical properties, studies have been conducted of the fundamental factors which control the properties of austenitic alloys having exceptionally high creep and rupture strengths in the temperature range of 1,200° F. and 1,500° F. Such a study has been completed on Inconel X (Technical Note 2385). Correlations of the mechanical properties with structural analyses were made. It appeared from the study that aging resulted in some improvement in rupture strength in the range from 100–1,000 hours rupture time at 1,200° F. This improvement was obtained apparently through increasing the resistance to creep prior to fracture. This is in contrast to results obtained with N-155 wherein the improvement appeared to be through the formation of a grain boundary phase.

As a part of the study of the conservation of strategic materials for aircraft engines, efforts are being directed toward the evaluation of substitute materials of reduced alloy content. Among the materials studied for sheet applications is AISI 310B (Technical Note 2162). One of the draw-backs of this material has been low ductility in the temperature range from 1,200° F. to 1,400° F. The objective of this research was to determine whether service at 1,700° F. to 1,800° F. would cause increased brittleness at 1,200° F. to 1,400° F. with resultant cracking during heating and cooling. Stress rupture tests were made in the 1,200° F. to 1,400° F. range with a few tests at 1,700° F. and 1,800° F. Among the variables considered were heat-to-heat reproducibility, the relative effects of annealing, and hot and cold working as initial treatments. It was found that the elongation at 1,300° F. was increased by prior heating from 1,700° F. to 2,000° F. In addition, it was determined that the cold-rolled stock had the lowest ductility.

Inasmuch as several of the important high-temperature alloys utilize both chromium and cobalt and because a number of the physical properties of an alloy are controlled by diffusion processes within the alloy in the solid state, an investigation was conducted to determine the diffusion coefficients of chromium in the alpha cobalt-chromium solid solutions (Technical Note 2218). The method employed consisted of pressure-welding cobalt and cobalt-chromium bars, annealing these joined bars at constant temperatures and determining the distribution of chromium through the diffusion zones thus formed. It was found that the diffusion coefficients at 1,000° C., 1,150° C., 1,300° C., and 1,360° C. were relatively constant. It was further found that chromium diffusivity from alpha cobalt-chromium alloys into cobalt is greater than chromium diffusivity from high-chromium alpha alloys to low-chromium alpha alloys for all concentration gradients studied.

Studies of the properties of molybdenum at high temperatures have been continued. The effects of swaging, recrystallization, and "test-section" area on the strength properties of sintered wrought molybdenum were determined at temperatures from 1,800° F. to 2,400° F. (Technical Note 2319). The amount of swaging on the specific bar sizes investigated had little effect on the tensile strengths, but increasing amounts of swaging progressively lowered the recrystallization temperature. Increasing the test-section area of a specimen had a negligible effect on the tensile strength. Wrought molybdenum had a 100-hour stress-rupture life of approximately $19,300 \pm 300$ p. s. i. at 1,800° F.

Studies of ceramic coatings for the protection of ceramals and less strategic alloys from high temperature oxidation and corrosion have led to the develop-

ment of a ceramic coating with a high chromium metal content. Evaluations of these coatings on a TiC base ceramal have demonstrated the ability of the coating to protect against oxidation as well as its ability to withstand small elongations and severe thermal shock (Technical Notes 2329 and 2386). It was found that in general the best protection was obtained in oxidation tests at the higher temperatures. Studies of the effects of firing time, firing temperature, and the number of coatings applied indicated that there is an optimum firing time and firing temperature. For the coating studied, two applications proved superior to one.

Studies of the effectiveness of ceramic coatings in preventing high temperature corrosion by lead-bromide vapors indicate that the ceramic coatings appeared to be inert to the PbBr_2 and successfully inhibited corrosion of the alloys studied for a 6-hour test period (Technical Note 2380). The alloys S-816, HS-21, Inconel, 19-9DL, and AISI 347 were tested (*a*) in an uncoated condition, (*b*) in a preoxidized condition, and (*c*) in a coated condition when exposed to the lead-bromide fumes in an air atmosphere at 1,350° F., 1,500° F., and 1,650° F.

As a part of the general program for the evaluation of ceramals, investigations have been completed of the bonding of TiC with Al, Be, Cu, Au, Fe, Pb, Mg, Mn, Pt, Ti, and V (Technical Note 2187). The test procedure consisted of placing powdered metal in a cup-shaped depression formed in a hot-pressed TiC test bar, inserting the specimen in a furnace at temperatures up to 3,650° F. Studies were made of the sectioned specimens to determine what kind of bonding, if any, took place. Of the elements studied, only nickel, cobalt, chromium, and silicon bonded with titanium carbide. There was some indication of a limited solubility of titanium carbide in nickel and cobalt.

A continuation of the study outlined above was conducted, utilizing zirconium carbide and columbium to determine the bonding mechanism (Technical Note 2198). The results indicated that the mechanism is one in which columbium atoms diffuse into the zirconium-carbide lattice, displace zirconium atoms, and form zirconium metal and a completely soluble columbium carbide. This results in a homogenous solid solution of the zirconium-carbide matrix and the columbium carbide. The size and the distribution of the metal phase was controlled by sintering time and temperature. In general, the specimen with a fine dispersion of metal had the highest strength.

Stresses Research

The failure of turbine and compressor blades due to vibrations has led to an increased interest in the de-

termination of the natural modes and frequencies. Assuming a compressor or turbine blade acts as a cantilever beam, a method has been developed for calculating the coupled modes (Report 1005). The method is based on the concept of Station Functions and permits the calculation of the modes and the frequencies of nonuniform cantilever beams vibrating in torsion, bending, and coupled bending-torsion motion. Results of the application of this method show that the effect of coupling between bending and torsion is to reduce the first natural frequency to a value below that which it would have if there were no coupling.

In the design of a turbine rotor it is desirable to know the detailed stress and strain distributions in the strain-hardening range and the amount of increase in load that can be sustained between the onset of yielding and failure. A simple method of solving the plane-stress problems with axial symmetry, employing the finite strain concept in the strain-hardening range, and based on the deformation theory of Hencky and Nadai, has been derived for the condition that the directions and ratios of the principal stresses remain constant during loading (Technical Note 2217). The results indicated that the ratios of the principal stresses remain essentially constant during loading and, therefore, the deformation theory is applicable to this type of problem. In general, it may be said that the deformation that can be accepted by the member before failure depends primarily on the maximum octahedral shear strain of the material.

In the design of high-stressed machine parts, a knowledge of the stress and strain concentration due to a hole and also of the distributions of stresses and strains in the strain-hardening range is desirable. As a continuation of the work reported above, a linearized solution has been obtained for the problem of plastic deformation of a thin plate with a circular hole in the strain-hardening range (Technical Note 2301). This solution is based on the deformation theory of plasticity for finite strains. The results indicate that the data obtained by the linearized method compare closely with those obtained without linearization. In addition, it was found that the solution for an ideally plastic material with the infinitesimal strain concept gives good approximate values of strains but not of stresses.

A further study of the distribution of stresses and strains in the strain-hardening range has led to a partly linearized solution of the plastic deformation of a rotating disk considering finite strains (Technical Note 2367). The results indicate that the variation of a parameter, which is determined from the octahedral shear stress-strain curve of the material, can be used as a gen-

eral criterion of the applicability of the deformation theory. In addition, it appeared that the rotating speed of a disk for a given maximum strain of the disk can be determined directly from the true tensile stress-strain curve of the material.

A continuation of the investigation reported in Report 1005 has been completed to determine the effect of twist on vibrations of cantilever beams (Technical Note 2300). It is shown that for a beam with a ratio of bending stiffness in the two principal directions equal to 144, the effect of coupling due to twisting is to raise the value of the first natural frequency by a very small amount, to decrease steadily the second frequency, and to lower the third frequency considerably.

A significant question to be considered in the specification of chemical composition and mechanical and thermal treatment of a rotor disk is the compromise between ductility and tensile strength. An investigation was conducted to determine the strength-reducing effects of several types of irregularities, various ductilities, and an optimum compromise between ductility and the tensile strength in the presence of such defects (Technical Note 2397). The strength of disks containing irregularities increased with increasing tensile strength independently of ductility at ductilities in excess of 14.9 percent elongation. The best compromise between ductility and tensile strength in material containing shrink porosity occurred at 6.8 percent elongation.

The application of X-ray diffraction techniques to a study of stress analysis indicates that the back reflection techniques will yield a reasonable strain accuracy under favorable conditions. Because the conventional experimental methods reduce the precision in determining interatomic spacing and restrict the analysis to those materials which yield a reasonably sharp diffraction pattern, a new technique and camera have been devised (Technical Note 2224). A calibration of the camera, using a gold powder standard having a reported atomic spacing of 0.91008A, yielded an accuracy of

atomic spacing of approximately $\pm 4 \times 10^{-5}$ A. A comparison of the new technique with the results obtained from the conventional equipment and techniques indicated that a more precise and detailed analysis of atomic strain could be obtained.

ROCKET RESEARCH

The major problems of rocket research are to select fuel and oxidant combinations that give large thrust per unit weight flow and volume flow and still have properties that permit their handling and use, to prevent failure of the engine from high temperatures attained, to achieve efficient combustion in the smallest possible unit, and to evaluate the optimum application of rocket propulsion.

Rocket Propellants

Investigations of rocket-propellant performance are usually preceded by theoretical analyses of the performance of various fuel-oxidant combinations. These analyses are used as a basis for selecting the propellants worthy of experimental evaluation. Compilation of the thermodynamic functions to be used in these analyses has been continued.

An analysis was made to show the relative importance of specific impulse and propellant density on missile performance. The analysis compares propellant combinations having different specific impulses and densities by determining the least gross weight of a missile to accomplish a given mission.

Rocket Combustion

A basic study of the hydraulic characteristics of injection systems was started as an aid in establishing the relation between injector method and combustion characteristics. Initial studies of two impinging jets of water revealed that the spray disintegrates into groups of drops which for the conditions existing in a rocket correspond to frequencies observed in combustion fluctuations. These results are reported in Technical Note 2349.

AIRCRAFT CONSTRUCTION

The NACA is continuing its efforts to enlarge the scope and the amount of its research on airframe construction problems. The need for such an increase in effort has been previously pointed out and the manner in which it is to be accomplished is under study.

Programs during the past year have included extensive laboratory and flight research and the continued collection of statistical data on gust loads encountered in regular airline operation. All airframe construction problems have been complicated by the increasing altitude and speeds of flight and these two factors can be

found at the root of most new structural design problems.

As in the past, a considerable amount of the NACA research on structural materials and structures was performed under contract at universities and other non-profit scientific organizations.

In accordance with the policy of holding technical conferences with representatives of the military services and the aircraft industry, a conference on aircraft loads and structures was held at the Langley Laboratory in the spring of 1951.

A description of the Committee's recent unclassified research on airframe construction is given in the following pages and is divided in four sections: Aircraft loads, structures, vibration and flutter, and aircraft structural materials.

AIRCRAFT STRUCTURES

Stress Distribution

Experimental investigations have shown that the stresses and distortions arising at the wing-fuselage juncture of swept wings are appreciably different from those occurring in straight wings. Some theoretical work showing the effects of sweep-back on the deflection of box beams under bending and torsion loadings had been published, but there was a need for an analysis giving the stresses at the triangular root section of a swept box beam. Accordingly, a method presented in Technical Note 2232 was developed for analyzing the triangular portion of the box beam and for establishing continuity between this section and the carry-through section, as well as the outboard portions of the box beam. The results obtained by this method were compared with previously published test data. The agreement was found to be fair, with the principal discrepancy being due to the fact that the method is based on a very simple type of idealized structure which prevents the appearance of shear lag in the results. An extension of the basic approach was therefore made in order that it might include the effects of shear lag.

Exact solutions for the stresses and deformations in wings subjected to torsion are not easily obtained. Oregon State College has therefore developed an approximate method of solving for the angle of twist, longitudinal stresses, and shear stresses in torsion boxes of rectangular, elliptical and airfoil cross sections. The results of these theoretical studies indicate trends for consideration in design.

For proper design of cut-outs that occur in stiffened-shell structures, a detailed knowledge of the redistribution of stresses around the cut-outs is necessary. An approximate theory had been developed for the analysis of stresses in torsion boxes containing large cut-outs which is adequate for the design of all components of such structures except the cut-out cover. This theory was extended in Technical Note 2290 to permit a more detailed calculation of the stresses in the cover. Numerical results were compared with experimental data and also with the results of a solution made by a more elaborate numerical procedure. The agreement was found to be satisfactory in all cases except those with very large cut-outs and flexible bulkheads.

For analysis of thin solid wings of small aspect ratio, such as might be used in supersonic airplanes and missiles, beam theory is no longer adequate. Wings of this

type are more nearly plates than beams and can be analyzed by plate theory. However, solutions to the partial-differential equations of plate theory are not readily obtained, especially for plates of arbitrary shape and loading. A method is presented in Technical Note 2369 for obtaining ordinary differential equations to replace the partial-differential equations. This simplified plate theory has been applied to specific problems involving static deflection, vibration, and buckling of cantilever plates.

One proposed method for increasing the storage space inside a wing is to replace the ribs with a number of vertical posts connecting the compression and tension covers of the wing box beam. A theoretical study of an idealized post-box configuration had indicated that the compression cover of a box beam could be stabilized by posts with an attendant weight saving. In order to check the validity of the theory developed, a beam was tested and the results reported in Technical Note 2414. The experimental buckling load was in good agreement with theory, but appreciable distortion of the beam cross section occurred before the buckling load was reached. It was concluded that a proper combination of posts and ribs may make a design that would be satisfactory with regard to both deformation and strength.

The skin on the upper surface of a wing is subjected to compression by the bending of the wing, and in most cases the addition of stiffening elements to the skin is required to enable the compression load to be carried. Direct-reading design charts have been prepared for 75S-T6 aluminum alloy flat compression panels having longitudinal extruded Z-section stiffeners. These charts, which are presented in Technical Note 2435, cover a wide range of propositions and make possible the direct determination of panel dimensions required to carry a given intensity of loading with a given skin thickness and effective length of panel. They also make possible the determination of the panel proportions having minimum weight and meeting the design conditions.

A type of sandwich plate widely used in airplane construction consists of a corrugated metal sheet riveted between two metal face sheets. In order to apply existing sandwich-plate theories to the analysis of the corrugated-core sandwich plate, a knowledge of certain elastic constants which describe the distortions of the sandwich plate under simple loadings is required. Formulas for evaluating these elastic constants for the corrugated-core type of sandwich have therefore been derived and presented in Technical Note 2289. Suggestions have also been made as to the method of extending existing sandwich-plate theory to make it strictly applicable to the unsymmetrical type of corrugated-core sandwich.

Pressurized cabins of high-altitude airplanes present many stress analysis problems unusual to the aircraft field. These problems have been reviewed by Stanford University and a report containing a summary of the available information on these problems has been prepared.

The ultimate load-carrying capacity of stiffened-shell structures which are subjected to combined loads is a subject on which little information is available. In order to investigate one phase of this problem, a series of five large stiffened circular cylinders was tested to failure under various combinations of torsion and compression. An interaction curve for the strength of stringers that fail by local crippling was determined, and data were obtained on stringer stresses for stiffened cylinders under combined loads. The test results and a method for estimating stiffener stresses are presented in Technical Note 2188.

Stability

In 1947 Shanley showed that it was possible for a straight column in the plastic stress range to start to bend at a load given by the Euler formula with the tangent modulus of elasticity substituted for Young's modulus of elasticity. He also pointed out the existence of a maximum column load which was greater than the tangent-modulus load. In order to determine the amount by which the maximum load could exceed the tangent-modulus buckling load, a study of column behavior in the plastic stress range was undertaken, the results of which are reported in Technical Note 2267. This study confirmed the existence of the tangent-modulus buckling load, and showed by a rigorous load-deflection analysis the relation of the maximum load to the stress-strain curve of the material.

A thin-walled fuselage may buckle either by deformation of the entire column (column buckling) or by deflection of its component webs and flanges (local buckling). Methods are available for the determination of the buckling load for each mode of failure, and it is common practice to assume that a column will fail at the lower of the two loads in a mode corresponding to this load. In reality, however, there is an interaction of these two modes of buckling so that the actual buckling load is smaller than that which is ordinarily used. An investigation has therefore been conducted by Cornell University for the purpose of determining the interaction effect between column and local buckling. The results of this study indicate that this effect is negligible for box sections and for common sizes of I, H, and channel sections but may be significant for sections possessing torsional instability, such as T and angle shapes.

A method has been developed by the National Bureau of Standards for computing the compressive load for lateral elastic instability of hat-section stringers and is presented in Technical Note 2272. Applying the method to a range of shapes and lengths of hat-section stringers shows that such stringers are unlikely to fail by lateral instability.

The use of swept wings and tail plan forms with ribs placed parallel to the flight path results in sheet panels that are parallelogram-shaped. Few data exist on the stability of such panels under the loadings that arise due to wing and tail bending. Accordingly, in Technical Note 2392 charts have been prepared giving theoretical compressive-buckling-stress coefficients for continuous flat sheet divided by nondeflecting supports into parallelogram-shaped panels. Over a wide range of panel aspect ratio, such panels are decidedly more stable than equivalent rectangular panels of the same area.

Thermal Effects

The determination of the structural effects of nonuniform temperature distributions, such as might be produced by aerodynamic heating, is rapidly becoming a problem of interest to aircraft designers because non-uniform temperature distributions have important and complex effects on the stresses and distortions of the structure. The calculation of the stresses due to non-uniform temperature distributions has been treated by several investigators, since the problem has long been of concern to power-plant designers; however, methods that would be directly applicable to the thermoelastic analysis of aircraft structures are not yet available. Two preliminary studies have therefore been completed on this subject. The first, reported in Technical Note 2240, considers the two effects of temperature changes on aircraft structures: The introduction of thermal stresses and distortions as a result of restrained thermal expansion, and the change in behavior of the structure resulting from the variation of elastic properties of materials with temperature. These effects have been illustrated by sample analyses of the stress and distortion distributions of simple box beams and by calculation of the stresses in a typical wing section. The analytical methods of the first study provide a relatively simple means of approximating the effects of temperature changes; however, they often yield inaccurate values for the secondary stresses in complicated structures, and in such cases some numerical approach is desirable. The second study, reported in Technical Note 2241, extends a previously published numerical method of stress analysis to include the effects of a nonuniform distribution of temperature. An illustrative analysis using this method was made of a two-cell box beam under the

combined action of vertical loads and a nonuniform temperature distribution.

Syracuse University has conducted an experimental investigation of the temperature and stress gradients resulting from the application of heat to the skin of a number of skin and spar cap combinations. The 75S-T6 aluminum alloy specimens with a range of skin thicknesses were heated at various rates to provide a better understanding of heat flow through high-speed aircraft wings.

Fatigue

An investigation is being conducted on the fatigue strength of full-scale airplane structures. The results, to date, indicate a surprisingly small amount of spread in the fatigue life even though all the fatigue cracks do not originate at the same point. Effective stress-concentration factors have been computed and compared with results from tests of small laboratory specimens. The importance of fatigue crack length on crack growth is pointed out in these tests. In connection with this work the structural damping of a typical airplane wing has been determined. These measurements were carried out by the shock-excitation method, with the resultant frequency of vibration being that of the wing fundamental bending frequency. The results are in agreement with those of other experimenters who have measured structural damping of smaller wing specimens. They indicate that the damping increases with amplitude of vibration.

AIRCRAFT LOADS

Steady Flight Loads

A simplified lifting-surface theory has been applied to the problem of evaluating span loading due to flap deflection for arbitrary wing planforms (Technical Note 2278). A procedure was developed from which the effects of flap deflection on span loading and associated aerodynamic characteristics can easily be computed for any wing that is symmetrical about the root chord and that has a straight quarter-chord line over the wing semispan. The effects of compressibility and spanwise variation of section lift-curve slope are taken into account by the procedure. The load distribution and lift due to flap deflection were prepared in chart form for a broad range of wing planforms for the case of straight tapered wings and a comparison between experimental values of flap effectiveness and values obtained from these charts show good agreement between theory and experiment.

Maneuvering Loads

As airplanes increase in size and weight, experimental data must be obtained to evaluate current maneuvering tail-load design requirements and to assist in extrapolation to the larger airplanes. The need for data arises from the variations in the response of the airplane as size increases, and also because a pilot's mental attitude and his control actions will vary according to the size and type of airplane. In view of the gaps in existing data a joint project for flight testing the Lockheed Constitution (design gross weight 184,000 pounds) was conducted by the Navy Department, Bureau of Aeronautics, and the NACA.

These tests, reported in Technical Note 2490, indicated that the analytical process of estimating the maneuvering tail loads for a large airplane need be little different from that for small airplanes. The flexibility of the tail had a minor effect on the estimated tail load and the resulting response of the airplane. Because of the relatively slow response of the airplane, the pilot was forced to anticipate the airplane response in applying control forces; for this reason the pilots often unknowingly obtained undesirably high load factors. The pilots were unaware of the critical tail loads arising from abrupt restoration of the control to neutral during a pull-up or sideslip. Pilots also expressed the opinion that the rolling pull-out maneuver felt similar to maneuvers required to recover from disturbances to flight in rough air.

Gust Loads

The dynamic behavior of an airplane wing during flight through rough air was investigated by flight tests on a twin-engine transport airplane. Results from flights made in clear-air turbulence were compared with results from slow pull-ups made in smooth air used as a quasi-static reference condition (Technical Note 2424).

As an extension of previous investigations of the effect of sweep on gust loads, tests were made in the Langley gust tunnel on a 60° swept-back-wing model to determine the effect of a large angle of sweep on gust loads. The results indicate that a simplified method of analysis, using a slope of the lift curve derived by the cosine law and strip theory to estimate the penetration effect is applicable to the prediction of gust loads on wings swept as much as 60°. These results as well as a summary curve representing the results of investigations with wing models having from 45° sweep-forward to 60° sweep-back are presented in Technical Note 2204.

An instrument to indicate atmospheric turbulence has been constructed and subjected to laboratory and flight tests. The design of the instrument was based on previous findings that the rapid fluctuations of the air speed in gusty air constituted a measure of atmospheric turbulence. The instrument consists essentially of a vented air-speed diaphragm in a sealed case with the lag characteristics of the vent chosen so that the diaphragm responds to the rapid changes in dynamic pressure in gusty air but remains insensitive to gradual changes resulting from normal air-speed changes. The results of flight and laboratory tests indicate that the principle of design was a practicable one, but further design refinement would be necessary if the instrument were to be used in routine airplane operations.

The effect of atmospheric turbulence on the loads encountered by aircraft in operational flight has been studied for some time by analyzing records secured with the NACA V-G (airspeed-acceleration) recorder. The instrument originally employed was troublesome to adjust and was limited in frequency response. An improved recorder has been developed which employs a viscous-damped accelerometer. Improved frequency response of the accelerometer has resulted and the necessity for adjusting damping in the field has been eliminated. The motion of the air-speed element has been made approximately linear with air speed; in addition, the element has been temperature-compensated. The new NACA oil-damped V-G recorder and its performance characteristics are described in Technical Note 2194, and the results of tests which demonstrate improvements to the older design are presented.

The NACA VGH recorder described in Technical Note 2265 is a flight instrument designed to collect gust-loads data on transport aircraft with the accuracy required for research purposes. Air speed, normal acceleration, and altitude are recorded on photographic paper as a function of time and in the usual applications 100 hours of continuous recording time is possible. A small acceleration-sensing unit used with the recorder can be mounted independently of the recorder. Although built principally for collecting statistical data on gusts encountered in scheduled airline operations, the instrument records data in a manner which permits many studies pertaining to airplane operations which have air speed, normal acceleration, and altitude as parameters.

Landing Loads

With the advent of thin wings for high-speed flight, the increased use of external stores, and the trend toward airplanes of larger size, a knowledge of the behavior of flexible wings during landing becomes increasingly important. An investigation of a model

consisting of a hull in combination with a flexible wing was made in the Langley impact basin, and the data are reported in Technical Note 2343. These data substantiated a theoretical method for considering the effect of structural flexibility on the applied hydrodynamic force.

The development of retractable lifting surfaces for imparting favorable hydrodynamic behavior to streamlined hulls has shown a trend toward the use of comparatively flat surfaces which blend with the bottom of a fuselage. Data on the impact loads and energy absorption of heavily loaded flat surfaces when landed at various trims and flight-path angles are reported in Technical Note 2330.

The validity of the reduced-mass method of representing wing-lift effects in free-fall drop tests of landing gears has been investigated and the results are reported in Technical Note 2400. The behavior of a small landing gear in the reduced-mass drop tests was compared with results obtained in a drop test where wing lift forces simulating airborne impact were mechanically applied to the test specimen during impact, and with results obtained in free-fall drop tests with full weight. Values of landing-gear load factor and ratio of shock-strut energy to impact energy obtained from reduced-mass drop tests were in fairly good agreement with the results obtained from simulated airborne impacts. The values of impact period and shock-strut effectiveness were generally lower in the reduced-mass drop tests than in the simulated airborne impacts, while strut stroke and mass travel were generally greater in the reduced-mass drop tests than in the simulated airborne impacts. The free-fall drop tests with full weight produced excessive values of load factor, impact period, strut stroke, mass travel, and impact energy, but values of strut effectiveness were in fairly good agreement with those obtained in the simulated airborne impacts.

VIBRATION AND FLUTTER

Increased effort is being expended upon flutter studies for transonic and supersonic speeds. Unusual configurations are also being investigated in order to keep pace with design trends for these speeds.

Unsteady Lift Theories

Calculations have been made of the forces on aerodynamic bodies in unsteady motion based upon the classical equations governing acoustic disturbances. These calculations have been greatly expedited by applying techniques which were developed in recent investigations of steady state supersonic theory. The build-up of lift and pitching moments on slender, triangular wings and wing sections undergoing sinking

and pitching motions were calculated for subsonic, sonic, and supersonic speeds. These results have been reported in Technical Notes 2256, 2387, and 2403. These transient responses provide the necessary information for the prediction of responses to oscillatory motions of arbitrary frequency and are therefore applicable to problems in airplane dynamics and flutter calculations.

Iterative Solution of Flutter

Existing methods of flutter analysis include the representative-section method, generalized coordinate methods, matrix methods, and operational methods. Wielandt has suggested an iterative transformation procedure which is well suited to the flutter problem and similar characteristic-value problems. Because both the original and the translated work of Wielandt are difficult to follow, an explanation of the iterative-transformation procedure is given in Technical Note 2346 and the application of the procedure to ordinary natural vibration problems and to flexure-torsion flutter problems is shown in numerical examples. It appears from this work that the method is of practical usefulness. Comparisons of computed results with experimental results and with results obtained with other methods of analysis have also been made.

Effect of Coupling Between Modes

In an analysis of flutter involving coupled vibrations, questions arise concerning the accuracy of using uncoupled modes or coupled modes, as the latter greatly complicate the analysis. An analysis of the Raleigh type based on coupled modal functions, was made for a wing with simulated engines, or tip tanks, representing a wing with high mass coupling. The wing considered was uniform, unswept, and high aspect ratio, and was mounted as a cantilever. The results of this analysis were compared with those of an analysis based on uncoupled modes as well as with experimental results. For the configurations studied, agreement with experimental data was obtained for both methods of analysis; therefore, the more complicated coupled-mode analysis may be unnecessary (Technical Note 2375).

Single Degree of Freedom Flutter

Beside the usual types of flutter involving coupled modes, the theoretical existence of a single-degree-of-freedom flutter has been indicated and is of interest in dynamic stability. Therefore it was deemed advisable to study the effects of some parameters affecting this single-degree-of-freedom oscillation. One such analytical study concerned the effect of Mach number and structural damping on single-degree-of-freedom pitching oscillations of a wing. Some experimental results

were compared with theory and good agreement was found for certain ranges of an inertia parameter (Technical Note 2396).

Sound

Propeller noise has been a problem even at subsonic tip speeds; hence the proposed use of propellers operating at supersonic tip speeds has caused some concern because of the severity of the associated noise problem. Very little information is available which would allow a prediction of noise levels. Therefore sound measurements have been made for a two-blade propeller with a tip Mach number of 1.30. Sound spectrums were obtained at both subsonic and supersonic tip speeds, and the measured data were compared with calculations by the Gutin theory.

AIRCRAFT STRUCTURAL MATERIALS

Fatigue of Metals

The project on fatigue at Battelle Memorial Institute mentioned in previous annual reports has continued, and significant progress has been made in fulfilling the purpose of the program, which is to provide information sufficiently comprehensive to permit its application to design and to the estimation of fatigue damage. The materials being investigated in this program are 24S-T3 and 75S-T6 aluminum alloys and SAE 4130 steel. Results previously reported on unnotched specimens are contained in Technical Note 2324. Fatigue data on notched specimens over a range of stress concentration factors of 1.5 to 5.0 are the subject of several other reports. Those on stress concentration factors of 2.0 and 4.0 are in Technical Note 2389, and those on stress concentration factors of 5.0 are given in Technical Note 2390. Data have also been obtained on specimens having a stress concentration factor of 1.5. This range of stress concentrations is considered to be that of most importance in aircraft from the fatigue viewpoint.

In the investigation of the statistical nature of fatigue at Carnegie Institute of Technology during the past year, the statistics of the fatigue fracture curves and endurance limits were determined for SAE 1050 steel, annealed Armco iron, and SAE 4340 steel in the quenched and tempered and quenched and spheroidized conditions. The effects of metallurgical factors such as composition, heat treatment, and cleanliness on fatigue behavior were shown by a statistical analysis of the experimental results. From this work it has been concluded that inclusions play the dominant role in the fatigue behavior of these materials. In addition, some other aspects of the fatigue problem were studied, among which were the dependence of statistical variations in fatigue life on stress level in the fracture range,

the statistics of the location of crack initiation, size effect, and understressing effects.

Axial fatigue tests were conducted at the National Bureau of Standards on flat sheet specimens of bare and alclad 24S-T3 aluminum alloy to determine the effect of frequency of loading on the fatigue strengths of these materials. The results of this investigation are contained in Technical Note 2231 and show that the fatigue strengths of the materials were slightly less when tested at 12 cycles per minute than when tested at 1,000 cycles per minute.

Another aspect of the fatigue problem was investigated at the Pennsylvania State College. The purpose of this project was to determine the influence of biaxial tensile stresses on the fatigue strength of 14S-T4 aluminum alloy when subjected to various ratios of biaxial stresses. These biaxial fatigue stresses were produced by applying simultaneously a pulsating internal pressure and fluctuating axial tensile load to a thin-walled tubular specimen. Fatigue strengths and S-N diagrams were obtained up to about 10,000,000 cycles for four principal stress ratios. The test results do not agree with any particular theory of failure but are sufficiently close to the maximum stress theory to permit its use in design.

Plasticity

In the 1950 annual report reference was made to an investigation at the Pennsylvania State College on the plastic biaxial stress-strain relations. The results of biaxial tensile stress tests on 75S-T6 material are published in Technical Note 2425, and the results of similar tests on 14S-T4 aluminum alloy were obtained during the past year. The main purpose of this investigation was to determine the validity of the plasticity theories and the correctness of the various assumptions made in these theories. To accomplish this purpose and to supplement the biaxial tensile test results the investigation was extended to include tests in which specimens were subjected to biaxial tension and compression. In these tests both constant and variable stress ratios were used. Although the data obtained do not agree with any of the current plasticity theories, it was found that the biaxial yield strength conforms most closely with the distortion energy theory.

For mathematical simplicity most plasticity theories assume solids to be isotropic and plastically incompressible. These assumptions fix the value of Poisson's ratio in the plastic range at 0.5. Physical considerations, however, make it apparent that the transition from the elastic to the plastic value of Poisson's ratio is gradual. An investigation of the variation in Poisson's ratio in the yield region of the stress-strain curve was completed by New York University during the past

year. It is in this region that the transition from the elastic to the plastic value is most pronounced. This transition was studied experimentally by a systematic system of tests on 14S-T6, 24S-T4, and 75S-T6 aluminum alloys under simple tensile and compressive stresses along three orthogonal axes. The data obtained on Poisson's ratio in the yield region as well as values of the plastic dilatation for strain deviations up to 0.6 percent are given in Technical Note 2561.

Creep

In the design of structures required to operate at elevated temperatures, knowledge of the characteristics of materials at these temperatures is essential. One of the phenomena associated with elevated temperatures is that of creep, the time-dependent distortion of metals under stress. It is felt that a better understanding of creep phenomena will allow the more efficient use of materials in design. The experimental data available, however, are in general not sufficiently detailed nor do they cover a wide enough range in conditions to be effective for use in design, and although several theories have been advanced to account for creep behavior, these theories are not complete enough to be of much help. A study of the fundamental mechanism of the creep of metals has been initiated, therefore, at the Battelle Memorial Institute. Before proceeding with the experimental work, Battelle conducted a survey of the existing knowledge in the field. This survey, released as Technical Note 2516, treats the creep of both single-crystal and polycrystalline metals. The modes of deformation in short-time, low-temperature tests are considered, as well as the various theories which purport to explain the behavior of metals during creep. Battelle is now obtaining experimental data on the creep of 2S aluminum alloy over a wide temperature range and comparing these data with available theories. Should the data conform with none of these, efforts will be made to develop an appropriate theory to try to account for all the observed phenomena.

Stress Corrosion

The Armour Research Foundation in its investigation of stress corrosion has developed equipment which will permit observation and recording of the relative potentials of various regions near the surface of a metallic specimen during corrosion. By means of this equipment it is possible to locate the anodic and cathodic areas on a corroding surface and identify such areas with the corresponding metallurgical feature. Technical Note 2523 presents a description of this equipment and describes the techniques developed for the interpretation of the data obtained. The equipment is now being used to investigate the stress corrosion

mechanism of a high-purity binary alloy, aluminum—4-percent copper.

Corrosion

The investigation of the corrosive properties of metals jointly sponsored by the NACA, Bureau of Aeronautics, and the Air Matériel Command has continued at the National Bureau of Standards. The purpose of this investigation is to determine the rate of corrosion of materials used in aircraft when exposed to marine atmosphere and tidewater. During the past year data were obtained from exposure tests on galvanized steel stitched-aluminum alloys. These are reported in Technical Note 2299. The effect of hot dimpling on the corrosion of aluminum alloys 75S-T6 and alclad 75S-T6 was evaluated and results were also obtained from corrosion tests of magnesium alloy ZK60.

Hydrogen Embrittlement

A fundamental study was conducted at the Battelle Memorial Institute on the mechanism by which hydrogen enters metals, particularly aircraft steels, causing hydrogen embrittlement during chemical or electrochemical action. Several known methods of controlling the inlet and exit of hydrogen in steel were correlated with the known chemical behavior of atomic and molecular hydrogen. By means of this correlation it was found possible to select suitable reagents which significantly decreased hydrogen permeability and embrittlement of steel during cathodic pickling operations in dilute sulfuric acid. The same data show that the diffusion of hydrogen in steel and the freedom of exit of hydrogen are important in determining the extent of embrittlement and also that these phenomena are chemical in nature rather than mechanical.

Adhesives

In the manufacture of aircraft the bonding of sandwich panels, consisting of aluminum alloys bonded to various core materials, and the bonding of aluminum to itself by means of adhesives offers several important advantages over other methods of fastening, and there is considerable interest at the present time in such processes. There is a need, however, for an adhesive adequate for use at elevated temperatures. This led to the investigation at Forest Products Laboratory on the effects of temperature on the bond strength of joints fabricated with adhesives. The evaluation of the performance of commercially available metal bonding adhesives in the temperature range of -70° to 600° F. has been completed. The evaluation of the relative merits of the materials is based upon the data obtained from tensile

tests of joints both when tested immediately upon reaching the test temperature and after several days' exposure at the test temperature.

Plastics

The investigation of the loss of strength of plastic glazing as a result of crazing has continued at the National Bureau of Standards. Since plastic glazing is used in the windows and canopies of airplanes, this property is of considerable interest to the aircraft industry. The meager information on the subject was expanded during the past year by two reports from the National Bureau of Standards. In one of these, Technical Note 2444, the loss in strength of several grades of polymethyl methacrylate due to stress solvent crazing is discussed. In this investigation tensile specimens were artificially crazed by applying benzene to the central portion of the specimen while it was under stress. Among the factors studied were the sheet-to-sheet variability of crazing specimens and the relative effect of a few large crazing cracks compared with more numerous finer cracks. Applying benzene to the specimens caused a loss of strength of approximately 30 percent in all of the materials.

An investigation was completed at the National Bureau of Standards on the strength of heat-resistant laminates up to 375° C. The flexural properties of samples of glass fabric laminates such as are used in radomes were determined for several loading cycles at elevated temperatures. The laminates tested were bonded with various resins, including unsaturated polyester, acrylic, silicone, phenolic, and melamine types. Tests of a silicone resin laminate showed it to be superior to the other laminates in retention of flexural properties at temperatures of 250° C. or higher (Technical Note 2266). At temperatures of 300° to 325° C. this laminate retained at least 30 percent of its initial flexural strength and over 50 percent of its initial modulus of elasticity.

Materials Evaluation

The light weight, high-strength, corrosion-resistant, and favorable elevated-temperature properties of titanium have aroused interest in the possibility of using this material for aircraft structural applications. In order to indicate the potentialities of this material, an evaluation of titanium including comparisons with several other aircraft materials has been made for typical structural applications. The procedure developed for this purpose provides a general method for determining the structural efficiency of a material for a particular structural application (Technical Note 2269).

OPERATING PROBLEMS

Investigations have been continued of the meteorological conditions adverse to the performance and safety of flight, means of accurately measuring air speed and static pressure at transonic and supersonic speeds, principles of design of aircraft systems for coping with natural icing conditions, and methods of improving aircraft safety. Research on atmospheric turbulence, aircraft ditching, and air speed measurement have been conducted by the Langley Aeronautical Laboratory, while studies of natural icing conditions, aircraft ice-protection systems, and aircraft crash fires have been conducted by the Lewis Flight Propulsion Laboratory. The Ames Aeronautical Laboratory has also done limited work in the field of aircraft icing. The work of the NACA laboratories has been supplemented by research conducted by several universities under contract to the NACA.

The practice of holding technical conferences with industry and military representatives was extended to the field of operating problems. A conference on some problems of aircraft operation was held at the Lewis Flight Propulsion Laboratory in the fall of 1950. The Committee on Operating Problems is responsible for the guidance of the operating problems research program. It is aided in specialized aspects of this program by its subcommittees, the Subcommittee on Meteorological Problems, the Subcommittee on Icing Problems, and the Subcommittee on Aircraft Fire Prevention.

AIRCRAFT OPERATION

Air Speed and Altitude Measurement

A systematic investigation of the static-pressure errors ahead of wings and fuselage noses of two low-wing airplanes was conducted in flight (Technical Note 2311). The variation of the static-pressure error with position of the static-pressure tube, located at several points from 0.25 to 2.0 chord lengths ahead of the wing tip and 0.5 to 1.5 fuselage diameters ahead of the nose, was determined over a speed range from the stall up to a maximum of 265 miles per hour. At low lift coefficients, the error decreased with increasing lengths ahead of the wing or fuselage, with variation in error over the lift-coefficient range being approximately the same for the $1\frac{1}{2}$ -diameter fuselage-nose installation and the 1-chord wing-tip installation.

For present-day high-performance airplanes having the capabilities of maneuvering to high angles of attack at supersonic speeds, the need has developed for rigid air-speed tubes which will measure total and static pressures correctly under these conditions. As a means of determining the optimum design of total-pressure

tubes, a series of wind-tunnel tests has been conducted to determine the effect of inclination of the air stream on various types of total-pressure tubes over a wide range of angle of attack through the subsonic and supersonic speed ranges. Preliminary tests on 39 tubes at subsonic speeds are reported in Technical Note 2331. Subsequent tests of 20 of these tubes at supersonic speeds are reported in Technical Note 2261. The results of the tests showed that the tube which remained insensitive over the greatest range of angle of attack, both at subsonic and supersonic speeds, was a shielded total-pressure tube designed for end mounting on a horizontal boom. The range of angle of attack over which this tube remained insensitive (to within 1 percent of the impact pressure) was $\pm 41.5^\circ$ at subsonic speeds and $\pm 37^\circ$ at supersonic speeds. From the results of the tests of the other tubes, the effects of such design factors as external shape, internal shape, and configuration of total-pressure entry were evaluated, and the effect of Mach number on the performance of the various tubes was determined. Studies were also conducted of effects of position and airplane configuration on the measurement of air speed and Mach number in the transonic speed range. The applicability of using pressures on a cone as a measure of angle of attack at transonic speeds was also investigated.

Ditching

The investigation of the ditching characteristics of civil transport-type aircraft was completed with the studies of the behavior of the Convair-Liner. The recommended ditching procedures follow those for other transport types, touchdown being made at fairly high angle of attack with flaps down and landing gear up. The behavior depends upon the manner in which the flaps fail, the maximum deceleration being somewhat greater if the flaps rotate to neutral rather than being carried away.

AERONAUTICAL METEOROLOGY

Prediction of Low-Level Clear-Air Turbulence

Preliminary investigation of the meteorological variables associated with turbulence in clear air within the earth's friction layer led to the development of a simple empirical relation for estimating the intensity of the turbulence. The correlation of the observed gust experience of an airplane with various combinations of meteorological variables showed that a simple product form of the intensity of solar radiation and the average wind shear layer below the first discontinuity in temperature lapse rate gave very good results for condi-

tions similar to those of the tests. This relation accounts for seasonal variations in turbulence intensity but, because of its simplicity, cannot discriminate between differences in turbulence intensity resulting from variations in flight altitude or diurnal variations in turbulence. The relationship has been established only for the limited conditions of terrain and meteorological situations during the flights in the vicinity of Wilmington, Ohio. Significant differences in these conditions will alter the relative importance of the meteorological variables considered.

Radar Detection of Thunderstorm Turbulence

Evaluation of the uses of airborne radar for detection of turbulent areas in thunderstorms was conducted by the Navy with the assistance of the Langley Laboratory staff. The presentation of areas of light rainfall and heavy rainfall on the radar PPI scope allowed preliminary correlation of the gust experience of an airplane with the indicated rainfall gradient. The limited results so far obtained indicated that possibilities of reducing the number and severity of the gusts likely to be encountered in a thunderstorm were good if the areas of high rainfall rate gradient were avoided.

Surface Wind Instruments

A wind-tunnel calibration of instruments for determining wind velocity and direction was conducted for the Signal Corps, United States Army. A three-cup anemometer was calibrated for measuring wind velocities up to 200 miles per hour. Two pressure-type combined wind-velocity and wind-direction indicators were also tested. One with four equally spaced orifices on the perimeter of a circular disk was tested up to 100 miles per hour, and another consisting of a pitot-static head mounted on a pivoted weather vane was tested up to 250 miles per hour.

AIRCRAFT ICING

Research is continuing at the Lewis Flight Propulsion Laboratory and the Ames Aeronautical Laboratory in an effort to establish the meteorological conditions that make up the icing cloud so that this knowledge may be applied toward the efficient design of aircraft ice-protection systems.

Characteristics of the Icing Cloud

The meteorological aspects of the icing cloud are being studied extensively in flight and laboratory investigations to determine what values of meteorological factors exist in the atmosphere and how these data can best be utilized to design efficient types of ice-protec-

tion systems for the various aircraft components. Tentative recommended values of meteorological factors to be considered in the design of aircraft ice-protection systems previously published (Technical Note 1855) have been supplemented with data collected by the NACA and other agencies during the 1948-49 and 1949-50 winter seasons and have been analyzed to present a method for the selection of design criteria. The data are summarized to give frequency of occurrence of icing conditions according to liquid-water content and drop size and further to indicate that statistical relations do exist between liquid-water content, mean effective droplet diameter, temperature, and pressure altitudes. A considerable amount of additional data is needed before a purely statistical analysis can be attempted; however, this accomplishment can be made possible only if the efforts to design suitable meteorological instruments are successful.

The Lewis Laboratory has developed and reported on a reliable pressure-type icing-rate meter that will automatically record icing rate, airspeed, altitude, and time in supercooled clouds. Evaluation tests on research and commercial aircraft indicate that the use of this instrument to collect a large amount of data for a statistical analysis is practical. The operation of the instrument is based on the creation of a differential pressure between an ice-free total-pressure system and a total-pressure system in which small total-pressure holes vented to static pressure are allowed to plug with ice accretion. At a fixed value of ice accretion, the differential pressure operates an electrical heater that de-ices the total-pressure holes. The resulting cyclic process varies with the intensity of the icing and serves as a measure of the icing rate. The simplicity of this operating principle permits automatic operation upon encountering an icing condition and relative freedom from maintenance and operating problems. The complete unit weighs only 18 pounds. Visual indications of the icing intensity are made available to the pilot by periodic flashing of a light on the instrument panel.

Another instrument previously developed as a cloud detector has been further investigated to provide a simple means for measuring liquid-water content in above- and below-freezing clouds. The instrument consists of a small cylindrical element so operated at high surface temperatures that the impingement of cloud droplets creates a significant drop in surface temperature. Preliminary calibrations of this instrument have indicated that improvements are expected to make it a reliable instrument for recording the important meteorological factor of liquid-water content.

Liquid-water content, droplet size, and temperature data measured in predominantly stratiform clouds during the last two icing seasons have been presented in

Technical Note 2306. The average liquid-water content of a cloud layer, as measured by the multicylinder technique, seldom exceeded two-thirds of that which could be released by adiabatic lifting. The horizontal and vertical extent of icing conditions and the relation of the existence of supercooled clouds to cyclonic areas and precipitation regions are indicated.

The Lewis Laboratory is continuing to make progress toward understanding of the fundamental processes involved in the formation of the icing cloud and in methods for its prediction. Part of this study has been a continuation of the investigation of spontaneous freezing temperatures of supercooled water droplets of the size found in the atmosphere. A statistical explanation of the previously reported investigation of spontaneous freezing temperatures of water droplets (Technical Note 2142) has been published in Technical Note 2234. The statistical theory is based on the presence of an assumed crystallization nuclei distribution suspended in the water and results indicate that small droplets may be supercooled to lower temperatures than the large droplets. The probable distribution curves of spon-

aneous freezing temperatures for water droplets of various sizes are given and compared with the experimental results.

Propeller Ice Protection

The Ames Laboratory has completed analytical and experimental investigations of the effect of ice formation on propeller performance (Technical Note 2212). The experimental measurements were made during flight in natural icing conditions, whereas the analysis consisted of an investigation of changes in blade section aerodynamic characteristics caused by ice formation and the resulting propeller efficiency changes. The results indicated that efficiency losses can generally be expected to be less than 10 percent with maximum losses as high as 20 percent to be expected only rarely. The effects of ice accretion on airfoil section characteristics at subcritical speeds, the effect of kinetic heating on the radial extent of ice formation, and the influence of the efficiency loss on the pilot's adjustment of engine and propeller controls are also discussed.

RESEARCH PUBLICATIONS

TECHNICAL REPORTS

- | | |
|--|--|
| <p>No.</p> <p>951. Use of Source Distributions for Evaluating Theoretical Aerodynamics of Thin Finite Wings at Supersonic Speeds. By John C. Evvard.</p> <p>952. Direct Method of Design and Stress Analysis of Rotating Disks with Temperature Gradient. By S. S. Manson.</p> <p>953. Attainable Circulation about Airfoils in Cascade. By Arthur W. Goldstein and Artur Mager.</p> <p>954. Two-Dimensional Compressible Flow in Centrifugal Compressors with Straight Blades. By John D. Stanitz and Gaylord O. Ellis.</p> <p>955. Application of Radial-Equilibrium Condition to Axial-Flow Compressor and Turbine Design. By Chung-Hua Wu and Lincoln Wolfenstein.</p> <p>956. Linearized Compressible-Flow Theory for Sonic Flight Speeds. By Max. A. Heaslet, Harvard Lomax, and John R. Spreiter.</p> <p>957. The Calculation of Downwash Behind Supersonic Wings with an Application to Triangular Plan Forms. By Harvard Lomax, Loma Sluder, and Max. A. Heaslet.</p> <p>958. Laminar Mixing of a Compressible Fluid. By Dean R. Chapman.</p> <p>959. One-Dimensional Flows of an Imperfect Diatomic Gas. By A. J. Eggers, Jr.</p> <p>960. Determination of Plate Compressive Strengths at Elevated Temperatures. By George J. Heimerl and William M. Roberts.</p> <p>961. The Application of Green's Theorem to the Solution of Boundary-Value Problems in Linearized Supersonic Wing Theory. By Max A. Heaslet and Harvard Lomax.</p> <p>962. The Aerodynamic Forces on Slender Plane- and Cruciform-Wing and Body Combinations. By John R. Spreiter.</p> | <p>No.</p> <p>963. Investigation with an Interferometer of the Turbulent Mixing of a Free Supersonic Jet. By Paul B. Goodrum, George P. Wood, and Maurice J. Brevoort.</p> <p>964. The Effects of Variations in Reynolds Number between 3.0×10^6 and 25.0×10^6 upon the Aerodynamic Characteristics of a Number of NACA 6-Series Airfoil Sections. By Laurence K. Loftin, Jr., and William J. Burnsall.</p> <p>965. The Longitudinal Stability of Elastic Swept Wings at Supersonic Speed. By C. W. Frick and R. S. Chubb.</p> <p>966. Flutter of a Uniform Wing with an Arbitrarily Placed Mass According to a Differential-Equation Analysis and a Comparison with Experiment. By Harry L. Runyan and Charles E. Watkins.</p> <p>967. Elastic and Plastic Buckling of Simply Supported Solid-Core Sandwich Plates in Compression. By Paul Seide and Elbridge Z. Stowell.</p> <p>968. Investigation at Low Speeds of the Effect of Aspect Ratio and Sweep on Rolling Stability Derivatives of Untapered Wings. By Alex Goodman and Lewis R. Fisher.</p> <p>969. Some Theoretical Low-Speed Span Loading Characteristics of Swept Wings in Roll and Sideslip. By John D. Bird.</p> <p>970. Theoretical Lift and Damping in Roll at Supersonic Speeds of Thin Sweptback Tapered Wings with Streamwise Tips, Subsonic Leading Edges, and Supersonic Trailing Edges. By Frank S. Malvestuto, Jr., Kenneth Margolis, and Herbert S. Ribner.</p> <p>971. Theoretical Stability Derivatives of Thin Sweptback Wings Tapered to a Point with Sweptback or Swept-forward Trailing Edges for a Limited Range of Supersonic Speeds. By Frank S. Malvestuto, Jr., and Kenneth Margolis.</p> <p>972. The Effect of Torsional Flexibility on the Rolling Characteristics at Supersonic Speeds of Tapered Unswept Wings. By Warren A. Tucker and Robert L. Nelson.</p> |
|--|--|

- No.
973. Flight Investigation of the Effect of Various Vertical-Tail Modifications on the Directional Stability and Control Characteristics of a Propeller-Driven Fighter Airplane. By Harold I. Johnson.
974. The Supersonic Axial-Flow Compressor. By Arthur Kantrowitz.
975. Small Bending and Stretching of Sandwich-Type Shells. By Eric Reissner.
976. Linear Theory of Boundary Effects in Open Wind Tunnels with Finite Jet Lengths. By S. Katzoff, Clifford S. Gardner, Leo Diesendruck, and Bertram J. Eisenstadt.
977. Frequency Response of Linear Systems from Transient Data. By Melvin E. Laverne and Aaron S. Boksenbom.
978. Method of Designing Cascade Blades with Prescribed Velocity Distributions in Compressible Potential Flows. By George R. Costello.
979. On Stability of Free Laminar Boundary Layer between Parallel Streams. By Martin Lessen.
980. General Algebraic Method Applied to Control Analysis of Complex Engine Types. By Aaron S. Boksenbom and Richard Hood.
981. Theoretical Analysis of Various Thrust-Augmentation Cycles for Turbojet Engines. By Bruce T. Lundin.
982. Icing-Protection Requirements for Reciprocating-Engine Induction Systems. By Willard D. Coles, Vern G. Rollin, and Donald R. Mulholland.
983. Line-Vortex Theory for Calculation of Supersonic Downwash. By Harold Mirels and Rudolph C. Haefeli.
984. Method for Determining the Frequency-Response Characteristics of an Element or System from the System Transient Output Response to a Known Input Function. By Howard J. Curfman, Jr. and Robert A. Gardiner.
985. A Radar Method of Calibrating Airspeed Installations on Airplanes in Maneuvers at High Altitudes and at Transonic and Supersonic Speeds. By John A. Zalovecik.
986. The Reversibility Theorem for Thin Airfoils in Subsonic and Supersonic Flow. By Clinton E. Brown.
987. Equilibrium Operating Performance of Axial-Flow Turbojet Engines by Means of Idealized Analysis. By John C. Sanders and Edward C. Chapin.
988. Comparative Drag Measurements at Transonic Speeds of Rectangular and Sweptback NACA 65-009 Airfoils Mounted on a Freely Falling Body. By Charles W. Mathews and Jim Rogers Thompson.
989. Critical Stress of Ring-Stiffened Cylinders in Torsion. By Manuel Stein, J. Lyell Sanders, Jr., and Harold Crate.
990. An Analysis of Supersonic Aerodynamic Heating with Continuous Fuel Injection. By E. B. Klunker and H. Reese Ivey.
991. The Application of the Statistical Theory of Extreme Values to Gust-Load Problems. By Harry Press.
992. Theoretical Comparison of Several Methods of Thrust Augmentation for Turbojet Engines. By Eldon W. Hall and E. Clinton Wilcox.
993. An Introduction to the Physical Aspects of Helicopter Stability. By Alfred Gessow and Kenneth B. Amer.
994. Analysis of Spanwise Temperature Distribution in Three Types of Air-Cooled Turbine Blade. By John N. B. Livingood and W. Byron Brown.
995. Blockage Corrections for Three-Dimensional-Flow Closed-Throat Wind Tunnels, with Consideration of the Effect of Compressibility. By John G. Herriot.
- No.
996. Free-Space Oscillating Pressures near the Tips of Rotating Propellers. By Harvey H. Hubbard and Arthur A. Regier.
997. Summary of Information Relating to Gust Loads on Airplanes. By Philip Donely.
998. Further Experiments on the Flow and Heat Transfer in a Heated Turbulent Air Jet. By Stanley Corrsin and Mahinder S. Uberoi.
999. Investigation of the NACA 4-(3)(08)-03 and NACA 4-(3)(08)-045 Two-Blade Propellers at Forward Mach Numbers to 0.725 to Determine the Effects of Compressibility and Solidity on Performance. By John Stack, Eugene C. Draley, James B. Delano, and Lewis Feldman.
1000. Calculation of the Aerodynamic Loading of Swept and Unswept Flexible Wings of Arbitrary Stiffness. By Franklin W. Diederich.
1001. Fundamental Effects of Aging on Creep Properties of Solution-Treated Low-Carbon N-155 Alloy. By D. N. Frey, J. W. Freeman, and A. E. White.
1002. On the Theory of Oscillating Airfoils of Finite Span in Subsonic Compressible Flow. By Eric Reissner.
1005. Analytical Determination of Coupled Bending-Torsion Vibrations of Cantilever Beams by Means of Station Functions. By Alexander Mendelson and Selwyn Gendler.

TECHNICAL NOTES ¹

- No.
2092. Approximate Aerodynamic Influence Coefficients for Wings of Arbitrary Plan Form in Subsonic Flow. By Franklin W. Diederich.
2111. A Study of Water Pressure Distributions during Landings with Special Reference to a Prismatic Model Having a Heavy Beam Loading and 30° Angle of Dead Rise. By Robert F. Smiley.
2117. Design and Applications of Hot-Wire Anemometers for Steady-State Measurements at Transonic and Supersonic Airspeeds. By Herman H. Lowell.
2120. Development and Preliminary Investigation of a Method of Obtaining Hypersonic Aerodynamic Data by Firing Models through Highly Cooled Gases. By Harold V. Soule and Alexander P. Sabol.
2123. Investigation of Turbulent Flow in a Two-Dimensional Channel. By John Laufer.
2124. Spectrums and Diffusion in a Round Turbulent Jet. By Stanley Corrsin and Mahinder S. Uberoi.
2126. Improvements in Heat Transfer for Anti-Icing of Gas-Heated Airfoils with Internal Fins and Partitions. By Vernon H. Gray.
2128. Investigation of 75-Millimeter-Bore Cylindrical Roller Bearings at High Speeds. I—Initial Studies. By E. Fred Macks and Zolton N. Nemeth.
2129. Method of Calculating the Lateral Motions of Aircraft Based on the Laplace Transform. By Harry E. Murray and Frederick C. Grant.
2130. Calculation of Transonic Flows Past Thin Airfoils by an Integral Method. By William Perl.
2131. Boundary-Layer Transition on a Cooled 20° Cone at Mach Numbers of 1.5 and 2.0. By Richard Scherrer.
2132. The Calculation of Modes and Frequencies of a Modified Structure from Those of the Unmodified Structure. By Edwin T. Kruszewski and John C. Houbolt.

¹ The missing numbers in the series of Technical Notes were released before or after the period covered by this report.

- No. 2133. Investigation of Separation of the Turbulent Boundary Layer. By G. B. Schubauer and P. S. Klebanoff.
2134. Comparison of Model and Full-Scale Spin Test Results for 60 Airplane Designs. By Theodore Berman.
2135. The Calculation of Downwash Behind Wings of Arbitrary Plan Form at Supersonic Speeds. By John C. Martin.
2136. Theory of Helicopter Damping in Pitch or Roll and a Comparison with Flight Measurements. By Kenneth B. Amer.
2137. An Analysis of Base Pressure at Supersonic Velocities and Comparison with Experiment. By Dean R. Chapman.
2138. Analytical and Experimental Investigation of Adiabatic Turbulent Flow in Smooth Tubes. By Robert G. Deissler.
2139. Effect of Variation in Rivet Diameter and Pitch on the Average Stress at Maximum Load for 24S-T3 and 75S-T6 Aluminum-Alloy, Flat, Z-Stiffened Panels that Fail by Local Instability. By Norris F. Dow and William A. Hickman.
2140. Theoretical Antisymmetric Span Loading for Wings of Arbitrary Plan Form at Subsonic Speeds. By John DeYoung.
2141. Charts for Estimating Downwash Behind Rectangular, Trapezoidal, and Triangular Wings at Supersonic Speeds. By Rudolph C. Haefeli, Harold Mirels, and John L. Cummings.
2142. Photomicrographic Investigation of Spontaneous Freezing Temperatures of Supercooled Water Droplets. By Robert G. Dorsch and Paul T. Hacker.
2143. Analysis of the Effects of Boundary-Layer Control on the Power-Off Landing Performance Characteristics of a Liaison Type of Airplane. By Elmer A. Horton, Laurence K. Loftin, Jr., and Stanley F. Racisz.
2144. Effect of Chemical Reactivity of Lubricant Additives on Friction and Surface Welding at High Sliding Velocities. By Edmond E. Bisson, Max A. Swikert, and Robert L. Johnson.
2145. Lift-Cancellation Technique in Linearized Supersonic-Wing Theory. By Harold Mirels.
2146. On the Effect of Subsonic Trailing Edges on Damping in Roll and Pitch of Thin Sweptback Wings in a Supersonic Stream. By Herbert S. Ribner.
2147. Some Conical and Quasi-Conical Flows in Linearized Supersonic-Wing Theory. By Herbert S. Ribner.
2148. Laminar-Boundary-Layer Heat-Transfer Characteristics of a Body of Revolution with a Pressure Gradient at Supersonic Speeds. By William R. Wimbrow and Richard Scherrer.
2149. Investigation of Boundary-Layer Control to Improve the Lift and Drag Characteristics of the NACA 65-415 Airfoil Section with Double Slotted and Plain Flaps. By Elmer A. Horton, Stanley F. Racisz, and Nicholas J. Paradiso.
2150. Flight Investigation of the Effect of Transient Wing Response on Measured Accelerations of a Modern Transport Airplane in Rough Air. By C. C. Shuffeberger and Harry C. Mickleboro.
2151. Estimation of the Damping in Roll of Supersonic-Leading-Edge Wing-Body Combinations. By Warren A. Tucker and Robert O. Piland.
2152. Shear Stress Distribution Along Glue Line Between Skin and Cap-Strip of an Aircraft Wing. By C. B. Norris and L. A. Ringelstetter.
- No. 2153. The Stability of the Compression Cover of Box Beams Stiffened by Posts. By Paul Seide and Paul F. Barrett.
2154. An Analysis of the Autorotative Performance of a Helicopter Powered by Rotor-Tip Jet Units. By Alfred Gessow.
2155. Calculated Engine Performance and Airplane Range for Variety of Turbine-Propeller Engines. By Tibor F. Nagey and Cecil G. Martin.
2156. Theoretical Calculations of the Lateral Force and Yawing Moment Due to Rolling at Supersonic Speeds for Sweptback Tapered Wings with Streamwise Tips. Supersonic Leading Edges. By Sidney M. Harmon and John C. Martin.
2157. Static and Impact Strengths of Riveted and Spot-Welded Beams of Alclad 14S-T6, Alclad 75S-T6, and Various Tempers of Alclad 24S Aluminum Alloy. By H. E. Grieshaber.
2158. A General Integral Form of the Boundary-Layer Equation for Incompressible Flow with an Application to the Calculation of the Separation Point of Turbulent Boundary Layers. By Neal Tetervin and Chia Chiao Lin.
2159. On the Particular Integrals of the Prandtl-Busemann Iteration Equations for the Flow of a Compressible Fluid. By Carl Kaplan.
2160. Measurements of Section Characteristics of a 45° Swept Wing Spanning a Rectangular Low-Speed Wind Tunnel as Affected by the Tunnel Walls. By Robert E. Dannenberg.
2161. Tables of Thermodynamic Functions for Analysis of Aircraft-Propulsion Systems. By Vearl N. Huff and Sanford Gordon.
2162. Investigation of Properties of AISI-Type 310B Alloy Sheet at High Temperatures. By E. E. Reynolds, J. W. Freeman, and A. E. White.
2163. Critical Stress of Plate Columns. By John C. Houbolt and Elbridge Z. Stowell.
2164. Axial-Momentum Theory for Propellers in Compressible Flow. By Arthur W. Vogeley.
2165. Method of Analysis for Compressible Flow through Mixed Flow Centrifugal Impellers of Arbitrary Design. By Joseph T. Hamrick, Ambrose Ginsburg, and Walter M. Osborn.
2166. Effect of Heat and Power Extraction on Turbojet-Engine Performance. II—Effect of Compressor-Outlet Air Bleed for Specific Modes of Engine Operation. By Frank E. Rom and Stanley L. Koutz.
2167. Sonic-Flow-Orifice Temperature Probe for High-Gas-Temperature Measurements. By Perry L. Blackshear, Jr.
2168. Experimental Investigation of the Effect of Vertical-Tail Size and Length and of Fuselage Shape and Length on the Static Lateral Stability Characteristics of a Model with 45° Sweptback Wing and Tail Surfaces. By M. J. Queijo and Walter D. Wolhart.
2169. Wind-Tunnel Investigation at Low Speed of a 45° Sweptback Untapered Semispan Wing of Aspect Ratio 1.59 Equipped with Various 25-Percent-Chord Plain Flaps. By Harold S. Johnson and John R. Hagerman.
2170. Effect of Initial Mixture Temperature on Flame Speeds and Blow-Off Limits of Propane—Air Flames. By Gordon L. Dugger.
2171. Investigation of a Two-Step Nozzle in the Langley 11-Inch Hypersonic Tunnel. By Charles H. McLellan, Thomas W. Williams, and Mitchel H. Bertram.

- No.
2172. Low-Speed Investigation of the Stalling of a Thin, Faired, Double-Wedge Airfoil with Nose Flap. By Leonard M. Rose and John M. Altman.
2173. Density Fields around a Sphere at Mach Numbers 1.30 and 1.62. By Paul B. Gooderum and George P. Wood.
2174. Comparison of the Experimental Pressure Distribution on an NACA 0012 Profile at High Speeds with that Calculated by the Relaxation Method. By James L. Amick.
2175. Effect of an Unswept Wing on the Contribution of Unswept-Tail Configurations to the Low-Speed Static and Rolling-Stability Derivatives of a Midwing Airplane Model. By William Letko and Donald R. Riley.
2176. An Analysis of the Normal Accelerations and Airspeeds of a Four-Engine Airplane Type in Postwar Commercial Transport Operations on Trans-Pacific and Caribbean-South American Routes. By Thomas L. Coleman and Paul W. J. Schumacher.
2177. Low-Speed Characteristics of Four Cambered, 10-Percent-Thick NACA Airfoil Sections. By George B. McCullough and William M. Haire.
2178. Method for Determining Optimum Division of Power between Jet and Propeller for Maximum Thrust Power of a Turbine-Propeller Engine. By Arthur M. Trout and Eldon W. Hall.
2179. Turning-Angle Design Rules for Constant Thickness Circular-Arc Inlet Guide Vanes in Axial Annular Flow. By Seymour Lieblein.
2180. Effectiveness of Molybdenum Disulfide as a Fretting-Corrosion Inhibitor. By Douglas Godfrey and Edmond E. Bisson.
2181. The Aerodynamic Forces and Moments on a 1/10-Scale Model of a Fighter Airplane in Spinning Attitudes as Measured on a Rotary Balance in the Langley 20-Foot Free-Spinning Tunnel. By Ralph W. Stone, Jr., Sanger M. Burk, Jr., and William Bihrlé, Jr.
2182. Analysis of Effect of Variations in Primary Variables on Time Constant and Turbine-Inlet-Temperature Overshoot of Turbojet Engine. By Marcus F. Heidmann.
2183. Analysis for Control Application of Dynamic Characteristics of Turbojet Engine with Tail-Pipe Burning. By Melvin S. Feder and Richard Hood.
2184. Investigation of Frequency-Response Characteristics of Engine Speed for a Typical Turbine-Propeller Engine. By Burt L. Taylor III, and Frank L. Oppenheimer.
2185. Analytical Determination of Coupled Bending-Torsion Vibrations of Cantilever Beams by Means of Station Functions. By Alexander Mendelson and Selwyn Gendler.
2186. Method for Determining Pressure Drop of Air Flowing through Constant-Area Passages for Arbitrary Heat-Input Distributions. By Benjamin Pinkel, Robert N. Noyes, and Michael F. Valerino.
2187. Bonding Investigations of Titanium Carbide with Various Elements. By Walter J. Engel.
2188. Experimental Investigation of Stiffened Circular Cylinders Subjected to Combined Torsion and Compression. By James P. Peterson.
2189. The Development and Performance of Two Small Tunnels Capable of Intermittent Operation at Mach Numbers between 0.4 and 4.0. By Walter F. Lindsey and William L. Chew.
2190. The Use of an Uncalibrated Cone for Determination of Flow Angles and Mach Numbers at Supersonic Speeds. By Morton Cooper and Robert A. Webster.
- No.
2191. Theoretical Investigation and Application of Transonic Similarity Law for Two-Dimensional Flow. By W. Perl and Milton M. Klein.
2192. A Survey of the Flow at the Plane of the Propeller of a Twin-Engine Airplane. By John C. Roberts and Paul F. Yaggy.
2193. Effect of Heat-Capacity Lag on a Variety of Turbine-Nozzle Flow Processes. By Robert B. Spooner.
2194. The NACA Oil-Damped V-G Recorder. By Israel Taback.
2195. A Flight Investigation and Analysis of the Lateral-Oscillation Characteristics of an Airplane. By Carl J. Stough and William M. Kauffman.
2196. Effect of Heat-Capacity Lag on the Flow through Oblique Shock Waves. By H. Reese Ivey and Charles W. Cline.
2197. Pressure Distribution and Damping in Steady Pitch at Supersonic Mach Numbers of Flat Swept-Back Wings Having All Edges Subsonic. By Harold J. Walker and Mary B. Ballantyne.
2198. Sintering Mechanism between Zirconium Carbide and Columbium. By H. J. Hamjian and W. G. Lidman.
2199. Wind-Tunnel Investigation at Low Speed of the Lateral Control Characteristics of an Unswept Untapered Semi-span Wing of Aspect Ratio 3.13 Equipped with Various 25-Percent-Chord Plain Ailerons. By Harold S. Johnson and John R. Hagerman.
2200. A Study of Second-Order Supersonic-Flow Theory. By Milton D. Van Dyke.
2201. Measurement of the Moments of Inertia of an Airplane by a Simplified Method. By Howard L. Turner.
2202. Effect of Heat and Power Extraction on Turbojet-Engine Performance. III—Analytical Determination of Effects of Shaft-Power Extraction. By Stanley L. Koutz, Reece V. Hensley, and Frank E. Rom.
2203. Boundary-Layer Measurements in 3.84- by 10-Inch Supersonic Channel. By Paul F. Brinich.
2204. Gust-Tunnel Investigation of a Wing Model with Semi-chord Line Swept Back 60°. By Harold B. Pierce.
2205. Theoretical Supersonic Characteristics of Inboard Trailing-Edge Flaps Having Arbitrary Sweep and Taper. Mach Lines behind Flap Leading and Trailing Edges. By Julian H. Kainer and Jack E. Marte.
2206. Graphical Method for Obtaining Flow Field in Two-Dimensional Supersonic Stream to Which Heat is Added. By I. Irving Pinkel and John S. Serafini.
2207. Analysis of Turbulent Free-Convection Boundary Layer on Flat Plate. By E. R. G. Eckert and Thomas W. Jackson.
2208. Analysis of Shear Strength of Honeycomb Cores for Sandwich Constructions. By Fred Werren and Charles B. Norris.
2209. Free Oscillations of an Atmosphere in Which Temperature Increases Linearly with Height. By C. L. Pekeris.
2210. Impact-Pressure Interpretation in a Rarefied Gas at Supersonic Speeds. By E. D. Kane and G. J. Maslach.
2211. An Approximate Method of Calculating Pressures in the Tip Region of a Rectangular Wing of Circular-Arc Section at Supersonic Speeds. By K. R. Czarnecki and James N. Mueller.
2212. The Effect of Ice Formations on Propeller Performance. By Carr B. Neel, Jr., and Loren G. Bright.
2213. Aerodynamic Coefficients for an Oscillating Airfoil with Hinged Flap, with Tables for a Mach Number of 0.7. By M. J. Turner and S. Rabinowitz.

- No. 2214. Formulas and Tables of Coefficients for Numerical Differentiation with Function Values Given at Unequally Spaced Points and Application to Solution of Partial Differential Equations. By Chung-Hua Wu.
2215. Compressibility Correction for Turning Angles of Axial-Flow Inlet Guide Vanes. By Seymour Lieblein and Donald M. Sandercock.
2216. Investigation of 75-Millimeter-Bore Cylindrical Roller Bearings at High Speeds. II—Lubrication Studies—Effect of Oil-Inlet Location, Angle, and Velocity for Single-Jet Lubrication. By E. Fred Macks and Zolton N. Nemeth.
2217. Analysis of Plane-Stress Problems with Axial Symmetry in Strain-Hardening Range. By M. H. Lee Wu.
2218. Diffusion of Chromium in a Cobalt-Chromium Solid Solutions. By John W. Weeton.
2219. The Dynamic Lateral Control Characteristics of Airplane Models Having Unswept Wings with Round- and Sharp-Leading-Edge Sections. By James L. Hassell and Charles V. Bennett.
2220. A Balsa-Dust Technique for Air-Flow Visualization and Its Application to Flow through Model Helicopter Rotors in Static Thrust. By Marion K. Taylor.
2221. Equations and Charts for the Rapid Estimation of Hinge-Moment and Effectiveness Parameters for Trailing-Edge Controls Having Leading and Trailing Edges Swept Ahead of the Mach Lines. By Kenneth L. Goin.
2222. A Method for the Determination of the Spanwise Load Distribution of a Flexible Swept Wing at Subsonic Speeds. By Richard B. Skoog and Harvey H. Brown.
2223. Investigation of the Flow through a Single-Stage Two-Dimensional Nozzle in the Langley 11-Inch Hypersonic Tunnel. By Charles H. McLellan, Thomas W. Williams, and Ivan E. Beckwith.
2224. Multiple-Film Back-Reflection Camera for Atomic Strain Studies. By Anthony B. Marmo.
2225. Bending and Buckling of Rectangular Sandwich Plates. By N. J. Hoff.
2226. Theoretical Analysis of Oscillations in Hovering of Helicopter Blades with Inclined and Offset Flapping and Lagging Hinge Axes. By M. Morduchow and F. G. Hinchey.
2227. Relation between Inflammables and Ignition Sources in Aircraft Environments. By Wilfred E. Scull.
2228. Effects of Modifications to the Leading-Edge Region on the Stalling Characteristics of the NACA 63-012 Airfoil Section. By John A. Kelly.
2229. The Effect of End Plates on Swept Wings at Low Speed. By John M. Riebe and James M. Watson.
2230. Synthesis and Purification of Alkyldiphenylmethane Hydrocarbons. 1—2-Methyldiphenylmethane, 3-Methyldiphenylmethane 2-Ethyldiphenylmethane, 4-Ethyldiphenylmethane and 4-Isopropyldiphenylmethane. By John H. Lamneck, Jr., and Paul H. Wise.
2231. Comparison of Fatigue Strengths of Bare and Alclad 24S-T3 Aluminum-Alloy Sheet Specimens Tested at 12 and 1,000 Cycles Per Minute. By Frank C. Smith, William C. Brueggeman, and Richard H. Harwell.
2232. Stress and Distortion Analysis of a Swept Box Beam Having Bulkheads Perpendicular to the Spars. By Richard R. Heldenfels, George W. Zender, and Charles Libove.
2233. Some Effects on Nonlinear Variation in the Directional Stability and Damping-in-Yawing Derivatives on the Lateral Stability of an Airplane. By Leonard Sternfield.
- No. 2234. Statistical Explanation of Spontaneous Freezing of Water Droplets. By Joseph Levine.
2235. The Boundary-Layer and Stalling Characteristics of the NACA 64A010 Airfoil Section. By Robert F. Peterson.
2236. Supersonic Flow around Circular Cones at Angles of Attack. By Antonio Ferri.
2237. Correlations of Heat-Transfer Data and of Friction Data for Interrupted Plane Fins Staggered in Successive Rows. By S. V. Manson.
2238. Effects on Longitudinal Stability and Control Characteristics of a B-29 Airplane of Variations in Stick-Force and Control-Rate Characteristics Obtained through Use of a Booster in the Elevator-Control System. By Charles W. Mathews, Donald B. Talmage, and James B. Whitten.
2239. Theoretical Investigation of Transonic Similarity for Bodies of Revolution. By W. Perl and Milton M. Klein.
2240. The Effect of Nonuniform Temperature Distributions on the Stresses and Distortions of Stiffened-Shell Structures. By Richard R. Heldenfels.
2241. A Numerical Method for the Stress Analysis of Stiffened-Shell Structures under Nonuniform Temperature Distributions. By Richard R. Heldenfels.
2242. Analytical Investigation of Turbulent Flow in Smooth Tubes with Heat Transfer with Variable Fluid Properties for Prandtl Number of 1. By Robert G. Deissler.
2243. Effect of Cell Shape on Compressive Strength of Hexagonal Honeycomb Structures. By L. A. Ringelstetter, A. W. Voss, and C. B. Norris.
2244. A Comparison of Theory and Experiment for High-Speed Free-Molecule Flow. By Jackson R. Stalder, Glen Goodwin, and Marcus O. Creager.
2245. Calculation of Compressible Potential Flow Past Slender Bodies of Revolution by an Integral Method. By Milton M. Klein and W. Perl.
2246. Method for Determining Distribution of Luminous Emitters in Cone of Laminar Bunsen Flame. By Thomas P. Clark.
2247. A Comparison of the Lateral Controllability with Flap and Plug Ailerons on a Sweptback-Wing Model Having Full-Span Flaps. By Powell M. Lovell, Jr.
2248. Analysis of the Effects of Design Pressure Ratio per Stage and Off-Design Efficiency on the Operating Range of Multistage Axial-Flow Compressors. By Melvyn Savage and Willard R. Westphal.
2249. The Spanwise Distribution of Lift for Minimum Induced Drag of Wings Having a Given Lift and a Given Bending Moment. By Robert T. Jones.
2250. An Analysis of the Applicability of the Hypersonic Similarity Law to the Study of Flow about Bodies of Revolution at Zero Angle of Attack. By Dorris M. Ehret, Vernon J. Rossow, and Victor I. Stevens.
2251. Effects of Mach Number up to 0.34 and Reynolds Number Up to 8×10^6 on the Maximum Lift Coefficient of a Wing of NACA 66-Series Airfoil Sections. By G. Chester Furlong and James E. Fitzpatrick.
2252. Formulas for Source, Doublet, and Vortex Distributions in Supersonic Wing Theory. By Harvard Lomax, Max. A. Heaslet, and Franklyn B. Fuller.
2253. On a Source-Sink Method for the Solution of the Prandtl-Busemann Iteration Equations in Two-Dimensional Compressible Flow. By Keith C. Harder and E. B. Klunker.
2254. Regenerator-Design Study and Its Application to Turbine-Propeller Engines. By S. V. Manson.

- No. 2255. Two-Dimensional Compressible Flow in Centrifugal Compressors with Logarithmic-Spiral Blades. By Gaylord O. Ellis and John D. Stanitz.
2256. Three-Dimensional, Unsteady-Lift Problems in High-Speed Flight—Basic Concepts. By Harvard Lomax, Max. A. Heaslet, and Franklyn B. Fuller.
2257. Temperature Distribution in Internally Heated Walls of Heat Exchangers Composed of Noncircular Flow Passages. By E. R. G. Eckert and George M. Low.
2258. Synthesis of Cyclopropane Hydrocarbons from Methylcyclopropyl Ketone. I—2-Cyclopropylpropene and 2-Cyclopropylpropane. By Vernon A. Slabey and P. H. Wise.
2259. Synthesis of Cyclopropane Hydrocarbons from Methylcyclopropyl Ketone. II—2-Cyclopropyl-1-Pentene, *cis* and *trans* 2-Cyclopropyl-2-Pentene and 2-Cyclopropylpentane. By Vernon A. Slabey and P. H. Wise.
2260. Synthesis and Purification of Some Alkylbiphenyls and Alkylbicyclohexyls. By Irving A. Goodman and Paul H. Wise.
2261. Wind-Tunnel Investigation of a Number of Total-Pressure Tubes at High Angles of Attack. Supersonic Speeds. By William Gracey, Donald E. Coletti, and Walter R. Russell.
2262. Rolling and Yawing Moments for Swept-Back Wings in Sideslip at Supersonic Speeds. By Seymour Lampert.
2263. The Use of a Luminescent Lacquer for the Visual Indication of Boundary-Layer Transition. By Jackson R. Stalder and Ellis G. Slack.
2264. Airfoil Profiles for Minimum Pressure Drag at Supersonic Velocities—General Analysis with Application to Linearized Supersonic Flow. By Dean R. Chapman.
2265. NACA VGH Recorder. By Norman R. Richardson.
2266. Strength of Heat-Resistant Laminates up to 375° C. By B. M. Axilrod and Martha A. Sherman.
2267. Inelastic Column Behavior. By John E. Duberg and Thomas W. Wilder, III.
2268. Tests of Two-Blade Propellers in the Langley 8-Foot High-Speed Tunnel to Determine the Effect on Propeller Performance of a Modification of Inboard Pitch Distribution. By James B. Delano and Melvin M. Carmel.
2269. A Structural-Efficiency Evaluation of Titanium at Normal and Elevated Temperatures. By George J. Heimerl and Paul F. Barrett.
2270. Theoretical Damping in Roll and Rolling Effectiveness of Slender Cruciform Wings. By Gaynor J. Adams.
2271. Further Study of Metal Transfer between Sliding Surfaces. By J. T. Burwell and C. D. Strang.
2272. Lateral Elastic Instability of Hat-Section Stringers under Compressive Load. By Stanley Goodman.
2273. Similarity Laws for Transonic Flow about Wings of Finite Span. By John R. Spreiter.
2274. Extension of the Theory of Oscillating Airfoils of Finite Span in Subsonic Compressible Flow. By Eric Reissner.
2275. A Survey of Stability Analysis Techniques for Automatically Controlled Aircraft. By Arthur L. Jones and Benjamin R. Briggs.
2276. Static and Fatigue Strengths of High-Strength Aluminum-Alloy Bolted Joints. By E. C. Hartmann, Marshall Holt, and I. D. Eaton.
2277. Effects of Compressibility on the Performance of Two Full-Scale Helicopter Rotors. By Paul J. Carpenter.
- No. 2278. Theoretical Symmetric Span Loading Due to Flap Deflection for Wings of Arbitrary Plan Form at Subsonic Speeds. By John DeYoung.
2279. Three-Dimensional Compressible Laminar Boundary-Layer Flow. By Franklin K. Moore.
2280. A Theoretical Analysis of the Effects of Fuel Motion on Airplane Dynamics. By Albert A. Schy.
2281. Detailed Computational Procedure for Design of Cascade Blades with Prescribed Velocity Distributions in Compressible Potential Flows. By George R. Costello, Robert L. Cummings, and John T. Sinnette, Jr.
2282. An Improved Approximate Method for Calculating Lift Distributions Due to Twist. By James C. Sivells.
2283. Method for Calculating Lift Distributions for Unswept Wings with Flaps or Ailerons by Use of Nonlinear Section Lift Data. By James C. Sivells and Gertrude C. Westrick.
2284. Lift, Pitching Moment, and Span Load Characteristics of Wings at Low Speed as Affected by Variations of Sweep and Aspect Ratio. By Edward J. Hopkins.
2285. Damping in Roll of Cruciform and Some Related Delta Wings at Supersonic Speeds. By Herbert S. Ribner.
2286. Preliminary Investigation of a New Type of Supersonic Inlet. By Antonio Ferri and Louis M. Nucci.
2287. An Investigation of Pure Bending in the Plastic Range When Loads Are Not Parallel to a Principal Plane. By Harry A. Williams.
2288. Estimation of Low-Speed Lift and Hinge-Moment Parameters for Full-Span Trailing-Edge Flaps on Lifting Surfaces with and without Sweepback. By Jules B. Dods, Jr.
2289. Elastic Constants for Corrugated-Core Sandwich Plates. By Charles Libove and Ralph E. Hubka.
2290. A Method for Calculating Stresses in Torsion-Box Covers with Cutouts. By Richard Rosecrans.
2291. Influence of Wall Boundary Layer upon the Performance of an Axial-Flow Fan Rotor. By Emanuel Boxer.
2292. Experimental Investigation of the Effects of Support Interference on the Drag of Bodies of Revolution at a Mach Number of 1.5. By Edward W. Perkins.
2293. The Physical Properties of Active Nitrogen in Low-Density Flow. By James M. Benson.
2294. Lift and Pitching Derivatives of Thin Sweptback Tapered Wings with Streamwise Tips and Subsonic Leading Edges at Supersonic Speeds. By Frank S. Malvestuto, Jr., and Dorothy M. Hoover.
2295. Chordwise and Compressibility Corrections to Slender-Wing Theory. By Harvard Lomax and Loma Sluder.
2296. Turbulent Boundary-Layer Temperature Recovery Factors in Two-Dimensional Supersonic Flow. By Maurice Tucker and Stephen H. Maslen.
2297. Effect of an Increase in Angle of Dead Rise on the Hydrodynamic Characteristics of a High-Length-Beam-Ratio Hull. By Walter E. Whitaker, Jr., and Paul W. Bryce, Jr.
2298. A Modification to Thin-Airfoil-Section Theory, Applicable to Arbitrary Airfoil Sections, to Account for the Effects of Thickness on the Lift Distribution. By David Graham.
2299. Exposure Tests of Galvanized-Steel-Stitched Aluminum Alloys. By Fred M. Reinhart.
2300. Analytical and Experimental Investigation of Effect of Twist on Vibrations of Cantilever Beams. By Alexander Mendelson and Selwyn Gendler.

- No.
2301. Linearized Solution and General Plastic Behavior of Thin Plate with Circular Hole in Strain-Hardening Range. By M. H. Lee Wu.
2302. A General Through-Flow Theory of Fluid Flow with Subsonic or Supersonic Velocity in Turbomachines of Arbitrary Hub and Casing Shapes. By Chung-Hua Wu.
2303. Correspondence Flows for Wings in Linearized Potential Fields at Subsonic and Supersonic Speeds. By Sidney M. Harmon.
2304. Effect of Heat and Power Extraction on Turbojet-Engine Performance. IV—Analytical Determination of Effects of Hot-Gas Bleed. By Stanley L. Koutz.
2305. An Analytical and Experimental Investigation of the Skin Friction of the Turbulent Boundary Layer on a Flat Plate at Supersonic Speeds. By Morris W. Rubesin, Randall C. Maydew, and Steven A. Varga.
2306. Meteorological Analysis of Icing Conditions Encountered in Low-Altitude Stratiform Clouds. By Dwight B. Kline and Joseph A. Walker.
2307. A Theoretical Method of Determining the Control Gear- ing and Time Lag Necessary for a Specified Damping of an Aircraft Equipped with a Constant-Time-Lag Au- topilot. By Ordway B. Gates, Jr., and Albert A. Schy.
2308. Vibratory Stresses in Propellers Operating in the Flow Field of a Wing-Nacelle-Fuselage Combination. By Vernon L. Rogallo, John C. Roberts, and Merritt R. Oldaker.
2309. Charts for Estimation of Longitudinal-Stability Deriva- tives for a Helicopter Rotor in Forward Flight. By Kenneth B. Amer and F. B. Gustafson.
2310. Generalization of Boundary-Layer Momentum-Integral Equations to Three-Dimensional Flows Including Those of Rotating System. By Artur Mager.
2311. Flight Investigation of the Variation of Static-Pressure Error of a Static-Pressure Tube with Distance Ahead of a Wing and a Fuselage. By William Gracey and El- wood F. Scheithauer.
2312. Bearing Strengths of Some Aluminum-Alloy Permanent- Mold Castings. By E. M. Finley.
2313. The Effects of Mass Distribution on the Low-Speed Dy- namic Lateral Stability and Control Characteristics of a Model with a 45° Sweptback Wing. By Donald E. Hewes.
2314. Effect of Quadratic Terms in Differential Equations of Atmospheric Oscillations. By C. L. Pekeris.
2315. Supersonic Lift and Pitching Moment of Thin Swept- back Tapered Wings Produced by Constant Vertical Ac- celeration. Subsonic Leading Edges and Supersonic Trailing Edges. By Frank S. Malvestuto, Jr., and Doro- thy M. Hoover.
2316. Wind-Tunnel Investigation at Low Speed of Lateral Con- trol Characteristics of an Untapered 45° Sweptback Semispan Wing of Aspect Ratio 1.59 Equipped with Various 25-Percent-Chord Plain Ailerons. By Harold S. Johnson and John R. Hagerman.
2317. Applications of Von Kármán's Integral Method in Super- sonic Wing Theory. By Chieh-Chien Chang.
2318. Full-Scale-Tunnel Investigation of the Static-Thrust Per- formance of a Coaxial Helicopter Rotor. By Robert D. Harrington.
2319. Some Properties of High-Purity Sintered Wrought Molyb- denum Metal at Temperatures up to 2,400° F. By R. A. Long, K. C. Dike, and H. R. Bear.
- No.
2320. Effects of Some Solution Treatments Followed by an Ag- ing Treatment on the Life of Small Cast Gas-Turbine Blades of a Cobalt-Chromium-Base Alloy. I—Effect of Solution-Treating Temperature. By C. Yaker and C. A. Hoffman.
2321. Analysis of Temperature Distribution in Liquid-Cooled Turbine Blades. By John N. B. Livingood and W. Byron Brown.
2322. Creep of Lead at Various Temperatures. By Peter W. Neurath and J. S. Koehler.
2324. Fatigue Strengths of Aircraft Materials. Axial-Load Fatigue Tests on Unnotched Sheet Specimens of 24S-T3 and 75S-T6 Aluminum Alloys and of SAE 4130 Steel. By H. J. Grover, S. M. Bishop, and L. R. Jackson.
2325. Development of Magnesium-Cerium Forged Alloys for Elevated-Temperature Service. By K. Grube, R. Kaiser, L. W. Eastwood, C. M. Schwartz, and H. C. Cross.
2326. Two-Dimensional Subsonic Compressible Flows Past Arbi- trary Bodies by the Variational Method. By Chi-Teh Wang.
2327. Fatigue Testing Machine for Applying a Sequence of Loads of Two Amplitudes. By Frank C. Smith, Darnley M. Howard, Ira Smith, and Richard Harwell.
2328. Method for Determining Pressure Drop of Monatomic Gases Flowing in Turbulent Motion through Constant- Area Passages with Simultaneous Friction and Heat Addition. By M. F. Valerino and R. B. Doyle.
2329. High-Temperature Protection of a Titanium-Carbide Cer- amal with a Ceramic-Metal Coating Having a High Chromium Content. By Dwight G. Moore, Stanley G. Benner, and William N. Harrison.
2330. Water-Landing Investigation of a Model Having Heavy Beam Loadings and 0° Angle of Dead Rise. By A. Ethelda McArver.
2331. Wind-Tunnel Investigation of a Number of Total-Pressure Tubes at High Angles of Attack. Subsonic Speeds. By William Gracey, William Letko, and Walter R. Russell.
2332. Analysis of the Effects of Wing Interference on the Tail Contributions to the Rolling Derivatives. By William H. Michael, Jr.
2333. Transient Aerodynamic Behavior of an Airfoil Due to Different Arbitrary Modes of Nonstationary Motions in a Supersonic Flow. By Chieh-Chien Chang.
2334. On Reflection of Shock Waves from Boundary Layers. By H. W. Liepmann, A. Roshko, and S. Dhawan.
2335. A Plan-Form Parameter for Correlating Certain Aerody- namic Characteristics of Swept Wings. By Franklin W. Diederich.
2336. Some Theoretical Characteristics of Trapezoidal Wings in Supersonic Flow and a Comparison of Several Wing- Flap Combinations. By Robert O. Piland.
2337. Approximate Calculation of Turbulent Boundary-Layer Development in Compressible Flow. By Maurice Tucker.
2338. Experimental Investigation of Localized Regions of Lam- inar-Boundary-Layer Separation. By William J. Bursnall and Laurence K. Loftin, Jr.
2339. Transonic Flow Past a Wedge Profile with Detached Bow Wave—General Analytical Method and Final Calculated Results. By Walter G. Vincenti and Cleo B. Wagoner.
2340. A Survey of Methods for Determining Stability Param- eters of an Airplane from Dynamic Flight Measure- ments. By Harry Greenberg.

- No.
2341. A Least Squares Curve Fitting Method with Applications to the Calculation of Stability Coefficients from Transient-Response Data. By Marvin Shinbrot.
2342. Evaluation of Packed Distillation Columns. I—Atmospheric Pressure. By Thaine W. Reynolds and George H. Sugimura.
2343. Comparison of Theoretical and Experimental Response of a Single-Mode Elastic System in Hydrodynamic Impact. By Robert W. Miller and Kenneth F. Merten.
2344. Method for Calculating Downwash Field Due to Lifting Surfaces at Subsonic and Supersonic Speeds. By Sidney M. Harmon.
2345. The Effect of an Arbitrary Surface-Temperature Variation along a Flat Plate on the Convective Heat Transfer in an Incompressible Turbulent Boundary Layer. By Morris W. Rubesin.
2346. An Iterative Transformation Procedure for Numerical Solution of Flutter and Similar Characteristic-Value Problems. By Myron L. Gossard.
2347. Effect of Aspect Ratio and Sweepback on the Low-Speed Lateral Control Characteristics of Untapered Low-Aspect-Ratio Wings Equipped with Retractable Ailerons. By Jack Fischel and John R. Hagerman.
2348. Effect of Aspect Ratio on the Low-Speed Lateral Control Characteristics of Unswept Untapered Low-Aspect-Ratio Wings. By Rodger L. Naeseth and William M. O'Hare.
2349. Fluctuations in a Spray Formed by Two Impinging Jets. By Marcus F. Heidmann and Jack C. Humphrey.
2350. On the Second-Order Tunnel-Wall-Constriction Corrections in Two-Dimensional Compressible Flow. By E. B. Klunker and Keith C. Harder.
2351. An Experimental Investigation of the Effect of Surface Heating on Boundary-Layer Transition on a Flat Plate in Supersonic Flow. By Robert W. Higgins and Constantine C. Pappas.
2352. Spin-Tunnel Investigation of the Effects of Mass and Dimensional Variations on the Spinning Characteristics of a Low-Wing Single-Vertical-Tail Model Typical of Personal-Owner Airplanes. By Walter J. Klinar and Jack H. Wilson.
2353. Charts and Tables for Use in Calculations of Downwash of Wings of Arbitrary Plan Form. By Franklin W. Diederich.
2354. A Numerical Approach to the Instability Problem of Monocoque Cylinders. By Bruno A. Boley, Joseph Kempner, and J. Mayers.
2355. Achromatization of Debye-Scherrer Lines. By Hans Ekstein and Stanley Siegel.
2356. Two-Dimensional Transonic Flow Past Airfoils. By Yung-Huai Kuo.
2357. Method for Calculation of Ram-Jet Performance. By John R. Henry and J. Buel Bennett.
2358. Effect of Vertical-Tail Area and Length on the Yawing Stability Characteristics of a Model Having a 45° Sweptback Wing. By William Letko.
2359. Floating Characteristics of a Plain and a Horn-Balanced Rudder at Spinning Attitudes as Determined from Rotary Tests on a Model of a Typical Low-Wing Personal-Owner Airplane. By William Bihrie, Jr.
2360. Effect of Tail Surfaces on the Base Drag of a Body of Revolution at Mach Numbers of 1.5 and 2.0. By J. Richard Spahr and Robert R. Dickey.
- No.
2361. Turbulence-Intensity Measurements in a Jet of Air Issuing from a Long Tube. By Barney H. Little, Jr., and Stafford W. Wilbur.
2362. Torsional Strength of Stiffened D-Tubes. By E. S. Kavanaugh and W. D. Drinkwater.
2363. On the Application of Mathieu Functions in the Theory of Subsonic Compressible Flow Past Oscillating Airfoils. By Eric Reissner.
2364. On Two-Dimensional Flow after a Curved Stationary Shock (with Special Reference to the Problem of Detached Shock Waves). By S. S. Shu.
2365. Analytical Evaluation of Aerodynamic Characteristics of Turbines with Nontwisted Rotor Blades. By William R. Slivka and David H. Silvern.
2366. Friction at High Sliding Velocities of Oxide Films on Steel Surfaces Boundary-Lubricated with Stearic-Acid Solutions. By Robert L. Johnson, Marshall B. Peterson, and Max A. Swikert.
2367. General Plastic Behavior and Approximate Solutions of Rotating Disk in Strain-Hardening Range. By M. H. Lee Wu.
2369. Torsion and Transverse Bending of Cantilever Plates. By Eric Reissner and Manuel Stein.
2370. Matrix Method of Determining the Longitudinal-Stability Coefficients and Frequency Response of an Aircraft from Transient Flight Data. By James J. Donegan and Henry A. Pearson.
2371. Possible Application of Blade Boundary-Layer Control to Improvement of Design and Off-Design Performance of Axial-Flow Turbomachines. By John T. Sinnette, Jr., and George R. Costello.
2372. Rectangular-Wind-Tunnel Blocking Corrections Using the Velocity-Ratio Method. By Rudolph W. Hensel.
2373. Practical Methods of Calculation Involved in the Experimental Study of an Autopilot and the Autopilot-Aircraft Combination. By Louis H. Smaus and Elwood C. Stewart.
2374. Effect of Initial Mixture Temperature on Flame Speed of Methane-Air, Propane-Air and Ethylene-Air Mixtures. By Gordon L. Dugger.
2375. On the Use of Coupled Modal Functions in Flutter Analysis. By Donald S. Woolston and Harry L. Runyan.
2376. Method for Analyzing Indeterminate Structures Stressed above Proportional Limit. By F. R. Steinbacher, C. N. Gaylord, and W. K. Rey.
2377. Effect of Fuel Immersion on Laminated Plastics. By W. A. Crouse, Margie Carickhoff and Margaret A. Fisher.
2378. Automatic Control Systems Satisfying Certain General Criteria on Transient Behavior. By Aaron S. Boksenbom and Richard Hood.
2379. An Investigation of the Effects of Jet-Outlet Cut-Off Angle on Thrust Direction and Body Pitching Moment. By James R. Blackaby.
2380. Effectiveness of Ceramic Coatings in Reducing Corrosion of Five Heat-Resistant Alloys by Lead-Bromide Vapors. By Dwight G. Moore and Mary W. Mason.
2381. Effect of Horizontal-Tail Location on Low Speed Static Longitudinal Stability and Damping in Pitch of a Model Having 45° Sweptback Wing and Tail Surfaces. By Jacob H. Lichtenstein.
2382. Effect of Horizontal-Tail Size and Tail Length on Low-Speed Static Longitudinal Stability and Damping in Pitch of a Model Having 45° Sweptback Wing and Tail Surfaces. By Jacob H. Lichtenstein.

- No.
2383. On a Solution of the Nonlinear Differential Equation for Transonic Flow Past a Wave-Shaped Wall. By Carl Kaplan.
2384. Preliminary Investigation of Wear and Friction Properties under Sliding Conditions of Materials Suitable for Cages of Rolling-Contact Bearings. By Robert L. Johnson, Max A. Swikert, and Edmond E. Bisson.
2385. Fundamental Aging Effects Influencing High-Temperature Properties of Solution-Treated Inconel X. By D. N. Frey, J. W. Freeman, and A. E. White.
2386. Studies of High-Temperature Protection of a Titanium-Carbide Ceramal by Chromium-Type Ceramic-Metal Coatings. By Dwight G. Moore, Stanley G. Benner and William N. Harrison.
2387. Three-Dimensional Unsteady Lift Problems in High-Speed Flight—The Triangular Wing. By Harvard Lomax, Max. A. Heaslet, and Franklyn B. Fuller.
2388. Axisymmetric Supersonic Flow in Rotating Impellers. By Arthur W. Goldstein.
2389. Fatigue Strengths of Aircraft Materials. Axial-Load Fatigue Tests on Notched Sheet Specimens of 24S-T3 and 75S-T6 Aluminum Alloys and of SAE 4130 Steel with Stress-Concentration Factors of 2.0 and 4.0. By H. J. Grover, S. M. Bishop, and L. R. Jackson.
2390. Fatigue Strengths of Aircraft Materials. Axial-Load Fatigue Tests on Notched Sheet Specimens of 24S-T3 and 75S-T6 Aluminum Alloys and of SAE 4130 Steel with Stress-Concentration Factor of 5.0. By H. J. Grover, S. M. Bishop, and L. R. Jackson.
2393. Flow through Cascades in Tandem. By William E. Spraglin.
2397. Influence of Tensile Strength and Ductility on Strengths of Rotating Disks in Presence of Material and Fabrication Defects of Several Types. By Arthur G. Holms, Joseph E. Jenkins, and Andrew J. Repko.
2398. Synthesis of Cyclopropane Hydrocarbons from Methylcyclopropyl Ketone. III—2-Cyclopropyl-1-Butene, *cis* and *trans* 2-Cyclopropyl-2-Butene, and 2-Cyclopropylbutane. By Vernon A. Slabey and Paul H. Wise.
2399. Applicability of the Hypersonic Similarity Rule to Pressure Distributions which Include the Effects of Rotation for Bodies of Revolution at Zero Angle of Attack. By Vernon J. Rossow.
2402. Construction and Use of Charts in Design Studies of Gas Turbines. By Sumner Alpert and Rose M. Litrenta.
2403. The Indicial Lift and Pitching Moment for a Sinking or Pitching Two-Dimensional Wing Flying at Subsonic or Supersonic Speeds. By Harvard Lomax, Max. A. Heaslet, and Loma Sluder.
2405. Investigation of NACA 64, 2-432 and 64, 3-440 Airfoil Sections with Boundary-Layer Control and an Analytical Study of Their Possible Applications. By Elmer A. Horton, Stanley F. Racisz, and Nicholas J. Paradiso.
2407. Method of Analysis for Compressible Flow Past Arbitrary Turbomachine Blades on General Surface of Revolution. By Chung-Hua Wu and Curtis A. Brown.
2408. Approximate Design Method for High-Solidity Blade Elements in Compressors and Turbines. By John D. Stanitz.
2409. Summary of Methods for Calculating Dynamic Lateral Stability and Response and for Estimating Lateral Stability Derivatives. By John P. Campbell and Marion O. McKinney.
- No.
2410. Analytical Investigation of Fully Developed Laminar Flow in Tubes with Heat Transfer with Fluid Properties Variable Along the Radius. By Robert G. Deissler.
2413. Flight Investigation of the Effect of Control Centering Springs on the Apparent Spiral Stability of a Personal-Owner Airplane. By John P. Campbell, Paul A. Hunter, Donald E. Hewes, and James B. Whitten.
2414. An Experimental Determination of the Critical Bending Moment of a Box Beam Stiffened by Posts. By Paul F. Barrett and Paul Seide.
2416. Theoretical Study of Some Methods for Increasing the Smoothness of Flight through Rough Air. By William H. Phillips and Christopher C. Kraft, Jr.
2418. Flow Separation Ahead of Blunt Bodies at Supersonic Speeds. By W. E. Moeckel.
2423. Theoretical Aerodynamic Characteristics of Bodies in a Free-Molecule-Flow Field. By Jackson R. Stalder and Vernon J. Zurick.
2424. Flight Investigation of the Effect of Transient Wing Response on Wing Strains of a Twin-Engine Transport Airplane in Rough Air. By Harry C. Mickleboro and C. C. Shuffelbarger.
2425. Plastic Stress-Strain Relations for 75S-T6 Aluminum Alloy Subjected to Biaxial Tensile Stresses. By Joseph Marin, B. H. Ulrich, and W. P. Hughes.
2431. Skin Friction of Incompressible Turbulent Boundary Layers Under Adverse Pressure Gradients. By Fabio R. Goldschmied.
2435. Direct-Reading Design Charts for 75S-T6 Aluminum-Alloy Flat Compression Panels Having Longitudinal Extruded Z-Section Stiffeners. By William A. Hickman and Norris F. Dow.
2438. Heat Transfer to Bodies in a High-Speed Rarefied-Gas Stream. By Jackson R. Stadler, Glen Goodwin, and Marcus O. Creager.
2441. Optical Methods Involving Light Scattering for Measuring Size and Concentration of Condensation Particles in Supercooled Hypersonic Flow. By Enoch J. Durbin.
2443. The Similarity Law for Hypersonic Flow about Slender Three-Dimensional Shapes. By Frank M. Hamaker, Stanford E. Neice, and A. J. Eggers, Jr.
2444. Effect of Stress-Solvent Crazing on Tensile Strength of Polymethyl Methacrylate. By B. M. Axilrod and Martha A. Sherman.
2490. Flight Investigation of Some Factors Affecting the Critical Tail Loads on Large Airplanes. By Harvey H. Brown.
2502. Examples of Three Representative Types of Airfoil-Section Stall at Low Speed. By George B. McCullough and Donald E. Gault.
2516. A Survey of Creep in Metals. By A. D. Schwoppe and L. R. Jackson.
2523. Rotogenerative Detection of Corrosion Currents. By Joseph B. McAndrew, William H. Colner, and Howard T. Francis.
2561. A Study of Poisson's Ratio in the Yield Region. By George Gerard and Sorrel Wildhorn.

TECHNICAL MEMORANDUMS¹

Citations to German reports in this list will use the following abbreviations where applicable:

ZWB—Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung des Generalluftzeugmeisters (German Central Publication Office for Aeronautical Reports).

FB—Forschungsbericht (Research Report).

UM—Untersuchungen und Mitteilungen (Reports and Memoranda).

No.

1241. Basic Differential Equations in General Theory of Elastic Shells. By V. S. Vlasov. From *Prikladnaya Matematika i Mekhanika*, Vol. 8, 1944, pp. 109–140.
1246. Hydrodynamic Properties of Planing Surfaces and Flying Boats. By N. A. Sokolov. From *Tsentrāl'nii Aerogidrodinamicheskii Institut*, Tsagi [CAHI Report] No. 149, 1932, pp. 1–39.
1248. Investigations of Lateral Stability of a Glide Bomb Using Automatic Control Having No Time Lag. By E. W. Sponder. From ZWB, FB 1819, May 1943.
1250. Subsonic Gas Flow Past a Wing Profile. By S. A. Christianovich and I. M. Yuriev. From *Prikladnaya Matematika i Mekhanika*, Vol. 11, No. 1, 1947, pp. 105–118.
1258. Flight Performance of a Jet Power Plant. III—Operating Characteristics of a Jet Power Plant as a Function of Altitude. By F. Weinig. From ZWB, FB 1743/3, Nov. 1, 1943.
1260. Exact Solutions of Equations of Gas Dynamics. By I. A. Kiebel. From *Prikladnaya Matematika i Mekhanika*, Vol. 11, 1947, pp. 193–198.
1261. On the Theory of the Propagation of Detonation in Gaseous Systems. By Y. B. Zeldovich. From *Zhurnal Eksperimentalnoi i Teoreticheskoi Fiziki*, Vol. 10, 1940, pp. 542–568.
1262. Turbulence and Heat Stratification. By H. Schlichting. From *Zeitschrift für Angewandte Mathematik und Mechanik*, Vol. 15, No. 6, December 1935, pp. 313–338.
1263. Contribution to the Problem of Buckling of Orthotropic Plates, with Special Reference to Plywood. By Wilhelm Thielemann. From *Forschungsgemeinschaft Halle* No. 150/19, Feb. 1945.
1268. Isentropic Phase Changes in Dissociating Gases and the Method of Sound Dispersion for the Investigation of Homogeneous Gas Reactions with Very High Speed. By Gerhard Damköhler. From *Zeitschrift für Elektrochemie*, Vol. 48, No. 2, 1942, pp. 62–82.
1269. Isentropic Phase Changes in Dissociating Gases and the Method of Sound Dispersion for the Investigation of Homogeneous Gas Reactions with Very High Speed. By Gerhard Damköhler. From *Zeitschrift für Elektrochemie*, Vol. 48, No. 3, 1942, pp. 116–131.
1271. Effect of Intense Sound Waves on a Stationary Gas Flame. By H. Hahnemann and L. Ehret. From *Zeitschrift für Technische Physik*, No. 10–12, 1943, pp. 228–242.
1272. Critical Velocities of Ultracentrifuges. By V. I. Sokolov. From *Zhurnal Tekhnicheskoi Fiziki* (U. S. S. R.), Vol. 16, No. 4, 1946, pp. 463–468.
1273. Resonance Sound Absorber with Yielding Wall. By S. N. Rzhavkin. From *Zhurnal Tekhnicheskoi Fiziki* (U. S. S. R.), Vol. 16, No. 4, 1946, pp. 381–394.
1276. Sideslip in a Viscous Compressible Gas. By V. V. Struminsky. From *Doklady, Akademii Nauk* (U. S. S. R.), Vol. 54, No. 9, 1946, pp. 769–772.

No.

1278. General Solution of Prandtl's Boundary-Layer Equation. By W. Mangler. From *Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht* 141, Oct. 1941, pp. 3–7.
1281. Unstable Capillary Waves on Surface of Separation of Two Viscous Fluids. By V. A. Borodin and Y. F. Dityakin. From *Prikladnaya Matematika i Mekhanika*, Vol. 13, No. 3, 1949, pp. 267–276.
1282. Theory of Flame Propagation. By Y. B. Zeldovich. From *Zhurnal Fizicheskoi Khimii* (U. S. S. R.), Vol. 22, 1948, pp. 27–49.
1283. Resistance of a Delta Wing in a Supersonic Flow. By E. A. Karpovich and F. I. Frankl. From *Prikladnaya Matematika i Mekhanika*, Vol. 11, No. 4, 1947, pp. 495–496.
1286. Method of Successive Approximations for the Solution of Certain Problems in Aerodynamics. By M. E. Shvets. From *Prikladnaya Matematika i Mekhanika*, Vol. 13, No. 3, 1949, pp. 257–266.
1287. Dependence of the Elastic Strain Coefficient of Copper on the Pre-Treatment. By W. Kuntze. From *Zeitschrift für Metallkunde*, Vol. 20, No. 4, 1928, pp. 145–150.
1288. The Diffusion of a Hot Air Jet in Air in Motion. By W. Szablewski. From *Luftfahrtforschungsanstalt Hermann Göring* (E. V.), Braunschweig, Sept. 1946.
1289. The Development of a Hollow Blade for Exhaust Gas Turbines. By H. Kohlmann. From ZWB, UM 788, Dec. 1943.
1291. On Stability and Turbulence of Fluid Flows. By Werner Heisenberg. From *Annalen der Physik*, Vol. 74, No. 15, 1924, pp. 577–627.
1292. Laws of Flow in Rough Pipes. By J. Nikuradse. From *VDI-Forschungsheft* 361, Supplement to "Forschung auf dem Gebiete des Ingenieurwesens," Series B, Vol. 4, July–August 1933.
1294. Exhaust Turbine and Jet Propulsion Systems. By K. Leist and E. Knörnschild. From ZWB, FB 1069, May 15, 1939.
1295. Gas Flow with Straight Transition Line. By L. V. Ovisiannikov. From *Prikladnaya Matematika i Mekhanika*, Vol. 13, 1949, pp. 537–542.
1296. On the Theory of Combustion of Initially Unmixed Gases. By Y. B. Zeldovich. From *Zhurnal Tekhnicheskoi Fiziki*, Vol. 19, No. 10, 1949, pp. 1199–1210.
1297. State and Development of Flutter Calculation. By A. Teichmann. From *Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht* 135, 1941, pp. 11–20.
1298. On the Problems of Gas Flow Over an Infinite Cascade Using Chaplygin's Approximation. By G. A. Bugaenko. From *Prikladnaya Matematika i Mekhanika*, Vol. 13, No. 4, 1949, pp. 449–456.
1299. Additional Measurements of the Drag of Surface Irregularities in Turbulent Boundary Layers. By W. Tillmann. From ZWB, UM 6619, Dec. 27, 1944.
1305. On Ionization and Luminescence in Flames. By E. Sängner, P. Goercke, and I. Bredt. From original manuscript received June 17, 1949.
1310. Correction Factors for Wind Tunnels of Elliptic Section with Partly Open and Partly Closed Test Section. By F. Riegels. From *Luftfahrtforschung*, Vol. 16, No. 1, 1939, pp. 26–30.
1312. Calculation of the Bending Stresses in Helicopter Rotor Blades. By P. de Guillenchmidt. From *Société Nationale de Constructions Aéronautiques du Centre, Rapport He3-0.03*, Dec. 23, 1948.

¹ The missing numbers in the series of Technical Memorandums were released before or after the period covered by this report.

OTHER TECHNICAL PAPERS BY STAFF MEMBERS

- Ault, G. Mervin, and Deutsch, G. C.: Review of NACA Research on Materials for Gas Turbine Blades. *SAE Quart. Trans.*, vol. 4, no. 3, July 1950, pp. 398-409; discussion, p. 409.
- Becker, John V.: Results of Recent Hypersonic and Unsteady Flow Research at the Langley Aeronautical Laboratory. *Jour. Appl. Physics*, vol. 21, no. 7, July 1950, pp. 619-628.
- Carpenter, Paul J.: Hovering and Low-Speed Performance and Control Characteristics of an Aerodynamic-Servocontrolled Helicopter Rotor System as Determined on the Langley Helicopter Tower. Paper 50-SA-30, A. S. M. E., 1950.
- Cohen, G., and Kuczynski, G. C.: Coefficient of Self-Diffusion of Copper. *Jour. Appl. Phys.*, vol. 21, no. 12, Dec. 1950, pp. 1339-40.
- Dedrick, J. H., and Kuczynski, G. C.: Electrical Conductivity Method for Measuring Self-Diffusion of Metals. *Jour. Appl. Phys.*, vol. 21, no. 12, Dec. 1950, pp. 1224-1225.
- Deissler, Robert G.: Investigation of Turbulent Flow and Heat Transfer in Smooth Tubes, Including the Effects of Variable Fluid Properties. *Trans. A. S. M. E.*, vol. 73, no. 2, Feb. 1951, pp. 101-107.
- Dryden, Hugh L.: A Guide to Recent Papers on the Turbulent Motion of Fluids. *Appl. Mech. Rev.*, vol. 4, no. 2, Feb. 1951, pp. 74-75.
- Dugger, Gordon L.: The Effect of Initial Mixture Temperature on Burning Velocity. *Jour. Am. Chem. Soc.*, vol. 73, no. 5, May 1951, p. 2398.
- Dugger, Gordon L.: Flame Propagation. I. The Effect of Initial Temperature on Flame Velocities of Propane-Air Mixtures. *Jour. Am. Chem. Soc.*, vol. 72, no. 11, Nov. 1950, pp. 5271-5274.
- Frick, C. W., and Chubb, R. S.: The Longitudinal Stability of Elastic Swept Wings at Supersonic Speed. *Jour. Aero. Sci.*, vol. 17, no. 11, Nov. 1950, pp. 691-704.
- Gerstein, Melvin; Levine, Oscar; and Wong, Edgar L.: Flame Propagation. II. The Determination of Fundamental Burning Velocities of Hydrocarbons by a Revised Tube Method. *Jour. Am. Chem. Soc.*, vol. 73, no. 1, Jan. 1951, pp. 418-422.
- Gessow, Alfred, and Amer, Kenneth B.: An Explanation of Some Important Stability Parameters That Influence Helicopter Flying Qualities. *Aero. Eng. Rev.*, vol. 9, no. 8, Aug. 1950, pp. 28-35.
- Goodman, Irving A., and Wise, Paul H.: Dicyclic Hydrocarbons. I—2-Alkylbiphenyls. *Jour. Am. Chem. Soc.*, vol. 72, no. 7, July 20, 1950, pp. 3076-3079.
- Goodman, Irving A., and Wise, Paul H.: Dicyclic Hydrocarbons. II—2-Alkylbicyclohexyls. *Jour. Am. Chem. Soc.*, vol. 73, no. 2, Feb. 1951, pp. 850-851.
- Gustafson, F. B.: Desirable Longitudinal Flying Qualities for Helicopters and Means to Achieve Them. *Aero. Eng. Rev.*, vol. 10, no. 6, June 1951, pp. 27-33.
- Hibbard, R. R., and Pinkel, B.: Flame Propagation. IV. Correlation of Maximum Fundamental Flame Velocity with Hydrocarbon Structure. *Jour. Am. Chem. Soc.*, vol. 73, no. 4, Apr. 1951, pp. 1622-1625.
- Himmel, Seymour C., and Krebs, Richard P.: The Effect of Changes in Altitude on the Controlled Behavior of a Gas-Turbine Engine. Preprint 320, Inst. Aero. Sci., 1951.
- Houbolt, John C.: A Recurrent Matrix Solution for the Dynamic Response of Elastic Aircraft. *Jour. Aero. Sci.*, vol. 17, no. 9, Sept. 1950, pp. 540-550, 594.
- Ivey, H. Reese: Hypersonic-Flow Analogy. Preprint 263, Inst. Aero. Sci., 1950.
- Jones, Robert T.: The Minimum Drag of Thin Wings in Frictionless Flow. *Jour. Aero. Sci.*, vol. 18, no. 2, Feb. 1951, pp. 75-81.
- Kuczynski, G. C., and Alexander, B. H.: Metallographic Study of Diffusion Interfaces. *Jour. Appl. Phys.*, vol. 22, no. 3, Mar. 1951, pp. 344-349.
- Lampert, Seymour: Conical Flow Methods Applied to Uniformly Loaded Wings in Subsonic Flow. *Jour. Aero. Sci.*, vol. 18, no. 2, Feb. 1951, pp. 107-114, 138.
- Landauer, Rolf: Conductivity of Cold-Worked Metals. *Phys. Rev.*, vol. 82, no. 4, May 15, 1951, pp. 520-521.
- Landauer, Rolf: Maximum Rate of Wave Function Amplitude Change. *Phys. Rev.*, vol. 82, no. 4, May 15, 1951, p. 549.
- Landauer, Rolf: Reflections in One-Dimensional Wave Mechanics. *Phys. Rev.*, vol. 82, no. 1, Apr. 1, 1951, pp. 80-83.
- McLellan, Charles H.: Exploratory Wind-Tunnel Investigation of Wings and Bodies at $M=6.9$. Preprint 322, Inst. Aero. Sci., 1951.
- Martin, J. C.: Retarded Potentials of Supersonic Flow. *Quart. Appl. Math.*, vol. VIII, no. 4, Jan. 1951, pp. 358-364.
- Nagey, Tibor F.: Four Turboprop Configurations. . . . How They Compare. *SAE Jour.*, vol. 59, no. 3, Mar. 1951, pp. 22-23.
- Neice, Stanford E., and Ehret, Dorris M.: Similarity Laws for Slender Bodies of Revolution in Hypersonic Flows. Preprint 321, Inst. Aero. Sci., 1951.
- Rodert, Lewis A.: Progress in Fire Prevention Following Crash. Preprint 265, Inst. Aero. Sci., 1950.
- Rogallo, Francis Melvin: Lateral Control of Personal Aircraft at High Lift Coefficients. *Aero. Eng. Rev.*, vol. 9, no. 8, Aug. 1950, pp. 18-22.
- Rogallo, Francis M.; Lowry, John G.; and Fischel, Jack: Lateral-Control Devices Suitable for Use with Full-Span Flaps. *Jour. Aero. Sci.*, vol. 17, no. 10, Oct. 1950, pp. 629-638, 652.
- Runyan, H. L.; Cunningham, H. J.; and Watkins, C. E.: Theoretical Investigation of Several Types of Single-Degree-of-Freedom Flutter. Preprint 323, Inst. Aero. Sci., 1951.
- Schwed, P.: Surface Diffusion in Sintering of Spheres on Planes. *Jour. Metals*, vol. 191, no. 3, Mar. 1951, pp. 245-246.
- Simon, Dorothy Martin: A Comparison of Quenching Distance and Inflammability Limit Data for Propane Air Flames. *Jour. Appl. Phys.*, vol. 22, no. 1, Jan. 1951, p. 103.
- Simon, Dorothy Martin: Flame Propagation. III. Theoretical Consideration of the Burning Velocities of Hydrocarbons. *Jour. Am. Chem. Soc.*, vol. 73, no. 1, Jan. 1951, pp. 422-425.
- Spakowski, A. E.; Evans, Albert; and Hibbard, R. R.: Determination of Aromatics and Olefins in Wide Boiling Petroleum Fractions. *Analytical Chem.*, vol. 22, no. 11, Nov. 1950, pp. 1419-1422.
- Spreiter, John R., and Sacks, Alvin H.: The Rolling Up of the Trailing Vortex Sheet and Its Effect on the Downwash behind Wings. *Jour. Aero. Sci.*, vol. 18, no. 1, Jan. 1951, pp. 21-32, 72.
- Stanitz, John D.: Analysis of the Exhaust Process in Four-Stroke Reciprocating Engines. *Trans. A. S. M. E.*, vol. 73, no. 3, Apr. 1951, pp. 319-329.
- Stevens, Victor I.: Hypersonic Research Facilities at the Ames Aeronautical Laboratory. *Jour. Appl. Physics*, vol. 21, no. 11, Nov. 1950, pp. 1150-1155.
- Wu, Chung H.: A General Theory of Three-Dimensional Flow with Subsonic and Supersonic Velocity in Turbomachines Having Arbitrary Hub and Casing Shapes. Paper 50-A-79, A. S. M. E., 1950.
- Wu, Chung-Hua, and Brown, Curtis A.: A Theory of the Direct and Inverse Problems of Compressible Flow Past Cascade of Arbitrary Airfoils. Preprint 326, Inst. Aero. Sci., 1951.
- Yntema, Robert T.: An Impulse-Momentum Method for Calculating Landing Gear Impact Conditions in Unsymmetrical Landings. Preprint 324, Inst. Aero. Sci., 1951.

Part II—COMMITTEE ORGANIZATION AND MEMBERSHIP

The act of Congress approved March 3, 1915, establishing the National Advisory Committee for Aeronautics, as amended (U. S. Code, Supplement IV, title 50, sec. 151), provides that the Committee shall consist of 17 members appointed by the President, and shall include "two representatives of the Department of the Air Force; two representatives of the Department of the Navy, from the office in charge of naval aeronautics; two representatives of the Civil Aeronautics Authority; one representative of the Smithsonian Institution; one representative of the United States Weather Bureau; one representative of the National Bureau of Standards; the Chairman of the Research and Development Board of the National Military Establishment; and not more than seven other members selected from persons acquainted with the needs of aeronautical science, either civil or military, or skilled in aeronautical engineering or its allied sciences." These latter seven members serve for terms of 5 years. The representatives of the Government organizations serve for indefinite periods. All members serve as such without compensation.

Hon. Donald W. Nyrop, Chairman of the Civil Aeronautics Board, was appointed by President Truman a member of the National Advisory Committee for Aeronautics under date of April 24, 1951, succeeding Hon. Delos W. Rentzel, who resigned from the Committee because of his appointment as Under Secretary of Commerce for Transportation.

In accordance with law, Hon. Walter G. Whitman was appointed a member of the Committee August 9, 1951, following his appointment as Chairman of the Research and Development Board. Professor Whitman succeeded Hon. William Webster, who resigned from the Committee in July, concurrently with his resignation as Chairman of the RDB.

Because of his retirement from the United States Air Force, the membership of Maj. Gen. Gordon P. Saville on the Committee was terminated July 31, 1951. General Saville was serving as Deputy Chief of Staff, Development, in the Air Force. Another termination of membership resulted from the resignation of Dr. Edward U. Condon as Director of the National Bureau of Standards, effective September 30, 1951. Successors to General Saville and Dr. Condon on the Committee have not been appointed.

In accordance with the regulations governing the organization of the Committee as approved by the President, the Chairman and Vice Chairman are elected

annually, as are also the Chairman and Vice Chairman of the Executive Committee.

On October 12, 1951, Dr. Jerome C. Hunsaker was reelected Chairman of the NACA and of the Executive Committee, and Dr. Alexander Wetmore and Dr. Francis W. Reichelderfer were reelected Vice Chairman of the NACA and Vice Chairman of the Executive Committee, respectively.

COMMITTEE ON AERODYNAMICS

Dr. Theodore P. Wright, Cornell University, Chairman.
Capt. Walter S. Diehl, U. S. N. (Ret.), Vice Chairman.
Dr. Albert E. Lombard, Jr., Office, Directorate of Research and Development, U. S. Air Force.
Col. Jack A. Gibbs, U. S. A. F., Air Matériel Command.
Mr. F. A. Loudon, Bureau of Aeronautics, Department of the Navy.
Capt. M. R. Kelley, U. S. N. (Ret.), Bureau of Ordnance.
Brig. Gen. Leslie E. Simon, U. S. A., Chief, Ordnance Research and Development Division.
Mr. Harold D. Hoekstra, Civil Aeronautics Administration.
Dr. Hugh L. Dryden (ex officio).
Mr. Floyd L. Thompson, NACA Langley Aeronautical Laboratory.
Mr. Russell G. Robinson, NACA Ames Aeronautical Laboratory.
Prof. Jesse W. Beams, University of Virginia.
Mr. Edward J. Horkey, North American Aviation, Inc.
Mr. Clarence L. Johnson, Lockheed Aircraft Corp.
Mr. John G. Lee, United Aircraft Corp.
Mr. Grover Loening.
Dr. Clark B. Millikan, California Institute of Technology.
Dr. W. Bailey Oswald, Douglas Aircraft Co., Inc.
Mr. George S. Schairer, Boeing Airplane Co.
Prof. E. S. Taylor, Massachusetts Institute of Technology.
Mr. Robert J. Woods, Bell Aircraft Corp.

Mr. Milton B. Ames, Jr., Secretary

Subcommittee on Fluid Mechanics

Dr. Clark B. Millikan, California Institute of Technology, Chairman.
Dr. Frank L. Wattendorf, DCS/Development, Hq., U. S. Air Force.
Mr. L. S. Wasserman, Wright Air Development Center.
Capt. W. S. Diehl, U. S. N. (Ret.).
Dr. E. Bromberg, Office of Naval Research, Department of the Navy.
Dr. Raymond J. Seeger, Naval Ordnance Laboratory.
Mr. Joseph Sternberg, Ballistic Research Laboratories, Aberdeen Proving Ground.
Dr. G. B. Schubauer, National Bureau of Standards.
Dr. Carl Kaplan, NACA Langley Aeronautical Laboratory.
Mr. John Stack, NACA Langley Aeronautical Laboratory.
Mr. Robert T. Jones, NACA Ames Aeronautical Laboratory.
Mr. Walter G. Vincenti, NACA Ames Aeronautical Laboratory.

Dr. John C. Evvard, NACA Lewis Flight Propulsion Laboratory.
 Dr. William Bollay, North American Aviation, Inc.
 Dr. Francis H. Clauser, The Johns Hopkins University.
 Prof. Howard W. Emmons, Harvard University.
 Dr. Hans W. Liepmann, California Institute of Technology.
 Dr. C. C. Lin, Massachusetts Institute of Technology.
 Dr. William R. Sears, Cornell University.

Mr. E. O. Pearson, Jr., Secretary

Subcommittee on High-Speed Aerodynamics

Mr. John G. Lee, United Aircraft Corp., Chairman.
 Col. Robert M. Wray, U. S. A. F., Wright Air Development Center.
 Mr. H. L. Anderson, Wright Air Development Center.
 Commander Sydney S. Sherby, U. S. N., Bureau of Aeronautics.
 Dr. George L. Shue, Naval Ordnance Laboratory.
 Mr. C. L. Poor, III, Ballistic Research Laboratories, Aberdeen Proving Ground.
 Mr. Robert R. Gilruth, NACA Langley Aeronautical Laboratory.
 Mr. John Stack, NACA Langley Aeronautical Laboratory.
 Mr. H. Julian Allen, NACA Ames Aeronautical Laboratory.
 Mr. Abe Silverstein, NACA Lewis Flight Propulsion Laboratory.
 Mr. Irving L. Ashkenas, Northrop Aircraft, Inc.
 Mr. Benedict Cohn, Boeing Airplane Co.
 Mr. David S. Lewis, Jr., McDonnell Aircraft Corp.
 Mr. Harold Luskin, Douglas Aircraft Co., Inc.
 Prof. John R. Markham, Massachusetts Institute of Technology.
 Mr. C. E. Pappas, Republic Aviation Corp.
 Dr. Allen E. Puckett, Hughes Aircraft Co.
 Mr. William C. Schoolfield, United Aircraft Corp.
 Mr. H. A. Storms, Jr., North American Aviation, Inc.
 Mr. R. H. Widmer, Consolidated Vultee Aircraft Corp.

Mr. Edward C. Polhamus, Secretary

Subcommittee on Stability and Control

Capt. Walter S. Diehl, U. S. N. (Ret.), Chairman.
 Mr. Melvin Shorr, Wright Air Development Center.
 Mr. Gerald G. Kayten, Bureau of Aeronautics, Department of the Navy.
 Mr. Philippe W. Newton, U. S. Army Ordnance Corps.
 Mr. John A. Carran, Civil Aeronautics Administration.
 Mr. Thomas A. Harris, NACA Langley Aeronautical Laboratory.
 Mr. Harry J. Goett, NACA Ames Aeronautical Laboratory.
 Mr. George S. Graff, McDonnell Aircraft Corp.
 Mr. Herbert Harris, Jr., Sperry Gyroscope Co., Inc.
 Mr. Stuart A. Krieger, Northrop Aircraft, Inc.
 Mr. W. F. Milliken, Jr., Cornell Research Foundation, Inc.
 Mr. Dale D. Myers, North American Aviation, Inc.
 Prof. Courtland D. Perkins, Princeton University.
 Prof. Robert C. Seamans, Jr., Massachusetts Institute of Technology.
 Mr. Ralph H. Shick, Consolidated Vultee Aircraft Corp.
 Mr. Charles Tilgner, Jr., Grumman Aircraft Engineering Corp.
 Prof. Fred E. Weick, Agricultural and Mechanical College of Texas.

Mr. Jack D. Brewer, Secretary

Subcommittee on Internal Flow

Mr. Philip A. Colman, Lockheed Aircraft Corp., Chairman.
 Mr. Joseph Flatt, Wright Air Development Center.
 Capt. Walter Haaser, U. S. A. F., Wright Air Development Center.
 Commander R. L. Duncan, U. S. N., Office of Naval Research.

Mr. Palmer R. Wood, Bureau of Aeronautics, Department of the Navy.

Mr. C. L. Zakhartchenko, National Bureau of Standards.
 Mr. John V. Becker, NACA Langley Aeronautical Laboratory.
 Mr. E. A. Mossman, NACA Ames Aeronautical Laboratory.
 Mr. DeMarquis D. Wyatt, NACA Lewis Flight Propulsion Laboratory.
 Mr. J. S. Alford, General Electric Co.
 Mr. Leo A. Geyer, Grumman Aircraft Engineering Corp.
 Mr. Henry H. Hoadley, United Aircraft Corp.
 Dr. William J. O'Donnell, Republic Aviation Corp.
 Mr. Otto P. Prachar, General Motors Corp.
 Mr. Ascher H. Shapiro, Massachusetts Institute of Technology.
 Mr. Maurice A. Sulkin, North American Aviation, Inc.

Mr. Edward C. Polhamus, Secretary.

Subcommittee on Propellers for Aircraft

Mr. Thomas B. Rhines, Hamilton Standard Division, United Aircraft Corp., Chairman.
 Mr. Anthony F. Dernbach, Wright Air Development Center.
 Mr. Daniel A. Dickey, Wright Air Development Center.
 Mr. Ivan H. Driggs, Bureau of Aeronautics, Department of the Navy.
 Mr. John C. Morse, Civil Aeronautics Administration.
 Mr. Eugene C. Draley, NACA Langley Aeronautical Laboratory.
 Mr. Robert M. Crane, NACA Ames Aeronautical Laboratory.
 Mr. George W. Brady, Curtiss Wright Corp.
 Mr. Frank W. Caldwell, United Aircraft Corp.
 Mr. Wilfred W. Davies, United Air Lines, Inc.
 Mr. Ralph R. LaMotte, General Motors Corp.
 Mr. E. B. Maske, Jr., Consolidated Vultee Aircraft Corp.
 Mr. Robert B. Smith, Douglas Aircraft Co., Inc.

Mr. Ralph W. May, Secretary.

Subcommittee on Seaplanes

Mr. Grover Loening, Chairman.
 Mr. Joseph A. Ellis, Wright Air Development Center.
 Mr. J. H. Herald, Wright Air Development Center.
 Mr. Charles J. Daniels, Bureau of Aeronautics, Department of the Navy.
 Mr. F. W. S. Locke, Jr., Bureau of Aeronautics, Department of the Navy.
 Mr. Charlie C. Garrison, Office of Naval Research, Department of the Navy.
 Commander W. C. Fortune, U. S. N., David Taylor Model Basin.
 Commander John A. Ferguson, U. S. N., Naval Air Test Center, Patuxent.
 Capt. Donald B. MacDiarmid, U. S. C. G., U. S. Coast Guard Air Station, San Diego.
 Mr. John B. Parkinson, NACA Langley Aeronautical Laboratory.
 Mr. Robert B. Cotton, All American Airways, Inc.
 Dr. K. S. M. Davidson, Stevens Institute of Technology.
 Mr. J. D. Pierson, the Glenn L. Martin Co.
 Mr. William R. Ryan, Edo Corp.
 Mr. E. G. Stout, Consolidated Vultee Aircraft Corp.
 Mr. Henry B. Suydam, Grumman Aircraft Engineering Corp.

Mr. Ralph W. May, Secretary.

Subcommittee on Helicopters

Mr. Richard H. Prewitt, Prewitt Aircraft Co., Chairman.
 Mr. Bernard Lindenbaum, Wright Air Development Center.

Capt. Paul Simmons, Jr., U. S. A. F., Wright Air Development Center.
 Mr. Raymond A. Young, Bureau of Aeronautics, Department of the Navy.
 Commander Frank A. Erickson, U. S. C. G., U. S. Naval Air Station, Patuxent.
 Lt. Col. Jack L. Marinelli, F. A., Army Field Forces.
 Mr. Burdell L. Springer, Civil Aeronautics Administration.
 Mr. R. B. Maloy, Civil Aeronautics Administration.
 Mr. Richard C. Dingeldein, NACA Langley Aeronautical Laboratory.
 Mr. F. B. Gustafson, NACA Langley Aeronautical Laboratory.
 Mr. Michael E. Gluhareff, Sikorsky Aircraft, United Aircraft Corp.
 Mr. Bartram Kelley, Bell Aircraft Corp.
 Mr. Rene H. Miller, Massachusetts Institute of Technology.
 Mr. Robert R. Osborn, McDonnell Aircraft Corp.
 Mr. F. N. Piasecki, Piasecki Helicopter Corp.
 Mr. N. M. Stefano, Hughes Aircraft Co.

Mr. Leslie E. Schneider, Secretary.

Special Subcommittee on the Upper Atmosphere

Dr. Harry Wexler, U. S. Weather Bureau, Chairman.
 Col. Benjamin G. Holzman, U. S. A. F., Office, Deputy Chief of Staff for Matériel, Department of the Air Force.
 Dr. Sverre Pettersen, Air Weather Service, U. S. Air Force.
 Capt. Walter S. Diehl, U. S. N. (Ret.).
 Dr. Homer E. Newell, Jr., Naval Research Laboratory.
 Dr. O. R. Wulf, California Institute of Technology.
 Dr. W. G. Brombacher, National Bureau of Standards.
 Mr. William J. O'Sullivan, NACA Langley Aeronautical Laboratory.
 Dr. Carl W. Gartlein, Cornell University.
 Dr. B. Gutenberg, California Institute of Technology.
 Dr. Harvey Hall, Office of Naval Research, Department of the Navy.
 Prof. Bernhard Haurwitz, New York University.
 Dr. Joseph Kaplan, University of California.
 Dr. W. W. Kellogg, The Rand Corp.
 Dr. Zdenek Kopal, Massachusetts Institute of Technology.
 Dr. C. L. Pekeris, the Institute for Advanced Study.
 Dr. J. A. Van Allen, The Johns Hopkins University.
 Dr. Harry W. Wells, Carnegie Institution of Washington.
 Dr. Fred L. Whipple, Harvard University.

Mr. Leslie E. Schneider, Secretary.

COMMITTEE ON POWER PLANTS FOR AIRCRAFT

Mr. Ronald M. Hazen, Allison Division, General Motors Corp., Chairman.
 Prof. E. S. Taylor, Massachusetts Institute of Technology, Vice Chairman.
 Col. Norman C. Appold, U. S. A. F., Chief, Power Plant Laboratory.
 Col. Paul F. Nay, U. S. A. F., Chief, Propulsion Branch, Aircraft Division.
 Capt. E. M. Condra, Jr., U. S. N., Bureau of Aeronautics.
 Mr. Stephen Rolle, Civil Aeronautics Administration.
 Dr. Hugh L. Dryden (ex officio).
 Mr. Abe Silverstein, NACA Lewis Flight Propulsion Laboratory.
 Mr. Frank W. Davis, Consolidated Vultee Aircraft Corp.
 Dr. Louis G. Dunn, California Institute of Technology.
 Mr. William M. Holaday, Socony Vacuum Oil Co., Inc.
 Mr. R. P. Kroon, Westinghouse Electric Corp.
 Mr. William C. Lawrence, American Airlines, Inc.

Mr. Wilton G. Lundquist, Wright Aeronautical Corp., Division of Curtiss-Wright Corp.
 Mr. Wright A. Parkins, Pratt and Whitney Aircraft Division, United Aircraft Corp.
 Mr. George S. Schairer, Boeing Airplane Co.
 Mr. Raymond W. Young, Reaction Motors, Inc.

Mr. William H. Woodward, Secretary.

Subcommittee on Aircraft Fuels

Dr. J. Bennett Hill, Sun Oil Co., Chairman.
 Lt. Col. J. H. Lee, U. S. A. F., Propulsion Branch, Aircraft Division.
 Maj. M. W. Shayeson, U. S. A. F., Wright Air Development Center.
 Comdr. Robert F. Wadsworth, U. S. N., Bureau of Aeronautics.
 Mr. Ralph S. White, Civil Aeronautics Administration.
 Dr. L. C. Gibbons, NACA Lewis Flight Propulsion Laboratory.
 Dr. D. P. Barnard, Standard Oil Co. of Indiana.
 Mr. A. J. Blackwood, Standard Oil Development Co.
 Mr. J. L. Cooley, California Research Corp.
 Mr. S. D. Heron, Ethyl Corp.
 Mr. W. M. Holaday, Socony-Vacuum Oil Co., Inc.
 Mr. C. R. Johnson, Shell Oil Co.
 Mr. Melvin H. Young, Wright Aeronautical Corp., Division of Curtiss-Wright Corp.
 Dr. W. E. Kuhn, The Texas Co.
 Mr. O. E. Rodgers, Aviation Gas Turbine Division, Westinghouse Electric Corp.
 Mr. Earle A. Ryder, Pratt and Whitney Aircraft Division, United Aircraft Corp.
 Mr. Harold M. Trimble, Phillips Petroleum Co.

Mr. Henry E. Alquist, Secretary.

Subcommittee on Combustion

Dr. Bernard Lewis, Bureau of Mines, Chairman.
 Dr. D. G. Samaras, Wright Air Development Center.
 Mr. Garrett L. Wander, Wright Air Development Force.
 Mr. N. N. Tilley, Bureau of Aeronautics, Department of the Navy.
 Mr. Edward H. Seymour, Office of Naval Research, Department of the Navy.
 Dr. Ernest F. Flock, National Bureau of Standards.
 Dr. Walter T. Olson, NACA Lewis Flight Propulsion Laboratory.
 Mr. Edmund D. Brown, Pratt and Whitney Aircraft Division, United Aircraft Corp.
 Dr. Alfred G. Cattaneo, Shell Development Co.
 Prof. Newman A. Hall, University of Minnesota.
 Dr. John P. Longwell, Standard Oil Development Co.
 Mr. A. J. Nerad, General Electric Co.
 Dr. Robert N. Pease, Princeton University.
 Mr. Edwin P. Walsh, Westinghouse Electric Corp.
 Prof. Glenn C. Williams, Massachusetts Institute of Technology.
 Dr. Kurt Wohl, University of Delaware.

Mr. Henry E. Alquist, Secretary.

Subcommittee on Lubrication and Wear

Mr. E. M. Phillips, General Electric Co., Chairman.
 Mr. J. Josteller, Wright Air Development Center.
 Mr. Robert C. Sheard, Wright Air Development Center.
 Mr. Charles C. Singleterry, Bureau of Aeronautics, Department of the Navy.
 Dr. William Zisman, Naval Research Laboratory, Department of the Navy.

Mr. Edmond E. Bisson, NACA Lewis Flight Propulsion Laboratory.
 Mr. Richard W. Blair, Wright Aeronautical Corp., Division of Curtiss-Wright Corp.
 Dr. John T. Burwell, Jr., Horizons, Inc.
 Dr. Merrell R. Fenske, Pennsylvania State College.
 Mr. Daniel Gurney, Marlin-Rockwell Corp.
 Dr. Robert G. Larsen, Shell Development Co.
 Mr. C. J. McDowall, Allison Division, General Motors Corp.
 Mr. Joseph Palsulich, Cleveland Graphite Bronze Co.
 Mr. Earle A. Ryder, Pratt and Whitney Aircraft Division, United Aircraft Corp.
 Mr. C. H. Stockdale, Westinghouse Electric Corp.
 Dr. Haakon Styri, SKF Industries, Inc.
 Mr. Arthur F. Underwood, General Motors Corp.
 Mr. W. Andrew Wright, Sun Oil Co.

Mr. William H. Woodward, Secretary.

Subcommittee on Engine Performance and Operation

Mr. Arnold H. Redding, Westinghouse Electric Corp., Chairman.
 Mr. Opie Chenoweth, Wright Air Development Center.
 Mr. C. C. Sorgen, Bureau of Aeronautics, Department of the Navy.
 Mr. Bruce T. Lundin, NACA Lewis Flight Propulsion Laboratory.
 Dr. John L. Barnes, North American Aviation, Inc.
 Prof. Gordon S. Brown, Massachusetts Institute of Technology.
 Mr. Dimitrius Gerdan, Allison Division, General Motors Corp.
 Mr. John Karanik, Grumman Aircraft Engineering Corp.
 Dr. Roy E. Marquardt, Marquardt Aircraft Co.
 Dr. William J. O'Donnell, Republic Aviation Corp.
 Mr. Erol F. Pierce, Wright Aeronautical Corp., Division of Curtiss-Wright Corp.
 Mr. Perry W. Pratt, Pratt and Whitney Aircraft Division, United Aircraft Corp.
 Mr. E. E. Stoeckly, General Electric Co.
 Mr. Lon Storey, Jr., Lockheed Aircraft Corp.
 Mr. Lee R. Woodworth, The Rand Corp.

Mr. Richard S. Cesaro, Secretary.

Subcommittee on Compressors and Turbines

Prof. Howard W. Emmons, Harvard University, Chairman.
 Mr. Ernest C. Simpson, Wright Air Development Center.
 Mr. Robert W. Pinnes, Bureau of Aeronautics, Department of the Navy.
 Commander R. L. Duncan, U. S. N., Office of Naval Research, Department of the Navy.
 Mr. John R. Erwin, NACA Langley Aeronautical Laboratory.
 Mr. Robert O. Bullock, NACA Lewis Flight Propulsion Laboratory.
 Mr. Kenneth Campbell, Wright Aeronautical Corp., Division of Curtiss-Wright Corp.
 Mr. Walter Doll, Pratt and Whitney Aircraft Division, United Aircraft Corp.
 Mr. Jack C. Fethers, Allison Division, General Motors Corp.
 Mr. R. S. Hall, General Electric Co.
 Dr. Frank E. Marble, California Institute of Technology.
 Mr. Charles A. Meyer, Westinghouse Electric Corp.
 Prof. George F. Wislicenus, The Johns Hopkins University.

Mr. Guy N. Ullman, Secretary.

Subcommittee on Heat-Resisting Materials

Mr. Arthur W. F. Green, General Motors Corp., Chairman.
 Mr. J. B. Johnson, Wright Air Development Center.
 Mr. Nathan E. Promisel, Bureau of Aeronautics, Department of the Navy.
 Mr. Benjamin Pinkel, NACA Lewis Flight Propulsion Laboratory.
 Mr. W. L. Badger, General Electric Co.
 Mr. Howard C. Cross, Battelle Memorial Institute.
 Mr. P. G. DeHuff, Jr., Westinghouse Electric Corp.
 Mr. Russell Franks, Union Carbide & Carbon Corp.
 Mr. Herbert J. French, International Nickel Co.
 Prof. Nicholas J. Grant, Massachusetts Institute of Technology.
 Mr. Alvin J. Herzig, Climax Molybdenum Co. of Michigan.
 Dr. L. H. Milligan, Norton Co.
 Dr. Gunther Mohling, Allegheny Ludlum Steel Corp.
 Mr. Rudolf H. Thielemann, Pratt and Whitney Aircraft Division, United Aircraft Corp.

Mr. William H. Woodward, Secretary.

Special Subcommittee on Rocket Engines

Dr. Maurice J. Zucrow, Purdue University, Chairman.
 Mr. C. W. Schnare, Wright Air Development Center.
 Commander K. C. Childers, Jr., U. S. N., Bureau of Aeronautics, Department of the Navy.
 Capt. Levering Smith, U. S. N., Naval Ordnance Test Station, Inyokern.
 Mr. Joseph L. Gray, Office of the Chief of Ordnance, Department of the Army.
 Mr. Paul R. Hill, NACA Langley Aeronautical Laboratory.
 Mr. John L. Sloop, NACA Lewis Flight Propulsion Laboratory.
 Mr. Richard B. Canright, California Institute of Technology.
 Mr. R. Bruce Foster, Bell Aircraft Corp.
 Mr. Stanley L. Gendler, The Rand Corp.
 Dr. George E. Moore, General Electric Co.
 Mr. Thomas E. Myers, North American Aviation, Inc.
 Mr. Jack H. Sheets, Curtiss-Wright Corp.
 Dr. Robert J. Thompson, Jr., M. W. Kellogg Co.
 Dr. Paul F. Winternitz, Reaction Motors, Inc.
 Mr. David A. Young, Aerojet Engineering Corp.

Mr. Henry E. Alquist, Secretary.

COMMITTEE ON AIRCRAFT CONSTRUCTION

Dr. Arthur E. Raymond, Douglas Aircraft Co., Inc., Chairman.
 Mr. R. L. Templin, Aluminum Co. of America, Vice Chairman.
 Mr. E. H. Schwartz, Wright Air Development Center.
 Capt. Robert S. Hatcher, U. S. N., Bureau of Aeronautics, Department of the Navy.
 Mr. Albert A. Vollmecke, Civil Aeronautics Administration.
 Dr. Hugh L. Dryden (ex officio).
 Dr. Henry J. E. Reid, NACA Langley Aeronautical Laboratory.
 Mr. Carlton Bioletti, NACA Ames Aeronautical Laboratory.
 Prof. Raymond L. Bisplinghoff, Massachusetts Institute of Technology.
 Prof. Emerson W. Conlon, University of Michigan.
 Dr. C. C. Furnas, Cornell Aeronautical Laboratory, Inc.
 Dr. Alexander A. Kartveli, Republic Aviation Corp.
 Mr. W. C. Mentzer, United Air Lines, Inc.
 Mr. George Snyder, Boeing Airplane Co.
 Mr. Luther P. Spalding, North American Aviation, Inc.
 Mr. Charles R. Strang, Douglas Aircraft Co., Inc.

Mr. Franklyn W. Phillips, Secretary.

Subcommittee on Aircraft Structures

Mr. Charles R. Strang, Douglas Aircraft Co., Inc., Chairman.
 Mr. J. W. Farrell, Wright Air Development Center.
 Mr. Joseph Kelley, Jr., Wright Air Development Center.
 Commander L. S. Chambers, U. S. N., Bureau of Aeronautics, Department of the Navy.
 Mr. Ralph L. Creel, Bureau of Aeronautics, Bureau of Aeronautics, Department of the Navy.
 Mr. William T. Shuler, Civil Aeronautics Administration.
 Mr. Samuel Levy, National Bureau of Standards.
 Dr. Eugene E. Lundquist, NACA Langley Aeronautical Laboratory.
 Prof. W. H. Gale, Massachusetts Institute of Technology.
 Mr. Ira G. Hedrick, Grumman Aircraft Engineering Corp.
 Dr. Nicholas J. Hoff, Polytechnic Institute of Brooklyn.
 Mr. Jerome F. McBrearty, Lockheed Aircraft Corp.
 Mr. Francis MeVay, Republic Aviation Corp.
 Mr. John H. Meyer, McDonnell Aircraft Corp.
 Prof. Ernest E. Sechler, California Institute of Technology.
 Mr. R. L. Templin, Aluminum Co. of America.

Mr. Melvin G. Rosche, Secretary.

Subcommittee on Aircraft Loads

Mr. George Snyder, Boeing Airplane Co., Chairman.
 Mr. Joseph H. Harrington, Wright Air Development Center.
 Mr. Edward J. Lunney, Wright Air Development Center.
 Commander Donald J. Hardy, U. S. N., Bureau of Aeronautics, Department of the Navy.
 Mr. John P. Wamser, Bureau of Aeronautics, Department of the Navy.
 Mr. Burdell L. Springer, Civil Aeronautics Administration.
 Mr. Philip Donely, NACA Langley Aeronautical Laboratory.
 Mr. Manley J. Hood, NACA Ames Aeronautical Laboratory.
 Mr. Albert Epstein, Republic Aviation Corp.
 Mr. F. D. Jewett, The Glenn L. Martin Co.
 Mr. Jerome F. McBrearty, Lockheed Aircraft Corp.
 Mr. Alfred I. Sibila, Chance Vought Aircraft, United Aircraft Corp.
 Mr. K. E. Van Every, Douglas Aircraft Co., Inc.

Mr. R. Fabian Goranson, Secretary.

Subcommittee on Vibration and Flutter

Prof. Raymond L. Bisplinghoff, Massachusetts Institute of Technology, Chairman.
 Mr. Howard A. Magrath, Wright Air Development Center.
 Mr. L. S. Wasserman, Wright Air Development Center.
 Capt. Walter S. Diehl, U. S. N. (Ret.).
 Mr. James E. Walsh, Bureau of Aeronautics, Department of the Navy.
 Mr. Robert Rosenbaum, Civil Aeronautics Administration.
 Mr. I. E. Garrick, NACA Langley Aeronautical Laboratory.
 Mr. Albert Erickson, NACA Ames Aeronautical Laboratory.
 Mr. Samuel S. Manson, NACA Lewis Flight Propulsion Laboratory.
 Dr. William E. Cox, Northrop Aircraft, Inc.
 Dr. J. M. Frankland, Chance Vought Aircraft, United Aircraft Corp.
 Mr. Martin Goland, Midwest Research Institute.
 Mr. E. B. Kinnaman, Boeing Airplane Co.
 Mr. Raymond A. Pepping, McDonnell Aircraft Corp.

Mr. Harvey H. Brown, Secretary.

Subcommittee on Aircraft Structural Materials

Mr. Edgar H. Dix, Jr., Aluminum Co. of America, Chairman.
 Mr. J. B. Johnson, Wright Air Development Center.
 Mr. James E. Sullivan, Bureau of Aeronautics, Department of the Navy.
 Dr. Gordon M. Kline, National Bureau of Standards.
 Mr. James E. Dougherty, Jr., Civil Aeronautics Administration.
 Mr. Frank B. Bolte, North American Aviation, Inc.
 Prof. Maxwell Gensamer, Columbia University.
 Mr. O. W. Loudenslager, Goodyear Aircraft Corp.
 Dr. J. C. McDonald, The Dow Chemical Co.
 Dr. Robert F. Mehl, Carnegie Institute of Technology.
 Mr. Paul P. Mozley, Lockheed Aircraft Corp.
 Mr. David G. Reid, Chance Vought Aircraft Division, United Aircraft Corp.
 Mr. D. H. Ruhnke, Republic Steel Corp.

Mr. Charles V. Krebs, Secretary.

COMMITTEE ON OPERATING PROBLEMS

Mr. William Littlewood, American Airlines, Inc., Chairman.
 Mr. Charles Froesch, Eastern Air Lines, Inc., Vice Chairman.
 Col. Baskin R. Lawrence, Jr., U. S. A. F., Wright Air Development Center.
 Col. J. Francis Taylor, Jr., U. S. A. F., Wright Air Development Center.
 Commander A. L. Maccubbin, U. S. N., Bureau of Aeronautics, Department of the Navy.
 Maj. Gen. William H. Tunner, U. S. A., Hq., Military Air Transport Service.
 Dr. F. W. Reichelderfer, Chief, U. S. Weather Bureau.
 Mr. George W. Haldeman, Civil Aeronautics Administration.
 Mr. Donald M. Stuart, Civil Aeronautics Administration.
 Dr. Hugh L. Dryden (ex officio).
 Mr. Melvin N. Gough, NACA Langley Aeronautical Laboratory.
 Mr. Eugene J. Manganiello, NACA Lewis Flight Propulsion Laboratory.
 Mr. P. R. Bassett, Sperry Gyroscope Co., Inc., Division of the Sperry Corp.
 Mr. M. G. Beard, American Airlines, Inc.
 Mr. John G. Borger, Pan American World Airways, Inc.
 Mr. Warren T. Dickinson, Douglas Aircraft Co., Inc.
 Mr. Robert E. Johnson, Wright Aeronautical Corp., Division of Curtiss-Wright Corp.
 Mr. Jerome Lederer.
 Dr. Ross A. McFarland, Harvard School of Public Health.
 Mr. W. C. Mentzer, United Air Lines, Inc.

Dr. T. L. K. Smull, Secretary.

Subcommittee on Meteorological Problems

Dr. F. W. Reichelderfer, U. S. Weather Bureau, Chairman.
 Brig. Gen. W. O. Senter, U. S. A. F., Air Weather Service.
 Maj. Frederic C. E. Oder, U. S. A. F., Air Force Cambridge Research Center.
 Capt. R. O. Minter, U. S. N., Naval Aerological Service.
 Dr. Ross Gunn, U. S. Weather Bureau.
 Mr. Delbert M. Little, U. S. Weather Bureau.
 Dr. Harry Wexler, U. S. Weather Bureau.
 Mr. Robert W. Craig, Civil Aeronautics Administration.
 Mr. George M. French, Civil Aeronautics Board.
 Mr. Harry Press, NACA Langley Aeronautical Laboratory.
 Mr. I. Irving Pinkel, NACA Lewis Flight Propulsion Laboratory.
 Dr. Horace R. Byers, University of Chicago.
 Mr. Martin B. Cahill, Northwest Airlines, Inc.

Mr. Allan C. Clark, Pan American World Airways, Inc.
 Mr. Joseph J. George, Eastern Air Lines, Inc.
 Prof. H. G. Houghton, Massachusetts Institute of Technology.
 Dean Athelstan F. Spilhaus, University of Minnesota.
 Mr. Frank C. White, Air Transport Association of America.
 Dr. E. J. Workman, New Mexico Institute of Mining and Technology.

Mr. Donald B. Talmage, Secretary.

Subcommittee on Icing Problems

Mr. Arthur A. Brown, Pratt and Whitney Aircraft, United Aircraft Corp.
 Mr. James E. DeRemer, Development, U. S. Air Force.
 Mr. Duane M. Patterson, Wright Air Development Center.
 Mr. Edward C. Y. Inn, Air Force Cambridge Research Center.
 Mr. Parker M. Bartlett, Bureau of Aeronautics, Department of the Navy.
 Mr. Harcourt C. Sontag, Bureau of Aeronautics, Department of the Navy.
 Mr. B. C. Haynes, U. S. Weather Bureau.
 Mr. Stephen Rolle, Civil Aeronautics Administration.
 Mr. I. Irving Pinkel, NACA Lewis Flight Propulsion Laboratory.
 Mr. Don O. Benson, Northwest Airlines, Inc.
 Mr. F. L. Boeke, North American Aviation, Inc.
 Dr. Wallace E. Howell, Mount Washington Observatory.
 Mr. Victor Hudson, Consolidated Vultee Aircraft Corp.
 Mr. David A. North, American Airlines, Inc.
 Mr. W. W. Reaser, Douglas Aircraft Co., Inc.
 Dr. Vincent J. Schaefer, General Electric Co.

Mr. Boyd C. Myers, II, Secretary.

Subcommittee on Aircraft Fire Prevention

Mr. Raymond D. Kelly, United Air Lines, Inc., Chairman.
 Capt. Howard A. Klein, U. S. A. F., Wright Air Development Center.

Mr. Frederick A. Wright, Wright Air Development Center.
 Maj. Charles H. McConnell, U. S. A. F., Technical Inspection and Flight Safety Research.
 Mr. Arthur V. Stamm, Bureau of Aeronautics, Department of the Navy.
 Mr. David L. Posner, Civil Aeronautics Administration.
 Mr. Harvey L. Hansberry, Civil Aeronautics Administration.
 Mr. Hugh B. Freeman, Civil Aeronautics Board.
 Mr. A. C. Hutton, National Bureau of Standards.
 Mr. Lewis A. Rodert, NACA Lewis Flight Propulsion Laboratory.
 Mr. E. M. Barber, The Texas Co.
 Mr. Allen W. Dallas, Air Transport Association of America.
 Mr. Harold E. Hoben, American Airlines, Inc.
 Mr. C. R. Johnson, Shell Oil Co.
 Mr. Gaylord W. Newton, ARO, Inc.
 Mr. Ivar L. Shogran, Douglas Aircraft Co., Inc.
 Mr. William I. Stieglitz, Republic Aviation Corp.
 Mr. Clem G. Trimbach, Cornell Aeronautical Laboratory, Inc.

Mr. Richard S. Cesaro, Secretary.

INDUSTRY CONSULTING COMMITTEE

Mr. Dwane L. Wallace, Cessna Aircraft Co., Chairman.
 Mr. William N. Allen, Boeing Airplane Co., Vice Chairman.
 Dr. Lynn L. Bollinger, Helio Aircraft Corp.
 Mr. Ralph S. Damon, Trans World Airlines, Inc.
 Mr. Robert E. Gross, Lockheed Aircraft Corp.
 Mr. Roy T. Hurley, Curtiss-Wright Corp.
 Mr. C. W. LaPierre, General Electric Co.
 Mr. Mundy I. Peale, Republic Aviation Corp.
 Mr. Earl F. Slick, Slick Airways, Inc.

Dr. T. L. K. Smull, Secretary.

Part III—FINANCIAL REPORT

Appropriations for the fiscal year 1951.—Funds and contract authority in the following amounts were appropriated for the Committee for the fiscal year 1951 in the General Appropriation Act, 1951, approved September 6, 1950, and the Second Supplemental Appropriation Act, 1951, approved January 6, 1951:

Salaries and expenses..... \$45, 750, 000
Construction and equipment of laboratory facilities:

Funds to continue financing of fiscal year 1949 program:

Langley Aeronautical Laboratory..... \$6, 000, 000
Lewis Flight Propulsion Laboratory..... 4, 000, 000

10, 000, 000

Funds to continue financing of fiscal year 1950 program:

Langley Aeronautical Laboratory..... \$2, 771, 988
Pilotless Aircraft Research Station..... 478, 012
Lewis Flight Propulsion Laboratory..... 1, 750, 000

5, 000, 000

Funds to start financing of fiscal year 1951 program:

Langley Aeronautical Laboratory..... \$200, 000
Ames Aeronautical Laboratory..... 300, 000
Lewis Flight Propulsion Laboratory..... 1, 818, 000

2, 318, 000

Total appropriated funds, fiscal year 1951.....

\$63, 068, 000

Contract authority for remaining obligations necessary to complete fiscal year 1951 program:

Langley Aeronautical Laboratory..... \$2, 300, 000
Ames Aeronautical Laboratory..... 8, 700, 000

Total contract authority, fiscal year 1951..... \$11, 000, 000

Obligations incurred against the fiscal year 1951 appropriated funds and contract authority are listed below, together with the unobligated balances remaining on August 31, 1951. The figures shown for salaries and expenses include the costs for personal services, travel, transportation, communication, utility services, printing and reproduction, contractual services, supplies and equipment.

Salaries and expenses:

NACA Headquarters..... \$1, 081, 842
Langley Aeronautical Laboratory..... 17, 631, 974
Pilotless Aircraft Research Station..... 803, 904

Salaries and expenses—Continued

High-Speed Flight Research Station..... \$919, 281
Ames Aeronautical Laboratory..... 7, 535, 318
Western Coordination Office..... 18, 485
Lewis Flight Propulsion Laboratory..... 16, 416, 186
Wright-Patterson Coordination Office..... 10, 424
Research contracts—educational institutions..... 779, 649
Services performed by National Bureau of Standards and Forest Products Laboratory... 209, 501
Unobligated balance..... 343, 436

Total appropriated funds, salaries and expenses.....

45, 750, 000

Construction and equipment of laboratory facilities:

Funds to continue financing of fiscal year 1949 program:

Langley Aeronautical Laboratory..... \$5, 858, 400
Lewis Flight Propulsion Laboratory..... 3, 999, 508
Unobligated balance..... 142, 092

10, 000, 000

Funds to continue financing of fiscal year 1951 program:

Langley Aeronautical Laboratory..... \$2, 755, 553
Pilotless Aircraft Research Station..... 473, 863
Lewis Flight Propulsion Laboratory..... 1, 708, 099
Unobligated balance..... 62, 485

5, 000, 000

Funds to start financing of fiscal year 1951 program:

Langley Aeronautical Laboratory..... \$198, 222
Ames Aeronautical Laboratory..... 300, 000
Lewis Flight Propulsion Laboratory..... 1, 676, 796
Unobligated balance..... 142, 982

2, 318, 000

Total appropriated funds, fiscal year 1951.....

\$63, 068, 000

Contract authority for remaining obligations necessary to complete fiscal year 1951 program:

Langley Aeronautical Laboratory..... \$1, 497, 744
Ames Aeronautical Laboratory..... 3, 177, 786
Unobligated balance..... 16, 324, 470

Total contract authority, fiscal year 1951..... \$11, 000, 000

¹ These unobligated balances remain available for obligation in the fiscal year 1952.

Appropriation for the Unitary Wind Tunnel Plan Act.—Funds in the amount of \$75,000,000 were appropriated in the Deficiency Appropriation Act, 1950, approved June 29, 1950, for the construction of wind tunnels authorized in the Unitary Wind Tunnel Plan Act of 1949 (Public Law 415, 81st Cong., approved October 27, 1949). These funds are available until expended. Allotments and obligations as of June 30, 1951, are as follows:

	Allotments	Obligations as of June 30, 1951
Langley Aeronautical Laboratory----	\$14,917,000	\$6,440,860
Ames Aeronautical Laboratory-----	27,227,000	9,904,944
Lewis Flight Propulsion Laboratory--	32,856,000	9,905,955
Total-----	\$75,000,000	\$26,251,759

Appropriations for the fiscal year 1952.—The major allotments of the funds appropriated for the Committee for the fiscal year 1952 in the Independent Offices Appropriation Act, 1952 approved August 31, 1951, are given below:

Salaries and Expenses-----	\$49,250,000
Construction and equipment of labora- tory facilities:	
Funds to complete financing of fiscal year 1949 program:	
Langley Aeronautical Labora- tory-----	\$372,800
Lewis Flight Propulsion Lab- oratory-----	550,000
	922,800

	Allotments	Obligations as of June 30, 1951
Construction and equipment of labora- tory facilities—Continued		
Funds to complete financing of fiscal year 1950 program:		
Langley Aeronautical Labora- tory-----	\$3,493,524	
Lewis Flight Propulsion Lab- oratory-----	1,500,000	
		\$4,993,524
Funds to continue financing of fiscal year 1951 program:		
Langley Aeronautical Labora- tory-----	\$1,233,676	
Ames Aeronautical Labora- tory-----	4,500,000	
		5,733,676
Funds to completely finance fiscal year 1952 program:		
Langley Aeronautical Labora- tory-----	\$730,000	
Pilotless Aircraft Research Station-----	90,000	
High-Speed Flight Research Station-----	4,000,000	
Ames Aeronautical Labora- tory-----	1,480,000	
Lewis Flight Propulsion Lab- oratory-----	350,000	
		6,650,000
Total appropriated funds, fiscal year 1952-----		\$67,600,000

REPORT 1003

CORRELATION OF PHYSICAL PROPERTIES WITH MOLECULAR STRUCTURE FOR SOME DICYCLIC HYDROCARBONS HAVING HIGH THERMAL-ENERGY RELEASE PER UNIT VOLUME¹

By P. H. WISE, K. T. SERIJAN, and I. A. GOODMAN

SUMMARY

As part of a program to study the correlation between molecular structure and physical properties of high-density hydrocarbons, the net heats of combustion, melting points, boiling points, densities, and kinematic viscosities of some hydrocarbons in the 2-*n*-alkylbiphenyl, 1,1-diphenylalkane, α , ω -diphenylalkane, 1,1-dicyclohexylalkane, and α , ω -dicyclohexylalkane series are presented. Comparisons are made on the following three bases:

1. As members of an homologous series in which the compounds have similar structures and differ in molecular weight
2. As isomers with the same molecular weight and molecular formula but different molecular structure

3. As compounds with the same carbon skeleton but different molecular formulas due to hydrogenation of the aromatic rings

The three series of aromatic hydrocarbons, 2-*n*-alkylbiphenyl, 1,1-diphenylalkane, and α , ω -diphenylalkane, did not show great differences in heat of combustion per unit volume. These series averaged 20 percent higher than typical aircraft fuels of AN-F-58 specification with respect to this property. The two series of dicyclohexyl hydrocarbons, 1,1-dicyclohexylalkane and α , ω -dicyclohexylalkane, had somewhat lower heats of combustion, but still averaged about 13 percent higher than AN-F-58 fuel.

Each series followed its own characteristic pattern in the relation of melting point and structure. The general trend was toward lower melting points by addition of a side chain to the parent hydrocarbon. With two exceptions, the dicyclohexyl compounds melted at lower temperatures than the analogous aromatic hydrocarbons.

The molecular weight had a greater influence on boiling point than the molecular structure. The α , ω -diphenylalkane series, however, showed a more rapid rate of increase in boiling point with increasing molecular weight than the other series of aromatic hydrocarbons.

In these three closely related homologous series of aromatic hydrocarbons, the molecular weight had a greater effect on viscosity than the molecular structure when the substituent groups were all normal alkyl in type. The cyclohexyl derivatives had viscosities 40 to 200 percent higher than the analogous aromatic hydrocarbons.

INTRODUCTION

High-speed aircraft are volume-limited because of the design requirements of thin wings and small fuselages; the fuel-storage space is therefore restricted. In evaluating fuels for such aircraft, one of the important properties is the heat of combustion per unit volume. Fuels with high heat of combustion per unit volume would be valuable in providing increased flight range.

A survey of the literature pertaining to properties of hydrocarbons was therefore made at the NACA Lewis laboratory with particular emphasis on the heat of combustion per unit volume. The hydrocarbons were compared as homologous series to determine which features of molecular structure might be expected to be associated with high heat of combustion per unit volume. The net heats of combustion of the normal paraffin, normal alkylcyclohexane, and normal alkylbenzene hydrocarbons of seven to fourteen carbon atoms vary from 824,000 to 977,000 Btu per cubic foot. For an equivalent number of carbon atoms, the comparison of net heat of combustion is: *n*-alkylbenzene > *n*-alkylcyclohexane > *n*-paraffin. It is evident that a ring structure increases the heat of combustion per unit volume and that the unsaturated ring in benzene has an advantage over the saturated cyclohexane ring. These values of net heat of combustion were calculated from data obtained in references 1 and 2.

Compounds from these three homologous series are principal constituents of the aircraft fuels AN-F-48, AN-F-32a, and AN-F-58, which are currently used and are described by the specifications in references 3, 4, and 5, respectively. Fuels that were purchased under these specifications and used at the Lewis laboratory were found to have the following values of net heat of combustion: AN-F-48, 823,000; AN-F-32a, 960,000; and AN-F-58, 894,000 Btu per cubic foot. These values vary depending upon the source of supply; they have been arbitrarily chosen, however, as a standard for comparing the hydrocarbons discussed herein.

The advantage of an aromatic ring in a molecule led to the investigation of the properties of biphenyl, diphenylmethane, and naphthalene. Each of these compounds has two aromatic rings per molecule, and the net heats of com-

¹ Supersedes NACA TN 2081, "Correlation of Physical Properties with Molecular Structure for Dicyclic Hydrocarbons. I—2-*n*-Alkylbiphenyl, 1,1-Diphenylalkane, α , ω -Diphenylalkane, 1,1-Dicyclohexylalkane, and α , ω -Dicyclohexylalkane, Series," by P. H. Wise, K. T. Serijan, and I. A. Goodman, 1950.

bustion (references 1 and 2) are 1,096,000; 1,110,000; and 1,210,000 Btu per cubic foot, respectively. These values represent an average increase of 30 percent over AN-F-48, 15 percent over AN-F-32a, and 20 percent over AN-F-58. On the basis of heat of combustion, these three hydrocarbons would be desirable fuels; however, they have relatively high melting points and are unsuitable for use as liquid fuels. If the introduction of various alkyl groups into these hydrocarbons would sufficiently lower the melting points without reducing the heat of combustion per unit volume, a superior fuel would be obtained. The data on many of these alkyl derivatives are incomplete and in some cases inaccurate; consequently, they cannot be used as a basis for a reliable analysis of the usefulness of these high-energy hydrocarbons as aircraft-engine fuels.

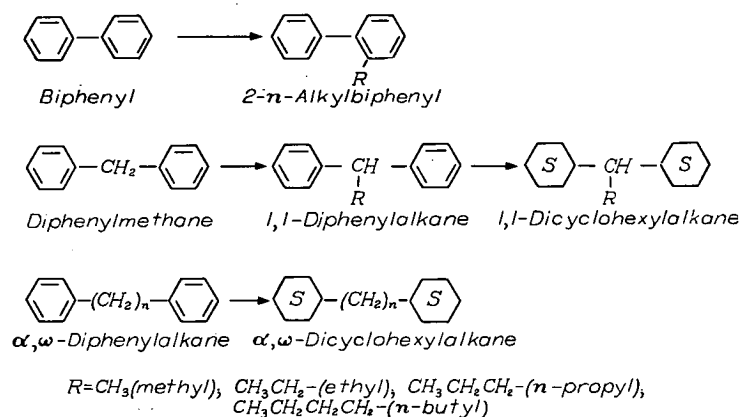


FIGURE 1.—Structure of hydrocarbons.

An investigation was therefore undertaken at the Lewis laboratory during 1945 to determine the effects of systematic alteration in structure of dicyclic-aromatic-type molecules on the following important properties of the fuel: net heat of combustion per unit volume, heat of combustion per unit weight, melting point, boiling point, and viscosity.

Comparisons are made on the following three bases:

1. As members of an homologous series in which the compounds have similar structures and differ in molecular weight
2. As isomers with the same molecular weight and molecular formula but different molecular structure
3. As compounds with the same carbon skeleton but different molecular formulas due to hydrogenation of the aromatic rings

METHOD AND APPARATUS

The compounds discussed represent five homologous series of hydrocarbons; namely, the 2-*n*-alkylbiphenyl; 1,1-diphenylalkane; 1,1-dicyclohexylalkane; α,ω -diphenylalkane; and α,ω -dicyclohexylalkane series. The structures of these hydrocarbons are illustrated in figure 1. The normal alkyl groups illustrated at the bottom of the figure were selected to study the effect of increasing the chain length of these substituent groups when they are attached to the parent

hydrocarbons, biphenyl and diphenylmethane. From diphenylmethane two series are developed, the 1,1-diphenylalkane series by adding the alkyl group as a side chain attached to the carbon atom between the phenyl groups and the α,ω -diphenylalkane series by lengthening the chain of carbon atoms between the two rings.

The saturated, or cyclohexyl, derivatives corresponding to the aromatic compounds are also illustrated. They are prepared by hydrogenation of the pure aromatic compounds. It is known that the saturated compounds generally have lower melting points and burn with less carbon formation than the corresponding aromatic hydrocarbons. On the other hand, saturated hydrocarbons have higher viscosities and lower net heats of combustion per unit volume than their aromatic counterparts. Inasmuch as many properties must be considered in selecting a fuel, it seemed advisable to study the saturated compounds to determine accurately the extent of the changes produced by hydrogenation.

With the exceptions noted in the tables of properties, the compounds were synthesized and purified, or purified from commercial stocks, at the Lewis laboratory; the properties were then determined. Before final measurement of the properties was made, precautions were taken to ascertain that the high purity essential for the accurate study of properties was attained.

The time-temperature melting curves were determined with a platinum resistance thermometer and a G-2 Mueller bridge with accessory equipment and by methods described in reference 6, and the melting points were determined from the curves according to the graphical method described in reference 7. Densities were determined by use of a gravimetric balance according to the method of reference 8, and the refractive indices were measured with a Bausch & Lomb precision oil-model five-plate instrument. The boiling points were determined by the use of a platinum resistance thermometer in an apparatus modified from that described in reference 9. The system was pressurized with dry air from a surge tank and held at constant pressure by adjusting a continuous bleed. The kinematic viscosities were determined in viscosimeters that had been calibrated with National Bureau of Standards standard viscosity samples H-5, H-7, D-7, or L-17. The A.S.T.M. procedure of reference 10 was followed. The net heats of combustion were determined according to reference 11 in an oxygen bomb calorimeter that had been calibrated on Bureau of Standards benzoic acid.

The magnitude of the uncertainties is estimated as follows: melting point, 0.02°C ; density, 0.00005 gram per milliliter; refractive index, 0.0002; boiling point, 0.1°C ; kinematic viscosity, 0.5 percent of determined value relative to 1.007 centistokes for water at 20°C ; and net heat of combustion, ± 100 Btu per pound, which is equivalent to 10 to 15 kilogram-calories per mole for these compounds.

The precision of measurements is: freezing point, $\pm 0.003^\circ \text{C}$; density, ± 0.00002 to ± 0.00003 gram per milliliter; refractive index, ± 0.0001 ; boiling point, $\pm 0.04^\circ \text{C}$; kinematic viscosity, 0.2 percent of determined value; and net heat of combustion, ± 60 Btu per pound.

DISCUSSION OF RESULTS

2-*n*-ALKYLBIPHENYL HYDROCARBONS

The physical properties of the 2-*n*-alkylbiphenyl hydrocarbons are presented in table I (a) and the heat-of-combustion data are given in table I (b). The data are plotted in figure 2. The comparison of molecular structure with net heat of combustion in Btu per cubic foot for the 2-*n*-alkylbiphenyl hydrocarbons is shown in figure 2 (a). The parent hydrocarbon, biphenyl, has the highest value, 1,096,000 Btu per cubic foot. The first member of the series, 2-methylbiphenyl, is slightly lower at 1,078,000 Btu per cubic foot. Each carbon atom that is subsequently added to lengthen the side chain causes some decrease in

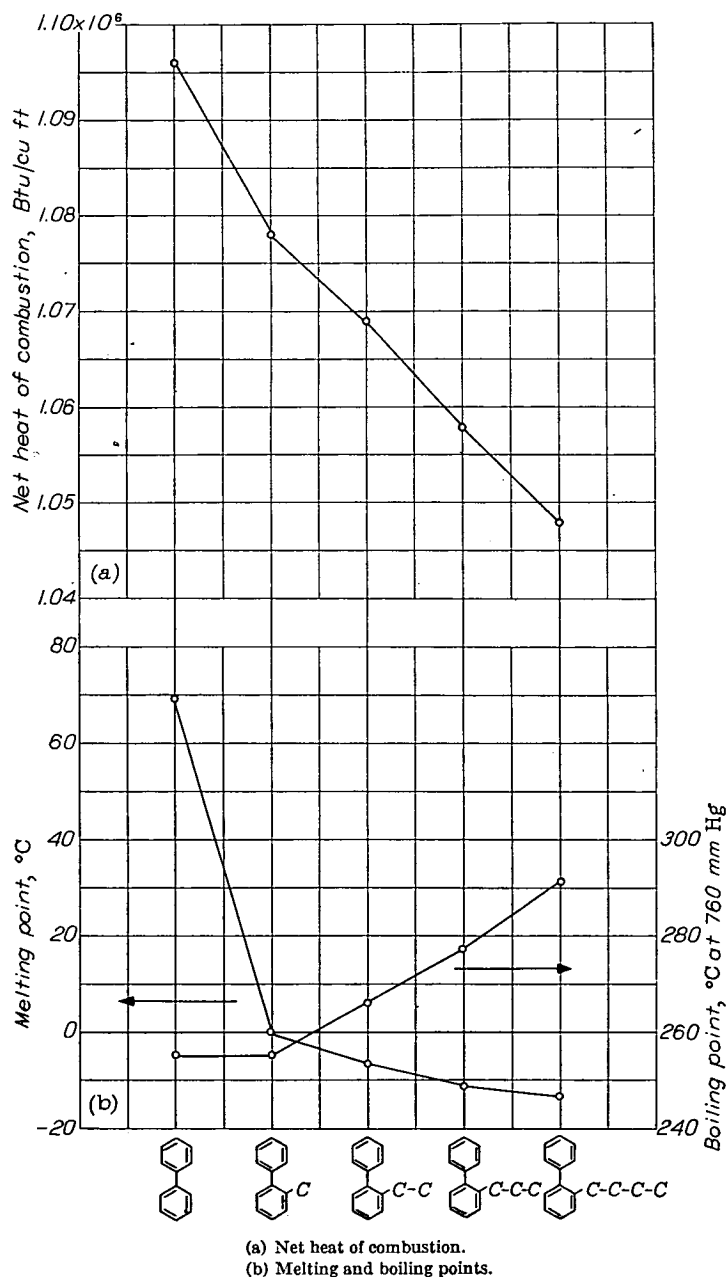


FIGURE 2.—Variation of properties with structure of 2-*n*-alkylbiphenyl hydrocarbons.

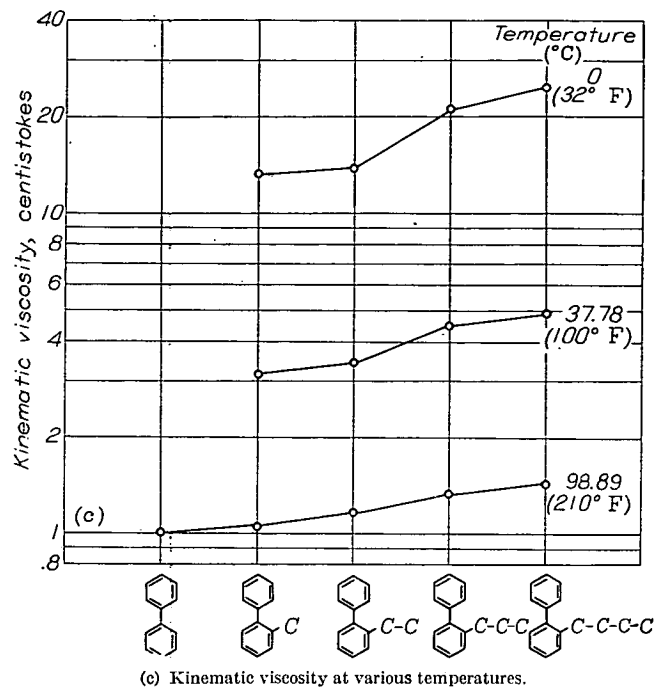


FIGURE 2.—Concluded. Variation of properties with structure of 2-*n*-alkylbiphenyl hydrocarbons.

the net heat of combustion per unit volume, and 2-*n*-butylbiphenyl with four carbon atoms in the side chain has a value of 1,048,000 Btu per cubic foot, which is a loss of 4.3 percent from biphenyl. This value is, however, still 17 percent higher than a typical AN-F-58 aircraft fuel.

The melting- and boiling-point comparisons of this series of hydrocarbons are shown in figure 2 (b). Biphenyl melts at 69.2° C, and each member of the series has a lower melting point as the molecular weight and side-chain length are increased. The fourth member of the series, 2-*n*-butylbiphenyl, melts at -13.71° C, a decrease of 83° C from the value for biphenyl.

The boiling-point curve shows the reverse trend, increasing from 255.0° to 291.20° C in the transition from the parent hydrocarbon to 2-*n*-butylbiphenyl.

The viscosities of these hydrocarbons at three temperatures are compared in figure 2 (c). The trend is the same as for the boiling point; an increase in molecular weight by lengthening the side chain is accompanied by an increase in viscosity. At 0° C (32° F), where the viscosity of biphenyl cannot be determined because the compound is a solid, there is an increase of 85 percent from 2-methylbiphenyl to 2-*n*-butylbiphenyl. The effect of temperature is even more pronounced. The average increase in viscosity for the individual members of the series is 200 percent when the temperature is lowered from 98.89° C (210° F) to 37.78° C (100° F) and about 1300 percent when the temperature is 0° C (32° F) compared with 98.89° C (210° F). No data were obtained below 0° C (32° F) because most of these compounds are crystalline solids at temperatures only a few degrees below this temperature.

1,1-DIPHENYLALKANE AND 1,1-DICYCLOHEXYLALKANE HYDROCARBONS

The properties of these two series of hydrocarbons are shown in table II and plotted in figure 3. By using data from these two series, it is possible to compare compounds that have like arrangement of the carbon skeleton and different hydrogen-carbon ratios. The data for the two series are therefore plotted in the same figures, with solid

lines representing the aromatic series and dashed lines, the saturated series.

The net heat of combustion in Btu per cubic foot is compared with the molecular structure in figure 3 (a). Biphenyl is considered to be the parent hydrocarbon and is compared with the other members of this series, as was the case with

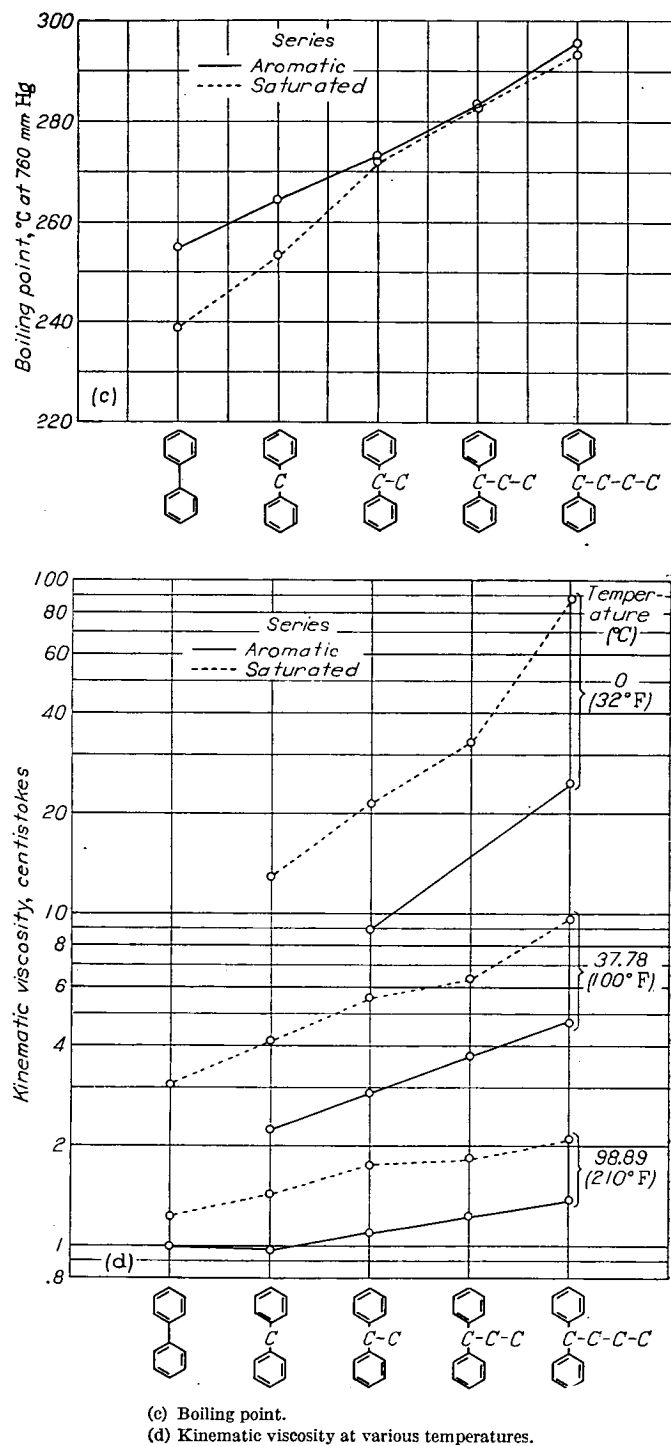
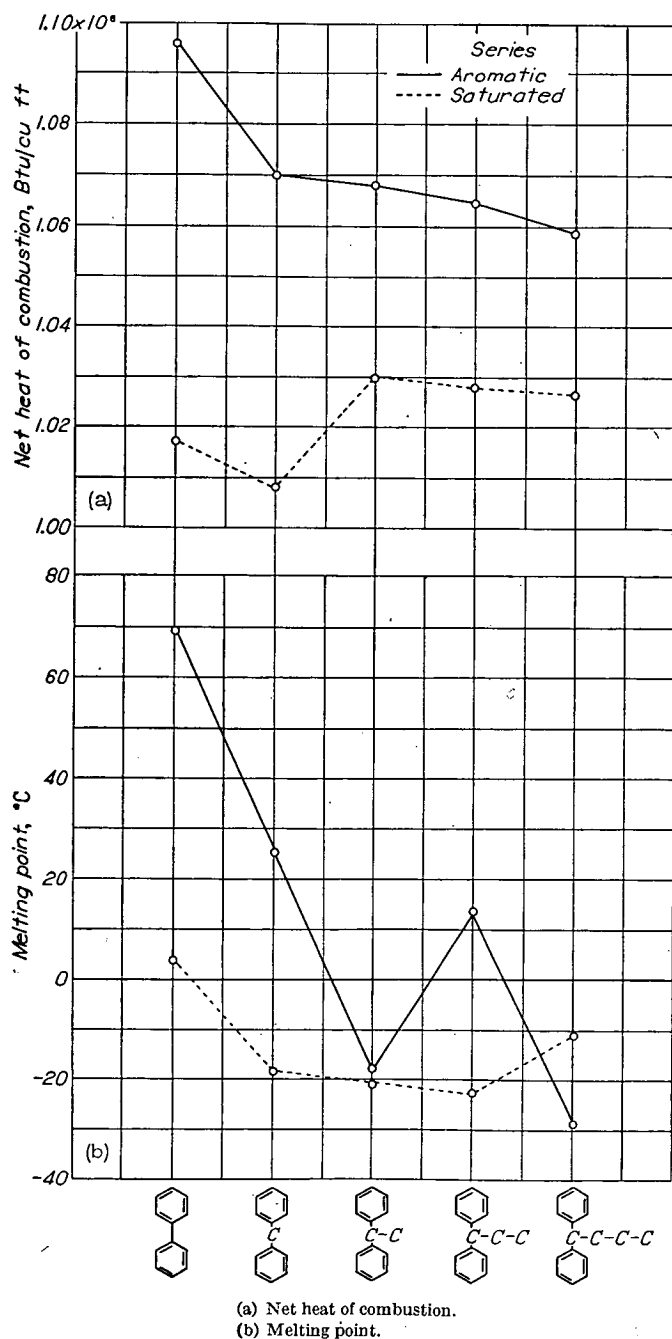


FIGURE 3.—Variation of properties with structure of 1,1-diphenyl and 1,1-dicyclohexylalkane hydrocarbons.

the 2-*n*-alkylbiphenyl hydrocarbons. The carbon atom added between the two rings in diphenylmethane results in a 2-percent loss in Btu per cubic foot. A small decrease in net heat of combustion per cubic foot is observed with each carbon atom that is added to the side chain. The over-all loss from biphenyl to 1,1-diphenylbutane is, however, only 3.3 percent.

In plotting the dicyclohexyl derivatives, each point indicates the value for the hydrogenated compound with the carbon skeleton corresponding to that of the aromatic compound immediately below the point on the figure. The average decrease in net heat of combustion per unit volume is 5 percent when each saturated compound is compared with its aromatic counterpart. This decrease is due to the lower density of the hydrogenated derivatives in comparison with the aromatic hydrocarbons.

The melting points of these two series of hydrocarbons are shown in figure 3 (b). The aromatic series shows a consistent decrease from biphenyl (melting point, 69.2° C) to 1,1-diphenylbutane (melting point, -28.4° C) except for one member, 1,1-diphenylpropane, which melts at 13.3° C. This melting point is higher than that of the compound preceding or following it in the series.

The saturated compounds melt at lower temperatures than the corresponding aromatic hydrocarbons except in the comparison of the butane derivatives where the saturated compound has a higher melting point than the diphenyl derivative.

The boiling points of the two series (fig. 3 (c)) show a regular increase with lengthening of the carbon chain. The difference between analogous members of the two series decreases with increasing chain length. The first members of the two series differ about 16° C; the last three pairs differ 1° to 3° C. The diphenyl compounds have consistently higher boiling points than the saturated derivatives.

The increase in viscosity due to hydrogenation of the aromatic nuclei to saturated rings is observed by comparing the viscosities of these two series in figure 3 (d). The saturated derivatives average 48 percent higher at 98.89° C (210° F) and 200 percent higher at 0° C (32° F). Both series show consistent increases in viscosity with increase in chain length.

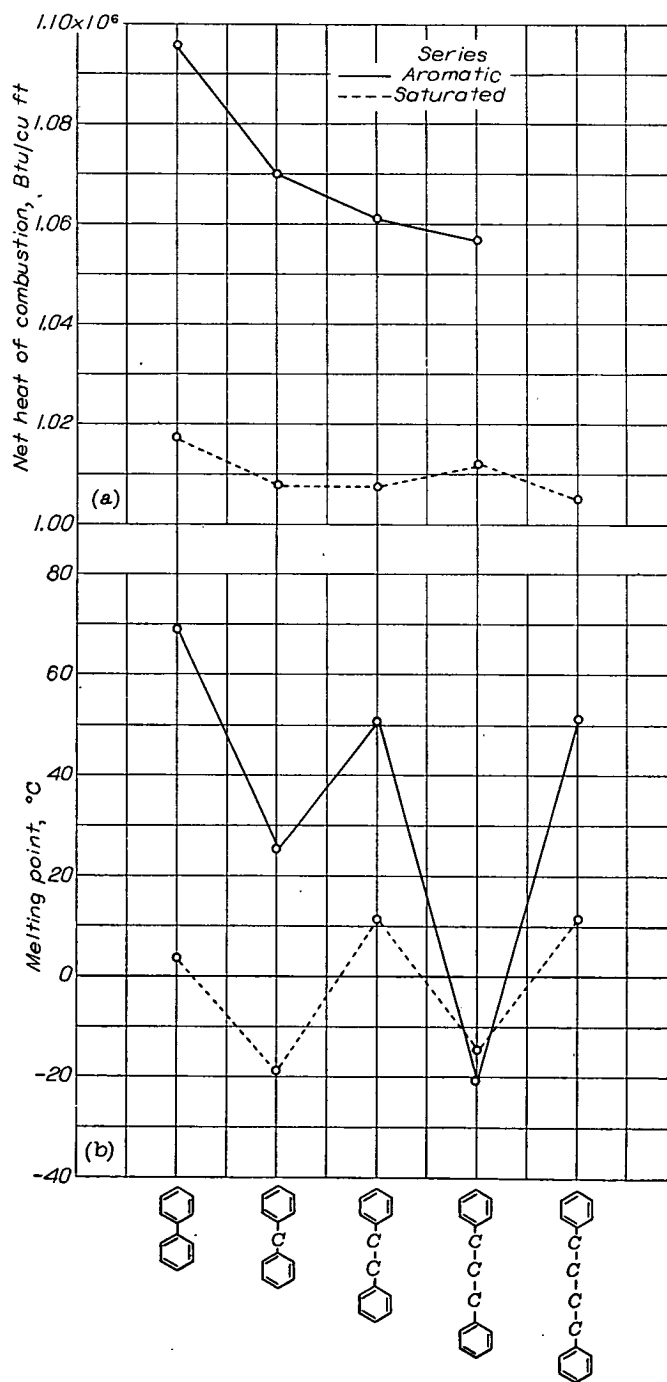
α,ω -DIPHENYLALKANE AND α,ω -DICYCLOHEXYLALKANE HYDROCARBONS

The properties of the two series of α,ω -dicyclicalkanes are tabulated in table III and plotted in figure 4. With the two α,ω -dicyclicalkane hydrocarbon series of compounds, it is again possible to compare the aromatic and the saturated derivatives. The data for net heat of combustion are shown in figure 4 (a). In the aromatic series, a decrease of 3.5 percent in Btu per cubic foot is observed as the carbon chain between the two rings is lengthened. The values for the saturated series are quite uniform and average about 5 percent lower than the aromatic hydrocarbons. The saturated compounds, however, average about 12 percent higher than an AN-F-58 aircraft fuel on the basis of heat of combustion per unit volume.

The two series follow similar patterns in melting-point behavior as the length of the carbon chain is increased

(fig. 4 (b)). The melting points of the saturated compounds are consistently lower than those of the corresponding aromatic hydrocarbons except for the 1,3-dicyclic compounds, in which case the aromatic derivative is slightly lower.

The boiling-point data for these compounds, shown in figure 4 (c), indicate uniform increases in both series as the carbon chain between the rings is lengthened. The boiling points of the aromatic compounds are consistently higher than those of the saturated derivatives with the corresponding carbon skeleton.



(a) Net heat of combustion.
(b) Melting point.

FIGURE 4.—Variation of properties with structure of α,ω -diphenylalkane and α,ω -dicyclohexylalkane hydrocarbons.

Comparison of viscosities in these series, plotted in figure 4 (d), is necessarily incomplete because the aromatic compounds have relatively high melting points and are solids at several of the temperatures of measurement. The aromatic compounds are liquids at 98.89° C (210° F), and the vis-

cosity increases slowly with increasing chain length. At this temperature, the viscosity values of the saturated compounds average 43 percent higher than the corresponding aromatic compounds. At the lower temperatures some data points are missing, but the trends are the same.

COMPARISON OF THREE AROMATIC SERIES

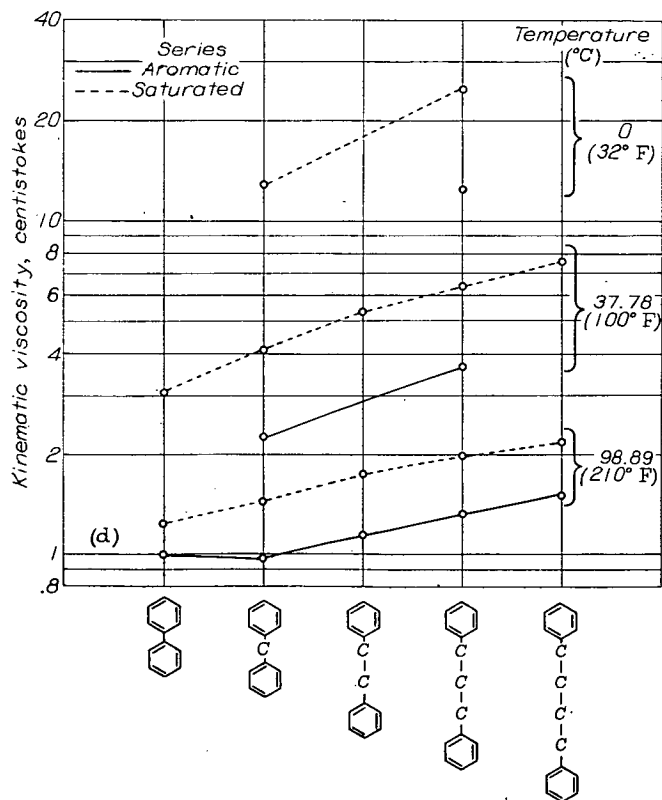
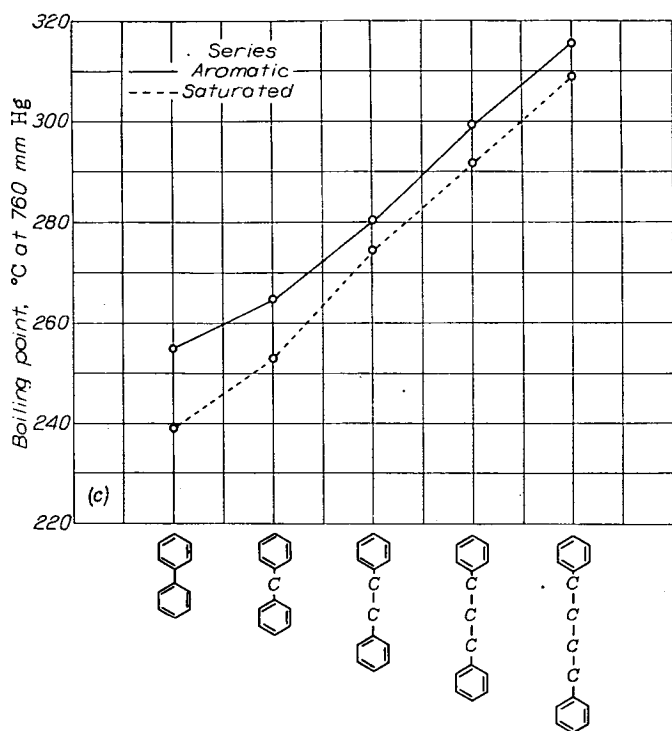
The comparisons that have been presented were made on the basis of homologous series in which the molecular structure of the individual members of a series differed only by the length of the aliphatic carbon chain in the molecule. Each series started from the common parent hydrocarbon, biphenyl. A comparison of the effect of structure on the properties can also be made by studying the properties of the hydrocarbons, one from each series, that have a common molecular formula. An example would be those with the formula $C_{14}H_{14}$, that is, 2-ethylbiphenyl, 1,1-diphenylethane, and 1,2-diphenylethane. In order to make this comparison, the properties of each series have been plotted in figure 5. The properties are plotted as the ordinates and the number of carbon atoms added to the parent hydrocarbon is plotted as the abscissa.

The effect of structure on heat of combustion per unit volume is shown in figure 5 (a). The ratio of Btu per cubic foot to the corresponding value for a typical AN-F-58 aircraft fuel (table I, footnote d) is used as the ordinate rather than the absolute value determined for the compounds. This ratio is a measure of the advantage to be gained in heat of combustion per unit volume by using fuels of this type. The 1,1-diphenylalkane series decreased less rapidly in heat of combustion per unit volume than the other two series as the carbon chain was lengthened. None of these series, however, decreased more than 6 percent from the parent hydrocarbon, and they are all about 20 percent higher than AN-F-58 with respect to this property.

The divergent behavior of these three series with respect to their melting points is illustrated in figure 5 (b). The 2-*n*-alkylbiphenyl series has increasingly lower melting points as the side chain is increased in length. The other series are characterized by alternate lower and higher melting points as the series are extended.

The curves of figure 5 (c) show that the molecular weight has more influence on boiling point than the structure, because the boiling points are quite closely grouped. The α,ω -diphenylalkane series does show, however, a more rapid rate of increase in boiling point with increase in chain length than the other two series. The boiling point of 1,3-diphenylpropane is higher than those of 1,1-diphenylbutane and 2-*n*-butylbiphenyl, which have one more carbon atom per molecule.

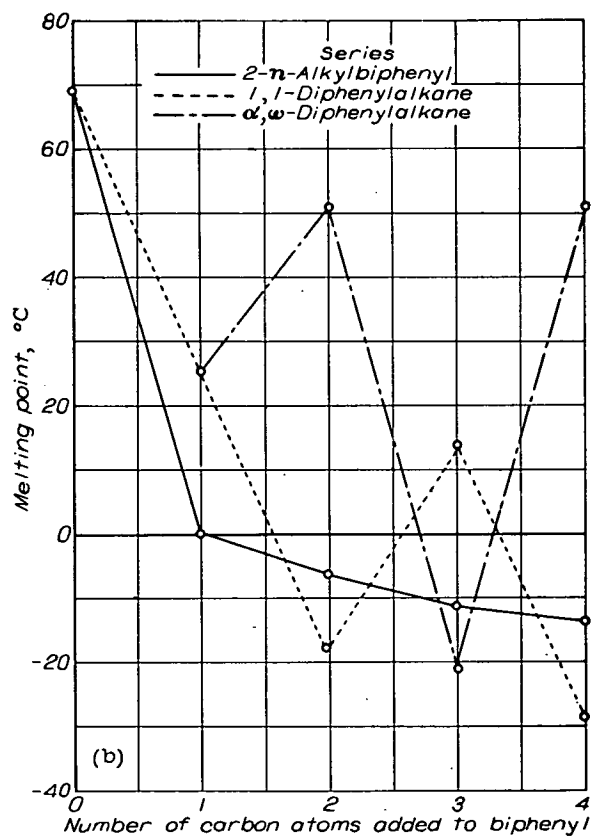
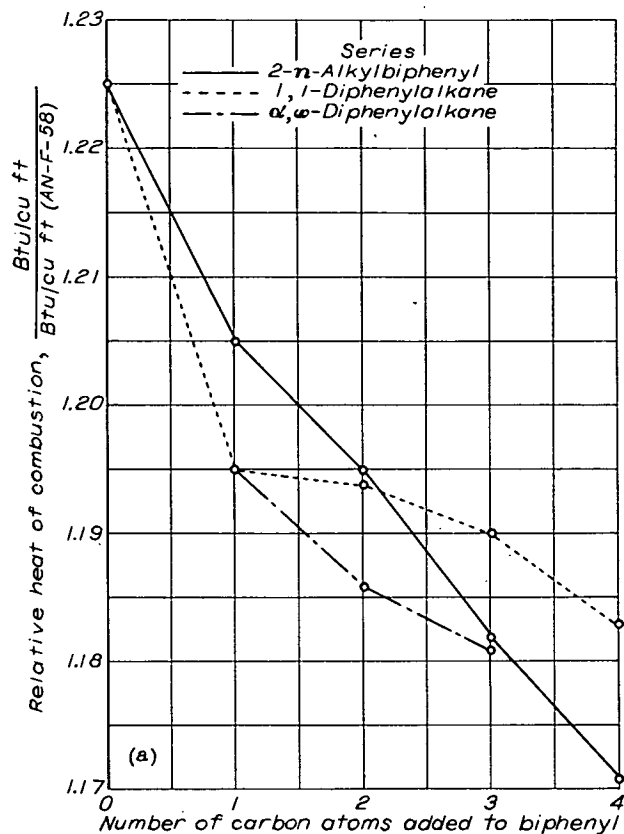
The viscosities of the three series of hydrocarbons at the three temperatures of measurement are plotted in figure 5 (d). The viscosities depend more upon the molecular weight than upon the molecular structure in these closely related series. As the molecular weight increases, the differences between the viscosity values of the isomers in the three series diminish.



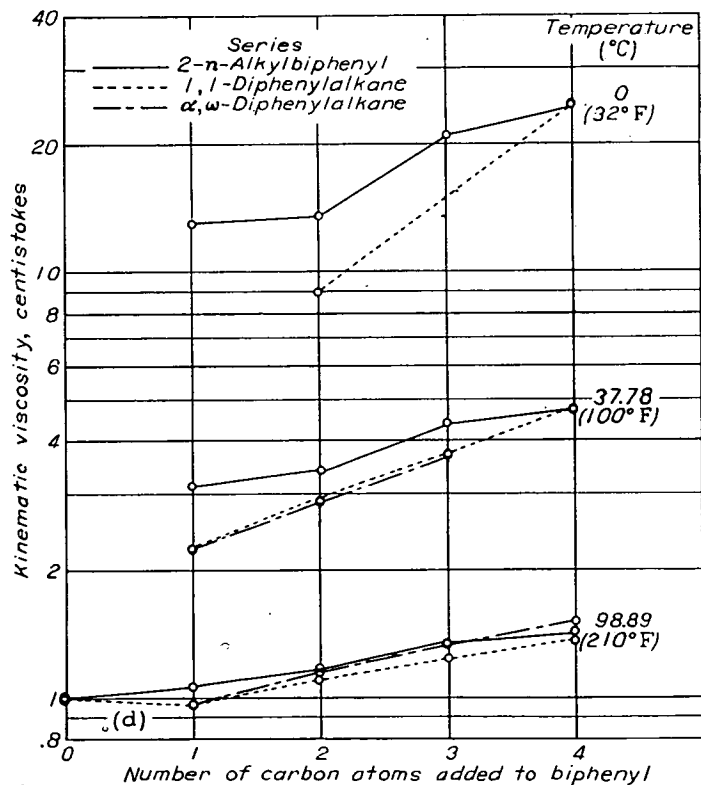
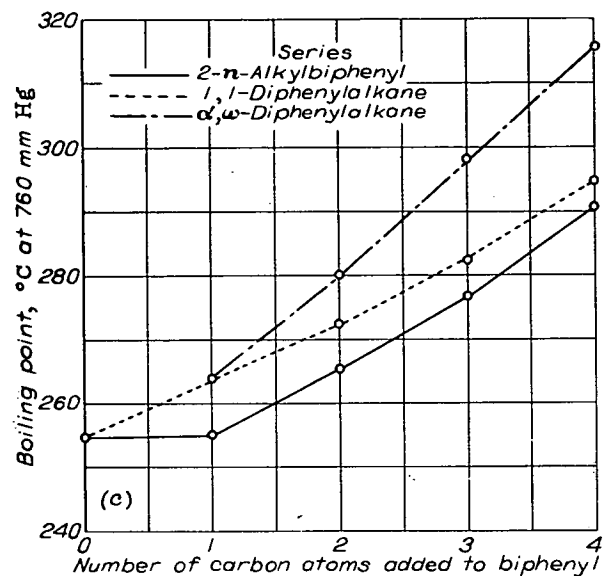
(c) Boiling point.

(d) Kinematic viscosity at various temperatures.

FIGURE 4.—Concluded. Variation of properties with structure of α,ω -diphenylalkane and α,ω -dicyclohexylalkane hydrocarbons.



(a) Net heat of combustion.
(b) Melting point.



(c) Boiling point.
(d) Kinematic viscosity at various temperatures.

FIGURE 5.—Comparison of properties and structural changes of three homologous series of hydrocarbons related to biphenyl.

SUMMARY OF RESULTS

A study of the variation of properties with change in molecular structure in the homologous series—2-*n*-alkylbiphenyl; 1,1-diphenylalkane; α,ω -diphenylalkane; 1,1-dicyclohexylalkane; and α,ω -dicyclohexylalkane—indicated the following trends:

1. The net heat of combustion per unit volume decreased an average of about 5 percent as the molecular weight was increased from the parent hydrocarbon, biphenyl, to 1,3-diphenylpropane and to the $C_{16}H_{18}$ isomers in the 2-*n*-alkylbiphenyl and 1,1-diphenylalkane hydrocarbon series. All three series averaged about 20 percent higher than an AN-F-58 aircraft fuel with respect to this property.

2. In the two series of hydrogenated compounds, 1,1-dicyclohexylalkane and α,ω -dicyclohexylalkane, the values of net heat of combustion per unit volume were somewhat lower than the analogous aromatic compounds and did not vary greatly throughout the series. The series averaged about 13 percent higher than an AN-F-58 fuel.

3. Each of the aromatic series followed a characteristic pattern of melting-point variation as the molecular weight was increased. The general trend was toward lower melting points as the size of the alkyl substituent was increased, but exceptions to this general statement were observed in the α,ω -diphenylalkane series and the 1,1-diphenylalkane series.

4. Hydrogenation of the aromatic nuclei to saturated rings lowered the melting point in all of the cases except two.

5. The boiling points of the investigated hydrocarbons were influenced more by their molecular weight than by molecular structure. The α,ω -diphenylalkane series, however, did show a more rapid rate of increase of boiling point with increase in molecular weight than the 1,1-diphenylalkane and 2-*n*-alkylbiphenyl series. The saturated derivatives had boiling points slightly lower than those of the corresponding diphenyl compounds.

6. In the three aromatic series, the viscosity increased uniformly with increase in molecular weight. The viscosity values of the isomers were nearly equal in many cases; it was therefore concluded that with the types of molecule studied the molecular weight influenced viscosity to a greater extent than did molecular structure. The saturated hydrocarbons had much higher viscosities than the corresponding diphenyl compounds, with the increases ranging from approximately 40 to 200 percent.

7. Of the compounds investigated, 1,1-diphenylbutane

had the lowest melting point, -28.4°C . This compound had a net heat of combustion of 1,059,000 Btu per cubic foot, which is 18 percent greater than the AN-F-58 fuel selected for comparison.

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, *June 20, 1949.*

REFERENCES

1. Rossini, Frederick D., Pitzer, Kenneth S., Taylor, William J., Ebert, Joan P., Kilpatrick, John E., Beckett, Charles W., Williams, Mary G., and Werner, Helene G.: *Selected Values of Properties of Hydrocarbons*. Circular C461, Nat. Bur. Standards, Nov. 1947.
2. Doss, M. P.: *Physical Constants of the Principal Hydrocarbons*. The Texas Co. (New York), 4th ed., 1943.
3. Anon.: *Army-Navy Aeronautical Specification Fuel; Aircraft Reciprocating Engine*. AN-F-48, Oct. 21, 1946.
4. Anon.: *Army-Navy Aeronautical Specification Fuel; grade JP-1, Aircraft Engine*. AN-F-32a, July 10, 1947.
5. Anon.: *Army-Navy Aeronautical Specification Fuel; Aircraft Turbine and Jet Engine*. AN-F-58, Dec. 12, 1947.
6. Glasgow, Augustus R., Jr., Krouskop, Ned C., Beadle, Joan, Axilrod, Gertrude D., and Rossini, Frederick D.: *Compounds Involved in Production of Synthetic Rubber*. *Anal. Chem.*, vol. 20, no. 5, May 1948, pp. 410-422.
7. Taylor, William J., and Rossini, Frederick D.: *Theoretical Analysis of Certain Time-Temperature Freezing and Melting Curves as Applied to Hydrocarbons*. *Jour. Res. Nat. Bur. Standards*, vol. 32, no. 5, May 1944, pp. 197-213.
8. Forziati, Alphonse F., Mair, Beveridge J., and Rossini, Frederick D.: *Assembly and Calibration of a Density Balance for Liquid Hydrocarbons*. *Jour. Res. Nat. Bur. Standards*, vol. 35, no. 6, Dec. 1945, pp. 513-519.
9. Quiggle, D., Tongberg, C. O., and Fenske, M. R.: *Apparatus for Boiling Point and Boiling Range Measurements*. *Ind. and Eng. Chem. (Anal. ed.)*, vol. 6, no. 6, Nov. 1934, pp. 466-468.
10. Anon.: *Tentative Method of Test for Kinematic Viscosity*. (Issued 1937; rev. 1938, 1939, 1942, 1946.) A.S.T.M. Designation: D 445-46T. A.S.T.M. Standards on Petroleum Products and Lubricants, Nov. 1948, pp. 255-265.
11. Anon.: *Standard Method of Test for Thermal Value of Fuel Oil*. (Adopted 1919; rev. 1939.) A.S.T.M. Designation: D 240-39. A.S.T.M. Standards on Petroleum Products and Lubricants, Nov. 1948, pp. 160-163.
12. Schiessler, Robert W., Herr, C. H., Rytina, A. W., Weisel, C. A., Fischl, F., McLaughlin, R. L., and Kuehner, H. H.: *The Synthesis and Properties of Hydrocarbons of High Molecular Weight—IV*. *Proc. Twelfth Mid-Year Meeting, Div. of Refining, Am. Petroleum Inst. (St. Louis)*, June 2-3, 1947.

TABLE I—PROPERTIES OF 2-*n*-ALKYLBIPHENYL HYDROCARBONS(a) Physical properties.
[S designates solid at indicated temperature.]

Hydrocarbon	Melting point		Boiling point at 760 mm		Density at 20° C (g/ml)	Kinematic viscosity (centistokes)			Refractive index n_D^{20}
	°C	°F	°C	°F		98.89° C (210° F)	37.78° C (100° F)	0° C (32° F)	
Biphenyl.....	69.2	156.6	255.0	491.0	1.041 ^a	0.992	S	S	S
2-Methylbiphenyl.....	-20	31.64	255.30	491.54	1.01134	1.06	3.18	13.3	1.5914
2-Ethylbiphenyl.....	-6.13	20.97	265.97	510.75	.99671	1.17	3.44	13.8	1.5805
2-Propylbiphenyl.....	-11.26	11.73	277.22	531.00	.98018	1.33	4.44	22.4	1.5696
2- <i>n</i> -Butylbiphenyl.....	-13.71 ^b	7.32 ^b	291.20	556.16	.96763	1.43	4.87	24.7	1.5604
	-9.65 ^b	14.63 ^b							

(b) Heats of combustion.

Hydrocarbon	Net heat of combustion			Ratio to AN-F-58	
	kcal/mole	Btu/lb	Btu/cu ft	Weight basis ^c	Volume basis ^d
Biphenyl.....	1443.5	16,850 ^a	1.096 × 10 ⁶	0.905	1.225
2-Methylbiphenyl.....	1598.2	17,100	1.078	.918	1.205
2-Ethylbiphenyl.....	1739.0	17,175	1.069	.922	1.195
2- <i>n</i> -Propylbiphenyl.....	1886.5	17,300	1.058	.929	1.182
2- <i>n</i> -Butylbiphenyl.....	2030.0	17,375	1.048	.933	1.171

^a Reference 2.^b Two different crystalline modifications.^c Based on experimentally determined value of 18,625 Btu/lb for fuel used at Lewis laboratory.^d The value of 894,000 Btu/cu ft for AN-F-58 was calculated from the value in footnote c using the experimentally determined value of 0.769₆₀⁶⁰ for the specific gravity of the fuel.

TABLE II—PROPERTIES OF 1,1-DICYCLICALKANE HYDROCARBONS

(a) Physical properties.
[S designates solid at indicated temperature.]

Hydrocarbon	Melting point		Boiling point at 760 mm		Density at 20° C (g/ml)	Kinematic viscosity (centistokes)			Refractive index n_D^{20}
	°C	°F	°C	°F		98.89° C (210° F)	37.78° C (100° F)	0° C (32° F)	
Diphenylalkanes									
Biphenyl.....	69. 2	156. 6	255. 0	491. 0	1. 041 ^a	0. 992	S	S	S
Diphenylmethane.....	25. 20	77. 36	264. 27	507. 69	1. 00592	. 965	2. 22	S	1. 5776
1,1-Diphenylethane.....	-18. 01	- . 42	272. 63	522. 73	. 99961	1. 10	2. 90	S, 8. 92	1. 5725
1,1-Diphenylpropane.....	13. 29	55. 92	283. 22	541. 80	. 98663	1. 23	3. 72	S	1. 5643
1,1-Diphenylbutane.....	-28. 38 ^b	-19. 08	294. 29	561. 72	. 97498	1. 38	4. 70	24. 8	1. 5568
	-25. 2 ^b	-13. 4							
Dicyclohexylalkanes									
Bicyclohexyl.....	3. 50	38. 30	238. 87	461. 97	0. 88604	1. 23	3. 07	S	1. 4796
Dicyclohexylmethane.....	-18. 69	-1. 64	252. 77	486. 99	. 87646	1. 43	4. 10	13. 0	1. 4763
1,1-Dicyclohexylethane.....	-20. 98	-5. 76	271. 17	520. 11	. 89309	1. 76	5. 59	21. 3	1. 4845
1,1-Dicyclohexylpropane.....	-23. 46	-10. 23	282. 31	540. 16	. 89299	1. 83	6. 41	32. 9	1. 4848
1,1-Dicyclohexylbutane.....	-10. 46	13. 17	292. 97	559. 35	. 89021	2. 10	9. 64	88. 3	1. 4843

(b) Heats of combustion.

Hydrocarbon	Net heat of combustion			Ratio to AN-F-58	
	kcal/mole	Btu/lb	Btu/cu ft	Weight basis ^c	Volume basis ^d
Diphenylalkanes					
Biphenyl.....	1443.5	16,850 ^a	1.096 × 10 ⁶	0.905	1.225
Diphenylmethane.....	1593.5	17,050	1.070	.916	1.196
1,1-Diphenylethane.....	1733.8	17,125	1.068	.920	1.194
1,1-Diphenylpropane.....	1886.5	17,300	1.065	.929	1.190
1,1-Diphenylbutane.....	2032.9	17,400	1.059	.934	1.183
Dicyclohexylalkanes					
Bicyclohexyl.....	1700.0	18,400	1.017 × 10 ⁶	0.988	1.137
Dicyclohexylmethane.....	1845.9	18,425	1.008	.989	1.127
1,1-Dicyclohexylethane.....	1997.5	18,500	1.030	.993	1.150
1,1-Dicyclohexylpropane.....	2138.8	18,475	1.028	.992	1.149
1,1-Dicyclohexylbutane.....	2285.8	18,500	1.027	.993	1.148

^a Reference 2.^b Two different crystalline modifications.^c Based on experimentally determined value of 18,625 Btu/lb for fuel used at Lewis laboratory.^d The value of 894,000 Btu/cu ft for AN-F-58 was calculated from the value in footnote c using the experimentally determined value of 0.769₆₀⁶⁰ for the specific gravity of the fuel.

REPORT 1003—NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TABLE III—PROPERTIES OF α,ω -DICYCLICALKANE HYDROCARBONS

(a) Physical properties.

[S designates solid at indicated temperature.]

Hydrocarbon	Melting point		Boiling point at 760 mm		Density at 20° C (g/ml)	Kinematic viscosity (centistokes)			Refractive index n_D^{20}
	°C	°F	°C	°F		98.89° C (210° F)	37.78° C (100° F)	0° C (32° F)	
Diphenylalkanes									
Biphenyl.....	69.2	156.6	255.0	491.0	1.041 ^a	0.992	S	S	S
Diphenylmethane.....	25.20	77.36	264.27	507.69	1.00592	.965	2.22	S	1.5776
1,2-Diphenylethane.....	51.16	124.09	280.49	536.88	S	1.14	S	S	S
1,3-Diphenylpropane.....	-20.78	-5.40	298 ^c	568 ^c	.9890 ^b	1.31	3.64	12.38	1.5594
1,4-Diphenylbutane.....	52.27	126.29	315.91	600.64	.97972	1.49	S	S	S
Dicyclohexylalkanes									
Bicyclohexyl.....	3.50	38.30	238.87	461.97	0.88604	1.23	3.07	S	1.4796
Dicyclohexylmethane.....	-18.69	-1.64	252.77	486.99	.87646	1.43	4.10	13.0	1.4763
1,2-Dicyclohexylethane.....	11.44	52.59	274.38	525.88	.87380	1.72	5.38	S	1.4759
1,3-Dicyclohexylpropane.....	-14.80	5.36	291.69	557.04	.87128	1.96	6.39	24.9	1.4752
1,4-Dicyclohexylbutane.....	11.62	52.92	309.0	588.2	.87027	2.19	7.58	S	1.4761

(b) Heats of combustion.

Hydrocarbon	Net heat of combustion			Ratio to AN-F-58	
	kcal/mole	Btu/lb	Btu/cu ft	Weight basis ^a	Volume basis ^f
Diphenylalkanes					
Biphenyl.....	1443.5	16,850 ^a	1.096×10 ⁶	0.905	1.225
Diphenylmethane.....	1593.5	17,050	1.070	.916	1.196
1,2-Diphenylethane.....	1741.5	17,200	1.061	.923	1.186
1,3-Diphenylpropane.....	1886.5	17,300	1.057	.929	1.181
1,4-Diphenylbutane.....	2032.9	17,400	(^d)	.929	-----
Dicyclohexylalkanes					
Bicyclohexyl.....	1700.0	18,400	1.017×10 ⁶	0.988	1.137
Dicyclohexylmethane.....	1845.9	18,425	1.008	.989	1.127
1,2-Dicyclohexylethane.....	1997.5	18,500	1.008	.993	1.127
1,3-Dicyclohexylpropane.....	2156.2	18,625	1.012	1.000	1.131
1,4-Dicyclohexylbutane.....	2285.7	18,500	1.005	.993	1.123

^a Reference 2.^b Extrapolated value from reference 12.^c Slight decomposition at this pressure.^d 1,4-Diphenylbutane is a solid at temperature (20° C) at which density measurements were made. No value is available in the literature, as in the case of biphenyl and 1,2-diphenyl ethane.^e Based on experimentally determined value of 18,625 Btu/lb for fuel used at Lewis laboratory.^f The value of 894,000 Btu/cu ft for AN-F-58 was calculated from the value in footnote c of table I using the experimentally determined value of 0.766^g for the specific gravity of the fuel.

REPORT 1004

A LIFT-CANCELLATION TECHNIQUE IN LINEARIZED SUPERSONIC-WING THEORY¹

By HAROLD MIRELS

SUMMARY

A lift-cancellation technique is presented for determining load distributions on thin wings at supersonic speeds. The loading on a wing having a prescribed plan form is expressed as the loading of a known related wing (such as a two-dimensional or a triangular wing) minus the loading of an appropriate cancellation wing.

A general expression is derived for the load distribution over a cancellation wing. The expression is valid when the plan-form edge (on the cancellation wing) separating a region of zero upwash from a region of known loading is everywhere subsonically inclined to the free stream. The boundary conditions can be satisfied for both subsonic leading and subsonic trailing plan-form edges on the prescribed wing.

The lift-cancellation technique can be used to find the loading on a large variety of wings. Applications to swept wings having curvilinear plan forms and to wings having reentrant side edges are indicated.

INTRODUCTION

The method of lift cancellation for obtaining the lift distribution on thin wings at supersonic speeds was first suggested in reference 1. The lift distribution on a given wing is determined by canceling excess lift, through the use of a "cancellation wing," on a related plan form having a known loading. This approach has been applied by several authors (for example, references 2 to 4). The expressions provided in reference 1 are applicable for wings that can be generated by the superposition of conical fields.

A procedure is presented in reference 5 for determining lift on a more general class of plan forms than can be handled by conical superposition. The method utilizes a surface distribution of doublets and an inversion by means of Abel's integral equation and is equivalent to a lift cancellation.

This report, prepared at the NACA Lewis laboratory, retains certain features of reference 5 (that is, the use of a surface distribution of doublets and an inversion by means of Abel's integral equation), whereas other features are simplified and generalized. The simplification consists in eliminating steps in the procedure for obtaining lift distributions. The generalization consists in determining a solution that can be made to satisfy the boundary conditions for either a subsonic leading edge or a subsonic trailing edge (Kutta condition). The method of reference 5 yields only the Kutta

solution. The lift-cancellation technique developed herein is illustrated by several examples.

In a concurrent investigation (reference 6), source distributions and integral-equation formulations are applied to obtain the loading on a special series of cancellation wings. Reference 7 employs some of these cancellation wings for the determination of lift and moments on swept wings.

THEORY

The usual assumptions of an inviscid fluid and small perturbations are made. The velocity field consists of the free-stream velocity U (taken in the positive x -direction) plus the perturbation velocities u , v , and w . The wing boundary conditions are specified in the $z=0$ plane.

The local lift coefficient ΔC_p may be expressed in terms of Δu ; that is,

$$\Delta C_p = \frac{p_B - p_T}{q} = \frac{2(u_T - u_B)}{U} = \frac{2\Delta u}{U} \quad (1)$$

(All symbols used in this report are defined in appendix A.) Inasmuch as the local lift coefficient is directly proportional to Δu , Δu will be referred to as "lift" in later developments.

LIFT-CANCELLATION METHOD

The lift distribution on a given wing is to be determined by canceling excess lift on a related wing with a known loading. The method is illustrated in figure 1. The wing for which the lift distribution is desired is shown in figure 1 (a). The solution can be expressed as the two-dimensional wing (fig. 1 (b)) minus a cancellation wing (fig. 1 (c)). The loading in region I of the cancellation wing equals the loading in the corresponding region of the two-dimensional wing and the upwash w in region II of the cancellation wing is zero. The two-dimensional wing minus the cancellation wing satisfies the boundary conditions for the flow about the given wing and is the desired solution.

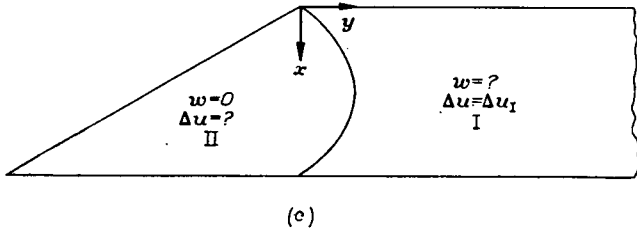
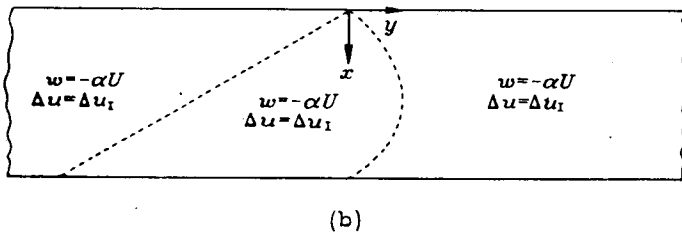
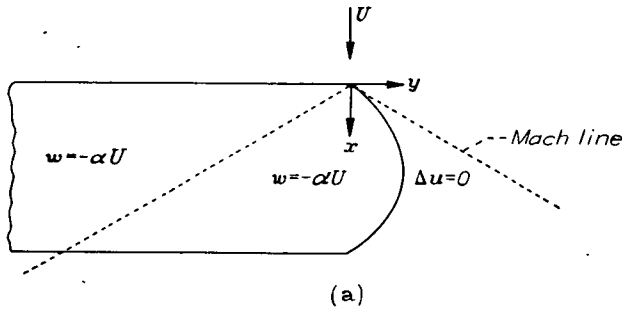
The fundamental problem in the lift-cancellation method is then to determine the lift in region II of a cancellation wing subject to the condition $w=0$ in this region and with the assumption of a known loading in region I. Solution of this problem is presented in the following sections.

DERIVATION OF LIFT-CANCELLATION EQUATIONS

The lift distribution in region II will be expressed in terms of quantities in region I.

¹ Supersedes NACA TN 2145, "Lift-Cancellation Technique in Linearized Supersonic-Wing Theory" by Harold Mirels, 1950.

Consider the cancellation wing shown in figure 2. The portion of the leading edge to the left of the origin coincides with a Mach line. The portion of the leading edge to the right (designated $r=r_1(s)$) is shown as a supersonic edge, although no restrictions as to a subsonic or supersonic edge are imposed. (A plan-form edge is subsonic or supersonic depending on whether the component of the free stream normal to the edge is subsonic or supersonic.) The line designated $r=r_2(s)$ separates region I and region II and is



(a) Given wing.
(b) Two-dimensional wing.
(c) Cancellation wing.

FIGURE 1.—Superposition to obtain lift on given wing by canceling lift on two-dimensional wing. (Given wing equals two-dimensional wing minus cancellation wing.)

assumed to be subsonically inclined to the free stream at all points. This line corresponds to a plan-form edge of the wing for which the lift distribution is desired.

General solution for $\Delta\varphi$ on cancellation wing.—The upwash field in the $z=0$ plane (associated with an arbitrary distribution of vorticity Δu and Δv) may be written, from reference 8,

$$w = \frac{\beta^2}{2\pi} \int \int_{\tau} \frac{[(y-y_0)\Delta v + (x-x_0)\Delta u] dx_0 dy_0}{[(x-x_0)^2 - \beta^2(y-y_0)^2]^{3/2}} \quad (2)$$

The symbol $\int$ designates the finite part of an infinite integral, as defined in reference 9. Application of the finite-part

concept to linearized supersonic-wing theory and the evaluation of the finite part of an infinite integral are discussed in references 8 and 10. For the present, it will suffice to state the fundamental definition of the finite part of an integral with a $3/2$ -power singularity, namely,

$$\int_a^x \frac{f(x_0) dx_0}{(x-x_0)^{3/2}} = \int_a^x \frac{[f(x_0) - f(x)] dx_0}{(x-x_0)^{3/2}} - \frac{2f(x)}{(x-a)^{1/2}} \quad (3)$$

By a transformation to the Mach coordinates of reference 11,

$$\left. \begin{aligned} x &= \frac{\beta}{M} (s+r) & r &= \frac{M}{2\beta} (x-\beta y) \\ y &= \frac{1}{M} (s-r) & s &= \frac{M}{2\beta} (x+\beta y) \\ \text{elemental area} &= \frac{2\beta}{M^2} dr ds \end{aligned} \right\} \quad (4)$$

equation (2) becomes

$$w = \frac{M}{8\pi} \int \int_{\tau} \frac{\left[(s-s_0) \frac{\partial \Delta\varphi}{\partial s_0} + (r-r_0) \frac{\partial \Delta\varphi}{\partial r_0} \right] dr_0 ds_0}{[(r-r_0)(s-s_0)]^{3/2}} \quad (5)$$

Upon substitution of the limits of integration, as indicated in figure 2,

$$w = \frac{M}{8\pi} \left\{ \int_0^s \frac{ds_0}{(s-s_0)^{3/2}} \int_{r_1(s_0)}^r \frac{\frac{\partial \Delta\varphi}{\partial r_0} dr_0}{(r-r_0)^{1/2}} + \int_0^s \frac{ds_0}{(s-s_0)^{1/2}} \int_{r_1(s_0)}^r \frac{\frac{\partial \Delta\varphi}{\partial s_0} dr_0}{(r-r_0)^{3/2}} \right\} \quad (6)$$

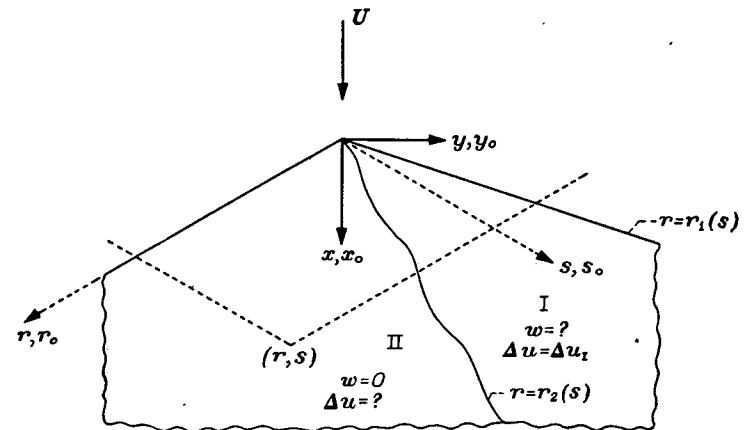


FIGURE 2.—Typical cancellation wing.

Integrating by parts, noting that $\Delta\varphi=0$ at $r_0=r_1(s_0)$, and recalling the definition of the finite part (equation (3)) yield

$$\begin{aligned} \int_{r_1(s_0)}^r \frac{\frac{\partial \Delta\varphi}{\partial r_0} dr_0}{(r-r_0)^{1/2}} &= \lim_{r_0 \rightarrow r} \left\{ \frac{\Delta\varphi}{(r-r_0)^{1/2}} \right\}_{r_1(s_0)} - \frac{1}{2} \int_{r_1(s_0)}^r \frac{\Delta\varphi dr_0}{(r-r_0)^{3/2}} \\ &= -\frac{1}{2} \int_{r_1(s_0)}^r \frac{\Delta\varphi dr_0}{(r-r_0)^{3/2}} \end{aligned} \quad (7)$$

Thus,

$$\int_0^s \frac{ds_o}{(s-s_o)^{3/2}} \int_{r_1(s_o)}^r \frac{\partial \Delta \varphi}{\partial r_o} \frac{dr_o}{(r-r_o)^{1/2}} \equiv -\frac{1}{2} \int_0^s \frac{ds_o}{(s-s_o)^{3/2}} \int_{r_1(s_o)}^r \frac{\Delta \varphi dr_o}{(r-r_o)^{3/2}} \quad (8)$$

Similarly, reversing the order of integration (with appropriate changes in limits of integration), integrating by parts, and then returning to the original order of integration establish the identity

$$\int_0^s \frac{ds_o}{(s-s_o)^{1/2}} \int_{r_1(s_o)}^r \frac{\partial \Delta \varphi}{\partial s_o} \frac{dr_o}{(r-r_o)^{3/2}} \equiv -\frac{1}{2} \int_0^s \frac{ds_o}{(s-s_o)^{3/2}} \int_{r_1(s_o)}^r \frac{\Delta \varphi dr_o}{(r-r_o)^{3/2}} \quad (9)$$

The right side of equations (8) and (9) are identical. Equation (6) can now be written as

$$w = -\frac{M}{8\pi} \int_0^s \frac{ds_o}{(s-s_o)^{3/2}} \int_{r_1(s_o)}^r \frac{\Delta \varphi dr_o}{(r-r_o)^{3/2}} \quad (10)$$

For points in region II, $w=0$ and equation (10) becomes

$$0 = \int_0^s \frac{ds_o}{(s-s_o)^{3/2}} \int_{r_1(s_o)}^r \frac{\Delta \varphi dr_o}{(r-r_o)^{3/2}} \quad (11a)$$

or

$$0 = \int_0^s \frac{G(r, s_o) ds_o}{(s-s_o)^{3/2}} \quad (11b)$$

where

$$G(r, s_o) = \int_{r_1(s_o)}^r \frac{\Delta \varphi dr_o}{(r-r_o)^{3/2}} \quad (12)$$

Equation (11b) is an integral equation for the unknown function $G(r, s_o)$. The solution (appendix B) is

$$G(r, s_o) = 0 \quad (13)$$

Thus,

$$\int_{r_1(s_o)}^r \frac{\Delta \varphi dr_o}{(r-r_o)^{3/2}} = \int_{r_1(s_o)}^{r_2(s_o)} \frac{\Delta \varphi_I dr_o}{(r-r_o)^{3/2}} + \int_{r_2(s_o)}^r \frac{\Delta \varphi_{II} dr_o}{(r-r_o)^{3/2}} = 0$$

or

$$\int_{r_2(s_o)}^r \frac{\Delta \varphi_{II} dr_o}{(r-r_o)^{3/2}} = - \int_{r_1(s_o)}^{r_2(s_o)} \frac{\Delta \varphi_I dr_o}{(r-r_o)^{3/2}} \quad (14)$$

The right side of equation (14) will be considered known. Equation (14) is then an integral equation for $\Delta \varphi_{II}$. The solution (appendix B) is

$$\Delta \varphi_{II} = \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta \varphi_I dr_o}{(r-r_o) \sqrt{r_2(s)-r_o}} \quad (15)$$

Equation (15) indicates that the doublet strength in region II, namely $\Delta \varphi_{II}$, can be obtained by a line integration along $s_o=s$ in region I. The geometric interpretation of the various terms in equation (15) is shown in figure 3 (a).

It can be shown, by expanding $\Delta \varphi_I$ about $r_o=r_2(s)$, that equation (15) yields a continuous solution ($\Delta \varphi_{II}=\Delta \varphi_I$) at $r=r_2(s)$. (A discontinuity in $\Delta \varphi$ implies a lifting line (reference 12) and is unrealistic.)

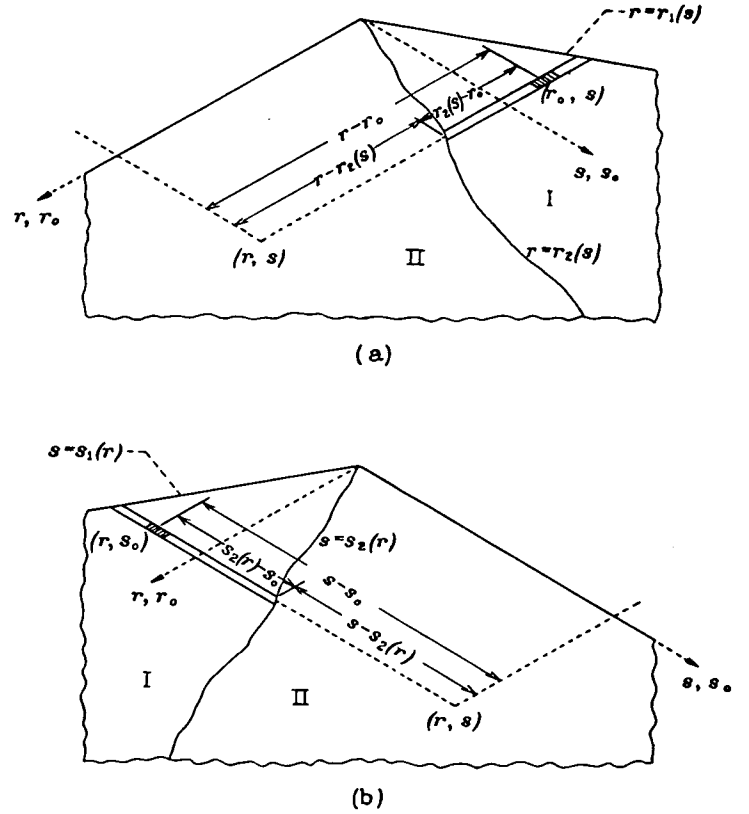


FIGURE 3.—Geometric interpretation of terms in equations (17a) and (17b).

General solution for load distribution on cancellation wing.—The load distribution in region II can be expressed as

$$\begin{aligned} \Delta u_{II} &= \frac{\partial \Delta \varphi_{II}}{\partial x} = \frac{\partial \Delta \varphi_{II}}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \Delta \varphi_{II}}{\partial s} \frac{\partial s}{\partial x} \\ &= \frac{M}{2\beta} \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) \Delta \varphi_{II} \end{aligned}$$

or, from equation (15),

$$\frac{2\beta}{M} \Delta u_{II} = \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) \left[\frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta \varphi_I dr_o}{(r-r_o) \sqrt{r_2(s)-r_o}} \right] \quad (16)$$

Differentiation yields (see appendix C)

$$\begin{aligned} \Delta u_{II} &= \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta u_I dr_o}{(r-r_o) \sqrt{r_2(s)-r_o}} - \\ &\quad \frac{1 - \frac{dr_2(s)}{ds}}{2\beta\pi \sqrt{r-r_2(s)}} \int_{r_1(s)}^{r_2(s)} \frac{(\beta \Delta u_I - \Delta v_I) dr_o}{\sqrt{r_2(s)-r_o}} \end{aligned} \quad (17a)$$

Equation (17a) is the desired expression for the lift distribution in region II in terms of quantities in region I.

Consider Δu_{II} to consist of two components, $\Delta u_{II}'$ and $\Delta u_{II}''$, where $\Delta u_{II}'$ and $\Delta u_{II}''$ are the first and second terms on the right side of equation (17a), respectively. Investigation of the integrals indicates that at $r=r_2(s)$, $\Delta u_{II}'=\Delta u_I$; whereas $\Delta u_{II}''$, in general, has a half-order singularity.

However, $\partial\Delta\varphi_I/\partial r_o$ must be continuous in the vicinity of $r_2(s)$. (The perturbation velocities on the basic wing can be discontinuous only along Mach lines or along plan-form edges. Inasmuch as $r=r_2(s)$ is neither of these cases, all derivatives of $\Delta\varphi_I$ must be continuous in the vicinity of $r=r_2(s)$.) When the Kutta condition is imposed, $\partial\Delta\varphi_{II}/\partial r_o$ is therefore also continuous (and bounded) in the neighborhood of $r=r_2(s)$. Then with the use of a mean value for $\partial\Delta\varphi_{II}/\partial r_o$,

$$\lim_{r \rightarrow r_2(s)} \left[\int_{r_2(s)}^r \frac{\frac{\partial\Delta\varphi_{II}}{\partial r_o} dr_o}{(r-r_o)^{1/2}} \right] = \lim_{r \rightarrow r_2(s)} \left[\left(\frac{\partial\Delta\varphi_{II}}{\partial r_o} \right)_{(r_2(s) < r_o < r)} \int_{r_2(s)}^r \frac{dr_o}{(r-r_o)^{1/2}} \right] = 0 \quad (22)$$

Therefore,

$$\int_{r_1(s)}^{r_2(s)} \frac{\frac{\partial\Delta\varphi_I}{\partial r_o} dr_o}{[r_2(s)-r_o]^{1/2}} = 0 \quad (23)$$

which was to be proved.

The solution for Δu_{II} that satisfies the Kutta condition at $r=r_2(s)$ is then, from equations (17a) and (23),

$$\Delta u_{II} = \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta u_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \quad (24a)$$

for the wing of figure 3 (a). Similarly,

$$\Delta u_{II} = \frac{\sqrt{s-s_2(r)}}{\pi} \int_{s_1(r)}^{s_2(r)} \frac{\Delta u_I ds_o}{(s-s_o)\sqrt{s_2(r)-s_o}} \quad (24b)$$

for the wing of figure 3 (b).

An alternative derivation of equations (24a) and (24b) (appendix D) indicates that only solutions satisfying the Kutta condition will result from the integral equation formulation of reference 5.

Sidewash in region II.—An expression for Δv_{II} can be obtained by differentiating equation (15) with respect to y . The result is

$$\Delta v_{II} = \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta v_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} + \frac{1 + \frac{dr_2(s)}{ds}}{2\pi\sqrt{r-r_2(s)}} \int_{r_1(s)}^{r_2(s)} \frac{\beta\Delta u_I - \Delta v_I}{\sqrt{r_2(s)-r_o}} dr_o$$

Similarly, for region II to the right of region I,

$$\Delta v_{II} = \frac{\sqrt{s-s_2(r)}}{\pi} \int_{s_1(r)}^{s_2(r)} \frac{\Delta v_I ds_o}{(s-s_o)\sqrt{s_2(r)-s_o}} + \frac{1 + \frac{ds_2(r)}{ds}}{2\pi\sqrt{s-s_2(r)}} \int_{s_1(r)}^{s_2(r)} \frac{\beta\Delta u_I + \Delta v_I}{\sqrt{s_2(r)-s_o}} ds_o$$

When the Kutta condition applies, these equations become, respectively,

$$\Delta v_{II} = \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta v_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}}$$

and

$$\Delta v_{II} = \frac{\sqrt{s-s_2(r)}}{\pi} \int_{s_1(r)}^{s_2(r)} \frac{\Delta v_I ds_o}{(s-s_o)\sqrt{s_2(r)-s_o}}$$

It should be noted that when $r=r_2(s)$ corresponds to a subsonic trailing edge, Δv_I , as well as Δv_{II} , is not generally known. The preceding expressions are therefore primarily useful for those problems where $r=r_2(s)$ corresponds to a subsonic leading edge.

APPLICATIONS

The loading in region II of a cancellation wing is given by the line integrals of equation (17a) or (17b). When the Kutta condition is imposed at a subsonic trailing edge, the expressions reduce to equations (24a) and (24b). These equations can be used to find the load distribution on a large variety of wings. Wings with curvilinear plan forms or arbitrary camber are examples. In each case, however, the solution for the related wing must be known.

The equations are applied in several illustrative examples. Only the solution associated with the cancellation wing is considered. The complete solution consists of the loading of the related wing minus the loading of the cancellation wing.

LEADING-EDGE AND SIDE-EDGE CANCELLATIONS

In leading-edge and side-edge cancellations, the lift to be canceled is upstream or to the side of the plan form for which the loading is desired (figs. 5 and 6).

Tip region of swept wing.—The loading in the region influenced by the side edge (II_a and II_b of fig. 5) of a swept wing having a subsonic leading and a supersonic trailing edge can be obtained by canceling excess lift on a triangular wing. The Kutta condition is applied across the portion of $r=r_2(s)$ influencing region II_b. The lift to be canceled in region I is (reference 13, equation (23))

$$\Delta u_I = \frac{H\theta^2 x}{\sqrt{\theta^2 x^2 - \beta^2 y^2}} = \frac{H\theta^2(s+r)}{\sqrt{\theta^2(s+r)^2 - (s-r)^2}} \quad (25)$$

where H and θ are constants defined in appendix A. The doublet distribution in region I_a, again from reference 13, is

$$\Delta\varphi_{I_a} = H\sqrt{\theta^2 x^2 - \beta^2 y^2} = \frac{\beta H}{M} \sqrt{\theta^2(s+r)^2 - (s-r)^2}$$

from which

$$\Delta v_{I_a} = \frac{\partial\Delta\varphi_{I_a}}{\partial y} = \frac{-H\beta^2 y}{\sqrt{\theta^2 x^2 - \beta^2 y^2}} = \frac{-H\beta(s-r)}{\sqrt{\theta^2(s+r)^2 - (s-r)^2}} \quad (26)$$

The sidewash distribution in region I_b (that is, Δv_{I_b}) could be found by an integration of the type indicated in equation (18b). A knowledge of Δv_{I_b} , however, is unnecessary in

the present problem because the Kutta condition is applied for region II_b.

The loading in region II_a is obtained by substituting equations (25) and (26), with r replaced by r_o , into equation (17a), which yields

$$\Delta u_{II_a} = \frac{H\theta^2\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{(s+r_o)dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}\sqrt{\theta^2(s+r_o)^2-(s-r_o)^2}} - \frac{H\left[1-\frac{dr_2(s)}{ds}\right]}{2\pi\sqrt{r-r_2(s)}} \int_{r_1(s)}^{r_2(s)} \frac{[\theta^2(s+r_o)+(s-r_o)]dr_o}{\sqrt{r_2(s)-r_o}\sqrt{\theta^2(s+r_o)^2-(s-r_o)^2}} \quad (27)$$

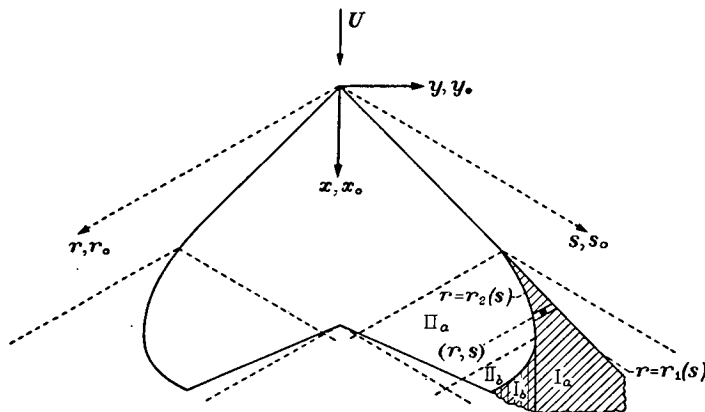


FIGURE 5.—Cancellation for obtaining loading in tip region of swept wing having supersonic trailing edges.

For region II_b, the Kutta condition applies and

$$\Delta u_{II_b} = \frac{H\theta^2\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{(s+r_o)dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}\sqrt{\theta^2(s+r_o)^2-(s-r_o)^2}} \quad (28)$$

Equations (27) and (28) reduce to elliptic integrals of the first, second, and third kinds upon transforming the variable of integration from r_o to ω_o according to the relation

$$r_o = \frac{1}{1+\theta} [(1-\theta)s + a_2\omega_o^2] \quad (29)$$

where

$$a_2 = (1+\theta)r_2(s) - (1-\theta)s$$

Equations (27) and (28) may then be written

$$\Delta u_{II_a} = \frac{H\theta^2\sqrt{r-r_2(s)}}{\pi\sqrt{s\theta}} \left\{ \frac{(s+r)(1+\theta)}{a} \left[\Pi\left(\frac{\pi}{2}, n, k\right) - \Pi(\phi, n, k) \right] - \left[F\left(\frac{\pi}{2}, k\right) - F(\phi, k) \right] \right\} - \frac{H\sqrt{s\theta}\left[1-\frac{dr_2(s)}{ds}\right]}{\pi\sqrt{r-r_2(s)}} \left\{ 2 \left[E\left(\frac{\pi}{2}, k\right) - E(\phi, k) \right] - \left[F\left(\frac{\pi}{2}, k\right) - F(\phi, k) \right] \right\} \quad (30)$$

and

$$\Delta u_{II_b} = \frac{H\theta^2\sqrt{r-r_2(s)}}{\pi\sqrt{s\theta}} \left\{ \frac{(s+r)(1+\theta)}{a} \left[\Pi\left(\frac{\pi}{2}, n, k\right) - \Pi(\phi, n, k) \right] - \left[F\left(\frac{\pi}{2}, k\right) - F(\phi, k) \right] \right\} \quad (31)$$

where

$$k = \sqrt{\frac{1-\theta}{4s\theta}} a_2 \quad a = (1+\theta)r - (1-\theta)s$$

$$\phi = \sin^{-1} \sqrt{\frac{a_1}{a_2}} \quad a_1 = (1+\theta)r_1(s) - (1-\theta)s$$

$$n = -\frac{a_2}{a} \quad a_2 = (1+\theta)r_2(s) - (1-\theta)s$$

Reentrant side edge.—A plan form has a reentrant edge if a line of constant y intersects the plan form at more than two points.

The load distribution in the region influenced by the reentrant side edge is to be determined for the wing (unshaded region) of figure 6. The side edge is first, for simplicity, the straight line $r=K_2s$, which is a subsonic trailing edge across which the Kutta condition is applied. The side edge then alternately becomes a subsonic leading and a subsonic trailing edge. The load distribution in region I is simply the Ackeret value $\Delta u_I = \frac{2\alpha U}{\beta}$, and $\Delta v_{I_a} = 0$. Regions II_a, II_b, and II_c are considered separately.

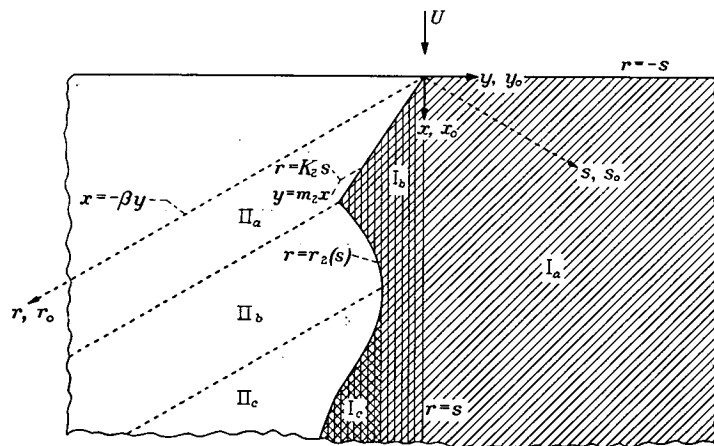


FIGURE 6.—Cancellation for obtaining loading in region influenced by reentrant side edge.

Region II_a:

From equation (24a) with $\Delta u_I = \frac{2\alpha U}{\beta}$,

$$\Delta u_{II_a} = \frac{\sqrt{r-K_2s}}{\pi} \int_{-s}^{K_2s} \frac{2\alpha U dr_o}{\beta(r-r_o)\sqrt{K_2s-r_o}} = \frac{4\alpha U}{\beta\pi} \tan^{-1} \sqrt{\frac{(K_2+1)s}{r-K_2s}} \quad (32a)$$

or, in x, y coordinates

$$\Delta u_{II_a} = \frac{4\alpha U}{\beta\pi} \tan^{-1} \sqrt{\frac{x+\beta y}{\beta(m_2x-y)}} \quad (32b)$$

Region II_b:

A knowledge of Δv_{I_b} is required. From equation (18b),

$$\Delta v_{I_b} = \frac{\partial}{\partial y} \left[\frac{4\alpha U}{\beta\pi} \int_{-\beta y}^{y/m_2} \tan^{-1} \sqrt{\frac{x_o+\beta y}{\beta(m_2x_o-y)}} dx_o + \frac{2\alpha U}{\beta} \int_{y/m_2}^x dx_o \right] = \frac{2\alpha U}{\sqrt{|\beta m_2|}} = 2\alpha U \sqrt{\frac{K_2+1}{K_2-1}} \quad (33)$$

The load distribution in region II_b is then, recalling that $\Delta v_{I_a} = 0$,

$$\begin{aligned} \Delta u_{II_b} &= \frac{\sqrt{r-r_2(s)}}{\pi} \int_{-s}^{r_2(s)} \frac{2\alpha U dr_o}{\beta(r-r_o)\sqrt{r_2(s)-r_o}} - \\ &\quad \frac{\left[1 - \frac{dr_2(s)}{ds}\right]}{2\beta\pi\sqrt{r-r_2(s)}} \left[\int_{-s}^s \frac{2\alpha U dr_o}{\sqrt{r_2(s)-r_o}} + \int_s^{r_2(s)} \left(2\alpha U - 2\alpha U \sqrt{\frac{K_2+1}{K_2-1}}\right) \frac{dr_o}{\sqrt{r_2(s)-r_o}} \right] \\ &= \frac{2\alpha U}{\beta\pi} \left(2 \tan^{-1} \sqrt{\frac{r_2(s)+s}{r-r_2(s)}} - \frac{\left[1 - \frac{dr_2(s)}{ds}\right]}{\sqrt{r-r_2(s)}} \left\{ \sqrt{r_2(s)+s} - \sqrt{\left(\frac{K_2+1}{K_2-1}\right) [r_2(s)-s]} \right\} \right) \end{aligned} \quad (34)$$

Region II_c:

Because the Kutta condition is applied,

$$\Delta u_{II_c} = \frac{4\alpha U}{\beta\pi} \tan^{-1} \sqrt{\frac{r_2(s)+s}{r-r_2(s)}} \quad (35)$$

TRAILING-EDGE CANCELLATION

The calculation of lift distributions on swept wings having subsonic trailing edges requires cancellation wings of the type shown in figure 7. These wings cancel that part of the lift of the basic triangular wing that is downstream of the trailing edge of the swept wing (references 3, 4, 6, and 7). The lift is specified in region I. The lift in regions II and III is to be determined subject to the conditions that $w=0$ and that the Kutta condition applies at $r=r_1(s)$ and $r=r_2(s)$.

The wing of figure 7 (a) differs from the previously discussed cases in that two unknown regions (II and III) are continuously interacting. A special treatment is required in order to obtain the loading in regions II and III. (See, for example, reference 6.) Approximate solutions can be obtained, however, using equations (24a) and (24b). For example, if the load at (r,s) in region II is desired, first assume that Δu_{III} is known. Then,

$$\Delta u_{II} = \frac{\sqrt{r-r_2(s)}}{\pi} \left[\int_{\frac{Mc_r}{2\beta}}^{r_1(s)} \frac{\Delta u_{III} dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} + \int_{r_1(s)}^{r_2(s)} \frac{\Delta u_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \right] \quad (36)$$

An expression for Δu_{III} is, by integration along lines of constant r_o (fig. 7 (a)),

$$\Delta u_{III} = \frac{\sqrt{s-s_1(r_o)}}{\pi} \left[\int_{\frac{Mc_r}{2\beta}}^{s_2(r_o)} \frac{\Delta u_{II} ds_o}{(s-s_o)\sqrt{s_1(r_o)-s_o}} + \int_{s_2(r_o)}^{s_1(r_o)} \frac{\Delta u_I ds_o}{(s-s_o)\sqrt{s_1(r_o)-s_o}} \right]$$

An approximate expression for Δu_{III} is then

$$\Delta u_{III} \approx \frac{\sqrt{s-s_1(r_o)}}{\pi} \int_{s_2(r_o)}^{s_1(r_o)} \frac{\Delta u_I ds_o}{(s-s_o)\sqrt{s_1(r_o)-s_o}} \quad (37)$$

Equation (36), which may now be written in terms of Δu_I by substituting equation (37) for Δu_{III} , becomes

$$\Delta u_{II} \approx \frac{\sqrt{r-r_2(s)}}{\pi} \left\{ \int_{\frac{Mc_r}{2\beta}}^{r_1(s)} \frac{dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \left[\frac{\sqrt{s-s_1(r_o)}}{\pi} \int_{s_2(r_o)}^{s_1(r_o)} \frac{\Delta u_I ds_o}{(s-s_o)\sqrt{s_1(r_o)-s_o}} \right] + \int_{r_1(s)}^{r_2(s)} \frac{\Delta u_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \right\} \quad (38)$$

The first term on the right side of equation (38) approximates the contribution of region III to the loading in region II. This term, as indicated in reference 6, is negligible for the commonly encountered Δu_I distributions (corresponding to steady lift, roll, or pitch) and $\frac{\beta dy_1(x)}{dx} > 0.5$. For those cases, equation (38) simplifies to

$$\Delta u_{II} \approx \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta u_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \quad (39)$$

The Kutta condition at $r=r_2(s)$ is satisfied by both equations

(38) and (39). Equation (39) can be reduced to elliptic integrals in canonical form by the substitution indicated in equation (29). The elliptic integrals may be avoided by expanding Δu_I in a Taylor's series about $r_o=r_2(s)$. Equation (39) can then be readily integrated, term by term, to yield a very good approximation for Δu_{II} in terms of algebraic and trigonometric functions.

When region I has a partly supersonic leading edge (fig. 7 (b)), it is possible to write exact expressions for the linearized load distribution in regions II and III. For example, the load at point (r,s) of figure 7 (b) is

$$\Delta u_{II} = \frac{\sqrt{r-r_2(s)}}{\pi} \left\{ \int_c^{r_1(s)} \frac{dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \left[\frac{\sqrt{s-s_1(r_o)}}{\pi} \int_{s_3(r_o)}^{s_1(r_o)} \frac{\Delta u_I ds_o}{(s-s_o)\sqrt{s_1(r_o)-s_o}} \right] + \int_{r_1(s)}^{r_2(s)} \frac{\Delta u_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \right\} \quad (40)$$

APPENDIX A

SYMBOLS

The following symbols are used in this report:

a	$=$	$(1+\theta)r-(1-\theta)s$
a_1	$=$	$(1+\theta)r_1(s)-(1-\theta)s$
a_2	$=$	$(1+\theta)r_2(s)-(1-\theta)s$
ΔC_p		local lift coefficient, $\frac{p_B-p_T}{q}$
c		constant
c_r		root chord of swept wing
$E(\phi, k)$		elliptic integral of second kind, $E(\phi, k) = \int_0^{\sin \phi} \frac{\sqrt{1-k^2\omega_o^2}}{\sqrt{1-\omega_o^2}} d\omega_o$
$F(\phi, k)$		elliptic integral of first kind, $F(\phi, k) = \int_0^{\sin \phi} \frac{d\omega_o}{\sqrt{(1-\omega_o^2)(1-k^2\omega_o^2)}}$
$G(r, s_o)$		function of r and s_o defined by equation (12)
H	$=$	$\frac{2\alpha U}{\beta E\left(\frac{\pi}{2}, \sqrt{1-\theta^2}\right)}$
K		slope of plan-form edge in r, s coordinates, dr/ds
k		modulus of elliptic integrals
M		Mach number
m		slope of plan-form edge in x, y coordinates, dy/dx
n		parameter of elliptic integral of third kind
p		local static pressure
q	$=$	$\frac{1}{2} \rho U^2$
r, r_o		Mach coordinate system (equation (4))
s, s_o		
U		free-stream velocity
u, v, w		perturbation velocities in x -, y -, and z -directions, respectively
Δu	$=$	$u_T - u_B$ (proportional to local lift)
Δv	$=$	$v_T - v_B$
x, x_o		Cartesian coordinate system
y, y_o		
z, z_o		

α		angle of attack
β	$=$	$\sqrt{M^2-1}$
δ		semivertex angle of triangular wing
θ	$=$	$\beta \tan \delta$
$\Pi(\phi, n, k)$		elliptic integral of third kind, $\int_0^{\sin \phi} \frac{d\omega_o}{(1+n\omega_o^2)\sqrt{(1-k^2\omega_o^2)(1-\omega_o^2)}}$
ρ		density
τ		area of integration
ϕ		amplitude of elliptic integrals
φ		perturbation velocity potential
$\Delta\varphi$		doublet strength, $\varphi_T - \varphi_B$
ω_o		integration variable
Regions:		
I		region on cancellation wing for which loading is specified
$I_a, I_b, \dots$		subdivisions of region I
II		region on cancellation wing for which $w=0$
$II_a, II_b, \dots$		subdivisions of region II
III		additional region on cancellation wing for which $w=0$
Special designations:		
$r=r_1(s)$		r as function of s along plan-form boundary 1
$s=s_1(r)$		s as function of r along plan-form boundary 1
$y=y_1(x)$		y as function of x along plan-form boundary 1
$x=x_1(y)$		x as function of y along plan-form boundary 1
$r=r_2(s)$		r as function of s along plan-form boundary 2
$s=s_2(r)$		s as function of r along plan-form boundary 2 and so forth.
Subscripts:		
1, 2, 3		refers to plan-form boundaries 1, 2, and 3, respectively
I, II		refers to regions I and II, respectively
B		bottom surface of $z=0$ plane
T		top surface of $z=0$ plane
o		variable of integration

APPENDIX B

SOLUTION OF INTEGRAL EQUATIONS

Consider the following integral equation (in the notation of the appendix in reference 5), where the function $f(x)$ is assumed known and the function $u(\xi)$ is to be determined:

$$f(x) = \int_a^x \frac{u(\xi) d\xi}{(x-\xi)^{3/2}} = \int_a^x \frac{[u(\xi) - u(x)] d\xi}{(x-\xi)^{3/2}} - \frac{2u(x)}{(x-a)^{1/2}} \quad (B1)$$

After an integration by parts, equation (B1) may be written

$$-\frac{f(x)}{2} - \frac{u(a)}{(x-a)^{1/2}} = \int_a^x \frac{u'(\xi) d\xi}{(x-\xi)^{1/2}} \quad (B2)$$

Equation (B2) is now an integral equation of the Abel type. The continuous solution for $u(\xi)$ is (reference 14)

$$u(z) = -\frac{1}{2\pi} \int_a^z \frac{f(x) dx}{(z-x)^{1/2}} \quad (B3)$$

evaluated at $z=\xi$. This result is presented in reference 5.

Equation (11b) corresponds to equation (B1) with $u(\xi) = G(r, s_o)$ and $f(x) = 0$. The solution for $G(r, s_o)$, according to equation (B3), is then

$$G(r, s_o) = 0 \quad (13)$$

Equation (14) corresponds to equation (B1) with

$$\begin{aligned} \xi &= r_o & a &= r_2(s_o) \\ x &= r & f(x) &= -\int_{r_1(s_o)}^{r_2(s_o)} \frac{\Delta\varphi_1 dr_o}{(r-r_o)^{3/2}} \end{aligned}$$

$$u(\xi) = \Delta\varphi_{II}$$

The solution for $\Delta\varphi_{II}$ according to equation (B3) is then

$$\Delta\varphi_{II} = \frac{1}{2\pi} \int_{r_2(s)}^z \frac{dr}{\sqrt{z-r}} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)^{3/2}} \quad (B4)$$

Reversing the order of integration and integrating yield

$$\Delta\varphi_{II} = \frac{\sqrt{z-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(z-r_o)\sqrt{r_2(s)-r_o}} \quad (B5)$$

Equation (B5), evaluated at $z=r$ and $s_o=s$, yields

$$\Delta\varphi_{II} = \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \quad (15)$$

The derivation of equation (15) is similar to that for equation (16) of reference 5.

APPENDIX C

DIFFERENTIATION TO OBTAIN Δu_{II}

The differentiation indicated in equation (16)

$$\frac{2\beta}{M} \Delta u_{II} = \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) \left[\frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \right] \quad (16)$$

is to be conducted.

First,

$$\begin{aligned} \frac{2\beta}{M} \Delta u_{II} = & \frac{1 - \frac{dr_2(s)}{ds}}{2\pi\sqrt{r-r_2(s)}} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} + \\ & \frac{\sqrt{r-r_2(s)}}{\pi} \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \end{aligned} \quad (C1)$$

Inasmuch as $\Delta\varphi_I$ is a function of r_o and s ,

$$\frac{\partial}{\partial r} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} = - \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)^2\sqrt{r_2(s)-r_o}} \quad (C2a)$$

and

$$\begin{aligned} \frac{\partial}{\partial s} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} = & \int_{r_1(s)}^{r_2(s)} \frac{\left(\frac{\partial \Delta\varphi_I}{\partial s} \right) dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} + \\ \lim_{r \rightarrow r_2(s)} \left\{ -\frac{1}{2} \frac{dr_2(s)}{ds} \int_{r_1(s)}^{r_o} \frac{\Delta\varphi_I dr_o}{(r-r_o)[r_2(s)-r_o]^{3/2}} + \right. \\ \left. \frac{[(\Delta\varphi_I)_{r_o=r_2(s)}]}{(r-r_o)\sqrt{r_2(s)-r_o}} \frac{dr_2(s)}{ds} \right\} - & \frac{[(\Delta\varphi_I)_{r_o=r_1(s)}]}{[r-r_1(s)]\sqrt{r_2(s)-r_1(s)}} \frac{dr_1(s)}{ds} \end{aligned} \quad (C2b)$$

However, $[(\Delta\varphi_I)_{r_o=r_1(s)}] = 0$ and, by integration by parts,

$$\begin{aligned} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)[r_2(s)-r_o]^{3/2}} = & -2 \int_{r_1(s)}^{r_2(s)} \frac{\left[\frac{\partial \Delta\varphi_I}{\partial r_o} + \frac{\Delta\varphi_I}{(r-r_o)} \right] dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} + \\ & \lim_{r \rightarrow r_1(s)} \left\{ \frac{2\Delta\varphi_I}{(r-r_o)\sqrt{r_2(s)-r_o}} \right\}_{r_1(s)} \end{aligned}$$

so that equation (C2b) can be written as

$$\begin{aligned} \frac{\partial}{\partial s} \int_{r_1(s)}^{r_2(s)} \frac{\Delta\varphi_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} = & \int_{r_1(s)}^{r_2(s)} \frac{\frac{\partial \Delta\varphi_I}{\partial s} dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} + \\ \frac{dr_2(s)}{ds} \int_{r_1(s)}^{r_2(s)} \frac{\left[\frac{\partial \Delta\varphi_I}{\partial r_o} + \frac{\Delta\varphi_I}{(r-r_o)} \right] dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} \end{aligned} \quad (C3)$$

Equations (C2a) and (C3) are substituted into equation (C1) and the integrals containing $\Delta\varphi_I$ are integrated by parts, reducing equation (C1) to

$$\begin{aligned} \Delta u_{II} = & \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta u_I dr_o}{(r-r_o)\sqrt{r_2(s)-r_o}} - \\ & \frac{1 - \frac{dr_2(s)}{ds}}{2\beta\pi\sqrt{r-r_2(s)}} \int_{r_1(s)}^{r_2(s)} \frac{(\beta\Delta u_I - \Delta v_I) dr_o}{\sqrt{r_2(s)-r_o}} \end{aligned} \quad (17a)$$

APPENDIX D

ALTERNATIVE DERIVATION OF SOLUTION SATISFYING KUTTA CONDITION AT $r=r_2(s)$

The integral equation formulation in terms of $\Delta\varphi$ (equation (14)) resulted in a solution that was continuous in $\Delta\varphi$ (equation (15)) but discontinuous, in general, in the derivative $\frac{\partial \Delta\varphi}{\partial x} = \Delta u$ (equation (17a)) at $r=r_2(s)$. In order to obtain

a solution continuous in Δu , an integral equation may be formulated that is similar to equation (14) but in terms of Δu rather than $\Delta\varphi$. The inversion shown in appendix B should result in a solution that is continuous in Δu but discontinuous in the derivatives of Δu at $r=r_2(s)$.

Consider equation (10) for the w distribution in the $z=0$ plane. This equation will be differentiated with respect to x using a technique introduced in reference 15 (equations (1) to (3) therein). The expression for w at any point (r,s) is, from equation (10),

$$w = -\frac{M}{8\pi} \iint \frac{\Delta\varphi dr_o ds_o}{(s-s_o)^{3/2}(r-r_o)^{3/2}} \quad (D1)$$

where τ is the area abc in figure 9 (a). The wing is moved upstream a distance dx (fig. 9 (b)), keeping the coordinate system fixed in space. The expression for the upwash at (r,s) now becomes

$$w + \frac{\partial w}{\partial x} dx = -\frac{M}{8\pi} \left[\int \int_{\tau} \left(\Delta\varphi + \frac{\partial \Delta\varphi}{\partial x_0} dx \right) \frac{dr_0 ds_0}{(s-s_0)^{3/2}(r-r_0)^{3/2}} + \int \int_{\Delta\tau} \frac{\Delta\varphi dr_0 ds_0}{(s-s_0)^{3/2}(r-r_0)^{3/2}} \right] \quad (D2)$$

The second term on the right side of equation (D2) is zero because $\Delta\varphi=0$ along the leading edge. Subtraction of equation (D1) from equation (D2) then yields

$$\frac{\partial w}{\partial x} = -\frac{M}{8\pi} \int \int_{\tau} \frac{\Delta u dr_0 ds_0}{(s-s_0)^{3/2}(r-r_0)^{3/2}} \quad (D3)$$

For points in region II of a cancellation wing, $\frac{\partial w}{\partial x}=0$. Thus, for the wing of figure 3 (a),

$$0 = \int_0^s \frac{ds_0}{(s-s_0)^{3/2}} \int_{r_1(s_0)}^r \frac{\Delta u dr_0}{(r-r_0)^{3/2}} \quad (D4)$$

This equation is the same as equation (11a) except that Δu replaces $\Delta\varphi$. The inversion by Abel's integral equation for Δu_{II} in terms of Δu_I then gives (from equation (15))

$$\Delta u_{II} = \frac{\sqrt{r-r_2(s)}}{\pi} \int_{r_1(s)}^{r_2(s)} \frac{\Delta u_I dr_0}{(r-r_0)\sqrt{r_2(s)-r_0}} \quad (24a)$$

Inasmuch as the integral equations of reference 5 are formulated in terms of Δu and are inverted by means of Abel's integral equation, only solutions satisfying the Kutta condition will result therein.

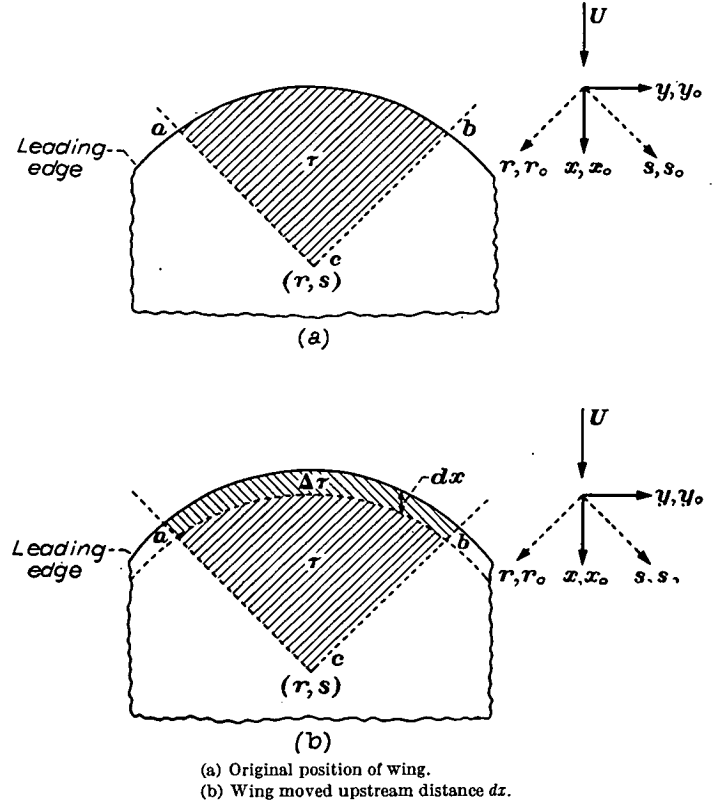


FIGURE 9.—Areas of integration relating to equations (D1) and (D2).

REFERENCES

1. Lagerstrom, P. A.: Linearized Supersonic Theory of Conical Wings—(Corrected Copy). NACA TN 1685, 1950.
2. Coleman, T. F.: Supersonic Lift Solutions Obtained by Extending the Simple Linearized Conical Flow Theory. Rep. No. CM-440, North American Aviation, Inc., Feb. 20, 1948. (Bumblebee Proj. Contract NOrd 9784.)
3. Cohen, Doris: The Theoretical Lift of Flat Swept-Back Wings at Supersonic Speeds. NACA TN 1555, 1948.
4. Cohen, Doris: Theoretical Loading at Supersonic Speeds of Flat Swept-Back Wings with Interacting Trailing and Leading Edges. NACA TN 1991, 1949.
5. Goodman, Theodore R.: The Lift Distribution on Conical and Nonconical Flow Regions of Thin Finite Wings in a Supersonic Stream. Jour. Aero. Sci., vol. 16, no. 6, June 1949, pp. 365-374.
6. Ribner, Herbert S.: Some Conical and Quasi-Conical Flows in Linearized Supersonic-Wing Theory. NACA TN 2147, 1950.
7. Ribner, Herbert S.: On the Effect of Subsonic Trailing Edges on Damping in Roll and Pitch of Thin Sweptback Wings in a Supersonic Stream. NACA TN 2146, 1950.
8. Robinson, A.: On Source and Vortex Distributions in the Linearized Theory of Steady Supersonic Flow. Rep. No. 9, College Aero. (Cranfield), Oct. 1947.
9. Hadamard, Jacques: Lectures on Cauchy's Problem in Linear Partial Differential Equations. Oxford Univ. Press (London), 1923, pp. 133-135.
10. Heaslet, Max. A., and Lomax, Harvard: The Use of Source-Sink and Doublet Distributions Extended to the Solution of Boundary Value Problems in Supersonic Flow. NACA Rep. 900, 1948. (Formerly NACA TN 1515.)
11. Evvard, John C.: Use of Source Distributions for Evaluating Theoretical Aerodynamics of Thin Finite Wings at Supersonic Speeds. NACA Rep. 951, 1950.
12. Mirels, Harold, and Haefeli, Rudolph C.: Line-Vortex Theory for Calculation of Supersonic Downwash. NACA Rep. 983, 1950. (Formerly NACA TN 1925.)
13. Lomax, Harvard, Sluder, Loma, and Heaslet, Max. A.: The Calculation of Downwash behind Supersonic Wings with an Application to Triangular Plan Forms. NACA Rep. 957, 1950. (Formerly NACA TNs 1620 and 1803.)
14. Whittaker, E. T., and Watson, G. N.: Modern Analysis. The MacMillan Co. (New York), 1943, p. 229.
15. Evvard, John C.: Theoretical Distribution of Lift on Thin Wings at Supersonic Speeds (An Extension). NACA TN 1585, 1948.

REPORT 1005

ANALYTICAL DETERMINATION OF COUPLED BENDING-TORSION VIBRATIONS OF CANTILEVER BEAMS BY MEANS OF STATION FUNCTIONS¹

By ALEXANDER MENDELSON and SELWYN GENDLER

SUMMARY

A method based on the concept of Station Functions is presented for calculating the modes and the frequencies of non-uniform cantilever beams vibrating in torsion, bending, and coupled bending-torsion motion. The method combines some of the advantages of the Rayleigh-Ritz and Stodola methods, in that a continuous loading function for the beam is used, with the advantages of the influence-coefficient method, in that the continuous loading function is obtained in terms of the displacements at a finite number of stations along the beam.

The Station Functions were derived for a number of stations ranging from one to eight. The deflections were obtained in terms of the physical properties of the beam and Station Numbers, which are general in nature and which have been tabulated for easy reference. Examples were worked out in detail; comparisons were made with exact theoretical results. For a uniform cantilever beam with n stations, the first n modes and frequencies were in good agreement with the theoretically exact values. The effect of coupling between bending and torsion was shown to reduce the first natural frequency to a value below that which it would have if there were no coupling.

INTRODUCTION

The failure of turbine and compressor blades due to vibrations has led to an increased interest in the study of the vibrations of these blades and in the determination of the natural modes and frequencies. In such theoretical studies, it is usually assumed that the compressor or turbine blade acts as a cantilever beam. The calculation of the uncoupled modes of arbitrarily shaped cantilever beams has been extensively investigated (references 1 to 4), but little work has as yet been done on calculating the coupled modes of such beams. If the geometry of the beam is such that coupling exists, the coupled modes are the actual vibrational modes that must be calculated.

Four general methods are currently in use for calculating uncoupled modes and frequencies of nonuniform beams. These methods are the Rayleigh-Ritz or energy method (reference 1), the Stodola method (references 5 and 6), the influence-coefficient method (references 4 and 7), and the integral-equation method (references 8 and 9). For each of these methods, computational work can usually be carried out in several ways. For example, by the use of influence coefficients the modes and frequencies can be determined by

Mykelstad's iteration procedure (reference 7) or by matrix methods (reference 4).

Any one of these methods can be extended to the calculation of coupled bending-torsion modes. The Rayleigh-Ritz method usually requires that the uncoupled modes be determined before the coupled modes can be computed. In applying either the Rayleigh-Ritz or the Stodola method, great difficulty is encountered in accurately determining the higher modes, because the lower modes must first be "swept out" by the use of exact orthogonality conditions (reference 10); the process will otherwise always converge back to the lowest mode. The same difficulties are encountered in the integral-equation method.

The influence-coefficient method reduces the problem to one having a finite number of degrees of freedom. The beam is divided into n intervals and a concentrated loading is assumed at the center of gravity of each interval. The solution of the resultant determinantal equation gives the first n modes. The accuracy of the higher modes is, however, very poor; only the first third of the modes and the first half of the frequencies are obtained within the usual engineering accuracy. Carrying along so many useless modes greatly increases the labor involved.

A straightforward accurate method for determining the coupled bending-torsion modes and the frequencies of non-uniform cantilever beams, together with applications of this method, was developed at the NACA Lewis laboratory during 1949 and is presented herein. This method is based on the use of Station Functions as first discussed in reference 11. Incorporated in the method are the advantages of the continuous-function deflections of the Rayleigh-Ritz and Stodola methods together with the advantages of the finite number of degrees of freedom of the influence-coefficient method. When the method is applied to a uniform beam, the first n roots of the resultant determinantal equation are amply accurate for engineering purposes.

The final determinantal equation is solved herein by matrix-iteration methods (reference 4). Any other convenient method may, however, be used and no knowledge of matrix algebra is needed to carry out the calculations by the matrix method. The work can be done by an inexperienced computer, as the only operations necessary for determining each mode are cumulative multiplication and division. In addition, for the case in which the coupling coefficient remains constant along the beam, a simple quadratic

¹ Supersedes NACA TN 2185, "Analytical Determination of Coupled Bending-Torsion Vibrations of Cantilever Beams by Means of Station Functions" by Alexander Mendelson and Selwyn Gendler, 1950.

formula and a series of curves are presented for determining the first coupled mode in terms of the uncoupled modes. Examples are developed in detail and comparisons with exact theoretical results are included.

THEORY

In the usual influence-coefficient methods for solving dynamical problems, a continuous body having an infinite number of degrees of freedom is replaced by a body having a finite number of degrees of freedom. Two principal assumptions are then made that introduce inaccuracies into the solutions, particularly in the higher modes: (1) The resultant of the inertia loads of all the infinitesimal masses in a finite interval passes through the center of gravity of that interval; and (2) a concentrated load that is the resultant of a distributed load produces the same deflection as the distributed load. An attempt has been made to reduce the error due to the second of these assumptions by the use of weighting matrices (reference 12). Although the accuracy is thereby increased, the effect of the first assumption is still great enough to introduce serious errors (reference 11).

In order to eliminate these assumptions, Rauscher (reference 11) introduced the concept of Station Functions. Instead of assuming the inertia loads to be concentrated at the centers of gravity of the intervals, the inertia loads and, consequently, the deflections are assumed to be continuous functions along the beam. The values of these continuous deflection functions at the reference stations must equal the deflections of the reference stations. The loading on the beam is therefore a continuous function of the deflections of the reference stations. Inasmuch as the deflections of the reference stations can be computed from the loading on the beam, which in turn is available from the deflections, the deflections are therefore obtained as functions of themselves. This procedure gives n homogeneous equations in the n deflections of the reference stations. The resultant determinantal equation has n roots for the frequency; it will be shown that for a uniform beam all these roots are sufficiently accurate for engineering purposes if the deflection functions are properly chosen. (For coupled bending-torsion vibrations, $2n$ homogeneous equations and $2n$ roots are obtained for n stations.)

The deflection functions used must satisfy the boundary conditions of the problem and also the condition that, at any reference station, the value of the function must equal the deflection of the reference station. Although it is always possible to find directly a single function that will satisfy these conditions, it is more convenient to obtain different component functions at each station and to add all these component functions together to give the complete deflection function. Rauscher (reference 11) calls these component deflection functions Station Functions. For example, the complete torsional deflection function for the beam will have the following form:

$$\theta(z) = \sum_{j=1}^n f_j(z) \theta_j$$

where

z dimensionless distance along beam

$\theta(z)$ torsional deflection at distance z from root

θ_j torsional deflection at j^{th} station

$f_j(z)$ Station Function in torsion associated with j^{th} station
(All symbols are defined in appendix A.)

Each Station Function must satisfy the boundary conditions of the problem and the following additional conditions: (1) At the reference station with which it is associated, the Station Function equals the deflection of that reference station; and (2) at all other reference stations, the Station Function equals zero. The sum of all these Station Functions will then give the complete deflection function for the beam. The Station Functions and corresponding loading functions are derived in appendix B for torsional vibrations,

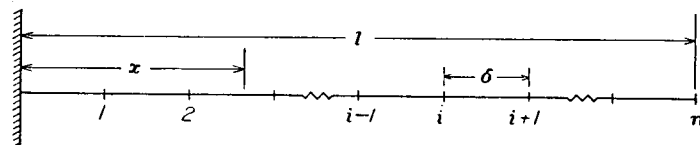


FIGURE 1.—Cantilever beam with n stations.

bending vibrations, and coupled bending-torsion vibrations of an arbitrary cantilever beam.

Torsional vibrations.—It is shown in appendix B that the torsional deflections of the reference stations for a beam divided into n intervals of length δ , as shown in figure 1, are given by the following system of equations:

$$\theta_i = \omega^2 \delta^2 \frac{I_0}{C_0} \sum_{j=1}^n \alpha_{ij} \theta_j \quad (1)$$

where

$$\alpha_{ij} = \sum_{k=1}^i \frac{1}{C_k} \left[I_k N_{jk} - (k-1) I_k M_{jk} + \sum_{r=k+1}^n I_r M_{jr} \right] \quad (2)$$

i and $j = 1, 2, \dots, n$

ω frequency of vibration

δ length of interval

I_0 mass moment of inertia per unit length about elastic axis at root section

I_k ratio of average mass moment of inertia per unit length of k^{th} interval to mass moment of inertia per unit length at root section

C_0 torsional stiffness of root section

C_k ratio of average torsional stiffness of k^{th} interval to torsional stiffness at root section

The Station Numbers N_{jk} and M_{jk} are functions only of the integers k, j , and n and are defined as

$$\left. \begin{aligned} N_{jk} &\equiv \int_{k-1}^k z f_j(z) dz \\ M_{jk} &\equiv \int_{k-1}^k f_j(z) dz \end{aligned} \right\} \quad (3)$$

where $f_j(z)$ represents the Station Functions derived in appendix B and is given by

$$f_j(z) = a_{1j} z + a_{2j} z^2 + \dots + a_{(n+1)j} z^{(n+1)} \quad (4)$$

The coefficients a_{ij} are determined in appendix B by satisfying the conditions on the Station Functions. The integrals in equations (3) are thus seen to be integrals of

simple polynomials and the limits of integration are integers. The Station Numbers N_{jk} and M_{jk} are therefore rational numbers, functions only of the integers n , k , and j . These numbers have been evaluated and are listed in tables I to VIII.

If the physical properties of the beam under consideration are known for each of the n intervals, C_k and I_k will be known. The Station Numbers N_{jk} and M_{jk} can be obtained from tables I to VIII. From equation (2), α_{ij} can then be easily calculated.

Equation (1) actually represents n homogeneous equations in the n unknown deflections θ_i . With $\frac{1}{\omega^2} \frac{C_0}{I_0 \delta^2} \equiv \lambda$, these equations can be written as follows:

$$\left. \begin{aligned} (\alpha_{11}-\lambda)\theta_1 + \alpha_{12}\theta_2 + \alpha_{13}\theta_3 + \dots + \alpha_{1n}\theta_n &= 0 \\ \alpha_{21}\theta_1 + (\alpha_{22}-\lambda)\theta_2 + \alpha_{23}\theta_3 + \dots + \alpha_{2n}\theta_n &= 0 \\ \alpha_{31}\theta_1 + \alpha_{32}\theta_2 + (\alpha_{33}-\lambda)\theta_3 + \dots + \alpha_{3n}\theta_n &= 0 \\ \dots & \\ \alpha_{n1}\theta_1 + \alpha_{n2}\theta_2 + \alpha_{n3}\theta_3 + \dots + (\alpha_{nn}-\lambda)\theta_n &= 0 \end{aligned} \right\} \cdot \quad (5)$$

For a nontrivial solution, the determinant of the coefficients must vanish and the characteristic equation becomes

$$\begin{vmatrix} \alpha_{11} - \lambda & \alpha_{12} & \alpha_{13} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} - \lambda & \alpha_{23} & \dots & \alpha_{2n} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} - \lambda & \dots & \alpha_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \alpha_{n2} & \alpha_{n3} & \dots & \alpha_{nn} - \lambda \end{vmatrix} = 0 \quad (6)$$

$$\text{or} \quad |\lambda I - [\alpha_{ij}]| = 0 \quad (6a)$$

where I is the identity matrix, and $[\alpha_{ij}]$ is the dynamical matrix.

Equation (6) can be solved for the n values of λ by any method available. The method used herein was to obtain the values of λ as the latent roots of the matrix $[\alpha_{ij}]$, which is actually the dynamical matrix for the problem. The mode shapes are obtained at the same time.

Bending vibrations.—The bending deflections for the beam shown in figure 1 are given by the following system of equations (appendix B):

$$y_i = \omega^2 \delta^4 \frac{m_0}{B_0} \sum_{j=1}^n \beta_{ij} y_j \quad (7)$$

where

$$\beta_{ij} \equiv \sum_{k=1}^i \frac{1}{B_k} \left(m_k (iP'_{jk} - Q'_{jk}) + \sum_{r=k+1}^n m_r \left\{ \left(i - k + \frac{1}{2} \right) N'_{jr} + \left[\frac{k^3 - (k-1)^3}{3} - \frac{(2k-1)i}{2} \right] M'_{jr} \right\} \right) \quad (8)$$

$i \text{ and } j = 1, 2, \dots, n$

m_0 mass per unit length of beam at root section

m_k ratio of average mass per unit length of k^{th} interval to mass per unit length at root section

B_0 bending stiffness at root section

B_k ratio of average bending stiffness of k^{th} interval to bending stiffness at root section

The Station Numbers M'_{jk} , N'_{jk} , P'_{jk} , and Q'_{jk} are functions only of the integers k , j , and n and are defined by

$$\left. \begin{aligned} P'_{jk} &\equiv \int_{k-1}^k \left[\frac{z^2}{2} - (k-1)z + \frac{1}{2} (k-1)^2 \right] g_j(z) dz \\ Q'_{jk} &\equiv \int_{k-1}^k \left[\frac{z^3}{6} - \frac{1}{2} (k-1)^2 z + \frac{1}{3} (k-1)^3 \right] g_j(z) dz \\ M'_{jk} &\equiv \int_{k-1}^k g_j(z) dz \\ N'_{jk} &\equiv \int_{k-1}^k z g_j(z) dz \end{aligned} \right\} \quad (9)$$

The Station Functions $g_j(z)$ are derived in appendix B and are given by

$$g_j(z) = b_{2j}z^2 + b_{3j}z^3 + b_{4j}z^4 + \dots + b_{(n+3)j}z^{(n+3)} \quad (10)$$

The integrals in equations (9) are thus seen to be integrals of simple polynomials. The Station Numbers M'_{jk} , N'_{jk} , P'_{jk} , and Q'_{jk} are rational numbers, functions only of the integers j , k , and n . These numbers have been evaluated and are listed in tables I to VIII.

If the physical properties of the beam are known for each of the n intervals, m_k and B_k will be known. The Station Numbers M'_{jk} , N'_{jk} , P'_{jk} , and Q'_{jk} are obtained from tables I to VIII; β_{ij} can then easily be calculated by equation (8).

The determinantal equation is:

$$\begin{vmatrix} \beta_{11}-\lambda & \beta_{12} & \beta_{13} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22}-\lambda & \beta_{23} & \dots & \beta_{2n} \\ \beta_{31} & \beta_{32} & \beta_{33}-\lambda & \dots & \beta_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ \beta_{n1} & \beta_{n2} & \beta_{n3} & \dots & \beta_{nn}-\lambda \end{vmatrix} = 0 \quad (11)$$

or

$$|\lambda I - [\beta_{ii}]| = 0 \quad (11a)$$

where

$$\lambda \equiv \frac{B_0}{m_0 \delta^4} \frac{1}{\omega^2}$$

The dynamical matrix is $[\beta_{ij}]$.

Coupled bending-torsion vibration.—The torsional and bending deflections due to coupled bending-torsion vibrations of a cantilever beam are given by (appendix B):

$$\left. \begin{aligned} \theta_i &= \omega^2 \frac{m_0}{B_0} \delta^4 \sum_{j=1}^n \left(\Gamma \alpha_{ij} \theta_j + \epsilon \Gamma \gamma_{ij} \frac{y_j}{r_0} \right) \\ \frac{y_i}{r_0} &= \omega^2 \frac{m_0}{B_0} \delta^4 \sum_{j=1}^n \left(\delta_{ij} \theta_j + \beta_{ij} \frac{y_j}{r_0} \right) \end{aligned} \right\} \quad (12)$$

where

$$\epsilon \equiv \frac{r_0^2}{r_{g0}^2}$$

$$\Gamma \equiv \frac{1}{\delta^2} \frac{I_0}{C_0} \frac{B_0}{m_0}$$

r_0 absolute magnitude of projection of distance from elastic axis to center of gravity on perpendicular to bending direction for root section

r_{z0} radius of gyration about elastic axis at root section

The quantities α_{ij} and β_{ij} are defined by equations (2) and (8). The quantities γ_{ij} and δ_{ij} are given by

$$\left. \begin{aligned} \gamma_{ij} &= \sum_{k=1}^i \frac{1}{C_k} \left[S_k N'_{jk} - (k-1) S_k M'_{jk} + \sum_{r=k+1}^n S_r M'_{jr} \right] \\ \delta_{ij} &= \sum_{k=1}^i \frac{1}{B_k} \left(S_k (i P_{jk} - Q_{jk}) + \sum_{r=k+1}^n S_r \left\{ \left(i - k + \frac{1}{2} \right) N_{jr} + \left[\frac{k^3 - (k-1)^3}{3} - \frac{(2k-1)i}{2} \right] M_{jr} \right\} \right) \end{aligned} \right\} \quad (13)$$

where

$$P_{jk} = \int_{k-1}^k \left[\frac{z^2}{2} - (k-1)z + \frac{1}{2} (k-1)^2 \right] f_j(z) dz$$

$$Q_{jk} = \int_{k-1}^k \left[\frac{z^3}{6} - \frac{1}{2} (k-1)^2 z + \frac{1}{3} (k-1)^3 \right] f_j(z) dz$$

and S_k is the ratio of the average static mass unbalance of the k^{th} interval to the static mass unbalance at the root section.

The Station Numbers P_{jk} and Q_{jk} are listed in tables I to VIII with the other Station Numbers. The determinantal equation becomes

$$\begin{vmatrix} \Gamma\alpha_{11} - \lambda & \Gamma\alpha_{12} & \dots & \Gamma\alpha_{1n} & \epsilon\Gamma\gamma_{11} & \epsilon\Gamma\gamma_{12} & \dots & \epsilon\Gamma\gamma_{1n} \\ \Gamma\alpha_{21} & \Gamma\alpha_{22} - \lambda & \dots & \Gamma\alpha_{2n} & \epsilon\Gamma\gamma_{21} & \epsilon\Gamma\gamma_{22} & \dots & \epsilon\Gamma\gamma_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \Gamma\alpha_{n1} & \Gamma\alpha_{n2} & \dots & \Gamma\alpha_{nn} - \lambda & \epsilon\Gamma\gamma_{n1} & \epsilon\Gamma\gamma_{n2} & \dots & \epsilon\Gamma\gamma_{nn} \\ \delta_{11} & \delta_{12} & \dots & \delta_{1n} & \beta_{11} - \lambda & \beta_{12} & \dots & \beta_{1n} \\ \delta_{21} & \delta_{22} & \dots & \delta_{2n} & \beta_{21} & \beta_{22} - \lambda & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \delta_{n1} & \delta_{n2} & \dots & \delta_{nn} & \beta_{n1} & \beta_{n2} & \dots & \beta_{nn} - \lambda \end{vmatrix} = 0 \quad (14)$$

or

$$|\lambda I - [\eta_{ij}]| = 0 \quad (14a)$$

where $[\eta_{ij}]$ is the dynamical matrix and I is the identity matrix.

The first n roots of equation (14) will give the first n coupled frequencies.

APPLICATIONS AND RESULTS

In applying the previously discussed method, it is necessary to determine for a given beam the elements α_{ij} , β_{ij} , γ_{ij} , and δ_{ij} of the dynamical matrices. These quantities will depend on the physical properties of the beam and on the number of stations chosen. If the physical properties of the beam are known, the quantities α_{ij} , β_{ij} , γ_{ij} , and δ_{ij} can be directly calculated from equations (2), (8), and (13). The numbers M_{jk} , N_{jk} , P_{jk} , Q_{jk} , M'_{jk} , N'_{jk} , P'_{jk} , and Q'_{jk} appearing in these equations depend on the number of stations n that are used and can be read directly from tables I to VIII for any given number of stations up to eight. Once these quantities have been calculated, equations (6), (11), or (14) can be solved for the frequencies by any method desired. The matrix-

iteration method used herein is simple and rapid and requires no particular computing skill. As will be indicated, however, the accuracy of equations (6), (11), and (14) is such that relatively few stations need be used, in which case it may be convenient to expand the determinants and to solve the resultant low-order algebraic equation.

In order to illustrate the accuracy, this method was applied to torsional vibrations, bending vibrations, and coupled vibrations of a uniform cantilever beam. The exact theoretical values for torsional vibrations and bending vibrations of uniform cantilevers are well known. The exact theoretical values for the coupled bending-torsion vibration of a uniform beam were calculated (appendix D). A comparison was then made between the values obtained by the method presented and the exact theoretical values. The number of stations used was 1, 2, and 3 ($n=1$, $n=2$, and $n=3$). The comparisons are summarized in table IX.

Torsional vibration.—For the case of a uniform beam, $C_k = I_k = 1$ and equation (2) becomes

$$\alpha_{ij} = \sum_{k=1}^i \left[N_{jk} - (k-1) M_{jk} + \sum_{r=k+1}^n M_{jr} \right] \quad (15)$$

The values of N_{jk} and M_{jk} are given in tables I to VIII. The table to be used depends on the choice of the number of stations.

Let $n=1$;

$$\therefore \alpha_{11} = N_{11}$$

From table I, $N_{11} = 5/12$,

$$\therefore \alpha_{11} = 5/12$$

and

$$\theta_1 = \frac{5}{12} l^2 \frac{I_0}{C_0} \omega^2 \theta_1$$

or

$$\omega^2 = \frac{12}{5} \frac{C_0}{I_0 l^2} = 2.400 \frac{C_0}{I_0 l^2}$$

$$\omega = 1.549 \sqrt{\frac{C_0}{I_0 l^2}}$$

The exact theoretical value for the first torsional frequency is

$$\omega = 1.571 \sqrt{\frac{C_0}{I_0 l^2}}$$

The percentage error is -1.4 when only one station is used.

The mode shape obtained by the method of Station Functions agrees well with the theoretical mode shape, as is shown in figure 2 (a).

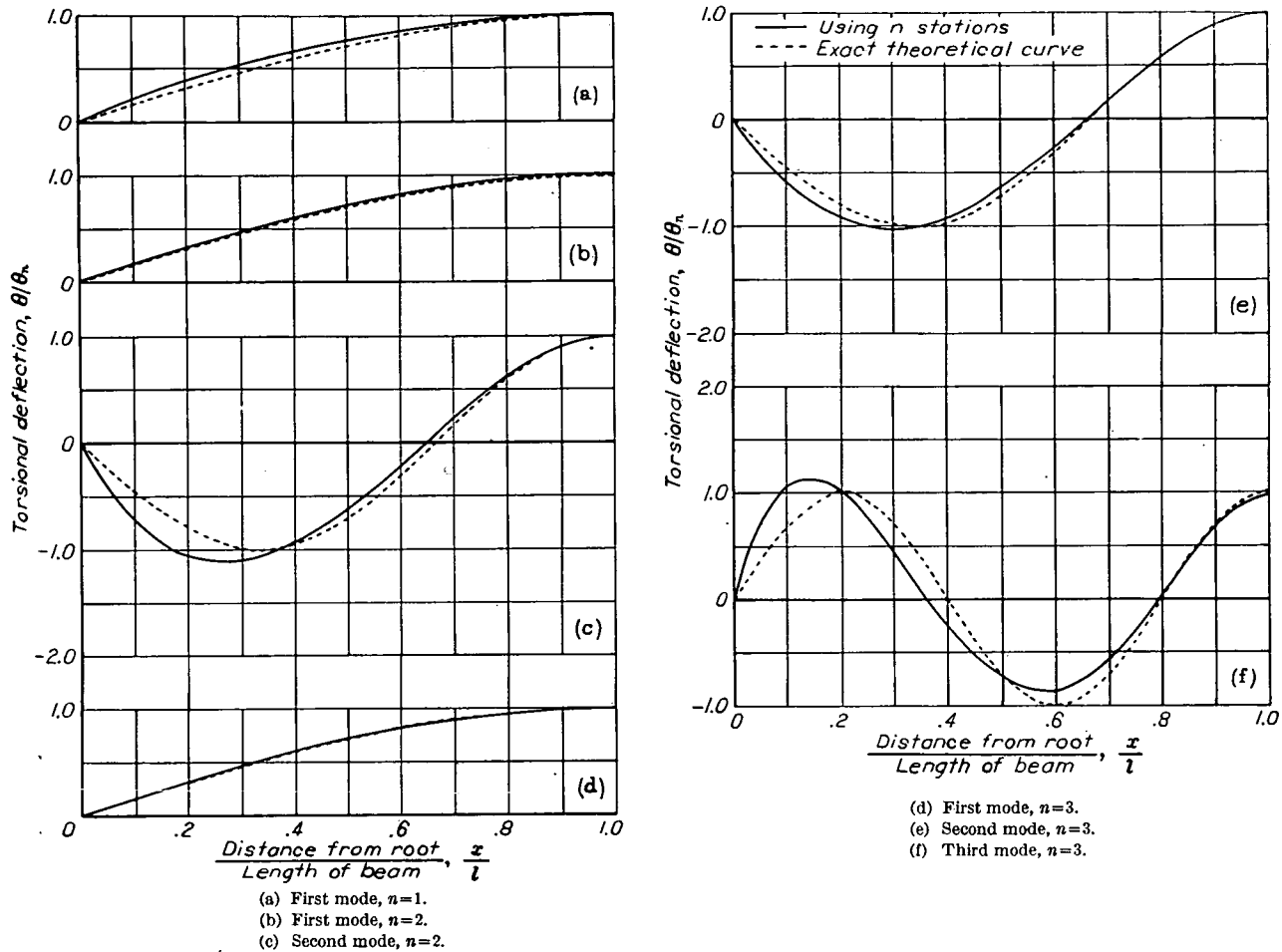
Let $n=2$; then by equation (15) and table II,

$$\alpha_{11} = N_{11} + M_{12} = \frac{8}{15} + \frac{5}{12} = \frac{57}{60}$$

$$\alpha_{12} = N_{21} + M_{22} = -\frac{31}{240} + \frac{29}{48} = \frac{57}{120}$$

$$\alpha_{21} = N_{11} + N_{12} = \frac{8}{15} + \frac{8}{15} = \frac{16}{15}$$

$$\alpha_{22} = N_{21} + N_{22} = -\frac{31}{240} + \frac{239}{240} = \frac{13}{15}$$

FIGURE 2.—Comparison of theoretical mode shapes with mode shapes obtained by taking n stations along the beam for torsional vibrations.

The determinantal equation then becomes

$$\begin{vmatrix} \frac{57}{60} - \lambda & \frac{57}{120} \\ \frac{16}{15} & \frac{13}{15} - \lambda \end{vmatrix} = 0$$

which gives

$$\lambda_1 = 1.6214$$

$$\lambda_2 = 0.1953$$

Therefore

$$\omega_1 = 1.571 \sqrt{\frac{C_0}{I_0 l^2}}$$

$$\omega_2 = 4.526 \sqrt{\frac{C_0}{I_0 l^2}}$$

The exact theoretical values are

$$\omega_1 = 1.571 \sqrt{\frac{C_0}{I_0 l^2}}$$

$$\omega_2 = 4.712 \sqrt{\frac{C_0}{I_0 l^2}}$$

The percentage errors of the first two modes, for only two stations, are found to be 0 and -4.

The mode shapes are shown in figures 2 (b) and 2 (c). Agreement of the first mode with the exact theoretical shape is excellent; the second mode agrees fairly well.

Let $n=3$; then by equation (15) and table III,

$$\begin{aligned} \alpha_{11} &= N_{11} + M_{12} + M_{13} = 0.945833 \\ \alpha_{12} &= N_{21} + M_{22} + M_{23} = 0.958333 \\ \alpha_{13} &= N_{31} + M_{32} + M_{33} = 0.520834 \\ \alpha_{21} &= N_{11} + N_{12} + 2M_{13} = 1.033333 \\ \alpha_{22} &= N_{21} + N_{22} + 2M_{23} = 1.883333 \\ \alpha_{23} &= N_{31} + N_{32} + 2M_{33} = 1.011113 \\ \alpha_{31} &= N_{11} + N_{12} + N_{13} = 1.012500 \\ \alpha_{32} &= N_{21} + N_{22} + N_{23} = 2.025000 \\ \alpha_{33} &= N_{31} + N_{32} + N_{33} = 1.387501 \end{aligned}$$

The determinantal equation is

$$\begin{vmatrix} 0.945833 - \lambda & 0.958333 & 0.520834 \\ 1.033333 & 1.883333 - \lambda & 1.011113 \\ 1.012500 & 2.025000 & 1.387501 - \lambda \end{vmatrix} = 0$$

The solutions are

$$\lambda_1 = 3.6474$$

$$\lambda_2 = 0.4093$$

$$\lambda_3 = 0.1599$$

Therefore

$$\omega_1 = 1.571 \sqrt{\frac{C_0}{I_0 l^2}}$$

$$\omega_2 = 4.689 \sqrt{\frac{C_0}{I_0 l^2}}$$

$$\omega_3 = 7.502 \sqrt{\frac{C_0}{I_0 l^2}}$$

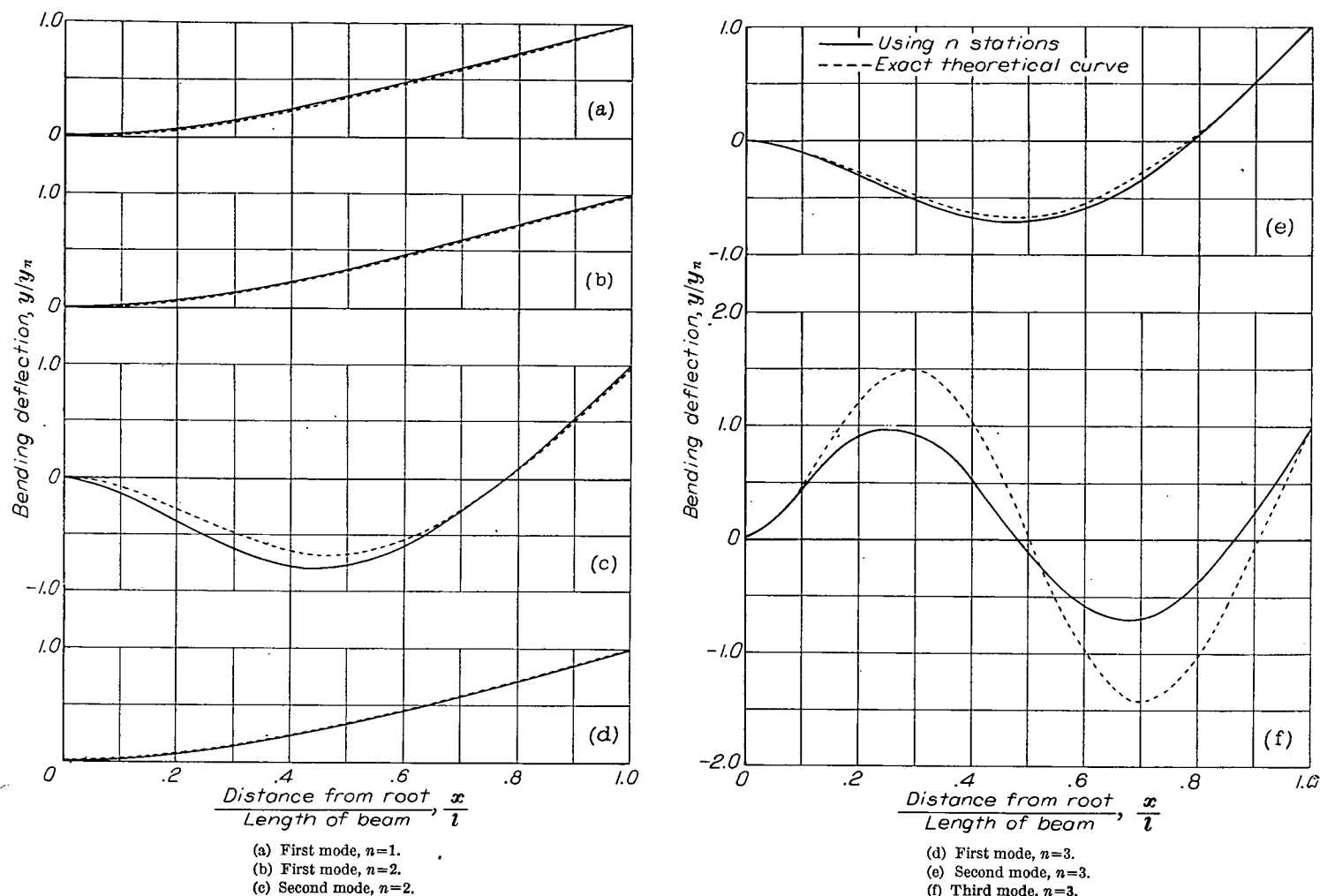


FIGURE 3.—Comparison of theoretical mode shapes with mode shapes obtained by taking n stations along the beam for bending vibrations.

The exact theoretical values are

$$\omega_1 = 1.571 \sqrt{\frac{C_0}{I_0 l^2}}$$

$$\omega_2 = 4.712 \sqrt{\frac{C_0}{I_0 l^2}}$$

$$\omega_3 = 7.854 \sqrt{\frac{C_0}{I_0 l^2}}$$

The percentage errors of the first three modes, calculated by use of three stations, are found to be 0, -0.5, and -4.5, respectively.

The mode shapes are shown in figures 2 (d) to 2 (f). The first two modes agree very well with the theoretical shapes; agreement of the third mode is fair.

This procedure can be carried out as shown for any number of stations desired.

Bending vibrations.—For a uniform beam, $B_k = m_k = 1$ and equation (8) becomes

$$\beta_{ij} = \sum_{k=1}^n \left\{ i P'_{jk} - Q'_{jk} + \sum_{r=k+1}^n \left[\left(i - k + \frac{1}{2} \right) N'_{jr} + \left(\frac{k^3 - (k-1)^3}{3} - \frac{(2k-1)}{2} i \right) M'_{jr} \right] \right\} \quad (16)$$

Let $n=1$;

$$\therefore \beta_{11} = P'_{11} - Q'_{11}$$

and from table I

$$\beta_{11} = \frac{71}{630} - \frac{31}{1008} = \frac{59}{720}$$

Therefore, from equation (7),

$$\omega = 3.493 \sqrt{\frac{B_0}{m_0 l^4}}$$

The exact theoretical value is

$$\omega = 3.516 \sqrt{\frac{B_0}{m_0 l^4}}$$

The percentage error for just one station is found to be -0.65:

The mode shape is shown in figure 3 (a) and is seen to agree very well with the theoretically exact shape.

Let $n=2$; then by equation (16) and table II,

$$\beta_{11} = P'_{11} - Q'_{11} + \frac{1}{2} N'_{12} - \frac{1}{6} M'_{12} = 0.422745$$

$$\beta_{12} = P'_{21} - Q'_{21} + \frac{1}{2} N'_{22} - \frac{1}{6} M'_{22} = 0.295925$$

$$\beta_{21} = 2P'_{11} + 2P'_{12} - Q'_{11} - Q'_{12} + \frac{3}{2} N'_{12} - \frac{2}{3} M'_{12} = 1.145167$$

$$\beta_{22} = 2P'_{21} + 2P'_{22} - Q'_{21} - Q'_{22} + \frac{3}{2} N'_{22} - \frac{2}{3} M'_{22} = 0.905530$$

The characteristic equation is

$$\begin{vmatrix} 0.422745 - \lambda & 0.295925 \\ 1.145167 & 0.905530 - \lambda \end{vmatrix} = 0$$

The roots are

$$\lambda_1 = 1.2943$$

$$\lambda_2 = 0.0339$$

$$\therefore \omega_1 = 3.516 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_2 = 21.71 \sqrt{\frac{B_0}{m_0 l^4}}$$

The exact theoretical values are

$$\omega_1 = 3.516 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_2 = 22.04 \sqrt{\frac{B_0}{m_0 l^4}}$$

The percentage errors for two stations are therefore found to be 0 for the first mode and -1.5 for the second mode. The mode shapes are plotted in figures 3 (b) and 3 (c). The first mode agrees excellently with the theoretically exact shape; the second mode agrees fairly well.

Let $n=3$; then by equation (16) and table III,

$$\beta_{11} = P'_{11} - Q'_{11} + \frac{1}{2} N'_{12} + \frac{1}{2} N'_{13} - \frac{1}{6} M'_{12} - \frac{1}{6} M'_{13} = 0.270604$$

$$\beta_{12} = P'_{21} - Q'_{21} + \frac{1}{2} N'_{22} + \frac{1}{2} N'_{23} - \frac{1}{6} M'_{22} - \frac{1}{6} M'_{23} = 1.009943$$

$$\beta_{13} = P'_{31} - Q'_{31} + \frac{1}{2} N'_{32} + \frac{1}{2} N'_{33} - \frac{1}{6} M'_{32} - \frac{1}{6} M'_{33} = 0.487441$$

$$\beta_{21} = 2P'_{11} + 2P'_{12} - Q'_{11} - Q'_{12} + \frac{3}{2} N'_{12} + 2N'_{13} - \frac{2}{3} M'_{12} - \frac{4}{3} M'_{13} = 0.648170$$

$$\beta_{22} = 2P'_{21} + 2P'_{22} - Q'_{21} - Q'_{22} + \frac{3}{2} N'_{22} + 2N'_{23} - \frac{2}{3} M'_{22} - \frac{4}{3} M'_{23} = 3.266250$$

$$\beta_{23} = 2P'_{31} + 2P'_{32} - Q'_{31} - Q'_{32} + \frac{3}{2} N'_{32} + 2N'_{33} - \frac{2}{3} M'_{32} - \frac{4}{3} M'_{33} = 1.689891$$

$$\beta_{31} = 3P'_{11} + 3P'_{12} + 3P'_{13} - Q'_{11} - Q'_{12} - Q'_{13} + \frac{5}{2} N'_{12} + 4N'_{13} - \frac{7}{6} M'_{12} - \frac{10}{3} M'_{13} = 0.985135$$

$$\beta_{32} = 3P'_{21} + 3P'_{22} + 3P'_{23} - Q'_{21} - Q'_{22} - Q'_{23} + \frac{5}{2} N'_{22} + 4N'_{23} - \frac{7}{6} M'_{22} - \frac{10}{3} M'_{23} = 5.822852$$

$$\beta_{33} = 3P'_{31} + 3P'_{32} + 3P'_{33} - Q'_{31} - Q'_{32} - Q'_{33} + \frac{5}{2} N'_{32} + 4N'_{33} - \frac{7}{6} M'_{32} - \frac{10}{3} M'_{33} = 3.204301$$

The characteristic equation is

$$\begin{vmatrix} 0.270604 - \lambda & 1.009943 & 0.487441 \\ 0.648170 & 3.266250 - \lambda & 1.689891 \\ 0.985135 & 5.822852 & 3.204301 - \lambda \end{vmatrix} = 0$$

The roots are

$$\lambda_1 = 6.5521$$

$$\lambda_2 = 0.1667$$

$$\lambda_3 = 0.0223$$

Therefore

$$\omega_1 = 3.516 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_2 = 22.04 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_3 = 60.20 \sqrt{\frac{B_0}{m_0 l^4}}$$

The exact values are

$$\omega_1 = 3.516 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_2 = 22.04 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_3 = 61.70 \sqrt{\frac{B_0}{m_0 l^4}}$$

The percentage errors for three stations are found to be 0, 0, and -2.4 , respectively. The modes are plotted in figures 3 (d) to 3 (f). The first two modes are seen to agree very well with the theoretical mode shape; agreement of the third mode is fair.

Coupled bending-torsion vibrations.—A uniform beam with the following constants was chosen:

$$\gamma = \frac{\omega_t^2}{\omega_b^2} = 38.56$$

$$\epsilon = 0.8$$

$$\Gamma = \frac{n^2}{193.2}$$

$$\epsilon\Gamma = \frac{n^2}{241.5}$$

The values of α_{ij} and β_{ij} are obtained as previously and are the same as given before for $n=1$, $n=2$, and $n=3$. Also, because $S_k = B_k = C_k = m_k = I_k = 1$, equations (13) become

$$\gamma_{ij} = \sum_{k=1}^i \left[N'_{jk} - (k-1)M'_{jk} + \sum_{r=k+1}^n M'_{jr} \right]$$

$$\delta_{ij} = \sum_{k=1}^i \left\{ i P_{jk} - Q_{jk} + \sum_{r=k+1}^n \left[\left(i - k + \frac{1}{2} \right) N_{jr} + \left(\frac{k^3 - (k-1)^3}{3} - \frac{2k-1}{2} i \right) M_{jr} \right] \right\}$$

Let $n=1$; then the determinant is

$$\begin{vmatrix} \Gamma\alpha_{11}-\lambda & \epsilon\Gamma\gamma_{11} \\ \delta_{11} & \beta_{11}-\lambda \end{vmatrix} = \begin{vmatrix} 0.002156-\lambda & 0.001196 \\ 0.111111 & 0.081944-\lambda \end{vmatrix} = 0$$

The roots are

$$\lambda_1 = 0.0837$$

$$\lambda_2 = 0.0005$$

$$\omega_1 = 3.46 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_2 = 44.7 \sqrt{\frac{B_0}{m_0 l^4}}$$

The procedure for calculating the exact theoretical values is derived in appendix D. The exact values are

$$\omega_1 = 3.49 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_2 = 20.6 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_3 = 49.1 \sqrt{\frac{B_0}{m_0 l^4}}$$

The percentage error for the first mode, calculated by use of one station, is -0.9 .

Let $n=2$; then the determinant is

$$\begin{vmatrix} \Gamma\alpha_{11}-\lambda & \Gamma\alpha_{12} & \epsilon\Gamma\gamma_{11} & \epsilon\Gamma\gamma_{12} \\ \Gamma\alpha_{21} & \Gamma\alpha_{22}-\lambda & \epsilon\Gamma\gamma_{21} & \epsilon\Gamma\gamma_{22} \\ \delta_{11} & \delta_{12} & \beta_{11}-\lambda & \beta_{12} \\ \delta_{21} & \delta_{22} & \beta_{21} & \beta_{22}-\lambda \end{vmatrix} = 0$$

Substituting the known values and solving for λ give for the first two roots

$$\lambda_1 = 1.3197$$

$$\lambda_2 = 0.0412$$

and the frequencies become

$$\omega_1 = 3.48 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_2 = 19.7 \sqrt{\frac{B_0}{m_0 l^4}}$$

The percentage errors for two stations are -0.3 for the first mode and -4.4 for the second mode.

This procedure can be carried out for any number of stations desired. For three stations, the frequencies obtained are

$$\omega_1 = 3.48 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_2 = 20.6 \sqrt{\frac{B_0}{m_0 l^4}}$$

$$\omega_3 = 48.2 \sqrt{\frac{B_0}{m_0 l^4}}$$

The percentage errors are -0.3 for the first mode, 0 for the second mode, and -1.8 for the third mode.

The results obtained by the method presented are seen to agree very well with the exact theoretical values.

These results are summarized in table IX, where a comparison is made with the results obtained for uncoupled bending and torsional vibrations by use of influence coefficients with weighted matrices (reference 12). The values using weighted matrices were taken from table I of reference 12. It can be seen that for a given number of stations, the results obtained by the method presented herein are considerably better than those obtained by using influence co-

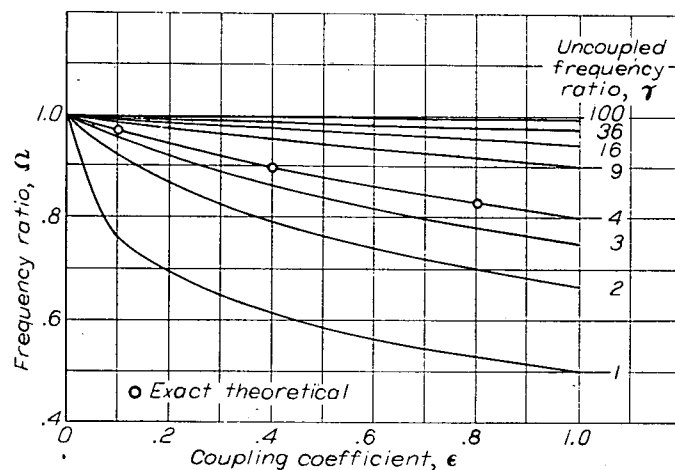


FIGURE 4.—Variation of frequency ratio Ω with coupling coefficient ϵ for several values of uncoupled frequency ratio γ .

efficients with weighted matrices. In general, it is indicated that for a uniform cantilever beam using n stations along the beam, the first $n-1$ frequencies and modes are in excellent agreement with exact theoretical values and even the n^{th} mode is given within the accuracy with which the physical properties of the material are known. For a tapered beam, more stations may be required, depending on the amount of taper. The number of stations required to give satisfactory accuracy is listed in table X. A comparison is made by using weighted influence coefficients; the values are taken from table II of reference 12.

The first vibrational frequency is given approximately by equation (C2) (appendix C) when coupling exists between bending and torsion; it is plotted in figure 4. In order to check these curves, the exact solution was obtained (appen-

dix D) for the ratio $(\omega_t/\omega_b)^2$ equal to 4 and was plotted on the same figure. The values given by equation (C2) are seen to be in excellent agreement with the theoretically exact values.

The effect of the coupling between bending and torsion is to reduce the first natural frequency below that which would exist if there were no coupling. This effect is shown in figure 4, wherein the value of Ω is always less than 1. This decrease in the first natural frequency due to coupling is, however, relatively unimportant in the practical range of $(\omega_t/\omega_b)^2 > 4$ and $\epsilon < 0.75$.

SUMMARY OF RESULTS

A method based on the use of Station Functions is presented for calculating uncoupled and coupled bending-torsion modes and frequencies of arbitrary continuous cantilever beams. The results of calculations made by this method

indicated that by the use of Station Functions derived herein, n modes and frequencies can be obtained with sufficient accuracy by using just n stations along the beam if the beam is uniform. For a tapered beam, more stations may be required, depending on the amount of taper. The amount of computational labor is markedly less than for other methods. The use of Station Numbers tabulated herein further reduces the amount of calculation necessary. The effect of coupling between bending and torsion is shown to reduce the first natural frequency to a value below that which it would have if there were no coupling.

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, *October 18, 1949.*

APPENDIX A

SYMBOLS

The following symbols are used in this report:

a_{ij}	coefficient in equation for Station Function in torsion	$q_b(z)$	bending loading function on beam
B	bending stiffness of beam, function of z	$q_t(z)$	torsional loading function on beam
B_0	bending stiffness at root section of beam	r	absolute magnitude of projection of distance from elastic axis to center of gravity on perpendicular to bending direction
B_k	ratio of average bending stiffness of k^{th} interval to bending stiffness of root section	r_{g0}	radius of gyration about elastic axis at root section
b_{ij}	coefficient in equation for Station Function in bending	r_0	absolute magnitude of projection of distance from elastic axis to center of gravity on perpendicular to bending direction for root section
C	torsional stiffness of beam, function of z	S	static mass unbalance, function of z , mr
C_0	torsional stiffness of root section of beam	S_0	static mass unbalance at root section, $m_0 r_0$
C_k	ratio of average torsional stiffness of k^{th} interval to torsional stiffness at root section	S_k	ratio of average of static mass unbalance at k^{th} section to static mass unbalance at root section
c_1, c_2, c_3	constants defined in appendix B	x	distance from root of beam, except where otherwise defined
$f_j(z)$	Station Function in torsion for j^{th} station (defined in text)	y	bending deflection, function of z
$g_j(z)$	Station Function in bending for j^{th} station (defined in text)	y_i	bending deflection at i^{th} station
I	mass moment of inertia per unit length of beam about elastic axis, function of z , except where otherwise defined	z	dimensionless distance along beam, x/δ
I_0	mass moment of inertia per unit length of beam about elastic axis at root section	$\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}, \eta_{ij}$	elements of dynamical matrix defined in text
I_k	ratio of average mass moment of inertia per unit length of k^{th} interval to mass moment of inertia per unit length at root section	Γ	$\frac{1}{\delta^2} \frac{I_0}{C_0} \frac{B_0}{m_0}$
i, j, k, n	station indices	γ	uncoupled frequency ratio, $(\omega_i/\omega_b)^2$
j, k, r	summation indices	δ	length of interval along beam between two stations
l	length of beam	ϵ	coupling coefficient, $(r_0/r_{g0})^2$
M_{jk}, N_{jk}, P_{jk}	Station Numbers (defined in text); function of indices j, k , and n	θ	torsional deflection, function of z
$Q_{jk}, M'_{jk}, N'_{jk}, P'_{jk}, Q'_{jk}$	of indices j, k , and n	θ_i	torsional deflection at i^{th} station
m	mass per unit length of beam, function of z	λ	root of frequency equation or characteristic root of dynamical matrix
m_0	mass per unit length of beam at root section	Ω	frequency ratio, $(\omega/\omega_b)^2$
m_k	ratio of average mass per unit length of k^{th} interval to mass per unit length at root section	ω	frequency of vibration
n	number of stations along beam	ω_b	frequency of uncoupled fundamental bending mode
		ω_t	frequency of uncoupled fundamental torsional mode
		$\ddot{}$	second derivative of deflection with respect to time

APPENDIX B

STATION FUNCTIONS AND DETERMINANTAL EQUATIONS

TORSIONAL VIBRATIONS

A schematic diagram of a cantilever beam divided into n intervals of length δ is shown in figure 1. The Station Functions for the torsional vibrations of such a beam must satisfy the following conditions:

At

$$z=0 \quad f_i(0)=0 \quad (B1)$$

$$z=n \quad f'_i(n)=0 \quad (B2)$$

$$z=i \quad f_i(i)=1 \quad (B3)$$

$$z=j \quad f_i(j)=0 \quad j \neq i \quad (B4)$$

where $f'(z)$ denotes the derivative with respect to z .

Equations (B1) and (B2) represent the boundary conditions that must be satisfied by a cantilever beam vibrating in torsion; equations (B3) and (B4) represent the further conditions imposed upon the Station Functions. These conditions will be satisfied by a function of the type

$$f_i(z) = a_{1i}z + a_{2i}z^2 + \dots + a_{(n+1)i}z^{(n+1)} \quad (B5)$$

where the coefficients a_{ij} must satisfy the following simultaneous equations obtained from conditions (B2), (B3), and (B4):

$$0 = a_{1i} + 2na_{2i} + 3n^2a_{3i} + \dots + (n+1)n^na_{(n+1)i} \quad (B2a)$$

$$1 = ia_{1i} + i^2a_{2i} + i^3a_{3i} + \dots + i^{(n+1)}a_{(n+1)i} \quad (B3a)$$

$$0 = ja_{1i} + j^2a_{2i} + j^3a_{3i} + \dots + j^{(n+1)}a_{(n+1)i} \quad j \neq i \quad (B4a)$$

The coefficients a_{ij} can be obtained by solving equations (B2a) to (B4a) and the functions $f_i(z)$ determined for each station. Equation (B5), however, can also be written in the following form:

$$f_i(z) = \frac{\prod_{j \neq i} (z-j)z(z-c_1)}{\prod_{j \neq i} (i-j)i(i-c_1)} \quad (B5a)$$

where Π represents the product for all values of j except $j=i$. The function in equation (B5a) obviously satisfies conditions (B1), (B3), and (B4) because it has zeros at all points specified by equation (B4), it equals 1 at the point specified by equation (B3), and it equals zero at the point specified by equation (B1). In order to satisfy condition (B2), the constant c_1 is determined by substitution of equation (B5a) into equation (B2).

$$c_1 = n \text{ for } i \neq n$$

$$c_1 = n \left(1 + \frac{1}{1 + \sum_{j \neq n} \frac{n}{n-j}} \right) \text{ for } i = n$$

Equation (B5) can be obtained from equation (B5a) by carrying out the indicated multiplications. The complete deflection function is then given by

$$\begin{aligned} \theta(z) &= f_1(z)\theta_1 + f_2(z)\theta_2 + \dots + f_n(z)\theta_n \\ &= \sum_{j=1}^n f_j(z)\theta_j \end{aligned} \quad (B6)$$

The continuous loading function $q_i(z)$ can now be written as

$$q_i(x) = I\omega^2\theta(z) = I\omega^2 \sum_{j=1}^n f_j(z)\theta_j \quad (B7)$$

A continuous loading function, which is a function of the deflections at the reference stations, has thus been obtained.

BENDING VIBRATIONS

The Station Functions for the bending vibrations of the beam shown in figure 1 must satisfy the following conditions: at

$$z=0 \quad g_i(0)=0 \quad (B8)$$

$$z=0 \quad g'_i(0)=0 \quad (B9)$$

$$z=n \quad g''_i(n)=0 \quad (B10)$$

$$z=n \quad g'''_i(n)=0 \quad (B11)$$

$$z=i \quad g_i(i)=1 \quad (B12)$$

$$z=j \quad g_i(j)=0 \quad j \neq i \quad (B13)$$

where $g'(z)$, $g''(z)$, and $g'''(z)$ denote the first, second, and third derivatives, respectively, of $g(z)$ with respect to z .

Equations (B8) to (B11) represent the boundary conditions that must be satisfied by a cantilever beam vibrating in bending and equations (B12) and (B13) represent the additional conditions imposed upon the Station Functions.

These conditions will be satisfied by functions of the type

$$g_i(z) = b_{2i}z^2 + b_{3i}z^3 + \dots + b_{(n+3)i}z^{(n+3)} \quad (B14)$$

where the coefficients b_{ij} must satisfy the following equations obtained from conditions (B10) to (B13):

$$0 = 2b_{2i} + 6nb_{3i} + \dots + (n+3)(n+2)n^{(n+1)}b_{(n+3)i} \quad (B10a)$$

$$0 = 6b_{3i} + 24nb_{4i} + \dots + (n+3)(n+2)(n+1)n^nb_{(n+3)i} \quad (B11a)$$

$$1 = i^2b_{2i} + i^3b_{3i} + \dots + i^{(n+3)}b_{(n+3)i} \quad (B12a)$$

$$0 = j^2b_{2i} + j^3b_{3i} + \dots + j^{(n+3)}b_{(n+3)i} \quad j \neq i \quad (B13a)$$

The coefficients can therefore be obtained from equations (B10a) to (B13a) and the functions $g_i(z)$ determined for

each station i . Equation (B14) can, however, be written in the following form:

$$g_i(z) = \frac{\prod_{j \neq i} (z-j) z^2 (z^2 + c_2 z + c_3)}{\prod_{j \neq i} (i-j) i^2 (i^2 + c_2 i + c_3)} \quad (\text{B14a})$$

where Π represents the product for all values of j except $j=i$. The function in equation (B14a) obviously satisfies conditions (B8), (B9), (B12), and (B13), because it has zeros at all points specified by conditions (B8), (B9), and (B13) and equals 1 at the point specified by equation (B12). In order to satisfy conditions (B10) and (B11), the constants c_2 and c_3 are determined by substitution of equation (B14a) into equations (B10) and (B11). The general forms for c_2 and c_3 are, however, complicated and it is easier to obtain the numerical values of these constants for each specific case. Equation (B14) can then be obtained from equation (B14a) by carrying out the indicated multiplications. The complete deflection function is then given by

$$y(z) = \sum_{j=1}^n g_j(z) y_j \quad (\text{B15})$$

The continuous bending loading function $q_b(z)$ can now be written as

$$q_b(z) = m \omega^2 y(z) = m \omega^2 \sum_{j=1}^n g_j(z) y_j \quad (\text{B16})$$

COUPLED BENDING-TORSION VIBRATIONS

The Station Functions for the coupled bending-torsion vibrations are the same as previously given for the bending vibrations and the torsion vibrations. The loading functions, however, are given as follows (reference 7):

$$\begin{aligned} q_i(z) &= I \omega^2 \theta(z) + S \omega^2 y(z) \\ &= \omega^2 \sum_{j=1}^n [I f_j(z) \theta_j + S g_j(z) y_j] \end{aligned} \quad (\text{B17})$$

and

$$\begin{aligned} q_b(z) &= S \omega^2 \theta(z) + m \omega^2 y(z) \\ &= \omega^2 \sum_{j=1}^n [S f_j(z) \theta_j + m g_j(z) y_j] \end{aligned} \quad (\text{B18})$$

DETERMINANTAL EQUATIONS AND DYNAMICAL MATRICES

Once the Station Functions and the corresponding loading functions have been determined, the deflections at the reference stations can be obtained in terms of the loading function. A homogeneous equation in the reference-station deflections for each station is thereby obtained. The determinant of the coefficients of the resultant set of homogeneous equations can be set equal to zero; the determinantal frequency equation is thus derived. The deflections at the reference stations are obtained by the well-known equations for obtaining influence coefficients.

Torsion.—The deflection at the station i due to the continuous loading $q_i(z)$ on the beam is given by

$$\theta_i = \delta^2 \int_0^i q_i(z) \int_0^z \frac{dz_1}{C} dz + \delta^2 \int_i^n q_i(z) \int_0^i \frac{dz_1}{C} dz \quad (\text{B19})$$

If C is assumed to have a constant value for each interval, these integrals may be written as the sum of integrals over each section. Equation (B19) then becomes

$$\theta_i = \frac{\delta^2}{C_0} \sum_{k=1}^i \frac{1}{C_k} \left[\int_{k-1}^k z q_i(z) dz + \int_{k-1}^k (1-k) q_i(z) dz + \int_k^n q_i(z) dz \right] \quad (\text{B20})$$

By substituting the relation

$$q_i(z) = \omega^2 I \sum_{j=1}^n f_j(z) \theta_j$$

and by assuming a constant value for I for each interval and changing the summation order,

$$\theta_i = \omega^2 \delta^2 \frac{I_0}{C_0} \sum_{j=1}^n \left\{ \sum_{k=1}^i \frac{1}{C_k} \left[I_k \int_{k-1}^k z f_j(z) dz - (k-1) I_k \int_{k-1}^k f_j(z) dz + \sum_{r=k+1}^n I_r \int_{r-1}^r f_j(z) dz \right] \right\} \theta_j \quad (\text{B21})$$

Let

$$\left. \begin{aligned} \int_{k-1}^k z f_j(z) dz &\equiv N_{jk} \\ \int_{k-1}^k f_j(z) dz &\equiv M_{jk} \end{aligned} \right\} \quad (\text{B22})$$

Then

$$\theta_i = \omega^2 \frac{I_0}{C_0} \delta^2 \sum_{j=1}^n \alpha_{ij} \theta_j \quad (\text{B23})$$

where

$$\alpha_{ij} \equiv \sum_{k=1}^i \frac{1}{C_k} \left[I_k N_{jk} - (k-1) I_k M_{jk} + \sum_{r=k+1}^n I_r M_{jr} \right] \quad (\text{B24})$$

If $C_k = I_k = 1$ (constant cross section), then

$$\alpha_{ij} = \sum_{k=1}^i \left[N_{jk} - (k-1) M_{jk} + \sum_{r=k+1}^n M_{jr} \right] \quad (\text{B25})$$

Let

$$\lambda \equiv \frac{C_0}{I_0 \omega^2 \delta^2} \quad (\text{B26})$$

Then

$$\lambda \theta_i = \sum_{j=1}^n \alpha_{ij} \theta_j \quad (\text{B23a})$$

and the characteristic equation is

$$|[\alpha_{ij}] - \lambda I| = 0 \quad (\text{B27})$$

where I is the identity matrix.

Bending.—The deflection at the station i due to the continuous loading $q_b(z)$ on the beam will be given by

$$\begin{aligned} y_i &= \delta^4 \int_0^i q_b(z) \int_0^z \frac{(z-z_1)(i-z_1)}{B} dz_1 dz + \\ &\quad \delta^4 \int_i^n q_b(z) \int_0^i \frac{(z-z_1)(i-z_1)}{B} dz_1 dz \end{aligned} \quad (\text{B28})$$

If B is assumed to have a constant value for each interval, these integrals may be written as the sum of integrals over each interval. Equation (B28) then becomes

$$y_i = \frac{\delta^4}{B_0} \sum_{k=1}^i \frac{1}{B_k} \left\{ i \int_{k-1}^k \left[\frac{z^2}{2} - (k-1)z + \frac{1}{2} (k-1)^2 \right] q_b(z) dz - \int_{k-1}^k \left[\frac{z^3}{6} - \frac{1}{2} (k-1)^2 z + \frac{1}{3} (k-1)^3 \right] q_b(z) dz + i \int_k^n \left[z - \frac{1}{2} (2k-1) \right] q_b(z) dz + \int_k^n \left[\frac{1}{2} (2k-1)z - \frac{k^3 - (k-1)^3}{3} \right] q_b(z) dz \right\} \quad (\text{B29})$$

By substituting the relation

$$q_b(z) = \omega^2 m \sum_{j=1}^n g_j(z) y_j \quad (\text{B30})$$

and by assuming a constant average value for m in each interval and changing the summation order,

$$y_i = \frac{\omega^2 m_0 \delta^4}{B_0} \sum_{j=1}^n \beta_{ij} y_j \quad (\text{B31})$$

where

$$\beta_{ij} = \sum_{k=1}^i \frac{1}{B_k} \left\{ m_k (i P'_{jk} - Q'_{jk}) + \sum_{r=k+1}^n m_r \left[\left(i - k + \frac{1}{2} \right) N'_{jr} + \left(\frac{k^3 - (k-1)^3}{3} - \frac{(2k-1)}{2} i \right) M'_{jr} \right] \right\} \quad (\text{B32})$$

$$\left. \begin{aligned} P'_{jk} &\equiv \int_{k-1}^k \left[\frac{z^2}{2} - (k-1)z + \frac{1}{2} (k-1)^2 \right] g_j(z) dz \\ Q'_{jk} &\equiv \int_{k-1}^k \left[\frac{z^3}{6} - \frac{1}{2} (k-1)^2 z + \frac{1}{3} (k-1)^3 \right] g_j(z) dz \\ N'_{jk} &\equiv \int_{k-1}^k z g_j(z) dz \\ M'_{jk} &\equiv \int_{k-1}^k g_j(z) dz \end{aligned} \right\} \quad (\text{B33})$$

For a uniform beam, $m_k = B_k = 1$ and equation (B32) becomes

$$\beta_{ij} = \sum_{k=1}^i \left(i P'_{jk} - Q'_{jk} + \sum_{r=k+1}^n \left\{ \left(i - k + \frac{1}{2} \right) N'_{jr} + \left[\frac{k^3 - (k-1)^3}{3} - \frac{(2k-1)}{2} i \right] M'_{jr} \right\} \right) \quad (\text{B32a})$$

Let

$$\lambda \equiv \frac{B_0}{\omega^2 \delta^4 m_0} \quad (\text{B34})$$

then the characteristic equation becomes

$$|[\beta_{ij}] - \lambda I| = 0 \quad (\text{B35})$$

where I is the identity matrix and β_{ij} is the dynamical matrix. In expanded form, equation (B35) becomes

$$\begin{vmatrix} \beta_{11} - \lambda & \beta_{12} & \dots & \beta_{1n} \\ \beta_{21} & \beta_{22} - \lambda & \dots & \beta_{2n} \\ \dots & \dots & \dots & \dots \\ \beta_{n1} & \beta_{n2} & \dots & \beta_{nn} - \lambda \end{vmatrix} = 0 \quad (\text{B35a})$$

where λ is a latent root of the matrix $[\beta_{ij}]$.

Coupled bending-torsion vibrations.—The deflections at station i are given as before by equations (B19) and (B28). The loading functions q_t and q_b are changed as follows:

$$\left. \begin{aligned} q_t(z) &= \omega^2 [I \theta(z) + S y(z)] \\ q_b(z) &= \omega^2 [S \theta(z) + m y(z)] \end{aligned} \right\} \quad (\text{B36})$$

If these two equations are substituted into equations (B19) and (B28) and the integrations are performed as previously, the following relation is obtained:

$$\left. \begin{aligned} \theta_i &= \frac{\omega^2 m_0 \delta^4}{B_0} \sum_{j=1}^n \left(\Gamma \alpha_{ij} \theta_j + \epsilon \Gamma \gamma_{ij} \frac{y_j}{r_0} \right) \\ \frac{y_i}{r_0} &= \frac{\omega^2 m_0 \delta^4}{B_0} \sum_{j=1}^n \left(\delta_{ij} \theta_j + \beta_{ij} \frac{y_j}{r_0} \right) \end{aligned} \right\} \quad (\text{B37})$$

where α_{ij} and β_{ij} are given in equations (B24) and (B32) and

$$\epsilon \equiv \frac{r_0^2}{r_{g0}^2}$$

$$\Gamma \equiv \frac{1}{\delta^2} \frac{I_0 B_0}{C_0 m_0}$$

$$\gamma_{ij} \equiv \sum_{k=1}^i \frac{1}{C_k} \left[S_k N'_{jk} - (k-1) S_k M'_{jk} + \sum_{r=k+1}^n S_r M'_{jr} \right]$$

$$\delta_{ij} \equiv \sum_{k=1}^i \frac{1}{B_k} \left\{ S_k [i P_{jk} - Q_{jk}] + \sum_{r=k+1}^n S_r \left[\left(i - k + \frac{1}{2} \right) N_{jr} + \left(\frac{k^3 - (k-1)^3}{3} - \frac{2k-1}{2} i \right) M_{jr} \right] \right\} \quad (\text{B38})$$

where

$$P_{jk} \equiv \int_{k-1}^k \left[\frac{z^2}{2} - (k-1)z + \frac{1}{2} (k-1)^2 \right] f_j(z) dz$$

$$Q_{jk} \equiv \int_{k-1}^k \left[\frac{z^3}{6} - \frac{1}{2} (k-1)^2 z + \frac{1}{3} (k-1)^3 \right] f_j(z) dz$$

the determinantal equation therefore is

$$|\lambda I - [\eta_{ij}]| = 0$$

where $[\eta_{ij}]$ is the dynamical matrix, the elements of which are as indicated in equation (B37). The matrix $[\eta_{ij}]$ is seen to be a $2n \times 2n$ matrix.

APPENDIX C

QUADRATIC FORMULA FOR FIRST COUPLED MODE

If only the first vibrational mode is desired, it is possible to obtain this mode approximately by coupling together the fundamental uncoupled bending mode with the fundamental uncoupled torsional mode to obtain a simple quadratic equation for the first coupled frequency. This equation is valid when the coupling coefficient ϵ is constant along the beam. The differential equations obtained by coupling the fundamental uncoupled torsional mode with the fundamental uncoupled bending mode are:

$$\left. \begin{aligned} m\ddot{y} + S\ddot{\theta} + m\omega_b^2 y &= 0 \\ S\ddot{y} + I\ddot{\theta} + I\omega_t^2 \theta &= 0 \end{aligned} \right\} \quad (C1)$$

where

m mass per unit length of beam, function of z
 S static mass unbalance, function of z

I mass moment of inertia about elastic axis, function of z
 ω_b frequency of uncoupled fundamental bending mode
 ω_t frequency of uncoupled fundamental torsional mode
 $\dots$ denotes differentiation twice with respect to time

These equations lead to a quadratic equation in the frequency ratio Ω , whose solution for the lowest frequency, provided ϵ is constant along the beam, is

$$\Omega \equiv \frac{\omega^2}{\omega_b^2} = \frac{1-\gamma}{2(1-\epsilon)} \left[1 - \sqrt{1 - \frac{4\gamma(1-\epsilon)}{(1-\gamma)^2}} \right] \quad (C2)$$

where

Ω frequency ratio, $(\omega/\omega_b)^2$
 γ uncoupled frequency ratio, $(\omega_t/\omega_b)^2$
 ϵ coupling coefficient, $(r/r_g)^2$

This quadratic has been plotted in figure 4 for values of ϵ ranging from 0 to 1 and values of $\gamma = (\omega_t/\omega_b)^2$ from 1 to 100.

APPENDIX D

EXACT SOLUTION FOR COUPLED BENDING-TORSION VIBRATIONS OF UNIFORM CANTILEVER BEAM

The differential equations for the equilibrium of an element of a beam vibrating in coupled bending-torsion vibrations can be put in the following dimensionless form:

$$\left. \begin{aligned} \frac{d^4 Y_1}{dx^4} &= \frac{ml^4}{B} \omega^2 Y_1 + \frac{ml^4}{B} \omega^2 Y_2 \\ \frac{d^2 Y_2}{dx^2} &= -\epsilon \frac{Il^2}{C} \omega^2 Y_1 - \frac{Il^2}{C} \omega^2 Y_2 \end{aligned} \right\} \quad (D1)$$

where

$$Y_1 \equiv y/r$$

$$Y_2 \equiv \theta$$

$$x \equiv \frac{\text{distance from root}}{l}$$

$$\epsilon = (r/r_g)^2$$

Now

$$\omega_b^2 = \frac{c_4 B}{ml^4}$$

$$\omega_t^2 = c_5 \frac{C}{Il^2}$$

where

$$c_4 = 12.36$$

$$c_5 = 2.467$$

Equations (D1) become

$$\left. \begin{aligned} \frac{d^4 Y_1}{dx^4} &= c_4 \Omega (Y_1 + Y_2) \\ \frac{d^2 Y_2}{dx^2} &= -\epsilon \frac{c_5 \Omega}{\gamma} Y_1 - \frac{c_5 \Omega}{\gamma} Y_2 \end{aligned} \right\} \quad (D2)$$

where

$$\Omega \equiv (\omega/\omega_b)^2$$

$$\gamma \equiv (\omega_t/\omega_b)^2$$

Let

$$\frac{dY_1}{dx} = Y_3$$

$$\frac{dY_3}{dx} = Y_4$$

$$\frac{dY_4}{dx} = Y_5$$

$$\frac{dY_2}{dx} = Y_6$$

$$\frac{dY_5}{dx} = c_4 \Omega (Y_1 + Y_2)$$

$$\frac{dY_6}{dx} = -\frac{c_5 \Omega}{\gamma} (\epsilon Y_1 + Y_2)$$

(D3)

Then

Equation (D3) can be written as the single matrix equation

$$\frac{d}{dx} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ c_4 \Omega & c_4 \Omega & 0 & 0 & 0 & 0 \\ -\frac{\epsilon c_5 \Omega}{\gamma} & -\frac{c_5 \Omega}{\gamma} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ Y_5 \\ Y_6 \end{bmatrix} \quad (D4)$$

or

$$\frac{dY}{dx} = AY \quad (D4a)$$

where Y and A are the matrices indicated.

The solution to the matrix equation (D4) is given by

$$Y = e^{Ax} Y_0 \quad (D5)$$

where Y_0 is a column of arbitrary constants.

From the boundary conditions

$$\begin{aligned} \text{at } x=0 \quad Y_1 = Y_2 = Y_3 = 0 \\ x=1 \quad Y_4 = Y_5 = Y_6 = 0 \end{aligned}$$

$$Y_0 = Y(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Y_4(0) \\ Y_5(0) \\ Y_6(0) \end{bmatrix}$$

If then Ω_{ij} is an element of the matrizant e^A , the boundary conditions give

$$\begin{vmatrix} \Omega_{44} & \Omega_{45} & \Omega_{46} \\ \Omega_{54} & \Omega_{55} & \Omega_{56} \\ \Omega_{64} & \Omega_{65} & \Omega_{66} \end{vmatrix} = 0 \quad (D6)$$

Equation (D6) is the frequency equation. It has an infinite number of roots for ω .In order to determine the elements Ω_{ij} , e^A must be evaluated. Use will be made of Sylvester's theorem (reference 13).The λ matrix of A is

$$\begin{bmatrix} -\lambda & 0 & 1 & 0 & 0 & 0 \\ 0 & -\lambda & 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & -\lambda & 1 & 0 \\ c_4\Omega & c_4\Omega & 0 & 0 & -\lambda & 0 \\ -\frac{c_5\Omega}{\gamma}\epsilon & -\frac{c_5\Omega}{\gamma} & 0 & 0 & 0 & -\lambda \end{bmatrix}$$

The characteristic equation $\Delta(\lambda) = 0$ is

$$\lambda^6 + \frac{c_5\Omega}{\gamma}\lambda^4 - c_4\Omega\lambda^2 - (1-\epsilon)c_4c_5\frac{\Omega^2}{\gamma} = 0 \quad (D7)$$

Equation (D7) is a cubic equation in λ^2 . Let the roots be

$$\lambda_1, -\lambda_1, \lambda_2, -\lambda_2, \lambda_3, -\lambda_3$$

Then by the confluent form of Sylvester's theorem,

$$e^A = \sum_{i=1}^r \frac{1}{(\alpha_i - 1)!} \frac{d^{\alpha_i - 1}}{d\lambda^{\alpha_i - 1}} \left[\frac{e^{\lambda F(\lambda)}}{\prod_{k \neq i} (\lambda - \lambda_k)^{\alpha_k}} \right]_{\lambda = \lambda_i} \quad (D8)$$

where $F(\lambda)$ is the adjoint matrix, r is the number of distinct roots, and α_i is the multiplicity of the i^{th} root.

If the roots are all distinct, this relation becomes

$$e^A = \sum_{i=1}^3 \frac{e^{\lambda_i F(\lambda_i)} - e^{-\lambda_i F(-\lambda_i)}}{2\lambda_i \prod_{j \neq i} (\lambda_i - \lambda_j)(\lambda_i + \lambda_j)} \quad (D9)$$

where the adjoint matrix $F(\lambda)$ is given by

$$F(\lambda) = - \begin{bmatrix} \lambda^5 + \frac{c_5\Omega}{\gamma}\lambda^3 & c_4\Omega\lambda & \lambda^4 + \frac{c_5\Omega}{\gamma}\lambda^2 & \lambda^3 + \frac{c_5\Omega}{\gamma}\lambda & \lambda^2 + c_5\frac{\Omega}{\gamma} & c_4\Omega \\ -\epsilon c_5\frac{\Omega}{\gamma}\lambda^3 & \lambda^5 - c_4\Omega\lambda & -\epsilon \frac{c_5\Omega}{\gamma}\lambda^2 & -\epsilon \frac{c_5\Omega}{\gamma}\lambda & -\epsilon \frac{c_5\Omega}{\gamma} & \lambda^4 - c_4\Omega \\ c_4\Omega\lambda^2 + (1-\epsilon)c_4c_5\frac{\Omega^2}{\gamma} & c_4\Omega\lambda^2 & \lambda^5 + \frac{c_5\Omega}{\gamma}\lambda^3 & \lambda^4 + \frac{c_5\Omega}{\gamma}\lambda^2 & \lambda^3 + \frac{c_5\Omega}{\gamma}\lambda & c_4\Omega\lambda \\ c_4\Omega\lambda^3 + (1-\epsilon)c_4c_5\frac{\Omega^2}{\gamma}\lambda & c_4\Omega\lambda^3 & c_4\Omega\lambda^2 + (1-\epsilon)\frac{c_4c_5\Omega^2}{\gamma} & \lambda^5 + \frac{c_5\Omega}{\gamma}\lambda^3 & \lambda^4 + \frac{c_5\Omega}{\gamma}\lambda^2 & c_4\Omega\lambda^2 \\ c_4\Omega\lambda^4 + (1-\epsilon)c_4c_5\frac{\Omega^2}{\gamma}\lambda^2 & c_4\Omega\lambda^4 & c_4\Omega\lambda^3 + (1-\epsilon)\frac{c_4c_5\Omega^2}{\gamma}\lambda & c_4\Omega\lambda^2 + (1-\epsilon)\frac{c_4c_5}{\gamma}\Omega^2 & \lambda^5 + \frac{c_5\Omega}{\gamma}\lambda^3 & c_4\Omega\lambda^3 \\ -\frac{\epsilon c_5\Omega}{\gamma}\lambda^4 & \frac{c_5\Omega}{\gamma}\lambda^4 + (1-\epsilon)\frac{c_4c_5\Omega^2}{\gamma} & -\epsilon \frac{c_5\Omega}{\gamma}\lambda^3 & -\epsilon \frac{c_5\Omega}{\gamma}\lambda^2 & -\epsilon \frac{c_5\Omega}{\gamma}\lambda & \lambda^5 - c_4\Omega\lambda \end{bmatrix} \quad (D10)$$

From equations (D9) and (D10), the elements Ω_{ij} are seen to be given by

$$\left. \begin{aligned} \Omega_{44} &= - \sum_{i=1}^3 \frac{\lambda_i^4 + \frac{c_5 \Omega}{\gamma} \lambda_i^2}{\prod_{j \neq i} (\lambda_i^2 - \lambda_j^2)} \cosh \lambda_i \\ \Omega_{45} &= - \sum_{i=1}^3 \frac{\lambda_i^4 + \frac{c_5 \Omega}{\gamma} \lambda_i^2}{\lambda_i \prod_{j \neq i} (\lambda_i^2 - \lambda_j^2)} \sinh \lambda_i \\ \Omega_{46} &= - \sum_{i=1}^3 \frac{c_4 \Omega \lambda_i^2}{\lambda_i \prod_{j \neq i} (\lambda_i^2 - \lambda_j^2)} \sinh \lambda_i \\ \Omega_{54} &= - \sum_{i=1}^3 \frac{c_4 \Omega \lambda_i^2 + \frac{c_4 c_5 \Omega^2}{\gamma} (1 - \epsilon)}{\lambda_i \prod_{j \neq i} (\lambda_i^2 - \lambda_j^2)} \sinh \lambda_i \\ \Omega_{55} &= \Omega_{44} \\ \Omega_{56} &= - \sum_{i=1}^3 \frac{c_4 \Omega \lambda_i^2}{\prod_{j \neq i} (\lambda_i^2 - \lambda_j^2)} \cosh \lambda_i \\ \Omega_{64} &= - \sum_{i=1}^3 \frac{-\epsilon c_5 \frac{\Omega}{\gamma} \lambda_i}{\prod_{j \neq i} (\lambda_i^2 - \lambda_j^2)} \sinh \lambda_i \\ \Omega_{65} &= - \sum_{i=1}^3 \frac{-\epsilon c_5 \frac{\Omega}{\gamma} \lambda_i}{\lambda_i \prod_{j \neq i} (\lambda_i^2 - \lambda_j^2)} \cosh \lambda_i \\ \Omega_{66} &= - \sum_{i=1}^3 \frac{\lambda_i^4 - c_4 \Omega}{\prod_{j \neq i} (\lambda_i^2 - \lambda_j^2)} \cosh \lambda_i \end{aligned} \right\} \quad (D11)$$

The value of the determinant in equation (D6) must be plotted against the frequency; the value of the frequency for which this determinant becomes zero is thereby obtained. This procedure involves first solving the cubic equation (D7)

for each assumed value of frequency parameter and then calculating the elements of the determinant from equations (D11). The process is evidently long and laborious.

REFERENCES

1. Timoshenko, S.: *Vibration Problems in Engineering*. D. Van Nostrand Co., Inc., 2d ed., 1937.
2. Burgess, C. P.: *The Frequencies of Cantilever Wings in Beam and Torsional Vibrations*. NACA TN 746, 1940.
3. Boukidis, N. A., and Ruggiero, R. J.: An Iterative Method for Determining Dynamic Deflections and Frequencies. *Jour. Aero. Sci.*, vol. 11, no. 4, Oct. 1944, pp. 319-328.
4. Duncan, W. J., and Collar, A. R.: A Method for the Solution of Oscillation Problems by Matrices. *Phil. Mag. and Jour. Sci.*, vol. 17, no. 115, ser. 7, May 1934, pp. 865-909.
5. Houbolt, John C., and Anderson, Roger A.: Calculation of Uncoupled Modes and Frequencies in Bending or Torsion of Non-uniform Beams. NACA TN 1522, 1948.
6. Den Hartog, J. P.: *Mechanical Vibrations*. McGraw-Hill Book Co., Inc., 2d ed., 1940, p. 188.
7. Mykelstad, N. O.: *Vibration Analysis*. McGraw-Hill Book Co., Inc., 1944.
8. Billington, A. E.: *The Vibrations of Stationary and Rotating Cantilevers with Special Reference to Turbine Blades*. Repts. SM. 109 and E. 61, Div. Aero., Council Sci. Ind. Res. (Australia), Jan. 1948.
9. White, Walter T.: An Integral-Equation Approach to Problems of Vibrating Beams. I. *Jour. Franklin Inst.*, vol. 245, no. 1, Jan. 1948, pp. 25-36; II, vol. 245, no. 2, Feb. 1948, pp. 117-133.
10. Fettis, Henry E.: The Calculation of Coupled Modes of Vibration by the Stodola Method. *Jour. Aero. Sci.*, vol. 16, no. 5, May 1949, pp. 259-271.
11. Rauscher, Manfred: Station Functions and Air Density Variations in Flutter Analysis. *Jour. Aero. Sci.*, vol. 16, no. 6, June 1949, pp. 345-354.
12. Bencoter, Stanley U., and Gossard, Myron L.: *Matrix Methods for Calculating Cantilever-Beam Deflections*. NACA TN 1827, 1949.
13. Frazer, R. A., Duncan, W. J., and Collar, A. R.: *Elementary Matrices*. Cambridge Univ. Press (London), 1938, pp. 78-85.

TABLE I—STATION NUMBERS

$n=1$

	$j \backslash k$	1
M	1	$\frac{2}{3}$
N		$\frac{5}{12}$
P		$\frac{3}{20}$
Q		$\frac{7}{180}$
M'		$\frac{2}{5}$
N'		$\frac{13}{45}$
P'		$\frac{71}{630}$
Q'		$\frac{31}{1008}$

TABLE II—STATION NUMBERS

$n=2$

	$j \backslash k$	1	2
M	1	$\frac{11}{12}$	$\frac{5}{12}$
N		$\frac{8}{15}$	$\frac{8}{15}$
P		0.183333	0.025000
Q		.046032	.029365
M'		.536364	.627273
N'		.367100	.851948
P'		.137933	.057955
Q'		.036616	.069733
M	2	$-\frac{13}{48}$	$\frac{29}{48}$
N		$-\frac{31}{240}$	$\frac{239}{240}$
P		-0.037500	0.143750
Q		-.008135	.181448
M'		-.060795	.448674
N'		-.034875	.758685
P'		-.011252	.118462
Q'		-.002614	.150415

TABLE III—STATION NUMBERS

 $n=3$

	$j \backslash k$	1	2	3
M	1	0.950000	0.450000	-0.050000
N		.545533	.587500	-.120833
P		.186310	.032143	-.005357
Q		.046577	.038244	-.011756
M'		.596268	.533205	-.097426
N'		.399646	.708205	-.239994
P'		.148013	.042560	-.012798
Q'		.038884	.050843	-.028318
M	2	-0.525000	0.725000	0.475000
N		-.241667	1.175000	1.091667
P		-.068452	.160714	.031548
Q		-.014583	.202083	.068750
M'		-.149356	.602896	.625418
N'		-.083406	.994860	1.475153
P'		-.026378	.143948	.057937
Q'		-.006034	.181698	.127659
M	3	0.235185	-0.153704	0.568519
N		.106019	-.231944	1.513426
P		.029563	-.023677	.139749
Q		.006222	-.028963	.316408
M'		.040630	-.072928	.445812
N'		.022325	-.111744	1.200133
P'		.006972	-.012081	.118007
Q'		.001579	-.014830	.267865

TABLE V—STATION NUMBERS

 $n=5$

	$j \backslash k$	1	2	3	4	5
M	1	1.097991	0.408755	-0.040898	0.019866	-0.013120
N		.608222	.527493	-.100112	.069159	-.058445
P		.202887	.026908	-.005074	.002776	-.001627
Q		.049943	.031910	-.011210	.008927	-.006836
M'		.649902	.492141	-.070298	.034939	-.024007
N'		.427616	.647530	-.172488	.121519	-.107639
P'		.156411	.036908	-.008903	.004824	-.003375
Q'		.040729	.043977	-.019678	.015509	-.014228
M	2	-0.539550	0.799339	0.493783	-0.089550	0.049339
N		-.373049	1.282044	1.145470	-.310549	.219544
P		-.103119	.169599	.037952	-.011949	.006007
Q		-.021583	.212792	.083245	-.038398	.025243
M'		-.255330	.699256	.528328	-.099723	.058134
N'		-.139170	1.138472	1.219101	-.345551	.260447
P'		-.043239	.157833	.041704	-.013166	.008078
Q'		-.009758	.198610	.091532	-.042299	.034055
M	3	0.762798	-0.313501	0.651687	0.575298	-0.126091
N		.329315	-.465823	1.718204	1.923065	-.559573
P		.089079	-.044602	.150488	.049347	-.014691
Q		.018334	-.054326	.340126	.157843	-.061693
M'		.197103	-.228783	.633812	.573549	-.137821
N'		.105126	-.344915	1.674943	1.916360	-.616234
P'		.032115	-.034985	.148520	.048803	-.018600
Q'		.007151	-.042759	.335798	.156076	-.078385
M	4	-0.548214	0.187897	-0.159325	0.576786	0.562897
N		-.233780	.276637	-.400446	2.109970	2.432887
P		-.062665	.025479	-.024868	.140460	.042295
Q		-.012809	.030950	-.055302	.468397	.177006
M'		-.117990	.117132	-.136452	.557359	.664560
N'		-.062435	.175188	-.344108	2.041363	2.901646
P'		-.018958	.017186	-.021868	.137234	.063512
Q'		-.004201	.020954	-.048664	.447980	.267043
M	5	0.214238	-0.069026	0.050904	-0.080137	0.525349
N		.090928	-.101352	.127410	-.283759	2.458336
P		.024280	-.009225	.007667	-.013645	.134478
Q		.004952	-.011196	.017031	-.044065	.573757
M'		.033722	-.031711	.032307	-.056459	.432107
N'		.017789	-.047311	.081140	-.200039	2.030260
P'		.003389	-.004593	.005002	-.009675	.116059
Q'		.001192	-.005596	.011120	-.031249	.495663

TABLE IV—STATION NUMBERS

 $n=4$

	$j \backslash k$	1	2	3	4
M	1	1.022222	0.429630	-0.051852	0.022222
N		.576455	.557937	-.127249	.076455
P		.194478	.029597	-.006581	.002612
Q		.048240	.035167	-.014547	.008359
M'		.623188	.511882	-.082891	.042276
N'		.413738	.676680	-.203719	.146954
P'		.152256	.039616	-.010651	.005795
Q'		.039818	.047267	-.023551	.018630
M	2	-0.647917	0.747917	0.518750	-0.085417
N		-.292857	1.207143	1.207143	-.292857
P		-.081920	.163021	.041295	-.009598
Q		-.017295	.204828	.090642	-.030688
M'		-.211987	.667412	.544025	-.112648
N'		-.116662	1.091462	1.269193	-.390585
P'		-.036502	.153469	.044508	-.015000
Q'		-.008281	.193310	.097745	-.048203
M	3	0.522222	-0.255556	0.633333	0.522222
N		.229365	-.381746	1.673810	1.729365
P		.062798	-.037401	.148512	.037202
Q		.013040	-.045624	.335791	.118397
M'		.122052	-.166738	.582158	.643846
N'		.065879	-.252823	1.545802	2.164827
P'		.020304	-.026235	.140822	.060554
Q'		.004551	-.032114	.318707	.194016
M	4	-0.221701	0.094850	-0.105961	0.543924
N		-.096544	.140724	-.267841	1.997206
P		-.026267	.013391	-.017322	.136803
Q		-.005428	.016301	-.038574	.446729
M'		-.035456	.042628	-.064723	.438962
N'		-.019023	.064205	-.164169	1.622066
P'		-.005836	.006481	-.010869	.117037
Q'		-.001303	.007917	-.024222	.382752

TABLE VI—STATION NUMBERS

 $n=6$

	$j \backslash k$	1	2	3	4	5	6
M	1	1.172073	0.391101	-0.032371	0.013323	-0.010149	0.008879
N		.638800	.501856	-.078978	.046300	-.045644	.048522
P		.210893	.024885	-.003894	.001823	-.001498	.001143
Q		.051551	.029221	-.008595	.005860	-.006322	.005949
M'		.676394	.474177	-.059129	.026582	-.018685	.015649
N'		.441269	.691067	-.144759	.092370	-.083901	.085903
P'		.160476	.034473	-.007337	.003631	-.002691	.002241
Q'		.041616	.041021	-.016206	.011674	-.011350	.011694
M	2	-1.066598	0.853106	0.468124	-0.070505	0.044513	-0.035782
N		-.466718	1.360105	1.081893	-.244062	.199948	-.195451
P		-.127634	.176358	-.034411	.009203	.006452	-.004561
Q		-.026505	.220969	.075401	-.020565	.027216	-.023740
M'		-.303948	.731991	.503742	-.085145	.049793	-.038661
N'		.164215	1.186681	1.169252	-.294736	.223322	-.212149
P'		-.050692	.162263	.038896	-.011103	.007043	-.005503
Q'		-.011384	.203987	.085309	-.035665	.020701	-.028708
M	3	1.150584	-0.404200	0.693794	0.546418	-0.133366	0.089627
N		.489124	-.597296	1.822457	1.822457	-.597296	.489124
P		.130870	.055956	.156259	.045293	-.018484	.011225
Q		.026719	.068060	.352909	.144808	-.077925	.058410
M'		.267118	.275585	.662165	.553477	-.126314	.083246
N'		.141177	.413819	1.745286	1.846431	-.564890	.456509
P'		.042839	-.041308	.152473	.045980	-.017100	.011713
Q'		.009490	-.050434	.344556	.146998	-.072063	.061097
M	4	-0.930965	0.273028	-0.194854	0.592473	0.624591	-0.171416
N		-.390902	.398997	-.488124	2.163786	2.720209	-.933437
P		-.103635	.036020	-.029594	.142229	.056698	-.020561
Q		.021011	.043690	-.065761	.464052	.238215	-.106938
M'		.1210538	.182630	-.150822	.598464	.604424	-.166643
N'		.110263	.271857	-.454522	2.185427	2.628772	-.912352
P'		-.033225	.026154	-.028221	.143453	.053359	-.022768
Q'		-.007320	.031846	-.062753	.468015	.224100	-.118729
M	5	0.581796	-0.156399	0.092907	-0.111954	0.537351	0.599157
N		.242612	-.228221	.231501	-.394888	2.509279	3.194001
P		.063996	-.020218	.013474	-.018261	.134578	.046991
Q		.012925	-.024496	.029896	-.058920	.574036	.243745
M'		.120308	-.097119	.081568	-.110987	.535339	.685021
N'		.062735	.144101	.204039	-.391731	2.499695	3.678582
P'		.018841	.013674	.012212	-.018226	.133988	.066470
Q'		.004140	-.016634	.027119	-.058815	.571521	.345986
M	6	-0.209220	0.054246	-0.030239	0.031561	-0.064035	0.510543
N		-.086982	.079042	-.075223	.110987	-.291427	2.902746
P		-.022893	.009657	-.004323	.004969	-.011243	.132560
Q		.004616	.008425	-.009587	.016022	-.047575	.698254
M'		-.033141	.025961	-.020586	.024431	-.049677	.425817
N'		-.017248	.038471	-.051417	.085977	-.225999	2.427820
P'		-.005173	.003631	-.003042	.003877	-.008676	.115150
Q'		-.001135	.004415	-.006753	.012502	-.036708	.606979

TABLE VII—STATION NUMBERS

 $n=7$

	$j \backslash k$	1	2	3	4	5	6	7
M	1	1. 243487	0. 376396	-0. 026266	0. 009112	-0. 005896	0. 006025	-0. 006513
N		. 667840	. 490602	- . 063889	. 031590	- . 026481	. 033195	- . 042160
P		. 218415	. 022882	- . 003069	. 001211	- . 000853	. 000925	- . 000563
Q		. 053049	. 027042	- . 006769	. 003890	- . 003599	. 004829	- . 005357
M'		. 702228	. 458303	- . 050122	. 019989	- . 012820	. 011370	- . 011099
N'		. 454474	. 597757	- . 122429	. 069352	- . 057533	. 062529	- . 072080
P'		. 164382	. 032358	- . 006088	. 002679	- . 001831	. 001689	- . 001614
Q'		. 042465	. 038456	- . 013441	. 008611	- . 007724	. 008815	- . 010037
M	2	-1. 321299	0. 905437	0. 446479	-0. 055674	0. 029812	-0. 027896	0. 023701
N		- . 570270	1. 435730	1. 028397	- . 192270	. 133730	- . 153603	. 185730
P		- . 154452	. 182773	. 031489	- . 007052	. 004233	- . 004239	. 003750
Q		- . 031847	. 228718	. 068932	- . 022639	. 017853	- . 022136	. 023463
M'		- . 357636	. 764868	. 485193	- . 071706	. 038132	- . 031029	. 028988
N'		- . 191650	1. 234950	1. 123273	- . 247832	. 170920	- . 170555	. 188216
P'		- . 058807	. 166641	. 036328	- . 009168	. 005346	- . 004565	. 004199
Q'		- . 013146	. 209297	. 079621	- . 029438	. 022542	- . 023823	. 026110
M	3	1. 672922	-0. 511106	0. 737737	0. 516672	-0. 104856	0. 081487	-0. 077078
N		. 701415	- . 751768	1. 931045	1. 718603	- . 468955	. 448232	- . 498585
P		. 185835	- . 069049	. 162183	. 040990	- . 014225	. 012162	- . 010056
Q		. 037666	- . 083877	. 366024	. 130958	- . 059956	. 063490	- . 062408
M'		. 355047	- . 329159	. 692136	. 532099	- . 108454	. 073511	- . 063669
N'		. 186093	- . 492459	1. 819566	1. 771836	- . 484684	. 403671	- . 413268
P'		. 056122	- . 048434	. 156613	. 042913	- . 014528	. 010624	- . 009166
Q'		. 012374	- . 059075	. 353730	. 137128	- . 061218	. 055439	- . 056987
M	4	-1. 605312	0. 409927	-0. 250374	0. 629063	0. 592219	-0. 182666	0. 144688
N		- . 664780	. 597643	- . 625274	2. 291470	2. 574726	-1. 002357	. 935220
P		- . 174510	. 052757	- . 037056	. 147488	. 051989	- . 026098	. 018551
Q		- . 035122	. 063907	- . 082279	. 480975	. 218352	- . 136173	. 115110
M'		- . 317567	. 247432	- . 216700	. 623577	. 584350	- . 157315	. 115706
N'		- . 164912	. 366953	- . 543417	2. 273022	2. 538692	- . 861833	. 750618
P'		- . 049381	. 034761	- . 033166	. 147041	. 050504	- . 021773	. 016464
Q'		- . 010526	. 042284	- . 073708	. 479557	. 212061	- . 113560	. 102355
M	5	1. 129029	-0. 264373	0. 134585	-0. 136596	0. 551252	0. 668960	-0. 220971
N		. 464325	- . 384008	. 334325	- . 480675	2. 570992	3. 589325	-1. 425675
P		. 121269	- . 033334	. 019010	- . 021700	. 136202	. 063551	- . 027154
Q		. 024312	- . 040333	. 042147	- . 069981	. 580855	. 330701	- . 168426
M'		. 234133	- . 168109	. 122950	- . 143020	. 568533	. 634247	- . 197770
N'		. 120971	- . 248408	. 306705	- . 503685	2. 649577	3. 397336	-1. 281188
P'		. 036084	- . 023166	. 017983	- . 022914	. 139081	. 057351	- . 027322
Q'		. 007887	- . 028149	. 039907	- . 073907	. 593022	. 300912	- . 169818
M	6	-0. 617807	0. 137401	-0. 064103	0. 054068	-0. 084474	0. 507772	0. 632193
N		- . 252961	. 199169	- . 158887	. 189539	- . 383331	2. 883613	4. 007039
P		- . 065852	. 017129	- . 008878	. 008211	- . 014243	. 130010	. 051386
Q		- . 013170	. 020712	- . 019672	. 026453	- . 060230	. 694756	. 318021
M'		- . 125078	. 086131	- . 058376	. 057414	- . 092005	. 516450	. 704705
N'		- . 064447	. 127049	- . 145336	. 201491	- . 417397	2. 931074	4. 491463
P'		- . 019183	. 011759	- . 008393	. 008826	- . 015450	. 131144	. 069349
Q'		- . 004186	. 014281	- . 018616	. 028441	- . 065330	. 690659	. 430362
M	7	0. 205449	-0. 044613	0. 020059	-0. 015852	0. 021382	-0. 053078	0. 498306
N		. 083943	- . 064608	. 049675	- . 055505	. 096798	- . 295079	3. 334065
P		. 021819	- . 005533	. 002756	- . 002373	. 003481	- . 009551	. 130932
Q		. 004358	- . 006689	. 006106	- . 007644	. 014713	- . 049981	. 820708
M'		. 033023	- . 022307	. 014642	- . 013579	. 018939	- . 044219	. 420133
N'		. 016993	- . 032878	. 036425	- . 047602	. 085721	- . 245636	2. 816717
P'		. 005053	- . 003033	. 002091	- . 002060	. 003074	- . 007854	. 114318
Q'		. 001102	- . 003682	. 004637	- . 006638	. 012992	- . 041094	. 716958

$n=8$

	j	k	1	2	3	4	5	6	7	8
M	1		1. 312192	0. 364019	-0. 021829	0. 006490	-0. 003545	0. 003081	-0. 003931	0. 005039
N			. 695399	. 462793	- . 052953	. 022453	- . 015893	. 016958	- . 025621	. 037691
P			. 225483	. 021401	- . 002485	. 000839	- . 000499	. 000464	- . 000622	. 000684
Q			. 054447	. 025255	- . 005476	. 002694	- . 002103	. 002422	- . 003869	. 004929
M'			. 727223	. 444367	- . 043004	. 015242	- . 008698	. 007082	- . 007522	. 008334
N'			. 467152	. 577362	- . 104816	. 052799	- . 038988	. 038933	- . 048943	. 062492
P'			. 168111	. 030533	- . 005119	. 002003	- . 001120	. 001045	- . 001145	. 001228
Q'			. 043271	. 036245	- . 011294	. 006433	- . 005146	. 005456	- . 007123	. 008863
M	2		-1. 600390	0. 955663	0. 428499	-0. 045077	0. 020351	-0. 016189	0. 019610	-0. 024337
N			-. 682210	. 150798	. 984086	- . 155335	. 091123	- . 089039	. 127790	- . 152002
P			-. 183160	. 188781	. 029123	- . 005550	. 002808	- . 002412	. 003082	- . 002859
Q			-. 037524	. 235968	. 063696	- . 017807	. 011838	- . 012593	. 019183	- . 023705
M'			-. 415706	. 797175	. 468735	- . 060778	. 028715	- . 021414	. 021627	- . 027560
N'			-. 221089	1. 282229	1. 085551	- . 209723	. 128559	- . 117660	. 140686	- . 174560
P'			-. 067464	. 170868	. 034087	- . 007611	. 003953	- . 003129	. 003272	- . 003442
Q'			-. 015018	. 214419	. 074659	- . 024428	. 016664	- . 016330	. 020358	- . 024694
M	3		2. 337500	-0. 630523	0. 780389	0. 491625	-0. 082648	0. 054514	-0. 060500	0. 071477
N			. 967933	- . 923581	2. 036155	1. 631308	- . 308956	. 299530	- . 394067	. 534419
P			. 254178	- . 083331	. 167794	. 037442	- . 010886	. 007975	- . 009418	. 009602
Q			. 051181	- . 101109	. 378439	. 119546	- . 045861	. 041626	- . 058605	. 069206
M'			. 462586	- . 388839	. 722423	. 512112	- . 091426	. 056607	- . 052469	. 054058
N'			. 240602	- . 579786	1. 894495	1. 702145	- . 408100	. 310711	- . 341167	. 405216
P'			. 072148	- . 056240	. 160734	. 040069	- . 012016	. 008120	- . 007864	. 007911
Q'			. 015838	- . 068533	. 302854	. 127977	- . 050617	. 042369	- . 048918	. 057107
M	4		-2. 630070	0. 593584	-0. 315718	0. 667195	0. 558819	-0. 143453	0. 132430	-0. 143916
N			-1. 075644	. 861856	- . 786292	2. 424356	- . 786292	. 861856	- . 1. 075644	. 0. 143916
P			-. 279850	. 074709	- . 045466	. 152881	- . 046981	. 020073	- . 020253	. 0. 19152
Q			-. 055949	. 090392	- . 101283	. 408324	- . 107214	. 104712	- . 126011	. 138032
M'			-. 406670	. 329808	- . 258293	. 607770	- . 561588	. 135711	- . 104892	. 100193
N'			-. 240468	. 487559	- . 646305	2. 367821	- . 2. 436346	. 743106	- . 681473	. 750848
P'			-. 071591	. 045534	- . 038518	. 150902	- . 047159	. 018614	- . 015449	. 014572
Q'			-. 015627	. 055337	- . 086223	. 491981	- . 197950	- . 097075	. 096084	- . 1

TABLE IX—COMPARISON OF RESULTS

Number of stations	Torsion			Bending			Coupled		
	$\omega_1 \sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_2 \sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_3 \sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_1 \sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_2 \sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_3 \sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_1 \sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_2 \sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_3 \sqrt{\frac{m_0 l^4}{B_0}}$
Station-Function method									
1	1.549			3.493			3.46		
2	1.571	4.526		3.516	21.71		3.48	19.7	
3	1.571	4.689	7.502	3.516	22.04	60.20	3.48	20.6	48.2
Weighted influence coefficients									
2	1.575	5.39		3.56	15.63				
4	1.571	4.73		3.52	22.80				
Exact theoretical value									
	1.571	4.712	7.854	3.516	22.04	61.70	3.49	20.6	49.1

TABLE X—STATIONS REQUIRED FOR SATISFACTORY ACCURACY

Method	Torsion			Bending		
	$\omega_1 \sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_2 \sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_3 \sqrt{\frac{I_0 l^2}{C_0}}$	$\omega_1 \sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_2 \sqrt{\frac{m_0 l^4}{B_0}}$	$\omega_3 \sqrt{\frac{m_0 l^4}{B_0}}$
Station Functions.....	1	3	4	1	2	3
Weighted influence coefficients.....	2	4		3	6	

ANALYSIS OF THRUST AUGMENTATION OF TURBOJET ENGINES BY WATER INJECTION AT COMPRESSOR INLET INCLUDING CHARTS FOR CALCULATING COMPRESSION PROCESSES WITH WATER INJECTION¹

By E. CLINTON WILCOX and ARTHUR M. TROUT

SUMMARY

A psychrometric chart having total pressure (sum of partial pressures of air and water vapor) as a variable, a Mollier diagram for air saturated with water vapor, and charts showing the thermodynamic properties of various air-water vapor and exhaust gas-water vapor mixtures are presented as aids in calculating the thrust augmentation of a turbojet engine resulting from the injection of water at the compressor inlet.

Curves are presented that show the theoretical performance of the augmentation method for various amounts of water injected and the effects of varying flight Mach number, altitude, ambient-air temperature, ambient relative humidity, compressor pressure ratio, and inlet-diffuser efficiency. Numerical examples, illustrating the use of the psychrometric chart and the Mollier diagram in calculating both compressor-inlet and compressor-outlet conditions when water is injected at the compressor inlet, are presented.

For a typical turbojet engine operating at sea-level zero flight Mach number conditions, the maximum theoretical ratio of augmented to normal thrust, when a nominal decrease in compressor efficiency with water injection was assumed, was 1.29. The ratio of augmented liquid consumption to normal fuel flow for these conditions, assuming complete evaporation, was 5.01. Both the augmented thrust ratio and the augmented liquid ratio increased rapidly as the flight Mach number was increased and decreased as the altitude was increased. Although the thrust augmentation possible from saturating the compressor-inlet air is very low at low flight speeds, appreciable gains in thrust are possible at high flight Mach numbers. For standard sea-level atmospheric temperature, the relative humidity of the atmosphere had a small effect on the augmented thrust ratio for all flight speeds investigated. For sea-level zero flight Mach number conditions, the augmented thrust ratio increased as the atmospheric temperature increased for low values of atmospheric relative humidity. Water injection therefore tends to overcome the loss in take-off thrust normally occurring at high ambient temperatures. For very high atmospheric relative humidities, the ambient temperature had only a small effect on the augmented thrust ratio.

INTRODUCTION

One method of augmenting the thrust of turbojet engines is to inject water or a nonfreezing mixture of water and alcohol into the air entering the compressor of the engine. The evaporation of this injected liquid extracts heat from the air and results, for the same compressor work input, in a higher compressor pressure ratio. This increased pressure

ratio is reflected throughout the engine and increases both the mass flow of gases through the engine and the exhaust-jet velocity; both factors increase the thrust produced by the engine.

An analysis of the evaporative cooling process can be conveniently divided into two phases: (1) The cooling occurring at constant pressure before the air enters the compressor, and (2) the additional cooling associated with further evaporation during the mechanical compression process. Analysis of both phases of the evaporative cooling process and the thrust augmentation produced for a typical turbojet engine were conducted at the NACA Lewis laboratory during 1947-48 and are described herein.

A specialized psychrometric chart that permits calculation of the constant-pressure evaporation process was developed and is presented. This psychrometric chart differs from the usual form, which is valid for only one value of total pressure (sum of partial pressures of air and water vapor), in that pressure is included as a variable and the temperature range has been greatly extended.

The evaporation of water during the mechanical compression process greatly complicates the calculation of this portion of the turbojet-engine cycle by conventional methods because of the trial-and-error solution involved. In order to provide a more convenient method for calculating a compression process during which evaporative cooling occurs, a Mollier diagram for air saturated with water vapor was developed. This Mollier diagram is similar to that presented in reference 1, but the ranges of temperature and pressure are extended to values suitable for use in turbojet-engine-compressor problems.

As a further aid in calculating the turbojet-engine cycle when water is injected at the compressor inlet, curves are presented giving the thermodynamic properties of the working fluid for both the compression and expansion processes. The use of both the psychrometric chart and the Mollier diagram is illustrated by means of numerical examples.

The performance of a turbojet engine utilizing thrust augmentation by water injection at the compressor inlet was evaluated over a range of flight Mach numbers and altitudes and for a range of water-injection rates. The effect on thrust augmentation of varying ambient-air temperature, relative humidity, compressor pressure ratio, and inlet-diffuser efficiency was also investigated. A comparison of the water-injection method of thrust augmentation with other thrust-augmentation methods is presented in references 2 and 3.

¹ Supersedes NACA TN 2104, "Theoretical Turbojet Thrust Augmentation by Evaporation of Water during Compression as Determined by Use of a Mollier Diagram" by Arthur M. Trout, 1950, and NACA TN 2105, "Turbojet Thrust Augmentation by Evaporation of Water Prior to Mechanical Compression as Determined by Use of Psychrometric Chart" by E. Clinton Wilcox, 1950.

ASSUMPTIONS AND METHODS OF ANALYSIS

In evaluating the performance of a turbojet engine utilizing water injection at the compressor inlet, the psychrometric chart was used to obtain the compressor-inlet conditions, the Mollier diagram was used to obtain the conditions at the end of the evaporation process in the compressor, and the remainder of the turbojet-engine cycle was calculated by conventional step-by-step methods.

The following discussion is a description of (a) the assumptions and the methods used in deriving the psychrometric chart and the Mollier diagram, and (b) the assumed design constants and component efficiencies of the turbojet engine used to illustrate performance with water injection at the compressor inlet. All symbols used in the analysis are defined in appendix A and the derivations of the equations necessary for obtaining the psychrometric chart and the Mollier diagram are presented in appendix B.

PSYCHROMETRIC CHART

The familiar psychrometric chart used in air-conditioning calculations is generally limited to values of total pressure (sum of partial pressures of air and water vapor) very near standard sea-level total pressure and to relatively low temperatures, usually less than 300° F. Inasmuch as the conditions of temperature and pressure at the compressor inlet of a turbojet engine vary widely over ranges not included in the usual psychrometric chart, the chart described herein was constructed to include these ranges.

In deriving the psychrometric chart, the following assumptions were made:

(a) The air and the water vapor are always at the same temperature.

(b) Dalton's law of partial pressures is valid for the mixture.

(c) The thermodynamic properties of water vapor are functions only of temperature for the range of vapor pressures considered.

(d) Air is a perfect gas.

By means of these assumptions, the theory of mixtures, the thermodynamic data for air and water vapor contained in references 4 and 5, respectively, and the equations derived in appendix B, a psychrometric chart with pressure as a variable and for temperatures up to 1500° R was constructed.

MOLLIER DIAGRAM

Because of the somewhat involved and tedious trial-and-error methods generally used for calculating compression processes during which water is evaporated, the Mollier diagram described in this report was prepared to provide a more direct solution of this particular problem.

The data presented in the Mollier diagram were calculated using the following assumptions:

(a) All values presented on the diagram are for a saturated mixture containing 1 pound of air.

(b) The water vapor and the air in the mixture are at the same temperature.

(c) Dalton's law of partial pressures is valid for the mixture.

(d) Air is a perfect gas.

The data presented in the Mollier diagram were calculated using the aforementioned assumptions, the thermodynamic properties data for air and water vapor presented in references 4 and 5, respectively, and the equations derived in appendix B.

Examples of the use of the psychrometric chart and the Mollier diagram are given in the subsequent section DISCUSSION OF CHARTS and in appendix C.

THRUST-AUGMENTATION CALCULATIONS

From an examination of experimental data of complete turbojet engines and of the compressor component alone, both operating with water injection at the compressor inlet, it is apparent that two prime factors cause deviation of the experimentally obtained results from the ideal process. These two factors are the decrease in compressor adiabatic efficiency that occurs with water injection and the failure of all the injected water to evaporate within the confines of the compressor. The decrease in compressor efficiency apparently is a function of the water-injection rate; however, the magnitude of the effect varies considerably with the particular compressor design. Because of the fundamental nature of the effect of compressor efficiency on the turbojet-engine cycle, its effect is relatively obscure and, for the present analysis, an arbitrary variation with water-injection rate was assumed that is representative of the effect experimentally observed in a current compressor. The theoretical results obtained are thus not expected to agree completely with experimental data obtained for engines embodying different compressor designs. The influence of this assumed decrease in compressor efficiency on the performance of the water-injection method of thrust augmentation will be indicated herein.

The degree of evaporation of water varies widely with the mode of injection and the engine operating conditions; however, insufficient data are available to determine the quantitative effects of these conditions on the degree of evaporation. Inasmuch as the thrust produced by the engine is primarily affected by the amount of water that is evaporated, the influence of incomplete evaporation can be evaluated simply by increasing the injected-water rate commensurate with the desired degree of evaporation. In the present analysis, all performance calculations have therefore been made by assuming complete evaporation.

The augmented thrust ratios experimentally obtained for engines embodying centrifugal-flow-type compressors agree quite well with those theoretically predicted herein for the condition of a saturated mixture throughout the compression process. For engines having axial-flow-type compressors, however, the efficacy of the water-injection method is considerably reduced because of centrifugal separation of the water from the air in passing through the compressor. In order to avoid the centrifugal-separation effect, the water must be injected at several stages throughout the compressor.

Inasmuch as considerable increase in thrust is attainable at high inlet temperatures or high flight Mach numbers merely by saturating the compressor-inlet air with water, the performance of the water-injection method has been determined for both the condition of saturation throughout the compression process and saturation only of the compressor-inlet air. The thrust increases obtained for the condition of saturation throughout the compression process are believed indicative of the increases possible for engines having centrifugal-flow compressors, whereas the thrust increases shown for the condition of saturated inlet air are more nearly equal to the possibilities of the water-injection method when applied to an engine having an axial-flow compressor and not utilizing interstage injection. Experimental investigations of interstage injection indicate that the thrust increases available for this method lie between those attainable for saturated inlet air and those obtained by saturation throughout the compression process. The decreased thrust augmentation obtainable with an engine having an axial-flow compressor utilizing interstage injection may be due to the increased sensitivity of the efficiency of this type compressor to water injection.

In order to determine the theoretical thrust augmentation made possible by evaporation of water during compression, step-by-step calculations of normal and augmented thrust for a typical turbojet engine operating under various flight conditions were made. The psychrometric chart was used to determine the compressor-inlet conditions and the Mollier diagram to calculate the conditions at the end of the evaporation process in the compressor with water injection. Equations used in the step-by-step calculations are given in appendix D. It was assumed that the inlet air was diffused to a sufficiently low velocity so that the static temperature at the compressor inlet did not appreciably differ from the total temperature.

The component efficiencies chosen for the typical turbojet engine were: inlet-diffuser adiabatic efficiency (except where otherwise noted), 0.85 up to a flight Mach number of 1.0, linearly decreasing to 0.75 at a flight Mach number of 2.0 (these values of diffuser efficiency correspond to total-pressure recovery ratios of 0.915 and 0.66 at flight Mach numbers of 1.0 and 2.0, respectively); compressor adiabatic efficiency, 0.80; turbine adiabatic efficiency, 0.85; and exhaust-nozzle adiabatic efficiency, 0.95. The compressor adiabatic efficiency is defined as the ratio of isentropic to actual work.

For the compression process with evaporative cooling, it must be recalled that a higher pressure ratio is obtained for a given isentropic work than is obtained with dry air. Inasmuch as experimental data of compressor performance with water injection indicate a decrease in compressor adiabatic efficiency, the compressor efficiency η_c was arbitrarily assumed to decrease according to the relation $\eta_c = 0.80 - \Delta X_c$, where ΔX_c is the change in water vapor-air ratio in the compressor. This decrease in compressor efficiency was assumed for all conditions of altitude and flight Mach number, except when it was desired to show the effect of this assumption on performance. A single-stage, centrifugal-flow compressor with a tip speed of 1500 feet per second

and a slip factor of 0.95 was assumed for all of the calculations except when it was desired to show the effect of normal compressor pressure ratio. For the assumed values of compressor tip speed, slip factor, and efficiency, the resulting compressor pressure ratio for sea-level zero flight Mach number conditions with no water injection was 4.61. The normal engine performance would be equally applicable to an engine having an axial-flow compressor with the same enthalpy rise, pressure ratio, and compressor efficiency. A combustion-chamber pressure loss of 3 percent and a turbine-inlet temperature of 2000° R were assumed. The ambient relative humidity was assumed to be 0.50 except when it was desired to show the effect of varying ambient relative humidity. All liquid water was assumed injected at the compressor inlet at a temperature of 519° R and the air was assumed to remain saturated as long as any liquid water was present. For conditions where all the injected water evaporated before the end of the compression process, that portion of the process occurring after evaporation was calculated from the thermodynamic relations for an adiabatic compression by using the appropriate thermodynamic properties of the resulting air-water vapor mixtures. The adiabatic efficiency of this portion of the compression process was assumed equal to that previously described for the compression process during which evaporation occurred. Calculations were made for flight Mach numbers up to 2.0 and for standard altitudes of sea level and 35,332 feet.

In order to simplify the calculations of the adiabatic compression occurring after the evaporation of all the injected water and the expansion processes occurring in the turbine and the exhaust nozzle, curves of the thermodynamic properties of air-water vapor and exhaust gas-water vapor mixtures were obtained. These curves were obtained from the theory of mixtures and the thermodynamic data for air, water vapor, and exhaust gases.

Because of the low temperatures occurring at high altitudes, a nonfreezing mixture of water and alcohol rather than water alone is usually employed for compressor-inlet injection. Although all the results presented herein are based on the injection of only water at the compressor inlet, they are probably equally applicable to the injection of water-alcohol mixtures because of the close agreement experimentally obtained between results for water- and for water-alcohol-mixture injection.

DISCUSSION OF CHARTS

PSYCHROMETRIC CHART

In the psychrometric chart (fig. 1), total enthalpy is plotted against dry-bulb temperature for various values of water-air ratio. Also included are curves of the ratio of relative humidity to relative pressure, where relative pressure is the ratio of static pressure to standard sea-level static pressure. Inasmuch as evaporation occurs at constant pressure, the value of relative pressure is unchanged by water injection. The curves of constant water-air ratio are approximately straight lines having a positive slope and the curves of constant ratio of relative humidity to relative

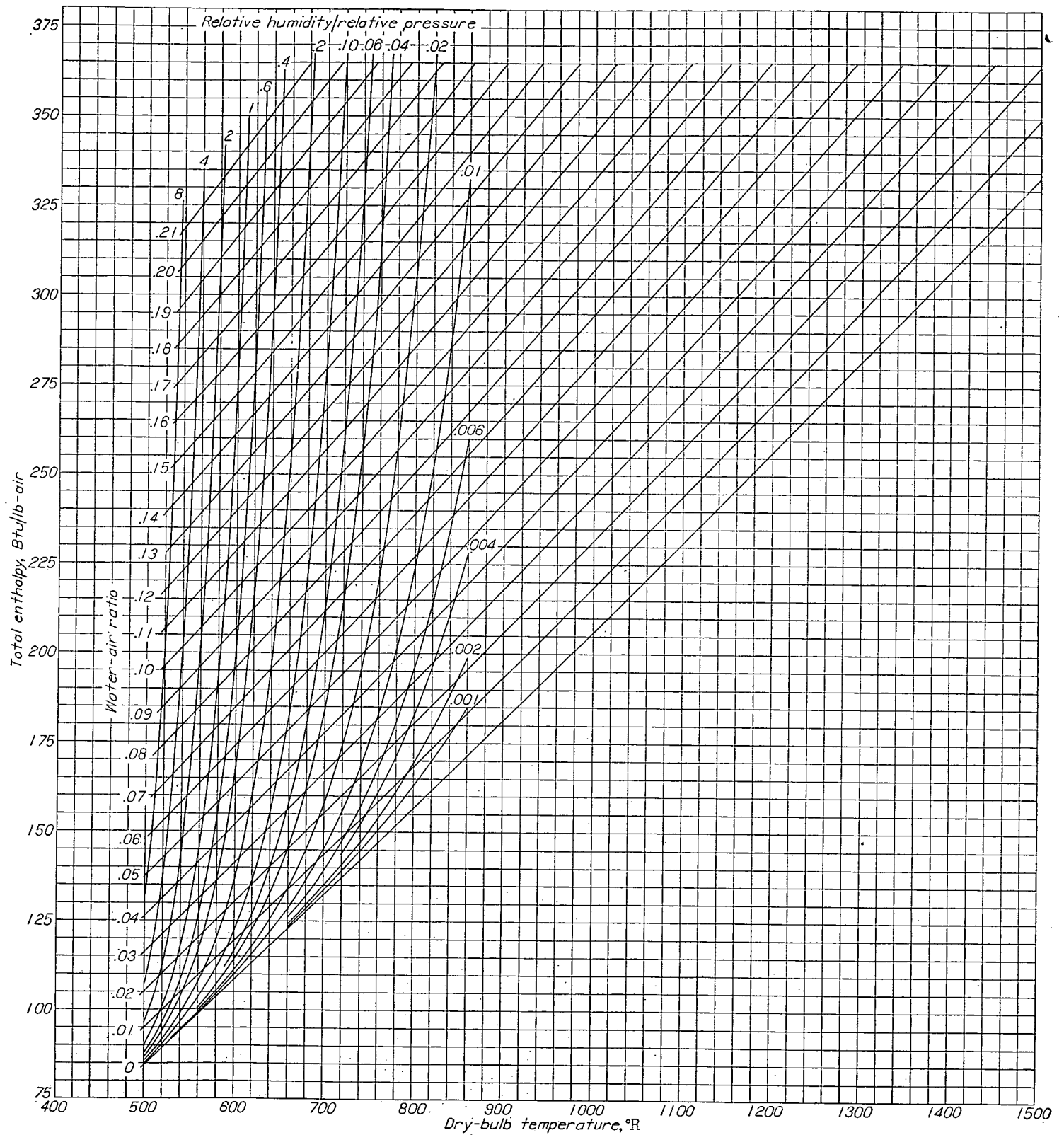


FIGURE 1.—Psychrometric chart. (A 23- by 25-inch print of this figure is available from NACA upon request.)

pressure are curved upward. The enthalpy scale was arbitrarily chosen so that the enthalpy of a saturated mixture of air and water vapor at a temperature of 519° R and a pressure of 14.696 pounds per square inch was 100 Btu per pound of air.

The temperature resulting after water evaporation may be obtained from figure 1 for a wide range of initial air temperatures and pressures. The temperature can be determined either for the evaporation of a given amount of water or for any final relative humidity.

In the construction of figure 1, the enthalpy of liquid water was assumed to be zero at 519° R. The chart values are thus directly applicable to cases where the water is injected at this temperature. For temperatures of liquid water only slightly different from 519° R, the errors involved in the chart are negligible; however, for extremely accurate work or for temperatures of injected water greatly different from 519° R, corrections must be made for the enthalpy of liquid water. These corrections are the same as the enthalpy corrections for the Mollier diagram discussed in appendix C, and consist in adding the enthalpy of liquid water at the start of the process and, if any liquid remains at the end of the process, subtracting the enthalpy of the remaining liquid.

The method of using the chart is indicated by the following illustrative cases:

Example 1.—For initially dry air at a temperature of 1260° R, assume that sufficient water is injected to give a final water-air ratio of 0.05. From figure 1, the total enthalpy is 271.6 Btu per pound of air for a temperature of 1260° R and a water-air ratio of 0. (For this example and all following examples, the values have been read from the large prints of figs. 1 or 2, which are available from the NACA on request and are more convenient to use when a high degree of accuracy is desirable.) Inasmuch as the evaporation occurs at constant total enthalpy, the temperature after evaporation can be determined from the initial total enthalpy and the final water-air ratio. For a total enthalpy of 271.6 Btu per pound of air and a water-air ratio of 0.05, the temperature is 1008° R. The temperature drop is then 252 F° for these particular conditions.

Example 2.—In order to illustrate the use of the chart in obtaining the temperature and the water-air ratio necessary to saturate air from a given set of initial conditions, initially dry air at a temperature of 1060° R and at twice standard sea-level static pressure is assumed. The enthalpy corresponding to a temperature of 1060° R and a water-air ratio of 0 is 221.0 Btu per pound of air. The value of relative humidity for saturation is 1.0 and for these conditions the value of relative pressure δ is 2.0. The temperature and the water-air ratio are 613° R and 0.0995, respectively, at an enthalpy of 221.0 Btu per pound of air and a ratio of relative humidity to relative pressure ϕ/δ of 0.5. For these conditions, the initial mixture may be cooled 447 F° by injecting 0.0995 pound of water per pound of air.

Example 3.—The use of the chart in obtaining temperatures resulting from water evaporation when some water is initially present in the air is the same as that previously described, except that the starting enthalpy is found from the initial water-air ratio and temperature. For example, for the conditions of temperature and pressure in example 2 and an initial water-air ratio of 0.02, the initial enthalpy is 247.0 Btu per pound of air. The saturation temperature for an enthalpy of 247.0 Btu per pound and a value of ϕ/δ of 0.5 is 620° R and the final water-air ratio is 0.122. For these conditions, cooling the initial mixture 440 F° is therefore possible by evaporating 0.102 pound of water per pound of air.

The effect of initial temperature and pressure on the amount of cooling possible from saturation can be readily seen from figure 1. For a constant pressure and consequent constant value of ϕ/δ ($\phi=1$), the amount of cooling possible increases as the initial temperature is raised. For a constant temperature, increasing the pressure decreases the value of ϕ/δ and, from figure 1, decreasing the value of ϕ/δ decreases the amount of cooling obtained. In general, when some water vapor is initially present, the cooling possible from saturation and the amount of water that must be injected to obtain saturation are somewhat less than those for dry air. In some instances, however, depending on the particular initial conditions, the amount of water that must be injected to obtain saturation is greater when some water vapor is initially present than for dry air, as illustrated in examples 2 and 3.

Example 4.—In order to illustrate the method of using the psychrometric chart when water is injected at a temperature other than 519° R and the enthalpy of the injected water is to be considered, assume that it is desired to saturate initially dry air at a temperature of 910° R and standard sea-level pressure with water injected at a temperature of 619° R. From figure 1, which is correct for water injected at 519° R, the initial enthalpy of the air at a temperature of 910° R is 183.5 Btu per pound. For a value of relative humidity divided by relative pressure of 1.0 (saturation at standard sea-level pressure) and an enthalpy of 183.5 Btu per pound, the temperature is 577° R and the water-air ratio is 0.075. If the water were injected at 519° R, cooling the air 333 F° would be possible by injecting sufficient water to obtain a water-air ratio of 0.075. For water not injected at 519° R, an estimate must be made of the final water-air ratio in order to determine the starting condition. For the present example, assume that the final water-air ratio is 0.08. From the insert of figure 2, the enthalpy of liquid water at a temperature of 619° R is 100 Btu per pound of water. The new starting enthalpy is now 183.5 plus 8.0 (water-air ratio, 0.08) or 191.5 Btu per pound. For a value of ϕ/δ of 1.0 and an enthalpy of 191.5 Btu per pound, the temperature is 579° R and the water-air ratio is 0.081. If water were injected at a temperature of 619° R, the cooling obtained from saturation would therefore be 331 F° and the required water-air ratio would be 0.081.

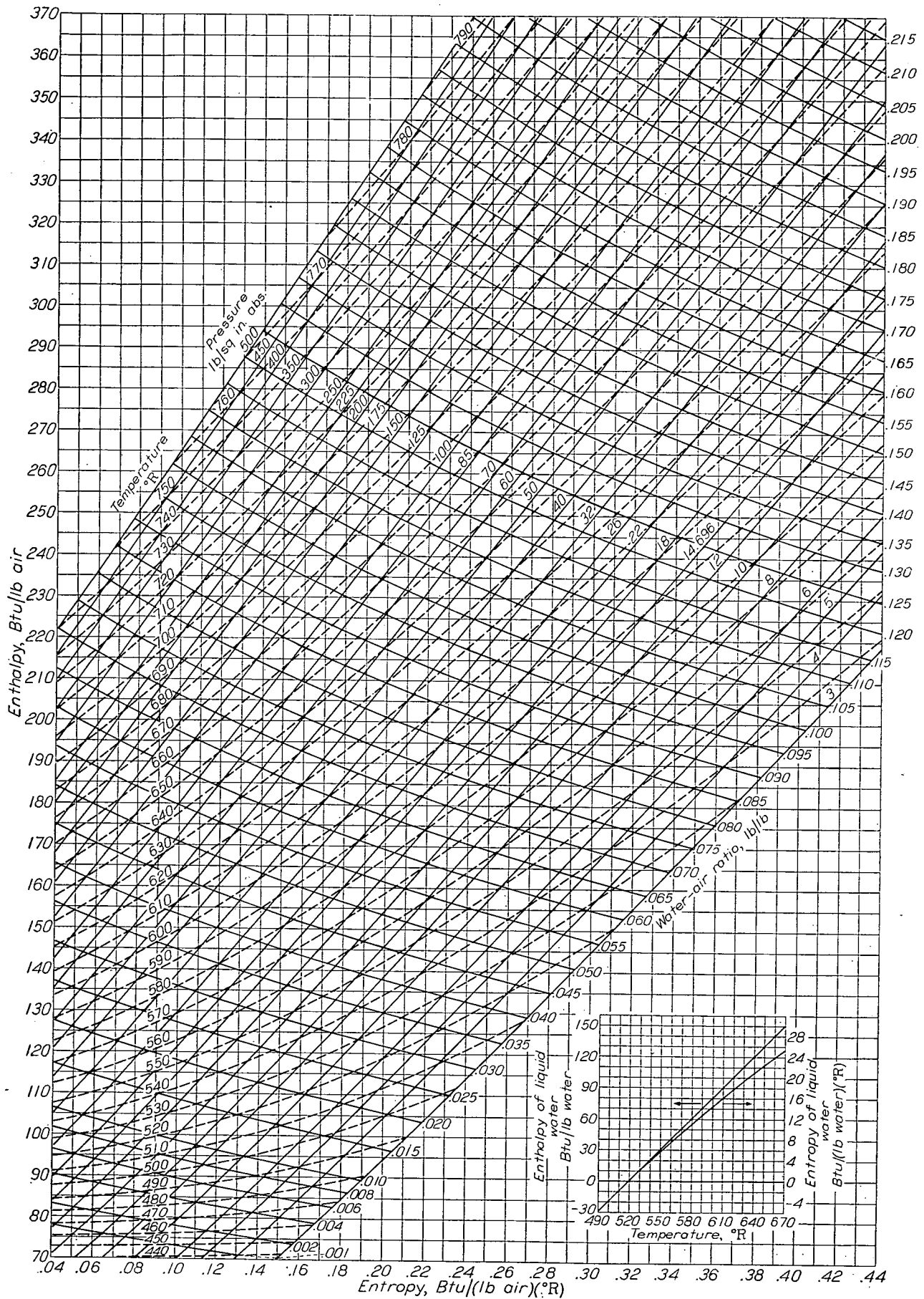


FIGURE 2.—Mollier diagram. (Ten 16-by 21-inch expanded sectional prints available from NACA upon request.)

MOLLIER DIAGRAM

The Mollier diagram (fig. 2) is so constructed that each point represents values for 1 pound of air plus the water vapor necessary for saturation. The abscissa is entropy (in Btu/(lb air)(°R)) and the ordinate is enthalpy (in Btu/lb air). The solid slanting lines with positive slope are lines of constant pressure in pounds per square inch. The broken lines with positive slope are lines of constant temperature (in °R). The solid lines of negative slope are lines of constant weight ratio of water vapor to air. Data are presented for pressures from 3 to 500 pounds per square inch and temperatures from 440° to 790° R. The entropy and the enthalpy of liquid water were assumed to be zero at 519° R and the bases for enthalpy and entropy of air were so fixed that at a temperature of 519° R and a pressure of 14.696 pounds per square inch the enthalpy of the saturated mixture is 100 Btu per pound of air and the entropy is 0.10 Btu/(lb air)(°R). When liquid water is present at a temperature other than 519° R, corrections for enthalpy and entropy must be made. These corrections are discussed in appendix B and a chart giving enthalpy and entropy of liquid water at various temperatures to be used for these corrections is shown on the Mollier diagram.

In order to utilize the Mollier diagram to calculate compressor-outlet conditions with evaporation of water during compression, knowledge of the conditions of the saturated air at the compressor inlet and the actual and isentropic work of the compression process is necessary. The conditions at the end of the saturated compression process may be obtained as follows:

A point is found on the Mollier diagram corresponding to the conditions of saturated air at the compressor inlet. An isentropic compression process is then followed on the diagram and a second point determined by moving vertically along a line of constant entropy until a value of enthalpy equal to the original enthalpy plus the enthalpy of the isentropic work of compression per pound of air (actual work/lb of air multiplied by adiabatic efficiency) is reached. The pressure read from the diagram at this second point is the pressure at the end of the compression process. The final temperature and water-air ratio for the compression process are read from the diagram at a third point, which is obtained by following a constant-pressure line from the second point until a value of enthalpy equal to the original enthalpy plus the enthalpy of the actual work of compression per pound of air is reached.

For a centrifugal-flow compressor, all the water to be evaporated during compression is added at the compressor inlet and the compressor enthalpy rise may be calculated per unit mass flow of the mixture of air and water vapor at the compressor outlet from considerations of tip speed and slip factor. Inasmuch as all values on the Mollier diagram are per unit mass flow of air, the value of enthalpy rise per unit mass flow of mixture at the compressor outlet must be converted to enthalpy rise per unit mass flow of air in order to use the chart. For an axial-flow compressor, if all the water is injected at the compressor inlet, centrifugal separation of the air and water greatly reduces the effects of water

injection and therefore injection of the water in several stages along the compressor length is desirable. If calculations for interstage water injection of this type are desired, the total enthalpy rise must be divided among the sections of the compressor between injection points and calculations similar to those for the centrifugal-flow compressor must be made for each section in order to allow for the change in mass flow from section to section and the corresponding change in enthalpy rise per unit mass flow of air.

When less water is injected than is necessary to saturate the air at the compressor outlet, the compression process is divided into two parts: (1) an adiabatic compression of a saturated mixture, which may be calculated by use of the Mollier diagram, and (2) further adiabatic compression of the resulting mixture of air and water vapor.

The following numerical example is presented to illustrate the method of using the Mollier diagram to calculate the compressor-outlet conditions for a saturated compression process:

Assume that a compressor having an adiabatic efficiency of 0.80 and imparting an enthalpy rise of 80 Btu per pound of air to the working fluid is operated with sufficient water injected at the compressor inlet at a temperature of 519° R to maintain saturation at all times. Further, assume that the inlet air has a relative humidity of 0.50 at a temperature of 530° R and a pressure of 14.7 pounds per square inch and that it is desired to find the compressor-outlet pressure P_3 , -outlet temperature T_3 , -outlet water vapor-air ratio X_3 , and the amount of water evaporated during the process.

The conditions at the compressor inlet after saturation can be obtained from the psychrometric chart (fig. 1). For the initial condition of ambient relative humidity ϕ of 0.50, temperature T_0 of 530° R, and pressure P_0 of 14.7 pounds per square inch, the water vapor-air ratio X_0 is 0.0077 and the enthalpy H_0 is 100 Btu per pound of air.

After saturation at constant pressure, from figure 1, the temperature T_1 is 519° R, the water vapor-air ratio X_1 is 0.0106, and the enthalpy H_1 is 100 Btu per pound of air.

From the Mollier diagram (fig. 2), for a temperature T_1 of 519° R and a pressure P_1 of 14.7 pounds per square inch, the entropy S_1 is 0.10 Btu/(lb air)(°R).

By increasing the enthalpy at constant entropy by an amount equal to the ideal work of compression, the actual pressure at the compressor outlet may be read from the diagram. The ideal work of compression is

$$\Delta H_{c,i} = \eta_c \Delta H_c = 0.80 \times 80 = 64 \text{ (Btu/lb air)}$$

and

$$H_{2,i} = H_1 + \Delta H_{c,i} = 100 + 64 = 164 \text{ (Btu/lb air)}$$

The value of pressure read from the diagram at an enthalpy of 164 Btu per pound of air and an entropy of 0.10 Btu/(lb air)(°R) is

$$P_3 = 70.7 \text{ (lb/sq in.)}$$

The enthalpy at the compressor outlet is equal to the enthalpy at the inlet plus the actual enthalpy change, or

$$H_3 = H_1 + \Delta H_c = 100 + 80 = 180 \text{ (Btu/lb air)}$$

From the Mollier diagram at a pressure of 70.7 pounds per square inch and an enthalpy of 180 Btu per pound of air, the values of temperature and water vapor-air ratio at the compressor outlet are

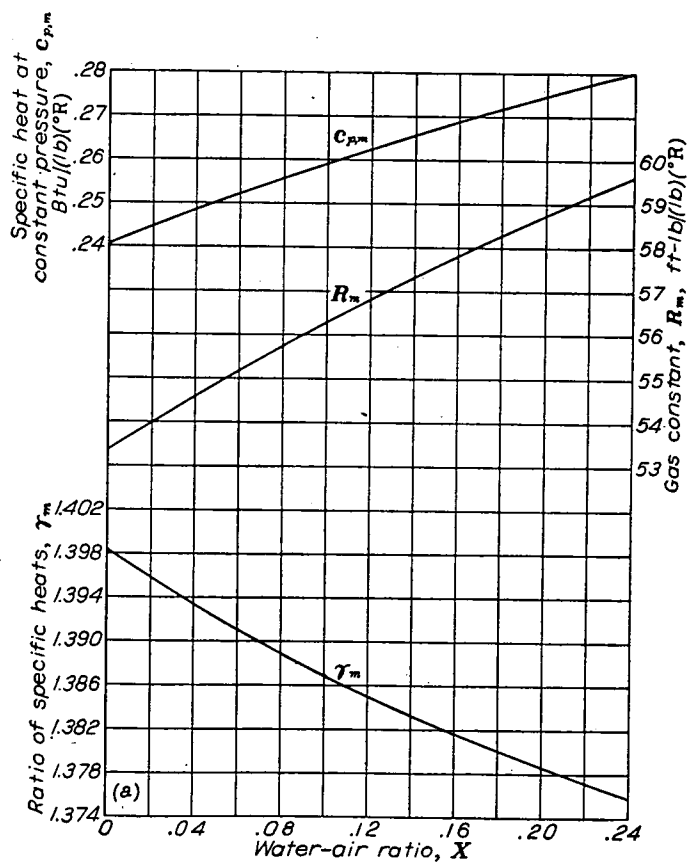
$$T_3 = 630^\circ \text{R and } X_3 = 0.0583$$

The amount of water evaporated is

$$X_3 - X_0 = (X_1 - X_0) + (X_3 - X_1) = (0.0106 - 0.0077) + (0.0583 - 0.0106) = 0.0506 \text{ (lb water/lb air)}$$

The amount of water vapor represented by $X_1 - X_0$ is that evaporated at the inlet prior to compression and the amount $X_3 - X_1$ represents that water evaporated during the mechanical compression process.

Additional examples presented in appendix C illustrate the use of the Mollier diagram in calculating a compression process when the water is injected at a temperature other than 519°R and when a given amount of water (less than the amount required to saturate the compressor-outlet air) is injected.



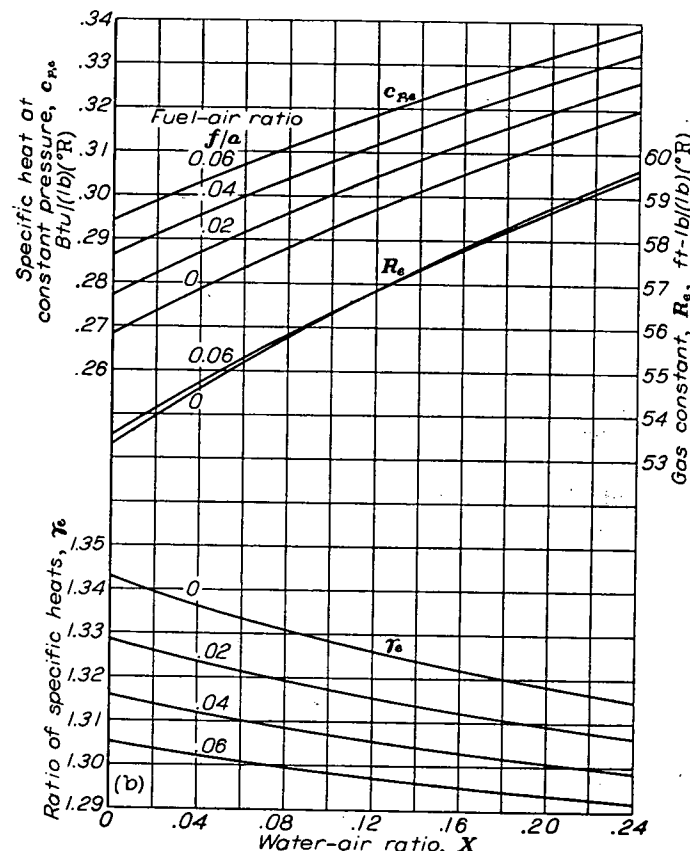
(a) Air-water vapor mixture. Temperature, 600°R .

FIGURE 3.—Thermodynamic properties of mixtures at pressure of 0 pound per square inch absolute.

MIXTURE PROPERTIES FOR COMPRESSOR

A chart of the thermodynamic properties of water vapor-air mixtures is shown in figure 3(a). The variation of specific heat at constant pressure with temperature and pressure is small in the range of interest for compressor calculations; figure 3(a) was therefore based on an average temperature of 600°R and zero pressure. Values of the ratio of specific heats γ_m , gas constant R_m , and specific heat at

constant pressure $c_{p,m}$ are plotted for various values of the water vapor-air ratio X . The data presented in this chart were used in calculating compressor-outlet conditions for those cases where less water was injected than the amount necessary to saturate the air at the compressor outlet. The use of figure 3(a) for a case of this type is illustrated in Sample Calculation II (appendix C). The thermodynamic data for air and water vapor used in obtaining the data of figure 3(a) were obtained from references 4 and 5, respectively, and the values presented are for 1 pound of mixture of air and water vapor.



(b) Exhaust gas-water vapor mixture. Temperature, 1650°R ; hydrogen-carbon ratio, 0.175.
FIGURE 3.—Concluded. Thermodynamic properties of mixtures at pressure of 0 pound per square inch absolute.

MIXTURE PROPERTIES FOR TURBINE AND JET

A chart of the thermodynamic properties of water vapor-exhaust gas mixtures for a temperature of 1650°R is shown in figure 3(b). Values for the ratio of specific heats γ_e , gas constant R_e , and specific heat at constant pressure $c_{p,e}$ are plotted for various values of the water vapor-air ratio X and the fuel-air ratio f/a . The thermodynamic data for water vapor and exhaust gases necessary for obtaining the data presented in figure 3(b) were obtained from references 5 and 6, respectively, and the values presented are for 1 pound of mixture of exhaust gas and water vapor. (A hydrogen-carbon ratio of the fuel of 0.175 was assumed in preparing these curves.) Although the actual values of γ_e and $c_{p,e}$ vary with temperature, the values given in figure 3(b) can be used for an appreciable temperature range with negligible error because of the compensating manner in which γ_e and $c_{p,e}$ enter the equations for an expansion process. For more accurate work, the method

presented in reference 7 may be used for calculating the expansion process.

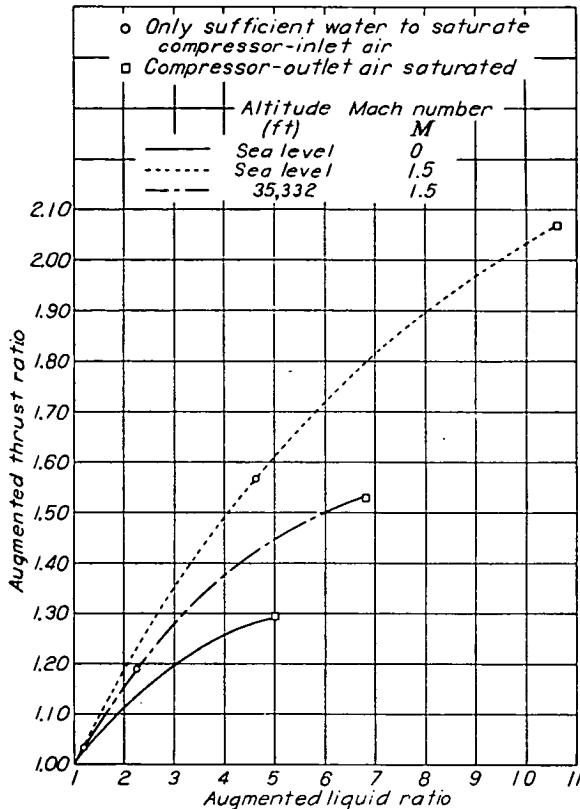


FIGURE 4.—Effect of flight conditions and ratio of augmented to normal liquid flow on thrust augmentation.

DISCUSSION OF AUGMENTATION RESULTS

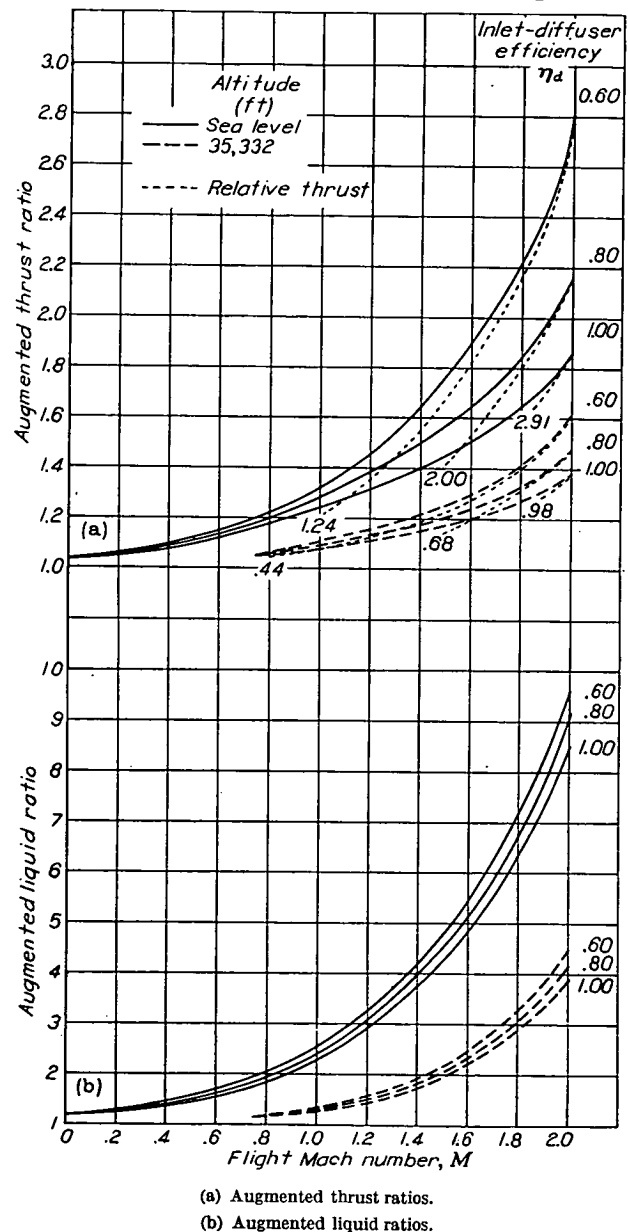
EFFECT OF WATER-INJECTION RATE

In figure 4, the augmented thrust ratio (ratio of augmented thrust to normal thrust) is shown as a function of the augmented liquid ratio (ratio of augmented total liquid flow to normal fuel flow) for three different operating conditions: flight Mach numbers of 0 and 1.5 at sea-level altitude and a flight Mach number of 1.5 at an altitude of 35,332 feet. The points on the curves indicated by circles represent the case where only sufficient water to saturate the compressor-inlet air is injected with no evaporation occurring during mechanical compression. The end points on the curves represent the condition where the air at the compressor outlet is saturated and indicate the maximum amounts of augmentation theoretically possible by evaporation of water before and during compression in the engine previously described and for the given flight conditions. As the augmented liquid ratio is increased by increasing the amount of injected water, the augmented thrust ratio increases but at a slightly decreasing rate. At sea-level, zero flight Mach number conditions for saturation only up to the compressor inlet, the augmented thrust ratio is 1.035 and the augmented liquid ratio is 1.18. As the augmented liquid ratio is increased by evaporating more water during mechanical compression, the augmented thrust ratio increases to 1.29 at an augmented liquid ratio of 5.01 for compressor-outlet saturation. For a given altitude, increasing the flight Mach number increases the compressor-inlet temperature, which permits the evaporation of more water and thereby increases the maximum

augmented thrust and augmented liquid ratios as well as increasing the augmented thrust ratio for a given augmented liquid ratio. At a sea-level flight Mach number of 1.5 for saturation of the compressor-outlet air, the augmented thrust ratio is 2.07 and the augmented liquid ratio is 10.66. Increasing the altitude for a given flight Mach number results in a decreased compressor-inlet temperature, thereby reducing the amount of water that may be evaporated. At an altitude of 35,332 feet and a flight Mach number of 1.5 for compressor-outlet saturation, the augmented thrust ratio is 1.53 at an augmented liquid ratio of 6.85.

EFFECT OF OPERATING CONDITIONS FOR SATURATED COMPRESSOR-INLET AND COMPRESSOR-OUTLET CONDITIONS

Effect of flight Mach number, altitude, and inlet-diffuser efficiency.—The augmented thrust and augmented liquid ratios are shown as functions of flight Mach number for altitudes of sea level and 35,332 feet and for various values of inlet-diffuser efficiency in figure 5 for compressor-inlet



(a) Augmented thrust ratios.

(b) Augmented liquid ratios.

FIGURE 5.—Effect of flight Mach number and altitude on augmented thrust and augmented liquid ratios for various values of inlet-diffuser efficiency. Sufficient water injected to saturate compressor-inlet air.

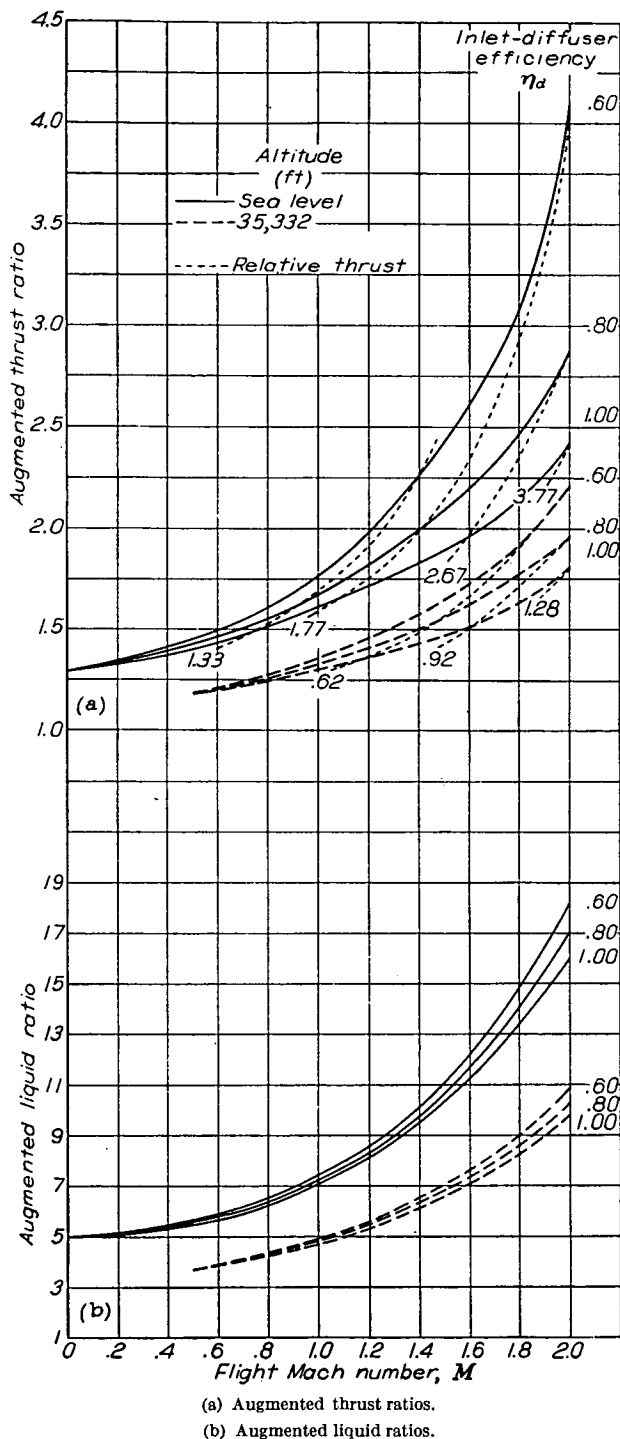


FIGURE 6.—Effect of flight Mach number and altitude on augmented thrust and augmented liquid ratios for various values of inlet-diffuser efficiency. Sufficient water injected to saturate compressor-outlet air.

saturation, and in figure 6 for compressor-outlet saturation. In order to indicate the magnitude of the thrust available, relative-thrust curves (constant values of the ratio of thrust to thrust available at sea-level, zero flight Mach number conditions for operation without water injection) are superimposed on the curves of figures 5(a) and 6(a). Both the augmented thrust ratio and the augmented liquid ratio increase rapidly as the flight Mach number is increased and decrease as the altitude is increased. Although the augmented thrust ratio available at a given flight Mach number increases at decreased values of inlet-diffuser efficiency, the

actual thrust available decreases because of the deterioration of the engine performance without water injection at the decreased values of inlet-diffuser efficiency.

For saturated compressor-inlet conditions (fig. 5(a)) and an inlet-diffuser adiabatic efficiency of 0.80, the augmented thrust ratio for sea-level zero flight Mach number is 1.035 and increases with increasing flight Mach number, reaching a value of 1.28 at a flight Mach number of 1.0 and 2.16 at a flight Mach number of 2.0. The corresponding augmented liquid ratios (fig. 5(b)) are 1.18, 2.40, and 9.20 for sea-level flight Mach numbers of 0, 1.0, and 2.0, respectively. For a flight Mach number of 2.0, increasing the altitude from sea level to 35,332 feet reduces the augmented thrust ratio from 2.16 to 1.48. The corresponding reduction in augmented liquid ratio is from 9.20 to 4.20. Although the thrust increases available from compressor-inlet saturation are very low at low flight Mach numbers, appreciable gains in thrust are possible at high flight speeds. Thrust augmentation by compressor-inlet-air saturation can therefore be used to good advantage not only for centrifugal-flow engines but also for axial-flow engines that require special injection systems to avoid centrifugal separation at higher injection rates.

For saturated compressor-outlet conditions and an inlet-diffuser adiabatic efficiency of 0.80, the augmented thrust ratio (fig. 6(a)) is 1.29, 1.67, and 2.88 for sea-level flight Mach numbers of 0, 1.0, and 2.0, respectively. The corresponding augmented liquid ratios (fig. 6(b)) are 5.01, 7.22, and 17.10, respectively. At a flight Mach number of 2.0, increasing the altitude from sea level to 35,332 feet reduces the augmented thrust ratio from 2.88 to 1.95 with a corresponding reduction in augmented liquid ratio from 17.10 to 10.30.

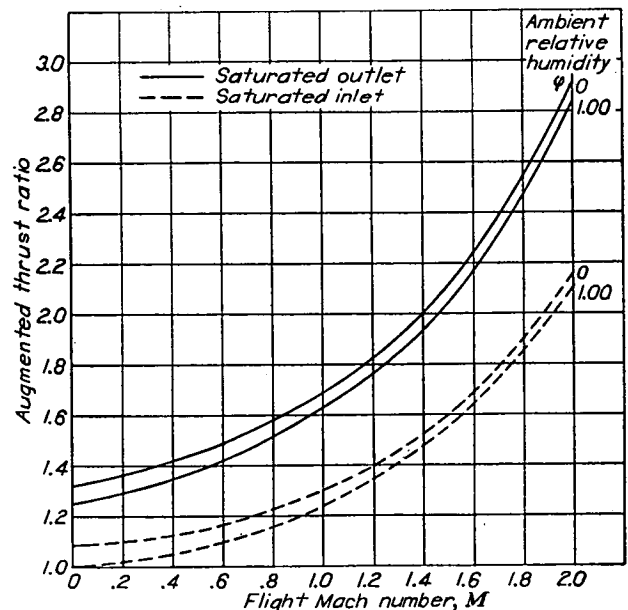


FIGURE 7.—Effect of atmospheric relative humidity on augmented thrust ratio for various flight Mach numbers. Altitude, sea level; inlet-diffuser efficiency, 0.80.

Effect of ambient relative humidity.—The augmented thrust ratio is shown in figure 7 as a function of flight Mach number for atmospheric relative humidities of 0 and 100 percent. The data shown are for standard sea-level condi-

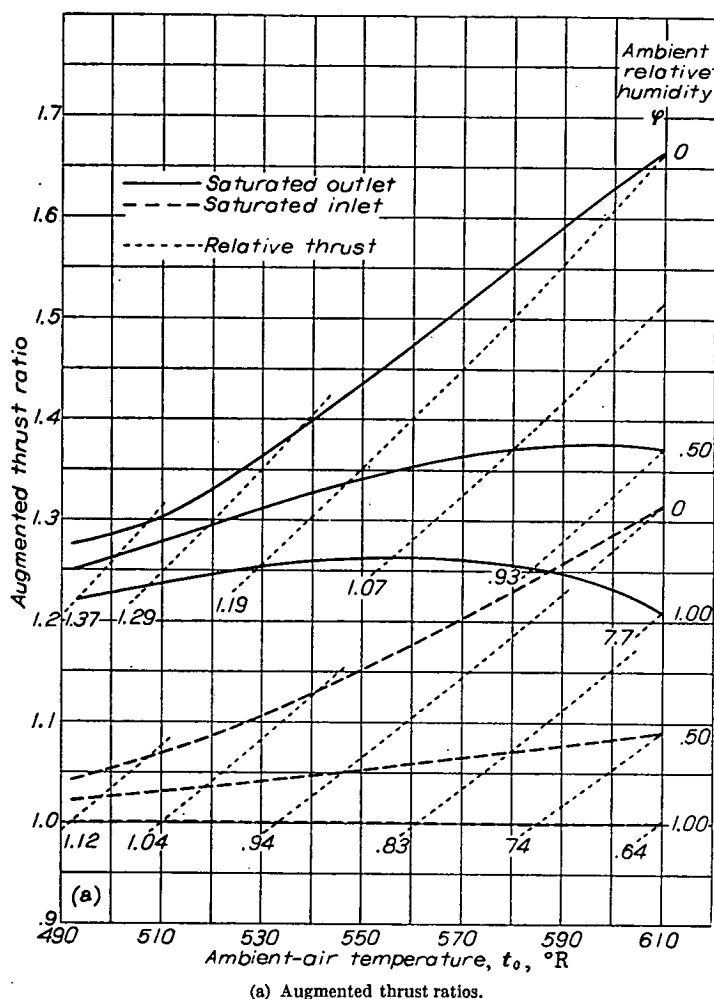


FIGURE 8.—Effect of ambient temperature on augmented thrust and augmented liquid ratios for various ambient relative humidities. Sea-level pressure; zero flight Mach number.

tions of temperature and pressure and an inlet-diffuser efficiency of 0.80. Results are presented for the case in which sufficient water is injected to saturate the compressor-inlet air and for the case in which the amount of water required to saturate the compressor-outlet air is injected. The ambient humidity has a relatively small effect on the augmented thrust ratio, especially at high flight Mach numbers.

Effect of ambient-air temperature for various relative humidities.—The effect of ambient temperature on augmented thrust and augmented liquid ratios for various ambient relative humidities is shown in figures 8(a) and 8(b), respectively. The data shown are for zero flight Mach number and standard sea-level atmospheric pressure. Results are shown for saturated compressor-inlet and -outlet conditions and relative-thrust curves showing constant values of the ratio of augmented thrust available to the thrust available from the engine operating without water injection at an atmospheric temperature of 519° R are included.

For an atmospheric relative humidity of 100 percent, no thrust augmentation may be obtained from compressor-inlet saturation inasmuch as no evaporative cooling is possible. As atmospheric relative humidity is decreased, the augmentation increases to about 8 percent for dry air at standard

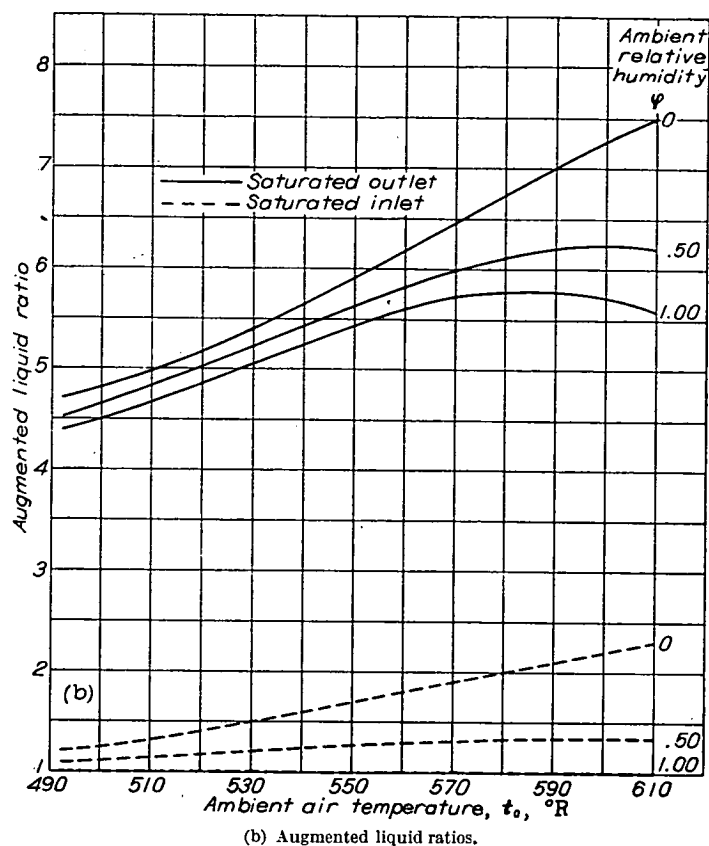


FIGURE 8.—Concluded. Effect of ambient temperature on augmented thrust and augmented liquid ratios for various ambient relative humidities. Sea-level pressure; zero flight Mach number.

sea-level temperature of 519° R. For low relative humidities, more evaporative cooling becomes possible as the ambient-air temperature is increased and at a temperature comparable to extreme summer heat (for example, 580° R) an augmented thrust ratio of 1.23 is available for dry air.

For saturated compressor-outlet air, the augmented thrust ratio for an atmospheric relative humidity of 1.00 is relatively unaffected by changes in atmospheric temperature. For an atmospheric relative humidity of 0.50, the augmented thrust ratio increases slightly and, for zero atmospheric relative humidity, a marked increase in the augmented thrust ratio occurs as the atmospheric temperature is increased. For an atmospheric temperature of 519° R, the augmented thrust ratio decreases from 1.33 to 1.24 as the atmospheric relative humidity increases from 0 to 1.00. At a temperature of 580° R, the decrease in augmented thrust ratio is from 1.55 to 1.26 for the same increase in atmospheric relative humidity.

As indicated by the relative-thrust curves in figure 8(a), the actual augmented thrust decreases as the atmospheric temperature is increased for all values of atmospheric relative humidity because of the marked decrease in normal engine thrust at increased compressor-inlet temperatures. The data of figure 8(a) indicate that thrust augmentation by water injection is one method for overcoming the marked decrease in take-off thrust accompanying high ambient-air temperatures. The augmented liquid ratios shown in figure 8(b) follow the same general trends as the augmented thrust ratios.

Effect of compressor efficiency.—As has been previously pointed out, the compressor efficiency has been decreased in

calculating the augmented engine performance with evaporation during mechanical compression in order to obtain closer agreement with experimental compressor performance with water injection. The effect of this decrease in compressor efficiency on the augmented thrust and augmented liquid ratios at various flight Mach numbers and for altitudes of sea level and 35,332 feet is shown in figure 9 for the condition of saturated compressor outlet. The results are shown for a constant value of compressor efficiency of 0.8 and for a value of compressor efficiency that has been decreased as indicated by the following relation:

$$\eta_c = 0.80 - \Delta X_c$$

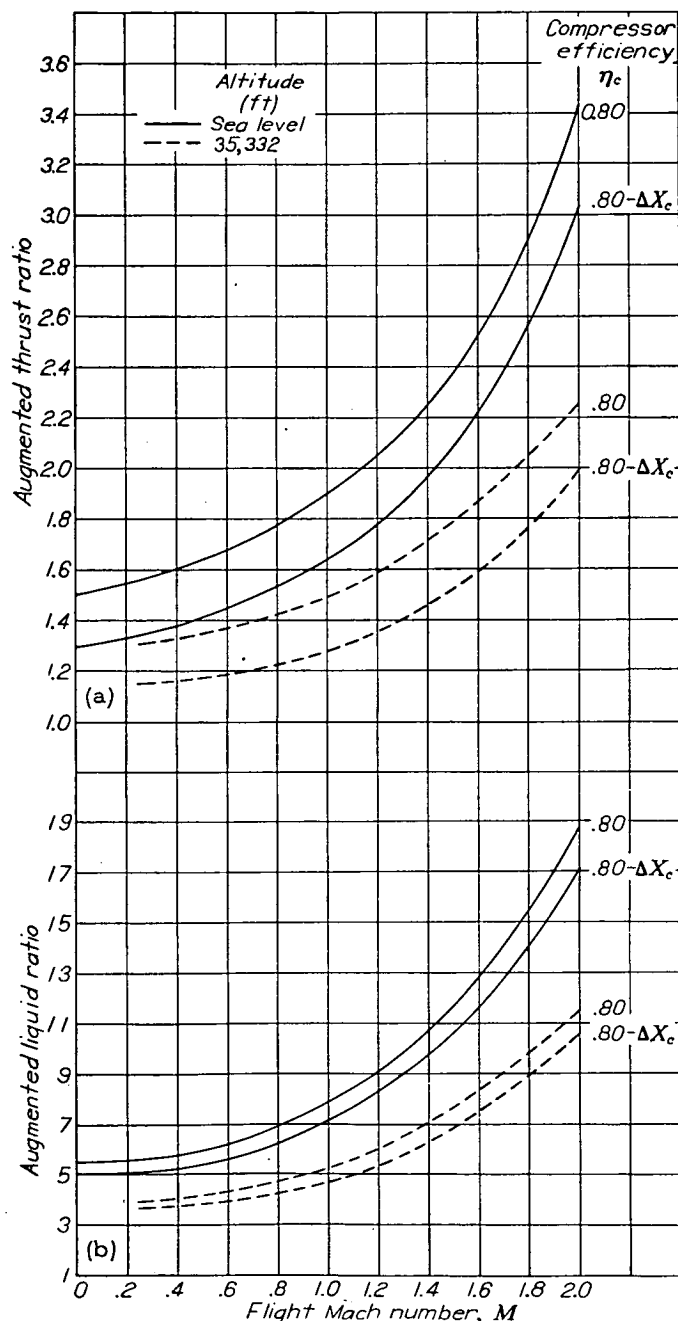


FIGURE 9.—Effect of compressor-efficiency assumption on augmented thrust and augmented liquid ratios for various flight conditions. Sufficient water injected to saturate compressor-outlet air.

where ΔX_c is equal to the change in water vapor-air ratio in the compressor.

From figure 9(a), it can be seen that the assumption of a constant compressor efficiency results in values of augmented thrust ratio considerably greater than the assumption of a compressor efficiency decreasing with the change in water vapor-air ratio in the compressor. For sea-level zero flight Mach number, the augmented thrust ratio is 1.29 when a decreased value of compressor efficiency is assumed, as compared with an augmented thrust ratio of 1.50 for the assumption of a constant compressor efficiency of 0.80. For a sea-level flight Mach number of 2.0, the corresponding values of augmented thrust ratio are 3.04 and 3.44. Similar results are obtained at an altitude of 35,332 feet. Additional calculations made for a constant compressor efficiency of 0.85 gave thrust ratios approximately equal to those for a constant efficiency of 0.80, which indicates that the magnitude of compressor efficiency has little effect on thrust augmentation if the efficiency for the normal and the augmented case is assumed to be the same.

The assumption of a constant value of compressor efficiency is shown in figure 9(b) to result in values of augmented liquid ratio somewhat higher than those obtained for the assumption of a decreased value of compressor efficiency.

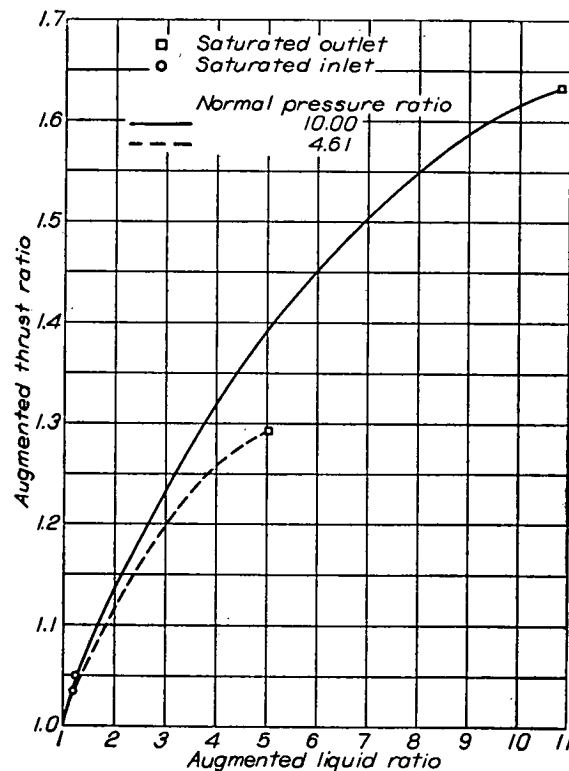


FIGURE 10.—Augmented thrust ratio as function of augmented liquid ratio for two normal compressor pressure ratios. Sea-level zero flight Mach number conditions.

Effect of normal compressor pressure ratio.—The augmented thrust ratio as a function of augmented liquid ratio for normal compressor pressure ratios of 4.61 and 10 is shown in figure 10. The data shown are for sea-level zero flight Mach number conditions. For any augmented liquid ratio, the engine having the high-pressure-ratio compressor gave the greatest augmented thrust ratio, with the difference

between the augmented thrust ratios for the low- and high-pressure-ratio engines increasing as the augmented liquid ratio increased. The use of a high-pressure-ratio compressor also permits operation at increased values of augmented liquid ratio and hence greater values of augmented thrust ratio. For a compressor pressure ratio of 4.61, the maximum value of augmented thrust ratio is 1.29 at an augmented liquid ratio of 5.01. For a compressor pressure ratio of 10, the maximum value of augmented thrust ratio is increased to 1.63 at an augmented liquid ratio of 10.84. The effect of compressor pressure ratio shown is based on a compressor efficiency reduced as a function of the amount of water evaporated, as previously mentioned. Because of the larger amounts of water that may be evaporated at the higher normal compressor pressure ratio, the reductions in compressor efficiency are correspondingly higher. The compressor efficiency was reduced from 0.80 for the normal condition to 0.70 for the full-augmented condition for the high-pressure-ratio compressor. Only experimental investigations of the effect of water injection on the efficiency of compressors having various pressure ratios will indicate the validity of this assumption. As previously mentioned, however, investigations of compressors having normal pressure ratios approximating that of the low-pressure-ratio compressor assumed in this investigation indicate the dependency of the compressor efficiency on the amount of water injected.

SUMMARY OF RESULTS

A psychrometric chart having total pressure as a variable and a Mollier diagram for a saturated mixture of air and water vapor were developed as aids in calculating the performance of compressors when water is injected at the compressor inlet. By use of the psychrometric chart and the Mollier diagram to calculate compressor performance, the performance of the water-injection method of thrust augmentation was evaluated over a range of flight and atmospheric conditions for a typical turbojet engine. The following results were obtained for an engine having a normal compressor pressure ratio of 4.61 at sea-level zero flight Mach number conditions when a nominal decrease in compressor efficiency with water injection was assumed:

1. For constant flight conditions, increasing the amount of water evaporated increased the ratio of augmented thrust to normal thrust and the ratio of augmented liquid flow to

normal fuel flow. For sea-level zero flight Mach number conditions and an atmospheric relative humidity of 0.50, increasing the amount of injected water from the amount required to saturate the compressor-inlet air to the amount required to saturate the compressor-outlet air increased the augmented thrust ratio from 1.035 to 1.29. The corresponding increase in augmented liquid ratio was from 1.18 to 5.01.

2. The augmented thrust and augmented liquid ratios increased rapidly as the flight Mach number was increased and decreased as the altitude increased. Although the thrust increases obtainable by compressor-inlet-air saturation were very small at low flight speeds, appreciable gains in thrust were possible at high flight Mach numbers. Thrust augmentation by compressor-inlet-air saturation can therefore be used to good advantage at high flight Mach numbers not only for centrifugal-flow engines but also for axial-flow engines that require special injection systems to avoid centrifugal separation at higher injection rates.

3. For standard atmospheric temperatures, the ambient relative humidity had only a small effect on the augmented thrust ratios produced at all flight speeds investigated.

4. For sea-level zero flight Mach number conditions and low ambient relative humidities, the augmented thrust ratio increased as the ambient-air temperature was increased. Thrust augmentation by water injection was therefore shown to be desirable for take-off use inasmuch as water injection tends to overcome the loss in take-off thrust normally occurring at high ambient temperatures. For very high atmospheric relative humidities, the effect of ambient temperature on augmented thrust ratio was small.

5. The augmented thrust ratio produced for a given set of conditions increased as the inlet-diffuser efficiency was decreased; however, because of the rapid deterioration of normal engine performance at decreased values of inlet-diffuser efficiency, the actual value of augmented thrust was decreased.

6. Increasing the normal engine compressor pressure ratio increased the maximum possible value of augmented liquid ratio and, hence, the maximum possible value of augmented thrust ratio.

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, June 1, 1950.

APPENDIX A

SYMBOLS

A	area, sq in.	v_g	specific volume of water vapor as given in reference 5 cu ft/lb
c_p	specific heat at constant pressure, Btu/(lb)(°R)	X	ratio of weight of water vapor to weight of air in mixture, lb water/lb air
F	net thrust, lb	ΔX	total amount of injected liquid water, lb water/lb air
f/a	fuel-air ratio, lb fuel/lb air	W	weight flow, lb/sec
g	acceleration due to gravity, 32.2 ft/sec ²	γ	ratio of specific heats
H	enthalpy, Btu/lb	δ	ratio of total pressure of mixture to standard sea-level static pressure
H_{corr}	corrected enthalpy for saturated mixture of air and water vapor plus liquid water not at 519° R, Btu/lb air	η	adiabatic efficiency
ΔH	enthalpy rise, Btu/lb	$\theta =$	$\frac{t_0}{519}$
h_a	enthalpy of air as given in reference 4, Btu/lb	ϕ	ambient relative humidity
h_f	enthalpy of liquid water as given in reference 5, Btu/lb	$\phi_a =$	$\int c_p dt/t$ for air, as given in reference 4, Btu/(lb)(°R)
h_g	enthalpy of water vapor as given in reference 5, Btu/lb	Subscripts:	
J	mechanical equivalent of heat, 778 ft-lb/Btu	a	air
K	compressor slip factor	c	compressor
M	flight Mach number	d	diffuser
P	total pressure, lb/sq in. absolute	e	exhaust gas—water vapor mixture
p	static and partial pressure, lb/sq in. absolute	f	liquid water
p_m	sum of partial pressures of air and water vapor, lb/sq in.	g	water vapor
R	gas constant, ft-lb/(lb)(°R)	i	ideal (isentropic)
S	entropy, Btu/(lb)(°R)	m	air—water vapor mixture
S_{corr}	corrected entropy for saturated mixture of air and water vapor plus liquid water not at 519° R, Btu/(lb air)(°R)	n	exhaust nozzle
s_f	entropy of liquid water as given in reference 5, Btu/(lb)(°R)	s	saturation
s_g	entropy of water vapor as given in reference 5, Btu/(lb)(°R)	t	turbine
T	total temperature, °R	0	ambient air
ΔT	total-temperature rise, F°	1	compressor inlet
t	static temperature, °R	2	end of portion of compression during which air re- mains saturated and after which no water is evapo- rated
Δt	static-temperature rise, F°	3	compressor outlet
V	flight velocity, ft/sec	4	turbine inlet
V_j	jet velocity, ft/sec	5	turbine outlet
V_T	compressor-rotor-tip velocity, ft/sec	6	exhaust-nozzle outlet
v	specific volume, cu ft/lb	(-)	between stations enclosed

Primed symbols are approximate values.

APPENDIX B

DERIVATION OF EQUATIONS FOR PSYCHROMETRIC CHART AND MOLLIER DIAGRAM

PSYCHROMETRIC CHART

The equations necessary for obtaining a psychrometric chart having total pressure (sum of partial pressures of air and water vapor) as a variable may be derived in the following manner:

From the general energy equation, the enthalpy of a mixture of air, liquid water, and water vapor per pound of air is

$$H_{a,0} + X_0 H_{g,0} + (X_1 - X_0) H_{f,0} = H_{a,1} + X_1 H_{g,1} \quad (B1)$$

When only the equilibrium between air and water vapor is considered, the term involving the enthalpy of liquid water $H_{f,0}$ disappears; in order to apply equation (B1) to a particular process, however, some assumption must be made as to the temperature for which the enthalpy of liquid water is zero. For the present analysis, the enthalpy of liquid water was assumed to be zero at 519° R. The results are therefore accurate for water injected at this temperature, with only small errors resulting for water injected at a somewhat different temperature. When it is assumed that the enthalpy of liquid water is zero, equation (B1) may be stated as

$$H_{a,0} + X_0 H_{g,0} = H_{a,1} + X_1 H_{g,1} \quad (B2)$$

The total enthalpy H for condition 0 or 1 is defined as

$$H = H_a + X H_g \quad (B3)$$

The water-air ratio X is evaluated as follows:

$$X = \frac{v_a}{v_g} = \frac{p_g}{R_g t_g} \frac{R_a t_a}{p_a} \quad (B4)$$

If the temperature of the air and vaporized water are equal and Dalton's law of partial pressure is valid, equation (B4) becomes

$$X = \frac{R_a}{R_g} \frac{p_g}{p_m - p_g} \quad (B5)$$

From the definition of relative humidity ϕ , relative pressure δ , and gas constant R ,

$$\phi = \frac{p_g}{p_{g,s}} \quad (B6)$$

$$\delta = \frac{p_m}{14.696} \quad (B7)$$

and

$$R_g = \frac{p_g v_g}{t_g} \quad (B8)$$

Substituting equations (B6) to (B8) in equation (B5) gives

$$X = \frac{R_a}{\left(\frac{p_g v_g}{t_g}\right)} \frac{\phi}{\delta} \frac{p_{g,s}}{14.696} \left(1 - \frac{\phi}{\delta} \frac{p_{g,s}}{14.696}\right) \quad (B9)$$

When the water-air ratio X is replaced in equation (B3) by the value expressed in equation (B9), the expression for total enthalpy H becomes

$$H = H_a + H_g \frac{R_a}{\left(\frac{p_g v_g}{t_g}\right)} \frac{\phi}{\delta} \frac{p_{g,s}}{14.696} \left(1 - \frac{\phi}{\delta} \frac{p_{g,s}}{14.696}\right) \quad (B10)$$

The data presented in figure 1 were obtained by means of equations (B3), (B9), (B10), and the thermodynamic data for air and water vapor contained in references 4 and 5, respectively. For convenience, the enthalpy of the saturated mixture of air and water vapor at a temperature of 519° R and a pressure of 14.696 pounds per square inch was arbitrarily fixed at 100 Btu per pound of air. In using equation (B10), the enthalpies of air and water vapor obtained from references 4 and 5, respectively, must be corrected in order to satisfy the conditions of zero enthalpy of liquid water at 519° R and the condition of an enthalpy of 100 Btu per pound of air for a saturated mixture of air and water vapor at a temperature of 519° R and a pressure of 14.696 pounds per square inch. The value of enthalpy of water vapor obtained from reference 5 must be decreased by 27.06 Btu per pound in order to satisfy the first condition, and the enthalpy of air obtained from reference 4 must be increased by 60.24 Btu per pound in order to satisfy the second condition. For very accurate work, the injection of water at a temperature other than 519° R can be corrected for simply by adding the enthalpy of liquid water to find the starting point of the process and, when liquid water remains, subtracting the enthalpy of this liquid water at the end of the process. These corrections are the same as the enthalpy corrections to the Mollier diagram discussed in appendix C.

The enthalpy and the specific volume of water vapor are functions of the vapor pressure of water as well as of the temperature. In the range of temperatures and pressures encountered at the compressor inlet of a turbojet engine, however, the enthalpy of water vapor and the product of vapor pressure and specific volume can be considered as functions only of temperature for all practical purposes. For any value of temperature, the enthalpy and the specific volume were evaluated for the maximum pressure that would exist for that temperature at the compressor inlet of a turbojet engine. This maximum pressure is the pressure that would be obtained by ideal ram compression from standard sea-level temperature and pressure to the desired temperature.

MOLLIER DIAGRAM

The general energy equation for a compression of 1 pound of air saturated with water vapor when no liquid is present at the end of compression becomes, similar to equation (B1),

$$H_{a,1} + X_1 H_{g,1} + (X_2 - X_1) H_{f,1} + \Delta H_c = H_{a,2} + X_2 H_{g,2} \quad (B11)$$

For the ideal conditions, the entropy remains constant; therefore

$$S_{a,1} + X_1 S_{g,1} + (X_2 - X_1) S_{f,1} = S_{a,2} + X_2 S_{g,2} \quad (B12)$$

If the enthalpy and the entropy of the excess liquid water are assigned values of zero at a particular temperature and the liquid water is introduced at this temperature, equations (B11) and (B12) become

$$H_{a,1} + X_1 H_{g,1} + \Delta H_c = H_{a,2} + X_2 H_{g,2} \quad (\text{B13})$$

and

$$S_{a,1} + X_1 S_{g,1} = S_{a,2} + X_2 S_{g,2} \quad (\text{B14})$$

In a saturated mixture of air and water vapor, the enthalpy per pound of air is

$$H = H_a + X H_g \quad (\text{B15})$$

and the entropy per pound of air is

$$S = S_a + X S_g \quad (\text{B16})$$

If equations (B15) and (B16) are plotted for a range of temperatures and pressures, the conditions of the mixture resulting from a compression of 1 pound of air that is continually saturated with water vapor may be determined. Equations (B15) and (B16) are the basis for the Mollier diagram presented. The base temperature for enthalpy and entropy of liquid water was chosen as 519° R. Thus, if any liquid water present is assumed to be at 519° R, no correction need be made for the enthalpy and the entropy of the liquid when using the Mollier diagram. The values of enthalpy and entropy were so adjusted that at standard sea-level conditions of temperature of 519° R and pressure of 14.696 pounds per square inch the enthalpy of the saturated mixture of air and water vapor is 100 Btu per pound of air and the entropy is 0.10 Btu/(lb air) (°R). The modified versions of equations (B15) and (B16), which were used in plotting the Mollier diagram, are

$$H = h_a + 60.24 + X(h_g - 27.06) \quad (\text{B17})$$

and

$$S = \phi_a - \frac{R_a}{J} \log_e \frac{(p_m - p_g)}{18.189} + X(s_g - 0.0536) \quad (\text{B18})$$

Values for h_a and ϕ_a were obtained from reference 4 and

are for a base temperature of 400° R; values for h_g and s_g were obtained from reference 5 and are for a base temperature of 32° F (492° R).

The weight ratio of water vapor to air in a saturated mixture may be determined by the relation

$$X = \frac{v_a}{v_g} \quad (\text{B19})$$

When the equation of state and Dalton's law of partial pressures are used, equation (B19) may be written

$$X = \frac{R_a t_a}{(p_m - p_g)(144 v_g)} \quad (\text{B20})$$

Equation (B20) was used for determining values of X for equations (B17) and (B18).

If liquid water is injected at any temperature other than 519° R, the enthalpy and the entropy of the water will not be zero and an error will be introduced when using the Mollier diagram. This error can be corrected by adjusting the initial enthalpy and entropy, as read from the diagram, to include the enthalpy and the entropy of the injected liquid water. The corrected enthalpy and entropy for the initial condition will be

$$H_{corr} = H + (h_f - 27.06)\Delta X \quad (\text{B21})$$

and

$$S_{corr} = S + (s_f - 0.0536)\Delta X \quad (\text{B22})$$

A similar correction, made by subtracting the enthalpy and the entropy of the liquid, can be performed at the end of the process if any liquid water remains. Values of H_f , which is $h_f - 27.06$, and S_f , which is $s_f - 0.0536$, are plotted in an insert on the Mollier diagram for various temperatures.

If liquid water is injected at a temperature different from the initial temperature of the saturated air, a slight increase in entropy occurs because of the mixing of the constituents as a new temperature equilibrium is reached. This increase in entropy may be neglected, however, without introducing appreciable error when using the Mollier diagram.

APPENDIX C

SAMPLE CALCULATIONS

The use of the Mollier diagram in calculating saturated compression processes is illustrated by the following sample calculations:

SAMPLE CALCULATION I

In calculating a compression process for an actual engine, the work input to the compressor is usually obtained on the basis of 1 pound of working fluid passing through the compressor. In order to use the Mollier diagram for a saturated compression process, the work input must be found on a basis of 1 pound of air (the unit on which the Mollier diagram is based). Corrections must also be made for the enthalpy and the entropy of the liquid water when the water injected is not at a temperature of 519° R if it is desired to maintain a high degree of accuracy in using the Mollier diagram. In order to illustrate these refinements, the following example is presented for an engine with a single-stage centrifugal compressor. The following conditions are assumed:

Ambient static temperature, t_0 , °R.....	519
Ambient static pressure, p_0 , lb/sq in.....	14.7
Flight Mach number, M	0.85
Flight velocity, V , ft/sec.....	949
Diffuser adiabatic efficiency, η_d	0.85
Ambient relative humidity, ϕ	0.50
Compressor-rotor-tip velocity, V_T , ft/sec.....	1500
Compressor slip factor, K	0.95
Compressor adiabatic efficiency, η_c	0.80

Enough liquid water is injected at the compressor inlet at a temperature of 540° R to saturate the air at this point and to keep the air saturated during the entire compression process. Find P_2 , T_2 , X_2 , and the amount of water evaporated.

The water-air ratio for saturated conditions at an ambient temperature of 519° R and a pressure of 14.7 pounds per square inch is, from the Mollier diagram,

$$X = 0.0106$$

For a relative humidity of 0.50, the value of X for the ambient air is

$$X_0 = \frac{0.0106}{2} = 0.0053$$

The static-temperature rise in the inlet diffuser due to ram compression is (value of $c_{p,m}$ obtained from fig. 3(a))

$$\Delta t_{(0-1)} = \frac{V^2}{2gJc_{p,m}} = \frac{(949)^2}{(2)(32.2)(778)(0.2416)} = 74.4 \text{ F}^\circ$$

and

$$T_1 = 519 + 74.4 = 593.4^\circ \text{ R}$$

When equation (B17) is evaluated for 593.4° R, the enthalpy at the compressor inlet without taking into account the enthalpy of the liquid water that is injected at this point is

$$\begin{aligned} H_1' &= h_a + 60.24 + X(h_g - 27.06) \\ &= 46.31 + 60.24 + 0.0053(1119.1 - 27.06) \\ &= 112.3 \text{ Btu/lb air} \end{aligned}$$

The compressor-inlet pressure is (γ_m obtained from fig. 3(a))

$$\begin{aligned} P_1 &= p_0 \left[\frac{\eta_d \Delta t_{(0-1)}}{t_0} + 1 \right]^{\frac{\gamma_m}{\gamma_m - 1}} \\ &= 14.7 \left[\frac{(0.85)(74.4)}{519} + 1 \right]^{\frac{1.398}{0.398}} \\ &= 22.0 \text{ lb/sq in.} \end{aligned}$$

When liquid water is injected at the compressor inlet, the temperature decreases but the pressure remains constant, as does the enthalpy if the enthalpy of the injected water is neglected. From the Mollier diagram at an enthalpy of 112.3 Btu per pound of air and a pressure of 22.0 pounds per square inch, the entropy at the compressor inlet, not considering the entropy of the liquid water, is

$$S_1' = 0.0949 \text{ Btu/(lb air)} (^\circ \text{R})$$

This point on the Mollier diagram could also have been determined from P_1 and T_1 where the value of T_1 is obtained from figure 1.

In order to obtain the enthalpy of the mixture in Btu per pound of air at the compressor inlet, correction must be made for the enthalpy of the liquid water that is injected at 540° R, which requires that the amount of water injected for saturation at the compressor outlet be known. Because X_2 is unknown, an approximate value X_2' must be assumed. The following approximation for sea-level pressure was obtained by inspection of calculations for a wide range of ambient-air temperatures, flight Mach numbers, and compressor pressure ratios and may be used to determine X_2' in order to eliminate or to simplify successive approximation methods:

$$X_2' = X_0 + 0.00065 \Delta H_{m,(0-2)} \theta^2$$

The enthalpy rise in the compressor is

$$\begin{aligned} \Delta H_{m,(1-2)} &= \frac{KV_T^2}{gJ} \\ &= \frac{(0.95)(1500)^2}{(32.2)(778)} \\ &= 85.3 \text{ Btu/lb mixture} \end{aligned}$$

and the enthalpy rise in the inlet diffuser is

$$\begin{aligned} \Delta H_{(0-1)} &= c_{p,a} \Delta t_{(0-1)} (1 + X_0) = (0.2416)(74.4)(1.0053) \\ &= 18.1 \text{ Btu/lb air} \end{aligned}$$

Therefore

$$X_2' = 0.0053 + 0.00065(18.1 + 85.3)(1)^2 = 0.0725$$

and

$$\Delta X'_{(0-2)} = 0.0725 - 0.0053 = 0.0672$$

These values will be used for X_2 and $\Delta X_{(0-2)}$ for the rest of the calculations.

From the insert on the Mollier diagram, the enthalpy and the entropy of the liquid water at 540° R are

$$H_f = 21.0 \text{ Btu/lb water}$$

and

$$S_f = 0.04 \text{ Btu/(lb water)} (^\circ\text{R})$$

The enthalpy and the entropy due to the liquid water are then

$$H_{f,1} = \Delta X_{(0-2)} H_f = (0.0672)(21.0) = 1.4 \text{ Btu/lb air}$$

and

$$S_{f,1} = \Delta X_{(0-2)} S_f = (0.0672)(0.04) = 0.0027 \text{ Btu/(lb air)} (^\circ\text{R})$$

The values for enthalpy and entropy at the compressor inlet corrected for the liquid water at a temperature of 540° R are

$$H_{1,corr} = H_1' + H_{f,1} = 112.3 + 1.4 = 113.7 \text{ Btu/lb air}$$

and

$$S_{1,corr} = S_1' + S_{f,1} = 0.0949 + 0.0027 = 0.0976 \text{ Btu/(lb air)} (^\circ\text{R})$$

When the enthalpy rise per pound of mixture in the compressor is known, it must be converted to enthalpy rise per pound of air in order to use the Mollier diagram. This conversion also requires a value for the water-air ratio at the compressor outlet. The ideal enthalpy rise per pound of air in the compressor is

$$\begin{aligned} \Delta H_{i,(1-2)} &= \Delta H_{m,(1-2)} (1 + X_2) \eta_c = (85.3)(1.0725)(0.80) \\ &= 73.2 \text{ Btu/lb air} \end{aligned}$$

and the final enthalpy for an isentropic compression is

$$H_{2,i} = H_{1,corr} + \Delta H_{i,(1-2)} = 113.7 + 73.2 = 186.9 \text{ Btu/lb air}$$

The compressor-outlet pressure P_2 read from the Mollier diagram at an enthalpy of 186.9 Btu per pound of air and an entropy of 0.0976 Btu/(lb air) ($^\circ\text{R}$) is

$$P_2 = 118.7 \text{ lb/sq in.}$$

The actual enthalpy per pound of air at the compressor outlet is

$$\begin{aligned} H_2 &= H_1 + \Delta H_{m,(1-2)} (1 + X_2) = 113.7 + \\ &\quad (85.3 \times 1.0725) = 205.2 \text{ Btu/lb air} \end{aligned}$$

When the Mollier diagram is read at an enthalpy of 205.2 Btu per pound of air and a pressure of 118.7 pounds per square inch, the temperature is

$$T_2 = 663.2^\circ \text{ R}$$

and

$$X_2 = 0.0730$$

This value of X_2 is very close to the approximate value that was used for X_2 and therefore no further correction is necessary. If this value of X_2 differed greatly from the assumed approximate value, it would be advisable to repeat the calculations using the value of X_2 just obtained.

The amount of water injected at the compressor inlet is

$$\Delta X_{(0-2)} = 0.0730 - 0.0053 = 0.0677 \text{ lb water/lb air}$$

In order to show the effect of corrections for liquid-water temperatures other than 519° R, the following table presents values of P_2 , T_2 , and X_2 that were obtained for liquid-water-injection temperatures of 519° R (no correction), 540° R (the sample calculation just completed), and for 620° R; all other conditions remained the same as in the sample calculation:

Temperature of injected water ($^\circ\text{R}$)	Pressure P_2 (lb/sq in.)	Temperature T_2 ($^\circ\text{R}$)	Water-air ratio X_2
519	119.2	662.7	0.0718
540	118.7	663.2	.0730
620	118.0	666.0	.0782

This table shows that, for most limits of accuracy for turbojet-cycle calculations, the enthalpy and the entropy of the liquid water may be neglected in the probable range of injected-water temperature.

SAMPLE CALCULATION II

The following example illustrates the calculation of compressor-outlet conditions when an amount of water less than the amount required for outlet saturation is injected at the compressor inlet.

The following conditions are assumed:

Ambient static temperature, t_0 , $^\circ\text{R}$	519
Ambient static pressure, p_0 , lb/sq in.	14.7
Flight velocity, V , ft/sec.....	0
Ambient relative humidity, ϕ	0.50
Compressor adiabatic efficiency, η_c	0.80
Compressor enthalpy rise, ΔH_c , Btu/lb mixture.....	85.3
Air weight flow, W_a , lb/sec.....	80.5
Water-injection rate at 519° R, lb/sec.....	4.0

From Sample Calculation I,

$$X_0 = 0.0053$$

and by evaluating equation (B17) for 519° R,

$$\begin{aligned} H_0 &= h_a + 60.24 X_0 (h_g - 27.06) \\ &= 28.53 + 60.24 + 0.0053(1087.6 - 27.06) \\ &= 94.4 \text{ Btu/lb air} \end{aligned}$$

From the Mollier diagram at an enthalpy of 94.4 Btu per pound of air and a pressure of 14.7 pounds per square inch, the entropy is

$$S_0 = 0.0890 \text{ Btu/(lb air)} (^\circ\text{R})$$

The change in water-air ratio ΔX due to the addition of the liquid water is

$$\Delta X_{(0-2)} = \frac{4}{80.5} = 0.0497$$

and

$$X_2 = X_0 + \Delta X_{(0-2)} = 0.0053 + 0.0497 = 0.0550$$

If the approximation from Sample Calculation II is used,

$$\Delta H_{m, (0-2)}' = \frac{\Delta X_{(0-2)}}{(0.00065)\theta^2} = \frac{0.0497}{(0.00065)1} = 76.5 \text{ Btu/lb mixture}$$

Using this approximate value of $\Delta H_{m, (0-2)}$ yields

$$\Delta H_{(0-2)}' = \Delta H_{m, (0-2)}'(1 + X_2) = (76.5)(1.0550) = 80.7 \text{ Btu/lb air}$$

Then

$$H_2' = H_0 + \Delta H_{(0-2)}' = 94.4 + 80.7 = 175.1 \text{ Btu/lb air}$$

and

$$H_{2, t}' = H_0 + \Delta H_{(0-2)}' \eta_c = 94.4 + (80.7)(0.80) = 159.0 \text{ Btu/lb air}$$

From the Mollier diagram at an enthalpy of 159.0 Btu per pound and an entropy of 0.0890 Btu per pound per $^{\circ}\text{R}$,

$$P_2' = 73.5 \text{ lb/sq in.}$$

From the Mollier diagram at a pressure of 73.5 pounds per square inch and an enthalpy of 175.1 Btu per pound air,

$$X_2' = 0.0542$$

This value of X_2' is lower than the desired value. If the value of $\Delta H_{m(0-2)}'$ is increased to 77.7 Btu per pound of air and the calculation repeated, the values obtained are

$$P_2 = 75.0 \text{ lb/sq in.}$$

$$X_2 = 0.0550$$

and

$$T_2 = 630.5^{\circ} \text{ R}$$

The rest of the compression process may be calculated as an adiabatic compression of the mixture of air and water vapor using the thermodynamic properties given in figure 3(a). From figure 3(a) at $X=0.0550$,

$$c_{p, m} = 0.2513 \text{ Btu/(lb)} (^{\circ}\text{R})$$

and

$$\gamma_m = 1.3916$$

Then

$$\Delta T_{(2-3)} = \frac{\Delta H_{m(2-3)}}{c_{p, m}} = \frac{(85.3 - 77.7)}{0.2513} = 30.2^{\circ} \text{ F}$$

and

$$\frac{P_3}{P_2} = \left[\left(\frac{\Delta T_{(2-3)}}{T_2} \right) \eta_c + 1 \right]^{\frac{\gamma_m}{\gamma_m - 1}} = \left[\left(\frac{30.2}{630.5} \right) 0.80 + 1 \right]^{\frac{1.3916}{0.3916}} = 1.143$$

The conditions at the compressor outlet are

$$P_3 = \left(\frac{P_3}{P_2} \right) P_2 = (1.143)(75.0) = 85.7 \text{ lb/sq in.}$$

$$T_3 = T_2 + \Delta T_{(2-3)} = 630.5 + 30.2 = 660.7^{\circ} \text{ R}$$

and

$$X_3 = X_2 = 0.0550$$

For a comparison of the effect of the amount of water evaporated on pressure and temperature at the compressor outlet, the results of Sample Calculation I (air saturated up to the compressor outlet for a water-injection temperature of 519° R), and of the cases of air saturated during one-half of the work input to the compressor, of air saturated up to the compressor inlet, and of no injection are shown in the following table:

($M=0.85$; $\Delta H_c=85.3 \text{ Btu/lb mixture}$; $\eta_c=0.80$)

Condition	Amount of water evaporated (lb water/lb air)	Outlet pressure P_3 (lb/sq in.)	Outlet temperature T_3 ($^{\circ}\text{R}$)
No injection.....	0	86.5	947
Air saturated up to compressor inlet.....	.0113	95.4	895
Air saturated through one-half of work input.....	.0384	108.0	778
Air saturated up to compressor outlet.....	.0665	119.2	663

APPENDIX D

EQUATIONS USED IN AUGMENTATION CALCULATIONS

The method of calculating compressor-outlet conditions when water is evaporated during compression is given in the section DISCUSSION OF CHARTS and illustrated in appendix C. The other equations used for the step-by-step calculations of engine performance are:

Inlet diffuser,

$$\frac{P_1}{P_0} = \left(\frac{\eta_d V^2}{2Jg c_{p,m} t_0} + 1 \right)^{\frac{\gamma_m}{\gamma_m - 1}}$$

Compressor (with no evaporation during compression),

$$\frac{P_3}{P_1} = \left(\frac{\eta_c K V_T^2}{T_1 g J c_{p,m}} + 1 \right)^{\frac{\gamma_m}{\gamma_m - 1}}$$

Turbine,

$$\frac{P_5}{P_4} = \left[1 - \frac{\Delta H_{m,c}(1 + X_3)}{\eta_t T_4 (1 + X_3 + f/a) c_{p,e}} \right]^{\frac{\gamma_e}{\gamma_e - 1}}$$

Thrust (for sonic jet velocity),

$$\frac{F}{A_t} = \frac{V_j W_t}{g A_t} + \frac{A_n}{A_t} (p_6 - p_0) - \frac{V W_a}{g A_t} (1 + X_0)$$

where

$$V_j = \sqrt{g \gamma_e R_e t_6}$$

$$t_6 = T_5 \left(\frac{2}{\gamma_e + 1} \right)$$

$$\frac{W_t}{A_t} = \left(\frac{P_4}{\sqrt{T_4 R_e / g \gamma_e}} \right) \left(\frac{2}{1 + \gamma_e} \right)^{\frac{\gamma_e + 1}{2(\gamma_e - 1)}}$$

$$\frac{A_n}{A_t} = \frac{W_t}{A_t} \frac{R_e t_6}{V_j p_6}$$

$$\frac{p_6}{P_5} = \left[1 - \left(\frac{\gamma_e - 1}{\gamma_e + 1} \right) \frac{1}{\eta_n} \right]^{\frac{\gamma_e}{\gamma_e - 1}}$$

and

$$\frac{W_a}{A_t} = \frac{W_t}{A_t} \left(\frac{1}{1 + X_3 + f/a} \right)$$

Thrust (for subsonic jet velocity),

$$\frac{F}{A_t} = V_j \frac{W_t}{g A_t} - \frac{V W_a}{g A_t} (1 + X_0)$$

where

$$V_j = \sqrt{2 \eta_n g R_e \frac{\gamma_e}{\gamma_e - 1} T_5 \left[1 - \left(\frac{p_0}{P_5} \right)^{\frac{\gamma_e - 1}{\gamma_e}} \right]}$$

and W_t/A_t and W_a/A_t are the same as for the case of sonic jet velocity. Values for $c_{p,m}$ and γ_m were obtained from figure 3(a) and values of $c_{p,e}$, R_e , and γ_e from figure 3(b). Values for f/a were obtained from reference 8.

REFERENCES

1. Hensley, Reece V.: Mollier Diagrams for Air Saturated With Water Vapor at Low Temperatures. NACA TN 1715, 1948.
2. Hall, Eldon W., and Wilcox, E. Clinton: Theoretical Comparison of Several Methods of Thrust Augmentation for Turbojet Engines. NACA Rep. 992, 1950.
3. Lundin, Bruce T.: Theoretical Analysis of Various Thrust-Augmentation Cycles for Turbojet Engines. NACA Rep. 981, 1950. (Formerly NACA TN 2083.)
4. Keenan, Joseph H., and Kaye, Joseph: Thermodynamic Properties of Air. John Wiley & Sons, Inc., 1945.
5. Keenan, Joseph H., and Keyes, Frederick G.: Thermodynamic Properties of Steam. John Wiley & Sons, Inc., 1936.
6. Pinkel, Benjamin, and Turner, L. Richard: Thermodynamic Data for the Computation of the Performance of Exhaust-Gas Turbines. NACA ARR 4B25, 1944.
7. English, Robert E., and Wachtl, William W.: Charts of Thermodynamic Properties of Air and Combustion Products from 300° to 3500° R. NACA TN 2071, 1950.
8. Turner, L. Richard, and Bogart, Donald: Constant-Pressure Combustion Charts Including Effects of Diluent Addition. NACA Rep. 937, 1949. (Formerly NACA TN's 1086 and 1655.)

REPORT 1007

HORIZONTAL TAIL LOADS IN MANEUVERING FLIGHT¹

By HENRY A. PEARSON, WILLIAM A. MCGOWAN, and JAMES J. DONEGAN

SUMMARY

A method is given for determining the horizontal tail loads in maneuvering flight. The method is based upon the assignment of a load-factor variation with time and the determination of a minimum time to reach peak load factor. The tail load is separated into various components. Examination of these components indicated that one of the components was so small that it could be neglected for most conventional airplanes; therefore, the number of aerodynamic parameters needed in this computation of tail loads was reduced to a minimum.

In order to illustrate the method, as well as to show the effect of the main variables, a number of examples are given.

Some discussion is given regarding the determination of maximum tail loads, maximum pitching accelerations, and maximum pitching velocities obtainable.

INTRODUCTION

The subject of maneuvering tail loads has received considerable attention both experimentally and theoretically. Theoretically, methods and solutions have been derived for determining the horizontal tail load following either a prescribed elevator motion (references 1 to 3) or an assigned load-factor variation (reference 4).

The first approach has been adopted into some of the load requirements where the type of elevator movement specified consists of linear segments whose magnitudes and rates of movement are governed by the assignment of a maximum initial elevator movement consistent with the pilot's strength. The rates of movement and the time the elevator is held before reversing are so adjusted that the design load factor will not be exceeded.

The results of reference 5 show, as is to be expected, that only when the aerodynamic force coefficients are accurately known from wind-tunnel tests can good agreement be obtained between measured and calculated tail loads. At the design stage, however, only general aerodynamic and geometric quantities are available and some of the more important stability parameters are not known accurately. Thus, the work involved in the solution for the tail load following a given elevator motion is not considered to be in keeping with the accuracy of the results obtained. Consequently, there appears to be a need for an abbreviated

design method of computing tail loads which, although incorporating approximations, will nevertheless be based on the theoretical considerations of the problem.

If the load-factor variation with time is specified and the corresponding tail load, elevator angles, and load distributions are subsequently determined, a simpler and equally rational approach to the tail-load problem can be made. Although this approach has been used to a limited degree (reference 4), several shortcomings have limited its use.

The purpose of this report is to develop further the load-factor or inverse approach and to present a method of computing horizontal tail loads which is comprehensive and generally simple. To this end, (1) the shape of the load-factor curve and the minimum time required to reach the peak load factor have been determined from an analysis of pull-up maneuvers that were available, (2) the minimum time required to reach the peak load factor has been determined from a theoretical analysis which is supported in some measure by statistical data obtained from a number of flight tests with airplanes of widely varying sizes, and (3) the equations relating the various quantities are presented.

SYMBOLS

b	wing span, feet; shape factor in equation (13)
b_t	tail span, feet
c	chord, feet
$\bar{c}$	mean aerodynamic wing chord, feet
C_L	lift coefficient (L/qS)
C_m	pitching-moment coefficient of airplane without horizontal tail (Mb/qS^2)
C_{m_t}	pitching-moment coefficient of isolated horizontal-tail surface
g	acceleration due to gravity, feet per second per second
I	pitching moment of inertia, slug-feet ²
k_r	radius of gyration about pitching axis, feet
K	empirical constant denoting ratio of damping moment of complete airplane to damping moment of tail alone
L	Lift, pounds
l	local lift at any spanwise station
m	airplane mass, slugs (W/g)
M	pitching moment, foot-pounds
n	airplane load factor at any instant

¹ Supersedes NACA TN 2078, "Horizontal Tail Loads in Maneuvering Flight" by Henry A. Pearson, William A. McGowan, and James J. Donegan, 1950.

N	maximum increment in load factor
q	dynamic pressure, pounds per square foot $\left(\frac{1}{2} \rho V^2\right)$
S	wing area, square feet
S_t	horizontal-tail area, square feet
t	time, seconds
t_1	time to reach peak of elevator deflection, seconds
V	airplane true velocity, feet per second
W	airplane weight, pounds
x_t	length from center of gravity of airplane to aerodynamic center of tail (positive for conventional airplanes), feet
y	spanwise dimension, feet
y^*	nondimensional spanwise dimension $\left(\frac{y}{b/2}\right)$
a, b, c	constants occurring in equations (13), (23), (26), and (30)
A, B, C, D, E	
K_1, K_2, K_3	constants occurring in basic differential equation (see equation (3))
λ	time to reach peak load factor, seconds
ρ	mass density of air, slugs per cubic foot
η_t	tail efficiency factor (q_t/q)
α	wing angle of attack, radians
$\bar{\alpha}$	average angle of attack of horizontal stabilizer, radians
α_t	tail angle of attack, radians
β	angle of sideslip, degrees
γ	flight-path angle, radians
θ	attitude angle, radians ($\alpha + \gamma$)
δ	elevator angle, radians
ϵ	downwash angle, radians $\left(\frac{d\epsilon}{d\alpha} \alpha\right)$
i_t	tail setting, radians

The notations $\dot{\alpha}$ and $\dot{\theta}$, $\ddot{\alpha}$ and $\ddot{\theta}$, and so forth, denote single and double differentiations with respect to t .

The symbol Δ represents an increment from the steady-flight datum value.

Subscripts:

0	initial or selected value
t	tail
max	maximum value
l_0	zero lift
geo	geometric
c	camber

METHODS

METHOD OF DETERMINING THE DYNAMIC TAIL LOAD

Basic equations of motion.—The simple differential equations for the longitudinal motion of an airplane for any

elevator deflection (see method given in reference 2) may be written as

$$m\dot{\gamma}V - \frac{dC_L}{d\alpha} \Delta\alpha q S - \left(\frac{dC_L}{d\delta}\right)_t \eta_t q S_t \Delta\delta = 0 \quad (1)$$

$$\frac{dC_m}{d\alpha} \Delta\alpha q \frac{S^2}{b} - \Delta L_t x_t + \frac{dC_{m_t}}{d\delta} \eta_t q \frac{S_t^2}{b_t} \Delta\delta - mk_Y^2 \ddot{\theta} = 0 \quad (2)$$

Equations (1) and (2) represent summations of forces perpendicular to the relative wind and of moments about the center of gravity. (See fig. 1 for direction of positive quantities.) Implicit in these equations are the following assumptions:

(1) In the interval between the start of the maneuver and the attainment of maximum loads, the flight-path angle does not change materially; therefore, the change in load factor due to flight-path change is small.

(2) At the Mach number for which computations are made, the aerodynamic derivatives are linear with angle of attack and elevator angle.

(3) The variation of speed during the maneuver may be neglected.

(4) Unsteady lift effects may be neglected.

By use of the relations $\theta = \gamma + \alpha$, $\dot{\theta} = \dot{\gamma} + \dot{\alpha}$, and $\ddot{\theta} = \ddot{\gamma} + \ddot{\alpha}$, equations (1) and (2) are reducible to the equivalent second-order differential equation

$$\ddot{\alpha} + K_1 \dot{\alpha} + K_2 \Delta\alpha = K_3 \Delta\delta \quad (3)$$

where

$$K_1 = \frac{\rho V}{2m} \left[\frac{dC_{L_t}}{d\alpha_t} \frac{S_t x_t^2}{k_Y^2} \eta_t \left(\frac{K}{\sqrt{\eta_t}} + \frac{d\epsilon}{d\alpha} \right) + \frac{dC_L}{d\alpha} S \right]$$

$$K_2 = -\frac{\rho V^2}{2m} \left\{ \frac{dC_m}{d\alpha} \frac{S^2}{k_Y^2 b} - \frac{dC_{L_t}}{d\alpha_t} \eta_t \frac{S_t x_t}{k_Y^2} \left[\left(1 - \frac{d\epsilon}{d\alpha} \right) + \frac{dC_L}{d\alpha} \frac{K}{\sqrt{\eta_t}} \frac{\rho S x_t}{2m} \right] \right\}$$

and

$$K_3 = \frac{\rho V^2}{2m} \left(-\frac{dC_{L_t}}{d\delta} \eta_t \frac{S_t x_t}{k_Y^2} + \frac{dC_{m_t}}{d\delta} \eta_t \frac{S_t^2}{b_t k_Y^2} - \frac{dC_{L_t}}{d\alpha_t} \frac{dC_{L_t}}{d\delta} \frac{K \eta_t^2}{\sqrt{\eta_t}} \frac{\rho x_t^2 S_t^2}{2m k_Y^2} \right)$$

In equations (1) and (2), $\Delta\alpha$, $\dot{\gamma}$, $\ddot{\theta}$, $\Delta\delta$, and ΔL_t will, in a given maneuver, vary with time. Using the relations between θ , γ , α , and their derivatives permits equation (2) to be rewritten as follows to give the increment in tail load:

$$\Delta L_t = \frac{dC_m}{d\alpha} \Delta\alpha q \frac{S^2}{b x_t} - \frac{mk_Y^2 \ddot{\alpha}}{x_t} - \frac{mk_Y^2 \ddot{\gamma}}{x_t} + \frac{dC_{m_t}}{d\delta} \eta_t q \frac{S_t^2}{b_t x_t} \Delta\delta \quad (4)$$

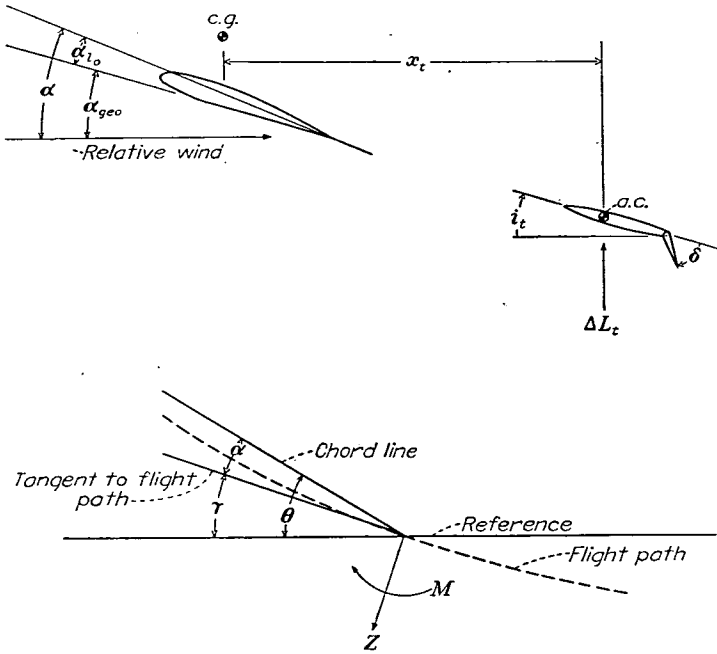


FIGURE 1.—Sign conventions employed. Positive directions shown.

In a still shorter form, equation (4) may be written as

$$\Delta L_t = \Delta L_{t_\alpha} + \Delta L_{t_{\ddot{\alpha}}} + \Delta L_{t_{\ddot{\gamma}}} + \Delta L_{t_\epsilon} \quad (5)$$

Equations (4) and (5) show that the tail-load increment (the increment above the steady-flight datum value) at any time is composed of four parts: ΔL_{t_α} , associated with the angle-of-attack change; $\Delta L_{t_{\ddot{\alpha}}}$, associated with angular acceleration about the flight path; $\Delta L_{t_{\ddot{\gamma}}}$, associated with angular acceleration of the flight path; and ΔL_{t_ϵ} , required to compensate for the moment introduced by change in camber of the horizontal-tail surface. The load ΔL_{t_ϵ} is generally small but in some extreme configurations may amount to 10 percent of the total increment and thus, for the present, it is retained in the development.

If the load-factor-increment variation with time Δn is known, then by the usual definition

$$\Delta n = \frac{dC_L}{d\alpha} \Delta \alpha \frac{q}{W/S} \quad (6)$$

so that

$$\Delta \alpha = \frac{\Delta n W/S}{\frac{dC_L}{d\alpha} q}$$

$$\dot{\alpha} = \frac{\dot{n} W/S}{\frac{dC_L}{d\alpha} q}$$

and

$$\ddot{\alpha} = \frac{\ddot{n} W/S}{\frac{dC_L}{d\alpha} q}$$

The following relation also exists between Δn and $\dot{\gamma}$:

$$\Delta n g = \dot{\gamma} V \quad (7)$$

so that

$$\ddot{\gamma} = \frac{\dot{n} g}{V} \quad (8)$$

When equations (6) and (8) are substituted into equations (4) and (5), the four tail-load components then become

$$\Delta L_{t_\alpha} = \frac{dC_m}{dC_L} \frac{W S}{b x_t} \Delta n \quad (9a)$$

$$\Delta L_{t_{\ddot{\alpha}}} = - \frac{W^2 k_Y^2}{g S q x_t} \frac{dC_L}{d\alpha} \ddot{n} \quad (9b)$$

$$\Delta L_{t_{\ddot{\gamma}}} = - \frac{W k_Y^2}{V x_t} \dot{n} \quad (9c)$$

$$\Delta L_{t_\epsilon} = \frac{dC_{m_t}}{d\delta} \eta_t q \frac{S_t^2}{b_t x_t} \Delta \delta \quad (9d)$$

Thus, if the variation of the load factor with time Δn and the geometric and aerodynamic characteristics of the airplane were known, the first three components of the tail load could be found immediately. The magnitude of the fourth component, that due to horizontal-tail camber, would follow from equation (3) in which the elevator angle is seen to be

$$\Delta \delta = \frac{\ddot{\alpha}}{K_3} + \frac{K_1}{K_3} \dot{\alpha} + \frac{K_2}{K_3} \Delta \alpha \quad (10)$$

Substitution into equation (10) of the values of $\Delta \alpha$, $\dot{\alpha}$, and $\ddot{\alpha}$ from equation (6) yields the value of the elevator angle at any instant

$$\Delta \delta = \frac{W/S}{K_3 \frac{dC_L}{d\alpha} q} (\ddot{n} + K_1 \dot{n} + K_2 \Delta n) \quad (11)$$

so that, finally, the fourth component is given as

$$\Delta L_{t_\epsilon} = \frac{dC_{m_t}}{d\delta} \eta_t \frac{S_t^2}{b_t x_t} \frac{W/S}{\frac{dC_L}{d\alpha} K_3} (\ddot{n} + K_1 \dot{n} + K_2 \Delta n) \quad (12)$$

The procedure outlined shows that the tail-load magnitude can be determined if the load-factor variation is known.

Types of load-factor variation.—The relation between the tail load, the geometric and aerodynamic characteristics,

and the load factor having been established, it is desirable to establish a load-factor variation which is reasonable as well as critical insofar as loads are concerned. The maximum value of load factor is usually specified; however, there are many possible variations for the shape. Regardless of the details of shape, the load factor may be considered to rise smoothly and continuously to a maximum, the rate of rise depending upon several variables. Beyond the maximum value of the load factor the return to initial conditions can, at the will of the pilot, be either gradual or rapid.

Experiments as well as theoretical studies have already indicated that the maneuver that combines maximum angular and linear accelerations causes critical loads in both the wing and tail. One such maneuver occurs when the maximum load factor is reached as rapidly as possible by using an initial elevator movement which is greater than that required to reach a given steady-trim value of the load factor. This initial elevator movement is followed by a rapid checking of the maneuver either by returning the elevator quickly to neutral or by reversing the controls.

The shape of the load-factor curve for such a maneuver may be expressed approximately by several analytic functions, one of which is

$$\Delta n = at^b e^{-ct} \quad (13)$$

By way of illustration, figure 2 shows details of the shape of the load-factor curve obtained with the use of equation (13) for which the constants have been adjusted so that an 8g peak is reached in 1 second. By further adjustment of the constants the load factor can, within certain limits, be made to rise to any specified peak and to diminish in any prescribed manner.

Because the positive slopes obtained from equation (13) are always greater than the negative slopes, the positive angular accelerations are greater than the negative ones. In general, this condition is true for most high g critical maneuvers performed by most classes of airplanes, but maneuvers may occasionally be performed for which the reverse may be true, particularly for small airplanes.

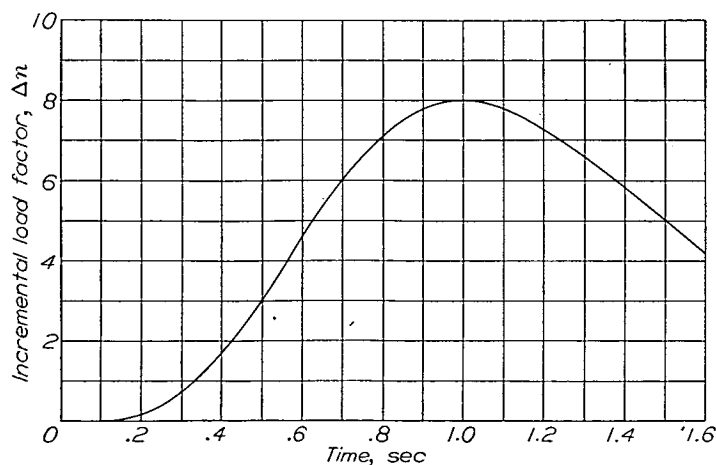


FIGURE 2.—Variation of load-factor increment. $\Delta n = 8(t)^{1/2} e^{-(t-1)}$.

Determination of constants.—From equations (9), (11), and (12) the required quantities relating to load factor are seen to be Δn , $\dot{n}$, and $\ddot{n}$. Since the increment Δn is to be given by

$$\Delta n = at^b e^{-ct} \quad (13)$$

then at maximum load factor

$$\dot{n} = 0 = \Delta n \left(\frac{b}{t} - c \right) \quad (14)$$

Thus $t = \frac{b}{c}$ at maximum load factor. Let $N = \Delta n_{max}$. Then

$$N = a \left(\frac{b}{c} \right)^b e^{-b} \quad (15)$$

so that

$$\frac{\Delta n}{N} = \left(\frac{t}{b/c} \right)^b e^{b - ct} \quad (16)$$

Let $\frac{b}{c} = \lambda$. Then

$$\frac{\Delta n}{N} = \left(\frac{t}{\lambda} \right)^b e^{b(1 - \frac{t}{\lambda})} \quad (17)$$

Equation (17) is in nondimensional form where λ is the time to reach the peak load factor and b is a constant.

When equation (17) is differentiated, the first and second derivatives become

$$\frac{\dot{n}\lambda}{N} = \frac{\Delta n}{N} b \left(\frac{1}{t/\lambda} - 1 \right) \quad (18)$$

and

$$\frac{\ddot{n}\lambda^2}{N} = \frac{\Delta n}{N} b^2 \left[\frac{\lambda^2}{t^2} \left(1 - \frac{1}{b} \right) - \frac{2\lambda}{t} + 1 \right] \quad (19)$$

In equations (17) to (19) the quantities N , λ , and b are now required in order to determine the variation of Δn , $\dot{n}$, and $\ddot{n}$. The value of N is immediately available from the required maneuver load factor; whereas the time to reach the peak load factor λ can be obtained from examination of available records or by specification. The constant b , as may be seen from equation (17), can best be described as a "shape" factor and has no particular physical significance.

The values of λ and b should be associated with a maneuver which produces maximum tail loads; therefore, the time λ to reach peak load factor should be the minimum possible consistent with possible pilot action and airplane response. The shape factor b should also be consistent with both of these.

In connection with the determination of the minimum time to reach peak load factor, the results shown in figure 3 for a typical airplane are informative. Figure 3 (a) shows the load-factor variation following several abrupt jump elevator movements. The load factor varies with the elevator position, but the time to reach peak load factor does not. Figure 3 (b) shows the load-factor variation for several abrupt hat-shape elevator impulses. Again the load factor is seen to

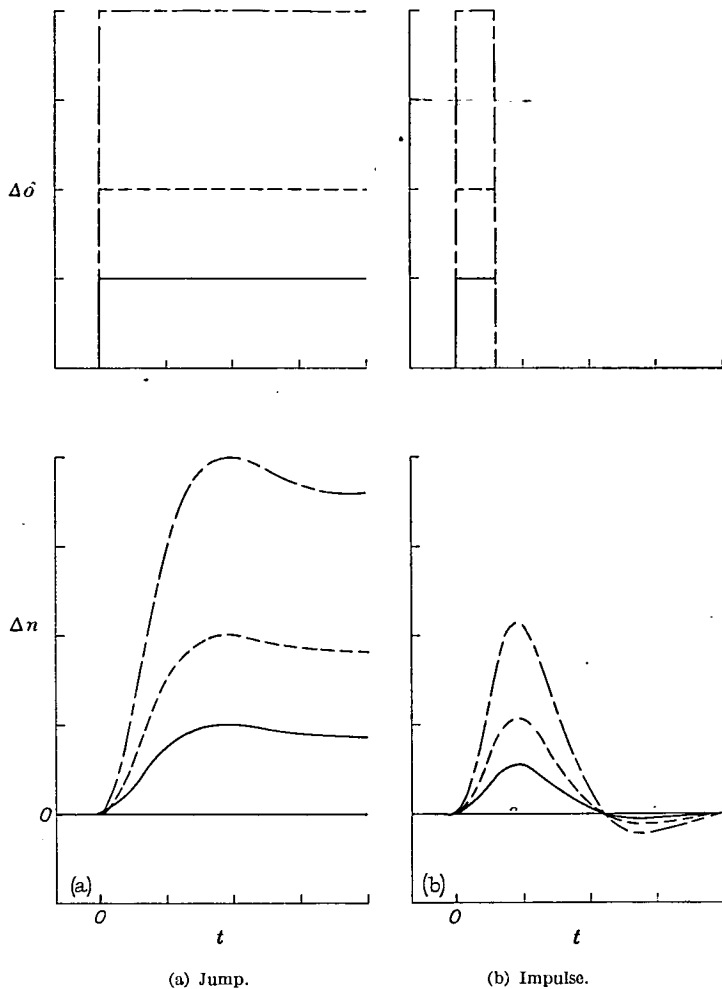


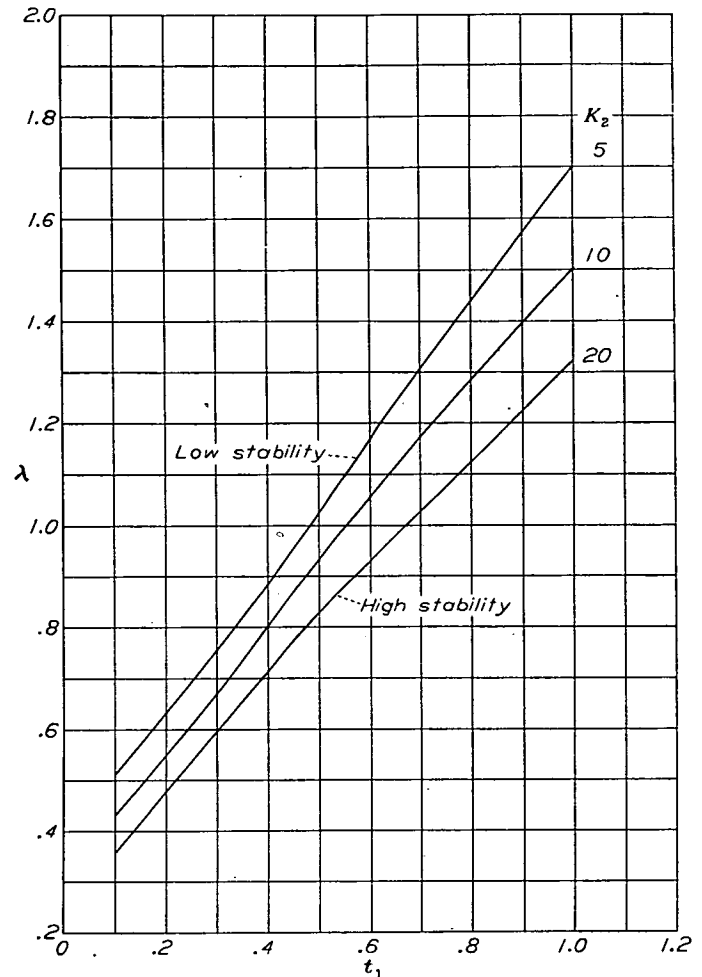
FIGURE 3.—Incremental-load factor variations following control movement.

vary with the amount of elevator deflection but the time to reach the peak value remains constant. Although the time to reach the peak load factor shown in figure 3 (b) remains constant, it is seen to be less than that shown in the previous case; therefore, an impulse elevator motion produces a smaller value of λ than the jump type.

Because of inertia and elasticity in the control system, the pilot cannot move the elevator instantaneously but requires some finite time t_1 to do so. A possible critical type of elevator impulse thus appears to be one which increases linearly to maximum and decreases at the same rate to zero. In order to determine the minimum time to reach peak load factor associated with such a variation, the equation of motion (equation (3)) has been solved for the triangular elevator impulse for airplanes of various static stabilities and damping.

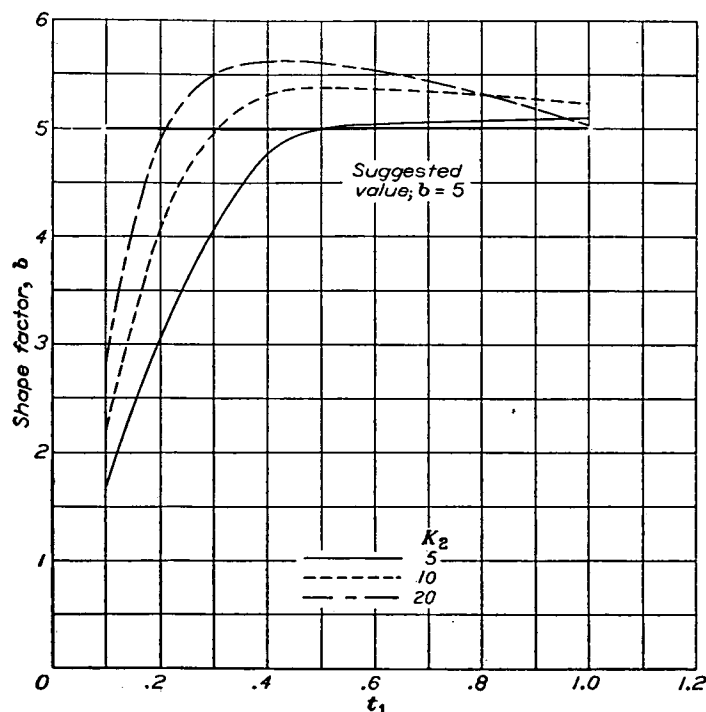
The results of the computations are given in figure 4 in which the minimum time λ to reach peak load factor is plotted against the time t_1 required to deflect the elevator.

For completeness the curves of figure 4 are labeled for the actual values of K_2 employed in the computation as well as for relative values of stability. By a series of computations the damping term K_1 was found, as was to be expected,

FIGURE 4.—Variation of λ with t_1 .

to have only a secondary effect on λ . The curves apply to an average value of the damping constant. The upper curve, labeled "low stability," should be associated with rearward center-of-gravity positions (that is, low static margin) in combination with one or both of the following: low dynamic pressure or heavy airplanes. The lower curve, labeled "high stability," would be associated with forward center-of-gravity positions in combination with one or both of the following: high dynamic pressure or light airplanes. It is seen that λ increases almost linearly with t_1 and also increases when the restoring forces are reduced, that is, when the stability is reduced.

A preliminary value of the shape factor b (required in equations (17) to (19)) was initially determined from flight records of typical impulse maneuvers by fitting curves of the type given by equation (13) through several points of the actual time histories and determining the constants. The results of this first step were then modified by the results of the same computations which had been made to determine λ , and the variation of b with t_1 given in figure 5 was obtained. Since the b factor is not found to be critical, an average value of 5.0 is suggested, although as a refinement the values from figure 5 may be used.

FIGURE 5.—Variation of shape factor b with t_1 .

The question of the value of t_1 to use is one which must be solved either from experience or from a knowledge of the characteristics of the controls and the control system. For conventional airplanes having the usual amounts of boost and no rate restrictors, the following values of t_1 are suggested as representative:

	t_1
Fighters or small civil airplanes with weight limit from about 500 to 12,000 pounds, seconds.....	0.20
Two-engine airplanes with weight limit from 25,000 to 45,000 pounds, seconds.....	0.25
Four-engine airplanes with weight limit from 50,000 to 80,000 pounds, seconds.....	0.30
Airplanes with weight limit above 100,000 pounds, seconds.....	0.40

The minimum time λ given in figure 4 was actually established separately from the adopted load-factor variation; therefore, in applying the inverse method, the derived elevator impulse would not be expected to agree in detail with the "tent" type impulse used in the derivation.

The first three tail-load components can now be computed by inserting the values of Δn , $\dot{n}$, and $\ddot{n}$ from equations (17) to (19) into equation (9) and using appropriate values of λ from figure 4. In order to facilitate this computation, curves of $\Delta n/N$, $\dot{n}\lambda/N$, and $\ddot{n}\lambda^2/N$ plotted against t/λ are given in figure 6 for the suggested value of $b=5$. Actually, in order to apply the results of figure 6 it is convenient to find first the components ΔL_{t_a} , and so forth, in terms of the nondimensional time t/λ and then to convert to time t in seconds. In order either to compute the fourth component or to obtain the elevator angles for use in chord loading, the constants K_1 , K_2 , and K_3 of equation (3) must also be known.

Thus, in terms of t/λ and the ordinates of figure 6, the various tail-load components are

$$\Delta L_{t_a} = \frac{dC_m}{dC_L} \frac{WSN}{bx_i} \quad (\text{Ordinate of fig. 6(a)}) \quad (20a)$$

$$\Delta L_{t_{\ddot{a}}} = \frac{-W^2 k_Y^2}{gSq x_i} \frac{N}{dC_L} \frac{1}{\lambda^2} \quad (\text{Ordinate of fig. 6(c)}) \quad (20b)$$

$$\Delta L_{t_{\dot{a}}} = \frac{-Wk_Y^2}{Vx_i} \frac{N}{\lambda} \quad (\text{Ordinate of fig. 6(b)}) \quad (20c)$$

$$\Delta L_{t_c} = \frac{dC_{m_t}}{d\delta} \eta_i \frac{S_i^2}{b_i x_i} \frac{W}{S} \frac{N}{K_3} \frac{dC_L}{d\alpha} \left[\frac{\text{Ordinate of fig. 6(c)}}{\lambda^2} + \frac{K_1(\text{Ordinate of fig. 6(b)})}{\lambda} + K_2(\text{Ordinate of fig. 6(a)}) \right] \quad (20d)$$

The constants K_1 , K_2 , and K_3 defined previously herein are the same as those given in reference 2, except for changed signs caused by specifying x_i as positive.

The conversion to time t is made by multiplying values of the base scale t/λ by λ .

Sample calculations for incremental tail loads.—The results of several examples are given to illustrate not only the method but also the effect of each of a number of variables on the incremental tail load of a fighter-type airplane, the geometric and aerodynamic characteristics of which are given in table I. In order to illustrate the effect of static stability, results have been computed for three center-of-gravity positions with the assumption that an 8g recovery is made at 19,100 feet from a vertical dive at an airspeed of 400 miles per hour. In order to illustrate the effect of the time of the elevator impulse on the tail load, computations were carried out at one of the center-of-gravity positions for several values of t_1 . The cases considered and the airplane characteristics are given in table I.

TABLE I.—AIRPLANE CHARACTERISTICS

(a) Geometric.	
Gross wing area, S , square feet.....	300
Gross horizontal-tail area, S_i , square feet.....	60
Airplane weight, W , pounds.....	12,000
Wing span, b , feet.....	41
Tail span, b_i , feet.....	16
Radius of gyration, k_Y , feet.....	6.4
Distance from aerodynamic center of airplane less tail to aerodynamic center of tail, x_i , feet:	
Center of gravity, 29 percent M. A. C.....	20.0
Center of gravity, 24 percent M. A. C.....	20.3
Center of gravity at aerodynamic center.....	21.0
(b) Aerodynamic.	
Slope of airplane lift curve, $dC_L/d\alpha$, radians.....	4.87
Slope of tail lift curve, $dC_{L_i}/d\alpha_i$, radians.....	3.15
Downwash factor, $d\epsilon/d\alpha$	0.54
Tail efficiency factor (q_i/q), η_i	1.00
Empirical airplane damping factor, K	1.1
Elevator effectiveness factor, $dC_{L_i}/d\delta$, radians.....	1.89
Rate of change of tail moment with camber due to elevator angle, $dC_{m_i}/d\delta$, radian.....	-0.57
Rate of change of moment coefficient with angle of attack for airplane less tail, $dC_{m_t}/d\alpha$, radians:	
Center of gravity, 29 percent M. A. C.....	0.625
Center of gravity, 24 percent M. A. C.....	0.403
Center of gravity at aerodynamic center.....	0.000

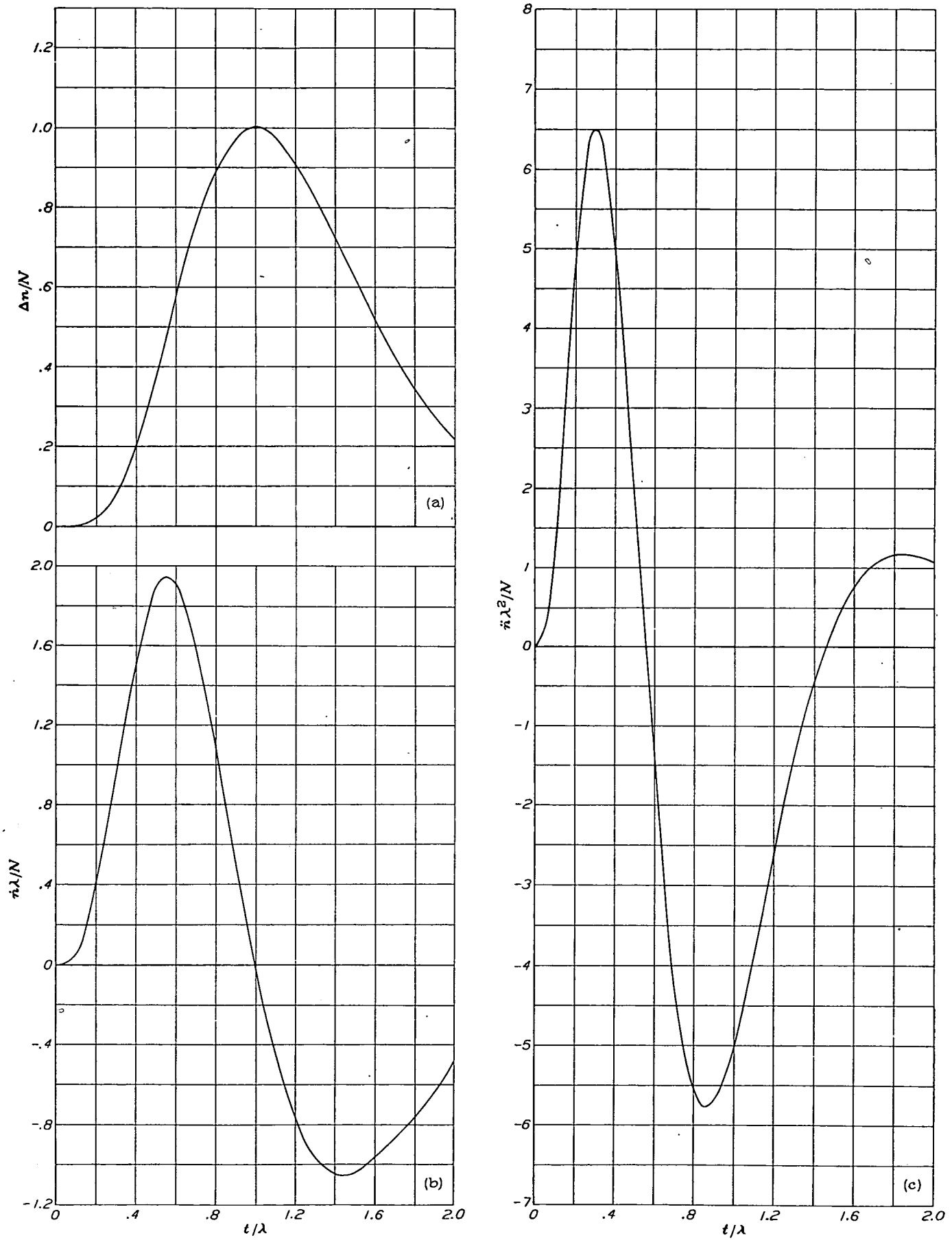


FIGURE 6.—Variation of incremental-load-factor curves with time ratio.

The specified conditions for the sample computations are given in table II. The computed results for tail components are given in figures 7 and 8. Figure 7 gives results for varying the center of gravity and figure 8 gives similar results for varying t_1 . The tail-load components are computed from equations (20) and the derived elevator angles from equation

(11). If the increment in tail load due to camber and the incremental elevator angle are not required, the values of K need not be computed and the computations are considerably shortened. Figures 7 and 8 show that a maximum error of only about 4 percent is introduced by this omission.

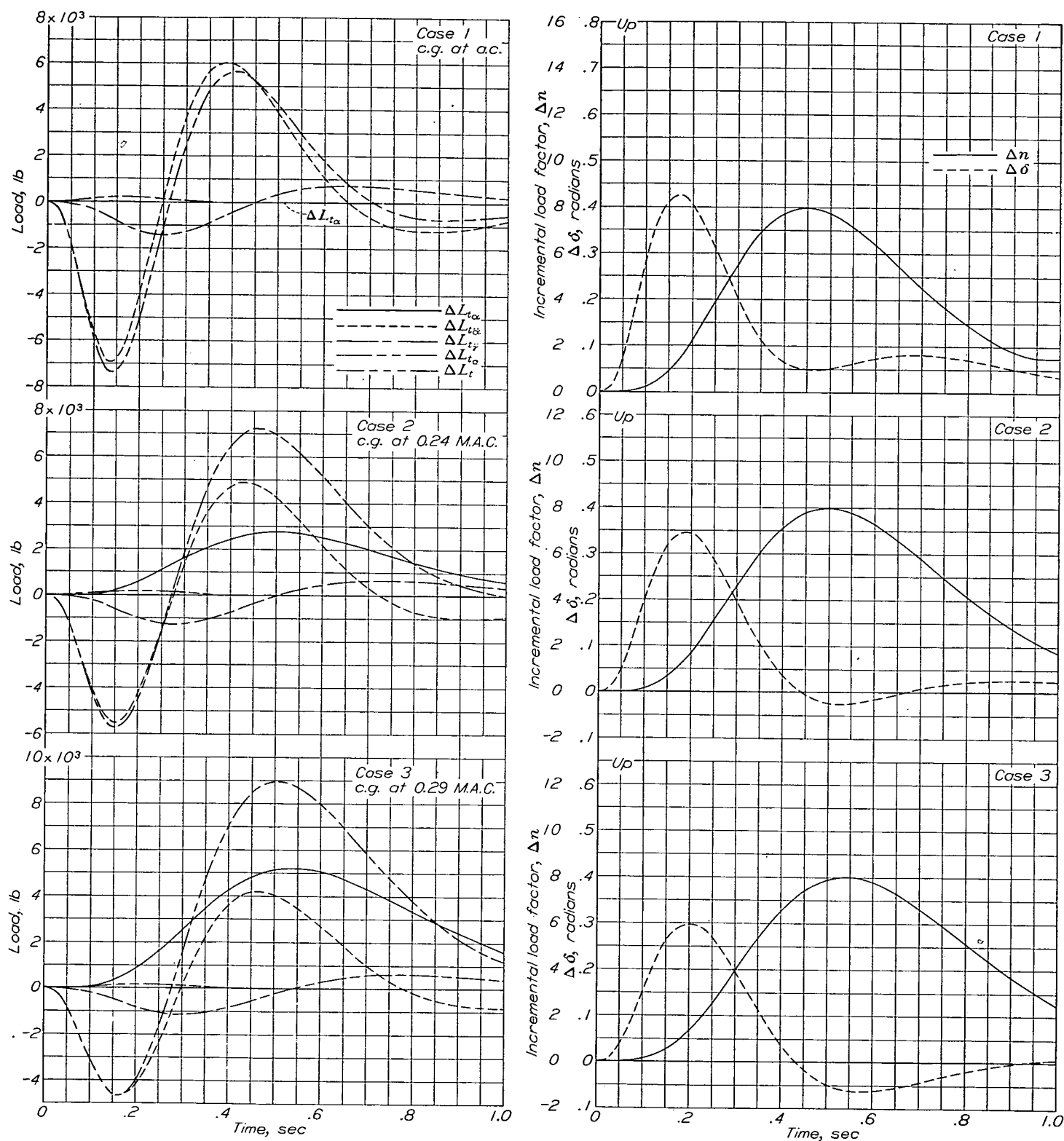


FIGURE 7.—Effect of center-of-gravity position on incremental-tail-load components.

TABLE II.—SPECIFIED CONDITIONS OF SAMPLE PROBLEM

Increment in load factor.....	8.0
Altitude, feet.....	19,100
Air density, slug per cubic foot.....	0.001306

Case	c. g. (percent M. A. C.)	t_1	K_1	K_2	K_3	λ (fig. 4)
1	a. c.	0.2	4.93	30.4	-33.4	0.45
2	24	.2	4.72	16.2	-32.2	.50
3	29	.2	4.61	8.45	-31.7	.56
4	24	.4	4.72	16.2	-32.2	.77
5	24	.6	4.72	16.2	-32.2	1.02

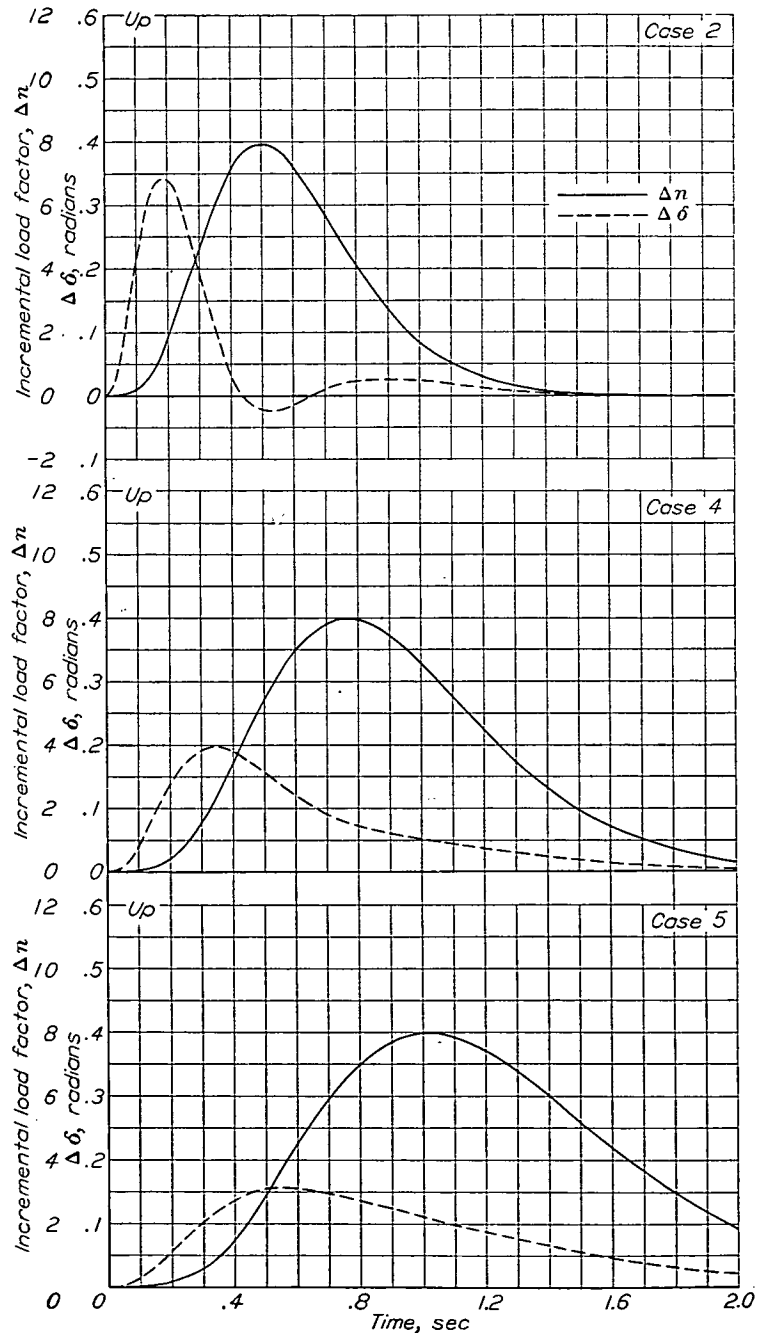
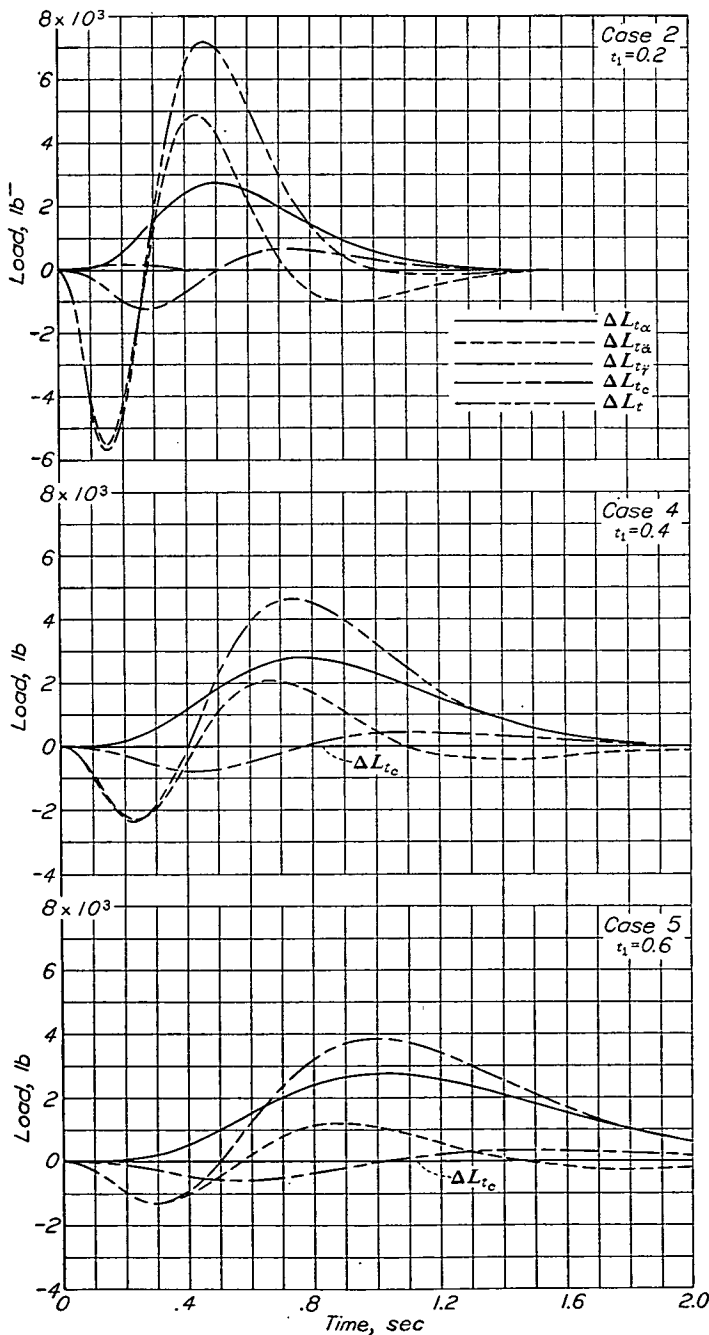
METHOD OF DETERMINING THE TOTAL TAIL LOAD

The initial or steady-flight tail load and elevator angles to which the computed incremental values are to be added

must also be determined. In steady flight, the horizontal tail furnishes the moment required to balance the moments from all other parts of the airplane so that the initial load may be written as

$$L_{t_0} = \frac{C_{m_0} q S c}{x_t} + \frac{dC_m}{dC_L} \frac{WS}{bx_t} \cos \gamma_0 \quad (21)$$

Thus the total tail load at any time in a maneuver is composed of the four previously mentioned parts plus the components given in equation (21). Only the first term of equation (21) represents a new type of load because the second term is a load of the type given by equation (9a) or equation (20a) and its effect may be immediately included in the computations by multiplying the ordinates of figure 6 by $N + \cos \gamma_0$ instead of by N .

FIGURE 8.—Effect of t_1 on incremental-tail-load components. Center of gravity, 0.24 $\bar{c}$.

The initial elevator angle required to balance the airplane in steady flight varies with airplane C_L and center-of-gravity position so that, in general, δ_0 must be obtained from wind-tunnel data. Without results of wind-tunnel tests, a rough rule which can be used as a guide at the design stage in determining the elevator position is that the final elevator setting will be so adjusted by repositioning of the stabilizer setting during acceptance tests that it will be near a zero position at the cruising speed and at the most prevalent center-of-gravity position.

METHOD OF DETERMINING MAXIMUM VALUES

Maximum tail loads and angular accelerations.—The method outlined enables a point-by-point evaluation to be made of the quantities that determine the tail load. Such detail may often be unnecessary and the procedure may be shortened by evaluating only those points near the load peaks or, alternatively, by accepting an approximation to the results. One such approximation which may be made is to balance the airplane at the combinations of load factor and angular acceleration which would result in maximum up and down tail loads.

Figure 7 shows that the maximum down tail load in a pull-up occurs near the start of the maneuver and before appreciable load factor is reached. This maximum load is practically coincident with the negative maximum in the $L_{t\alpha}$ tail-load component. Since, for a given configuration, this component increases as the center of gravity is moved forward and since the steady-flight down load increases with speed, the maximum down tail load in a pull-up occurs at the highest design speed in combination with the most forward center-of-gravity position.

Figures 7 and 8 show that at the time of the maximum down-tail-load increment the elevator is near but has not quite reached its peak position. Also at the time of maximum up-tail-load increment the elevator is near its zero position, although it may be on either side of this position depending upon the stability and the time t_i . These results suggest that the maximum down load for the elevator and the hinge brackets would occur with the airplane center of gravity well forward and at the start of the maneuver. The maximum load for the stabilizer is likely to occur at the peak load factor.

Figure 7 also shows that the up tail load occurs near the peak of the component $L_{t\alpha}$ as well as near the positive maximum peak in the component $L_{t\alpha}$. Since the component $L_{t\alpha}$ increases as the center of gravity is moved rearward and since a decrease in speed generally reduces the initial down load, the maximum up tail load occurs at the upper left-hand corner of the V - n diagram for the most rearward center-of-gravity position.

The maximum tail load in a pull-up maneuver may be written as

$$L_{t_{max}} = \frac{C_{m0}qSc}{x_t} + \frac{dC_m}{dC_L} \frac{WS}{bx_t} (n + \cos \gamma_0) - \frac{I\ddot{\theta}}{x_t} \quad (22)$$

where the sum of the second and third terms is to be a maximum in the maneuver. From the previous discussion

the load-factor increment at maximum down load is nearly zero and at maximum up load it is nearly equal to N so that if the positive and negative values of $\ddot{\theta}_{max}$ can be determined, a relatively simple method for determining maximum loads is available.

Since by definition $\ddot{\theta} = \ddot{\alpha} + \ddot{\gamma}$, an expression for angular acceleration can be derived from equations (6) and (7) and written in the form

$$\ddot{\theta} = \frac{W/S}{C_{L\alpha}q} \ddot{n} + \frac{g}{V} \dot{n}$$

The maximum angular acceleration can be approximated by

$$\ddot{\theta}_{max} = B \frac{W/S}{\frac{dC_L}{d\alpha} q} \frac{N}{\lambda^2} + C \frac{gN}{V\lambda} \quad (23)$$

For the maximum positive pitching acceleration, B is the maximum positive ordinate in figure 6 (c) and C is the ordinate of figure 6 (b) at a value of t/λ for which B was determined. Thus, B is 6.5 and C is 0.95 for this example.

For the maximum negative pitching acceleration, B is the maximum negative ordinate in figure 6(c) and C is the ordinate of figure 6(b) at a value of t/λ for which B was determined. Thus, B is -5.8 and C is 0.80. For use in equation (23) the values of λ for the maneuver are available from figure 4 and the other quantities are available from the conditions of the problem. The maximum loads can be given by the following equations:

For maximum up tail load in the pull-up:

$$L_{t_{max+}} = \frac{C_{m0}qSc}{x_t} + \frac{dC_m}{dC_L} \frac{WS}{bx_t} (N + 1.0) + \frac{I_Y N}{x_t \lambda} \left(\frac{5.8W/S}{\frac{dC_L}{d\alpha} q \lambda} - \frac{0.8g}{V} \right) \quad (24a)$$

For maximum down tail load in the pull-up:

$$L_{t_{max-}} = \frac{C_{m0}qSc}{x_t} + \frac{dC_m}{dC_L} \frac{WS}{bx_t} \cos \gamma - \frac{I_Y N}{x_t \lambda} \left(\frac{6.5W/S}{\frac{dC_L}{d\alpha} q \lambda} + \frac{0.95g}{V} \right) \quad (24b)$$

For push-downs to limit load factor, equations (24a) and (24b) still apply with changed signs for N and changed directions for $L_{t_{max+}}$ and $L_{t_{max-}}$. A question arises as to whether the maximum down tail load at the start of a pull-up with forward center-of-gravity position is greater than that which would occur when pulling up from a negative load-factor condition with the center of gravity in the most rearward position. This can be determined only by computing both cases and seeing which is the larger.

Maximum value of angular velocity.—The maximum value of the pitching angular velocity in the pull-up may also be found in a manner similar to that used to obtain the maximum angular acceleration. Since $\dot{\theta} = \dot{\alpha} + \dot{\gamma}$ and the relations involving these quantities in terms of load factor

are given by equations (6) and (7), the following equation may be written:

$$\dot{\theta} = \frac{nW/S}{dC_L} + \frac{\Delta n}{V} \frac{g}{q} \quad (25)$$

The maximum angular velocity may be approximated by

$$\dot{\theta} = D \frac{N}{\lambda} \frac{W/S}{dC_L} + E \frac{Ng}{V} \quad (26)$$

where D is the maximum positive ordinate in figure 6 (b) and E is the ordinate of figure 6 (a) at a value of t/λ for which D was determined. Thus D , for this example, is 1.95 and E is 0.48.

In the steady turn or pull-up at constant g , the angular velocity is usually given by the expression $\dot{\theta} = 1.0 \frac{gN}{V}$. The difference between the factor 1.0 of this expression and the factor 0.48 of equation (26) is more than made up by the angle-of-attack component of the angular velocity.

APPROXIMATE METHOD OF DETERMINING LOAD DISTRIBUTION

Symmetrical loading.—The spanwise distribution of the total load can be formulated with various degrees of exactness. If information regarding details of the angle-of-attack distribution across the span were available, then an exact solution could be obtained for the loading with the use of existing lifting-surface methods. The following method may be used as a first approximation to the solution.

From the total tail load, the total tail lift coefficient C_{L_t} can readily be found. The average effective angle of attack $\bar{\alpha}$ of the stabilizer portion is given in the definition

$$C_{L_t} = \int_0^1 c_{l_\alpha} \bar{\alpha} \frac{c}{c} dy^* + \int_0^1 c_{l_\delta} (\delta_0 + \Delta\delta) \frac{c}{c} dy^* \quad (27)$$

where only $\bar{\alpha}$ is assumed as unknown and c_{l_α} and c_{l_δ} may be taken as the rates of change of section lift coefficient with α and δ , respectively.

Thus, for constant elevator angle across the span,

$$\bar{\alpha} = \frac{C_{L_t} - (\delta_0 + \Delta\delta) \int_0^1 c_{l_\delta} \frac{c}{c} dy^*}{\int_0^1 c_{l_\alpha} \frac{c}{c} dy^*} \quad (28)$$

In a practical case both integrals in equation (28) need be evaluated only once for a given configuration and Mach number. A plot of $\bar{\alpha}$ against C_{L_t} with δ as a parameter would be useful in further computations. With $\bar{\alpha}$ known as a function of C_{L_t} and δ , the local lift at any spanwise station is then obtained from the expression

$$l = c_{l_t} q c = [c_{l_\alpha} \bar{\alpha} + c_{l_\delta} (\delta_0 + \Delta\delta)] q c \quad (29)$$

Unsymmetrical loading.—Up to this point the total loads have been assumed to be symmetrical about the airplane center line, whereas, in reality, the load may have an unsymmetrical part. The sources of this dissymmetry may be due to uneven rigging, differences in elasticity between the two sides, or to effects of slipstream, rolling, and sideslip. The first two sources are usually inadvertent ones while the last two are difficult to determine without either wind-tunnel tests or a knowledge of how the airplane will be operated. Present design rules regarding dissymmetry of tail load are concerned more with providing adequate design conditions for the rear of the fuselage than with recognizing that at the maximum critical tail load some dissymmetry may exist.

Tests in the Langley full-scale tunnel (reference 6) and flight tests (reference 7) of a fighter-type airplane, as well as unpublished flight tests of another fighter-type airplane, indicate that the tail-load dissymmetry varies linearly with angle of sideslip so that the difference in lift coefficient between the two sides of the tail can be given as

$$C_{L_{Right}} - C_{L_{Left}} = A\beta \quad (30)$$

The average values of A per degree found for the two fighter-type airplanes are approximately 0.01. No similar values are available for larger airplanes nor for tail surfaces having appreciable dihedral.

In maneuvers of the type considered herein it is doubtful that angles of sideslip larger than 3° would be developed at the time the maximum tail load is reached. If the value of the sideslip angle at the time of maximum tail load can be established, equations (27) to (29) are easily modified to include this effect, provided that the approximate value of A is known.

Chordwise loading.—The chordwise distribution can be determined for any one spanwise station in either of two ways. One way for design work is outlined in reference 8. A knowledge of the airfoil section and the quantities contained in equation (29) suffices for this determination.

If pressure-distribution data are available for a similar section with flaps, an alternative way would be to distribute the load chordwise according to the two-dimensional pressure diagrams with the use of the computed values of section lift coefficient and elevator angle.

DISCUSSION

The method presented is another approach to the determination of tail loads. From the results given in figures 7 and 8, it can be seen that the camber component L_c is so small that for all practical cases it may be omitted with considerable simplification in the computation of tail loads. This omission reduces to a minimum the number of aerodynamic parameters needed to compute the tail loads.

It is possible, in the application of the present method with the use of the suggested values of t_1 , that the derived elevator angles may not be within the pilot's capabilities. Since it must be assumed that all airplanes, to be satisfactory, should have sufficient control to reach their design load boundaries, such an occurrence requires only that the time to reach elevator peak deflection t_1 be increased so as to reduce the elevator angle. The results of figure 8, in which the time t_1 is varied, furnish a useful guide for determining the increase t_1 that might be required.

If sufficient information is available, it is recommended that existing lifting-surface methods be used in determining the spanwise distribution of the total load; however, if information of the angle-of-attack distribution across the span is not known, the method presented may be used as a first approximation.

Along some of the boundaries of the V - n diagram tail buffeting may occur. Measurements show that buffeting usually occurs along the line of maximum lift coefficient and again along a high-speed buffet line which is associated with a compressibility or force break on some major part of the airplane. All airplanes are subject to buffeting at the design conditions associated with the left-hand corner of the V - n diagram. Only high-speed or high-altitude airplanes or both are capable of reaching the other boundary. Measurements show that the oscillatory buffeting loads may be so high that the designer should at least be cognizant of them at the design stage.

The maximum angular acceleration varies inversely with airspeed and directly with the load factor with the contribution due to acceleration in angle of attack likely to be more important than the angular acceleration of the flight path. A somewhat similar variation is indicated for the maximum angular velocity (equation (26)) where it is seen by direct substitution that the part due to angle of attack is likely to be larger than the part due to the angular velocity of the flight path.

CONCLUDING REMARKS

A simple method has been presented for determining the horizontal tail loads in maneuvering flight with the use of a prescribed incremental load-factor variation.

The incremental tail load was separated into four components representing α , $\ddot{\alpha}$, $\dot{\gamma}$, and c . The camber component L_{t_c} is so small that for most conventional airplanes it may be neglected; therefore, the number of aerodynamic parameters needed in this computation of tail loads was reduced to a minimum.

An approximate method is presented for predicting maximum angular accelerations and maximum angular velocities.

The method indicates that maximum tail loads in a pull-up occur at forward center-of-gravity positions and early in the maneuver. The maximum down tail loads in a pull-up occur at the highest design speed in combination with the most forward center-of-gravity position. The maximum up tail load occurs at the upper left-hand corner of the V - n diagram for the most rearward center-of-gravity positions.

LANGLEY AERONAUTICAL LABORATORY,

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,

LANGLEY FIELD, VA., February 9, 1950.

REFERENCES

1. Schmidt, W., and Clasen, R.: Luftkräfte auf Flügel und Höhenleitwerk bei einem Höhenruderausschlag $\beta_H = \beta_{H_{max}} \cdot k t e^{1-k t}$ im waagerechten Geradeausflug. Jahrb. 1937 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich & Berlin), pp. I 169 - I 173.
2. Pearson, Henry A.: Derivation of Charts for Determining the Horizontal Tail Load Variation with Any Elevator Motion. NACA Rep. 759, 1943.
3. Kelley, Joseph, Jr., and Missall, John W.: Maneuvering Horizontal Tail Loads. AAF TR No. 5185, Air Technical Service Command, Army Air Forces, Jan. 25, 1945.
4. Dickinson, H. B.: Maneuverability and Control Surface Strength Criteria for Large Airplanes. Jour. Aero. Sci., vol. 7, no. 11, Sept. 1940, pp. 469-477.
5. Matheny, Cloyce E.: Comparison between Calculated and Measured Loads on Wing and Horizontal Tail in Pull-Up Maneuvers. NACA ARR L5H11, 1945.
6. Sweberg, Harold H., and Dingeldein, Richard C.: Effects of Propeller Operation and Angle of Yaw on the Distribution of the Load on the Horizontal Tail Surface of a Typical Pursuit Airplane. NACA ARR 4B10, 1944.
7. Garvin, John B.: Flight Measurements of Aerodynamic Loads on the Horizontal Tail Surface of a Fighter-Type Airplane. NACA TN 1483, 1947.
8. Anon.: Chordwise Air-Load Distribution. ANC-1(2), Army-Navy-Civil Committee on Aircraft Design Criteria, Oct. 28, 1942.

REPORT 1008

A SMALL-DEFLECTION THEORY FOR CURVED SANDWICH PLATES¹

By MANUEL STEIN and J. MAYERS

SUMMARY

A small-deflection theory that takes into account deformations due to transverse shear is presented for the elastic-behavior analysis of orthotropic plates of constant cylindrical curvature with considerations of buckling included. The theory is applicable primarily to sandwich construction.

INTRODUCTION

The usual sandwich plate as used in aircraft construction consists of a light-weight, low-stiffness core material bonded or riveted between two high-stiffness cover sheets. The elastic behavior of such plates under loading cannot be analyzed by conventional plate and shell theories in general since these theories neglect deformations due to transverse shear, an effect which may be of great importance in sandwich construction.

Many authors have considered transverse shear deflections in analyzing the elastic behavior of flat sandwich plates by means of small-deflection theories (see, for example, references 1 to 4). Most of this work has been concerned with sandwich plates of the isotropic type (for example, Metalite, cellular-cellulose-acetate core). In reference 3, however, sandwich plates of the orthotropic type are also considered (for example, corrugated core).

The treatment of curved sandwich plates in the literature has not been as general as that accorded flat sandwich plates, although several specific studies of the curved isotropic sandwich plate have been published. These studies have covered (a) simply supported, slightly curved isotropic sandwich plates under compressive end loading (reference 1), (b) axially symmetric buckling of a simply supported isotropic sandwich cylinder in compression (reference 1), and (c) a nonbuckling small-deflection theory for isotropic sandwich shells which takes into account not only deflections due to shear but also the effects of core compression normal to the faces (reference 5).

The need for a general theory for curved sandwich plates which is applicable to orthotropic as well as isotropic types and which includes both nonbuckling and buckling effects has led to the development of the theory presented in this report. This theory, which takes into account deflections due to

transverse shear, covers those types of sandwich plates having constant cylindrical curvature, similar properties on the average above and below the middle surface, and essentially constant core thickness.

SYMBOLS

D_s	flexural stiffness of isotropic sandwich plate, inch-pounds $\left(\frac{E_s t_s h^2}{2(1-\mu^2)}\right)$
D	flexural stiffness of ordinary plate, inch-pounds $\left(\frac{Et^3}{12(1-\mu^2)}\right)$
D_x, D_y	flexural stiffnesses of orthotropic plate in axial and circumferential directions, inch-pounds
D_{xy}	twisting stiffness of orthotropic plate in xy -plane, inch-pounds
D_{Q_x}, D_{Q_y}	transverse shear stiffnesses of orthotropic plate in axial and circumferential directions, pounds per inch
D_Q	transverse shear stiffness of isotropic sandwich plate, pounds per inch
E	Young's modulus for ordinary plate, pounds per square inch
E_s	Young's modulus for faces of isotropic sandwich plate, pounds per square inch
E_x, E_y	extensional stiffness of orthotropic plate in axial and circumferential directions, pounds per inch
G_{xy}	shear stiffness of orthotropic plate in xy -plane, pounds per inch
$L_E, L_E^{-1}, L_D, \nabla^2, \nabla^4, \nabla^{-4}$	mathematical operators defined in section entitled "Theoretical Derivations"
M_x, M_y	bending moments on plate cross sections perpendicular to x - and y -axes, respectively, inch-pounds per inch
M_{xy}	twisting moments on cross sections perpendicular to x - and y -axes, inch-pounds per inch
N_x, N_y	resultant normal forces in x - and y -directions, pounds per inch
N_{xy}	resultant shearing force in xy -plane, pounds per inch
q	lateral loading, pounds per square inch

¹ Supersedes NACA TN 2017, "A Small-Deflection Theory for Curved Sandwich Plates" by Manuel Stein and J. Mayers, 1950.

Q_x, Q_y	resultant shearing forces in yz -plane and xz -plane, respectively, pounds per inch
h	depth of isotropic sandwich plate measured between middle surfaces of faces, inches
r	constant radius of curvature of plate, inches
t	thickness of ordinary plate, inches
t_s	thickness of face of isotropic sandwich plate, inches
u, v, w	displacements in x -, y -, z -directions, respectively, of a point in middle surface of plate, inches
x, y, z	rectangular coordinates
γ_{xy}	shear strain in xy -plane
ϵ_x, ϵ_y	normal strains in axial and circumferential directions
μ	Poisson's ratio for ordinary plate
μ_x, μ_y	Poisson's ratios for orthotropic plate, defined in terms of curvatures
μ'_x, μ'_y	Poisson's ratios for orthotropic plate, defined in terms of normal strains

THEORETICAL DERIVATIONS

GENERAL THEORY

In developing the equations of equilibrium for the orthotropic curved plate element, shown in figure 1, the basic assumptions made are that the materials are elastic, that the deflections are small compared with the plate thickness, and that the thickness is small compared with the radius of curvature. The last assumption implies that the shear forces N_{xy} and N_{yx} are equal and that the twisting moments M_{xy} and M_{yx} are equal.

Eleven basic equations.—As in ordinary curved-plate theory, 11 equations exist for orthotropic curved plates (considering deflections due to shear) from which the displacements acting in the plate can be determined. The 11 equations consist of 5 equilibrium equations, 3 equations

relating resultant forces to strains, and 3 equations relating resultant moments with curvatures and twist.

The first five equations, expressing force equilibrium in the x - and y -directions, moment equilibrium about the x - and y -axes, and force equilibrium in the z -direction, are

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \quad (1a)$$

$$\frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \quad (1b)$$

$$Q_x - \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = 0 \quad (1c)$$

$$Q_y - \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} = 0 \quad (1d)$$

$$\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \left(\frac{1}{r} + \frac{\partial^2 w}{\partial y^2} \right) + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + q = 0 \quad (1e)$$

It should be noted that in these equations, higher-order terms have been neglected in accordance with considerations similar to those of reference 6.

For the orthotropic curved plate, the relations between the resultant middle-surface forces and the middle-surface strains are (see appendix)

$$N_x = \frac{E_x}{1 - \mu'_x \mu'_y} \left[\frac{\partial u}{\partial x} + \mu'_y \left(\frac{\partial v}{\partial y} - \frac{w}{r} \right) \right] \quad (2a)$$

$$N_y = \frac{E_y}{1 - \mu'_x \mu'_y} \left(\frac{\partial v}{\partial y} - \frac{w}{r} + \mu'_x \frac{\partial u}{\partial x} \right) \quad (2b)$$

$$N_{xy} = G_{xy} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \quad (2c)$$

From reference 3, the corresponding relations between resultant moments and curvatures and twist are

$$M_x = -\frac{D_x}{1 - \mu_x \mu_y} \left[\frac{\partial^2 w}{\partial x^2} - \frac{1}{D_{Q_x}} \frac{\partial Q_x}{\partial x} + \mu_y \left(\frac{\partial^2 w}{\partial y^2} - \frac{1}{D_{Q_y}} \frac{\partial Q_y}{\partial y} \right) \right] \quad (3a)$$

$$M_y = -\frac{D_y}{1 - \mu_x \mu_y} \left[\frac{\partial^2 w}{\partial y^2} - \frac{1}{D_{Q_y}} \frac{\partial Q_y}{\partial y} + \mu_x \left(\frac{\partial^2 w}{\partial x^2} - \frac{1}{D_{Q_x}} \frac{\partial Q_x}{\partial x} \right) \right] \quad (3b)$$

$$M_{xy} = \frac{1}{2} D_{xy} \left(2 \frac{\partial^2 w}{\partial x \partial y} - \frac{1}{D_{Q_y}} \frac{\partial Q_y}{\partial x} - \frac{1}{D_{Q_x}} \frac{\partial Q_x}{\partial y} \right) \quad (3c)$$

Equations (1), (2), and (3) are the 11 basic equations necessary for determining the forces, moments, and deflections acting in the plate. The number of equations can be reduced to five, however, by substituting equations (2) and (3) into equations (1). In this manner, five differential equations are obtained for determining the resultant transverse shear forces Q_x and Q_y and the displacements u , v , and w .

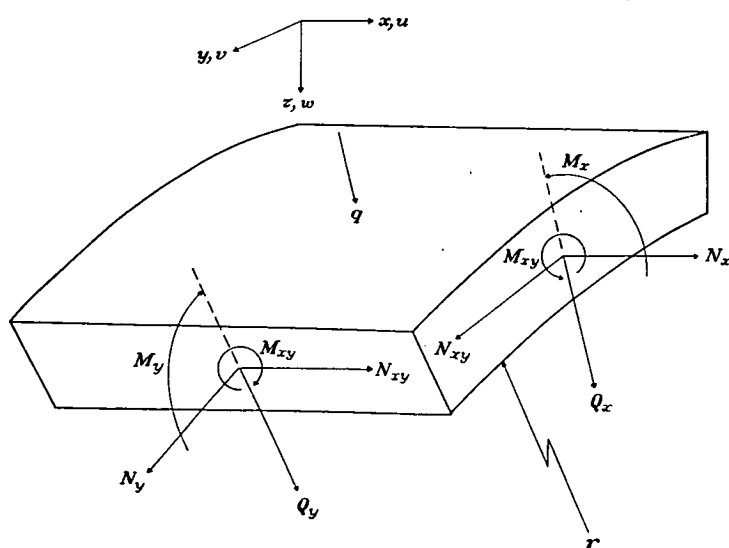


FIGURE 1.—Forces and moments acting on curved plate element.

The 11 basic equations presented are not restricted to deflection problems alone but may be applied to buckling problems as well by considering the changes that occur during buckling and modifying equations (1) accordingly. For equilibrium of the curved plate element after buckling, equations (1) can be written with N_x , N_{xy} , N_y , Q_x , Q_y , M_x , M_{xy} , M_y , and w replaced by $N_{x_0} + N_{x_1}$, $N_{xy_0} + N_{xy_1}$, $N_{y_0} + N_{y_1}$, $Q_{x_0} + Q_{x_1}$, $Q_{y_0} + Q_{y_1}$, $M_{x_0} + M_{x_1}$, $M_{xy_0} + M_{xy_1}$, $M_{y_0} + M_{y_1}$, and $w_0 + w_1$, respectively, where the subscript 0 refers to values prior to buckling and the subscript 1 refers to changes in these values that occur during buckling. For equilibrium of the curved plate element prior to buckling the following equations apply:

$$\frac{\partial N_{x_0}}{\partial x} + \frac{\partial N_{xy_0}}{\partial y} = 0$$

$$\frac{\partial N_{y_0}}{\partial y} + \frac{\partial N_{xy_0}}{\partial x} = 0$$

$$Q_{x_0} - \frac{\partial M_{x_0}}{\partial x} + \frac{\partial M_{xy_0}}{\partial y} = 0$$

$$Q_{y_0} - \frac{\partial M_{y_0}}{\partial y} + \frac{\partial M_{xy_0}}{\partial x} = 0$$

$$\frac{\partial Q_{x_0}}{\partial x} + \frac{\partial Q_{y_0}}{\partial y} + N_{x_0} \frac{\partial^2 w_0}{\partial x^2} + N_{y_0} \left(\frac{1}{r} + \frac{\partial^2 w_0}{\partial y^2} \right) + 2N_{xy_0} \frac{\partial^2 w_0}{\partial x \partial y} + q = 0$$

Subtracting the previous equations from equations (1) (as modified) gives the following equilibrium equations which apply to buckling problems:

$$\frac{\partial N_{x_1}}{\partial x} + \frac{\partial N_{xy_1}}{\partial y} = 0 \quad (4a)$$

$$\frac{\partial N_{y_1}}{\partial y} + \frac{\partial N_{xy_1}}{\partial x} = 0 \quad (4b)$$

$$Q_{x_1} - \frac{\partial M_{x_1}}{\partial x} + \frac{\partial M_{xy_1}}{\partial y} = 0 \quad (4c)$$

$$Q_{y_1} - \frac{\partial M_{y_1}}{\partial y} + \frac{\partial M_{xy_1}}{\partial x} = 0 \quad (4d)$$

$$\begin{aligned} & \frac{\partial Q_{x_1}}{\partial x} + \frac{\partial Q_{y_1}}{\partial y} + N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{x_1} \frac{\partial^2 (w_0 + w_1)}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + \\ & N_{y_1} \left[\frac{1}{r} + \frac{\partial^2 (w_0 + w_1)}{\partial y^2} \right] + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} + 2N_{xy_1} \frac{\partial^2 (w_0 + w_1)}{\partial x \partial y} = 0 \end{aligned} \quad (4e)$$

In equation (4e) the terms $N_{x_1} \frac{\partial^2 w_1}{\partial x^2}$, $N_{y_1} \frac{\partial^2 w_1}{\partial y^2}$, and $N_{xy_1} \frac{\partial^2 w_1}{\partial x \partial y}$ may be neglected since they will be small compared with $N_{x_0} \frac{\partial^2 w_1}{\partial x^2}$, $N_{y_0} \frac{\partial^2 w_1}{\partial y^2}$, and $N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y}$. Also, if the deflection

prior to buckling is zero or constant as occurs for many problems (for example, axial compression, hydrostatic pressure), all derivatives of w_0 vanish. For this type of problem equation (4e) becomes

$$\frac{\partial Q_{x_1}}{\partial x} + \frac{\partial Q_{y_1}}{\partial y} + N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + \frac{N_{y_1}}{r} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} = 0 \quad (4e')$$

The six equations relating changes in middle-surface resultant forces with buckling strains and changes in moment with buckling distortions are identical with equations (2) and (3) with the subscript 1 added to N_x , N_{xy} , N_y , Q_x , Q_y , M_x , M_{xy} , M_y , u , v , and w .

The 11 equations, given by equations (4) and equations (2) and (3) (with subscript 1), apply to buckling problems in general (with equation (4e) or (4e') as required) and can be used to obtain the critical values of the loads acting on the plate. As is shown in the next section, however, for the case in which the deflection prior to buckling is zero or constant, the 11 equations can be suitably combined to yield 3 equations in w_1 , Q_{x_1} , and Q_{y_1} , a form convenient for application to plates of sandwich construction.

Reduction to three equations for buckling problems in which the deflection prior to buckling is zero or constant.—The reduction of the 11 equations to 3 equations in w_1 , Q_{x_1} , and Q_{y_1} is achieved in several steps as follows:

By differentiating equation (4c) with respect to x , equation (4d) with respect to y , and adding the results to obtain the relationship

$$\frac{\partial Q_{x_1}}{\partial x} + \frac{\partial Q_{y_1}}{\partial y} = \frac{\partial^2 M_{x_1}}{\partial x^2} - 2 \frac{\partial^2 M_{xy_1}}{\partial x \partial y} + \frac{\partial^2 M_{y_1}}{\partial y^2}$$

equation (4e') may be rewritten as

$$\begin{aligned} & \frac{\partial^2 M_{x_1}}{\partial x^2} - 2 \frac{\partial^2 M_{xy_1}}{\partial x \partial y} + \frac{\partial^2 M_{y_1}}{\partial y^2} + \frac{N_{y_1}}{r} + N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + \\ & N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} = 0 \end{aligned} \quad (4e'')$$

Next, equations (2) and (3) (with subscript 1) are substituted into the equilibrium equations (4a) to (4d) and (4e'') to give

$$\frac{\partial^2 u_1}{\partial x^2} + \mu'_{xy} \frac{\partial^2 v_1}{\partial x \partial y} - \frac{\mu'_{xy}}{r} \frac{\partial w_1}{\partial x} + (1 - \mu'_{xy} \mu'_{yx}) \frac{G_{xy}}{E_x} \left(\frac{\partial^2 u_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial x \partial y} \right) = 0 \quad (5)$$

$$\frac{\partial^2 v_1}{\partial y^2} - \frac{1}{r} \frac{\partial w_1}{\partial y} + \mu'_{yx} \frac{\partial^2 u_1}{\partial x \partial y} + (1 - \mu'_{xy} \mu'_{yx}) \frac{G_{xy}}{E_y} \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 u_1}{\partial x \partial y} \right) = 0 \quad (6)$$

$$\begin{aligned} & Q_{x_1} + \frac{D_x}{1 - \mu_x \mu_y} \left(\frac{\partial^3 w_1}{\partial x^3} - \frac{1}{D_{q_x}} \frac{\partial^2 Q_{x_1}}{\partial x^2} + \mu_y \frac{\partial^3 w_1}{\partial x \partial y^2} - \frac{\mu_y}{D_{q_y}} \frac{\partial^2 Q_{y_1}}{\partial x \partial y} \right) + \\ & \frac{1}{2} D_{xy} \left(2 \frac{\partial^3 w_1}{\partial x \partial y^2} - \frac{1}{D_{q_x}} \frac{\partial^2 Q_{x_1}}{\partial y^2} - \frac{1}{D_{q_y}} \frac{\partial^2 Q_{y_1}}{\partial x \partial y} \right) = 0 \end{aligned} \quad (7)$$

$$Q_{v_1} + \frac{D_y}{1 - \mu_x \mu_y} \left(\frac{\partial^3 w_1}{\partial y^3} - \frac{1}{D_{Q_y}} \frac{\partial^2 Q_{v_1}}{\partial y^2} + \mu_x \frac{\partial^3 w_1}{\partial x^2 \partial y} - \frac{\mu_x}{D_{Q_x}} \frac{\partial^2 Q_{x_1}}{\partial x \partial y} \right) + \frac{1}{2} D_{xy} \left(2 \frac{\partial^3 w_1}{\partial x^2 \partial y} - \frac{1}{D_{Q_x}} \frac{\partial^2 Q_{x_1}}{\partial x \partial y} - \frac{1}{D_{Q_y}} \frac{\partial^2 Q_{v_1}}{\partial x^2} \right) = 0 \quad (8)$$

$$L_D w_1 - \frac{E_y}{r(1 - \mu'_x \mu'_y)} \left(\frac{\partial v_1}{\partial y} - \frac{w_1}{r} + \mu'_x \frac{\partial u_1}{\partial x} \right) - \left(N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} \right) - \frac{1}{D_{Q_x}} \left[\frac{D_x}{1 - \mu_x \mu_y} \frac{\partial^3 Q_{x_1}}{\partial x^3} + \left(\frac{\mu_x D_y}{1 - \mu_x \mu_y} + D_{xy} \right) \frac{\partial^3 Q_{x_1}}{\partial x \partial y^2} \right] - \frac{1}{D_{Q_y}} \left[\frac{D_y}{1 - \mu_x \mu_y} \frac{\partial^3 Q_{v_1}}{\partial y^3} + \left(\frac{\mu_y D_x}{1 - \mu_x \mu_y} + D_{xy} \right) \frac{\partial^3 Q_{v_1}}{\partial x^2 \partial y} \right] = 0 \quad (9)$$

where L_D is the linear differential operator defined by

$$L_D = \frac{D_x}{1 - \mu_x \mu_y} \frac{\partial^4}{\partial x^4} + \left(\frac{\mu_y D_x}{1 - \mu_x \mu_y} + 2D_{xy} + \frac{\mu_x D_y}{1 - \mu_x \mu_y} \right) \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{D_y}{1 - \mu_x \mu_y} \frac{\partial^4}{\partial y^4}$$

At this point, six equations have been eliminated and, therefore, five equations remain—equations (5), (6), (7), (8), and (9) in u_1 , v_1 , w_1 , Q_{x_1} , and Q_{v_1} .

A further reduction in the number of equations and unknowns is effected by first solving equations (5) and (6) to obtain relations from which u_1 and v_1 can be determined and then substituting for u_1 and v_1 in equation (9). The expressions obtained by solving equations (5) and (6), in accordance with the rules governing the multiplication of linear operators, are

$$r L_E u_1 = \mu'_y \frac{G_{xy}}{E_y} \frac{\partial^3 w_1}{\partial x^3} - \frac{G_{xy}}{E_x} \frac{\partial^3 w_1}{\partial x \partial y^2} \quad (10)$$

and

$$r L_E v_1 = \left(1 - \mu'_y \frac{G_{xy}}{E_y} \right) \frac{\partial^3 w_1}{\partial x^2 \partial y} + \frac{G_{xy}}{E_x} \frac{\partial^3 w_1}{\partial y^3} \quad (11)$$

where L_E is the linear differential operator defined by

$$L_E = \frac{G_{xy}}{E_y} \frac{\partial^4}{\partial x^4} + \left(1 - \mu'_x \frac{G_{xy}}{E_x} - \mu'_y \frac{G_{xy}}{E_y} \right) \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{G_{xy}}{E_x} \frac{\partial^4}{\partial y^4}$$

The relationships given by equations (10) and (11) may be written in a form more suitable for substitution into equation (9) by differentiating equation (10) with respect to x , equation (11) with respect to y , and then, symbolically solving the equations for $\frac{\partial u_1}{\partial x}$ and $\frac{\partial v_1}{\partial y}$, respectively, to give

$$\frac{\partial u_1}{\partial x} = L_E^{-1} \left(\frac{\mu'_y}{r} \frac{G_{xy}}{E_y} \frac{\partial^4 w_1}{\partial x^4} - \frac{G_{xy}}{r E_x} \frac{\partial^4 w_1}{\partial x^2 \partial y^2} \right) \quad (12)$$

$$\frac{\partial v_1}{\partial y} = L_E^{-1} \left(\frac{1 - \mu'_y \frac{G_{xy}}{E_y}}{r} \frac{\partial^4 w_1}{\partial x^2 \partial y^2} + \frac{G_{xy}}{r E_x} \frac{\partial^4 w_1}{\partial y^4} \right) \quad (13)$$

where L_E^{-1} is defined by $L_E^{-1}(L_E w_1) = L_E(L_E^{-1} w_1) = w_1$. The inverse operator L_E^{-1} is similar to the inverse operator ∇^{-4}

defined in reference 7, and, as is shown subsequently, L_E^{-1} reduces to ∇^{-4} for the special case of the isotropic plate.

Substituting the expressions for $\frac{\partial u_1}{\partial x}$ and $\frac{\partial v_1}{\partial y}$ from equations

(12) and (13) into equation (9) and replacing $\frac{w_1}{r}$ by

$L_E^{-1} \left(L_E \frac{w_1}{r} \right)$ results in the following equation:

$$L_D w_1 + \frac{G_{xy}}{r^2} L_E^{-1} \frac{\partial^4 w_1}{\partial x^4} - \left(N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} \right) - \frac{1}{D_{Q_x}} \left[\frac{D_x}{1 - \mu_x \mu_y} \frac{\partial^3 Q_{x_1}}{\partial x^3} + \left(\frac{\mu_x D_y}{1 - \mu_x \mu_y} + D_{xy} \right) \frac{\partial^3 Q_{x_1}}{\partial x \partial y^2} \right] - \frac{1}{D_{Q_y}} \left[\frac{D_y}{1 - \mu_x \mu_y} \frac{\partial^3 Q_{v_1}}{\partial y^3} + \left(\frac{\mu_y D_x}{1 - \mu_x \mu_y} + D_{xy} \right) \frac{\partial^3 Q_{v_1}}{\partial x^2 \partial y} \right] = 0 \quad (14)$$

At this stage, the original 11 equations have been reduced to the 3 equations (14), (7), and (8), in the 3 unknowns w_1 , Q_{x_1} , and Q_{v_1} .

For most problems, equations (14), (7), and (8), together with proper boundary conditions, can determine the elastic stability criteria for an orthotropic curved plate subjected to middle-surface loadings. It should be noted, however, that the three equations are not sufficient if boundary conditions are specified on the displacements u_1 and v_1 . For boundary conditions on u_1 and v_1 , as well as w_1 , equations (10) and (11) must also be employed. When boundary conditions are not specified on u_1 and v_1 (the case when only equations (14), (7), and (8) are used), certain boundary conditions are implied, nevertheless, by equations (10) and (11), consistent with the expression for w_1 . A discussion of similar implied boundary conditions on u_1 and v_1 is included in reference 7.

SPECIAL CASES OF BUCKLING EQUATIONS

Isotropic curved sandwich plate with non-direct-stress-carrying core.—For the isotropic sandwich plate with non-direct-stress-carrying core, the physical constants bear the following relationships to those of the orthotropic plate:

$$D_{Q_x} = D_{Q_y} = D_Q$$

$$\mu_x = \mu_y = \mu'_x = \mu'_y = \mu$$

$$D_x = D_y = D_s(1 - \mu^2)$$

$$D_{xy} = D_s(1 - \mu)$$

$$E_x = E_y = 2E_s t_s$$

$$G_{xy} = \frac{E_s t_s}{1 + \mu}$$

These relationships permit equation (14) to be simplified as follows:

$$D_s \nabla^4 w_1 + \frac{2t_s E_s}{r^2} \nabla^{-4} \frac{\partial^4 w_1}{\partial x^4} - \left(N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} \right) - \frac{D_s}{D_Q} \nabla^2 \left(\frac{\partial Q_{x_1}}{\partial x} + \frac{\partial Q_{v_1}}{\partial y} \right) = 0 \quad (15)$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

$$\nabla^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}$$

and ∇^{-4} is defined by $\nabla^{-4}(\nabla^4 w_1) = \nabla^4(\nabla^{-4} w_1) = w_1$.

In this case, however, equations (7) and (8) are not needed to obtain the quantity $\frac{\partial Q_{x_1}}{\partial x} + \frac{\partial Q_{y_1}}{\partial y}$, since this quantity can be found more conveniently from equation (4e'). From equation (4e'), therefore,

$$\frac{\partial Q_{x_1}}{\partial x} + \frac{\partial Q_{y_1}}{\partial y} = - \left(\frac{N_{v_1}}{r} + N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} \right)$$

Substituting for $\frac{\partial Q_{x_1}}{\partial x} + \frac{\partial Q_{y_1}}{\partial y}$ in equation (15) gives

$$D_s \Delta^4 w_1 + \frac{2t_s E_s}{r^2} \nabla^{-4} \frac{\partial^4 w_1}{\partial x^4} - \frac{D_s}{D_q} \nabla^2 \left(- \frac{N_{v_1}}{r} \right) - \left(1 - \frac{D_s}{D_q} \nabla^2 \right) \left(N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} \right) = 0 \quad (16)$$

The term $-N_{v_1}/r$, which appears in equation (14) for the orthotropic plate as $\frac{G_{xy}}{r^2} L_E^{-1} \frac{\partial^4 w_1}{\partial x^4}$, reduces to $\frac{2t_s E_s}{r^2} \nabla^{-4} \frac{\partial^4 w_1}{\partial x^4}$ for the case of the isotropic plate. If this result is used in equation (16), the equation of equilibrium for the isotropic curved sandwich plate with non-direct-stress-carrying core becomes

$$D_s \nabla^4 w_1 + \left(1 - \frac{D_s}{D_q} \nabla^2 \right) \left[\frac{2t_s E_s}{r^2} \nabla^{-4} \frac{\partial^4 w_1}{\partial x^4} - \left(N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} \right) \right] = 0 \quad (17)$$

If the radius is taken infinite, equation (17) becomes equivalent to equation (71) of reference 4.

Isotropic curved plate, deflections due to shear neglected.—The present theory can be reduced to a known theory for ordinary curved plates by appropriate substitutions for the physical constants. For an ordinary plate, the physical constants become

$$D_{Q_x} = D_{Q_y} = \infty \quad (\text{no shear deflections})$$

$$\mu_x = \mu_y = \mu'_x = \mu'_y = \mu$$

$$D_x = D_y = D(1 - \mu^2)$$

$$D_{xy} = D(1 - \mu)$$

$$E_x = E_y = Et$$

$$G_{xy} = \frac{Et}{2(1 + \mu)}$$

Upon substitution of these constants into equation (14), the resulting equation becomes independent of equations (7) and (8) and the equilibrium equation of the ordinary curved plate, therefore, is given by

$$D \nabla^4 w_1 + \frac{Et}{r^2} \nabla^{-4} \frac{\partial^4 w_1}{\partial x^4} - \left(N_{x_0} \frac{\partial^2 w_1}{\partial x^2} + N_{y_0} \frac{\partial^2 w_1}{\partial y^2} + 2N_{xy_0} \frac{\partial^2 w_1}{\partial x \partial y} \right) = 0 \quad (18)$$

Equation (18) is equivalent to the modified equilibrium equation for ordinary curved plates presented in reference 7.

CONCLUDING REMARKS

A theory has been developed for analyzing the elastic behavior of orthotropic curved plates, that takes into account the effect of deflections due to shear and requires the use of 12 physical constants to characterize the plate. Seven of the physical constants appearing in the equations of equilibrium are directly associated with the flat-orthotropic-plate theory presented in NACA Rep. 899. The remaining five physical constants are included in the present theory to account for the stretching under loading of the middle surface of the curved plate.

For each type of orthotropic plate, the 12 physical constants may be evaluated either from the geometry of the cross sections and the properties of the materials used or by direct tests conducted on sample specimens. Because two reciprocal relationships exist (see appendix), only 10 of the constants need be determined independently.

The theory presented in this report does not take into account the compressibility of the sandwich plate in a direction normal to the faces. Such an effect does not enter into flat-sandwich-plate theory but might be of importance in certain types of curved sandwich plates where the elastic constants of the core material are small compared with those of the face material.

For practical sandwiches of the end-grain-balsa or corrugated-core types, order-of-magnitude considerations lead to the conclusion that the effect of core compressibility will be negligible as regards both buckling loads and deflections. For sandwiches with less stiff cores—for example, cellular cellulose acetate—the effect of core compressibility will be more important. Even for such cores, however, in the case of all the numerical examples given in NACA TN 1832, the effect of core compressibility is negligible in comparison with the effect of transverse shear deformations for sandwich-type circular cylindrical shells. The present theory, in which the core is assumed to be incompressible in a direction normal to the faces, appears, therefore, to be applicable to most practical sandwich plates.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., November 22, 1949.

APPENDIX

DERIVATION OF MIDDLE-SURFACE FORCE-DISTORTION RELATIONSHIPS

The orthotropic curved plate (effects of transverse shear being considered) is characterized by 12 physical constants, 7 of which are associated with flat plates, as presented in reference 3. The remaining five constants enter the present theory because of the additional stretching strain developed under loading in the middle surface of the curved plate. In this appendix the five additional constants are defined, and expressions for the resultant forces, involving these constants, are derived.

Physical constants.—The seven flat-plate constants are the flexural stiffnesses D_x and D_y , the flexural Poisson ratios μ_x and μ_y , the twisting stiffness D_{xy} , and the transverse shear stiffnesses D_{Q_x} and D_{Q_y} . As derived in reference 3, the first four of these constants are related by

$$\mu_x D_y = \mu_y D_x$$

The five additional constants appearing in the curved-plate theory are the extensional stiffnesses E_x and E_y , the extensional Poisson ratios μ'_x and μ'_y , and the shearing stiffness G_{xy} . The first four constants are found by a procedure similar to that used in reference 3 to be related by

$$\mu'_x E_y = \mu'_y E_x$$

As a result of these two reciprocal relationships, only 10 of the 12 physical constants need be determined independently.

The five additional physical constants are defined in the same manner as the flat-plate constants of reference 3—that is, by considering the effect of imposing particular loading conditions on the element shown in figure 1. To obtain E_x , for example, only the middle-surface forces N_x are assumed to be acting on the element. As a result of this loading, the strain ϵ_x is induced in the middle surface.

The stiffness E_x is then defined by the relation $E_x = \frac{N_x}{\epsilon_x}$ when only N_x is acting.

The Poisson effect of the forces N_x acting on the element is to introduce a strain ϵ_y , negative with respect to ϵ_x , in the middle surface. The constant μ'_x is then defined by the relation $\mu'_x = -\frac{\epsilon_y}{\epsilon_x}$ when only N_x is acting.

In a similar manner, E_y , μ'_y , and G_{xy} are defined as $E_y = \frac{N_y}{\epsilon_y}$ when only N_y is acting, $\mu'_y = -\frac{\epsilon_x}{\epsilon_y}$ when only N_y is acting, and $G_{xy} = \frac{N_{xy}}{\gamma_{xy}}$.

Resultant forces.—The relations between the elastic middle-surface strains and forces, satisfying the foregoing definitions, can be written as

$$\left. \begin{aligned} \epsilon_x &= \frac{N_x}{E_x} - \mu'_y \frac{N_y}{E_y} \\ \epsilon_y &= \frac{N_y}{E_y} - \mu'_x \frac{N_x}{E_x} \\ \gamma_{xy} &= \frac{N_{xy}}{G_{xy}} \end{aligned} \right\} \quad (A1)$$

The three strain equations can be solved for N_x , N_y , and N_{xy} in terms of the strains to give

$$\left. \begin{aligned} N_x &= \frac{E_x}{1 - \mu'_x \mu'_y} (\epsilon_x + \mu'_y \epsilon_y) \\ N_y &= \frac{E_y}{1 - \mu'_x \mu'_y} (\epsilon_y + \mu'_x \epsilon_x) \\ N_{xy} &= G_{xy} \gamma_{xy} \end{aligned} \right\} \quad (A2)$$

Substituting the expressions for the middle-surface strains of a cylindrical section in terms of middle-surface displacements

$$\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} - \frac{w}{r} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}$$

into equation (A2) gives

$$\left. \begin{aligned} N_x &= \frac{E_x}{1 - \mu'_x \mu'_y} \left[\frac{\partial u}{\partial x} + \mu'_y \left(\frac{\partial v}{\partial y} - \frac{w}{r} \right) \right] \\ N_y &= \frac{E_y}{1 - \mu'_x \mu'_y} \left(\frac{\partial v}{\partial y} - \frac{w}{r} + \mu'_x \frac{\partial u}{\partial x} \right) \\ N_{xy} &= G_{xy} \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \end{aligned} \right\} \quad (A3)$$

These equations are used in the derivation of the equilibrium equations.

REFERENCES

1. Leggett, D. M. A., and Hopkins, H. G.: Sandwich Panels and Cylinders under Compressive End Loads. R. & M. No. 2262, British A.R.C., 1942.
2. Bijlaard, P. P.: On the Elastic Stability of Sandwich Plates. I-II. Koninklijke Nederlandsche Akad. Wetenschappen. Reprinted from Proc., vol. L, nos. 1 and 2, 1947, pp. 79-87, 186-193.
3. Libove, Charles, and Batdorf, S. B.: A General Small-Deflection Theory for Flat Sandwich Plates. NACA Rep. 899, 1948.
4. Reissner, Eric: Finite Deflections of Sandwich Plates. Jour. Aero. Sci., vol. 15, no. 7, July 1948, pp. 435-440.
5. Reissner, Eric: Small Bending and Stretching of Sandwich-Type Shells. NACA TN 1832, 1949.
6. Donnell, L. H.: Stability of Thin-Walled Tubes under Torsion. NACA Rep. 479, 1933.
7. Batdorf, S. B.: A Simplified Method of Elastic-Stability Analysis for Thin Cylindrical Shells. NACA Rep. 874, 1947.

REPORT 1009

INVESTIGATION OF FRETTING BY MICROSCOPIC OBSERVATION¹

By DOUGLAS GODFREY

SUMMARY

An experimental investigation, using microscopic observation and color motion photomicrographs of the action, was conducted to determine the cause of fretting. Glass and other noncorrosive materials, as well as metals, were used as specimens. A very simple apparatus vibrated convex surfaces in contact with stationary flat surfaces at frequencies of 120 cycles or less than 1 cycle per second, an amplitude of 0.001 inch, and a load of 0.2 pound.

The observations and analysis led to the conclusions that fretting was initiated by the loosening, due to inherent adhesive forces, of finely divided and apparently virgin material, and that its initiation is independent of vibratory motion or high sliding speeds. The fretting of platinum, glass, quartz, ruby, and mica relegated the role of oxidation as a cause to that of a secondary factor. Fretting occurred to clean nonmetals and metals readily, and glass microscope slides and steel balls provided an excellent method for visual studies.

INTRODUCTION

Fretting, defined as surface failure that may occur when closely fitting metal surfaces experience slight relative motion, damages many machine parts subject to vibration. The phenomenon of fretting (also known as fretting corrosion and friction oxidation) is principally characterized by surface stain, pitting, and the generation of oxides, as described in detail in references 1 and 2. The determination of the cause of fretting has been the object of several research programs. The research reported in reference 2 supports Tomlinson's theory of molecular attrition (reference 3) that proposes that the cause of fretting is a physical action, or specifically that the disintegration of the metal surface is due to "molecular plucking." More recently, chemical action has been reported to be of primary importance

(reference 4). Other investigators (for example, references 5 and 6) have advanced electrolytic, abrasion, welding, fatigue, and "vibrational decay" theories to explain the action that produces fretting. These theories are incompatible or divergent and more experimental evidence is needed before any one of them may be generally accepted.

The failure of certain aircraft parts due to fretting has produced a new incentive toward solving the problem of the prevention of the phenomenon. Antifriction bearings subject to rotational oscillation and side-thrust vibration experience the greatest amount of fretting, and such parts as connecting rods, knuckle pins, splined shafts, and clamped and bolted flanges suffer deleterious effects (reference 1).

The research reported herein was conducted at the NACA Lewis laboratory during 1949 to observe and to analyze the action of fretting in its initial stages in order to determine its cause. Inasmuch as the limits of mechanical variables such as load, frequency, and amplitude have been established (reference 2), this investigation was limited to microscopic observation, debris analysis, and surface analysis of fretting areas. The nonmetallic and noncorrosive materials, glass, quartz, ruby, and mica, and the metal platinum were used as specimens because their fretting would indicate whether the primary action was physical or chemical; in addition, the nonmetallic materials eliminated the possibility of ordinary welding, simplified the debris analysis, and in some cases permitted microscopic observation or color motion photomicrographs of the action (reference 7). The color motion photomicrographs describe fretting more effectively than words or black-and-white photomicrographs.

Fretting was induced by a very simple apparatus, which vibrated a convex (usually spherical) surface in contact with a flat surface at frequencies of 120 cycles or less than 1 cycle per second, an amplitude of 0.001 inch, and a load of 0.2 pound.

¹ Supersedes NACA TN 2039, "Investigation of Fretting Corrosion by Microscopic Observation" by Douglas Godfrey, 1950.

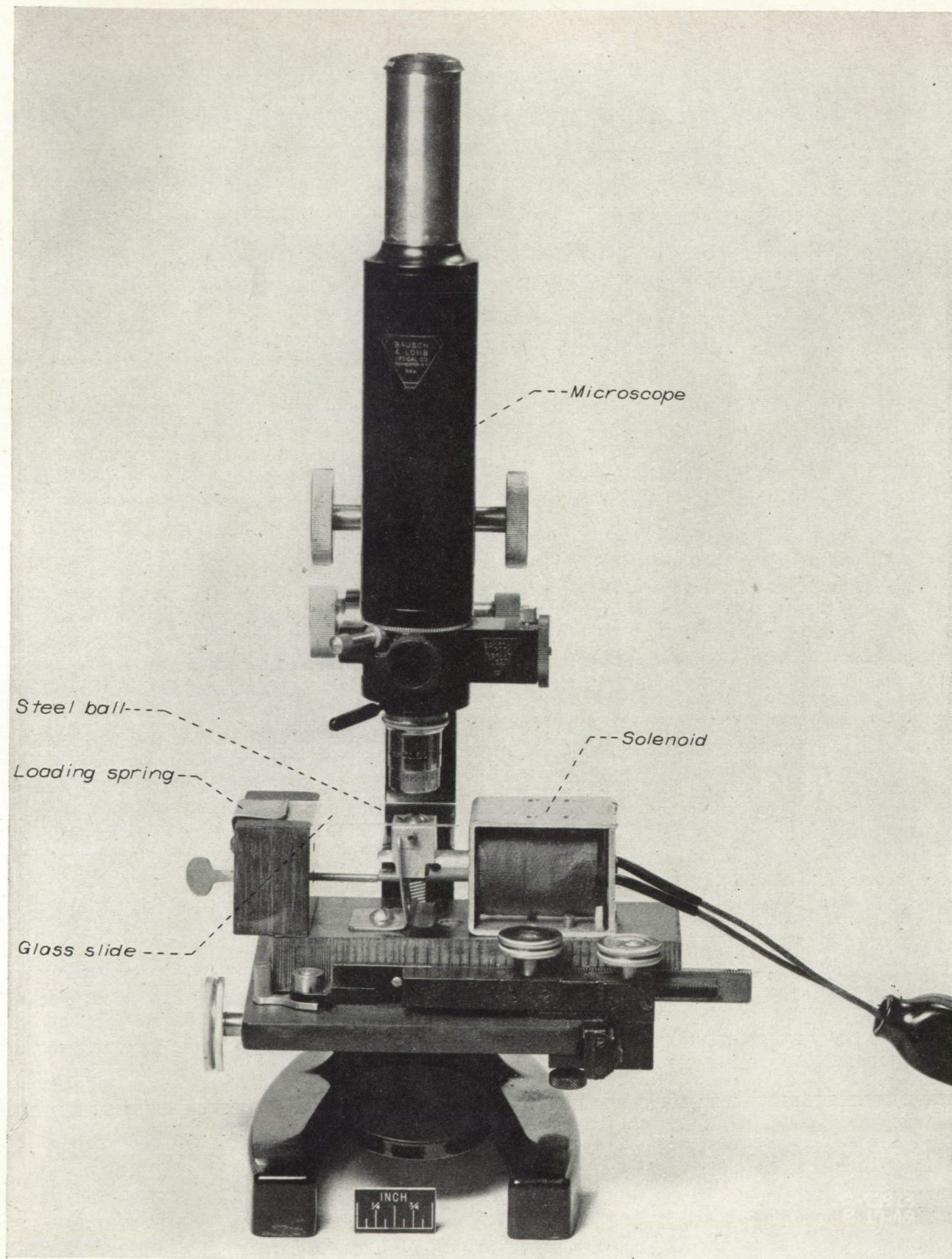


FIGURE 1.—Fretting apparatus.

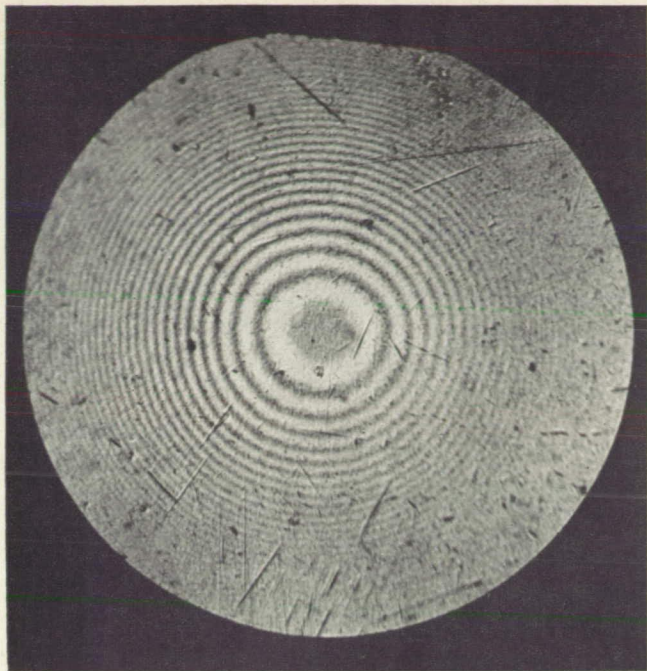


FIGURE 2.—Photomicrograph with vertical illumination and red filter of chrome-alloy steel ball through glass slide showing interference bands surrounding contact area. $\times 100$.

APPARATUS, MATERIALS, AND PROCEDURE

A photograph of the apparatus is shown in figure 1. The vibrating action of 120 cycles per second with an amplitude of 0.001 inch was induced by a solenoid. The specimen with the convex surface was firmly attached to the plunger of the solenoid, which was horizontal. The specimen with the flat surface was held stationary at one end and loaded vertically by a flat spring. The spring caused the flat surface to exert a load of approximately 0.2 pound on the convex surface.

The apparatus could be attached to the mechanical stage of a microscope, which permitted microscopic observation of the contact area with vertical or oblique illumination. With transparent flat specimens, the contact area could readily be located because it was surrounded by concentric interference bands (fig. 2). Amplitude was measured by noting the distance between the extreme positions of a scratch mark on the vibrating surface. The experiments conducted at 120 cycles per second were divided into three groups: (1) the preliminary metal against metal, (2) the metal against nonmetal, and (3) the nonmetal against nonmetal. No general material survey was made and the number of metals used was limited.

In the metal-against-metal group, mild steel, chrome steel, stainless steel, copper, and aluminum were vibrated against mild steel essentially to ascertain whether the apparatus would induce fretting with satisfactory reproducibility. In the metal-against-nonmetal group, the five metals plus platinum were each vibrated against glass microscope slides, mica, and Lucite and extensive observations were made with $\frac{1}{2}$ -inch-diameter chrome-alloy steel (SAE 52100) balls and the glass slides. Glass was used as one of the sur-

faces because it is rigid, noncorrosive, and transparent. These properties were desirable to permit observation of the action as well as to simplify debris analysis. The statement is made in reference 2 that glass not only produces fretting on steel but is itself attacked; the ability of glass to pick up metal during sliding is demonstrated in reference 8. In the third group, glass, quartz, ruby, mica, and Lucite were vibrated against one another. All the materials were shaped into convex or flat pieces as required and finished to a surface roughness of 1 to 3 root mean square by fine grinding and polishing.

A group of experiments that are considered to be $\frac{1}{2}$ -cycle experiments were conducted with an adapter on the apparatus, which provided for sliding the glass microscope slide over the surface of the stationary ball at a speed of approximately 0.002 inch per second or less. The motion was caused by hand-turning a fine screw to which the glass was attached. The direction was reversed each 0.001 inch, and the frequency was considered 1 cycle per second or less. A slow-motion vibration was thus simulated and the action that occurs in $\frac{1}{2}$ cycle was observed.

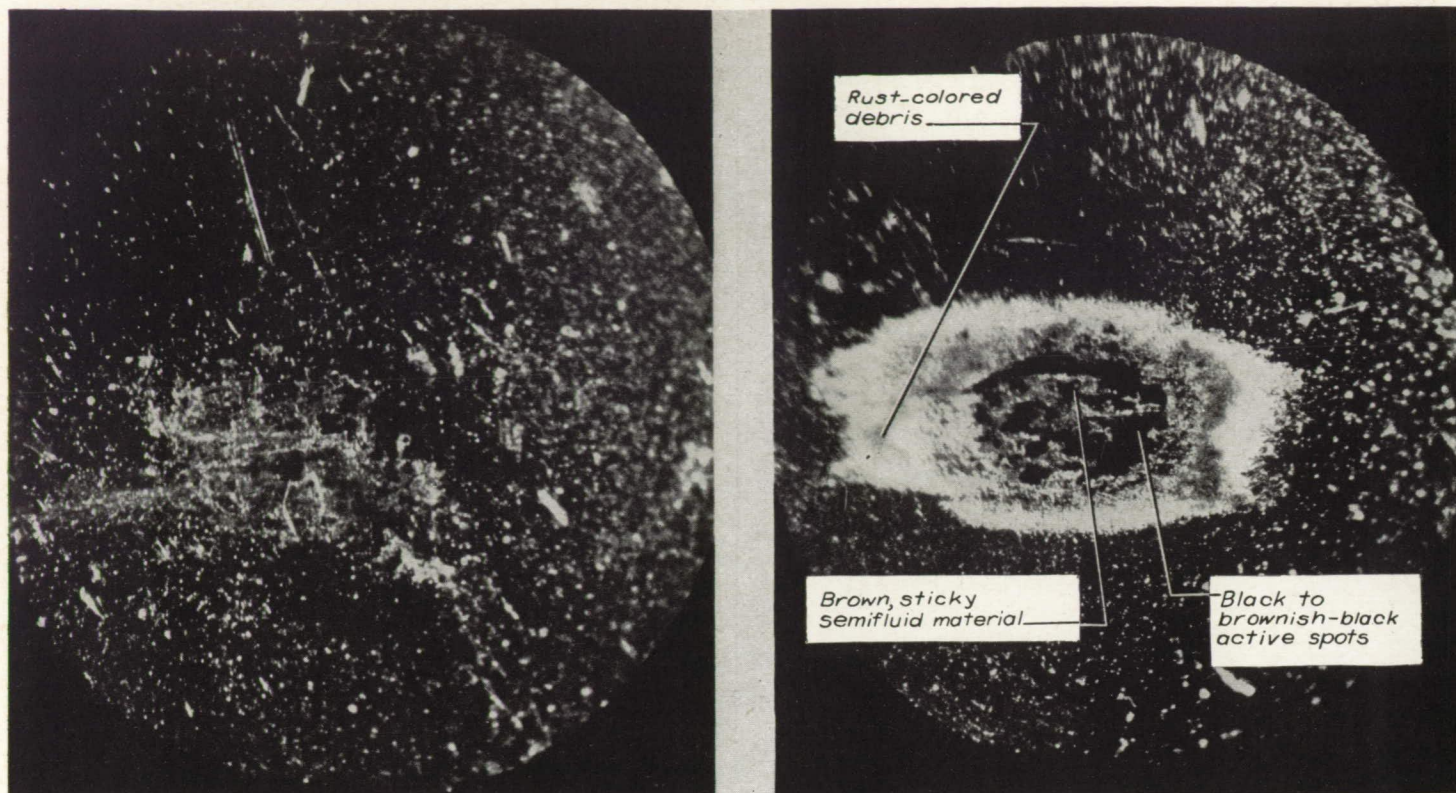
All specimens were thoroughly cleaned immediately before being mounted in the apparatus. The metal cleaning consisted in degreasing in an organic solvent followed by scrubbing with surgical cotton in a paste of levigated alumina, a thorough rinse in tap water, a rinse in 95-percent alcohol, and drying in air. Particular effort was made to wash off all adhering alumina. The glass, ruby, quartz, and mica specimens were cleaned in fresh sodium dichromate-sulfuric acid cleaning solution followed by rinse in tap water and distilled water and oven drying. The Lucite was wiped clean with lens tissue.

The number of cycles that any pair of specimens had experienced was determined by measuring the length of time current was supplied to the solenoid. In those cases in which the flat specimens were not transparent, the vibration was induced for periods of 1, 2, 4, 8, 16, 32, 60, 120, 180, 300, and 600 seconds cumulatively. After each period, the flat specimen was removed and the spots on the convex specimen and on the flat specimen were each microscopically examined. Photomicrographs were taken in some cases to aid in the description of the action. When transparent specimens were used, the complete action could be viewed or photographed with ease. The vibration could be stopped at any time, however, and the contact area viewed or photographed at rest.

RESULTS AND DISCUSSION

PRELIMINARY METAL AGAINST METAL

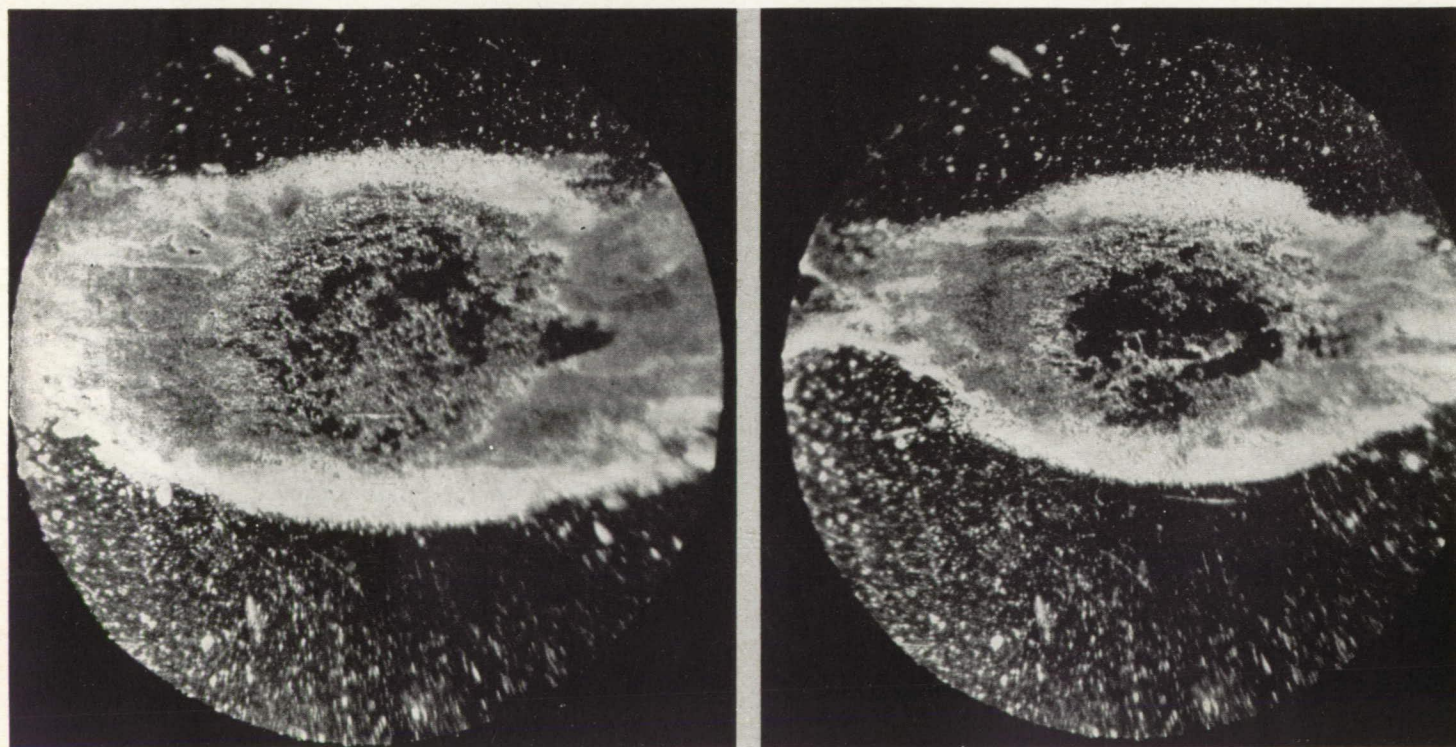
In the preliminary metal-against-metal experiments, each of the several pairs of specimens investigated exhibited fretting within approximately 300 seconds and produced the characteristic debris and wear area. The pattern of the spot on the flat surface was always similar to the pattern of the spot on the convex surface. These experiments revealed that load, amplitude, and frequency were such



(a) Cycles, 30.

b) Cycles, 120.

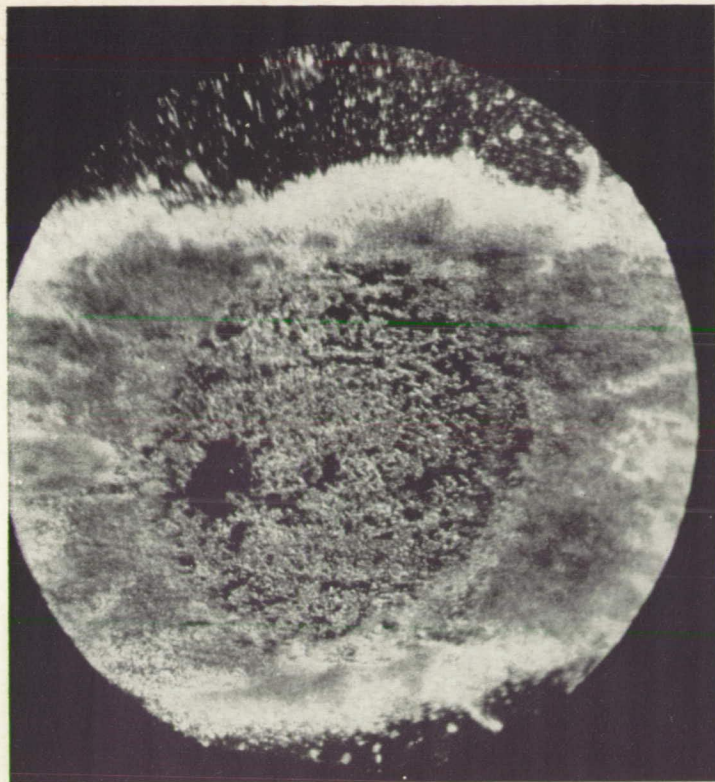
FIGURE 3.—Photomicrographs showing progressive stages in fretting between glass microscope slide and chrome-alloy steel ball. $\times 120$.



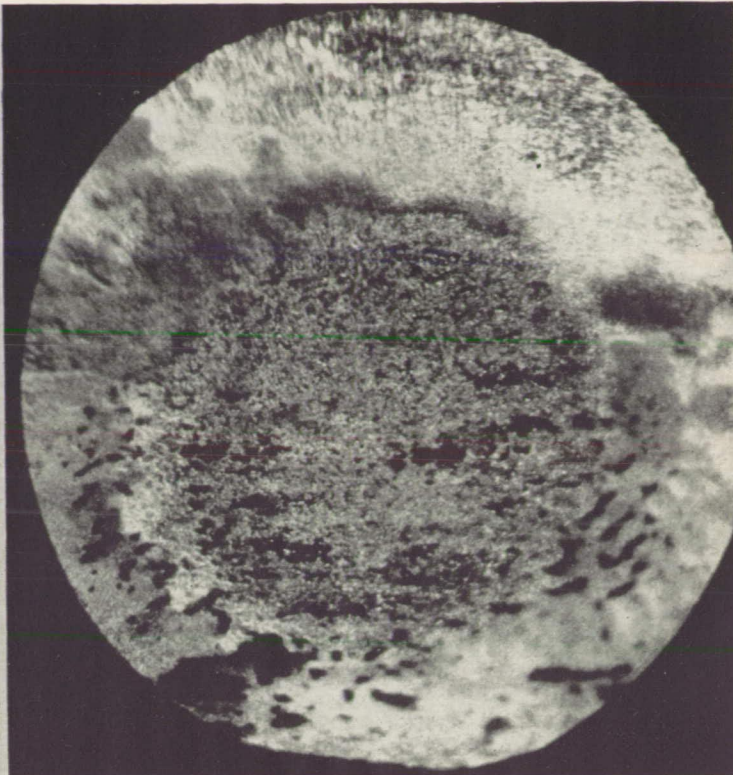
(c) Cycles, 240.

(d) Cycles, 720.

FIGURE 3.—Continued. Photomicrographs showing progressive stages in fretting between glass microscope slide and chrome-alloy steel ball. $\times 120$.



(e) Cycles, 1200.



(f) Cycles, 3600.

FIGURE 3.—Concluded. Photomicrographs showing progressive stages in fretting between glass microscope slide and chrome-alloy steel ball. $\times 120$.

as to produce fretting quickly and that the amount of debris and the size of the wear spot could be reproduced for any given pair in the same number of cycles.

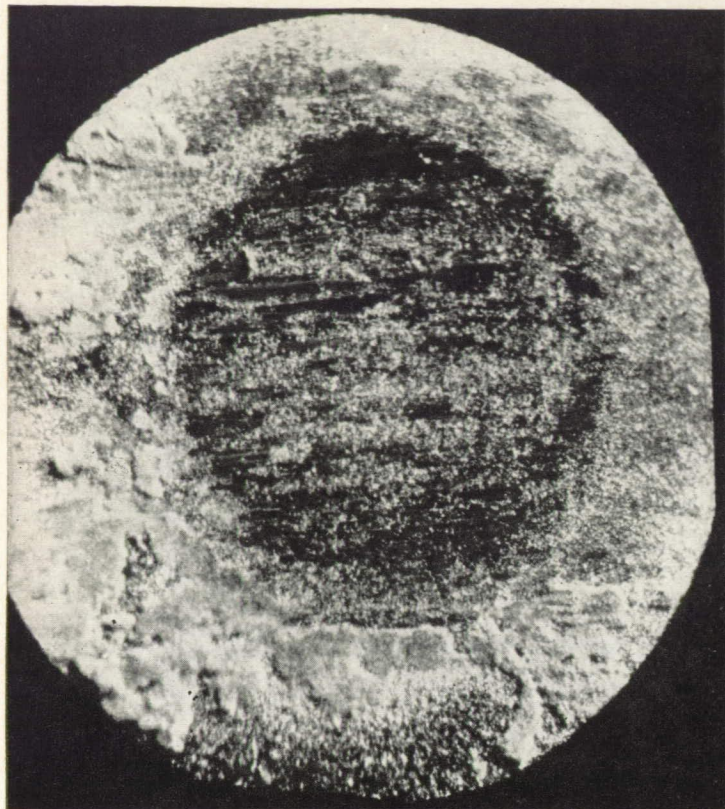
The introduction of a mineral oil film increased the time required to produce fretting but did not otherwise appreciably alter the results.

METAL AGAINST NONMETAL

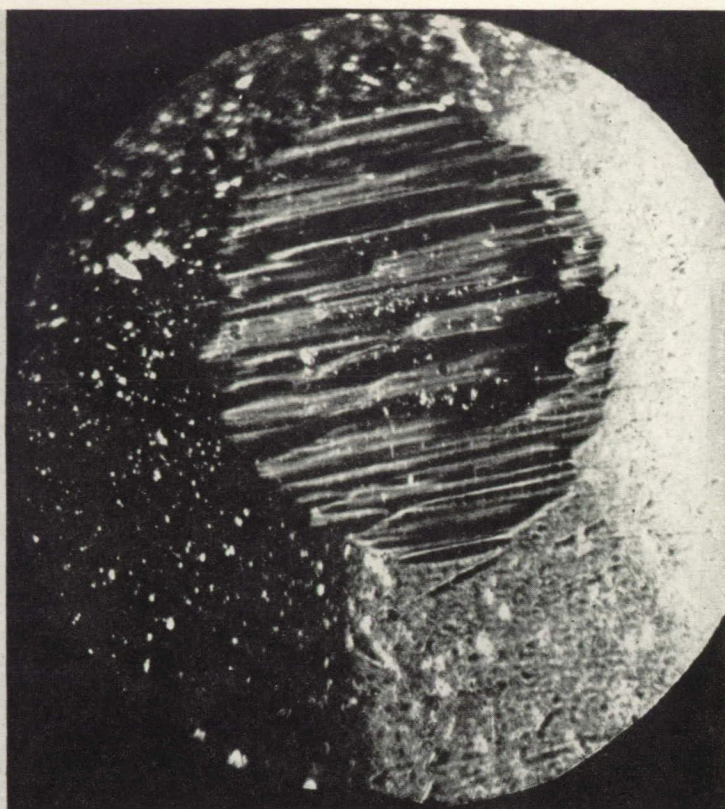
Photomicrographs of the contact area at various stages of the fretting caused by vibrating a chrome-alloy steel ball against a glass specimen are shown in figure 3. Within $\frac{1}{4}$ second (30 cycles), fretting was evident as a small black irregular spot on the ball, which steadily grew in size and became discontinuous (figs. 3 (a) and 3 (b)). A brown, sticky semifluid oxide was then generated within the contact area, spread over the glass, and adhered tenaciously to it. This oxide shares the contact area with the black material (figs. 3 (b) to 3 (f)). A rust-colored fine dry oxide was also formed that was extruded from the area in increasing quantities and accumulated just outside the rubbing area, as evident in figures 3 (b) to 3 (f). Active black spots shifted position as the phenomenon progressed because of disintegration of peaks bearing the load for the moment. Color motion photomicrographs of this action can be seen in the aforementioned film.

After approximately 40,000 cycles, the action stabilized to

a steady-growth condition with continued generation of material in the black spots surrounded by brown semifluid oxide and the formation of fine powder-like oxide. The black color of the active areas indicated the presence of finely divided iron; this indication was further confirmed by subsequent change to dark brown and then to rust brown. If the vibration was stopped, the trapped black material could be seen to gradually turn brown. Finely divided iron, ferrous oxide FeO , and ferrosferic oxide Fe_3O_4 are black, whereas ferric oxide Fe_2O_3 and its hydrated form $\text{Fe}_2\text{O}_3 \cdot \text{H}_2\text{O}$ are reddish. The fretting debris of steel was identified by electron diffraction as Fe_2O_3 and the color changes that occurred suggest the progressive oxidation of the iron. Fretting is apparently initiated by the loosening, due to inherent adhesive forces, of extremely finely divided and apparently virgin material that is extruded and reacts with oxygen simultaneously. Examination of the ball after the experiment revealed a flat oxide-encrusted spot (fig. 4 (a)), which on subsequent cleaning was revealed to be an abraded flat spot (fig. 4 (b)). The sticky oxide on the glass was removed with difficulty but completely with acids, and a striated cracked area was revealed. The striations could not be brought into sharp focus with the microscope, indicating appreciable depth and irregularity (fig. 5 (a)). If the action was continued for longer periods, the glass fragmented locally (fig. 5 (b)) and was pitted.



(a) Before cleaning.



(b) After cleaning.

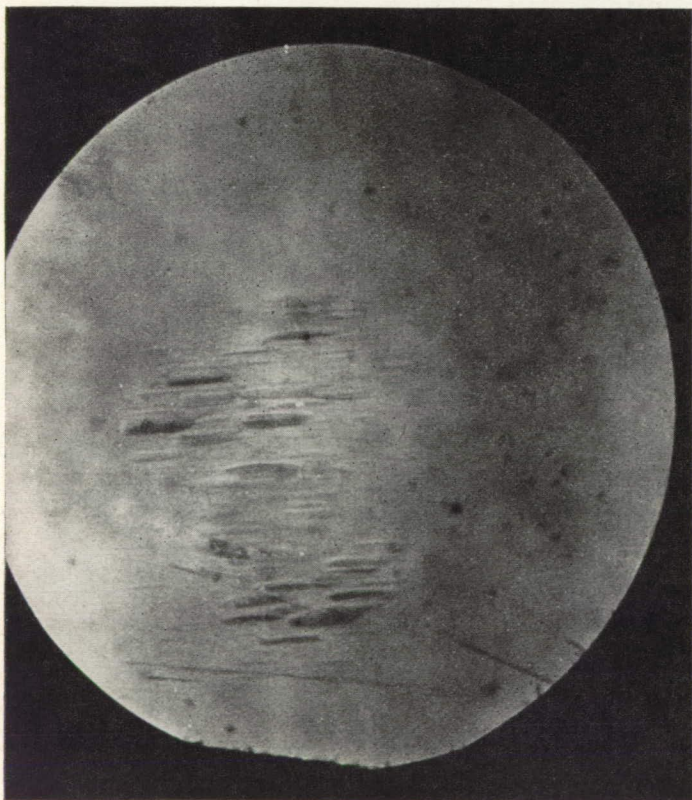
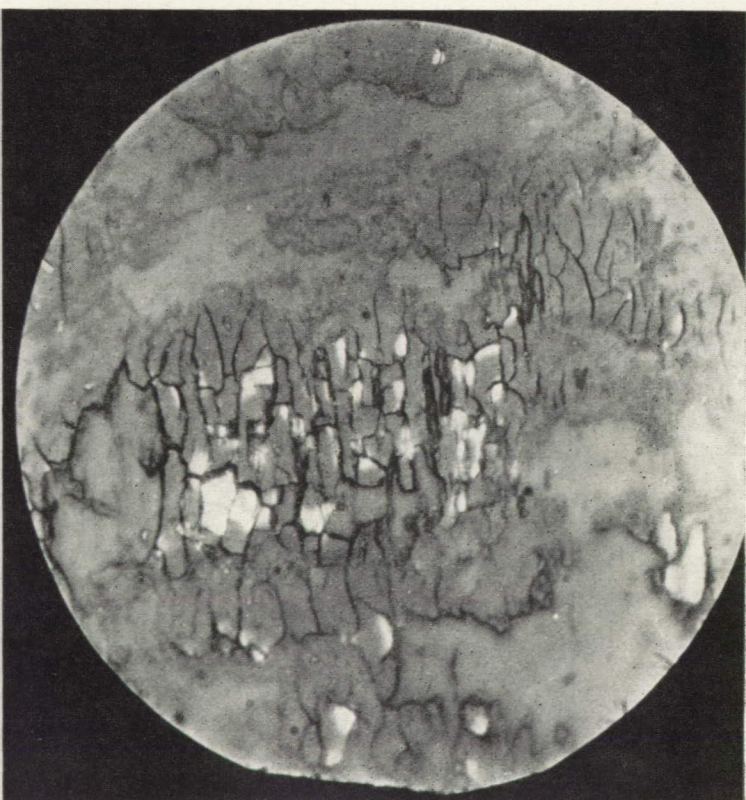
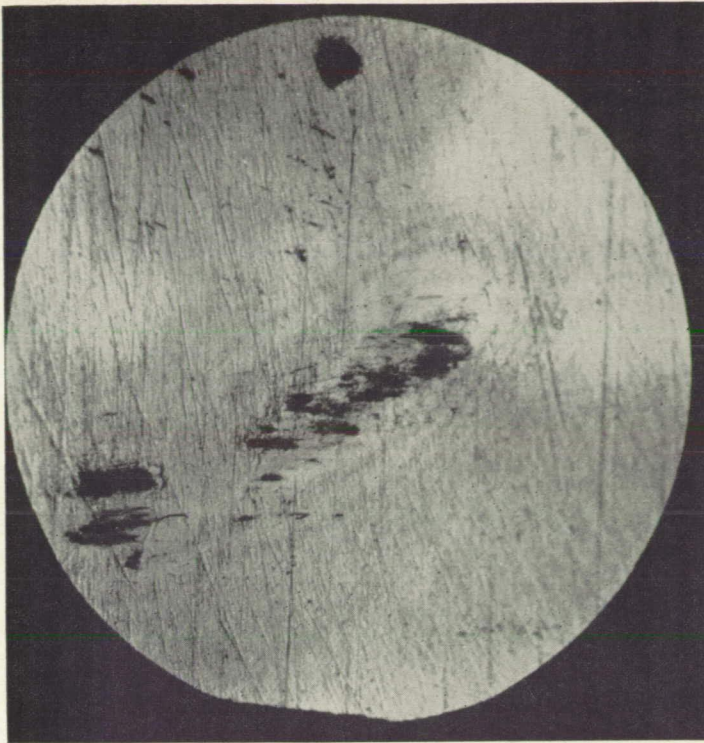
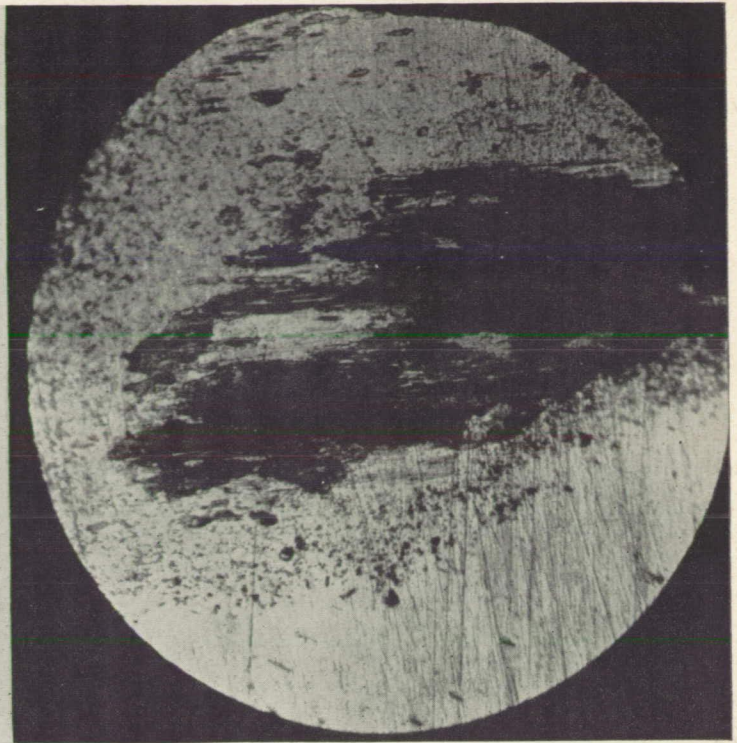
FIGURE 4.—Photomicrographs showing fretting area on chrome-alloy steel ball after 3600 cycles. $\times 120$.(a) Early stage, $\times 120$.(b) Later stage, $\times 250$.

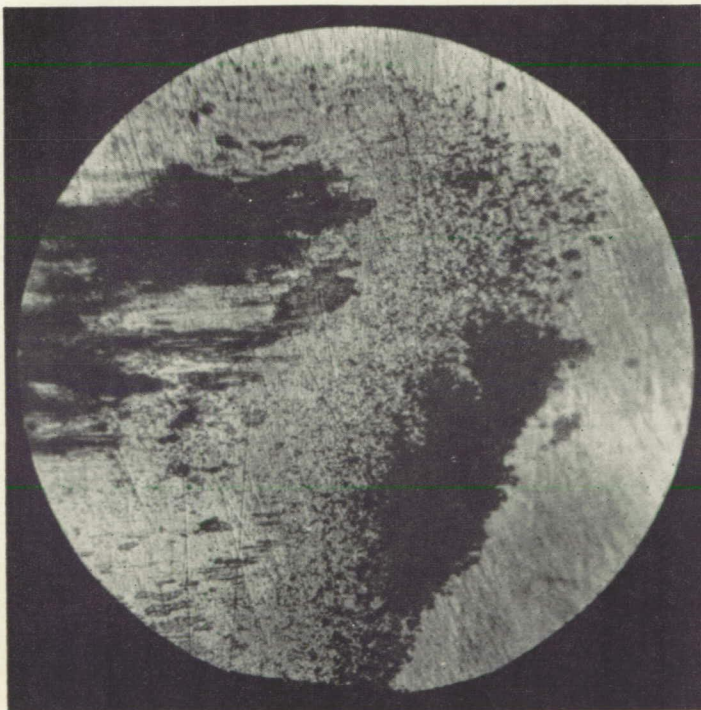
FIGURE 5.—Photomicrographs of cleaned wear area on glass showing striations and cracking produced by vibration of contacting chrome-alloy steel ball.



(a) Early stage.



(b) Advanced stage.

FIGURE 6.—Photomicrographs of convex platinum surface through glass slide showing wear and debris produced. $\times 120$.

(c) Debris accumulation outside field of view of figure 6 (b).

FIGURE 6.—Concluded. Photomicrographs of convex platinum surface through glass slide showing wear and debris produced. $\times 110$.

The introduction of a film of pure mineral oil between the surfaces increased the time for the first evidence of fretting

action 50 times. This delay also provided a slow-motion study of the fretting process and showed that oxides were produced in the same manner in the presence of oil. Once the phenomena were well under way, the modifying effect of the oil was reduced.

Results obtained when other metals were vibrated against glass slides supported the explanation of fretting deduced from the steel-glass experiments. In one experiment, flame-cleaned platinum foil supported by a steel ball was vibrated in contact with clean glass. The generation of a black powder in the characteristic way was observed (fig. 6). This powder remained black and that which adhered to the glass could be completely removed only by a 4-hour treatment in aqua regia. Color, acid resistance, electron-diffraction patterns, and source indicated that the material was finely divided platinum (platinum black). Because platinum will not oxidize in air at any temperature, the theory that oxidation is a primary cause of fretting is not supported.

The action of a stainless-steel (SAE 30705) convex surface against the flat glass was similar to that of the chrome-alloy steel balls except that the number and the extent of the active black areas were greater. Copper produced a brownish-black material that also adhered tenaciously to the glass. At times, large particles of metallic copper could be seen to leave the contact area. During the third minute of vibration, the contact area assumed the greenish tint characteristic of copper oxide. Both these metals were pitted and abraded and the glass was striated.

Under the conditions of the $\frac{1}{2}$ -cycle experiment, two types of action were observed. If the surface of the ball was previously roughened with 2/0 emery paper, the glass would polish the contact area on the ball and become smeared with the semifluid oxide, which would again tenaciously adhere to the glass. The amount of debris accumulated in three $\frac{1}{2}$ cycles of vibration is shown in figure 7 (a).

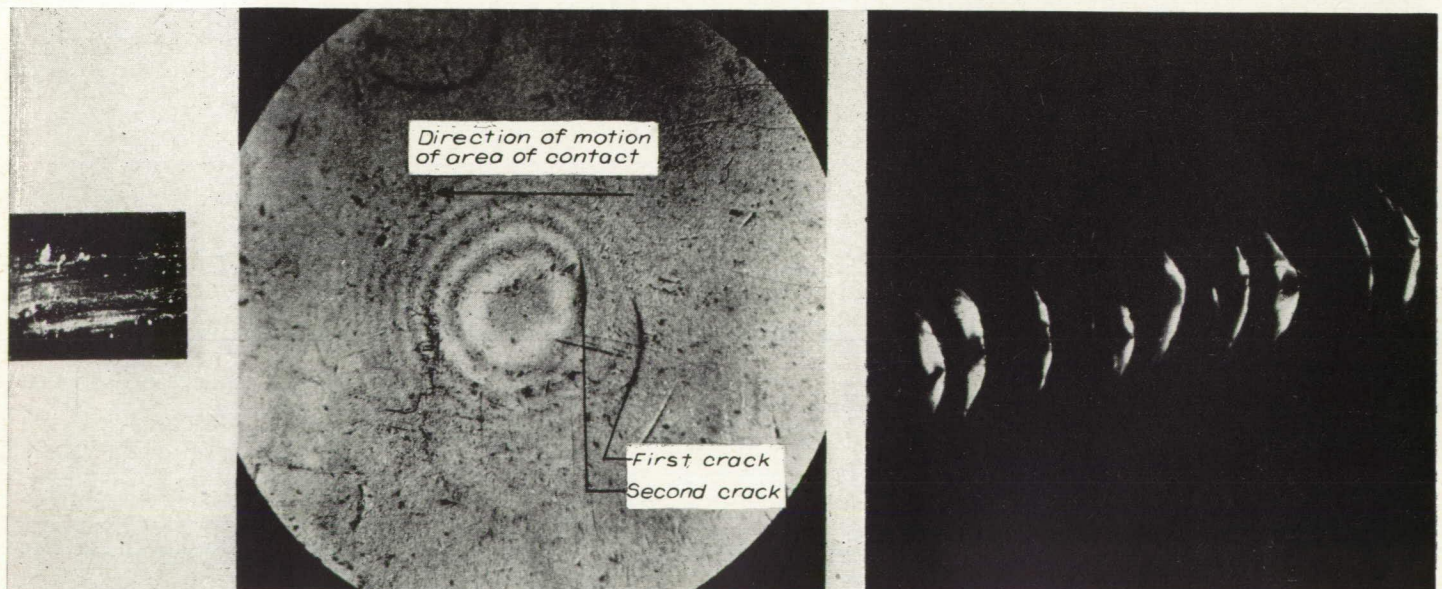
An increase in frictional force was apparent when the oxide was present between the surfaces. This greater motivating force was required to continue relative sliding because a non-lubricating material was present between the surfaces and tenaciously adhered to each. The oxide itself was sheared and the two new faces thus formed became the sliding surfaces. If a ball with its original polished surface was used, much less oxide would be smeared, but the movement would cause the glass to grip the ball and as the relative sliding continued the glass would visibly and audibly crack. The cracking occurred at the trailing edge of the contact area. A photomicrograph of the glass in the process of forming a second crack is shown in figure 7 (b). The row of cracks produced in the glass during continued sliding is shown in figure 7 (c). The smearing of the metallic oxide onto the glass and the cracking of the glass are apparently due to a great amount of adhesion permitted by the extreme cleanliness of the specimens. The surface finish on the ball determines which phenomenon will occur. A rough surface supports the load on a number of peaks, the average particle of which is relatively exposed. Sliding of the smooth glass over these peaks results in disintegration of the peaks and subsequent oxidation of the removed metal. A smooth polished surface on the ball supports the load on a few broad undulations, the material of which is secure. Sliding of the smooth glass over these contact areas results in failure of the glass rather than the metal. This concept is in agreement with reference 8, which presents data to show that the amount of

material transferred from base to rider increased with increase in surface roughness if the rider was harder than the base. With the steel ball considered as the rider and the steel harder than the glass, some incongruity may exist in this comparison in that the glass picked up material from the steel.

The glass appeared to plow into the aluminum, scoring it heavily. Large metallic pieces were mixed with a grayish-black powder. Some of the debris adhered tenaciously to the glass in the characteristic manner. The powder was insoluble in nitric acid but soluble in alkali, indicating that aluminum was still present. Mica and Lucite always wore away the metal surfaces and produced very great amounts of flocculent debris. Observation of the action of the metals was difficult because of the great dilution of the metal debris by the voluminous debris from the flat specimen. This experiment indicated that no property peculiar to glass was inducing the action.

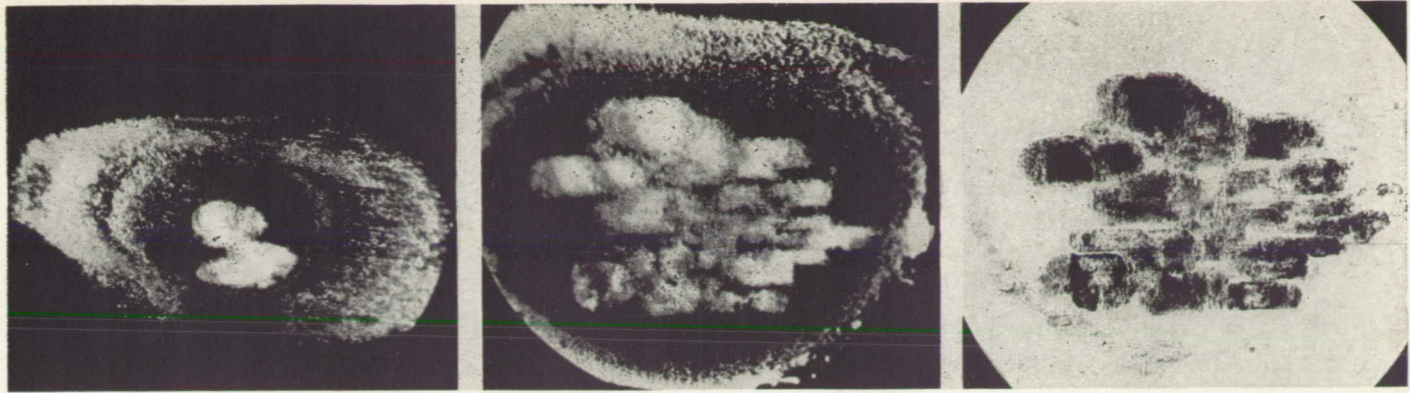
NONMETALS AGAINST NONMETALS

The independence of fretting from chemical action was indicated in a group of experiments involving the nonmetallic, nonoxidizing materials with glass vibrated against quartz, ruby, or mica. By vibrating a ruby with a convex surface against a glass slide and observing the action under a microscope, surface disintegration could be seen to occur. The glass, after 2 to 3 seconds of rubbing, suddenly collapsed locally and distributed debris about the contact area, leaving a pit. This action shifted the bearing area to the edge of the pit and the disintegration was continued (fig. 8). An early stage of disintegration with two pits formed is shown in figure 8 (a), whereas the advanced stage of 20 or more pits is shown in figure 8 (b). The ruby was simply further polished, that is, the microscopic scratches were erased. Quartz against glass acted the same as ruby against glass. Mica



(a) View of glass slide showing amount of debris transferred from roughened steel. (b) View of ball through glass showing second crack being formed in glass on trailing edge of area of contact. (c) View of glass slide showing resultant row of cracks.

FIGURE 7.—Photomicrographs showing effects of $\frac{1}{2}$ cycle vibration between glass and steel surface. $\times 120$.



(a) Early stage.

(b) Advanced stage.

(c) Advanced stage after removal of debris.

FIGURE 8.—Photomicrographs showing results of fretting between ruby and glass. $\times 120$.

against a glass ball and Lucite against a glass ball also behaved similarly. In all cases, the debris was produced, extruded, and accumulated in the same physical manner observed with metal debris, except that no change in color took place. Fretting occurred to mica in the least number of cycles, the number required being progressively greater for Lucite, glass, quartz, and ruby, which is also the order of increasing hardness of the materials. The tendency for fretting between the various nonmetals and glass and the amount of debris generated therefore appear to be inversely proportional to the hardness of the nonmetal.

CONCLUDING REMARKS

The observations made indicate that fretting can occur between two nonmetals. The susceptibility to fretting of nonoxidizing materials such as glass, quartz, ruby, mica, and platinum relegates the role of oxidation as a cause to that of a secondary factor and indicates that finely divided and apparently virgin material is first produced. In the metals, this production of virgin metal is suggested by the color changes through which the metal passed as it oxidized. The oxides are therefore considered a byproduct of the initial action. The spontaneity of the oxidation is probably due to the increased chemical activity of the particles. This increase is called mechanical activation in reference 9, which also states that the increase is more than that produced by cleanliness or reduced particle size. The oxide formation was induced at extremely low speeds and light loads where the dissipation of frictional heat is sufficiently rapid to minimize the occurrence of local hot spots. The possibility of fretting being caused by rubbing off a regenerative high-temperature-forming oxide film is therefore unlikely.

The conclusion is made in reference 2 that alternating slip is a necessary condition for fretting. The production of a smear on the glass surface in $\frac{1}{2}$ cycle and the cracking of glass on smooth metal indicate that alternation of motion is unnecessary to initiate fretting, although alternation does give it its usual characteristics. This idea is supported by the

observation that fretting starts with the sliding or slip. Lowering the frequency only decreases the rate at which fretting occurs. This relation indicates that the basic cause of fretting is independent of frequency.

The metal transfer caused by sliding one metal over another demonstrated in references 8 and 10 was attributed to adhesion. Similarly, in the experiments reported herein the action observed, namely, removal of material, cracking, and general surface destruction, was attributed to adhesion, which leads to the conclusion that fretting is a manifestation of the adhesion component of friction.

RESULTS AND CONCLUSIONS

An experimental investigation was conducted to determine the cause of fretting. Glass and other noncorrosive materials were vibrated in contact with one another and with metals and microscopic observation was made of the action. Convex surfaces were vibrated in contact with stationary flat surfaces at frequencies of 120 cycles or less than 1 cycle per second, an amplitude of 0.001 inch, and a load of 0.2 pound.

The following results and conclusions were indicated:

1. Fretting is initiated by the loosening, due to inherent adhesive forces, of extremely finely divided and apparently virgin material that is extruded from the contact area and reacts with oxygen simultaneously.
2. The fretting of platinum, glass, quartz, ruby, and mica relegated the role of oxidation as a cause to that of a secondary factor.
3. Fretting readily occurred between clean nonmetals and metals. Steel balls vibrating in contact with glass microscope slides provided an excellent method for observing fretting.
4. The initiating of fretting is independent of vibratory motion or high sliding speeds.

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, August 31, 1949.

REFERENCES

1. Anon.: The Corrosion Handbook, Herbert H. Uhlig, ed. John Wiley & Sons, Inc., 1948, pp. 590-597.
2. Tomlinson, G. A., Thorpe, P. L., and Gough, H. J.: An Investigation of the Fretting Corrosion of Closely Fitting Surfaces. *Proc. Inst. Mech. Eng.*, vol. 141, 1939, pp. 223-249.
3. Tomlinson, G. A.: A Molecular Theory of Friction. *Phil. Mag. and Jour. Sci., Suppl.*, ser. 7, vol. 7, no. 46, June 1929, pp. 905-939.
4. Sakmann, B. W., and Rightmire, B. G.: An Investigation of Fretting Corrosion under Several Conditions of Oxidation. NACA TN 1492, 1948.
5. Morton, Hudson T.: Friction Oxidation. The Fafnir Bearing Co. (New Britain, Conn.).
6. Dies, Kurt: Fretting Corrosion, a Chemical-Mechanical Phenomenon. *Eng. Digest*, vol. 5, no. 11, Nov. 1944, p. 324. (From *V. D. I. Zeitschr.*, vol. 87, no. 29-30, July 1943, pp. 475-476.)
7. Godfrey, Douglas: Preliminary Investigation of Fretting. NACA Tech. Film 23. (Available upon request from NACA Headquarters Office, Washington, D. C.)
8. Sakmann, B. W., Burwell, J. T., Jr., and Irvine, J. W., Jr.: Measurements of the Adhesion Component in Friction by Means of Radioactive Indicators. *Jour. Appl. Phys.*, vol. 15, no. 6, June 1944, pp. 459-473.
9. Shaw, M. C.: Mechanical Activation—A Newly Developed Chemical Process. *Jour. Appl. Mech.*, vol. 15, no. 1, March 1948, pp. 37-44.
10. Bowden, F. P., Moore, A. J. W., and Tabor, D.: The Ploughing and Adhesion of Sliding Metals. *Jour. Appl. Phys.*, vol. 14, no. 2, Feb. 1943, pp. 80-91.

REPORT 1010

A RECURRENCE MATRIX SOLUTION FOR THE DYNAMIC RESPONSE OF AIRCRAFT IN GUSTS¹

By JOHN C. HOUBOLT

SUMMARY

A systematic procedure is developed for the calculation of the structural response of aircraft flying through a gust by use of difference equations and matrix notation. The use of difference equations in the solution of dynamic problems is first illustrated by means of a simple-damped-oscillator example. A detailed analysis is then given which leads to a recurrence matrix equation for the determination of the response of an airplane in a gust. The method takes into account wing bending and twisting deformations, fuselage deflection, vertical and pitching motion of the airplane, and some tail forces. The method is based on aerodynamic strip theory, but compressibility and three-dimensional aerodynamic effects can be taken into account approximately by means of over-all corrections. Either a sharp-edge gust or a gust of arbitrary shape in the spanwise or flight directions may be treated. In order to aid in the application of the method to any specific case, a suggested computational procedure is included.

The possibilities of applying the method to a variety of transient aircraft problems, such as landing, are brought out. A brief review of matrix algebra, covering the extent to which it is used in the analysis, is also included.

INTRODUCTION

In the problem of an airplane flying through gusts, accurate predictions of stresses are not always obtained if the interaction between aerodynamic loads and structural deformations is not considered. The present report gives a method for determining the dynamic response of aircraft in gusts in which this interaction is considered. An approach is employed which is a departure from the usual modal type of solution. The time derivatives in the integrodifferential equations of motion of the airplane are replaced by appropriate difference expressions and use is made of matrix notation to express conveniently the conditions of equilibrium at a number of points along the wing span. The result is a systematic procedure which is complete and general in form. The airplane is assumed to be free to translate and pitch. Wing bending, wing twist, and fuselage flexibility are all included. Tail forces due to vertical motion, angle of attack, and gust penetration are also included in the analysis.

With the method, a gust with any gradient in the direction of flight or along the span may be treated. The method is

based on aerodynamic strip theory, but over-all compressibility and aspect-ratio corrections may be included without difficulty, if desired. One such over-all correction is indicated.

In the first part of the report the method of using difference equations in the solution of dynamic problems is illustrated by an example in which the transient response of a simple oscillator is determined. The analysis for the determination of the response of an airplane in a gust is then given. In the following section a computational procedure is suggested. This section is not intended to describe or add to the understanding of the analysis, but by following the directions indicated, the response of any airplane may be found without following through the complete details of the analysis.

Supplementary definitions and derivations are presented in appendixes. Appendix A summarizes the various matrix coefficients and matrices that are used in the analysis, appendix B gives a derivation of the difference equations, appendix C gives a derivation of the flexibility matrices, appendix D gives a derivation of a recurrence equation for evaluating the form of Duhamel's integral which involves an exponential kernel, and appendix E presents a review of the fundamentals of matrix algebra. It is suggested that those not familiar with matrix algebra read appendix E before reading the analysis.

SYMBOLS

a	distance between leading edge of wing and elastic axis
a_1	coefficient used in unsteady lift function for sudden change in angle of attack
A	aspect ratio of wing
A_t	aspect ratio of horizontal tail
b	semispan of wing
c	chord of wing
c_o	chord at wing midspan
c_{o_t}	midspan chord of tail
c_t	mean aerodynamic chord of tail
e	distance between mass center of wing cross section and elastic axis of wing, positive when elastic axis lies forward of mass center
E	Young's modulus of elasticity

¹ Supersedes NACA TN 2060, "A Recurrence Matrix Solution for the Dynamic Response of Aircraft in Gusts" by John C. Houbolt, 1950.

F	suddenly applied force	w	deflection of elastic axis of wing, positive upward, or deflection of mass oscillator
G	shear modulus of elasticity	w_f	deflection of fuselage, positive upward
i	integers 0, 1, 2, 3, 4, and 5 used to designate stations (for most part used as parenthetical numbers, that is, $w(3)$ is deflection at station 3)	W_1	fuselage modal function, zero at wing elastic axis and unity at tail one-quarter-chord location
I	bending moment of inertia	x	distance along fuselage measured from wing elastic axis, positive in rearward direction
J	torsional stiffness constant	x_n	distance from foremost part of nose to elastic axis
k	radius of gyration of wing mass about elastic axis or elastic spring constant	x_t	distance from elastic axis to one-quarter-chord location on tail
l	length of section associated with a spanwise station	y	distance along wing measured from center of airplane
L	aerodynamic lift over interval l on wing	z	ratio of dynamic deflection to static deflection
L_f	shear force transmitted to wing by fuselage	α_t	angle of attack of horizontal tail, positive in the stalling direction
L_g	aerodynamic lift over interval l on wing due to gust	β	forward-speed and aspect-ratio factor for wing ($m_A \pi \rho U$) or coefficient of damping for spring oscillator
L_{g_t}	one-half aerodynamic lift on tail due to gust	β_t	forward-speed and aspect-ratio factor for tail
L_t	one-half total aerodynamic lift on tail		$\left(\frac{1}{2} m_{A_t} \pi \rho S_t U\right)$
L_1	part of aerodynamic lift over interval l on wing (see equation (16))	γ	exponential coefficient in Φ function associated with time t
L_2	part of aerodynamic lift over interval l on wing (see equation (17))		$\left(\gamma = \frac{2U}{c_o} \lambda\right)$
m	mass of beam included in interval l or concentrated mass in spring oscillator	δ	coefficient of fuselage modal function
$\bar{m}$	mass m including apparent mass effect $\left(m + \frac{\pi \rho l c^2}{4}\right)$	ϵ	time interval
m_A	assumed over-all compressibility and aspect-ratio correction for wing $\left(\frac{A}{2 + A\sqrt{1-M^2}}\right)$	λ	exponential coefficient in Φ function associated with variable s
m_{A_t}	assumed over-all compressibility and aspect-ratio correction for horizontal tail $\left(\frac{A_t}{2 + A_t\sqrt{1-M^2}}\right)$	λ_t	dimensionless interval between $i-1$ and i stations ($\lambda_t b$ is actual length)
$\bar{m}_e$	mass moment m_e including apparent mass effect $\left(m_e + \frac{\pi \rho l c^3}{4} \left(\frac{1}{2} - \frac{a}{c}\right)\right)$	ρ	mass density of air
m_f	mass of fuselage per unit length	φ	angle of twist of wing, positive in stalling direction
$\bar{m}k^2$	mass polar moment of inertia mk^2 including apparent mass effects $\left(mk^2 + \frac{\pi \rho l c^4}{4} \left(\frac{1}{2} - \frac{a}{c}\right)^2 + \frac{\pi \rho l c^4}{128}\right)$	ψ	function which denotes growth of lift on rigid airfoil entering sharp-edge gust (used without subscript to indicate function for wing and with subscript t used to indicate function for tail)
M	Mach number or aerodynamic moment over interval l about elastic axis of wing	ω_f	natural frequency associated with W_1 , radians per second
M_f	moment transmitted to wing by fuselage	$I(t)$	unit-step function
n	integers 0, 1, 2, 3, and so forth to designate number of time intervals passed	$1-\Phi$	function which denotes growth of lift on airfoil following sudden change in angle of attack (used without subscript to indicate function for wing and with subscript t used to indicate function for tail)
p	normal load acting at a station	$\begin{bmatrix} & \end{bmatrix}$	square matrix
p_f	fuselage inertia loading per unit length	$\begin{bmatrix} & \\ & \end{bmatrix}$	rectangular matrix
q	torsional load acting at a station	$\begin{bmatrix} \\ & \end{bmatrix}$	column matrix
s	distance traveled by wing in half-chords $\left(\frac{2U}{c_o} t\right)$, where midspan chord c_o is used as reference chord	$\begin{bmatrix} \\ & \\ & \end{bmatrix}$	row matrix
Δs	distance interval in half-chords $\left(\frac{2U}{c_o} \epsilon\right)$	Subscripts:	
S_t	horizontal-tail area	t	tail
t, τ	time, zero at beginning of gust penetration	0, 1, 2, 3, . . . n	number of time intervals passed
U	forward velocity of flight	0, 1, 2, 3, 4, 5 or i	station (however, station is usually given as parenthetical number, such as $w(3)$ for deflection at station 3); i is also used as general subscript in appendix A
v	vertical velocity of gust		

All the terms, coefficients, and matrices not defined in this section are defined in appendix A.

Dots are used to indicate derivatives with respect to time; for example, $\frac{\partial w}{\partial t} = \dot{w}$ or $\frac{\partial w}{\partial \tau} = \dot{w}$.

ANALYSIS

TRANSIENT RESPONSE OF A SIMPLE DAMPED OSCILLATOR

In order to illustrate the use of difference equations and to test the accuracy of the procedure that is to be used in the solution of the more complicated gust problems, the solution of a simple problem having a known analytical solution is first presented. The problem is to compute the response of the damped oscillator shown in figure 1 to a suddenly applied force. The differential equation of motion of this system due to the suddenly applied force is

$$m\ddot{w} + \beta\dot{w} + kw = FI(t) \quad (1)$$

By use of difference equations this differential equation may be transformed into an equation which involves deflection ordinates at several successive values of time. Probably the most commonly used difference equations are the following (see appendix B for derivation):

$$\dot{w}_n = \frac{w_{n+1} - w_{n-1}}{2\epsilon} \quad (2)$$

$$\ddot{w}_n = \frac{w_{n+1} - 2w_n + w_{n-1}}{\epsilon^2} \quad (3)$$

which give the derivatives at the intermediate of three successive ordinates. Although these equations are quite adequate for the oscillator problem of the present report, they cannot be used in the gust analysis which follows. Rather, for reasons which are brought out in a subsequent part of the analysis, equations that give the derivatives at the end ordinate of several successive ordinates must be used. If only three successive ordinates are used, the derivatives so found are not accurate enough to be useful. If a fourth ordinate is added, however, derivatives may be taken at the end ordinate with accuracies which are comparable to those given by equations (2) and (3). Such derivatives are derived also in appendix B and are given by the equations:

$$\dot{w}_n = \frac{11w_n - 18w_{n-1} + 9w_{n-2} - 2w_{n-3}}{6\epsilon} \quad (4)$$

$$\ddot{w}_n = \frac{2w_n - 5w_{n-1} + 4w_{n-2} - w_{n-3}}{\epsilon^2} \quad (5)$$

Although either equations (2) and (3) or equations (4) and (5) may be used in the solution of this oscillator problem, only equations (4) and (5) will be used, since only these equations can be used in the gust-problem solution presented in this report.

If the derivatives in equation (1) are replaced by the

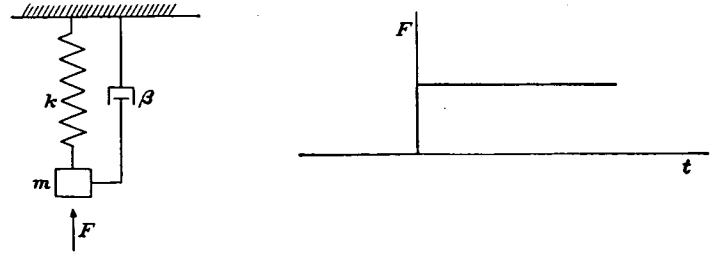


FIGURE 1.—Damped oscillator and suddenly applied force.

difference equations (4) and (5), the following equation is obtained:

$$\left(2 + \frac{11}{6} \frac{\beta\epsilon}{m} + \frac{k}{m} \epsilon^2\right) w_n = \left(5 + \frac{3\beta\epsilon}{m}\right) w_{n-1} - \left(4 + \frac{9\beta\epsilon}{6m}\right) w_{n-2} + \left(1 + \frac{2\beta\epsilon}{6m}\right) w_{n-3} + \frac{\epsilon^2 F}{m} \quad (6)$$

This equation may be said to be the difference equation of motion. If the more general case of a variable applied force were being considered, the factor F in this equation would be replaced by F_n , the value of the force present at the time $t = n\epsilon$.

If a specific case is now considered, in which $\frac{k}{m} = 400$, $\frac{\beta}{2m} = 2$, $\epsilon = 0.01$, $F = 1$, and the notation $z = \frac{w}{F/k}$ (ratio of dynamic deflection to static deflection) is used, equation (6) becomes

$$z_n = 0.018927 + 2.42272 z_{n-1} - 1.92114 z_{n-2} + 0.47949 z_{n-3} \quad (7)$$

This equation may be regarded as a recurrence formula; the value z_n may be interpreted as the deflection to come and may be found easily from the three preceding deflections z_{n-1} , z_{n-2} , and z_{n-3} . Then with the newly found value z_n and with z_{n-1} and z_{n-2} , the next deflection can be found, and so on. This process thus gives a step-by-step derivation of the time history of deflection and may be carried out as far as is desired. Of course the process must start with known initial values of z . These values can be found with the aid of the initial conditions of the problem by means of the following approach.

The difference equations for the first and second derivatives at the third ordinate of four successive ordinates are (see appendix B)

$$\dot{w}_n = \frac{1}{6\epsilon} (2w_{n+1} + 3w_n - 6w_{n-1} + w_{n-2})$$

$$\ddot{w}_n = \frac{1}{\epsilon^2} (w_{n+1} - 2w_n + w_{n-1})$$

If these equations are taken to apply at $t=0$ ($n=0$), they become

$$\dot{w}_0 = \frac{1}{6\epsilon} (2w_1 + 3w_0 - 6w_{-1} + w_{-2}) \quad (8)$$

$$\ddot{w}_0 = \frac{1}{\epsilon^2} (w_1 - 2w_0 + w_{-1}) \quad (9)$$

For the problem under consideration the primary initial conditions are that, at $t=0$, the displacement and velocity are zero. By use of equation (1) or by reasoning from Newton's second law, a secondary initial condition can be established—that is, the acceleration immediately following the application of the unit force must be $1/m$. In equation form these conditions are

$$w_0=0$$

$$\dot{w}_0=0$$

$$\ddot{w}_0=\frac{1}{m}$$

By substitution of these values into equations (8) and (9) and by use of the notation $z=\frac{w}{F/k}$, the following relations can be found to exist between the ordinates:

$$\left. \begin{aligned} z_0 &= 0 \\ z_{-2} &= 0.24 - 8z_1 \\ z_{-1} &= 0.04 - z_1 \end{aligned} \right\} \quad (10)$$

Substitution of these values into equation (7), with n set equal to 1, gives an equation from which z_1 , the deflection at $t=\epsilon$, may be evaluated. Three successive deflections can now be established: the deflection at $t=\epsilon$, the zero deflection at $t=0$, and a fictitious deflection for $t=-\epsilon$ given by equation (10). In the real problem no deflection exists for t less than zero; the assumption that a deflection does exist before t is zero is simply a means for allowing the recurrence formula, equation (7), to apply at the origin as well as at later values of time. Furthermore, no violation is made of the conditions under consideration because, mathematically, the response after $t=0$ is not influenced by the response that may exist before $t=0$, so long as the initial conditions are satisfied. The process just described for treating the initial conditions is actually not different from the procedure commonly employed in difference-equation approaches, in which exterior points near a region under consideration are written in terms of the interior points by means of the boundary conditions.

With the initial values of deflection thus established the

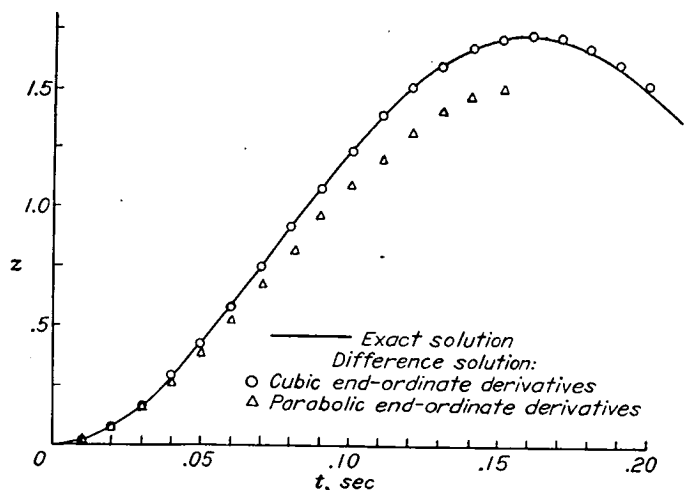


FIGURE 2.—Comparison of exact and difference-equation solutions for response of damped oscillator.

step-by-step evaluation of succeeding deflections proceeds in a straight-forward manner—that is, equation (7) is now evaluated for $n=2$, then for $n=3$, and so on. The response of the oscillator for the physical constants listed previously is given in figure 2. The comparison between the difference solution shown in this figure and the exact solution of equation (1) is seen to be good. As a matter of interest, the solution is also shown in this figure that is obtained by the use of the parabolic end-ordinate derivative which involves only three successive ordinates. The agreement in this case is seen to be quite bad. If equations (2) and (3) had been used, on the other hand, the difference solution (in this case for w_{n+1}) would correspond to that given for the cubic end-ordinate derivative.

RECURRENCE MATRIX EQUATION FOR RESPONSE OF AN AIRPLANE IN A GUST

In order to help the reader to obtain a perspective of what is to be covered in this section, the following basic phases of the analysis are given:

- (1) The assumptions are stated.
- (2) The coordinate system and displacements are defined.
- (3) The aerodynamic lift and moment are defined.
- (4) The normal and torsional dynamic loadings (inertia forces, aerodynamic forces, and fuselage forces) on the wing are derived.
- (5) The equations of elastic deformation—wing vertical motion, wing rotation, and fuselage bending—are given.
- (6) The dynamic loadings on the wing are transformed into difference equations.
- (7) The equations of elastic deformation and the difference equations for loading are combined to give the recurrence matrix equation for response.

In succeeding sections the initial response is determined, the method for evaluating the gust forces is shown, and the method for computing the loads and stresses is indicated.

Assumptions.—In this analysis an attempt is made to obtain the simplest and most direct solution to the problem with a minimum of simplifying assumptions. The case treated is that of an airplane having an essentially straight wing and penetrating a gust of known structure. The tail is considered to penetrate subsequently the same gust as does the wing. The assumptions made are as follows:

Assumptions pertaining to elasticity and airplane motion:

- (1) The usual assumptions of engineering beam theory are made.
- (2) The fuselage is free to pitch and move vertically. The portion of the fuselage in front of the elastic axis of the wing is assumed for convenience to be rigid. The portion of the fuselage rearward of the elastic axis is assumed flexible, and the elastic deflection is assumed to be given by a single modal function.
- (3) The tail is assumed rigid.

Assumptions pertaining to aerodynamic forces:

- (1) Aerodynamic strip theory applies. Three-dimensional effects, however, may be taken into account approximately by means of over-all corrections. Some such corrections are indicated.
- (2) The gust force and forces due to vertical and pitching motion are the only tail forces considered. Other forces of known character may be included, however, if desired.

(3) Aerodynamic forces on the fuselage are neglected.

Coordinate system and displacements.—Position on the airplane is denoted by an orthogonal system of axes. The origin is at the intersection of the wing elastic axis with the plane of symmetry of the airplane: the w -axis runs positive upward, the x -axis runs along the fuselage positive in the rearward direction, and the y -axis runs spanwise. The wing semispan is considered to be divided into six, not necessarily equal, sections, with a station point at the middle of each section. (See fig. 3.) More or fewer stations could be chosen, but it is believed that six is a fair compromise between the amount of labor involved in setting up a solution and the accuracy desired. The interval between stations is designated by the number of the station at the outboard end of the interval. Station 0 is located near the wing root and generally may be located where the fuselage intersects the wing. In this way the concentrated forces of the fuselage are allowed to act through station 0. The other five stations are then located in any convenient manner so as to fall at concentrated mass locations or at points which represent the average of distributed masses, station 5 being nearest the tip. The total mass within a section is assumed to be concentrated at the station point, and the average of the section geometry (chord, elastic axis position, and so on) is assumed to apply. In this way the wing is assumed to be a beam subject to six load concentrations and as such will have a linear moment variation between each station. The further assumption is made that the $\frac{1}{EI}$ variation is linear between each station. With these assumptions for the EI variation and concentrated load locations, equations for deflection at each station point may be derived (appendix C) by direct analytical treatment.

The displacements of the cross section at each station of the wing are given as the deflection of and rotation about the wing elastic axis as shown in figure 4. The fuselage displacements are shown in figure 5 and are given by the equations:

$$w_f = w(0) - \varphi(0)x \quad (11)$$

for the forward section and

$$w_f = w(0) - \varphi(0)x + W_1\delta \quad (12)$$

for the rearward section. The function W_1 is taken as the fundamental mode of vibration of the fuselage and tail assembly, when the fuselage is considered to be clamped as a cantilever beam at the elastic-axis location of the wing, and is given in terms of a unit deflection at the $\frac{1}{4}$ -chord position on the tail. With this function to represent the elastic deformation of the fuselage the deflection and angle of attack of the tail are found with the aid of equation (12) to be

$$w_f(x_t) = w(0) - \varphi(0)x_t + \delta \quad (13)$$

$$\alpha_t = -\left.\frac{dw_f}{dx}\right|_{x=x_t} = \varphi(0) - \delta\theta_1 \quad (14)$$

where

$$\theta_1 = \left.\frac{dW_1}{dx}\right|_{x=x_t}$$

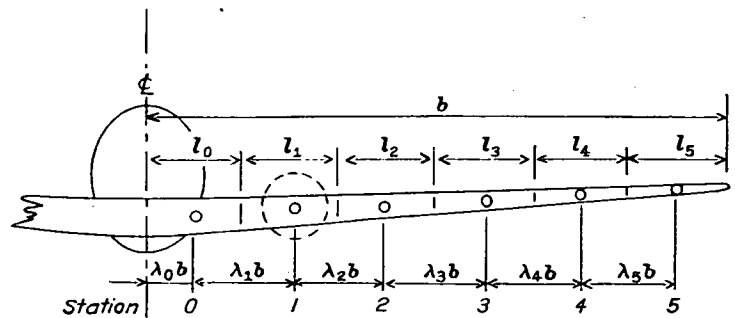


FIGURE 3.—Division of wing into sections.

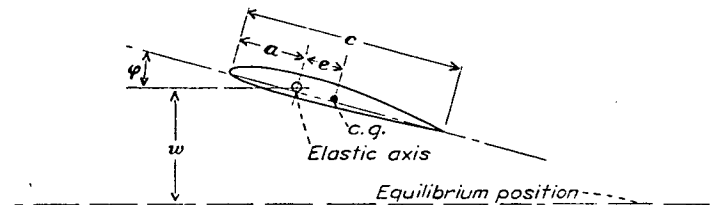


FIGURE 4.—Displacements of a wing cross section.

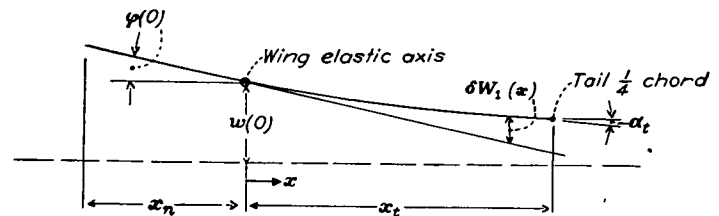
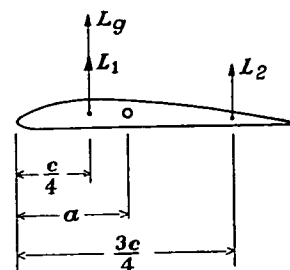


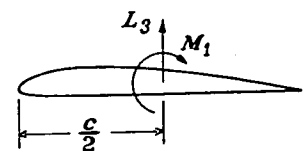
FIGURE 5.—Coordinate system for fuselage displacement.

Aerodynamic lift and moment.—Before going into the details of the analysis it is felt worthwhile to define and describe the nature of the aerodynamic forces to which the wing is subjected. These forces originate from two sources: they arise directly from the gust encountered, and they arise from the ensuing airplane motion. The equations for the aerodynamic lift and moment that develop are herein set up in a convenient form on the basis of work given in references 1 to 4. In these investigations various methods for separating the lift forces have been used, but the particular method for separating these forces is not important so long as they are taken into account properly.

In the present report the aerodynamic lift and moment are considered to be composed of two parts: one part, designated as the lift or moment due to circulation, which includes all lift forces or moments exclusive of aerodynamic inertia effects and the other part, which is due solely to these inertia effects. These lift forces and moments can be resolved into the force systems acting on the airfoil as shown in sketch 1 (forces due to circulation) and sketch 2 (inertia force and moment).



SKETCH 1.



SKETCH 2.

The force L_g is the lift force developed by the gust and corresponds to the gust force that would develop on the airfoil considered rigid and restrained against vertical motion. All the other forces occur as a result of motion of the airfoil. These forces, as well as the gust force, are given for an interval l of the span by the equations: For the forces due to circulation,

$$L_g = m_A \pi \rho c l U \int_0^t \frac{\partial v}{\partial \tau} \psi(t-\tau) d\tau \quad (15)$$

$$L_1 = m_A \pi \rho c l U \int_0^t \left[U \dot{\Phi} - \ddot{w} + c \left(\frac{3}{4} - \frac{a}{c} \right) \ddot{\psi} \right] [1 - \Phi(t-\tau)] d\tau \quad (16)$$

$$\bar{L}_2 = \frac{m_A \pi \rho l c^2}{4} U \dot{\Phi} \quad (17)$$

and for the inertia force and moment,

$$L_3 = \frac{\pi \rho l c^2}{4} \left[-\ddot{w} + \left(\frac{c}{2} - a \right) \ddot{\psi} \right] \quad (18)$$

$$M_1 = -\frac{\pi \rho l c^4}{128} \ddot{\psi} \quad (19)$$

where

m_A factor which can be used to introduce over-all compressibility and aspect-ratio corrections; in this report the factor is assumed to be given by

$$\frac{A}{2 + A\sqrt{1-M^2}}$$

$1-\Phi$ lift function which denotes the growth of lift on an airfoil following a sudden change in angle of attack

ψ lift function which denotes the growth of lift on a rigid airfoil entering a sharp-edge gust

The functions $1-\Phi$ and ψ and the correction m_A are established as follows. In reference 5, approximate equations are derived which give the lift-coefficient form of the growth of lift on a finite wing following a sudden change in angle of attack or due to the penetration of a sharp-edge gust. The equations may conveniently be considered as the product of a factor, which may be regarded as a lift-curve slope, and an unsteady lift function, designated by $1-\Phi$ for the function due to the angle-of-attack change and by ψ for the function due to the sharp-edge gust. These unsteady lift functions are shown in figures 6 and 7 and are given by the following equations: For the $1-\Phi$ functions

$$(1-\Phi)_{A=3} = 1 - 0.283e^{-0.540s} \quad (20a)$$

$$(1-\Phi)_{A=6} = 1 - 0.361e^{-0.381s} \quad (20b)$$

$$(1-\Phi)_{A=\infty} = 1 - 0.165e^{-0.045s} - 0.335e^{-0.300s} \quad (20c)$$

and for the ψ functions

$$\psi_{A=3} = 1 - 0.679e^{-0.558s} - 0.227e^{-3.20s} \quad (21a)$$

$$\psi_{A=6} = 1 - 0.448e^{-0.290s} - 0.272e^{-0.725s} - 0.193e^{-3.00s} \quad (21b)$$

$$\psi_{A=\infty} = 1 - 0.236e^{-0.058s} - 0.513e^{-0.364s} - 0.171e^{-2.42s} \quad (21c)$$

$$\psi_{A=\infty} = 1 - 0.500e^{-0.130s} - 0.500e^{-s} \quad (22)$$

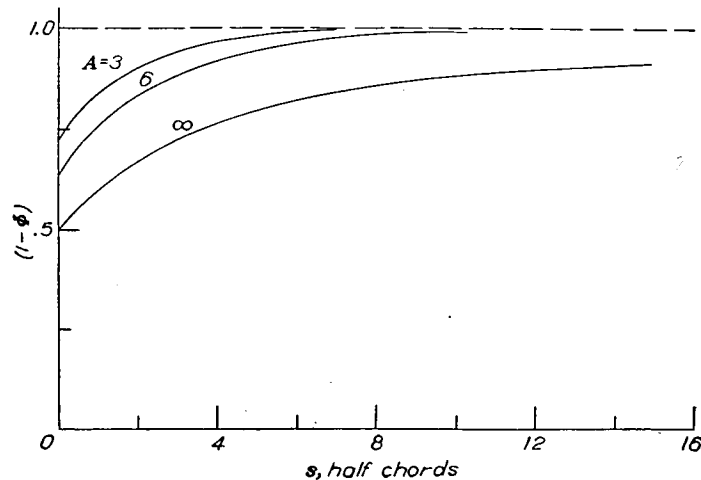


FIGURE 6.—Lift functions for sudden change in angle of attack. (See equations (20).)

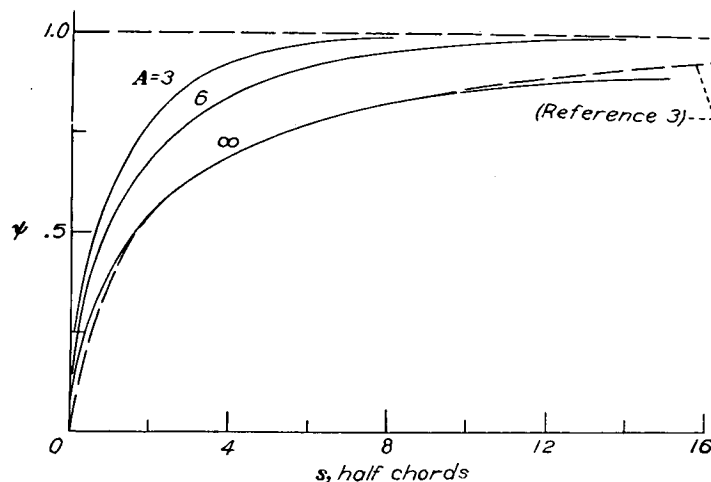


FIGURE 7.—Lift functions for wings entering a sharp-edge gust. (See equations (21) and (22).)

Equations (21) are based on equations of reference 5; whereas equation (22) is the ψ function that is suggested for wings of infinite aspect ratio in reference 3. Inspection of equations (20) shows that the Φ function for aspect ratios 3 and 6 is given by a single exponential term. It is probable that the Φ function for higher aspect ratios, say 10 and even 20, may also be given to a sufficient approximation by a single exponential term. Therefore, the assumption is made that in general Φ may be represented by an equation of the form

$$\Phi = a_1 e^{-\lambda s} \quad (23)$$

Interpolation, for example, of the curves in figure 6 shows that the Φ function for aspect ratio 10 might be approximated by the equation:

$$\Phi_{A=10} = 0.41e^{-0.3s} \quad (24)$$

The analysis does not necessarily limit Φ to a single exponential term. Other terms could be added with some increase in labor, but the degree of refinement obtained is not expected to add much to the over-all accuracy of the solution.

Although the functions given by equations (20) to (22) are known to approximate the true functions quite well over a large range in s ($s = \frac{2U}{c_0} t$), the ψ functions given by

equations (21) do not vanish, as they should, when $t=0$. When used in the computational procedures given hereinafter, these functions, therefore, are to be taken as zero when $t=0$. Another point to note is that the variable s is given in terms of a reference chord c_0 ; thus this variable as applied to the wing is different, in general, from the variable as applied to the tail.

Examination of the values of lift-curve slope, which were stated to be present in the equations taken from reference 5, reveals that they may be approximated with good accuracy by the product of 2π and the often-used aspect-ratio correction $\frac{A}{A+2}$ for steady incompressible flow. In the present report it is assumed that compressibility and aspect-ratio corrections can be made by replacing this aspect-ratio correction by a compressible aspect-ratio correction defined by $\frac{A'}{A'+2}$, where $A' = A\sqrt{1-M^2}$, and by multiplying this correction by the Glauert-Prandtl Mach number correction $\frac{1}{\sqrt{1-M^2}}$ to give the product m_A . The procedure then for taking into account three-dimensional and compressibility effects in the present analysis is to determine m_A from the forward speed and aspect ratio of the wing and to use the $1-\Phi$ and ψ functions, equations (20) to (24), for the aspect ratio which is nearest that of the wing.

Some word of explanation of equation (16) might be worthwhile at this point. The $\Phi(t-\tau)$ function is associated with the lift forces which are due to the wake. Without this term the equation would yield the steady lift corresponding to the instantaneous values of angle of attack and vertical velocity. If equation (16) is integrated by parts and the conditions are stipulated that w , $\dot{w}$, φ , and $\dot{\varphi}$ are all zero at $t=0$, the following equation may be found:

$$L_1 = \beta c l \dot{\Phi}_0 w - (1 - \Phi_0) \beta c l \dot{w} + \beta c l U \left[1 - \Phi_0 - \frac{c}{U} \left(\frac{3}{4} - \frac{a}{c} \right) \dot{\Phi}_0 \right] \varphi + (1 - \Phi_0) \beta c^2 l \left(\frac{3}{4} - \frac{a}{c} \right) \dot{\varphi} + \beta c l \int_0^t w \ddot{\Phi}(t-\tau) d\tau - \beta c l U \int_0^t \varphi \dot{\Phi}(t-\tau) d\tau - \beta c^2 l \left(\frac{3}{4} - \frac{a}{c} \right) \int_0^t \varphi \ddot{\Phi}(t-\tau) d\tau \quad (25)$$

where β has been introduced as a forward-speed and aspect-ratio parameter defined by the equation

$$\beta = m_A \pi \rho U \quad (26)$$

With reference to equation (23), Φ_0 and $\dot{\Phi}_0$ in equation (25) would have the values

$$\Phi_0 = a_1$$

$$\dot{\Phi}_0 = -\frac{2U}{c_0} \lambda a_1$$

The form of L_1 given by equation (25) is presented because this form is more convenient to use in the present analysis.

For this analysis the total lift and moment acting at the elastic-axis location are desired. For the present, the total lift L and moment M of the forces due to circulation are found; the inertia force and moment are to be treated

separately. Summation of all the lift forces due to circulation and summation of the moments of these forces about the elastic axis gives the desired equations for the aerodynamic lift and moment acting on the airfoil over an interval l as follows:

$$L = L_1 + L_2 + L_3 \quad (27)$$

$$M = \left(a - \frac{c}{4} \right) L_1 - \left(\frac{3c}{4} - a \right) L_2 + \left(a - \frac{c}{4} \right) L_3 \quad (28)$$

The loading on the wing.—The normal and torsional dynamic loads that are held in equilibrium by the elastic restoring forces of the wing may be found by considering all the aerodynamic and inertia forces that act on the wing. The mass situated at any station (see fig. 4) can be shown to have an inertia normal force equal to

$$-m\ddot{w} + me\ddot{\varphi}$$

and an inertia torsional moment about the elastic axis equal to

$$me\ddot{w} - mk^2\ddot{\varphi}$$

If the aerodynamic forces and moments (see equations (18), (19), (27), and (28)) are added to these inertia loadings, the total normal and torsional loadings on the wing at each station are found to be given, respectively, by the equations:

$$p = -m\ddot{w} + me\ddot{\varphi} + L + L_3$$

$$q = me\ddot{w} - mk^2\ddot{\varphi} + M - \left(\frac{c}{2} - a \right) L_3 + M_1$$

The terms L_3 and M_1 ordinarily would appear with the aerodynamic lift and moment values but are treated separately so that they can be combined with the structural mass terms. If use is made of equations (18) and (19), the loading equations become

$$p = -\bar{m}\ddot{w} + \bar{m}e\ddot{\varphi} + L \quad (29)$$

$$q = \bar{m}e\ddot{w} - \bar{m}k^2\ddot{\varphi} + M \quad (30)$$

where

$$\bar{m} = \left(m + \frac{\pi \rho l c^2}{4} \right)$$

$$\bar{m}e = \left[me + \frac{\pi \rho l c^3}{4} \left(\frac{1}{2} - \frac{a}{c} \right) \right]$$

$$\bar{m}k^2 = \left[mk^2 + \frac{\pi \rho l c^4}{4} \left(\frac{1}{2} - \frac{a}{c} \right)^2 + \frac{\pi \rho l c^4}{128} \right]$$

The terms appearing with the structural mass quantities in the definitions of $\bar{m}$, $\bar{m}e$, and $\bar{m}k^2$ are the terms which are commonly associated with apparent mass effects.

The value of the shear forces L_f and the moment M_f transmitted to the wing by the fuselage can be found in the following manner. From equations (11) and (12) the values of the inertia loading on the forward and rearward sections of the fuselage can be shown to be given, respectively, by the equations:

$$p_f = -m_f [\ddot{w}(0) - \ddot{\varphi}(0)x] \quad (31)$$

$$p_f = -m_f [\ddot{w}(0) - \ddot{\varphi}(0)x + \ddot{\delta} W_1] \quad (32)$$

Integration of these inertia loadings over the length of the fuselage and addition of the aerodynamic tail load $2L_t$ give the value of the total load transmitted to the wing; one-half of this load is designated by L_f and is assumed to act at station 0, the other half being considered to act through the corresponding station on the other half of the wing. Integration of the moment of the inertia loading about the elastic-axis location and addition of the moment $-2x_t L_t$ of the tail forces give the total moment due to the fuselage; one-half of the moment is designated M_f and acts at station 0. The values of L_f and M_f thus found can be given by the equations:

$$L_f = -M_1 \ddot{w}(0) + M_2 \ddot{\varphi}(0) - M_3 \ddot{\delta} + L_t \quad (33)$$

$$M_f = M_2 \ddot{w}(0) - M_4 \ddot{\varphi}(0) + M_5 \ddot{\delta} - x_t L_t \quad (34)$$

where the M_i 's are considered to be generalized masses defined as follows:

$$\left. \begin{aligned} M_1 &= \frac{1}{2} \int_{x_n}^{x_t} m_f dx \\ M_2 &= \frac{1}{2} \int_{x_n}^{x_t} m_f x dx \\ M_3 &= \frac{1}{2} \int_0^{x_t} m_f W_1 dx \\ M_4 &= \frac{1}{2} \int_{x_n}^{x_t} m_f x^2 dx \\ M_5 &= \frac{1}{2} \int_0^{x_t} m_f x W_1 dx \\ M_6 &= \frac{1}{2} \int_0^{x_t} m_f W_1^2 dx \end{aligned} \right\} \quad (35)$$

The generalized mass constant M_6 , although not appearing in equations (33) or (34), is included in this group because it occurs in a subsequent part of the analysis. In the derivation of equation (34), the aerodynamic moment of the tail about the tail $\frac{1}{4}$ -chord position is neglected since it is considered to be small in comparison with the value $x_t L_t$. The lift on the tail L_t can be found by application of equation (27) to the tail surface. In this case the Φ function appropriate to the tail should be chosen and the values of displacement w and φ should be replaced by $w_f(x_t)$ and α_t , the values of deflection and angle of attack at the tail $\frac{1}{4}$ -chord position. These values are given by equations (13) and (14).

Matrix equation of equilibrium.—The problem of computing the response may be considered to be one of the determination of the deflection and rotation of a beam which is subjected to normal and torque loadings. In differential

form, the bending and rotational displacements are related to the normal and torque loadings by the well-known expressions:

$$\frac{\partial^2}{\partial y^2} EI \frac{\partial^2 w}{\partial y^2} = p \quad (36)$$

$$-\frac{\partial}{\partial y} GJ \frac{\partial \varphi}{\partial y} = q \quad (37)$$

where in this instance p and q are the loadings per unit length of beam. In addition to these two equations which are considered to apply to the wing, an equation for computing the elastic deformations of the fuselage may be found; this equation may be found in the following manner. The rearward part of the fuselage is considered to be a cantilever beam subjected to the inertia loading given by equation (32) and the tail force $2L_t$. If equation (36) is applied to the fuselage and use is made of equations (12) and (32), the following equation for fuselage bending results:

$$\delta \frac{\partial^2}{\partial x^2} EI_f \frac{\partial^2 W_1}{\partial x^2} = -m_f [\ddot{w}(0) - \ddot{\varphi}(0)x + \ddot{\delta} W_1] + 2L_t \quad (38)$$

in which L_t must be treated properly as a concentrated load and I_f is the bending moment of inertia of the fuselage. Since W_1 represents a vibration modal function, the following relation exists:

$$\frac{\partial^2}{\partial x^2} EI_f \frac{\partial^2 W_1}{\partial x^2} = m_f \omega_f^2 W_1$$

where ω_f is the frequency of vibration associated with W_1 . Equation (38) may therefore be written

$$\delta m_f \omega_f^2 W_1 = -m_f [\ddot{w}(0) - \ddot{\varphi}(0)x + \ddot{\delta} W_1] + 2L_t$$

Multiplication of this equation through by W_1 and integration between 0 and x_t results in the following equation for fuselage bending

$$\omega_f^2 M_6 \delta = -M_3 \ddot{w}(0) + M_5 \ddot{\varphi}(0) - M_6 \ddot{\delta} + L_t \quad (39)$$

where M_3 , M_5 , and M_6 are defined by equations (35).

Equations (36), (37), and (39), when the loadings given by equations (29) and (30) are considered, are seen to be rather involved integrodifferential equations but describe completely the motion of the airplane. The problem is to find functions w , φ , and δ which satisfy these equations and which satisfy both the boundary conditions and the initial conditions.

The problem of finding the w and φ functions may be simplified considerably by reducing the rather complicated equations of motion to a simplified and systematic algebraic form. The first step (see appendix C) is to replace the differential equations (36) and (37) for wing deflection and wing rotation by the following simple matrix equations:

$$[A] |w| = |p| \quad (40)$$

$$[B] |\varphi| = |q| \quad (41)$$

The matrices in these equations are defined in appendix C (see equations (C22) and (C23) and equations (C29) and (C30), respectively) and have been derived on the basis that the displacements along the semispan are given at six stations.

Equations (40) and (41) and the fuselage deflection coefficient δ are now combined in a single matrix equation of the form indicated as follows:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & [A] & 0 \\ 0 & 0 & [B] \end{bmatrix} \begin{bmatrix} \delta \\ |w| \\ |\varphi| \end{bmatrix} = \begin{bmatrix} 0 \\ |p| \\ |q| \end{bmatrix} \quad (42)$$

This form is chosen because it will be useful subsequently. With the notation given in appendix A, equation (42) may be abbreviated to the form:

$$[C] \begin{bmatrix} \delta \\ w \\ \varphi \end{bmatrix} = [P] \quad (43)$$

This equation may be regarded as the loading matrix equation of equilibrium; it relates the loadings to the displacements by linear simultaneous equations. The boundary conditions are automatically satisfied when this equation is used because they had to be taken into account when the submatrices $[A]$ and $[B]$ were derived. Only the initial conditions remain to be satisfied and these are treated separately in a subsequent section.

Transformation of the loading equations into difference form.—The loading equations are now simplified by replacing the time derivatives by difference equations. If equation (5) is used to replace the derivative in equations (29) and (30), the values of the loading at the n th time interval are found to be

$$p_n = -\frac{\bar{m}}{\epsilon^2} (2w_n - 5w_{n-1} + 4w_{n-2} - w_{n-3}) + \frac{\bar{m}e}{\epsilon^2} (2\varphi_n - 5\varphi_{n-1} + 4\varphi_{n-2} - \varphi_{n-3}) + L_n \quad (44)$$

$$q_n = \frac{\bar{m}e}{\epsilon^2} (2w_n - 5w_{n-1} + 4w_{n-2} - w_{n-3}) - \frac{\bar{m}k^2}{\epsilon^2} (2\varphi_n - 5\varphi_{n-1} + 4\varphi_{n-2} - \varphi_{n-3}) + M_n \quad (45)$$

The values L_n and M_n are found by determining the expressions for L_1 , L_2 , and L_g at $t=n\epsilon$ (see equations (27) and (28)). Of these L_1 is the most complicated, since it (see equation (25)) involves three unsteady lift integrals of the Duhamel type. Fortunately, however, a rather simple recurrence relation can be developed which allows the calculation of the value of these integrals at a given time interval directly from the value at the previous time interval. This derivation is presented in appendix D and is made possible because the Φ function is of an exponential form. (Where the Φ function is given by more than one exponential term, a recurrence

relation for each term may be written.) From the derivation in appendix D, therefore, the value of the three integrals at the n th time interval may be given as follows:

$$F_n + \frac{\beta c l \epsilon}{2} \ddot{\Phi}_0 w_n - \frac{\beta c l \epsilon}{2} \left[U \dot{\Phi}_0 + c \left(\frac{3}{4} - \frac{a}{c} \right) \ddot{\Phi}_0 \right] \varphi_n \quad (46)$$

where

$$F_n = e^{-\gamma \epsilon} F_{n-1} + g w_{n-1} + g' \varphi_{n-1}$$

in which g and g' are defined by equations (A5) in appendix A. With this expression to replace the value of the integrals in equation (25), the value of L_{1n} may be written

$$\begin{aligned} L_{1n} = & \beta c l \left(\dot{\Phi}_0 + \frac{\epsilon}{2} \ddot{\Phi}_0 \right) w_n - (1 - \Phi_0) \beta c l \dot{w}_n + \\ & \beta c l \left[U(1 - \Phi_0) - \frac{\epsilon U}{2} \dot{\Phi}_0 - \left(\dot{\Phi}_0 + \frac{\epsilon}{2} \ddot{\Phi}_0 \right) c \left(\frac{3}{4} - \frac{a}{c} \right) \right] \varphi_n + \\ & (1 - \Phi_0) \beta c^2 l \left(\frac{3}{4} - \frac{a}{c} \right) \dot{\varphi}_n + F_n \end{aligned} \quad (47)$$

With the use of difference equation (4), this equation may be transformed finally into the form:

$$\begin{aligned} L_{1n} = & d_0 w_n + d_1 w_{n-1} + d_2 w_{n-2} + d_3 w_{n-3} + d_0' \varphi_n + \\ & d_1' \varphi_{n-1} + d_2' \varphi_{n-2} + d_3' \varphi_{n-3} + F_n \end{aligned} \quad (48)$$

where the d 's are defined in appendix A. Likewise, from equations (4), (17), and (26), L_{2n} may be written

$$L_{2n} = \frac{\beta l c^2}{24 \epsilon} (11 \varphi_n - 18 \varphi_{n-1} + 9 \varphi_{n-2} - 2 \varphi_{n-3})$$

If L_{1n} , L_{2n} , and the value L_{gn} of the gust force at $t=n\epsilon$ are introduced into equation (44), the value of p at the n th time interval can be shown to be given by the equation:

$$\begin{aligned} p_n = & \eta_0 w_n + \eta_1 w_{n-1} + \eta_2 w_{n-2} + \eta_3 w_{n-3} + \eta_0' \varphi_n + \\ & \eta_1' \varphi_{n-1} + \eta_2' \varphi_{n-2} + \eta_3' \varphi_{n-3} + F_n + L_{gn} \end{aligned} \quad (49)$$

where the η 's are coefficients which are given by equations (A3) in appendix A. In a similar manner, the equation for q (equation (45)) can be reduced to the form:

$$\begin{aligned} q_n = & \nu_0 w_n + \nu_1 w_{n-1} + \nu_2 w_{n-2} + \nu_3 w_{n-3} + \nu_0' \varphi_n + \nu_1' \varphi_{n-1} + \\ & \nu_2' \varphi_{n-2} + \nu_3' \varphi_{n-3} + \left(a - \frac{c}{4} \right) F_n + \left(a - \frac{c}{4} \right) L_{gn} \end{aligned} \quad (50)$$

where the ν 's are given by equations (A4) in appendix A.

The value of aerodynamic lift acting at the tail $\frac{1}{4}$ -chord L_t is found most conveniently by applying equation (49) to one-half of the tail surface. This application is made by modifying the η coefficients as follows: The mass value m is set equal to zero, $\frac{a}{c}$ is taken as $\frac{1}{4}$, c is replaced by c_t , and

β_{cl} is replaced by β_t , defined as the forward-speed and aspect-ratio parameter of the tail by the equation:

$$\beta_t = \frac{1}{2} m_{A_t} \pi \rho U S_t \quad (51)$$

In addition, w and φ are replaced by the deflection and rotation of the tail given by equations (13) and (14). With these substitutions the value of L_{t_n} is found to be

$$\begin{aligned} L_{t_n} = & f_0 w(0)_n + f_1 w(0)_{n-1} + f_2 w(0)_{n-2} + f_3 w(0)_{n-3} + f_0' \varphi(0)_n + \\ & f_1' \varphi(0)_{n-1} + f_2' \varphi(0)_{n-2} + f_3' \varphi(0)_{n-3} + \bar{f}_0 \delta_n + \bar{f}_1 \delta_{n-1} + \\ & \bar{f}_2 \delta_{n-2} + \bar{f}_3 \delta_{n-3} + F_{t_n} + L_{g_{t_n}} \end{aligned} \quad (52)$$

where

$$F_{t_n} = e^{-\gamma_t \epsilon} F_{t_{n-1}} + j w(0)_{n-1} + j' \varphi(0)_{n-1} + \bar{j} \delta_{n-1} \quad (53)$$

$L_{g_{t_n}}$ is one-half the tail gust force at $t = n\epsilon$ and the f 's and j 's are defined by equations (A7) and (A11), respectively, in appendix A.

With equation (52) and difference equation (5), equations (33) and (34) for L_f and M_f and equation (39) for fuselage bending may be reduced readily to the following form:

$$\begin{aligned} L_{f_n} = & \gamma_0 w(0)_n + \gamma_1 w(0)_{n-1} + \gamma_2 w(0)_{n-2} + \gamma_3 w(0)_{n-3} + \gamma_0' \varphi(0)_n + \\ & \gamma_1' \varphi(0)_{n-1} + \gamma_2' \varphi(0)_{n-2} + \gamma_3' \varphi(0)_{n-3} + \bar{\gamma}_0 \delta_n + \bar{\gamma}_1 \delta_{n-1} + \\ & \bar{\gamma}_2 \delta_{n-2} + \bar{\gamma}_3 \delta_{n-3} + F_{t_n} + L_{g_{t_n}} \end{aligned} \quad (54)$$

$$\begin{aligned} M_{f_n} = & \mu_0 w(0)_n + \mu_1 w(0)_{n-1} + \mu_2 w(0)_{n-2} + \mu_3 w(0)_{n-3} + \mu_0' \varphi(0)_n + \\ & \mu_1' \varphi(0)_{n-1} + \mu_2' \varphi(0)_{n-2} + \mu_3' \varphi(0)_{n-3} + \bar{\mu}_0 \delta_n + \bar{\mu}_1 \delta_{n-1} + \\ & \bar{\mu}_2 \delta_{n-2} + \bar{\mu}_3 \delta_{n-3} - x_t F_{t_n} - x_t L_{g_{t_n}} \end{aligned} \quad (55)$$

$$\begin{aligned} 0 = & r_0 w(0)_n + r_1 w(0)_{n-1} + r_2 w(0)_{n-2} + r_3 w(0)_{n-3} + r_0' \varphi(0)_n + \\ & r_1' \varphi(0)_{n-1} + r_2' \varphi(0)_{n-2} + r_3' \varphi(0)_{n-3} + \bar{r}_0 \delta_n + \bar{r}_1 \delta_{n-1} + \\ & \bar{r}_2 \delta_{n-2} + \bar{r}_3 \delta_{n-3} + F_{t_n} + L_{g_{t_n}} \end{aligned} \quad (56)$$

where the γ 's, μ 's, and r 's are given by equations (A8) to (A10) in appendix A.

The complete set of loading equations as well as the fuselage bending equations are now available in difference form. Equations (49) and (50) apply at each spanwise station and in addition the values of L_f and M_f must be introduced at station 0. The coefficients η , ν , γ , and so forth are seen to involve only the physical properties of the airplane structure, the forward-speed and aspect-ratio parameters given by equations (26) and (51), certain constants derived from the unsteady lift function, and the time interval. The time interval ϵ that is chosen should be fairly small in comparison with the natural period of the fundamental mode in bending of the wing. To serve as a guide an interval chosen near 1/30 of the estimated period of vibration of the fundamental mode appears to be quite satisfactory. Of course, some caution should be observed

in the choice of this interval if the airplane is near a critical condition where some mode other than the fundamental may predominate. For example, if the airplane is flying near the flutter speed, the characteristic frequency of the response may be near the natural torsional frequency of the wing. The time interval should be modified accordingly.

Recurrence matrix equation for response.—Equations (49), (50), (54), and (55) for loading, equation (56) for fuselage bending, and the equilibrium equation (43) may now be combined to give the recurrence matrix equation for response. In order to simplify the process of combining these equations, only the abbreviated or symbolic form of the matrices which occur are used. The definitions of these matrices are given, unless otherwise stated, in a complete group in appendix A.

Application of equations (49) and (50) to each of the spanwise stations and of equations (54) and (55) to station 0 leads to a set of loading equations which may be put in the matrix form given by the following equations:

$$\begin{aligned} |p|_n = & |\bar{\gamma}_0| \delta_n + |\bar{\gamma}_1| \delta_{n-1} + |\bar{\gamma}_2| \delta_{n-2} + |\bar{\gamma}_3| \delta_{n-3} + [\eta_0] |w|_n + \\ & [\eta_1] |w|_{n-1} + [\eta_2] |w|_{n-2} + [\eta_3] |w|_{n-3} + [\eta_0'] |\varphi|_n + \\ & [\eta_1'] |\varphi|_{n-1} + [\eta_2'] |\varphi|_{n-2} + [\eta_3'] |\varphi|_{n-3} + |F| + |L_g| \Big|_n + \\ & |1| (F_t + L_{g_t})_n \end{aligned} \quad (57)$$

$$\begin{aligned} |q|_n = & |\bar{\mu}_0| \delta_n + |\bar{\mu}_1| \delta_{n-1} + |\bar{\mu}_2| \delta_{n-2} + |\bar{\mu}_3| \delta_{n-3} + [\nu_0] |w|_n + \\ & [\nu_1] |w|_{n-1} + [\nu_2] |w|_{n-2} + [\nu_3] |w|_{n-3} + [\nu_0'] |\varphi|_n + \\ & [\nu_1'] |\varphi|_{n-1} + [\nu_2'] |\varphi|_{n-2} + [\nu_3'] |\varphi|_{n-3} + \left[a - \frac{c}{4} \right] |F| + \\ & |L_g| \Big|_n + |x_t| (F_t + L_{g_t})_n \end{aligned} \quad (58)$$

where

$$|F|_n = e^{-\gamma_t \epsilon} |F|_{n-1} + [g] |w|_{n-1} + [g'] |\varphi|_{n-1} \quad (59)$$

$$(F_t)_n = e^{-\gamma_t \epsilon} (F_t)_{n-1} + j w(0)_{n-1} + j' \varphi(0)_{n-1} + \bar{j} \delta_{n-1} \quad (60)$$

Equations (57) and (58) and equation (56) may now be combined to form the following matrix equation:

$$\begin{aligned} \begin{bmatrix} 0 \\ |p| \\ |q| \end{bmatrix}_n = & \begin{bmatrix} \bar{\gamma}_0 & [r_0] & [r_0'] \\ |\bar{\gamma}_0| & [\eta_0] & [\eta_0'] \\ |\bar{\mu}_0| & [\nu_0] & [\nu_0'] \end{bmatrix} \begin{bmatrix} \delta \\ |w| \\ |\varphi| \end{bmatrix}_n + \begin{bmatrix} \bar{\gamma}_1 & [r_1] & [r_1'] \\ |\bar{\gamma}_1| & [\eta_1] & [\eta_1'] \\ |\bar{\mu}_1| & [\nu_1] & [\nu_1'] \end{bmatrix} \begin{bmatrix} \delta \\ |w| \\ |\varphi| \end{bmatrix}_{n-1} + \\ & \begin{bmatrix} \bar{\gamma}_2 & [r_2] & [r_2'] \\ |\bar{\gamma}_2| & [\eta_2] & [\eta_2'] \\ |\bar{\mu}_2| & [\nu_2] & [\nu_2'] \end{bmatrix} \begin{bmatrix} \delta \\ |w| \\ |\varphi| \end{bmatrix}_{n-2} + \begin{bmatrix} \bar{\gamma}_3 & [r_3] & [r_3'] \\ |\bar{\gamma}_3| & [\eta_3] & [\eta_3'] \\ |\bar{\mu}_3| & [\nu_3] & [\nu_3'] \end{bmatrix} \begin{bmatrix} \delta \\ |w| \\ |\varphi| \end{bmatrix}_{n-3} + \\ & \begin{bmatrix} 1 & 0 \\ |1| & [I] \\ |x_t| & \left[a - \frac{c}{4} \right] \end{bmatrix} \begin{bmatrix} F_t + L_{g_t} \\ |F| + |L_g| \\ \end{bmatrix}_n \end{aligned} \quad (61)$$

For simplicity, this equation may be abbreviated to the form:

$$[P]_n = [S_0] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_n + [S_1] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-1} + [S_2] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-2} + [S_3] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-3} + [R] \begin{Bmatrix} \bar{F} \\ \bar{L}_g \end{Bmatrix}_n \quad (62)$$

where

$$[\bar{F}]_n = [e] [\bar{F}]_{n-1} + [W] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-1} \quad (63)$$

and the matrix $[\bar{L}_g]_n$ is defined in the section entitled "Derivation of Gust Forces."

Substitution of equation (62) in equation (43) gives

$$[C] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_n = [S_0] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_n + [S_1] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-1} + [S_2] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-2} + [S_3] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-3} + [R] \begin{Bmatrix} \bar{F} \\ \bar{L}_g \end{Bmatrix}_n \quad (64)$$

With the use of the notations

$$[D] = [C] - [S_0] \quad (65)$$

and

$$[Q]_n = [S_1] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-1} + [S_2] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-2} + [S_3] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_{n-3} + [R] \begin{Bmatrix} \bar{F} \\ \bar{L}_g \end{Bmatrix}_n \quad (66)$$

equation (64) may be written simply

$$[D] \begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_n = [Q]_n \quad (67)$$

Multiplying through by the reciprocal of $[D]$ gives finally the equation

$$\begin{Bmatrix} \delta \\ w \\ \varphi \end{Bmatrix}_n = [D]^{-1} [Q]_n \quad (68)$$

This equation gives the displacements that apply at time n in terms of the displacements that occurred at several preceding values of time (see equations (63) and (66) for the definitions of $[\bar{F}]_n$ and $[Q]_n$).

From equation (68) the complete response of the airplane can be computed once the character of the gust is known. The matrix of gust-force values $[\bar{L}_g]_n$ can be determined by the procedure given in the section entitled "Derivation of Gust Forces." The initial conditions (treated in the following section) are used to obtain three successive initial sets of the displacements. With these sets of displacements the next set may be obtained by application of equation (68). With the newly found set and the preceding sets of displacements, the next set may then be found, and so forth, until a sufficient time history of the displacements is found from which maximum loading conditions may be determined.

The reason for using the difference form of the derivatives as given by equations (4) and (5) might now be given. Equation (64) may be considered a differential equation, since the matrix $[C]$ contains the spanwise derivative matrices $[A]$ and $[B]$ and may be likened to the differential equation which relates the load to the deflection for a beam. The unknowns are the deflections at time n . The right-hand terms correspond to the loading, the first term being the only one which is not known since it contains the unknown deflection. The subsequent inversion of the matrix $[D]$ leads to, in effect, the solution to this differential equation and, in the beam analogy, corresponds to the integration of the loading to obtain the deflection. When numerical methods are used, the deflection may be computed with good accuracy by integration of the loading. On the other hand, if the difference equations which apply at an interior ordinate had been used, the matrix $[C]$ would have appeared as an operator on one of the known deflections on the right-hand side of the equation. Effectively, its operation would be to differentiate a known deflection, and in the beam analogy this operation corresponds to the attempt to obtain the load which caused a given deflection. This process, however, cannot be done with accuracy when numerical methods are used because of the difficulty encountered in the form of small differences of large numbers. The difference equations which apply at an outer ordinate and, consequently, lead to an integration process, therefore, have to be used.

DERIVATION OF THE INITIAL RESPONSE

As has been mentioned, some initial values of deflection are needed before equation (68) can be used. This section shows how these values are obtained. The airplane, just before gust penetration, is considered to be in level flight, and all displacements which follow are given relative to this level-flight condition. The initial conditions are that the vertical displacements, vertical velocity, wing rotation, and angular velocity are all zero. The gust force can be shown to start from zero and, therefore, by Newton's second law the additional initial condition can be established that the acceleration must be zero at the start of the response. These conditions can be satisfied, and the beginning of the response can be found by means of the analysis which follows.

The initial response is assumed to be given in terms of four successive ordinates, denoted by w_{-2}, w_{-1}, w_0 , and w_1 ; the w_0 ordinate is considered, as in the case of the damped oscillator, to locate the origin of time. The first and second derivatives at the w_0 ordinate are given by equations (8) and (9), respectively. By virtue of the initial conditions (the vanishing of the deflection, velocity, and accelerations at $t=0$), the ordinate w_0 and the derivatives given by equations (8) and (9) must be zero; therefore, the ordinates w_{-2} and w_{-1} are found to be related to the ordinate w_1 by the following relations:

$$w_{-2} = -8w_1 \quad (69)$$

$$w_{-1} = -w_1 \quad (70)$$

These relations are general and must apply for deflection and rotation at each of the spanwise stations and for the fuselage deflection as well; that is, the displacements at $t = -2\epsilon$ must be minus eight times the displacements at $t = \epsilon$, and the displacements at $t = -\epsilon$ must be the negative of those at $t = \epsilon$. Substituting these conditions in equation (64), taking n as equal to 1, and using the condition that the displacements are zero at $t=0$ give the following matrix equation in terms of the displacement at $t = \epsilon$ alone:

$$[D] + [S_2] + 8[S_3] \begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix}_1 = [R] \bar{L}_1 \quad (71)$$

The term $|\bar{F}|_1$ is zero and therefore does not appear in this equation. Solution of this equation gives the values of the displacements that occur at $t = \epsilon$ ($n=1$).

The three sets of initial displacements required to proceed with equation (68) are thus known: the set of deflections found at $t = \epsilon$, the zero set at $t = 0$, and the set at $t = -\epsilon$ given by equation (70), or simply the negative of the displacements which were found at $t = \epsilon$. In the actual case no displacements are present at $t = -\epsilon$, but these displacements may be regarded as being of a fictitious nature the only purpose of which is to allow the step-by-step evaluation of the displacements to be started easily:

DERIVATION OF GUST FORCES

The matrix $|\bar{L}_g|_n$ which appears in the response equation (68) is derived as follows. From equation (15) and the notation of equation (26), the total gust force acting over a station section at the n th time interval may be given by the equation

$$L_{g_n} = \beta c l \int_0^n \frac{\partial v}{\partial \tau} \psi(n\epsilon - \tau) d\tau \quad (72)$$

The integral in this expression is also of the Duhamel type and since the ψ function is expressed by exponential terms (see equations (21)), the integral may be evaluated quickly by a method similar to that developed in appendix D. The procedure of computing the gust force by this equation and then the response is not recommended, however, since a complete response evaluation would have to be made for each gust considered. Instead the procedure recommended is to compute the response due to a sharp-edge gust; then

with this response the response to any gust may be found directly by superposition.

Thus for the case of a sharp-edge gust, equation (72) reduces simply to

$$L_{g_n} = \beta c l v \psi_n \quad (73)$$

where ψ_n is the value of the ψ function at $t = n\epsilon$. This equation when applied to each of the spanwise stations leads directly to the matrix equation for gust force:

$$|L_g|_n = \beta \begin{vmatrix} c_0 l_0 v_0 \\ c_1 l_1 v_1 \\ c_2 l_2 v_2 \\ c_3 l_3 v_3 \\ c_4 l_4 v_4 \\ c_5 l_5 v_5 \end{vmatrix} \psi_n \quad (74)$$

If the gust is uniform in the spanwise direction, the v 's in this equation will all be equal.

In a similar manner, one-half the gust force acting on the tail due to a sharp-edge gust may be shown to be

$$L_{g_{t_n}} = \beta_{t_n} v_0 \psi_{t_n} \quad (75)$$

where the gust gradient is assumed to be the same as for station 0 and ψ_{t_n} is the value of the ψ function for the tail. This equation and equation (74) may now be combined to give the desired matrix $|\bar{L}_g|_n$ as follows:

$$|\bar{L}_g|_n = \begin{vmatrix} L_{g_t} \\ |L_g|_n \end{vmatrix} = \begin{vmatrix} \beta_{t_n} v_0 & 0 \\ 0 & \beta c_0 l_0 v_0 \\ 0 & \beta c_1 l_1 v_1 \\ 0 & \beta c_2 l_2 v_2 \\ 0 & \beta c_3 l_3 v_3 \\ 0 & \beta c_4 l_4 v_4 \\ 0 & \beta c_5 l_5 v_5 \end{vmatrix} \begin{vmatrix} \psi_t \\ \psi \end{vmatrix}_n \quad (76)$$

In the application of this equation it should be kept in mind that L_{g_t} does not begin to act until the tail starts to penetrate the gust. The time interval at which penetration starts may be taken as the interval nearest to the quantity $\frac{x_t}{\epsilon U}$.

COMPUTATION OF LOADS AND STRESSES

Once the time history of the displacements has been found from equation (68), the normal or torque loading on the wing can be found with little additional work. If the notation of equation (66) is used, equation (62) may be written

$$|P|_n = [S_0] \begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix}_n + |Q|_n \quad (77)$$

This equation shows that the loading matrix $|P|$ may be found by adding an easily computed matrix to the matrix $|Q|$, the value of which will have already been determined in the response calculation. The loading matrix $|P|$ is remembered

matrix $[\bar{F}]$ takes into account the forces which develop due to the "wake effect," and $[\bar{L}_g]$ is the gust-force matrix which is derived in step (3). Equation (78) is seen to give the displacements that occur at time n in terms of the displacements which occurred at the times $n-1$, $n-2$, and $n-3$.

The initial response:

(8) By use of the matrices given in step (6) and the gust forces which apply at $n=1$, set up the following matrix equation:

$$[[D] + [S_2] + 8[S_3]] \begin{bmatrix} \delta \\ w \\ \varphi_1 \end{bmatrix} = [R][\bar{L}_g]_1 \quad (79)$$

The term $[\bar{F}]_1$ does not appear in this equation because it is zero.

(9) Solve equation (79) for the displacements. Any convenient method, such as the Crout method (see reference 6), may be used. The displacements found will be the value of displacements that apply at $t=\epsilon$ or $n=1$.

The response:

(10) The response may now be found by successive application of equation (78). The response at $n=1$ has been found in step (9); the response at $n=2$ is next to be determined. The values of the displacements in the $n-2$ term of the response equation are all taken to be zero (initial condition), and the values in the $n-3$ term are taken as the negative of those found in step (9). The gust forces to use are those which apply at $n=2$. The deflections that apply at $n=2$ are then found by matrix algebra. For convenience the column matrix $[Q]$ is evaluated first, and then multiplication of this column matrix by the reciprocal of the $[D]$ matrix gives the deflections at $n=2$. With the newly found deflections at $n=2$ and the deflections at $n=1$ and $n=0$, the deflections at $n=3$ can be found, and so forth. This process is continued until the wing bending appears to be the most pronounced.

Wing loading:

(11) With the deflections known, the value of wing loading in bending or in torsion can be computed directly from equation (77). The stresses at any point can then be computed from the wing loading by ordinary static means. Since the loading may be computed at any value of time, the complete stress history of any point on the structure may be computed.

EXAMPLE

As an illustration of the method of analysis given in the present report, the response of a typical two-engine airplane due to a sharp-edge gust is determined. For brevity the fuselage is assumed rigid and only vertical displacement and wing bending are considered. The weight variation over the wing semispan and the equivalent-weight concentrations are shown in figure 8. In this figure are shown also the station locations and the interval covered by each station section. The solution is made for a forward velocity of flight of about 210 miles per hour and a gust velocity of 10 feet per second. In tables 2, 3, and 4 are listed, respectively, the various physical characteristics and the factors which

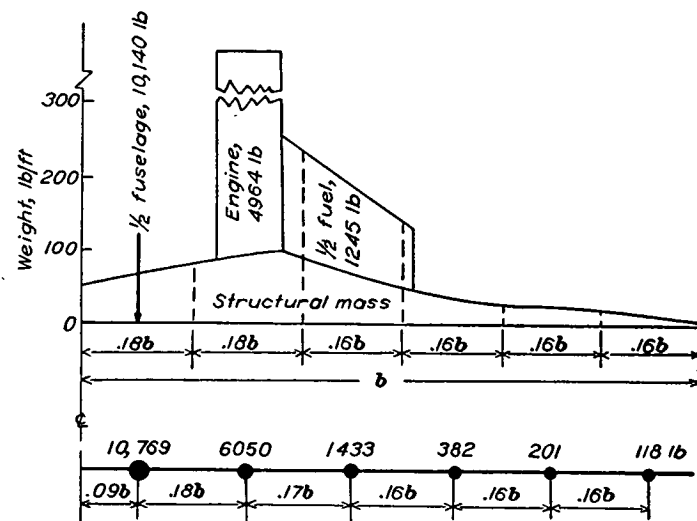


FIGURE 8.—Weight distribution and equivalent concentrations for example two-engine aircraft.

TABLE 2.—PHYSICAL CHARACTERISTICS AND UNSTEADY LIFT FACTORS FOR EXAMPLE AIRPLANE

b , in.	560
c_o , in.	154
EI_o/b^3 , lb/in.	165
ρ , lb/ft ³	0.0765
U , mph	210
U , in./sec.	3700
v , in./sec.	120
ϵ , sec.	0.01
Δs , half-chords	0.48052
M	0.276
A	10
m_A	0.861
β	0.001147
λ	0.381
γ	18.3078
$e^{-\gamma\epsilon}$	0.832703
$\Phi_0 = a_1$	0.361
$\dot{\Phi}_0$	-6.60912
$\ddot{\Phi}_0$	120.9983

TABLE 3.— ψ ORDINATES AND GUST-FORCE MATRIX FOR EXAMPLE AIRPLANE

n	ψ (Equation (22))	n	ψ (Equation (22))
0	0	10	0.72820
1	.22105	11	.74595
2	.36744	12	.76215
3	.46716	13	.77707
4	.53741	14	.79085
5	.58888	15	.80373
6	.62830	16	.81574
7	.65980	17	.82696
8	.68594	18	.83748
9	.70840		

$$|L_g| = 120 \quad \begin{bmatrix} 17.8404 \\ 15.7552 \\ 12.1811 \\ 10.5295 \\ 8.77455 \\ 7.01964 \end{bmatrix} \psi_n$$

come from the unsteady lift function, the values of the ψ function and the gust-force matrix, and the matrix elements that are required for the solution (steps (1) to (5)). The Φ function for an aspect ratio of 6 was chosen (see equation (20b)); and the ψ function for an aspect ratio of infinity (equation (22)) was used.

The $[A]$ matrix as computed from the values of λ and I listed in table 4 is shown in table 5 (a). In the computation of the η values shown in table 4 for station 0, the fuselage was treated as a concentrated wing mass. This treatment is allowable since the fuselage is assumed rigid and further saves the work of computing the γ values (see equations (A8)). The $[[A]-[S_0]]$ or $[D]$ matrix, which in this case applies only to bending and vertical displacement, is shown in table 5 (b). The equation which is formed from equation (78) (step (7)) and which involves the reciprocal of $[D]$ and the $[S_i]$ and $[R]$ matrices is shown in table 6. The equation for computing the initial response (step (8)) is shown in table 7.

The solution to these equations is shown in figure 9 in which deflection in inches is plotted against spanwise station points for various intervals of time. For clarity the deflections for the odd intervals have been left off. From these curves the consequent wing bending and the manner in which the airplane is swept upward by the gust can be seen. The time histories of the loads (equation (77)) that occur at each of the spanwise stations are shown in figure 10. These curves indicate the presence of some second-mode excitation in the response. The stresses that occur at stations 0, 1, and 2 are shown in figure 11. The presence of second-mode excitation is not readily discernible from the stress curves.

TABLE 4.—MATRIX ELEMENTS FOR EXAMPLE AIRPLANE

Station	λ	I	c	l	η_0	η_1	η_2	η_3	g
0	0.09	2700	154	101	-56.019712×10^4	139.84200×10^4	-111.77100×10^4	27.93800×10^4	17.9752
1	.18	1870	136	101	-31.594032	78.802027	-62.951014	15.733559	15.2697
2	.17	1100	118	90	-7.570015	18.783512	-14.956756	3.735946	12.2731
3	.16	520	102	90	-2.109675	5.151851	-4.060925	1.012428	10.6091
4	.16	225	85	90	-1.150062	2.773208	-2.168104	$.539690$	8.84085
5	.16	67.5	68	90	$-.698450$	1.664566	-1.291283	$.320952$	7.07268

TABLE 5.—THE $[A]$ AND $[D]$ MATRICES FOR EXAMPLE AIRPLANE(a) The $[A]$ matrix

82,192.75	-133,410.07	61,949.726	-12,599.08	2,094.4210	-237.6981
-133,410.07	258,299.66	-172,806.94	56,197.447	-9,339.8380	1,059.7383
61,959.73	-172,806.94	194,219.495	-108,709.511	28,579.472	-3,242.2365
-12,599.080	56,197.447	-108,709.511	103,953.971	-47,410.326	8,567.4988
2,094.4210	-9,339.8380	28,579.472	-47,410.326	36,607.4681	-10,531.197
-237.6981	1,059.7383	-3,242.2365	8,567.4988	-10,531.197	4,383.89451

(b) The $[D]$ matrix

642,389.82	-133,410.07	61,959.726	-12,599.080	2,094.4210	-237.6981
-133,410.07	574,239.980	-172,806.94	56,197.447	-9,339.8380	1,059.7383
61,959.726	-172,806.94	269,919.640	-108,709.511	28,579.472	-3,242.2365
-12,599.080	56,197.447	-108,709.511	125,050.721	-47,410.326	8,567.4988
2,094.4210	-9,339.8380	28,579.472	-47,410.326	48,108.0880	-10,531.197
-237.6981	1,059.7383	-3,242.2365	8,567.4988	-10,531.197	11,368.3945

TABLE 6.—RECURRENCE EQUATION FOR RESPONSE OF EXAMPLE AIRPLANE

$$\begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix}_n = \frac{1}{10,000} \begin{bmatrix} 0.016448152 & 0.003274436 & -0.002535913 & -0.002352327 & -0.0008201844 & 0.0003284241 \\ 0.003274436 & 0.022344322 & 0.01467594 & 0.002077166 & -0.002939533 & -0.002117350 \\ -0.002535913 & 0.01467594 & 0.069848456 & 0.06246892 & 0.02103792 & -0.009089948 \\ -0.002352327 & 0.002077166 & 0.062468920 & 0.19074974 & 0.15523588 & 0.01762344 \\ -0.0008201844 & -0.002939533 & 0.02103792 & 0.15523588 & 0.40588417 & 0.26526111 \\ 0.0003284241 & -0.002117350 & -0.009089948 & 0.01762344 & 0.26526111 & 1.10968865 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix}_{n-1} + Q$$

where

$$Q = \frac{1}{10,000} \begin{bmatrix} 139.84200 & 78.802027 & 18.783512 & 5.151851 & 2.773208 & 1.664566 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix}_{n-1} + \begin{bmatrix} -111.77100 & -62.951014 & -14.956756 & -4.060925 & -2.168104 & -1.291283 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix}_{n-2} + \begin{bmatrix} 27.938000 & 15.733559 & 3.735946 & 1.012428 & 0.539690 & 0.320952 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix}_{n-3} + \frac{1}{10,000} [|F| + |L_g|]_n$$

In which

$$|F|_n = 0.832703 |F|_{n-1} + \begin{bmatrix} 17.9752 \\ 15.2697 \\ 12.2731 \\ 10.6091 \\ 8.84085 \\ 7.07268 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix}_{n-1}$$

TABLE 7.—EQUATION FOR INITIAL RESPONSE OF EXAMPLE AIRPLANE

$$10,000 \begin{bmatrix} 175.9720 & -13.34101 & 6.195973 & -1.259908 & 0.2094421 & -0.02376981 \\ -13.34101 & 120.3415 & -17.28069 & 5.619745 & -0.9339838 & 0.1059738 \\ 6.195973 & -17.28069 & 41.92278 & -10.87095 & 2.857947 & -0.3242237 \\ -1.259908 & 5.619745 & -10.87095 & 16.54357 & -4.741033 & 0.8567499 \\ 0.2094421 & -0.9339838 & 2.857947 & -4.741033 & 6.960225 & -1.053120 \\ -0.02376981 & 0.1059738 & -0.3242237 & 0.8567499 & -1.053120 & 2.413172 \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix}_1 = 120(0.22105) \begin{bmatrix} 17.8404 \\ 15.7552 \\ 12.1811 \\ 10.5295 \\ 8.77455 \\ 7.01964 \end{bmatrix}$$

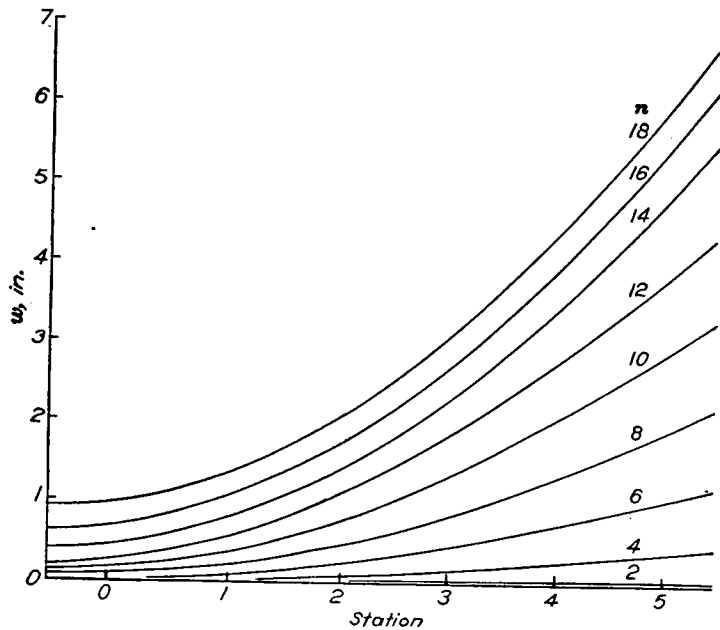


FIGURE 9.—Response of example airplane due to 10-foot-per-second sharp-edge gust.
 $U=210$ miles per hour.

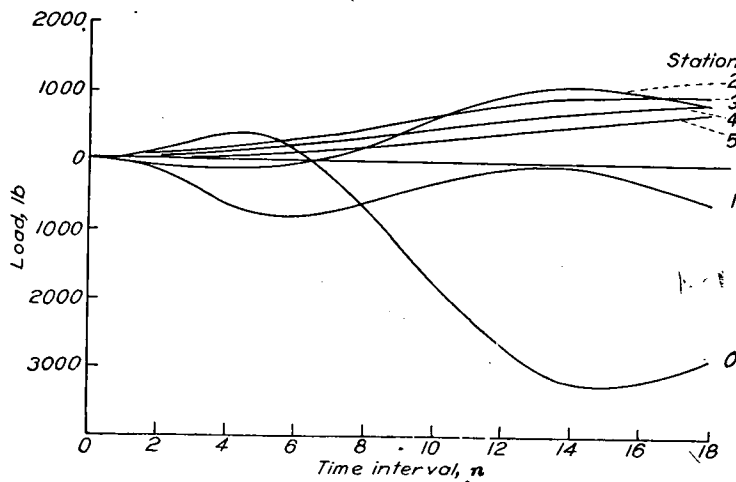


FIGURE 10.—Time history of station loads for example airplane.

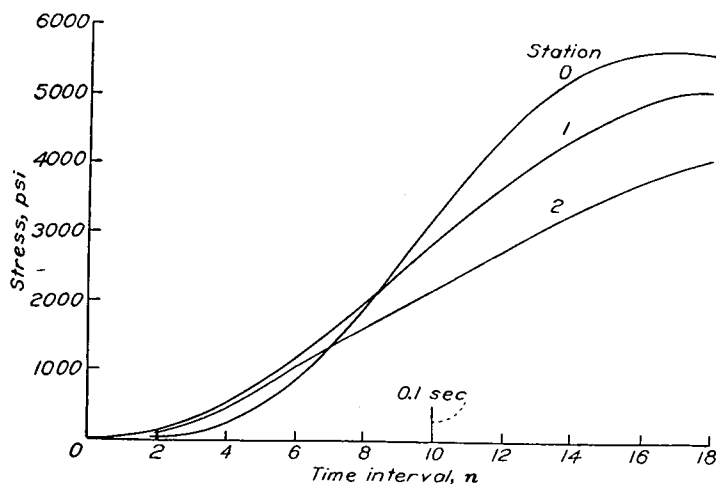


FIGURE 11.—Bending stress developed in example airplane due to 10-foot-per-second sharp-edge gust.

DISCUSSION

A method for computing the stresses and structural action of an airplane flying through a gust has been given. The method is based on aerodynamic strip theory, but over-all corrections for compressibility and three-dimensional effects can be made as is indicated by a suggested correction procedure. Some tail forces are included in the analysis and others might equally well be included if their character is known.

The analysis as given is general enough to include the wing bending and twisting flexibilities and the fuselage flexibility. In a good many cases that may be considered, however, the last two of the flexibilities may prove to be of negligible importance. Some investigators have indicated (see reference 1) that, unless the forward speed of the airplane approaches the flutter or divergence speed of the wing, the torsional deformations do not have to be included. Likewise, in cases in which the fuselage is rather stiff, the effect of fuselage flexibility on the response may be rather small. In such cases in which either or both of these flexibilities may be ignored, the analysis is, of course, simplified and shortened. The example presented in the previous section illustrates this point. In the present state of understanding of gust-response analysis, enough information is not available to indicate definitely when and when not to include the various flexibilities of the aircraft structure. The analysis in the present report may provide a useful means to assess their importance. The extent, for example, to which coupling exists between wing bending and wing torsion in any particular case may be seen by comparing the displacements obtained from the coupling terms with the displacements obtained from the noncoupling terms.

Both the symmetrical and antisymmetrical types of gusts can be handled by the analysis given in the present report. In general, the symmetrical gust is expected to produce the most severe stress condition, and therefore only the matrices which apply for a symmetrical case have been given. These matrices were derived by using the boundary conditions for the symmetrical deformation of a free-free beam. The case of an antisymmetrical gust can be treated by replacing these matrices by the ones which apply for the antisymmetrical deformation of a free-free beam. The case of a general unsymmetrical gust can be handled by first breaking the gust into two parts—a symmetrical part and an antisymmetrical part—and then treating each part independently.

It might be of interest at this point to compare briefly the matrix method to a modal-function solution. One of the chief disadvantages of the modal-function solution is that the modes and frequencies of natural vibration of the structure have to be computed in advance. Then, a large number of integrals which involve these modes have to be determined in order to set up the problem. In the present analysis the physical properties of the airplane are used directly in the setting-up of the problem. Further, in order to make the

modal solution practical the higher modes must be dropped and only the basic or fundamental modes can be used. Hence, the success of the analysis depends to a large degree on how well single modal functions, one mode each for bending and torsion, can represent the airplane distortion. In the analysis of the present report the distortions are found for all practical purposes as the correct values at a number of spanwise stations, at least to within the accuracy to which the aerodynamic and structural parameters are known. Also, in this analysis, probably the most difficult operation is the inversion of the matrix $[D]$, which is actually not a very involved operation, especially when done by the quick and systematic procedure afforded by the Crout method (reference 6).

The present report indicates the methods for determining the response for both a sharp-edge gust and a gust of arbitrary shape. Probably the best approach, however, is to compute only the response for a sharp-edge gust, since the response for any arbitrary gust may thereafter be computed by means of Duhamel's integral. To follow such a procedure would also save a great amount of work in the evaluation of the gust forces.

One of the important features of the method of analysis presented is that it is not restricted to the gust problem. The approach used may be used to treat other problems of a similar nature. The landing problem can be handled by simply replacing the distributed gust force by the concentrated landing forces. In the landing problem also, the problem is set up much more easily since the aerodynamic terms do not ordinarily have to be included. However, the

landing problem in which aerodynamic forces are included may be solved by this method if desired. The approach used herein may also be used to solve the problem of the release of heavy objects such as bombs. This problem could be considered the inverse of the gust problem; a load is released rather than encountered. Some maneuvering problems, such as the sudden deflection of the ailerons, and a number of other transient problems might also be treated by an approach similar to that given in the present report.

CONCLUDING REMARKS

A method for computing the stresses and structural response of an aircraft flying through a gust has been presented. The method is based on aerodynamic strip theory, but compressibility and three-dimensional effects can be taken into account approximately by means of over-all corrections. The method takes into account wing bending and twisting deformations, fuselage deflection, vertical and pitching motion of the airplane, and some tail forces. A sharp-edge gust or a gust of arbitrary shape in the spanwise or flight directions may be treated. A suggested computational procedure is given to aid in the application of the method to any specific case.

The possibilities of applying the method to a variety of transient aircraft problems, such as landing, are brought out.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *January 19, 1950.*

APPENDIX A

DEFINITIONS OF MATRICES USED IN ANALYSIS

For convenience in presentation, most of the matrices and matrix elements that are used in the analysis are defined in this appendix. The matrices are presented without derivations, but their origin should become evident by a study of the analysis.

Matrices.—The various matrices that are used in the analysis are defined as follows for the case in which the wing semispan is divided into six sections (the elements which are used in the matrices being defined in the subsequent section):

$$|w| = \begin{vmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{vmatrix}$$

$$|\varphi| = \begin{vmatrix} \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \\ \varphi(5) \\ \varphi(6) \end{vmatrix}$$

$$\begin{vmatrix} \delta \\ w \\ \varphi \end{vmatrix} = \begin{vmatrix} \delta \\ |w| \\ |\varphi| \end{vmatrix}$$

$$|p| = \begin{vmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \end{vmatrix}$$

$$|q| = \begin{vmatrix} q(0) \\ q(1) \\ q(2) \\ q(3) \\ q(4) \\ q(5) \end{vmatrix}$$

$$|P| = \begin{vmatrix} 0 \\ |p| \\ |q| \end{vmatrix}$$

$$\begin{Bmatrix} [A] \\ [B] \end{Bmatrix} \left. \vphantom{\begin{Bmatrix} [A] \\ [B] \end{Bmatrix}} \right\} \text{See appendix C for definitions.}$$

$$[C] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & [A] & 0 \\ 0 & 0 & [B] \end{bmatrix}$$

$$[D] = [C] - [S_0]$$

$$[\eta_i] = \begin{bmatrix} \eta_i(0) + \gamma_i & & & & \\ & \eta_i(1) & & & \\ & & \eta_i(2) & & \\ & & & \eta_i(3) & \\ & & & & \eta_i(4) \\ & & & & & \eta_i(5) \end{bmatrix}$$

$$[\eta_i'] = \begin{bmatrix} \eta_i'(0) + \gamma_i' & & & & \\ & \eta_i'(1) & & & \\ & & \eta_i'(2) & & \\ & & & \eta_i'(3) & \\ & & & & \eta_i'(4) \\ & & & & & \eta_i'(5) \end{bmatrix}$$

$$[\nu_i] = \begin{bmatrix} \nu_i(0) + \mu_i & & & & \\ & \nu_i(1) & & & \\ & & \nu_i(2) & & \\ & & & \nu_i(3) & \\ & & & & \nu_i(4) \\ & & & & & \nu_i(5) \end{bmatrix}$$

$$[\nu_i'] = \begin{bmatrix} \nu_i'(0) + \mu_i' & & & & \\ & \nu_i'(1) & & & \\ & & \nu_i'(2) & & \\ & & & \nu_i'(3) & \\ & & & & \nu_i'(4) \\ & & & & & \nu_i'(5) \end{bmatrix}$$

$$|\bar{\gamma}_i| = \begin{bmatrix} \bar{\gamma}_i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|\bar{\mu}_i| = \begin{bmatrix} \bar{\mu}_i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[r_i] = [r_i \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$[r'_i] = [r'_i \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$[S_i] = \begin{bmatrix} \bar{r}_i & [r_i] & [r'_i] \\ |\bar{\gamma}_i| & [\eta_i] & [\eta'_i] \\ |\bar{\mu}_i| & [\nu_i] & [\nu'_i] \end{bmatrix}$$

$$|1| = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$|x_i| = \begin{bmatrix} -x_i \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$[I] = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \end{bmatrix}$$

$$\left[a - \frac{c}{4} \right] = \begin{bmatrix} \left(a - \frac{c}{4} \right)_0 & & & & & \\ & \left(a - \frac{c}{4} \right)_1 & & & & \\ & & \left(a - \frac{c}{4} \right)_2 & & & \\ & & & \left(a - \frac{c}{4} \right)_3 & & \\ & & & & \left(a - \frac{c}{4} \right)_4 & \\ & & & & & \left(a - \frac{c}{4} \right)_5 \end{bmatrix}$$

$$[R] = \begin{bmatrix} 1 & 0 \\ |1| & [I] \\ |x_i| & \left[a - \frac{c}{4} \right] \end{bmatrix}$$

$$[e] = \begin{bmatrix} e^{-\gamma_i \epsilon} & & & & & \\ & e^{-\gamma_i \epsilon} & & & & \\ & & e^{-\gamma_i \epsilon} & & & \\ & & & e^{-\gamma_i \epsilon} & & \\ & & & & e^{-\gamma_i \epsilon} & \\ & & & & & e^{-\gamma_i \epsilon} \end{bmatrix}$$

$$[j] = [j \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$[j'] = [j' \ 0 \ 0 \ 0 \ 0 \ 0]$$

$$[g] = \begin{bmatrix} g(0) & & & & & \\ & g(1) & & & & \\ & & g(2) & & & \\ & & & g(3) & & \\ & & & & g(4) & \\ & & & & & g(5) \end{bmatrix}$$

$$[g'] = \begin{bmatrix} g'(0) & & & & & \\ & g'(1) & & & & \\ & & g'(2) & & & \\ & & & g'(3) & & \\ & & & & g'(4) & \\ & & & & & g'(5) \end{bmatrix}$$

$$[W] = \begin{bmatrix} \bar{j} & [j] & [j'] \\ 0 & [g] & [g'] \end{bmatrix}$$

Matrix elements.—The matrix elements which appear in the matrices defined in the previous section are expressed for convenience in terms of the following common factors:

$$\left. \begin{aligned} \beta &= m_A \pi \rho U & \Phi_0 &= a_1 \\ \gamma &= \frac{2U}{c_0} \lambda & \dot{\Phi}_0 &= -\gamma a_1 \\ & & \ddot{\Phi}_0 &= \gamma^2 a_1 \end{aligned} \right\} \quad (\text{A1})$$

in which the last four are associated with the Φ function for the wing. (See equation (23).) With these factors the elements that must be computed at each spanwise station are as follows:

$$\left. \begin{aligned} d_0 &= -\frac{11}{6\epsilon} (1-\Phi_0) \beta c l + \beta c l \left(\dot{\Phi}_0 + \frac{1}{2} \ddot{\Phi}_0 \epsilon \right) \\ d_1 &= \frac{3}{\epsilon} (1-\Phi_0) \beta c l \\ d_2 &= -\frac{3}{2\epsilon} (1-\Phi_0) \beta c l \\ d_3 &= \frac{1}{3\epsilon} (1-\Phi_0) \beta c l \\ d_0' &= \frac{11}{6\epsilon} (1-\Phi_0) \beta c^2 l \left(\frac{3}{4} - \frac{a}{c} \right) + \beta c l \left[U(1-\Phi_0) - \frac{1}{2} \dot{\Phi}_0 U \epsilon - \left(\dot{\Phi}_0 + \frac{1}{2} \ddot{\Phi}_0 \epsilon \right) c \left(\frac{3}{4} - \frac{a}{c} \right) \right] \\ d_1' &= -\frac{3}{\epsilon} (1-\Phi_0) \beta c^2 l \left(\frac{3}{4} - \frac{a}{c} \right) \\ d_2' &= \frac{3}{2\epsilon} (1-\Phi_0) \beta c^2 l \left(\frac{3}{4} - \frac{a}{c} \right) \\ d_3' &= -\frac{1}{3\epsilon} (1-\Phi_0) \beta c^2 l \left(\frac{3}{4} - \frac{a}{c} \right) \end{aligned} \right\} \quad (\text{A2})$$

$$\left. \begin{aligned} \eta_0 &= -\frac{2\overline{m}}{\epsilon^2} + d_0 & \eta_0' &= \frac{2\overline{m}\epsilon}{\epsilon^2} + d_0' + \frac{11}{24\epsilon} \beta c^2 l \\ \eta_1 &= \frac{5\overline{m}}{\epsilon^2} + d_1 & \eta_1' &= -\frac{5\overline{m}\epsilon}{\epsilon^2} + d_1' - \frac{3}{4\epsilon} \beta c^2 l \\ \eta_2 &= -\frac{4\overline{m}}{\epsilon^2} + d_2 & \eta_2' &= \frac{4\overline{m}\epsilon}{\epsilon^2} + d_2' + \frac{9}{24\epsilon} \beta c^2 l \\ \eta_3 &= \frac{\overline{m}}{\epsilon^2} + d_3 & \eta_3' &= -\frac{\overline{m}\epsilon}{\epsilon^2} + d_3' - \frac{1}{12\epsilon} \beta c^2 l \end{aligned} \right\} \quad (\text{A3})$$

$$\left. \begin{aligned} \nu_0 &= \frac{2\overline{m}\epsilon}{\epsilon^2} + d_0 \left(a - \frac{c}{4} \right) \\ \nu_1 &= -\frac{5\overline{m}\epsilon}{\epsilon^2} + d_1 \left(a - \frac{c}{4} \right) \\ \nu_2 &= \frac{4\overline{m}\epsilon}{\epsilon^2} + d_2 \left(a - \frac{c}{4} \right) \\ \nu_3 &= -\frac{\overline{m}\epsilon}{\epsilon^2} + d_3 \left(a - \frac{c}{4} \right) \\ \nu_0' &= -\frac{2\overline{m}\epsilon^2}{\epsilon^2} + d_0' \left(a - \frac{c}{4} \right) - \frac{11}{24\epsilon} \beta c^3 l \left(\frac{3}{4} - \frac{a}{c} \right) \\ \nu_1' &= \frac{5\overline{m}\epsilon^2}{\epsilon^2} + d_1' \left(a - \frac{c}{4} \right) + \frac{3}{4\epsilon} \beta c^3 l \left(\frac{3}{4} - \frac{a}{c} \right) \\ \nu_2' &= -\frac{4\overline{m}\epsilon^2}{\epsilon^2} + d_2' \left(a - \frac{c}{4} \right) + \frac{9}{24\epsilon} \beta c^3 l \left(\frac{3}{4} - \frac{a}{c} \right) \\ \nu_3' &= \frac{\overline{m}\epsilon^2}{\epsilon^2} + d_3' \left(a - \frac{c}{4} \right) + \frac{1}{12\epsilon} \beta c^3 l \left(\frac{3}{4} - \frac{a}{c} \right) \end{aligned} \right\} \quad (\text{A4})$$

$$\left. \begin{aligned} g &= \beta c l \ddot{\Phi}_0 \epsilon e^{-\gamma \epsilon} \\ g' &= \beta c l \epsilon e^{-\gamma \epsilon} \left[-U \dot{\Phi}_0 - \ddot{\Phi}_0 c \left(\frac{3}{4} - \frac{a}{c} \right) \right] \end{aligned} \right\} \quad (\text{A5})$$

The coefficients which must be computed for the fuselage and tail are expressed in part in terms of the following common factors:

$$\left. \begin{aligned} \beta_t &= \frac{1}{2} m_{A_t} \pi \rho U S_t & \Phi_{t_0} &= a_{1_t} \\ \gamma_t &= \frac{2U}{c_{0_t}} \lambda_t & \dot{\Phi}_{t_0} &= -\gamma_t a_{1_t} \\ & & \ddot{\Phi}_{t_0} &= \gamma_t^2 a_{1_t} \end{aligned} \right\} \quad (\text{A6})$$

in which the last four are associated with the Φ functions appropriate to the tail. Also used are the generalized masses given by equations (35) and the value θ_1 as given in equation (14). With these factors the coefficients for the tail and fuselage are as follows:

$$\left. \begin{aligned} f_0 &= -\frac{11}{6\epsilon} (1-\Phi_{t_0}) \beta_t + \beta_t \left(\dot{\Phi}_{t_0} + \frac{1}{2} \ddot{\Phi}_{t_0} \epsilon \right) \\ f_1 &= \frac{3}{\epsilon} (1-\Phi_{t_0}) \beta_t \\ f_2 &= -\frac{3}{2\epsilon} (1-\Phi_{t_0}) \beta_t \\ f_3 &= \frac{1}{3\epsilon} (1-\Phi_{t_0}) \beta_t \end{aligned} \right\} \quad (\text{A7a})$$

$$\left. \begin{aligned} f_0' &= \frac{11}{6\epsilon} (1 - \Phi_{t_0}) \left(x_t + \frac{1}{2} c_t \right) \beta_t + \beta_t \left[U(1 - \Phi_{t_0}) - \frac{1}{2} \dot{\Phi}_{t_0} U \epsilon - \left(\dot{\Phi}_{t_0} + \frac{1}{2} \ddot{\Phi}_{t_0} \epsilon \right) \left(x_t + \frac{1}{2} c_t \right) \right] + \frac{11}{24\epsilon} \beta_t c_t \\ f_1' &= -\frac{3}{\epsilon} (1 - \Phi_{t_0}) \left(x_t + \frac{1}{2} c_t \right) \beta_t - \frac{3}{4\epsilon} \beta_t c_t \\ f_2' &= \frac{3}{2\epsilon} (1 - \Phi_{t_0}) \left(x_t + \frac{1}{2} c_t \right) \beta_t + \frac{9}{24\epsilon} \beta_t c_t \\ f_3' &= -\frac{1}{3\epsilon} (1 - \Phi_{t_0}) \left(x_t + \frac{1}{2} c_t \right) \beta_t - \frac{1}{12\epsilon} \beta_t c_t \end{aligned} \right\} \quad (\text{A7b})$$

$$\left. \begin{aligned} \bar{f}_0 &= -\frac{11}{6\epsilon} (1 - \Phi_{t_0}) \left(1 + \frac{1}{2} c_t \theta_1 \right) \beta_t - \beta_t \left[U \theta_1 (1 - \Phi_{t_0}) - \frac{1}{2} \dot{\Phi}_{t_0} U \epsilon \theta_1 - \left(\dot{\Phi}_{t_0} + \frac{1}{2} \ddot{\Phi}_{t_0} \epsilon \right) \left(1 + \frac{1}{2} c_t \theta_1 \right) \right] - \frac{11}{24\epsilon} \beta_t c_t \theta_1 \\ \bar{f}_1 &= \frac{3}{\epsilon} (1 - \Phi_{t_0}) \left(1 + \frac{1}{2} c_t \theta_1 \right) \beta_t + \frac{3}{4\epsilon} \beta_t c_t \theta_1 \\ \bar{f}_2 &= -\frac{3}{2\epsilon} (1 - \Phi_{t_0}) \left(1 + \frac{1}{2} c_t \theta_1 \right) \beta_t - \frac{9}{24\epsilon} \beta_t c_t \theta_1 \\ \bar{f}_3 &= \frac{1}{3\epsilon} (1 - \Phi_{t_0}) \left(1 + \frac{1}{2} c_t \theta_1 \right) \beta_t + \frac{1}{12\epsilon} \beta_t c_t \theta_1 \end{aligned} \right\} \quad (\text{A7c})$$

$$\left. \begin{aligned} \gamma_0 &= -\frac{2M_1}{\epsilon^2} + f_0 \\ \gamma_1 &= \frac{5M_1}{\epsilon^2} + f_1 \\ \gamma_2 &= -\frac{4M_1}{\epsilon^2} + f_2 \\ \gamma_3 &= \frac{M_1}{\epsilon^2} + f_3 \end{aligned} \right\} \quad (\text{A8a})$$

$$\left. \begin{aligned} \gamma_0' &= \frac{2M_2}{\epsilon^2} + f_0' \\ \gamma_1' &= -\frac{5M_2}{\epsilon^2} + f_1' \\ \gamma_2' &= \frac{4M_2}{\epsilon^2} + f_2' \\ \gamma_3' &= -\frac{M_2}{\epsilon^2} + f_3' \end{aligned} \right\} \quad (\text{A8b})$$

$$\left. \begin{aligned} \bar{\gamma}_0 &= -\frac{2M_3}{\epsilon^2} + \bar{f}_0 \\ \bar{\gamma}_1 &= \frac{5M_3}{\epsilon^2} + \bar{f}_1 \\ \bar{\gamma}_2 &= -\frac{4M_3}{\epsilon^2} + \bar{f}_2 \\ \bar{\gamma}_3 &= \frac{M_3}{\epsilon^2} + \bar{f}_3 \end{aligned} \right\} \quad (\text{A8c})$$

$$\left. \begin{aligned} \mu_0 &= \frac{2M_2}{\epsilon^2} - x_i f_0 \\ \mu_1 &= -\frac{5M_2}{\epsilon^2} - x_i f_1 \\ \mu_2 &= \frac{4M_2}{\epsilon^2} - x_i f_2 \\ \mu_3 &= -\frac{M_2}{\epsilon^2} - x_i f_3 \end{aligned} \right\} \quad (\text{A9a})$$

$$\left. \begin{aligned} \mu_0' &= -\frac{2M_4}{\epsilon^2} - x_i f_0' \\ \mu_1' &= \frac{5M_4}{\epsilon^2} - x_i f_1' \\ \mu_2' &= -\frac{4M_4}{\epsilon^2} - x_i f_2' \\ \mu_3' &= \frac{M_4}{\epsilon^2} - x_i f_3' \end{aligned} \right\} \quad (\text{A9b})$$

$$\left. \begin{aligned} \bar{\mu}_0 &= \frac{2M_5}{\epsilon^2} - x_i \bar{f}_0 \\ \bar{\mu}_1 &= -\frac{5M_5}{\epsilon^2} - x_i \bar{f}_1 \\ \bar{\mu}_2 &= \frac{4M_5}{\epsilon^2} - x_i \bar{f}_2 \\ \bar{\mu}_3 &= -\frac{M_5}{\epsilon^2} - x_i \bar{f}_3 \end{aligned} \right\} \quad (\text{A9c})$$

$$\left. \begin{aligned} r_0 &= -\frac{2M_3}{\epsilon^2} + f_0 \\ r_1 &= \frac{5M_3}{\epsilon^2} + f_1 \\ r_2 &= -\frac{4M_3}{\epsilon^2} + f_2 \\ r_3 &= \frac{M_3}{\epsilon^2} + f_3 \end{aligned} \right\} \quad (\text{A10a})$$

$$\left. \begin{aligned} r_0' &= \frac{2M_5}{\epsilon^2} + f_0' \\ r_1' &= -\frac{5M_5}{\epsilon^2} + f_1' \\ r_2' &= \frac{4M_5}{\epsilon^2} + f_2' \\ r_3' &= -\frac{M_5}{\epsilon^2} + f_3' \end{aligned} \right\} \quad (\text{A10b})$$

$$\left. \begin{aligned} \bar{r}_0 &= -\frac{2M_6}{\epsilon^2} + \bar{f}_0 - \omega_f^2 M_6 \\ \bar{r}_1 &= \frac{5M_6}{\epsilon^2} + \bar{f}_1 \\ \bar{r}_2 &= -\frac{4M_6}{\epsilon^2} + \bar{f}_2 \\ \bar{r}_3 &= \frac{M_6}{\epsilon^2} + \bar{f}_3 \end{aligned} \right\} \quad (\text{A10c})$$

$$\left. \begin{aligned} j &= \beta_i \ddot{\Phi}_{t_0} \epsilon e^{-\gamma_i \epsilon} \\ j' &= \beta_i \epsilon e^{-\gamma_i \epsilon} \left[-U \dot{\Phi}_{t_0} - \ddot{\Phi}_{t_0} \left(x_i + \frac{1}{2} c_i \right) \right] \\ \bar{j} &= \beta_i \epsilon e^{-\gamma_i \epsilon} \left[U \dot{\Phi}_{t_0} \theta_1 + \ddot{\Phi}_{t_0} \left(1 + \frac{1}{2} c_i \theta_1 \right) \right] \end{aligned} \right\} \quad (\text{A11})$$

APPENDIX B

DERIVATION OF DIFFERENCE EQUATIONS

In this appendix the parabolic and cubic difference equations for the first and second derivatives of a function are derived.

Parabolic equations.—For the parabolic difference equation, consider the function shown in figure 12(a). This function is assumed to be replaced by the arc of a parabola which passes through the three ordinates a , b , and c . It can be verified readily that such a curve can be given by the equation

$$w = \frac{a}{2} \left(\frac{y}{\epsilon} - 1 \right) \left(\frac{y}{\epsilon} - 2 \right) - b \frac{y}{\epsilon} \left(\frac{y}{\epsilon} - 2 \right) + \frac{c}{2} \frac{y}{\epsilon} \left(\frac{y}{\epsilon} - 1 \right) \quad (\text{B1})$$

The first and second derivatives of this equation at $y = \epsilon$ are given by the equations

$$\left. \frac{dw}{dy} \right]_{y=\epsilon} = \frac{c-a}{2\epsilon} \quad (\text{B2})$$

$$\left. \frac{d^2w}{dy^2} \right]_{y=\epsilon} = \frac{c-2b+a}{\epsilon^2} \quad (\text{B3})$$

These equations are the standard difference equations for the first and second derivatives of a function. The derivatives are purposely taken at the middle of the three ordinates because the resulting equations are suitable for use in the simplification of many problems. If the derivative had been taken at an outer ordinate, the approximation afforded would not be accurate enough to be useful.

Cubic equations.—The cubic difference equations may be derived in a manner similar to that for the parabolic equations. In this case four successive ordinates are used. (See fig. 12(b).) The function is replaced by a third-degree curve which is given by the equation

$$w = -\frac{a}{6} \left(\frac{y}{\epsilon} - 1 \right) \left(\frac{y}{\epsilon} - 2 \right) \left(\frac{y}{\epsilon} - 3 \right) + \frac{b}{2} \frac{y}{\epsilon} \left(\frac{y}{\epsilon} - 2 \right) \left(\frac{y}{\epsilon} - 3 \right) - \frac{c}{2} \frac{y}{\epsilon} \left(\frac{y}{\epsilon} - 1 \right) \left(\frac{y}{\epsilon} - 3 \right) + \frac{d}{6} \frac{y}{\epsilon} \left(\frac{y}{\epsilon} - 1 \right) \left(\frac{y}{\epsilon} - 2 \right) \quad (\text{B4})$$

Because of the increase in accuracy that results from the use of a higher-degree curve, the first and second derivatives may be taken at an outer ordinate with an accuracy which is about equivalent to that given by equations (B2) and (B3). The derivatives at $y = 3\epsilon$ are

$$\left. \frac{dw}{dy} \right]_{y=3\epsilon} = \frac{11d-18c+9b-2a}{6\epsilon} \quad (\text{B5})$$

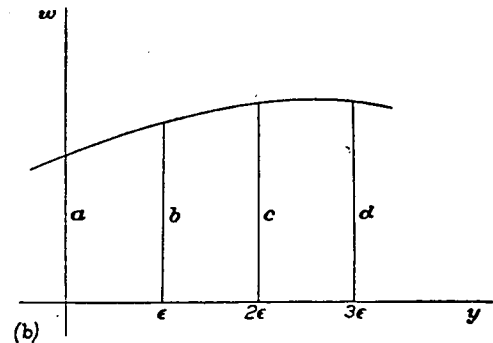
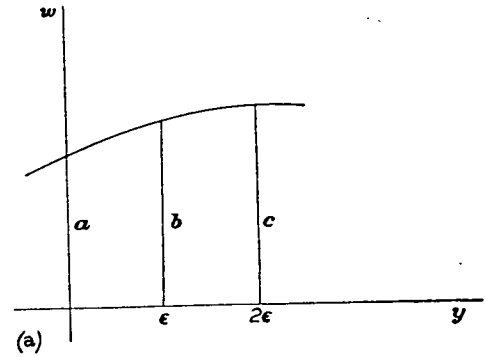
$$\left. \frac{d^2w}{dy^2} \right]_{y=3\epsilon} = \frac{2d-5c+4b-a}{\epsilon^2} \quad (\text{B6})$$

If taken at the third of the four ordinates, the derivatives are

$$\left. \frac{dw}{dy} \right]_{y=2\epsilon} = \frac{2d+3c-6b+a}{6\epsilon} \quad (\text{B7})$$

$$\left. \frac{d^2w}{dy^2} \right]_{y=2\epsilon} = \frac{d-2c+b}{\epsilon^2} \quad (\text{B8})$$

Equations (B5) and (B6) are used in the derivation of the response equation for an airplane in a gust. Equations (B7) and (B8) are useful in the derivation of the initial response.



(a) Parabolic.
(b) Cubic.

FIGURE 12.—Functional notation used in the derivation of parabolic and cubic difference equations.

APPENDIX C

DERIVATION OF MATRIX EQUATIONS OF EQUILIBRIUM

In this appendix the matrix equations

$$[A]|w| = |p| \quad (C1)$$

$$[B]|\varphi| = |q| \quad (C2)$$

for symmetrical bending and twisting of a free-free beam under normal and torsional loads are derived.

Bending.—In accordance with the assumptions made in this report the wing semispan is considered to be divided into six sections with a station point at the center of each section (see fig. 3). The inertia force of the mass and the aerodynamic force that develops over each section is in turn assumed to be concentrated at the respective station points. The wing is thus effectively a beam bending under six concentrated loads and, as such, will have a linearly varying moment between each station. The following general equation for the moment between the i and $i+1$ stations may therefore be written:

$$M = a_i + b_i y \quad (C3)$$

where

$$a_i = \left[1 + \frac{y(i)}{b\lambda_{i+1}} \right] M(i) - \frac{y(i)}{b\lambda_{i+1}} M(i+1)$$

$$b_i = \frac{1}{b\lambda_{i+1}} [M(i+1) - M(i)]$$

in which $y(i)$ is the abscissa to the i station.

The wing is further assumed to have a linear $1/EI$ variation between stations with the correct value of $1/EI$ at each station. This type of variation would lead to an EI curve which follows very closely the true stiffness curve of the wing and which of course has the correct values of EI at each station. A general equation for $1/EI$ may therefore also be written; thus,

$$\frac{1}{EI} = c_i + d_i y \quad (C4)$$

where

$$c_i = \left[1 + \frac{y(i)}{b\lambda_{i+1}} \right] \frac{1}{EI(i)} - \frac{y(i)}{b\lambda_{i+1}} \frac{1}{EI(i+1)}$$

$$d_i = \frac{1}{b\lambda_{i+1}} \left[\frac{1}{EI(i+1)} - \frac{1}{EI(i)} \right]$$

With equation (C3) and equation (C4) the well-known expression relating deflection to moment for a beam may be written

$$\frac{d^2 w}{dy^2} = \frac{M}{EI} = (a_i + b_i y)(c_i + d_i y) \quad (C5)$$

The deflection may be found most conveniently from this equation by use of the engineering beam theorem which states that the deflection of one point on a beam relative to the tangent of the deflection curve at another point is equal to the moment about the displaced point of the M/EI diagram between the two points. In this case symmetrical loading is being considered and therefore the boundary condition at the center line is that the slope must be zero; the deflection of each station relative to this point therefore may be readily computed. Fortunately, because of the convenient analytical representation of M/EI , these deflections may be found by exact integration. The deflection, for example, at station 4 due to the M/EI variation between stations i and $i+1$ may be given by the expression:

$$\int_{y(i)}^{y(i+1)} (a_i + b_i y)(c_i + d_i y)[y(4) - y] dy$$

Consideration of all the expressions of this sort leads to the total deflection of each station relative to the wing center line. From this deflection the more useful deflection relative to station 0 can be readily determined. The values of the deflection thus obtained are found to be expressible by the following matrix equation:

$$\begin{bmatrix} \bar{w}(1) \\ \bar{w}(2) \\ \bar{w}(3) \\ \bar{w}(4) \\ \bar{w}(5) \end{bmatrix} = \frac{b^2}{EI(0)} \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ a_{31} & a_{32} & a_{33} & a_{34} & 0 \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} M(0) \\ M(1) \\ M(2) \\ M(3) \\ M(4) \end{bmatrix} \quad (C6)$$

where the matrix elements are defined by the equations:

$$\left. \begin{aligned} a_{11} &= \lambda_0 \lambda_1 + \lambda_1^2 A_1 \\ a_{21} &= \lambda_0(\lambda_1 + \lambda_2) + \lambda_1^2 A_1 + \lambda_1 \lambda_2 B_1 \\ a_{31} &= \lambda_0(\lambda_1 + \lambda_2 + \lambda_3) + \lambda_1^2 A_1 + \lambda_1(\lambda_2 + \lambda_3) B_1 \\ a_{41} &= \lambda_0(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) + \lambda_1^2 A_1 + \lambda_1(\lambda_2 + \lambda_3 + \lambda_4) B_1 \\ a_{51} &= \lambda_0(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) + \lambda_1^2 A_1 + \lambda_1(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) B_1 \end{aligned} \right\} \quad (C7a)$$

$$\left. \begin{aligned} a_{12} &= \lambda_1^2 C_1 \\ a_{22} &= \lambda_1^2 C_1 + \lambda_1 \lambda_2 D_1 + \lambda_2^2 A_2 \\ a_{32} &= \lambda_1^2 C_1 + \lambda_1(\lambda_2 + \lambda_3) D_1 + \lambda_2^2 A_2 + \lambda_2 \lambda_3 B_2 \\ a_{42} &= \lambda_1^2 C_1 + \lambda_1(\lambda_2 + \lambda_3 + \lambda_4) D_1 + \lambda_2^2 A_2 + \lambda_2(\lambda_3 + \lambda_4) B_2 \\ a_{52} &= \lambda_1^2 C_1 + \lambda_1(\lambda_2 + \lambda_3 + \lambda_4 + \lambda_5) D_1 + \lambda_2^2 A_2 + \lambda_2(\lambda_3 + \lambda_4 + \lambda_5) B_2 \end{aligned} \right\} \quad (C7b)$$

$$\left. \begin{aligned} a_{23} &= \lambda_2^2 C_2 \\ a_{33} &= \lambda_2^2 C_2 + \lambda_2 \lambda_3 D_2 + \lambda_3^2 A_3 \\ a_{43} &= \lambda_2^2 C_2 + \lambda_2 (\lambda_3 + \lambda_4) D_2 + \lambda_3^2 A_3 + \lambda_3 \lambda_4 B_3 \\ a_{53} &= \lambda_2^2 C_2 + \lambda_2 (\lambda_3 + \lambda_4 + \lambda_5) D_2 + \lambda_3^2 A_3 + \lambda_3 (\lambda_4 + \lambda_5) B_3 \end{aligned} \right\} \quad (C7c)$$

$$\left. \begin{aligned} a_{34} &= \lambda_3^2 C_3 \\ a_{44} &= \lambda_3^2 C_3 + \lambda_3 \lambda_4 D_3 + \lambda_4^2 A_4 \\ a_{54} &= \lambda_3^2 C_3 + \lambda_3 (\lambda_4 + \lambda_5) D_3 + \lambda_4^2 A_4 + \lambda_4 \lambda_5 B_4 \end{aligned} \right\} \quad (C7d)$$

$$\left. \begin{aligned} a_{45} &= \lambda_4^2 C_4 \\ a_{55} &= \lambda_4^2 C_4 + \lambda_4 \lambda_5 D_4 + \lambda_5^2 A_5 \end{aligned} \right\} \quad (C7e)$$

in which

$$\left. \begin{aligned} A_i &= \frac{1}{4} \frac{I(0)}{I(i-1)} + \frac{1}{12} \frac{I(0)}{I(i)} \\ B_i &= \frac{1}{3} \frac{I(0)}{I(i-1)} + \frac{1}{6} \frac{I(0)}{I(i)} \\ C_i &= \frac{1}{12} \frac{I(0)}{I(i-1)} + \frac{1}{12} \frac{I(0)}{I(i)} \\ D_i &= \frac{1}{6} \frac{I(0)}{I(i-1)} + \frac{1}{3} \frac{I(0)}{I(i)} \end{aligned} \right\} \quad (C8)$$

The moment $M(5)$ does not appear in equation (C6) because the boundary condition is used so that the moment at station 5 is zero. For convenience equation (C6) may be given in the abbreviated form:

$$|\bar{w}| = \frac{b^2}{EI(0)} [H_1] |M| \quad (C9)$$

From the five loads $p(1)$, $p(2)$, $p(3)$, $p(4)$, and $p(5)$, the moment at each station may be found. The equations relating the moments to the loads can be shown to be given by the matrix equation:

$$\begin{bmatrix} M(0) \\ M(1) \\ M(2) \\ M(3) \\ M(4) \end{bmatrix} = b \begin{bmatrix} \lambda_1 & \lambda_1 + \lambda_2 & \lambda_1 + \lambda_2 + \lambda_3 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 & \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ 0 & \lambda_2 & \lambda_2 + \lambda_3 & \lambda_2 + \lambda_3 + \lambda_4 & \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 \\ 0 & 0 & \lambda_3 & \lambda_3 + \lambda_4 & \lambda_3 + \lambda_4 + \lambda_5 \\ 0 & 0 & 0 & \lambda_4 & \lambda_4 + \lambda_5 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix} \begin{bmatrix} p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \end{bmatrix} \quad (C10)$$

which can be abbreviated simply

$$|M| = b [H_2] |p| \quad (C11)$$

Substitution of equation (C11) into equation (C9) gives

$$|\bar{w}| = \frac{b^3}{EI(0)} [H_1] [H_2] |p| \quad (C12)$$

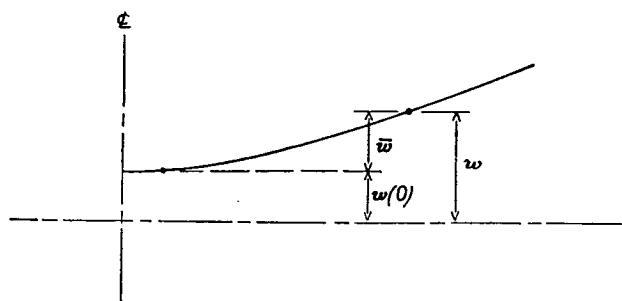
Multiplication through by the reciprocal of $\frac{b^3}{EI(0)} [H_1] [H_2]$

results in the equation:

$$\frac{EI(0)}{b^3} [[H_1] [H_2]]^{-1} |\bar{w}| = |p| \quad (C13)$$

This equation thus gives the loads in terms of the deflection of each station relative to station 0. In the case under consideration, however, the wing is a free body capable of motion through space and therefore to set up properly the

equations of motion the deflection must be given relative to a fixed datum line. This datum line is most conveniently located as the position of the wing prior to action of the disturbing loads. Consideration of sketch 3



SKETCH 3.

will show therefore that the following relation must exist:

$$\bar{w} = w - w(0) \quad (C14)$$

Substitution of this equation into equation (C13) gives

$$\frac{EI(0)}{b^3} [[H_1][H_2]]^{-1} |w - w(0)| = |p| \quad (C15)$$

To aid in the derivation $\frac{EI(0)}{b^3} [[H_1][H_2]]^{-1}$ is now written in the general form:

$$\frac{EI(0)}{b^3} [[H_1][H_2]]^{-1} = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{12} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{13} & b_{23} & b_{33} & b_{34} & b_{35} \\ b_{14} & b_{24} & b_{34} & b_{44} & b_{45} \\ b_{15} & b_{25} & b_{35} & b_{45} & b_{55} \end{bmatrix} \quad (C16)$$

Thus with this equation, equation (C15) may be transformed to the form:

$$\begin{bmatrix} b_{01} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{02} & b_{12} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{03} & b_{13} & b_{23} & b_{33} & b_{34} & b_{35} \\ b_{04} & b_{14} & b_{24} & b_{34} & b_{44} & b_{45} \\ b_{05} & b_{15} & b_{25} & b_{35} & b_{45} & b_{55} \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix} = \begin{bmatrix} p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \end{bmatrix} \quad (C17)$$

where

$$\left. \begin{aligned} b_{01} &= -(b_{11} + b_{12} + b_{13} + b_{14} + b_{15}) \\ b_{02} &= -(b_{12} + b_{22} + b_{23} + b_{24} + b_{25}) \\ b_{03} &= -(b_{13} + b_{23} + b_{33} + b_{34} + b_{35}) \\ b_{04} &= -(b_{14} + b_{24} + b_{34} + b_{44} + b_{45}) \\ b_{05} &= -(b_{15} + b_{25} + b_{35} + b_{45} + b_{55}) \end{aligned} \right\} \quad (C18)$$

Equation (C17) is noted to express all the loads except $p(0)$ in terms of the six deflections. An additional equation in which $p(0)$ is expressed also in terms of the deflections may be established by use of the condition that all the loads acting on the wing semispan must add up to equal zero; that is,

$$p(0) + p(1) + p(2) + p(3) + p(4) + p(5) = 0 \quad (C19)$$

This condition automatically satisfies the two boundary conditions that the shear must be zero at the tip and center line of the wing. Thus if the five equations represented by equation (C17) are added, and use is made of equations (C18) and (C19), the following equation results:

$$b_{00}w(0) + b_{01}w(1) + b_{02}w(2) + b_{03}w(3) + b_{04}w(4) + b_{05}w(5) = p(0) \quad (C20)$$

where

$$b_{00} = -(b_{01} + b_{02} + b_{03} + b_{04} + b_{05}) \quad (C21)$$

This equation may now be combined with equation (C17) to give finally

$$\begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} & b_{05} \\ b_{01} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} \\ b_{02} & b_{12} & b_{22} & b_{23} & b_{24} & b_{25} \\ b_{03} & b_{13} & b_{23} & b_{33} & b_{34} & b_{35} \\ b_{04} & b_{14} & b_{24} & b_{34} & b_{44} & b_{45} \\ b_{05} & b_{15} & b_{25} & b_{35} & b_{45} & b_{55} \end{bmatrix} \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(3) \\ w(4) \\ w(5) \end{bmatrix} = \begin{bmatrix} p(0) \\ p(1) \\ p(2) \\ p(3) \\ p(4) \\ p(5) \end{bmatrix} \quad (C22)$$

This equation is thus the desired matrix equation which relates the normal loads to the deflection. If the square matrix is denoted by $[A]$, the equation may be abbreviated conveniently to the form

$$[A] |w| = |p| \quad (C23)$$

which is the form used in the text. (See equation (40).)

As an aid in computational work, a summary of the steps involved in the determination of $[A]$ is given to close this section:

(1) From the I values at the respective stations, compute the coefficients given by equations (C8).

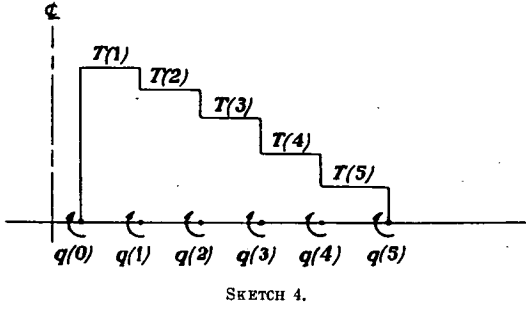
(2) With these coefficients determine the matrix elements given by equations (C7). These elements form the matrix $[H_1]$ which is defined by equations (C6) and (C9).

(3) Multiply the $[H_1]$ matrix by the $[H_2]$ matrix, which is defined by equations (C10) and (C11). The result should be a symmetrical matrix; this property serves as a very useful computational check.

(4) Invert the $\frac{b^3}{EI(0)} [[H_1][H_2]]$ matrix. This matrix should also be symmetrical. (The Crout method (reference 6) serves as a rather quick and useful means for performing the inversion.)

(5) Add the columns of the inverted matrix and place the negative of these sums at the top of their respective columns such as to form a new row of matrix elements. Then add these sums and place the negative of the sum as the first matrix element of the newly formed row. A new column headed by this value is thus in the making. Fill in the remainder of the column with the respective elements of the new row; that is, the appropriate values should be inserted to make the matrix symmetrical. This final matrix is the desired [A] matrix.

Torsion.—For the torsional case the torque loads q are assumed to be concentrated at the stations just as in the case for the normal loads p . Consideration then of the following example torque diagram (sketch 4)

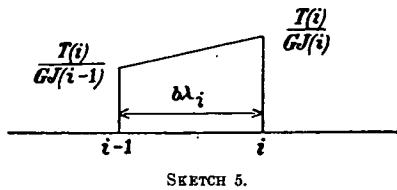


will show that the following equations must apply:

$$\left. \begin{aligned} q(0) &= -T(1) \\ q(1) &= T(1) - T(2) \\ q(2) &= T(2) - T(3) \\ q(3) &= T(3) - T(4) \\ q(4) &= T(4) - T(5) \\ q(5) &= T(5) \end{aligned} \right\} \quad (C24)$$

where $T(i)$ represents the total torque present in the i interval. No torque exists between the wing center line and station 0.

To aid in the derivation, the assumption is made that $1/GJ$ varies linearly between stations. A typical T/GJ diagram between, say, the $i-1$ and the i stations would appear as in sketch 5:



From the differential relation $\frac{d\varphi}{dy} = \frac{T}{GJ}$, the fact may be observed that the change in angle of twist between two stations is equal to the area of the T/GJ diagram between the two stations; therefore,

$$\varphi(i) - \varphi(i-1) = \frac{b\lambda_i}{2} \left[\frac{T(i)}{GJ(i-1)} + \frac{T(i)}{GJ(i)} \right] \quad (C25)$$

If the notation

$$j_i = \frac{2G}{b\lambda_i} \frac{1}{\frac{1}{J(i-1)} + \frac{1}{J(i)}} \quad (C26)$$

is employed, equation (C25) may be written

$$T(i) = j_i [\varphi(i) - \varphi(i-1)] \quad (C27)$$

Application of this equation to each of the spanwise stations gives the following equations for T :

$$\left. \begin{aligned} T(1) &= j_1 [\varphi(1) - \varphi(0)] \\ T(2) &= j_2 [\varphi(2) - \varphi(1)] \\ T(3) &= j_3 [\varphi(3) - \varphi(2)] \\ T(4) &= j_4 [\varphi(4) - \varphi(3)] \\ T(5) &= j_5 [\varphi(5) - \varphi(4)] \end{aligned} \right\} \quad (C28)$$

Substitution now of these equations into equations (C24) gives the desired equations relating the torque loads to the angle of twist. The equations thus found can be given in the matrix form:

$$\begin{bmatrix} j_1 & -j_1 & 0 & 0 & 0 & 0 \\ -j_1 & (j_1 + j_2) & -j_2 & 0 & 0 & 0 \\ 0 & -j_2 & (j_2 + j_3) & -j_3 & 0 & 0 \\ 0 & 0 & -j_3 & (j_3 + j_4) & -j_4 & 0 \\ 0 & 0 & 0 & -j_4 & (j_4 + j_5) & -j_5 \\ 0 & 0 & 0 & 0 & -j_5 & j_5 \end{bmatrix} \begin{bmatrix} \varphi(0) \\ \varphi(1) \\ \varphi(2) \\ \varphi(3) \\ \varphi(4) \\ \varphi(5) \end{bmatrix} = \begin{bmatrix} q(0) \\ q(1) \\ q(2) \\ q(3) \\ q(4) \\ q(5) \end{bmatrix} \quad (C29)$$

which can be abbreviated to

$$[B] |\varphi| = |q| \quad (C30)$$

the form used in the test. (See equation (41).) Thus all that is involved in the computation of the matrix [B] is the evaluation of the matrix elements by means of equation (C26).

APPENDIX D

RECURRENCE EQUATION FOR THE EVALUATION OF DUHAMEL'S INTEGRAL INVOLVING AN EXPONENTIAL KERNEL

The derivation of a rather simple recurrence relation for the step-by-step evaluation of the three unsteady lift integrals appearing in equation (25) is presented. This derivation is made possible because the kernels of the integrals are expressible in exponential form.

From equation (23) the first and second derivatives of the Φ function may be written

$$\dot{\Phi} = -\frac{2U}{c_0} \lambda a_1 e^{-\lambda \frac{2U}{c_0} t} = \dot{\Phi}_0 e^{-\gamma t} \quad (D1)$$

$$\ddot{\Phi} = \frac{4U^2}{c_0^2} \lambda^2 a_1 e^{-\lambda \frac{2U}{c_0} t} = \ddot{\Phi}_0 e^{-\gamma t} \quad (D2)$$

where

$$\gamma = \lambda \frac{2U}{c_0}$$

$$\dot{\Phi}_0 = -\gamma a_1$$

$$\ddot{\Phi}_0 = \gamma^2 a_1$$

With these equations the three integrals of equation (25) may be combined conveniently into the following single integral denoted by I_t :

$$I_t = \int_0^t \left\{ \ddot{\Phi}_0 \beta c l w - \left[\dot{\Phi}_0 \beta c l U + \ddot{\Phi}_0 \beta c l \left(\frac{3}{4} - \frac{a}{c} \right) \right] \varphi \right\} e^{-\gamma(t-\tau)} d\tau \quad (D3)$$

For convenience the notation

$$Y = \left\{ \ddot{\Phi}_0 \beta c l w - \left[\dot{\Phi}_0 \beta c l U + \ddot{\Phi}_0 \beta c l \left(\frac{3}{4} - \frac{a}{c} \right) \right] \varphi \right\} \quad (D4)$$

is introduced and thus equation (D3) becomes

$$I_t = \int_0^t Y e^{-\gamma(t-\tau)} d\tau$$

or

$$I_t = e^{-\gamma t} \int_0^t Y e^{\gamma \tau} d\tau \quad (D5)$$

Mathematically, the integral in this equation may be interpreted to represent the area under the function given as a product of Y and $e^{\gamma \tau}$. In accordance with numerical evaluation processes, the interval 0 to t may be divided into a number of time stations of interval ϵ . The product of Y and $e^{\gamma \tau}$ may then be found at each of the time stations and from these products the area under the curve may be determined in first approximation by the trapezoidal method of determining areas. Thus, if the time station n corresponds to time t , the expression for I_t may be approximated as follows:

$$I_t \approx I_n = \epsilon e^{-\gamma n \epsilon} \left[Y_1 e^{\gamma \epsilon} + Y_2 e^{\gamma 2 \epsilon} + \dots + Y_{n-1} e^{\gamma(n-1)\epsilon} + \frac{1}{2} Y_n e^{\gamma n \epsilon} \right] \quad (D6)$$

where Y_0 does not appear since the initial conditions are used that the deflection w and rotation φ are zero at $t=0$, and therefore Y_0 is zero. (See equation (D4).) More accurate methods, such as Simpson's method, could be used for determining the area under the curve, but because of the small interval chosen the consequent increase in accuracy is negligible. If the notation

$$F_n = \epsilon e^{-\gamma n \epsilon} [Y_1 e^{\gamma \epsilon} + Y_2 e^{\gamma 2 \epsilon} + \dots + Y_{n-1} e^{\gamma(n-1)\epsilon}] \quad (D7)$$

is introduced, equation (D6) may be written simply

$$I_n = F_n + \frac{\epsilon}{2} Y_n \quad (D8)$$

If equation (D5) is expanded similarly, only for an upper limit of $t - \epsilon$, the expanded result would be

$$I_{n-1} = \epsilon e^{-\gamma(n-1)\epsilon} \left[Y_1 e^{\gamma \epsilon} + Y_2 e^{\gamma 2 \epsilon} + \dots + Y_{n-2} e^{\gamma(n-2)\epsilon} + \frac{1}{2} Y_{n-1} e^{\gamma(n-1)\epsilon} \right] \quad (D9)$$

By analogy with equation (D7), however,

$$F_{n-1} = \epsilon e^{-\gamma(n-1)\epsilon} [Y_1 e^{\gamma \epsilon} + Y_2 e^{\gamma 2 \epsilon} + \dots + Y_{n-2} e^{\gamma(n-2)\epsilon}] \quad (D10)$$

and therefore equation (D9) becomes

$$I_{n-1} = F_{n-1} + \frac{\epsilon}{2} Y_{n-1} \quad (D11)$$

A study of equations (D7) and (D10) shows that the following relation must exist:

$$F_n = e^{-\gamma \epsilon} F_{n-1} + \epsilon e^{-\gamma \epsilon} Y_{n-1} \quad (D12)$$

Now, if equation (D4) is used to rewrite Y_n and Y_{n-1} in equations (D8) and (D12), the value of I_n may be given finally by the equation:

$$I_n = F_n + \frac{1}{2} \ddot{\Phi}_0 \beta c l \epsilon w_n - \frac{1}{2} \beta c l \epsilon \left[U \dot{\Phi}_0 + c \left(\frac{3}{4} - \frac{a}{c} \right) \ddot{\Phi}_0 \right] \varphi_n \quad (D13)$$

where

$$F_n = e^{-\gamma \epsilon} F_{n-1} + \ddot{\Phi}_0 \epsilon e^{-\gamma \epsilon} \beta c l w_{n-1} - \beta c l \epsilon e^{-\gamma \epsilon} \left[U \dot{\Phi}_0 + c \left(\frac{3}{4} - \frac{a}{c} \right) \ddot{\Phi}_0 \right] \varphi_{n-1} \quad (D14)$$

The value of the unsteady lift integrals is thus given by equation (D13). As regards the analysis given in the present report, w_{n-1} and φ_{n-1} are the values of deflection and rotation which have, say, just been determined from the recurrence equation for response. The value F_{n-1} was also established and therefore F_n can be determined as a definite quantity. The value I_n is thus seen to be given in terms of the known F_n and in terms of w_n and φ_n which are the next values to be evaluated from the recurrence equation.

APPENDIX E

MATRIX ALGEBRA.

This appendix is written for those not familiar with matrix notations or matrix methods. All the matrix algebra necessary for the understanding of this report is described hereinafter by way of examples.

Matrix definition.—Some of the basic types of matrices are illustrated by the following arbitrary matrices which are of the third order:

The column matrix

$$\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

The row matrix

$$\begin{bmatrix} 2 & -3 & 1 \end{bmatrix}$$

The square matrix

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix}$$

The diagonal matrix

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The identity matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Element definition.—Each of the terms that appear in a matrix is defined as an element. Its position is usually denoted in a row by the number of terms from the left and in a column by the number of terms from the top.

Matrix addition.—The addition of two matrices produces a single matrix. Addition is performed by simply adding together corresponding elements. For example,

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} + \begin{bmatrix} 4 & -1 & 0 \\ 0 & 3 & 2 \\ 2 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 6 & -4 & 1 \\ 1 & 5 & 0 \\ 1 & -1 & 2 \end{bmatrix}$$

Multiplication of a matrix by a scalar number.—In the multiplication of a matrix by a scalar number every element in the matrix is multiplied by the number. For example,

$$2 \begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -6 & 2 \\ 2 & 4 & -4 \\ -2 & -2 & 6 \end{bmatrix}$$

Multiplication of a column matrix by a row matrix.—The product of a column matrix and a row matrix is equal to the sum of the products of the corresponding elements. For example,

$$\begin{bmatrix} 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -4 \end{bmatrix} = (2 \times 2) + (-3 \times 1) + [1 \times (-4)] = -3$$

Multiplication of a column matrix by a square matrix.—The multiplication of a column matrix by a square matrix produces a column matrix. Consider the following set of three simultaneous equations:

$$\left. \begin{aligned} 2y_1 - 3y_2 + y_3 &= a_1 \\ y_1 + 2y_2 - 2y_3 &= a_2 \\ -y_1 - y_2 + 3y_3 &= a_3 \end{aligned} \right\} \quad (\text{E1})$$

The procedure adopted in matrix algebra is to write these equations in the matrix form

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (\text{E2})$$

where the multiplication of the $|y|$ column matrix by each row in the square matrix produces the respective elements in the $|a|$ column matrix. (See multiplication of a column matrix by a row matrix.)

In order to simplify the presentation of an analysis, the symbolic or abbreviated matrix form is used quite often. The symbolic form of equation (E2) is simply

$$[M] |y| = |a| \quad (\text{E3})$$

The determination of $|a|$ by the multiplication of $|y|$ by $[M]$ is illustrated with arbitrary values of y , say $y_1=4$, $y_2=5$, and $y_3=6$, by the following equation:

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} (2 \times 4) + (-3 \times 5) + (1 \times 6) \\ (1 \times 4) + (2 \times 5) + (-2 \times 6) \\ (-1 \times 4) + (-1 \times 5) + (3 \times 6) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 9 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (\text{E4})$$

Multiplication of a square matrix by a square matrix.—The multiplication of two square matrices produces a square matrix. Multiplication is performed by letting the multiplying matrix operate, as in the preceding section, on each of the successive columns in the matrix being multiplied to produce corresponding successive columns in the product matrix. For example,

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & -2 & 3 \\ 1 & 3 & -2 \\ -1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -12 & 14 \\ 5 & 2 & -5 \\ -5 & 2 & 5 \end{bmatrix} \quad (\text{E5})$$

Order of multiplication.—In general the commutative multiplication law of ordinary algebra does not hold in matrix methods; that is,

$$|A| |B| \neq |B| |A|$$

Therefore, whenever the product of several matrices is indicated, these matrices must be multiplied together without changing their order.

Matrix partitioning and submatrices.—A matrix may be partitioned or divided at will into smaller matrices. For example, the left-hand side of equation (E4) may be partitioned as follows:

$$\left[\begin{array}{c|cc} 2 & -3 & 1 \\ \hline - & - & - \\ 1 & 2 & -2 \\ \hline -1 & -1 & 3 \end{array} \right] \left| \begin{array}{c} 4 \\ 5 \\ 6 \end{array} \right|$$

The matrices which are formed by the dividing lines are called submatrices. These submatrices may be treated as though they were elements when matrix operations are performed. For example, with the notation

$$\begin{aligned} a &= \begin{bmatrix} -3 & 1 \end{bmatrix} \\ b &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ c &= \begin{bmatrix} 2 & -2 \\ -1 & 3 \end{bmatrix} \\ d &= \begin{bmatrix} 5 \\ 6 \end{bmatrix} \end{aligned}$$

the multiplication of the foregoing partitioned matrix is as follows:

$$\left[\begin{array}{c|c} 2 & a \\ \hline - & - \\ b & c \end{array} \right] \left| \begin{array}{c} 4 \\ d \end{array} \right| = \left| \begin{array}{c} 8+ad \\ 4b+cd \end{array} \right|$$

The reciprocal of a matrix and the identity matrix.—By ordinary algebraic methods the formal operation involved in the solution for x of the equation

$$mx = a$$

is the multiplication through by the reciprocal of m ; thus,

$$x = m^{-1}a$$

The same formal operation may be applied to matrix equations. For example, the solution for $|y|$ in equation (E3) is simply

$$|y| = [M]^{-1}|a|$$

where $[M]^{-1}$ is the reciprocal, or the inverse, of $[M]$.

The reciprocal of a matrix is found as the matrix which satisfies either of the equivalent equations

$$[M]^{-1} [M] = [I]$$

$$[M] [M]^{-1} = [I]$$

where $[I]$ is the identity matrix. For equations (E2) and (E3), the reciprocal of $[M]$ is found as the matrix which satisfies the equation

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 2 & -2 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

If this equation is considered in relation to equations (E1), (E2), and (E5), the values of b_1 , b_2 , and b_3 would simply be values of y_1 , y_2 , and y_3 which satisfy equation (E1) for $a_1=1$, $a_2=0$, and $a_3=0$; c_1 , c_2 , and c_3 would be the values for $a_1=0$, $a_2=1$, and $a_3=0$; and d_1 , d_2 , and d_3 would be the values for $a_1=0$, $a_2=0$, and $a_3=1$. For this example, the solutions are

$$b_1 = \frac{1}{3} \quad c_1 = \frac{2}{3} \quad d_1 = \frac{1}{3}$$

$$b_2 = -\frac{1}{12} \quad c_2 = \frac{7}{12} \quad d_2 = \frac{5}{12}$$

$$b_3 = \frac{1}{12} \quad c_3 = \frac{5}{12} \quad d_3 = \frac{7}{12}$$

The Crout method (reference 6) provides a very quick and convenient method for determining these solutions.

The determination of y by the operation $[M]^{-1}$ on $|a|$ is illustrated as follows for $a_1 = -1$, $a_2 = 2$, and $a_3 = 9$:

$$\frac{1}{12} \begin{bmatrix} 4 & 8 & 4 \\ -1 & 7 & 5 \\ 1 & 5 & 7 \end{bmatrix} \left| \begin{array}{c} -1 \\ 2 \\ 9 \end{array} \right| = \left| \begin{array}{c} 4 \\ 5 \\ 6 \end{array} \right| \quad (\text{E11})$$

The operation performed by this equation can be seen to be the inverse operation of equation (E4).

REFERENCES

1. Goland, M., Luke, Y. L., and Kahn, E. A.: Prediction of Wing Loads Due to Gusts Including Aero-Elastic Effects. Part I—Formulation of the Method. AF TR No. 5706, Air Materiel Command, U. S. Air Force, July 21, 1947.
2. Sears, William R.: Some Aspects of Non-Stationary Airfoil Theory and Its Practical Application. Jour. Aero. Sci., vol. 8, no. 3, Jan. 1941, pp. 104–108.
3. Sears, William R., and Sparks, Brian O.: On the Reaction of an Elastic Wing to Vertical Gusts. Jour. Aero. Sci., vol. 9, no. 2, Dec. 1941, pp. 64–67.
4. Jenkins, E. S., and Pancu, C. D. P.: Dynamic Loads on Airplane Structures. Preprint for presentation at the SAE National Aeronautic and Air Transport Meeting. April 13–15, 1948.
5. Jones, Robert T.: The Unsteady Lift of a Wing of Finite Aspect Ratio. NACA Rep. 681, 1940.
6. Crout, Prescott D.: A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients. Trans. AIEE, vol. 60, 1941, pp. 1235–1240. (Abridged as Marchant Methods MM-182, Sept. 1941, Marchant Calculating Machine Co., Oakland, Calif.)

REPORT 1011

DYNAMICS OF A TURBOJET ENGINE CONSIDERED AS A QUASI-STATIC SYSTEM¹

By EDWARD W. OTTO and BURT L. TAYLOR, III¹

SUMMARY

A determination of the dynamic characteristics of a typical turbojet engine with a centrifugal compressor, a sonic-flow turbine-nozzle diaphragm, and fixed-area exhaust nozzle is presented. A generalized equation for the transient behavior of the engine was developed; this equation was then verified by calculations using compressor- and turbine-performance charts extrapolated from equilibrium operating data and by experimental data obtained from an engine operated under transients in fuel flow.

The results indicate that a linear differential equation for engine acceleration as a function of fuel flow and engine speed for operation near a steady-state operating condition can be written. The coefficients of this equation can be obtained either from actual transient data or with a fair degree of accuracy from the steady-state performance maps of the compressor and turbine and can be corrected for altitude in the same manner that steady-state performance data are corrected.

INTRODUCTION

In a turbojet engine, the various engine variables such as speed and thrust respond comparatively slowly to changes in fuel flow—much more slowly than reciprocating-engine variables such as torque respond to changes in manifold pressure. Any control mechanism charged with the responsibility of keeping turbojet-engine variables at a set value is faced with the problem of changing the fuel flow in response to errors in the controlled variable at a rate that is fast enough to bring the variable back to the set value as quickly as possible but not so fast that excessive turbine temperatures are encountered. The same problem is encountered when the control is asked to change the speed or thrust from one set value to another. Thus, to insure the design of a successful control the designer must have information regarding the response rate of the particular engine variable that he is interested in controlling.

A study of the dynamic behavior of engine speed, made at the NACA Lewis laboratory in 1948, is presented herein.

This variable rather than thrust has been studied because, in the turbojet engine, speed is almost analogous to thrust and can be more easily and more accurately measured. Obviously, other engine variables may possess as many advantages for use as a thrust-control parameter. A previous investigation at this laboratory has indicated, however, that speed is the best parameter available because no other engine variable exhibits as many advantages as does speed. At the present time speed is used almost universally as a thrust-control parameter in turbojet engines.

It was felt that the needs of the control designer would be best served if the dynamic behavior of engine speed could be expressed in mathematical form and if altitude corrections to this expression could be indicated. Also it is desirable to know whether or not the engine dynamics can be predicted within reason from the steady-state performance charts of the engine components. Accordingly, the study contained herein was pursued as follows: A differential equation relating speed and fuel flow was derived by considering the thermodynamic and flow processes to be quasi-static. Altitude corrections were then applied to this equation in the same manner as steady-state altitude corrections are applied. The coefficients of the terms of the differential equation were obtained by plotting accelerating torque as a function of fuel flow and speed from calculations involving the use of the steady-state performance charts of the engine components. A typical engine was then operated under certain transients of fuel flow to obtain experimental data for a verification of both the differential equation derived and the coefficients obtained from steady-state component data.

ANALYSIS

In a turbojet engine of the type under consideration, an increase in fuel flow at constant speed causes an increase in turbine-inlet temperature, which results in an increase in turbine torque. The difference between the turbine torque and the torque absorbed by the compressor then accelerates the engine according to the following equation:

$$\alpha = \frac{Q}{I}$$

¹ Supersedes NACA TN 2091, "Dynamics of Turbojet Engine Considered as Quasi-Static System" by Edward W. Otto and Burt L. Taylor, III, 1950.

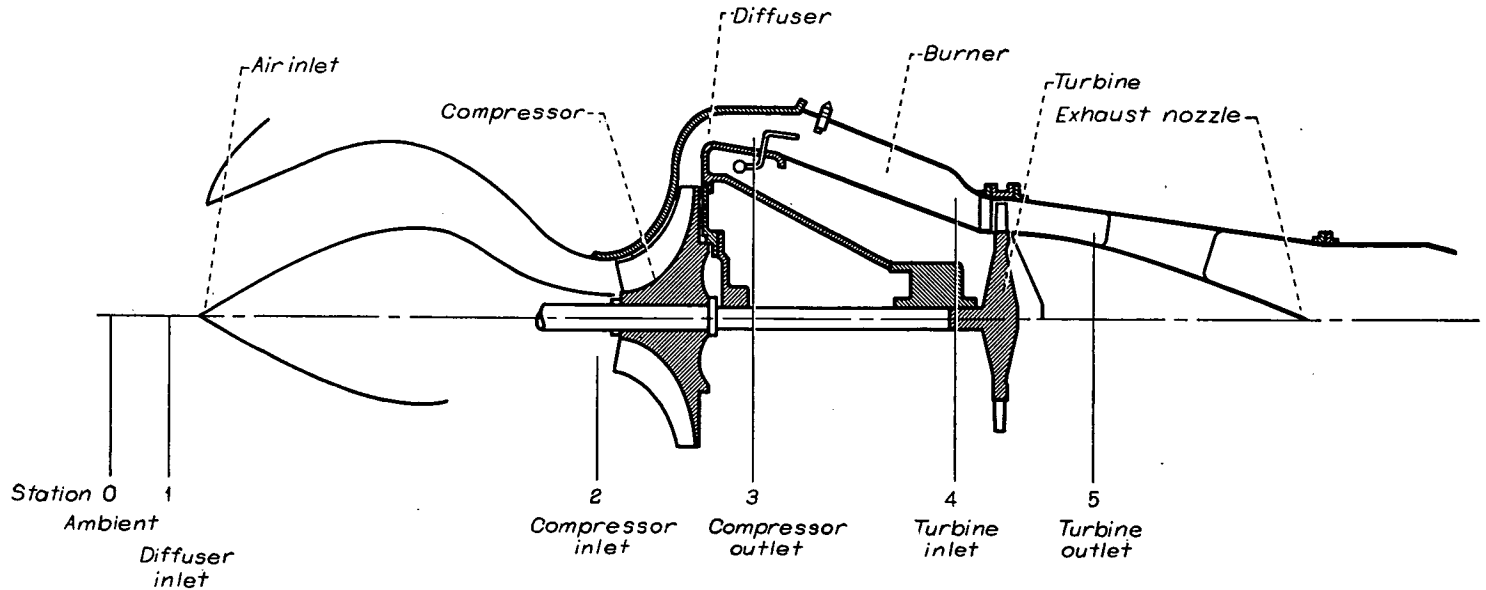


FIGURE 1.—Typical configuration of turbojet engine.

where α is the angular acceleration, Q is the difference between the instantaneous torque output of the turbine and the instantaneous torque absorbed by the compressor, and I is the polar moment of inertia of all rotating parts. Thus the effect of a change in fuel flow on engine acceleration is proportional to the effect of a change in fuel flow on the difference between torque output of the turbine and torque absorbed by the compressor. From a general consideration of the mechanics of the system, therefore, a differential equation relating speed and fuel flow can be written. This equation can then be correlated with data from a typical engine of the type under consideration. This procedure is demonstrated with an engine consisting of the components shown in figure 1.

DEVELOPMENT OF GENERALIZED ENGINE EQUATION

For the equilibrium running conditions of turbojet engines, various engine parameters may be represented by general functional forms of the one independent variable, effective fuel flow; that is, $N=f(W_f\eta_b)$, $T_4=g(W_f\eta_b)$, and so forth. (All symbols are defined in appendix A.) The development of general functional forms relating the variables during transient conditions of operation requires some hypothesis concerning the thermodynamic and flow processes during these periods. The hypothesis made herein is that the processes are quasi-static; that is, they act like a continuous series of equilibrium states. Each point of the dynamic state is an equilibrium condition for the flow and thermodynamic processes. It follows that a point reached by each component of the system during the transient corresponds to some equilibrium point for that component. Furthermore, the point encountered by the complete engine, being a sum of the components, is also some equilibrium point for the complete engine.

Without altering the turbojet-engine geometry, equilibrium operating points other than the equilibrium operating conditions can be obtained only by applying an outside

load to the engine. The action of an external load creates an additional independent variable for the dynamic state. This action may be expressed in terms of some engine variable, which allows this variable to be considered as the additional independent variable. For transient conditions of operation, various dependent variables of the engine may therefore be considered as functions of the effective fuel flow and another engine variable.

Because this analysis is concerned with the transient behavior of engine speed, this variable was chosen as the additional independent variable for the dynamic state. Unbalanced torque was chosen as the dependent variable because of its relation to engine speed. The assumed functional relation is therefore:

$$Q=f(W_f\eta_b, N) \quad (1)$$

This function may be expanded about an initial point i as follows:

$$\begin{aligned} Q=f(W_f\eta_b, N) &= Q_i + \left[\frac{\partial Q}{\partial (W_f\eta_b)} \right]_i [W_f\eta_b - (W_f\eta_b)_i] + \\ &\quad \left(\frac{\partial Q}{\partial N} \right)_i (N - N_i) + \frac{\left[\frac{\partial^2 Q}{\partial (W_f\eta_b)^2} \right]_i [W_f\eta_b - (W_f\eta_b)_i]^2}{2!} + \\ &\quad \frac{2 \left[\frac{\partial^2 Q}{\partial (W_f\eta_b) \partial N} \right]_i [W_f\eta_b - (W_f\eta_b)_i] (N - N_i)}{2!} + \\ &\quad \frac{\left(\frac{\partial^2 Q}{\partial N^2} \right)_i (N - N_i)^2}{2!} + \dots \end{aligned} \quad (2)$$

When the deviation from the point i is sufficiently small, the terms containing products of higher derivatives and products or powers of small numbers become negligible and only the

first three terms of equation (2) are of significant value. Furthermore, if the point i is a steady-state point o , the initial value of accelerating torque Q_i is zero.

When the deviations from the steady state are denoted by Δ , the significant terms of equation (2) can be written as

$$Q = \left[\frac{\partial Q}{\partial (W_f \eta_b)} \right]_o \Delta(W_f \eta_b) + \left(\frac{\partial Q}{\partial N} \right)_o \Delta N \quad (3)$$

The accelerating torque can also be expressed as a function of engine acceleration as

$$Q = \frac{I d(N - N_o)}{dt} = I \frac{d(\Delta N)}{dt} \quad (4)$$

Substitution of equation (4) in equation (3) and rearrangement give

$$I \frac{d(\Delta N)}{dt} = \left[\frac{\partial Q}{\partial (W_f \eta_b)} \right]_o \Delta(W_f \eta_b) + \left(\frac{\partial Q}{\partial N} \right)_o \Delta N \quad (5)$$

Equation (5) represents the dynamic behavior of engine speed in that it gives engine acceleration as a function of fuel flow and speed.

A graphical representation of equation (5) for given altitude and ram conditions is shown in figure 2 (a). The $Q=0$ line is the steady-state operating line. The slope of any constant-speed line of figure 2(a) at $Q=0$ is $\left[\frac{\partial Q}{\partial (W_f \eta_b)} \right]_o$ and the slope of any constant-fuel-flow line of figure 2 (a) at $Q=0$ is $\left(\frac{\partial Q}{\partial N} \right)_o$. Equation (5) can therefore be rewritten as

$$I \frac{d(\Delta N)}{dt} = a_o \Delta(W_f \eta_b) - b_o \Delta N \quad (6)$$

where a_o is the numerical value of the slope of the torque-

fuel-flow curves at constant speed and b_o is the numerical value of the slope of the torque-speed curves at constant fuel flow.

Equation (6) holds only for constant altitude and ram conditions. Because speed, fuel-flow, and torque can all be corrected for altitude, however, equation (6) can be generalized to be true for all altitude conditions. By applying the standard corrections to the speed, fuel-flow, and torque terms in equation (3), this equation can be rewritten in terms of corrected quantities:

$$\begin{aligned} \frac{Q}{\delta_2} = & \left[\frac{\partial \left(\frac{Q}{\delta_2} \right)}{\partial \left(\frac{W_f \eta_b}{\delta_2 \sqrt{\theta_2}} \right)} \right]_o \left[\frac{W_f \eta_b}{\delta_2 \sqrt{\theta_2}} - \left(\frac{W_f \eta_b}{\delta_2 \sqrt{\theta_2}} \right)_o \right] \frac{1}{\delta_2 \sqrt{\theta_2}} + \\ & \left[\frac{\partial \left(\frac{Q}{\delta_2} \right)}{\partial \left(\frac{N}{\sqrt{\theta_2}} \right)} \right]_o (N - N_o) \frac{1}{\sqrt{\theta_2}} \end{aligned} \quad (7)$$

Equation (4) may be written as follows after both sides of the equation are divided by δ_2 and $\sqrt{\theta_2}$ and terms are collected:

$$\frac{Q}{\delta_2} = \frac{I \sqrt{\theta_2}}{\delta_2} \frac{d(\Delta N)}{dt \left(\frac{\sqrt{\theta_2}}{\delta_2} \right)} \quad (8)$$

When equation (8) is substituted in equation (7) the following equation results:

$$\frac{I \sqrt{\theta_2}}{\delta_2} \frac{d(\Delta N)}{dt \left(\frac{\sqrt{\theta_2}}{\delta_2} \right)} = \left[\frac{\partial \left(\frac{Q}{\delta_2} \right)}{\partial \left(\frac{W_f \eta_b}{\delta_2 \sqrt{\theta_2}} \right)} \right]_o \frac{\Delta(W_f \eta_b)}{\delta_2 \sqrt{\theta_2}} + \left[\frac{\partial \left(\frac{Q}{\delta_2} \right)}{\partial \left(\frac{N}{\sqrt{\theta_2}} \right)} \right]_o \frac{\Delta N}{\sqrt{\theta_2}} \quad (9)$$

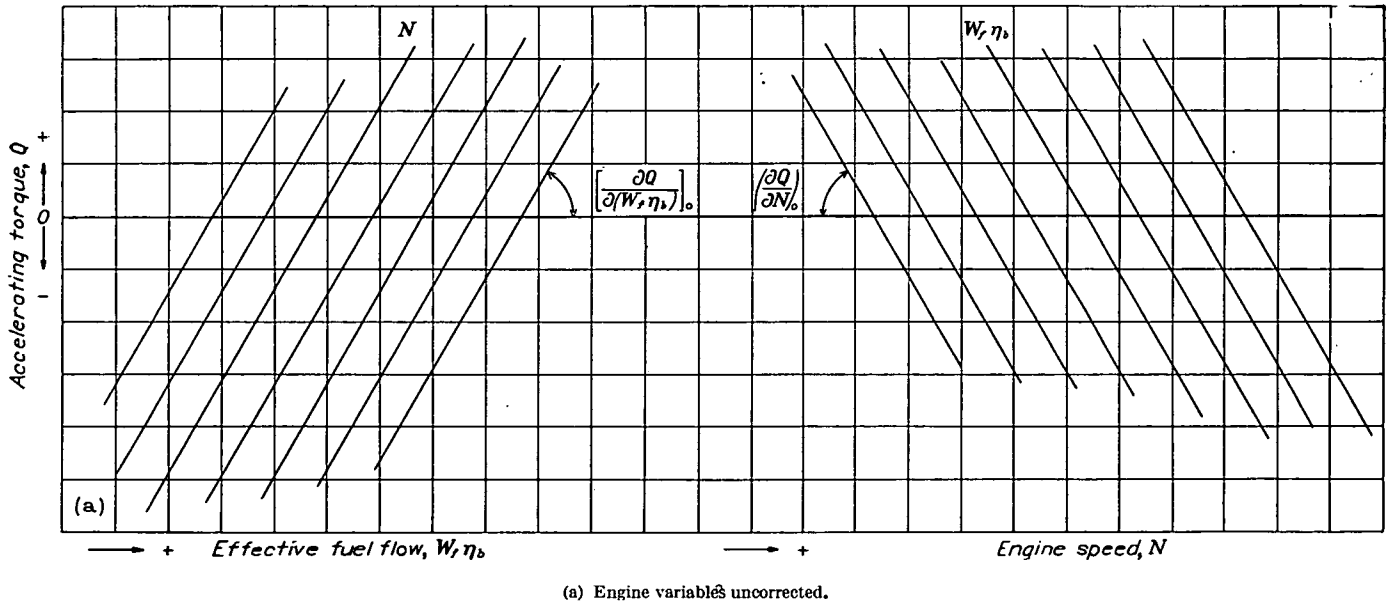
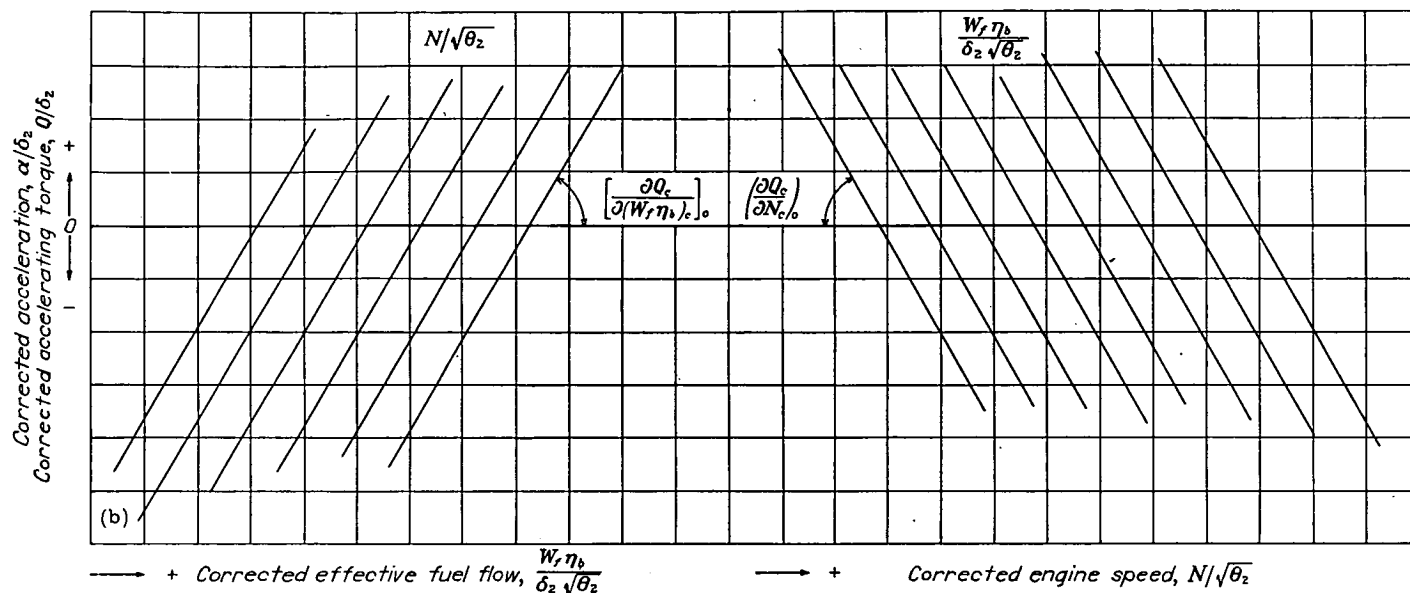


FIGURE 2.—Graphical representation of engine differential equation for condition of constant partials.



(b) Engine variables corrected to compressor-inlet conditions relative to NACA standard atmospheric conditions at sea level.
 FIGURE 2.—Concluded. Graphical representation of engine differential equation for condition of constant partials.

By designating all the corrected quantities with a subscript c and by substituting the corrected curve slopes (fig. 2 (b)) for the partials, equation (9) may be written as

$$\frac{I\sqrt{\theta_2}}{\delta_2} \frac{d(\Delta N_c)}{dt} = a_{o,c} \Delta(W_f \eta_b)_c - b_{o,c} \Delta N_c \quad (10)$$

Equation (10) represents the dynamic behavior of the engine in terms of corrected engine variables at constant ram.

An examination of equation (10) reveals several pertinent facts concerning the behavior of engine acceleration under altitude conditions. First, the coefficients $a_{o,c}$ and $b_{o,c}$ are corrected quantities, which are nearly independent of altitude although they do depend on ram. Second, the burner efficiency affects the dynamic behavior of the engine. Burner efficiency may have a considerable effect on engine acceleration because the efficiency may change substantially under transients in fuel flow. Last, equation (10) indicates that the effective moment of inertia $I\sqrt{\theta_2}/\delta_2$ increases with increasing altitude. Thus the permissible temperature-limited acceleration is reduced with increasing altitude.

Equation (10) may be rearranged into the usual form for a differential equation describing a simple first-order lag system:

$$\frac{I\sqrt{\theta_2}}{\delta_2 b_{o,c}} \frac{d(\Delta N_c)}{dt} + \Delta N_c = \frac{a_{o,c}}{b_{o,c}} \Delta(W_f \eta_b)_c \quad (11)$$

With the equation written in this form, the coefficient of the derivative term is the time constant τ for the system. Equation (11) shows that the effect of increased altitude is to increase the engine time constant above its sea-level value by the factor $\sqrt{\theta_2}/\delta_2$.

EVALUATION OF ENGINE CONSTANTS FROM STEADY-STATE CHARACTERISTICS

In order to evaluate the slopes a_o and b_o , calculations were made for a typical turbojet engine with a centrifugal compressor, a sonic-flow turbine-nozzle diaphragm, and a fixed-

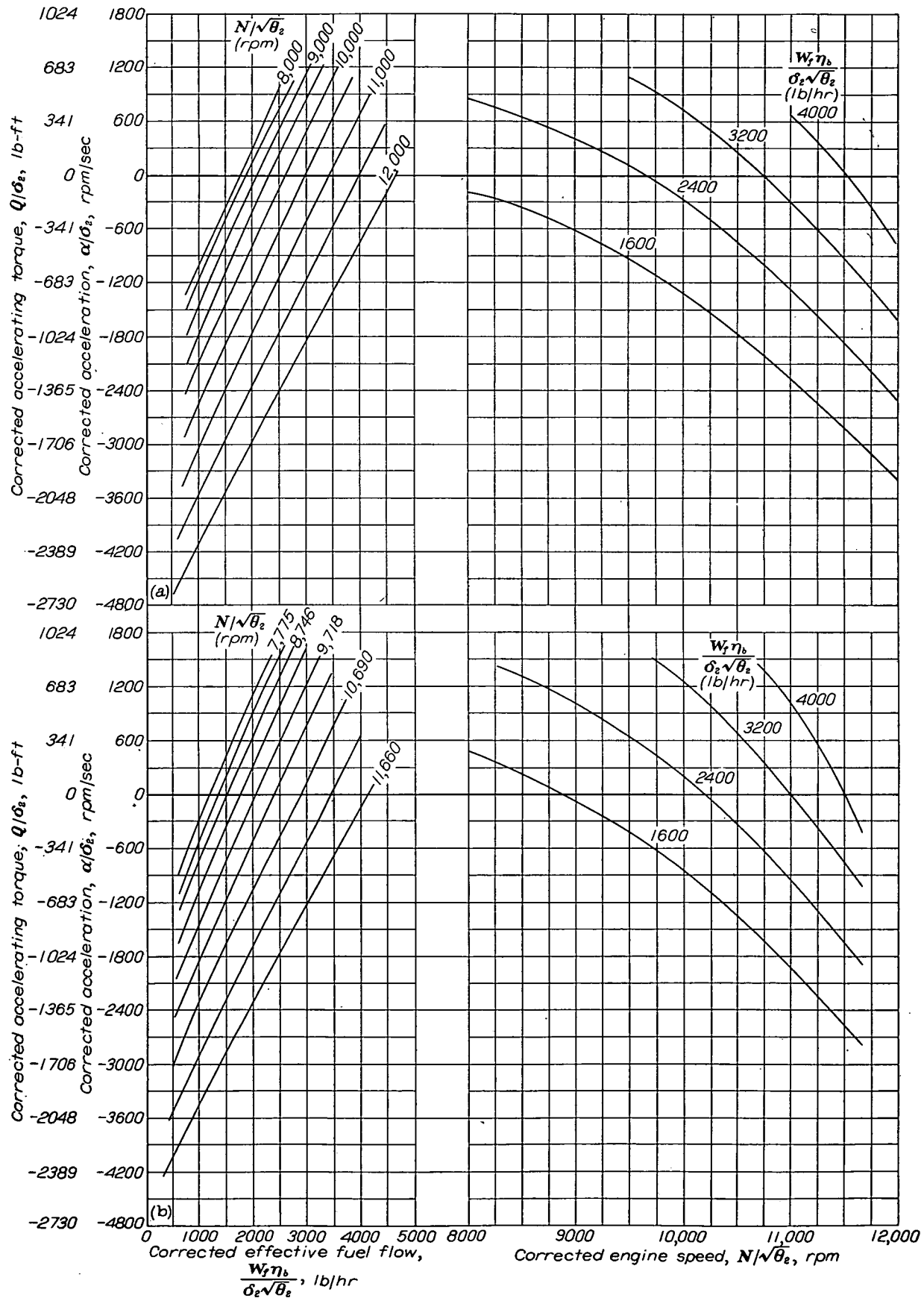
area exhaust nozzle by use of the compressor-performance charts and the thermodynamic relations in the engine. The method of making these calculations is presented in appendix B and is briefly outlined as follows: At constant ambient conditions, the compressor torque and the turbine torque were determined for a number of combinations of the parameters, turbine-inlet temperature and engine speed—parameters that are common to both the compressor and the turbine in a direct-coupled engine. The calculations were made by assuming constant compressor slip factor, burner efficiency, fuel-air ratio, burner pressure ratio, turbine efficiency, nozzle efficiency, and percentage accessory and gear loss. The difference in torque between the compressor and the turbine at any speed was considered an accelerating torque, and the parameter turbine-inlet temperature was expressed in terms of fuel flow. From these data, acceleration could be plotted in terms of engine speed and fuel flow for a set of constant ambient conditions. Because all the engine parameters may be corrected for altitude, the calculations were carried through in terms of corrected quantities.

The results of these calculations for two ram conditions corresponding to 0 and 340 miles per hour at sea level are shown in figure 3. The curves of the left-hand plot are essentially straight lines with variable spacing that cross-plot to the curved lines shown on the right-hand side. The curves of the left-hand side are nearly straight lines because of the constant efficiencies assumed in the calculations. If variable efficiencies are assumed, the shape of the curves will be changed depending on the variation of efficiencies assumed.

For the constant efficiencies assumed, the slope a_o is nearly constant over the power range and for large deviations from the steady-state region. The slope b_o , however, changes substantially over the power range and for large deviations from the steady-state region. This change in b_o indicates that some of the terms dropped from the expansion (equation (1)) may have significant value, especially for large deviations from the steady-state line.

Because the product of the moment of inertia and the reciprocal of b_o is the engine time constant, figure 3 indicates that the engine time constant depends to a considerable

extent on the operating point of the engine, which is roughly defined by the engine speed and altitude. The effect of ram on both a_o and b_o is rather small.



(a) Ram pressure ratio, 1.0.

(b) Ram pressure ratio, 1.2.

FIGURE 3.—Relation among acceleration or accelerating torque, effective fuel flow, and engine speed corrected to compressor-inlet conditions relative to NACA standard atmospheric conditions at sea level as calculated from steady-state component data.

EVALUATION OF ENGINE CONSTANTS FROM TRANSIENT DATA

In order to determine the validity of the preceding analysis, an engine of the type for which the calculations were made was operated under conditions in which the fuel flow was rapidly changed through a series of magnitudes in order to obtain as large a region as possible of nonequilibrium operation. In attempting to impose operation of this nature on the engine, it was necessary to include components in the fuel system by means of which the fuel flow could be rapidly changed from one value to another. The system used consisted of two fuel lines from the pump to the fuel manifold arranged in parallel. The regular engine throttle was installed in one line and another engine throttle in series with a solenoid valve was installed in the other line. With the solenoid valve closed, the regular engine throttle was adjusted to give a selected lower engine speed. With the solenoid valve open, the auxiliary throttle was adjusted to give a selected higher engine speed. The fuel flows for any two steady-state engine speeds could thus be established and nonequilibrium conditions were obtained by opening or closing the solenoid valve.

Fuel flow was measured by means of a thin-plate orifice in the main fuel line with a differential-pressure gage to measure the pressure drop across the orifice. Engine speed was

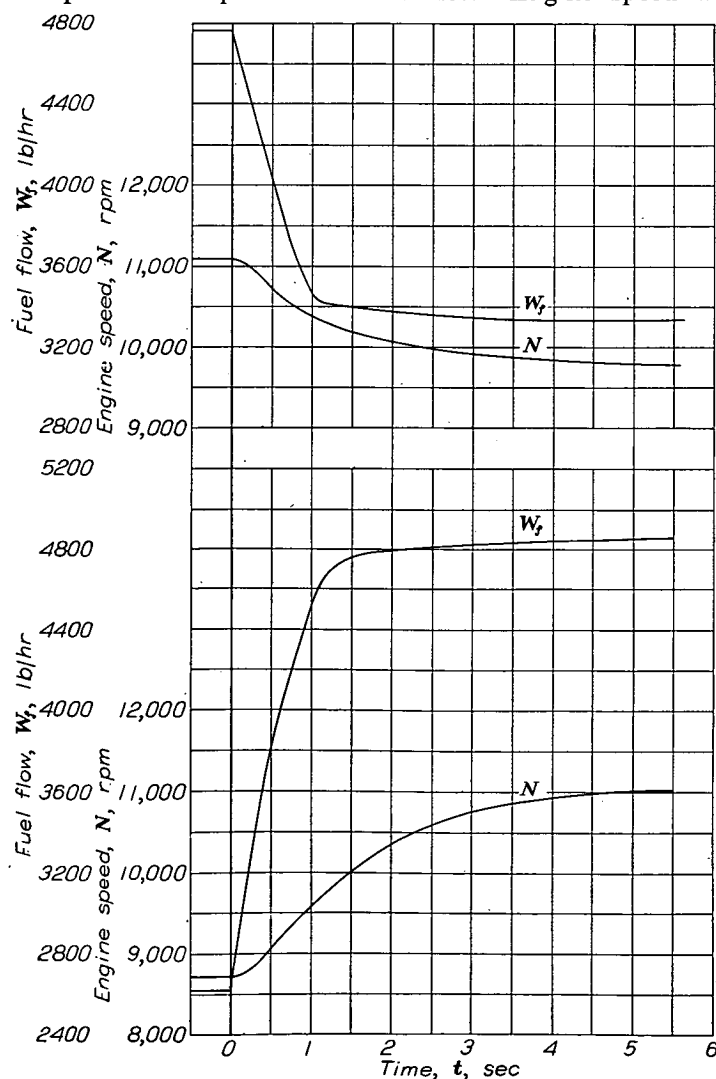


FIGURE 4.—Typical time-domain curves of fuel flow and engine speed for engine transient experiments.

measured by a standard electric tachometer indicating revolutions per minute and time was measured by a clock reading in thousandths of a minute. The data were recorded simultaneously by a motion-picture camera at the rate of 12 frames per second. A preliminary investigation of the response rate of the fuel-flow and speed-measuring instrumentation showed that it was capable of following much more rapid rates of change than those encountered during the engine runs.

Typical examples of the imposed transients of fuel flow for accelerations and decelerations together with the resulting speed-time curves are shown in figure 4. The experimental data are shown in figure 5 in the same form as the calculated data in figure 3 except that, for the experimental data, the actual fuel flow is used instead of the effective fuel flow. The left-hand side of figure 5 was obtained from the data as follows: At a given speed, the acceleration and the fuel flow corresponding to the same instant in time were measured. This procedure was repeated for a number of transient runs resulting from various magnitudes of fuel-flow change. The right half of figure 5 was obtained by cross-plotting the faired curves of the left-hand side. The grouping of the data points from several runs indicates that a unique relation exists among torque, fuel flow, and speed, regardless of the manner in which the point is reached.

Near the steady-state region, the experimental data follow the results of the analysis and the calculations in that the slope $a_{o,c}$ is virtually constant over the power range and the slope $b_{o,c}$ varies substantially over the power range as indicated in figure 5. For accelerations in excess of about ± 5 percent of rated speed per second, the engine begins to exhibit large deviations from the calculated curves.

EFFECT OF ENGINE CONSTANTS ON CONTROLLED ENGINE

In matching controls to engines, the damping ratio is one of the criteria used to evaluate the degree of matching. Both engine coefficients and control coefficients appear in this damping term. For example, if it is assumed that a control operating on speed error to vary fuel flow according to the following equation

$$W_f = \int \beta(N_s - N) dt + \epsilon(N_s - N) + \sigma \frac{d(N_s - N)}{dt}$$

is applied to the engine described by equation (11), the combined equation for the system becomes

$$\left(\frac{I\theta_2}{a_{o,c}\eta_b} + \sigma\sqrt{\theta_2} \right) N_c'' + \left(\frac{b_{o,c}\delta_2\sqrt{\theta_2}}{a_{o,c}\eta_b} + \epsilon\sqrt{\theta_2} \right) N_c' + \beta\sqrt{\theta_2}N_c = \beta N_s + \epsilon N_s' + \sigma N_s''$$

where the primes signify derivatives with respect to time. The damping ratio for this expression is

$$\frac{\frac{b_{o,c}\delta_2}{a_{o,c}\eta_b} + \epsilon}{2\sqrt{\beta \left(\frac{I\sqrt{\theta_2}}{a_{o,c}\eta_b} + \sigma \right)}} \quad (12)$$

Thus if it is desired to maintain a constant value of the damping term, such as critical damping, it is evident from an examination of equation (12) that if an engine coefficient varies, some control coefficient must be varied in such a

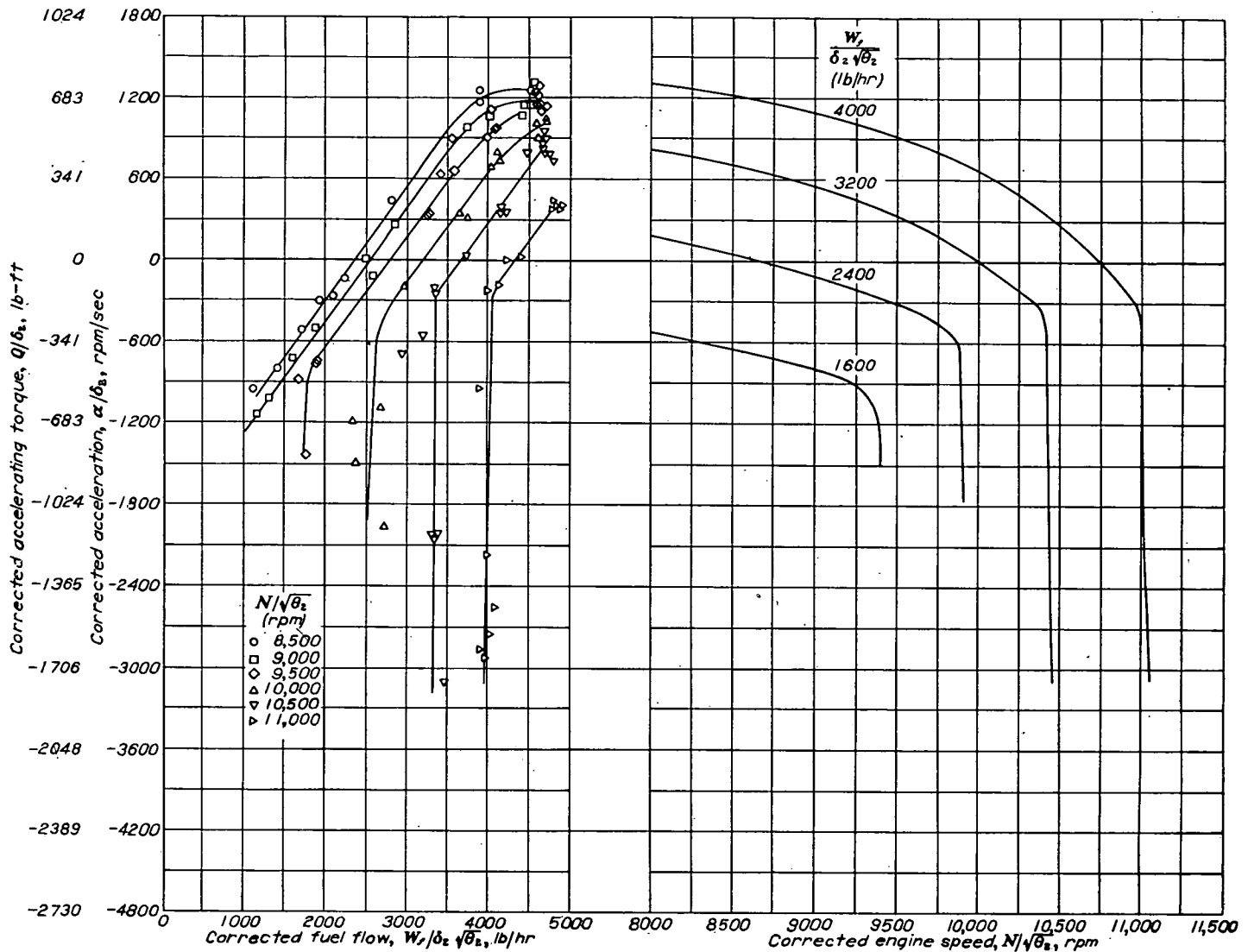


FIGURE 5.—Relation among acceleration or accelerating torque, fuel flow, and engine speed corrected to compressor-inlet conditions relative to NACA standard atmospheric conditions at sea level as obtained from experimental data for a ram pressure ratio of 1.0

manner that the damping ratio remains substantially constant. Figures 6 and 7 are plots of $a_{o,c}$ and $b_{o,c}$, respectively, as functions of engine speed for the calculated data and the experimental data. The slope $a_{o,c}$ is virtually constant and will have little effect on the damping ratio but $b_{o,c}$ may change the damping ratio considerably for various operating points. If the variation is considered too great, it will be necessary to vary a control coefficient to compensate for the variation in $b_{o,c}$. Figure 7 indicates that $b_{o,c}$ varies almost directly with engine speed, which would

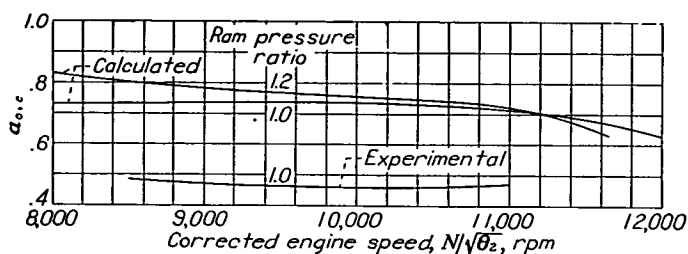


FIGURE 6.—Values of partial of torque with respect to fuel flow or effective fuel flow at steady-state point plotted against engine speed corrected to compressor-inlet conditions relative to NACA standard atmospheric conditions at sea level for calculated and experimental data.

indicate that one method of compensating for the variation of $b_{o,c}$ is to vary a control coefficient as a function of engine speed.

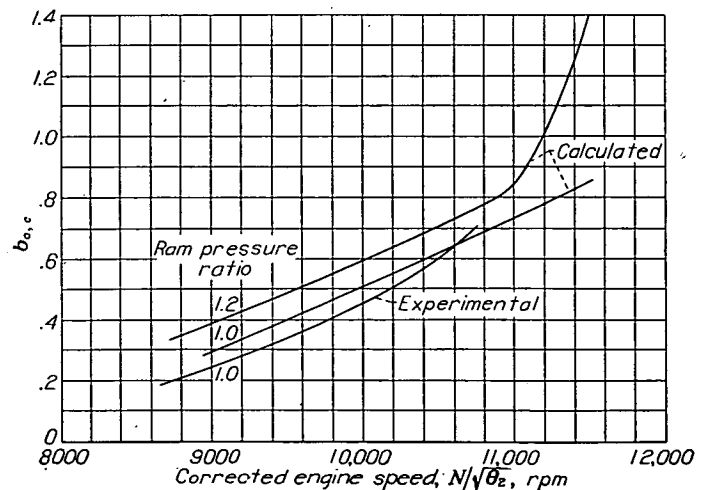


FIGURE 7.—Values of partial of torque with respect to engine speed at steady-state point plotted against engine speed corrected to compressor-inlet conditions relative to NACA standard atmospheric conditions at sea level for calculated and experimental data.

A plot of the corrected engine time constant as a function of engine speed is shown in figure 8. The time constant for this engine varies through a 4 to 1 range over the power range of the engine.

DISCUSSION

In general, the correlation between the calculated results and the experimental results is considered good and would be better if an approximation of burner efficiency were included in the calculations. This correlation indicates that for an engine of the type investigated, operating near the steady-state region, a good approximation of the dynamic characteristics can be made as soon as the component characteristics are available. Thus an engine may be

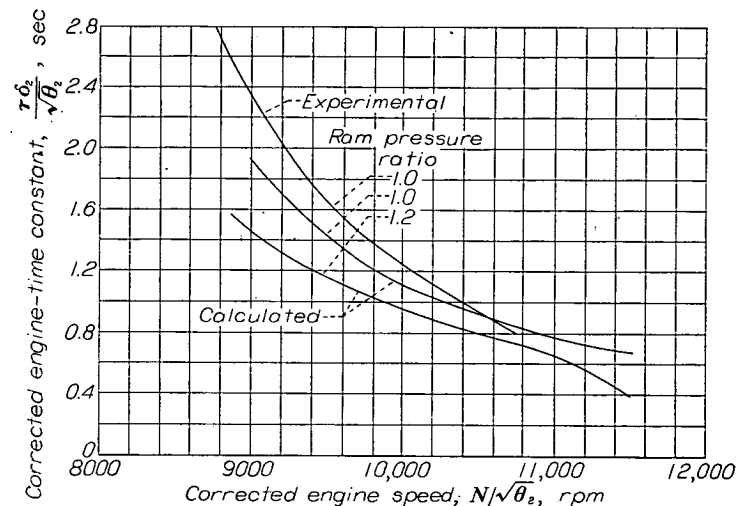


FIGURE 8.—Values of corrected engine time constant at steady-state point plotted against engine speed corrected to compressor-inlet conditions relative to NACA standard atmospheric conditions at sea level for calculated and experimental data.

designed, at least to a certain extent, to have good transient characteristics as well as good steady-state performance characteristics or at least a compromise may be reached between transient and steady-state performance if the two conflict in certain configurations.

Although the correlation in the steady-state regions is good, the correlation in the region of large accelerations can be regarded as only fair and in the region of large decelerations as poor. A possible explanation for the poorer correlation in the accelerating region is that the engine is approaching the acceleration blow-out region in the burners, which is a region of apparently low burner efficiencies. Temperature data taken during the experiments indicated that when excessive amounts of fuel were suddenly added combustion occurred through the turbine and the tail cone, which would result in low burner efficiencies during these periods.

The large deviations between calculated and observed values of deceleration seem to have no obvious explanation. The very large decelerations observed would indicate that very large reductions in torque output of the turbine occurred. These reductions can be only partly accounted for by the normal steady-state effect of a reduction in temperature on turbine torque. The reduction in temperature that did occur may possibly have caused a rotation of the

turbine-entrance-velocity vector relative to the blade to a condition of negative angle of attack. Such a condition corresponds to a high value of turbine-velocity ratio for which the turbine efficiency is rapidly falling. The existence of a negative angle of attack together with the increased air flow and consequently increased compressor torque that results from a reduction in temperature at constant speed could conceivably cause large decelerating torques. The existence of a negative angle of attack would also explain the position of the sharp breaks in the curves of figure 5, because at low speeds the steady-state angle of attack is greater than at high speeds, which are near the design condition; therefore a greater reduction in temperature at constant speed is necessary to cause negative angles of attack and a consequent sharp change in curve slope. The preceding discussion would seem to indicate a possible explanation for the sharp change of slope for deceleration points but does not explain the vertical parts of the curves, which indicate that various decelerations occur with the same turbine-inlet temperature. A check of the data, however, indicated that along the vertical part of a constant-speed line of figure 5 the deceleration value was consistent with the rate of change of fuel flow; that is, the deceleration value increased with increasing rates of change of fuel flow. Thus, if the burner efficiency reduced as a function of the rate of change of fuel flow so that the temperature was reduced, the vertical lines of figure 5 seem logical.

CONCLUSIONS

The following conclusions are drawn from an analysis of the dynamic behavior of, and from transient data obtained from, a typical turbojet engine with a centrifugal compressor, a sonic-flow turbine-nozzle diaphragm, and a fixed-area exhaust nozzle:

1. Accelerating torque is a function of fuel flow and engine speed. A linear differential equation approximately represents the transient behavior of the engine speed for accelerations less than approximately 5 percent of rated speed per second.

2. The transient equation of the engine may be obtained from the steady-state performance of the engine components to a degree of accuracy that will permit the prediction of the engine transient characteristics as soon as the component characteristics are available.

3. A transient equation for engine speed may be derived including the effects of altitude by the use of corrected engine variables in the derivation.

4. The manner in which the control constants should be varied to compensate for the changes in engine characteristics with altitude and with engine speed may be predicted from an analysis of the combined engine and control equation.

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, July 27, 1949.

APPENDIX A

SYMBOLS

The following symbols have been used throughout this report:

A	exhaust-nozzle area, sq ft
a	partial of torque with respect to either effective or actual fuel flow at constant engine speed $\left(\frac{\partial Q}{\partial(W_f \eta_b)} \text{ or } \frac{\partial Q}{\partial \dot{W}_f}\right)$
b	partial of torque with respect to engine speed at constant values of either actual or effective fuel flow $\left(\frac{\partial Q}{\partial N}\right)$
$c_{p,c}$	average specific heat for gas passing through compressor (assumed at 0.243 Btu/(lb) (°F))
$c_{p,\tau}$	average specific heat for gas passing through turbine (assumed at 0.276 Btu/(lb) (°F))
H_f	lower heating value of fuel, 18,400 Btu/lb
h	enthalpy, Btu/lb
I	polar moment of inertia of rotating parts, $5.427 \left(\frac{2\pi}{60}\right)$ (lb-ft) (sec) (min) (rad)/revolution
N	engine speed, rpm
N_s	set value of engine speed, rpm
P	total pressure, lb/sq ft
p	static pressure, lb/sq ft
Q	accelerating torque, lb-ft
Q_c	torque required by compressor, lb-ft

Q_T	torque output of turbine, lb-ft
T	total temperature, °R
t	time, sec
W_a	air flow, lb/sec
W_f	fuel flow, lb/hr
W_g	total gas flow, lb/sec
α	acceleration of engine speed, rpm/sec
β	coefficient of integral control component, $\frac{\text{lb/hr}}{(\text{rpm}) (\text{sec})}$
γ	ratio of specific heats
δ_2	altitude pressure ratio, $P_2/14.7$
ϵ	coefficient of proportional control component, $\frac{\text{lb/hr}}{\text{rpm}}$
η_b	burner efficiency, percent
η_T	turbine efficiency (assumed at 83 percent), percent
θ_2	altitude temperature ratio, $T_2/518.6$
σ	coefficient of derivative control component, $\frac{(\text{lb/hr}) \text{ sec}}{\text{rpm}}$
τ	engine time constant, sec
Subscripts:	
c	corrected
i	any engine operating point
o	any steady-state operating point
0	ambient
2	compressor inlet
3	compressor outlet
4	turbine inlet
5	turbine outlet

APPENDIX B

ENGINE-TRANSIENT-CHARACTERISTIC CALCULATIONS

In general, the method used to calculate the characteristics of an engine operating under transient conditions involves determining the difference between compressor torque and turbine torque for assumed values of engine speed, turbine-inlet temperature, ambient conditions, and ram pressure ratio. Figure 9 is a chart of typical compressor characteristics obtained from unpublished data. For assumed values of ram pressure ratio, ambient conditions, engine speed, and turbine-inlet temperature, corresponding values of compressor pressure ratio and corrected air flow may be determined. The torque required by the compressor for the assumed conditions may then be calculated from the following equations:

$$\frac{T_3}{T_2} = 1 + 0.5079 \times 10^{-8} \left(\frac{N}{\sqrt{\theta_2}} \right)^2 \quad (\text{B1})$$

$$Q_c = c_{p,c} (T_3 - T_2) \frac{W_a (778) (60)}{2\pi N} \quad (\text{B2})$$

Equation (B1) is developed for an assumed slip factor of

0.925 and a compressor rotor diameter of 2.5 feet. The definitions of the symbols used are given in appendix A.

The equation for the torque output of the turbine is similar to equation (B2) in that it involves the temperature drop through the turbine, the gas flow, and the speed. The speed and the gas flow are known from the assumed conditions and the resulting compressor point. The temperature drop through the turbine, however, must be determined for the operating point. In the absence of a complete turbine-performance map, general thermodynamic relations defining typical turbine operation must be utilized. A relation between turbine pressure ratio P_4/P_5 and turbine temperature ratio $\sqrt{T_4/T_5}$ is determined by assuming a constant turbine efficiency as expressed by the following equation:

$$\eta_T = \frac{1 - \frac{T_5}{T_4}}{1 - \left(\frac{P_5}{P_4} \right)^{\frac{\gamma-1}{\gamma}}} \quad (\text{B3})$$

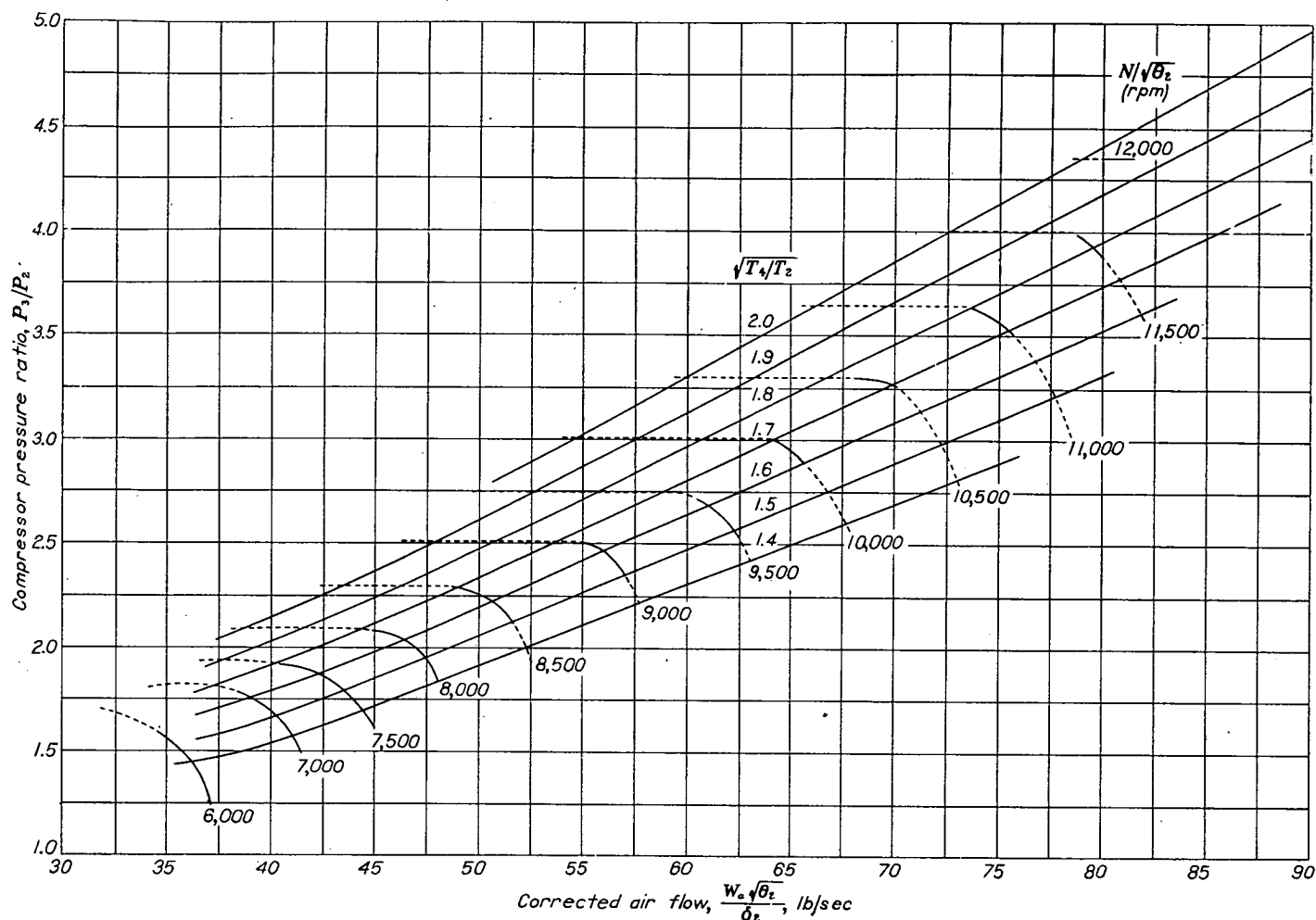


FIGURE 9.—Compressor characteristics corrected to compressor-inlet temperature and pressure relative to NACA standard atmospheric conditions at sea level. Curves of constant ratio of turbine-inlet temperature to compressor-inlet temperature for engine are superimposed on compressor-characteristic curves. (Obtained from unpublished data.)

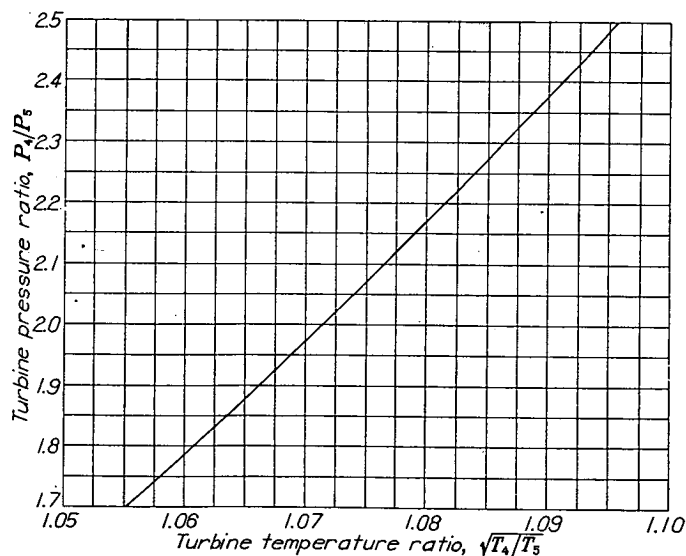


FIGURE 10.—Relation between turbine total-pressure ratio and total-temperature ratio at constant turbine efficiency of 0.83.

For this analysis η_T was assumed at 83 percent, and γ_4 was assumed at 1.33. The resulting relation between $\sqrt{T_4/T_5}$ and P_4/P_5 is plotted in figure 10.

Because critical flow exists in the turbine nozzles over the

power range, the corrected gas flow through the turbine nozzles will have a constant value. For this engine, unpublished data have indicated that the corrected gas flow will have a value expressed by the following equation:

$$\frac{W_g \sqrt{\left(\frac{T_4}{519}\right) \left(\frac{\gamma_4}{1.4}\right)}}{\left(\frac{P_4}{14.7}\right) \left(\frac{\gamma_4}{1.4}\right)} = 41.6 \text{ lb/sec} \quad (\text{B4})$$

This equation may be expanded as follows:

$$\frac{W_g \sqrt{\frac{T_5}{519}}}{A \frac{P_5}{14.7}} = \frac{41.6 \sqrt{\frac{\gamma_4}{1.4}} \left(\frac{P_4}{P_5}\right)}{A \sqrt{\frac{T_4}{T_5}}}$$

For an effective exhaust-nozzle area of 1.87 square feet and with γ_4 assumed constant at 1.33, the preceding equation becomes

$$\frac{W_g \sqrt{\theta_5}}{A \delta_5} = \frac{21.75 \frac{P_4}{P_5}}{\sqrt{\frac{T_4}{T_5}}} \quad (\text{B5})$$

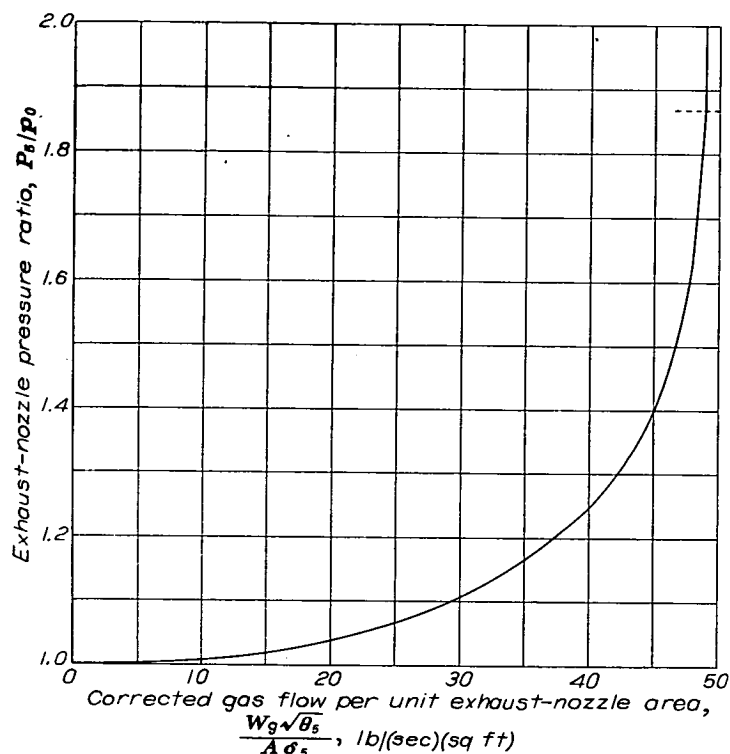


FIGURE 11.—Variation of exhaust-nozzle pressure ratio with gas flow per unit exhaust-nozzle area corrected to total pressure and temperature in the exhaust nozzle relative to NACA standard atmospheric conditions at sea level. (Obtained from unpublished data.)

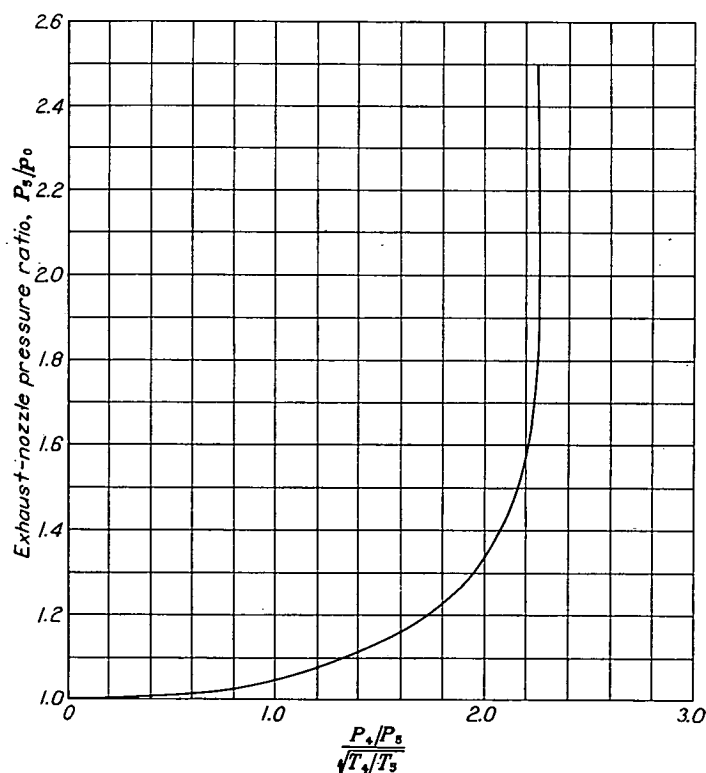


FIGURE 12.—Relation of exhaust-nozzle pressure ratio and turbine total-pressure- and temperature-ratio parameters for exhaust-nozzle area of 1.87 square feet.

From equation (B5) and figure 11, which is plotted from unpublished data, values can be obtained for a plot of P_5/p_0 as a function of $(P_4/P_5)/\sqrt{T_4/T_5}$ for an exhaust-nozzle area of 1.87 square feet. This relation is shown in figure 12.

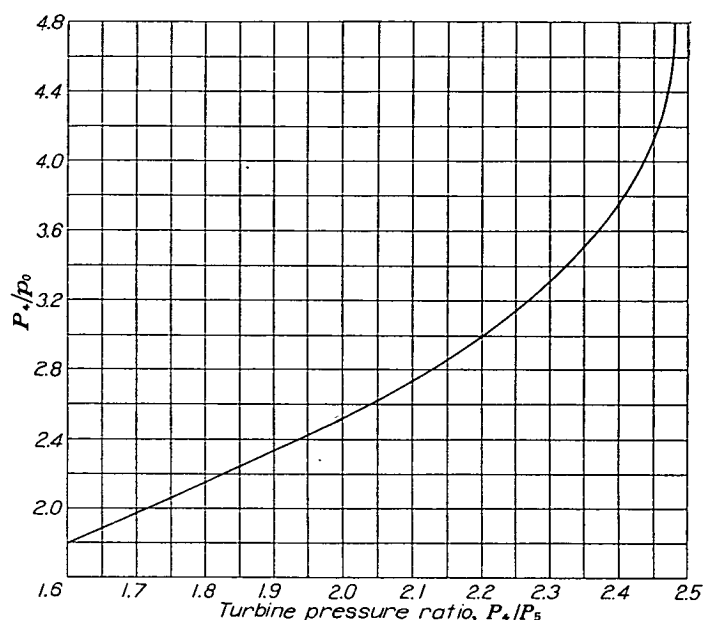


FIGURE 13.—Relation between engine-pressure ratio and turbine-pressure ratio for exhaust-nozzle area of 1.87 square feet.

Figures 10 and 12 can then be used to obtain P_4/p_0 as a function of P_4/P_5 . This relation is shown in figure 13.

The turbine torque for each condition for which compressor torque was computed can now be found. In this analysis, it was assumed that P_4/P_5 was constant at 0.95, which determines P_4 for each compressor point, and that W_g/W_a was constant at 1.015, which determines W_g for each compressor point. With the use of the value of P_4 obtained by applying one of these assumed relations to the value of P_3 obtained from the compressor-torque calculations, $\sqrt{T_4/T_5}$ can now be obtained through the use of figures 13 and 10, successively. The turbine torque corresponding to the assumed conditions of ram pressure ratio, ambient conditions, engine speed, and turbine-inlet temperature can then be computed from the following equation:

$$Q_T = c_{p,T}(T_4 - T_5) \frac{W_o(778)(60)}{2\pi N} \quad (B6)$$

The accelerating torque can then be calculated from the following equation in which 1 percent of the turbine torque or power was assumed lost to the accessories and to gear and bearing friction:

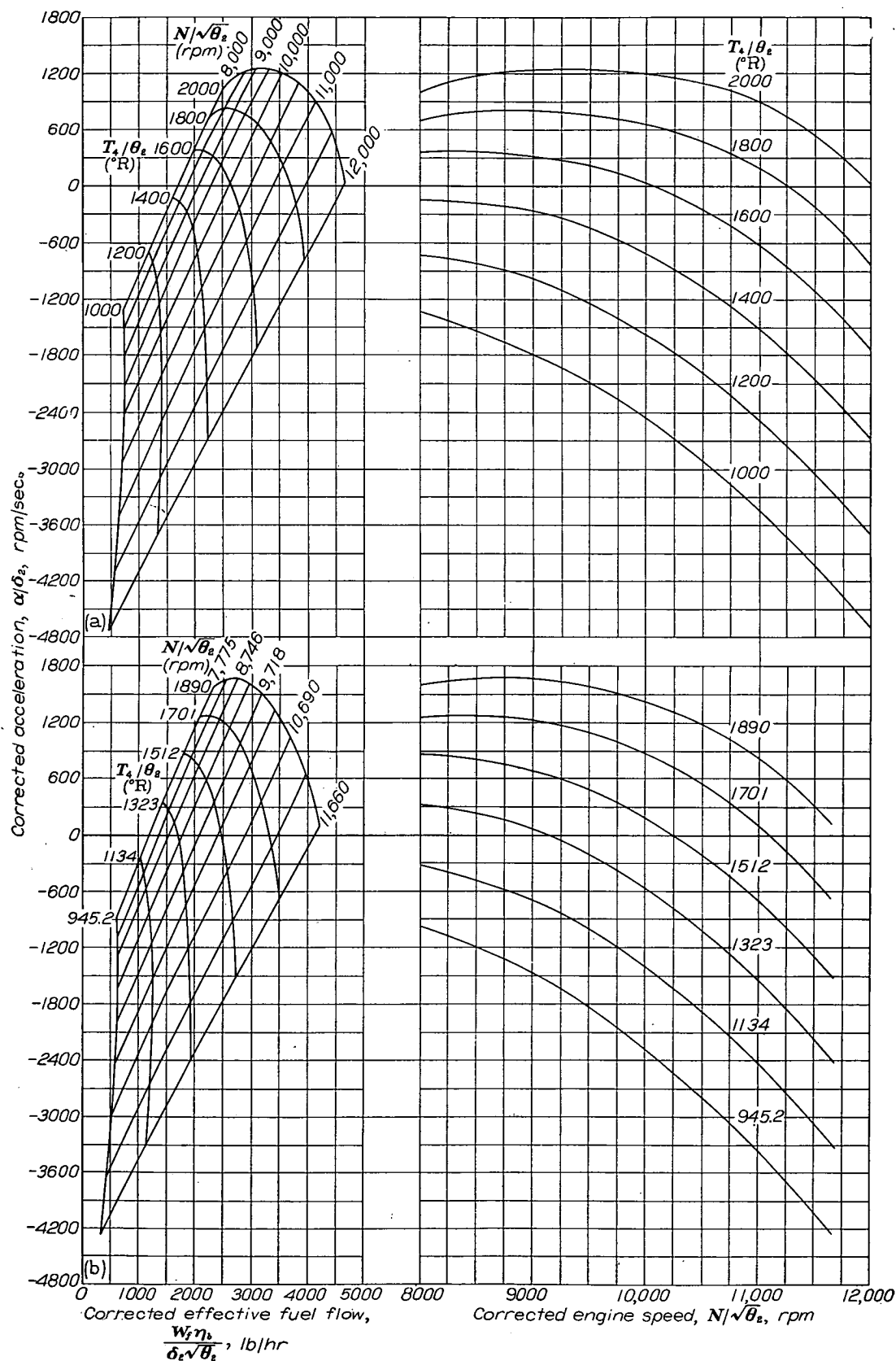
$$Q = 0.99 Q_T - Q_C = I\alpha \quad (B7)$$

A plot of angular acceleration as a function of engine speed, turbine-inlet temperature, and effective fuel flow $W_f\eta_b$ for ram conditions corresponding to an airplane velocity of 0 and 340 miles per hour at sea level, corrected to NACA standard sea-level conditions is shown in figure 14. The effective fuel flows were obtained by use of the following equation, which assumes no time lag between temperature and fuel flow:

$$W_f\eta_b = \frac{W_a(h_4 - h_3)3600}{H_f} \quad (B8)$$

The values of h_4 and h_3 corresponding to the assumed temperature T_4 and the temperature T_3 calculated from equation (B1) can be found from a table of enthalpies. Figure 3

was obtained by plotting the constant-speed lines of figure 14 and then cross-plotting to obtain the constant-fuel-flow curves.



(a) Ram pressure ratio, 1.0.

(b) Ram pressure ratio, 1.2.

FIGURE 14.—Relation among acceleration, effective fuel flow, turbine-inlet temperature, and engine speed corrected to compressor-inlet conditions relative to NACA standard atmospheric conditions at sea level as calculated from steady-state data.

REPORT 1012

INVESTIGATION OF THE NACA 4-(5)(08)-03 AND NACA 4-(10)(08)-03 TWO-BLADE PROPELLERS AT FORWARD MACH NUMBERS TO 0.725 TO DETERMINE THE EFFECTS OF CAMBER AND COMPRESSIBILITY ON PERFORMANCE¹

By JAMES B. DELANO

SUMMARY

As part of a general investigation of propellers at high forward speeds, tests of two-blade propellers having the NACA 4-(5)(08)-03 and NACA 4-(10)(08)-03 blade designs were made in the Langley 8-foot high-speed tunnel through a range of blade angle from 20° to 60° for forward Mach numbers from 0.165 to 0.70 to determine the effect of camber and compressibility on propeller characteristics. Results previously reported for similar tests of a two-blade propeller having the NACA 4-(3)(08)-03 blade design are included for comparison.

Blades of high design camber were more efficient than blades of low design camber for operation at high power loadings. The blade of highest camber gave efficiencies 15 to 25 percent higher than the efficiencies of the low-camber and medium-camber blades for high power loadings at advance ratios corresponding to take-off and climb at low Mach numbers. The NACA 4-(5)(08)-03 propeller generally gave peak efficiencies 2 to 5 percent higher than those for the NACA 4-(3)(08)-03 propeller and 3 to approximately 12 percent higher than those for the NACA 4-(10)(08)-03 propeller. These higher efficiencies were due mainly to reduced compressibility losses. At the design blade angle of 45°, the critical tip Mach number for maximum efficiency was 0.01 higher for the NACA 4-(5)(08)-03 propeller than for the NACA 4-(3)(08)-03 propeller, which began to show compressibility losses at a tip Mach number of approximately 0.90. The NACA 4-(10)(08)-03 propeller began to show compressibility losses at a tip Mach number as low as 0.70 but, because of the large power-absorbing capacity of this propeller, produced about 46 percent more thrust than the NACA 4-(5)(08)-03 propeller for a high-speed operating condition corresponding to a tip Mach number of 0.86, a forward Mach number of 0.53, and an advance ratio of 2.48.

INTRODUCTION

Many airplanes now take off and climb with propellers at least partly stalled, and the tendency to use increasing powers may aggravate a condition that is already serious. Flight at high altitudes also may necessitate propeller operation at high lift coefficients, which would increase possible stall and compressibility effects and result in a reduction of propeller efficiency.

The National Advisory Committee for Aeronautics has attempted to improve propeller performance by conducting a general investigation of propellers at high forward speeds.

This investigation included the effects on propeller characteristics of compressibility, blade solidity, and blade-section camber. The research program included tests of propellers of a sufficient range of blade forms to make possible the study of changes in blade shapes that might be required as a consequence of compressibility effects.

The effects of compressibility and solidity on performance as determined from tests of the NACA 4-(3)(08)-03 and NACA 4-(3)(08)-045 two-blade propellers (reference 1) constituted the initial phase of a general investigation of propellers at high forward speeds. The effects of camber and compressibility on performance as determined from tests of the NACA 4-(5)(08)-03 and NACA 4-(10)(08)-03 two-blade propellers constituted the second phase of the investigation and are presented herein. These results are compared with results of reference 1 for the NACA 4-(3)(08)-03 propeller, in order to indicate the effects of section design camber for propellers operating over a wide range of forward Mach number. These three blade designs cover the practical range of blade section camber.

SYMBOLS

B	number of blades
b	blade width, feet
b/D	blade width ratio
c_d	blade-section profile-drag coefficient
c_l	blade-section lift coefficient
c_{l_d}	blade-section design lift coefficient
C_P	power coefficient $\left(\frac{P}{\rho n^3 D^5}\right)$
C_T	thrust coefficient $\left(\frac{T}{\rho n^2 D^4}\right)$
$C_{T_{max}}$	maximum thrust coefficient
D	propeller diameter, feet
G	Goldstein tip-correction factor
h	maximum thickness of blade section, feet
h/b	blade thickness ratio
J	advance ratio (V_0/nD)
M	tunnel-datum (forward) Mach number (tunnel-empty Mach number uncorrected for tunnel-wall constraint)
M_t	helical tip Mach number $\left(M\sqrt{1+\left(\frac{\pi}{J}\right)^2}\right)$
n	propeller rotational speed, revolutions per second
P	power absorbed by the propeller, foot-pounds per second

¹ Supersedes NACA ACR L5F15, "Investigation of Two-Blade Propellers at High Forward Speeds in the NACA 8-Foot High-Speed Tunnel. III—Effects of Camber and Compressibility NACA 4-(5)(08)-03 and NACA 4-(10)(08)-03 Blades" by James B. Delano, 1945.

P_c	power disk-loading coefficient $\left(\frac{P}{\frac{1}{2}\rho V_0^3 S}\right)$
R	propeller tip radius, feet
r	blade-section radius, feet
S	propeller disk area, square feet $\left(\frac{\pi D^2}{4}\right)$
T	propulsive thrust of propeller, pounds
T_c	thrust disk-loading coefficient $\left(\frac{T}{\rho V^2 D^2}\right)$
V	tunnel-datum velocity (tunnel-empty velocity uncorrected for tunnel-wall constraint), feet per second
V_0	equivalent free-air velocity (tunnel-datum velocity corrected for tunnel-wall constraint), feet per second
r/R	blade-section station
α_i	induced angle of attack, degrees $\left(\tan^{-1} \frac{\sigma c_l}{4G \sin \phi}\right)$
β	section blade angle, degrees
$\beta_{0.75R}$	section blade angle at 0.75 tip radius, degrees
$\gamma = \tan^{-1} \frac{c_d}{c_l}$	degrees
η	propulsive efficiency $\left(\frac{C_T}{C_P} J\right)$
η_{max}	maximum propulsive efficiency
ρ	air density, slugs per cubic foot
σ	total blade-section solidity $(Bb/2\pi r)$
ϕ	aerodynamic helix angle, degrees $(\phi_0 + \alpha_i)$
ϕ_0	geometric helix angle, degrees $\left(\tan^{-1} \frac{JR}{\pi r}\right)$

APPARATUS, METHODS, AND TESTS

The apparatus and methods described in reference 1 were used in the present investigation. The investigation was conducted in the Langley 8-foot high-speed tunnel. A photograph showing the model setup is given as figure 1.

The blades of the propellers investigated were designed for three-blade propellers to produce minimum induced energy losses (profile drag assumed equal to zero) at a blade angle of approximately 45° at the 0.7-radius station. The blade sections are late-critical-speed sections of the NACA 16 series (reference 2); methods and principles employed in the design of the blades are discussed in reference 3. The blades differ only in design camber and are designated as NACA 4-(3)(08)-03 (low camber, reference 1), NACA 4-(5)(08)-03

(medium camber), and NACA 4-(10)(08)-03 (high camber).

The designation numbers describe the propellers. The number (or numbers) of the first group is the diameter in feet; the number (or numbers) of the second group (enclosed within the first set of parentheses) is the design lift coefficient (in tenths) of the blade section at the 0.7-radius station; the numbers of the third group (enclosed within the second set of parentheses) are the thickness ratio of the blade section at the 0.7-radius station; and the numbers of the fourth group are the blade solidity expressed as the ratio of the blade chord at the 0.7-radius station to the circumference of the circle having a radius 0.7 of the propeller tip radius. The NACA 4-(10)(08)-03 propeller thus has a diameter of 4 feet and the blade section at the 0.7-radius station has a design lift coefficient of 1.0, a thickness ratio of 0.08, and a blade solidity of 0.03.

Blade-form curves for the propellers are presented in figure 2. A photograph of one of these blades and a comparison of the sections at the 0.7-radius station are given as figure 3.

The range of this investigation was the same, within power limitations, as that of reference 1. The range of blade angle and tunnel-datum Mach number is given in table I.

REDUCTION OF DATA

The data have been reduced to the usual thrust and power coefficients and efficiency and have been corrected for the propulsive effects of the cowling and spinner and for tunnel-wall constraint. The tunnel-wall constraint necessitated a velocity correction to free-air conditions and a model-drag correction because of the buoyancy effect. The methods involved in making these corrections are discussed in reference 1.

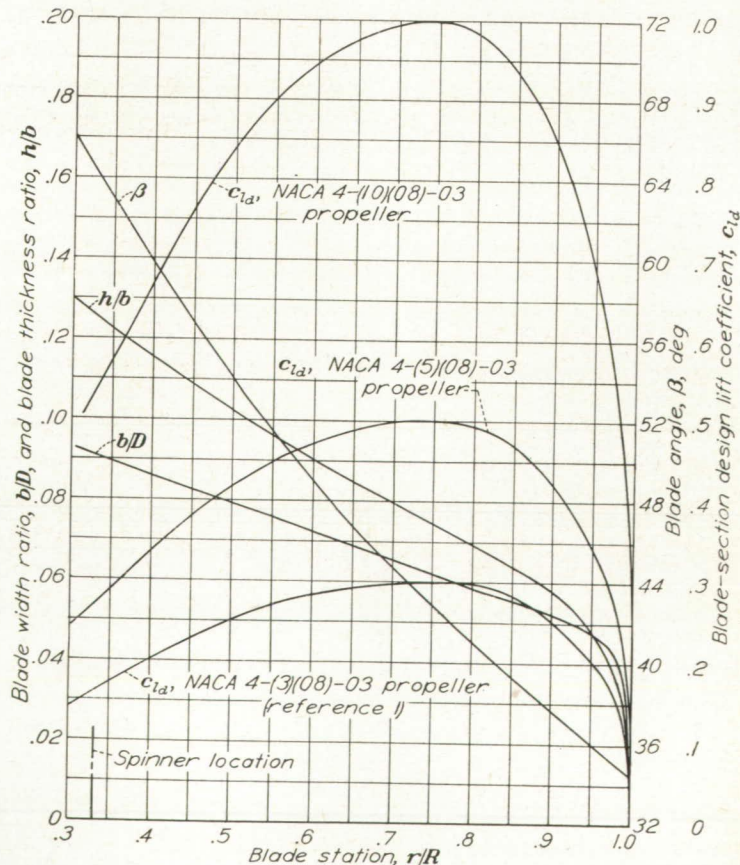


FIGURE 2.—Blade-form curves.

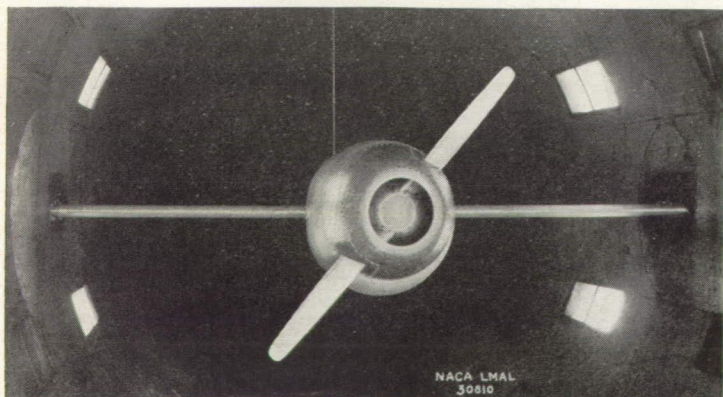


FIGURE 1.—Setup for testing propellers in the Langley 8-foot high-speed tunnel.

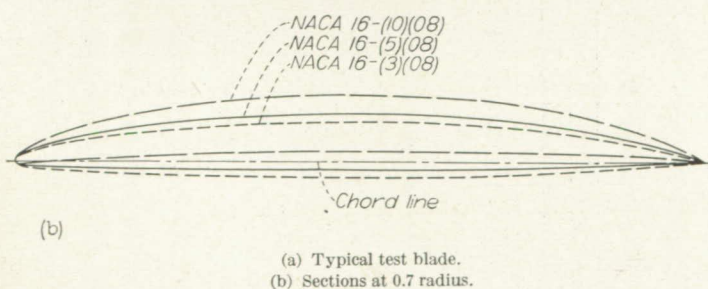
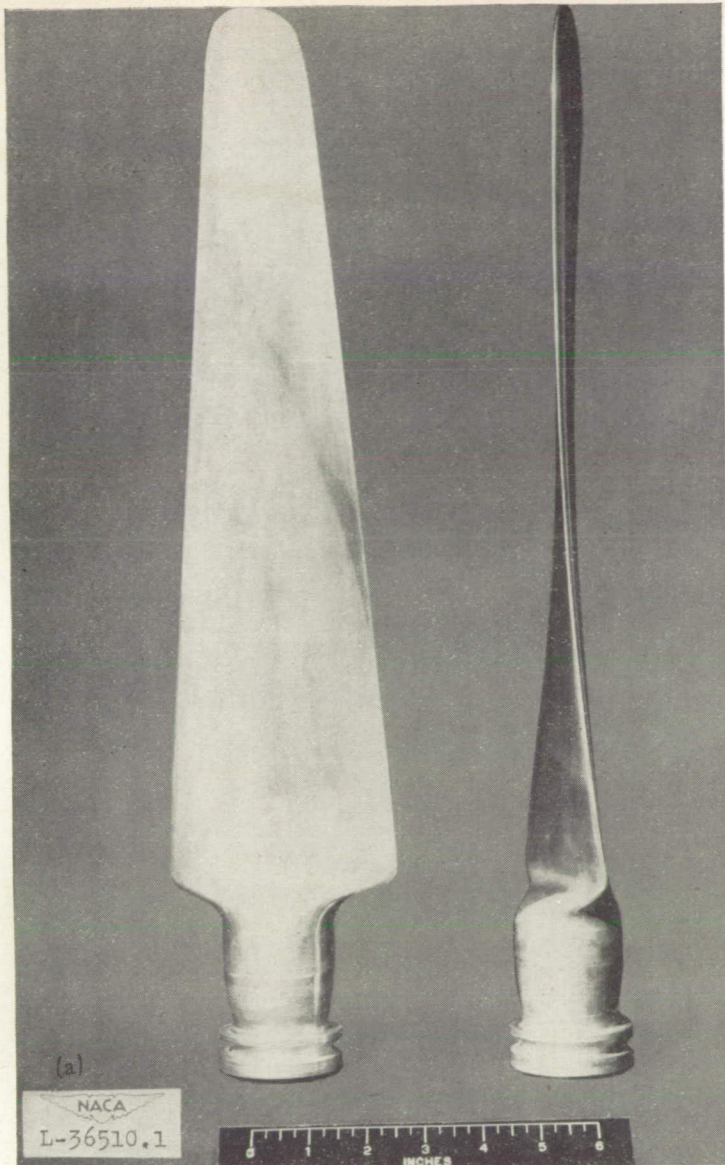


FIGURE 3.—Typical test blade and comparison of sections at 0.7 radius.

Thrust.—The thrust coefficient was determined from the propulsive thrust, that is, the net measured force minus the drag of the model without the propeller and minus the thrust due to the buoyancy effect (see reference 1).

Velocity correction due to tunnel-wall constraint.—The equivalent free-stream velocity corresponding to the thrust and torque of the propeller measured at each rotational speed differs from the tunnel-datum velocity (tunnel empty) because of the flow constraint produced by the tunnel walls.

TABLE I.—TEST RANGE OF BLADE ANGLE AND MACH NUMBER

Tunnel-datum (forward) Mach number, M	Blade angle at 0.75 radius, $\beta_{0.75R}$ (deg)									
	α 20	25	30	35	40	45	50	55	60	
0.165	α 20		30	35	40	45	50	55	60	
.23		25	30	35	40	45	50	55	60	
.35		β 25	30	35	40	45	50	55	60	
.43				35	40	45	50	55	60	
.53					40	45	50	55	60	
.60						45	50	55	60	
.65							50	55	60	
.675								55	60	
.70									60	
.725										60

^a Except for NACA 4-(10)(08)-03 propeller.

^b Except for NACA 4-(3)(08)-03 and NACA 4-(5)(08)-03 propellers.

The velocity correction, which has been applied to the calculation of advance ratio, is presented in figure 4 as the ratio of free-air velocity to the tunnel-datum velocity (tunnel empty) as a function of the thrust disk-loading coefficient. The tunnel-wall correction was found to be dependent only on the thrust disk-loading coefficient for the range of tunnel speed and propeller operation used in this investigation.

The tunnel-datum Mach number has not been corrected for tunnel-wall constraint. For the range of velocity shown in figure 4, the factors required to correct the tunnel-datum velocity and tunnel-datum Mach numbers to the free-stream condition are essentially equal.

RESULTS AND DISCUSSION

The basic characteristics for the NACA 4-(5)(08)-03 and NACA 4-(10)(08)-03 two-blade propellers are presented in figures 5 and 6, respectively. For each value of the tunnel-datum Mach number the propeller thrust coefficient, power coefficient, and efficiency are plotted against advance ratio. The variation of tip Mach number with advance ratio is also included. As used in this report, the tunnel-datum Mach number M is not corrected for the effects of tunnel-wall constraint. The free-stream Mach number can be obtained by applying the tunnel-wall corrections presented in figure 4 to the tunnel-datum Mach number. Similarly, the corrected tip Mach number can also be obtained.

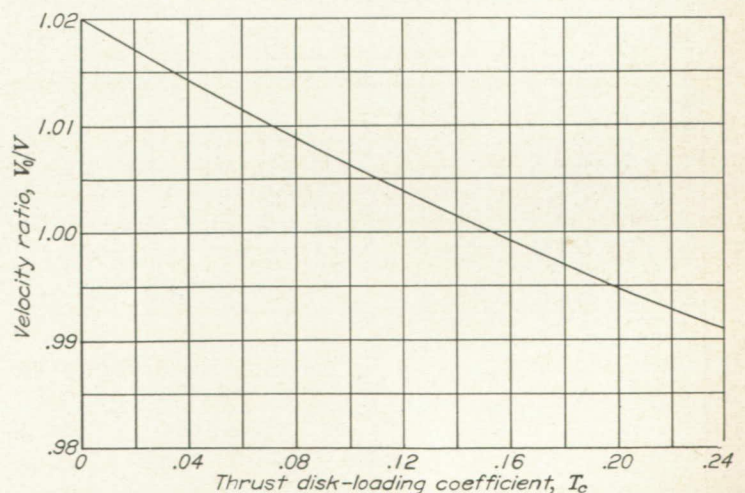


FIGURE 4.—Tunnel-wall interference correction.

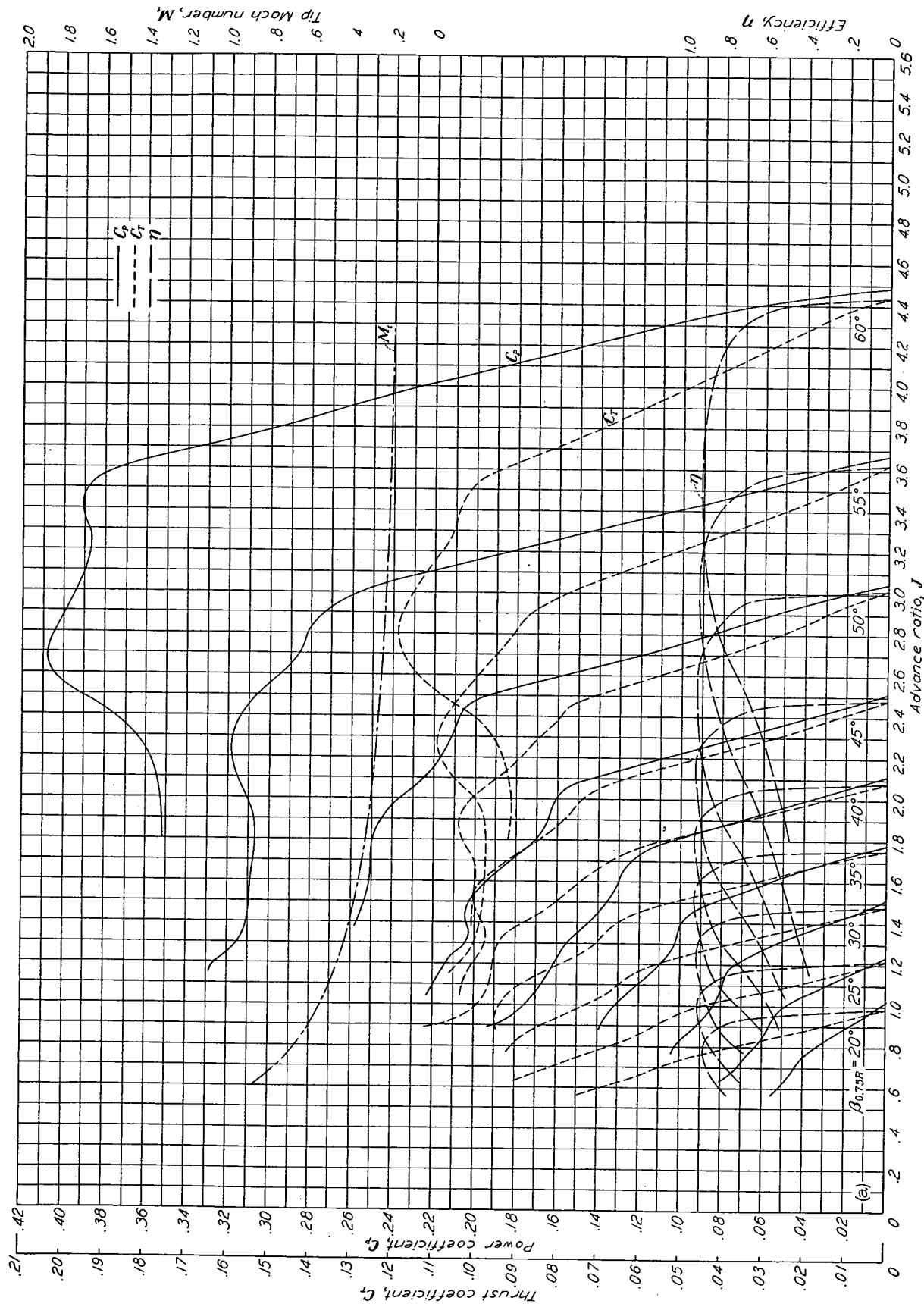
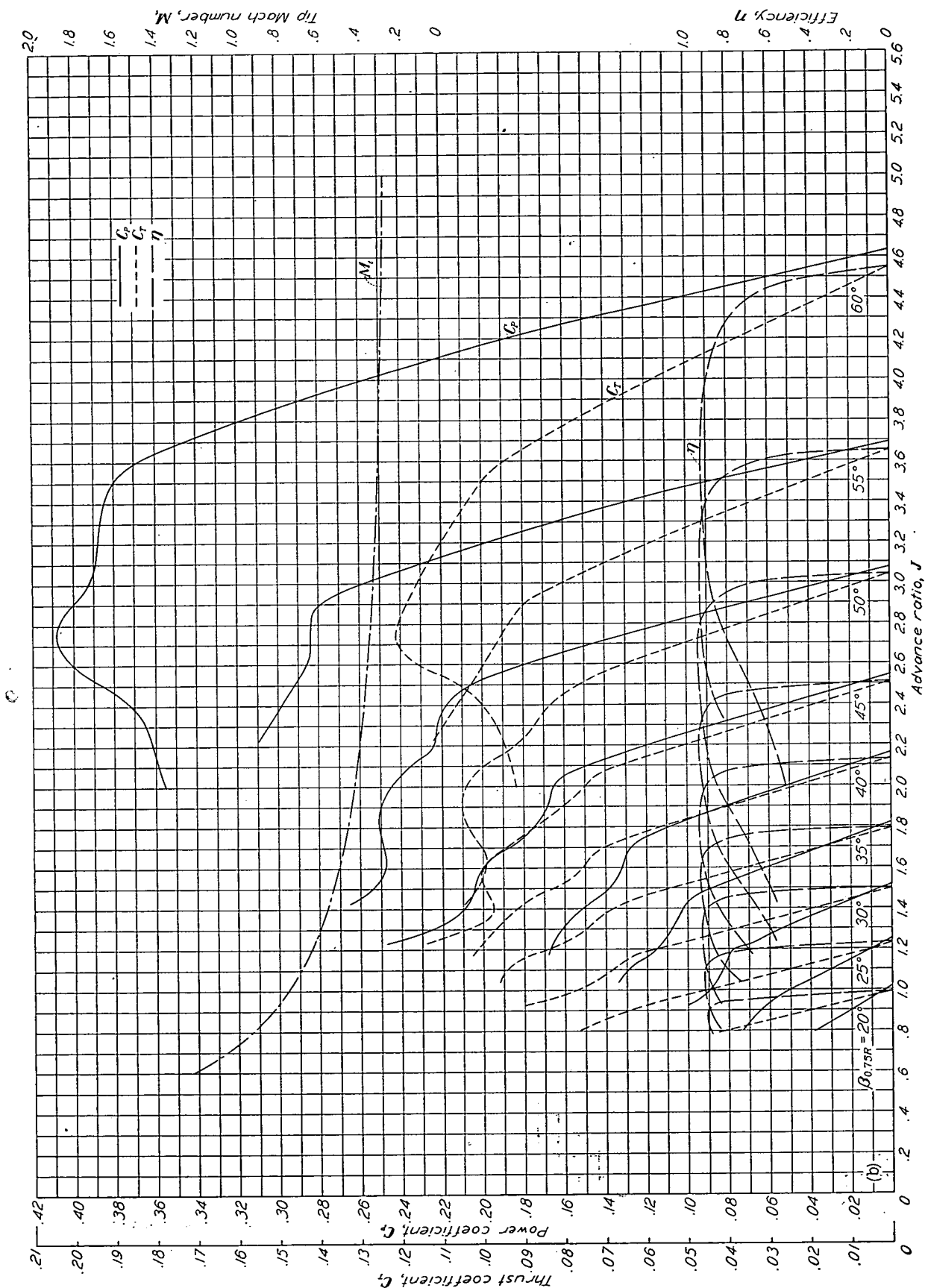
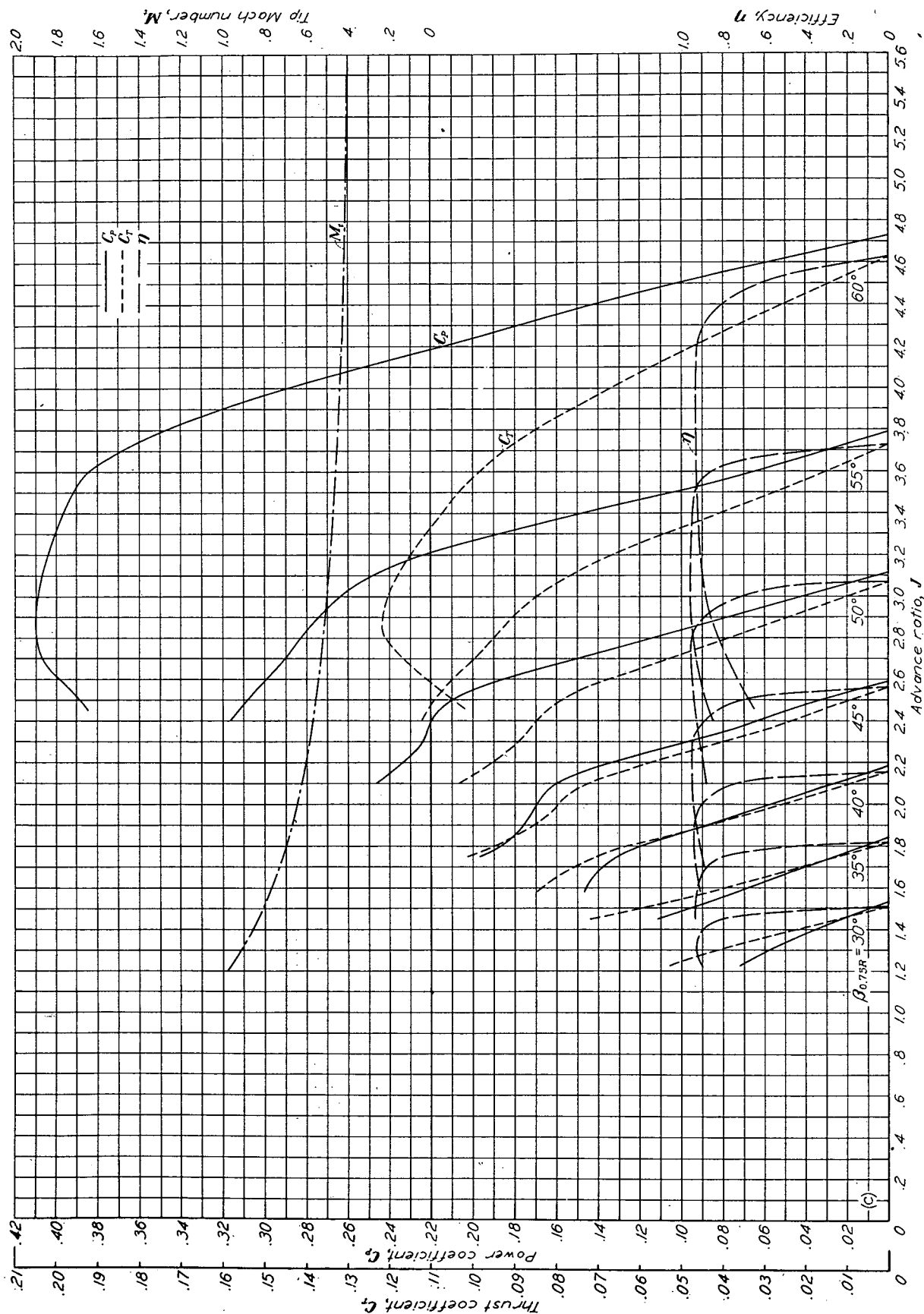
(a) $M=0.165$.

FIGURE 5.—Characteristics for the NACA 4-(5)(08)-03 propeller.



(b) $M=0.23$.
FIGURE 5.—Continued.



(c) $M=0.35$.
FIGURE 5.—Continued.

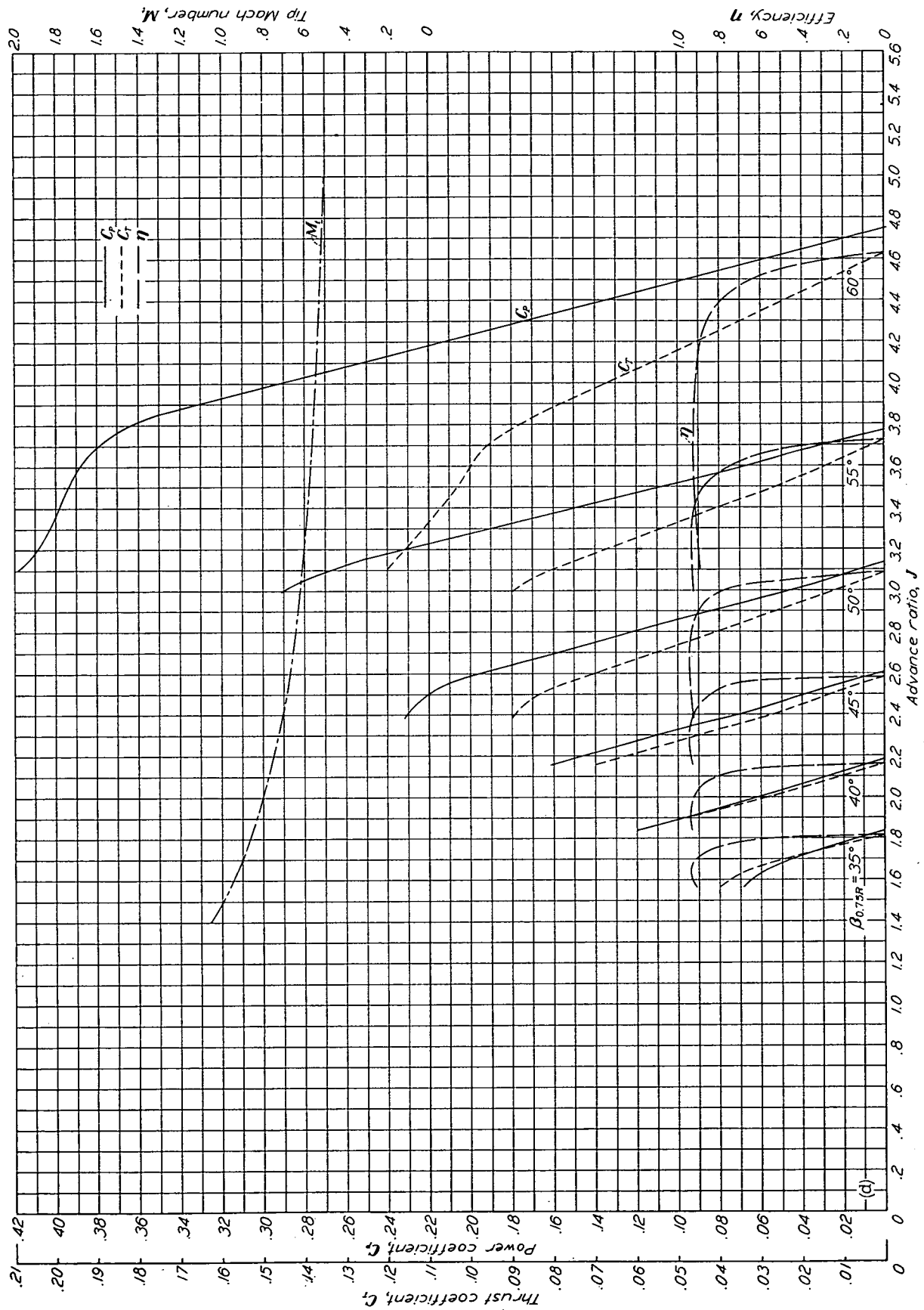
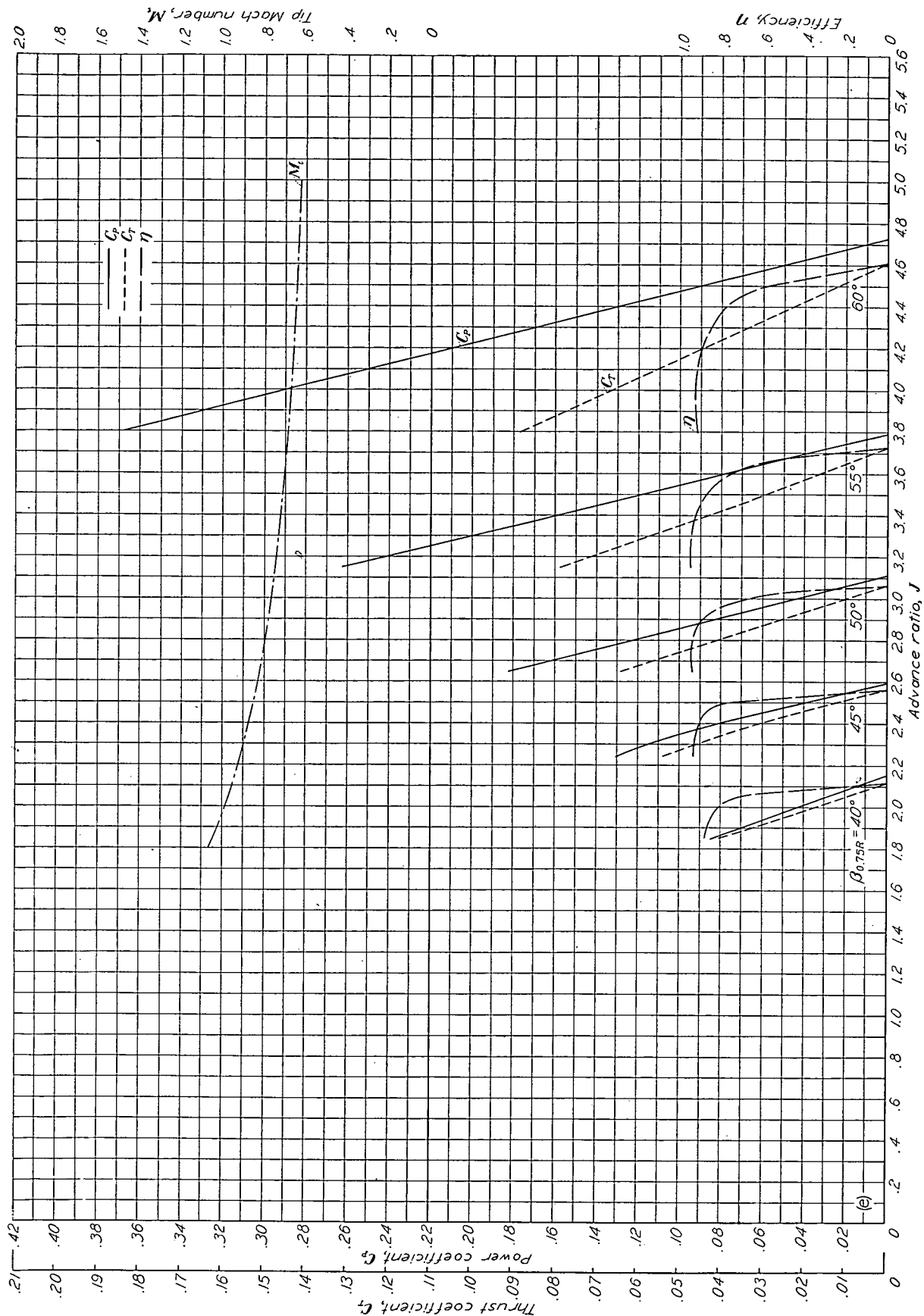
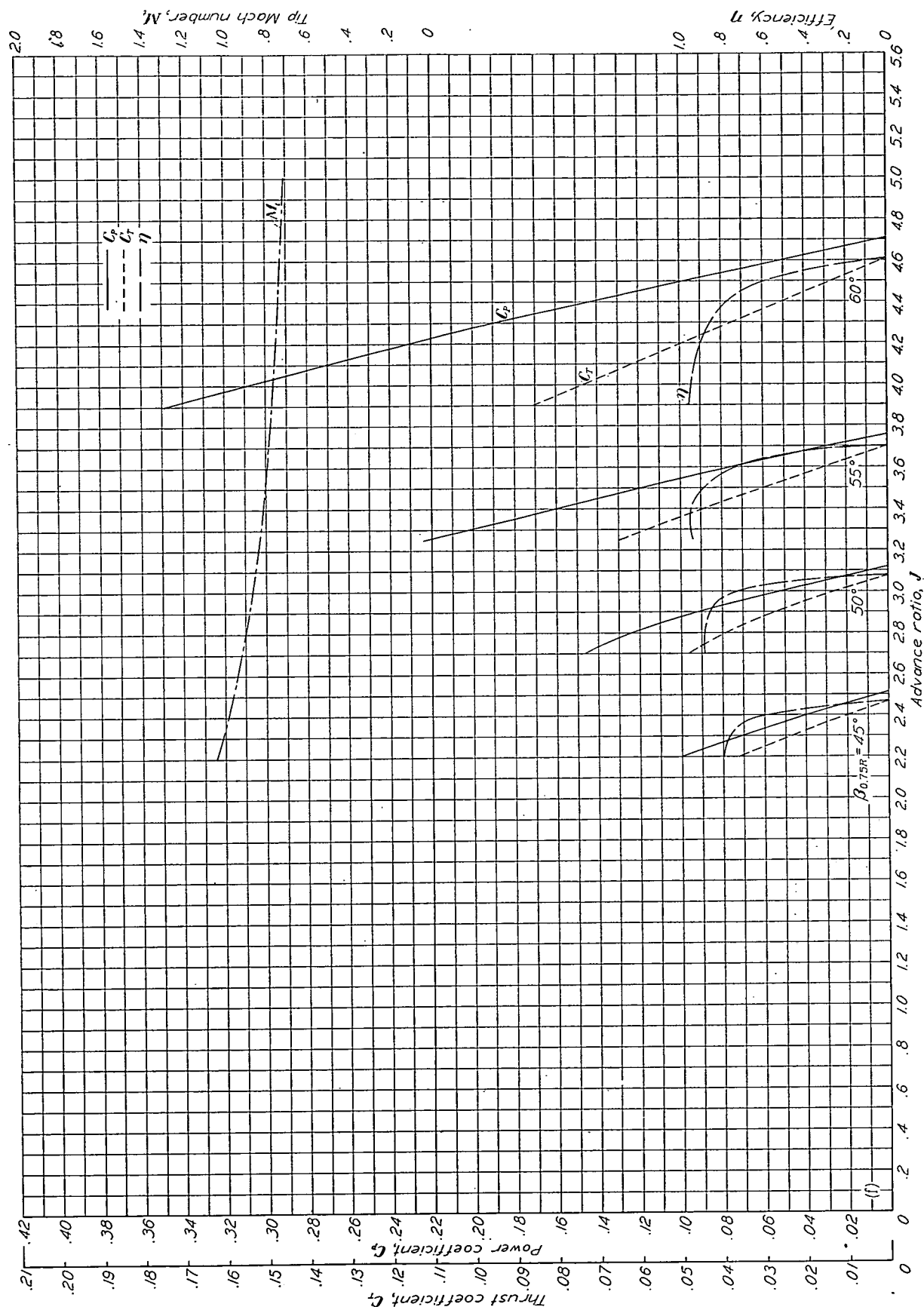
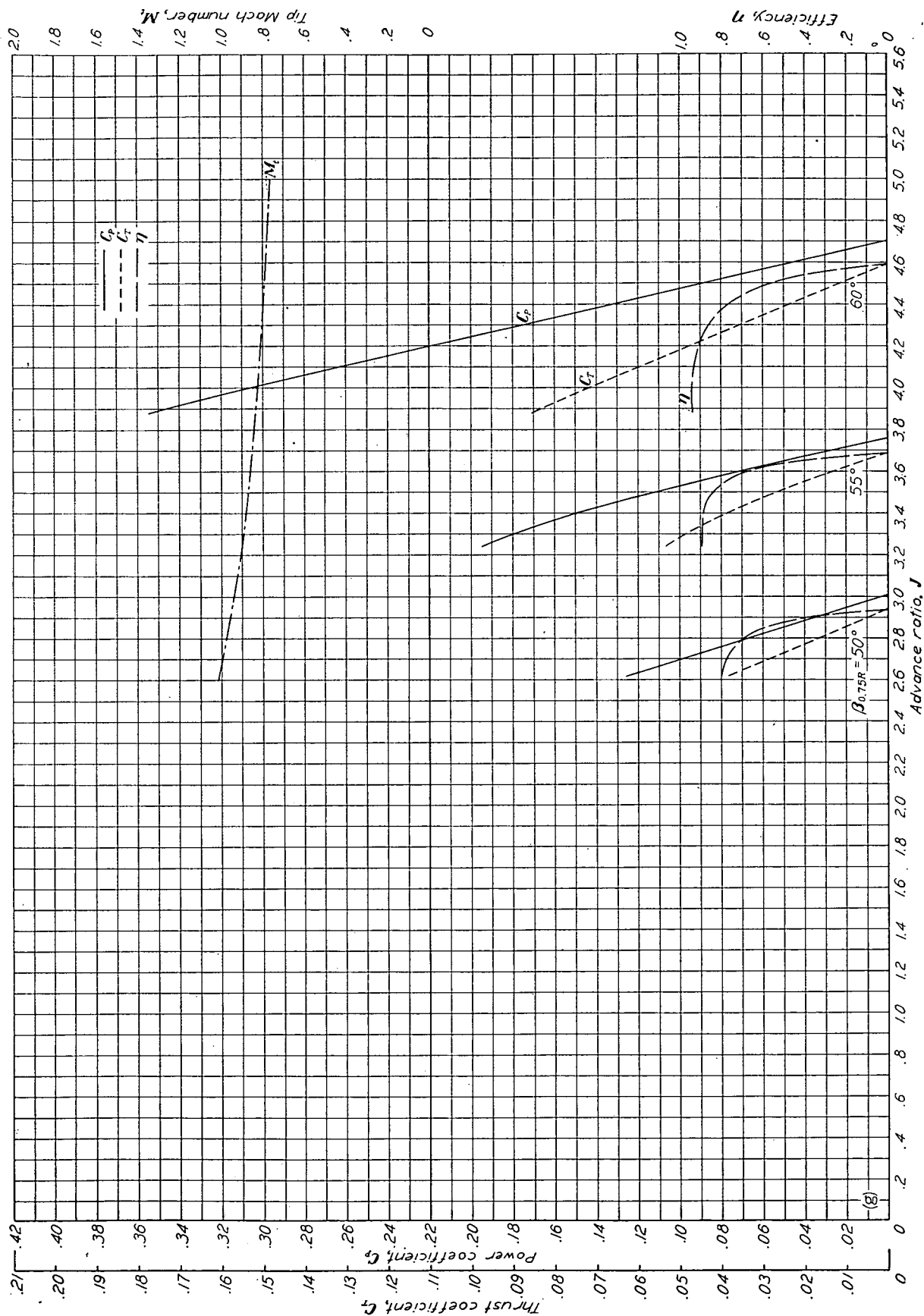


FIGURE 5.—Continued.

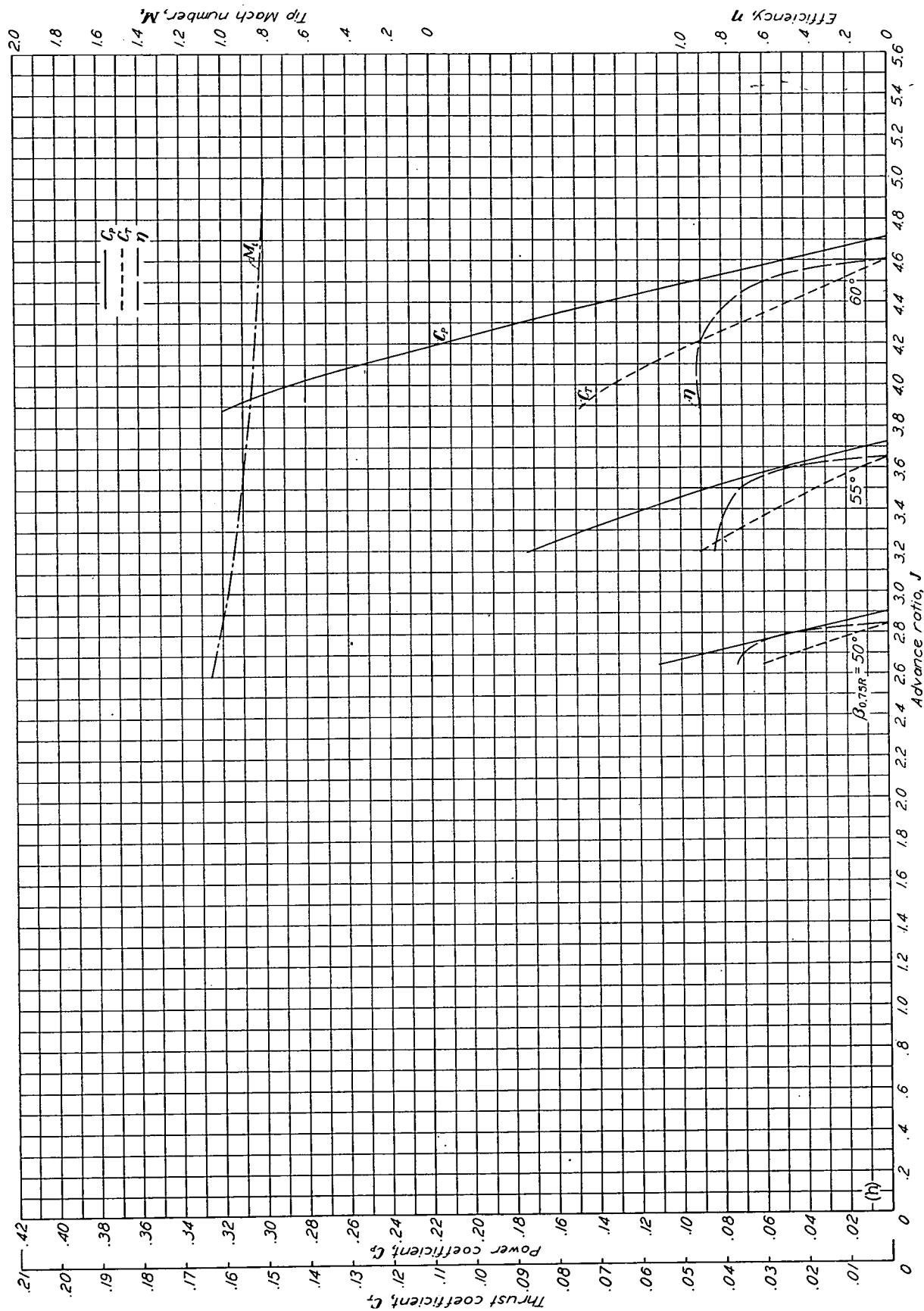


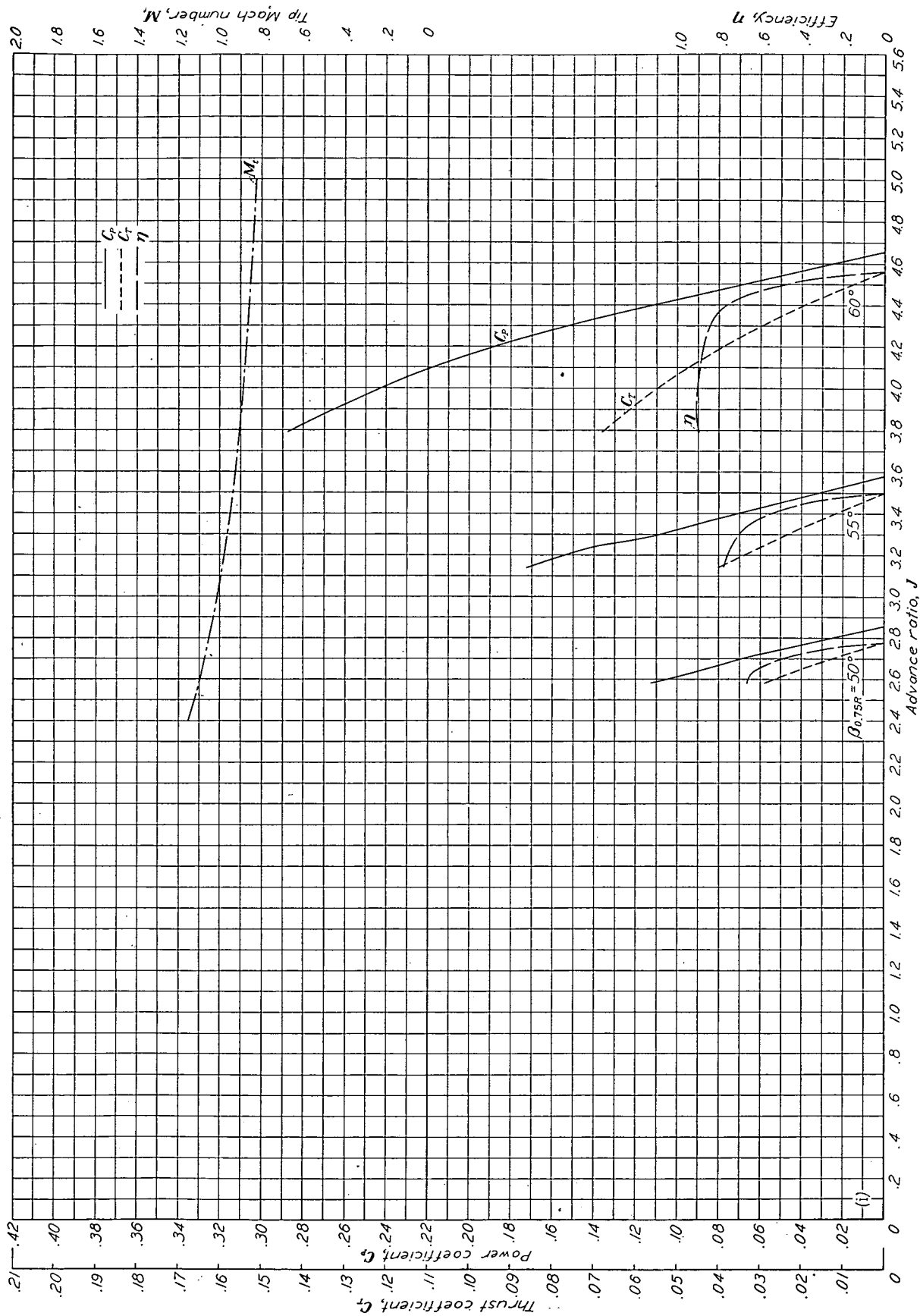
(e) $M=0.53$.
FIGURE 5.—Continued.


 (f) $M=0.60$.
 FIGURE 5.—Continued.

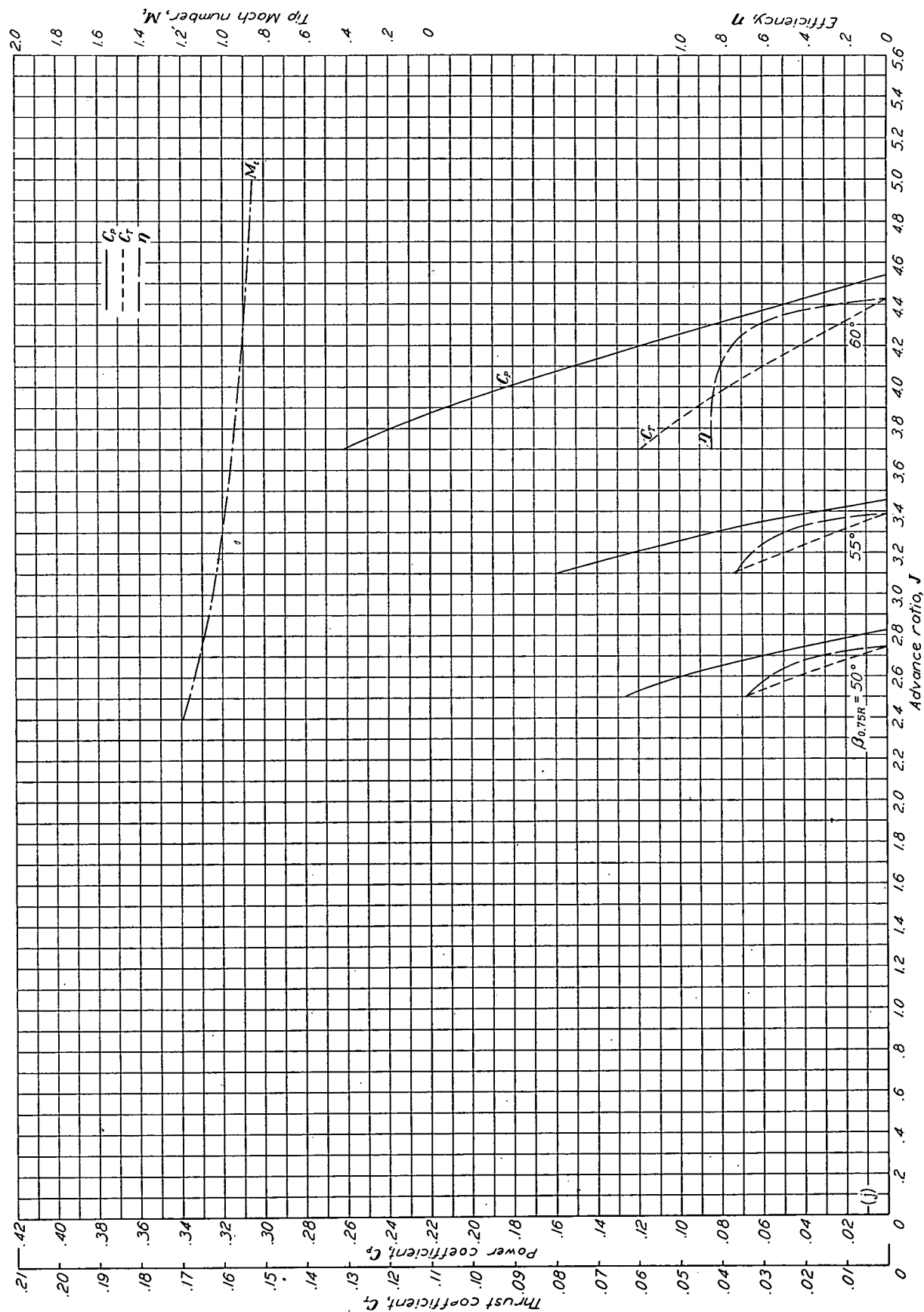


(g) $M = 0.65$.
FIGURE 5.—Continued.


 (h) $M = 0.675$.
FIGURE 5—Continued.



(1) $M=0.70$.
FIGURE 5.—Continued.


 (i) $M=0.725$.
 FIGURE 5.—Concluded.

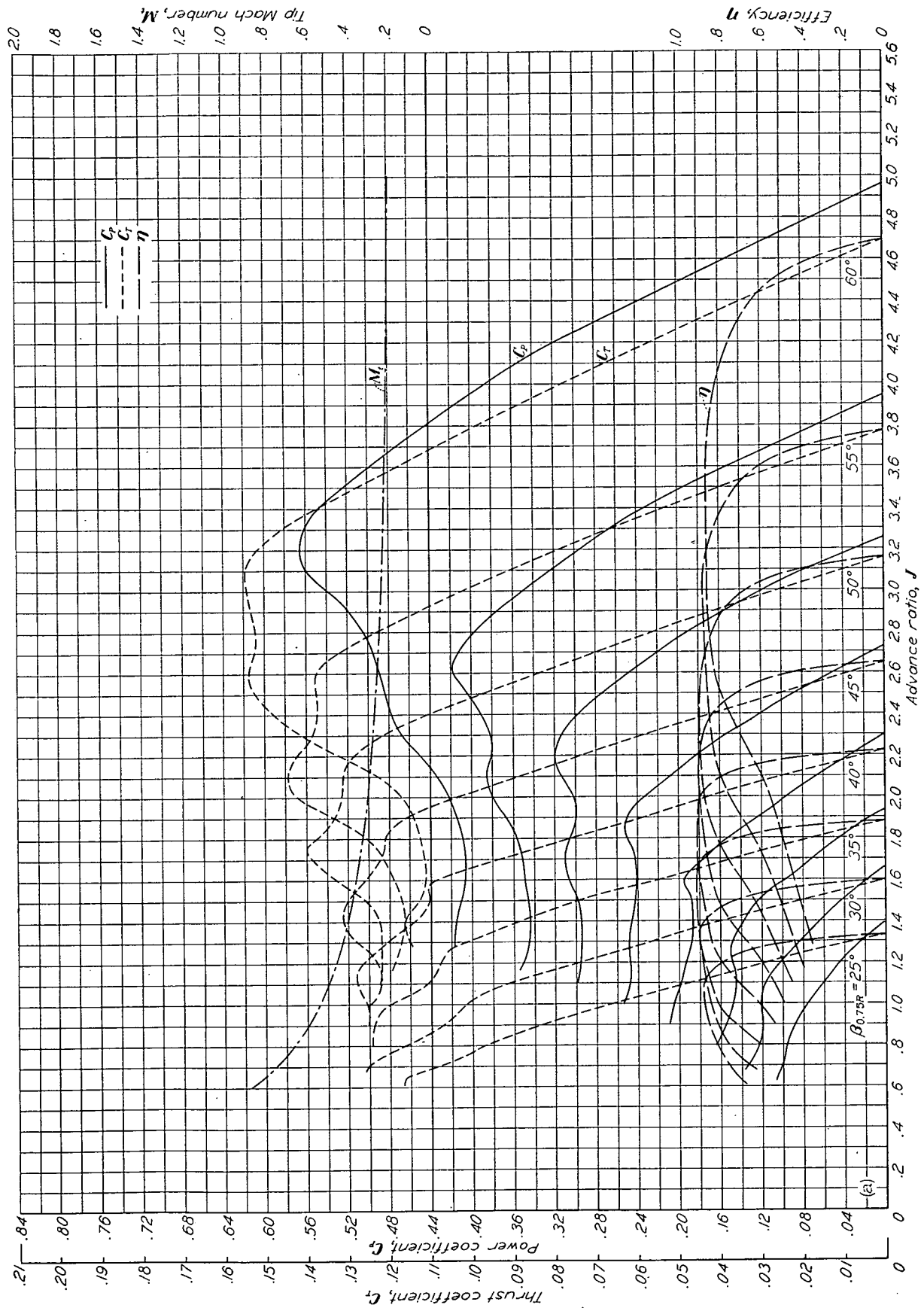
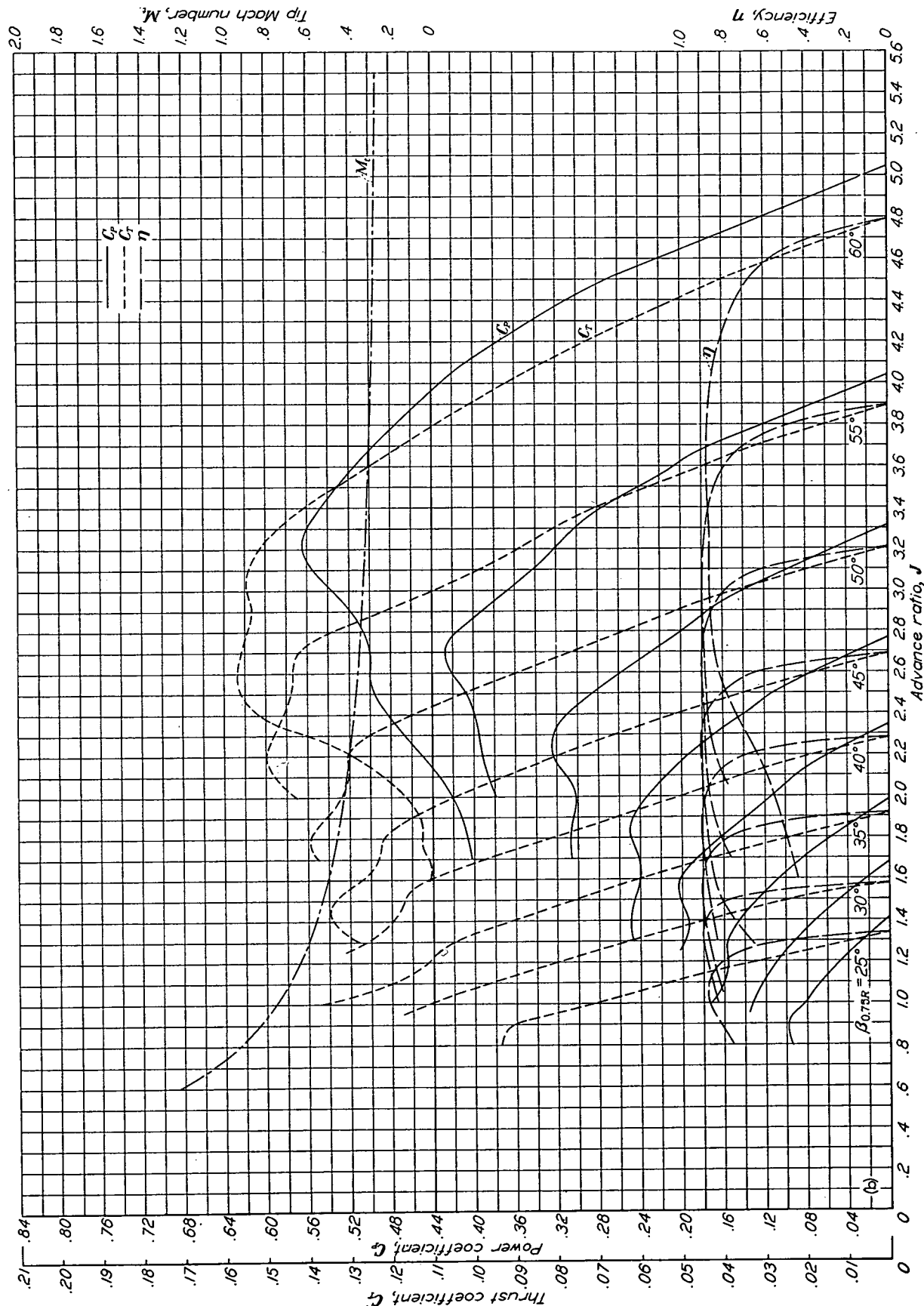
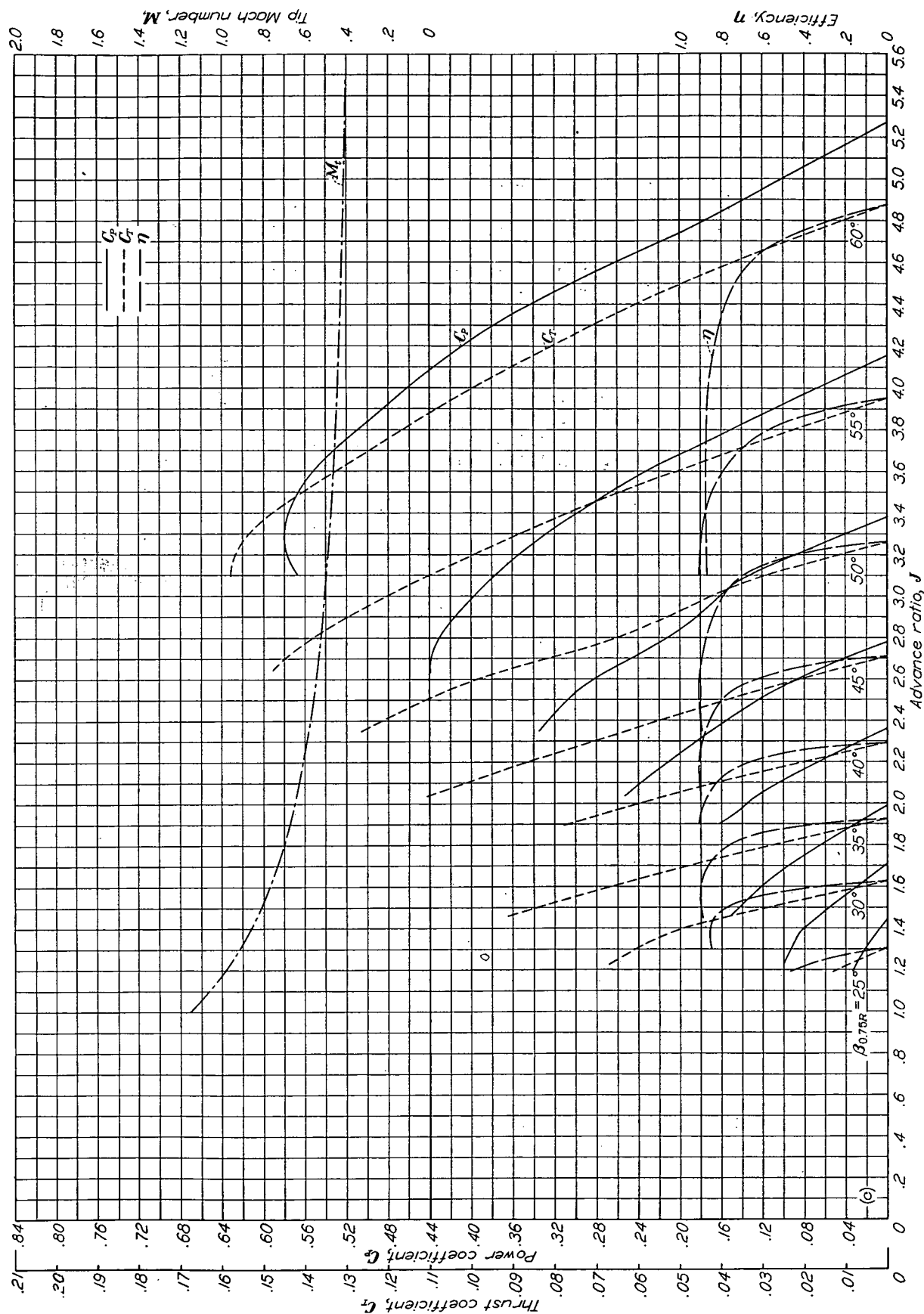


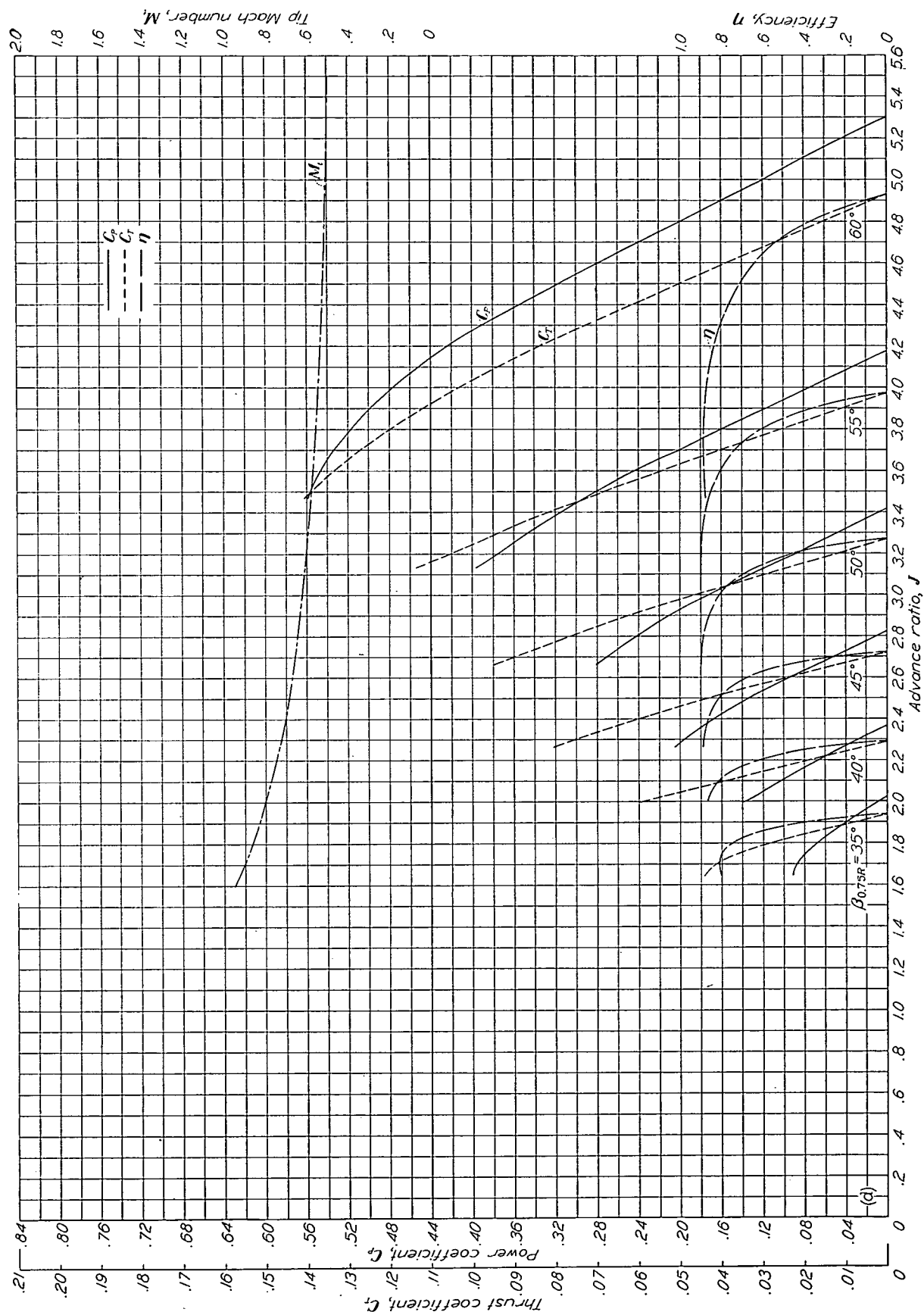
FIGURE 6.—Characteristics for the NACA 4-(10)(08)-48 propeller.
(a) $M_t = 0.165$.



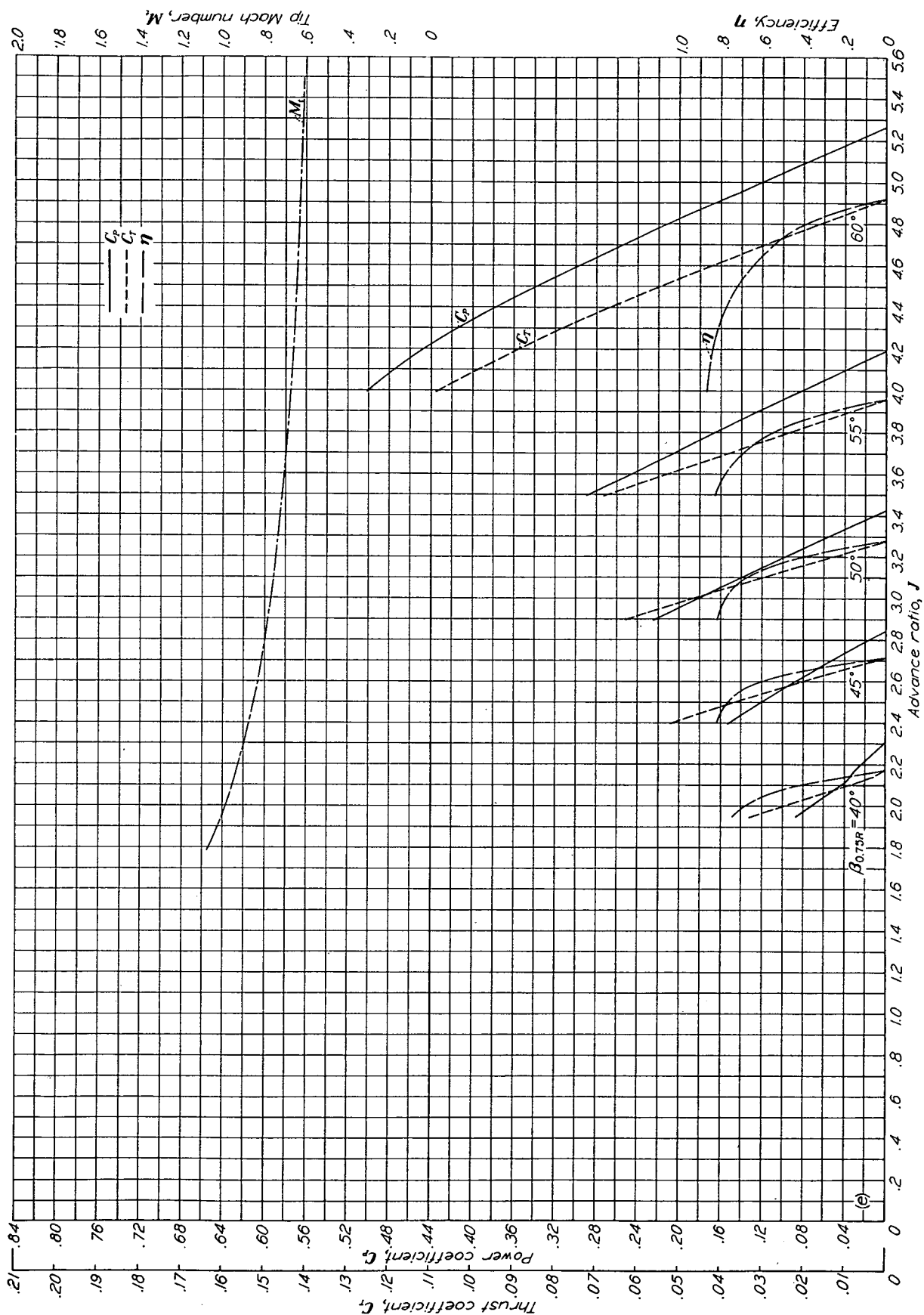
(b) $M=0.23$.
FIGURE 6.—Continued.



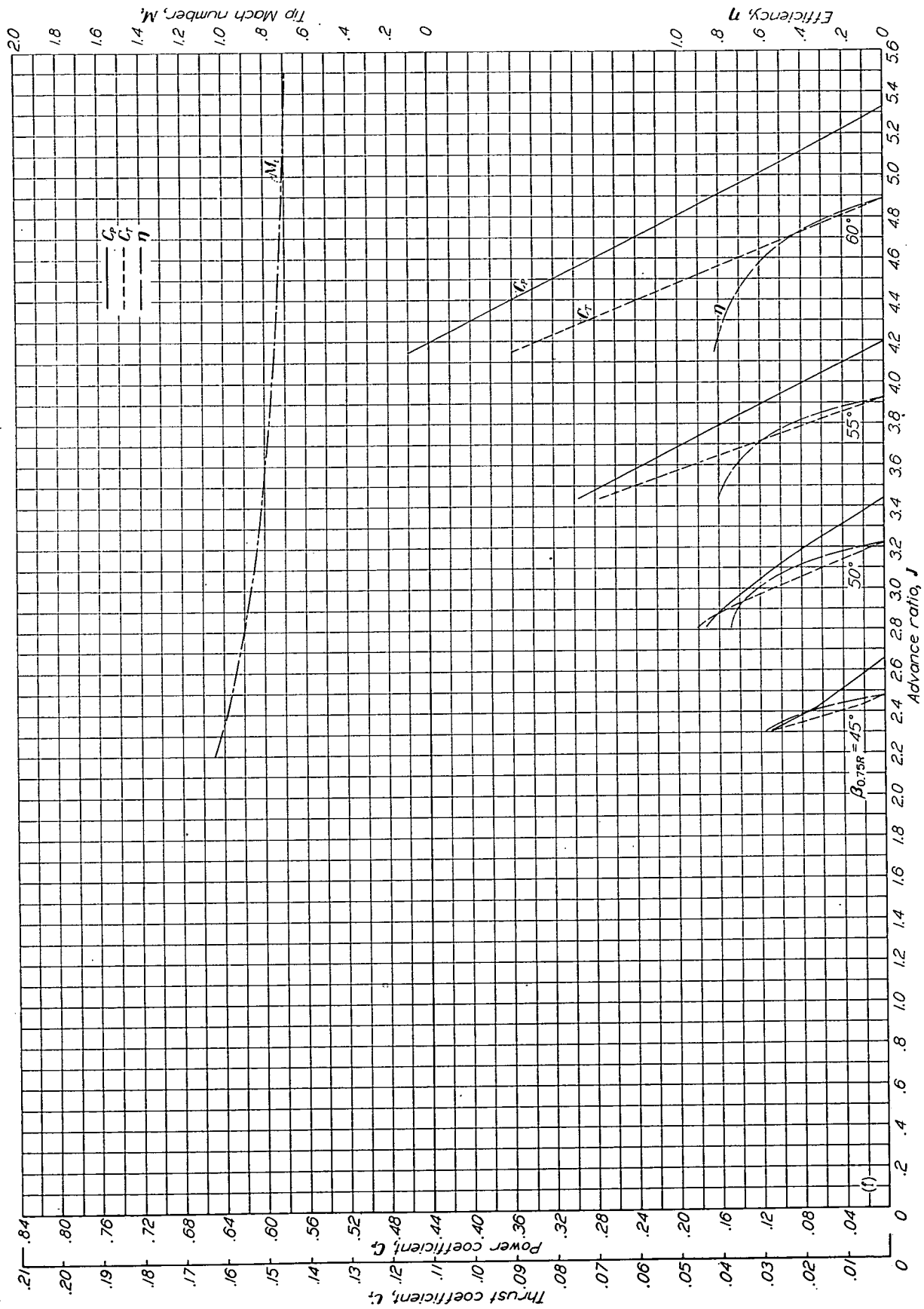
(c) $M = 0.35$.
FIGURE 6.—Continued.



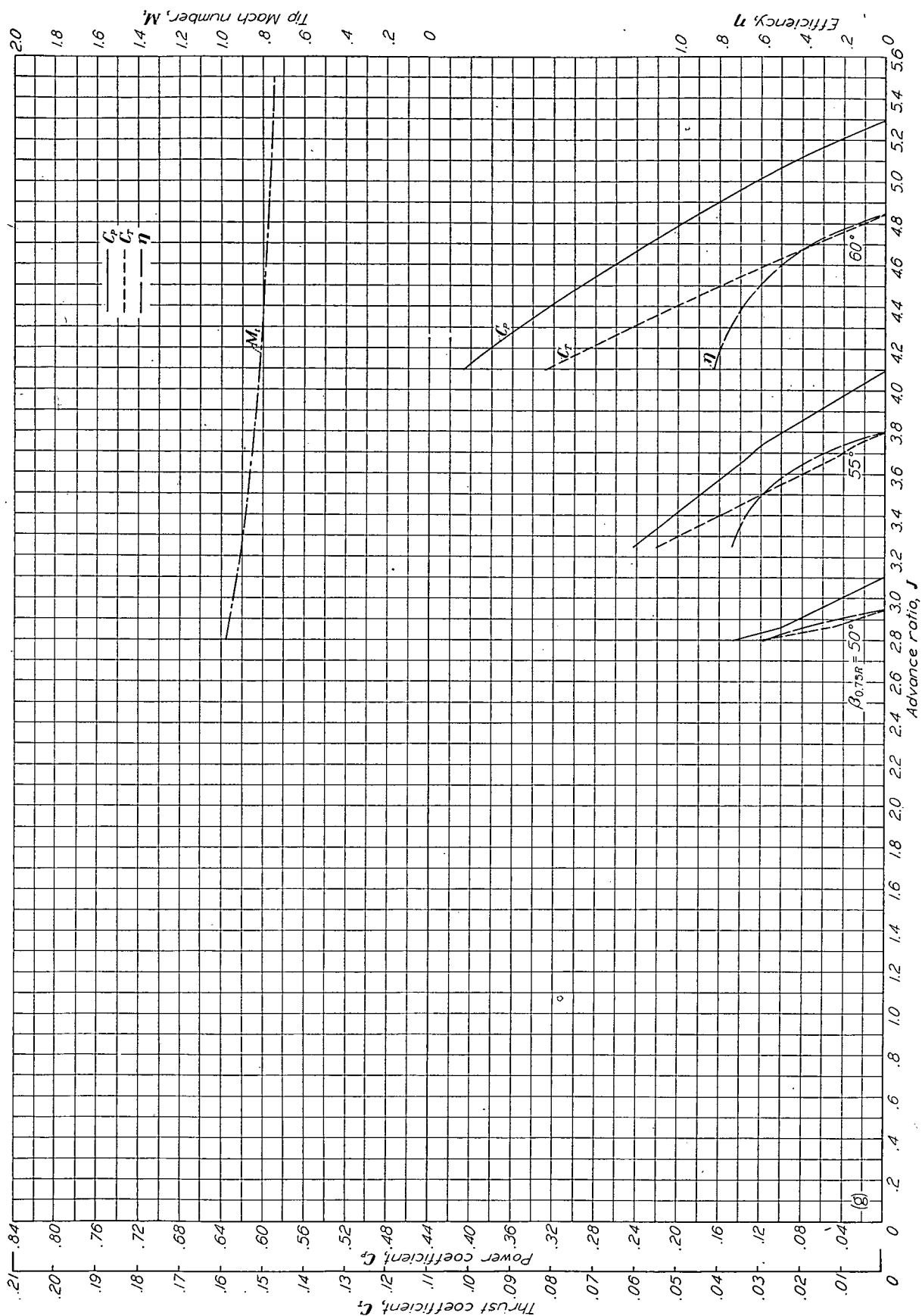
(d) $M_t = 0.43$.
FIGURE 6.—Continued.



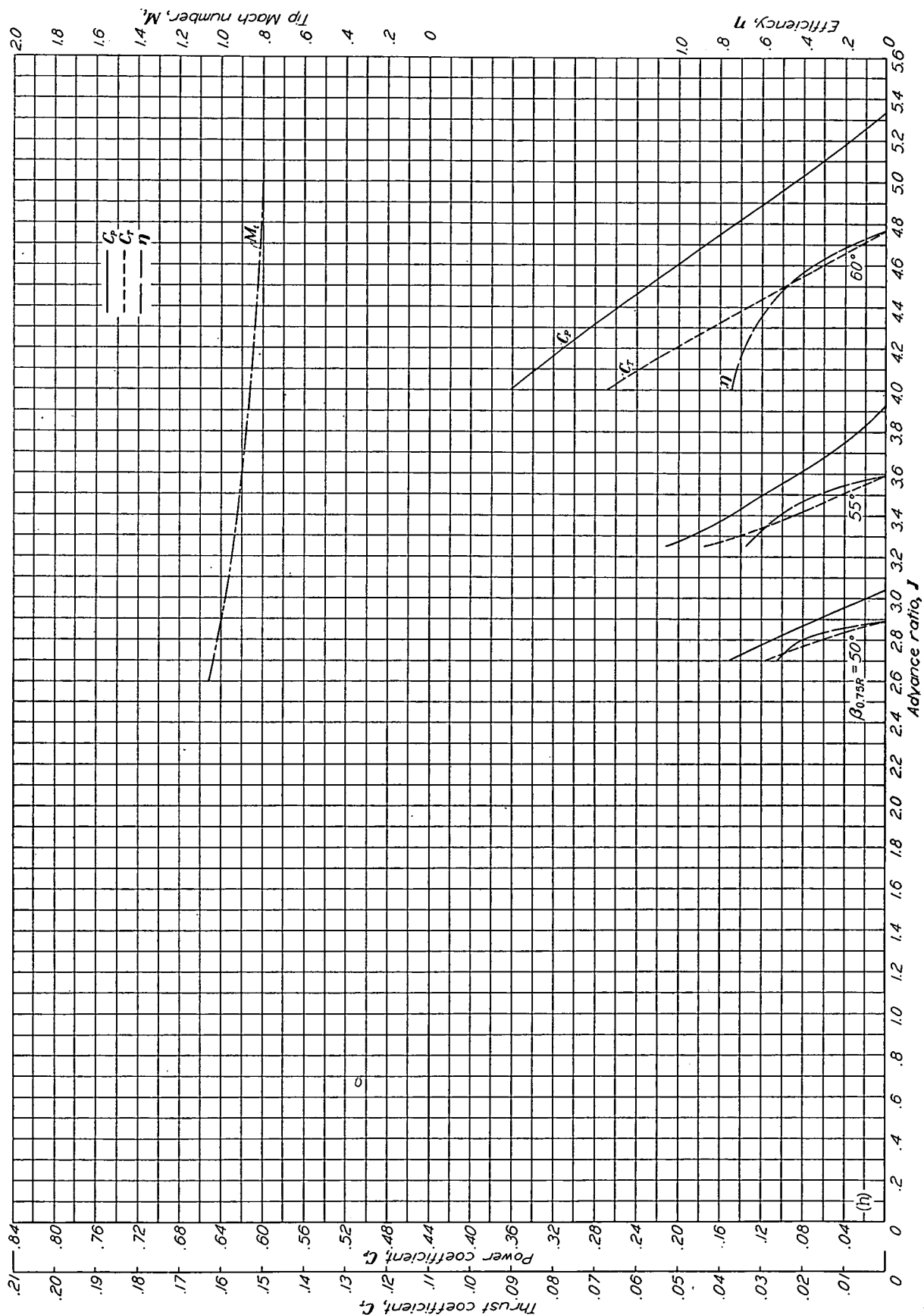
(e) $M = 0.53$.
FIGURE 6.—Continued.



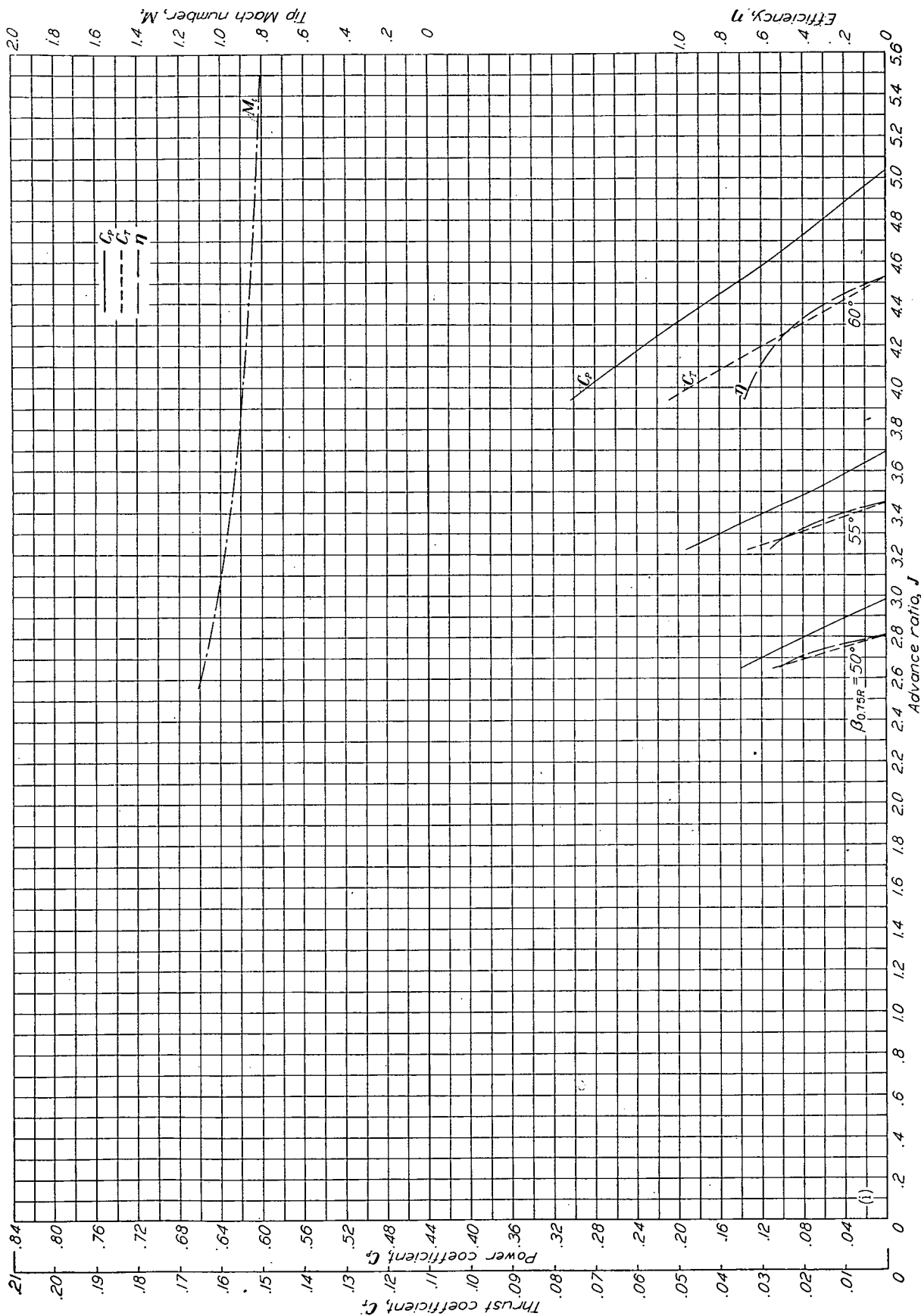
(1) $M=0.60$.
FIGURE 6.—Continued.



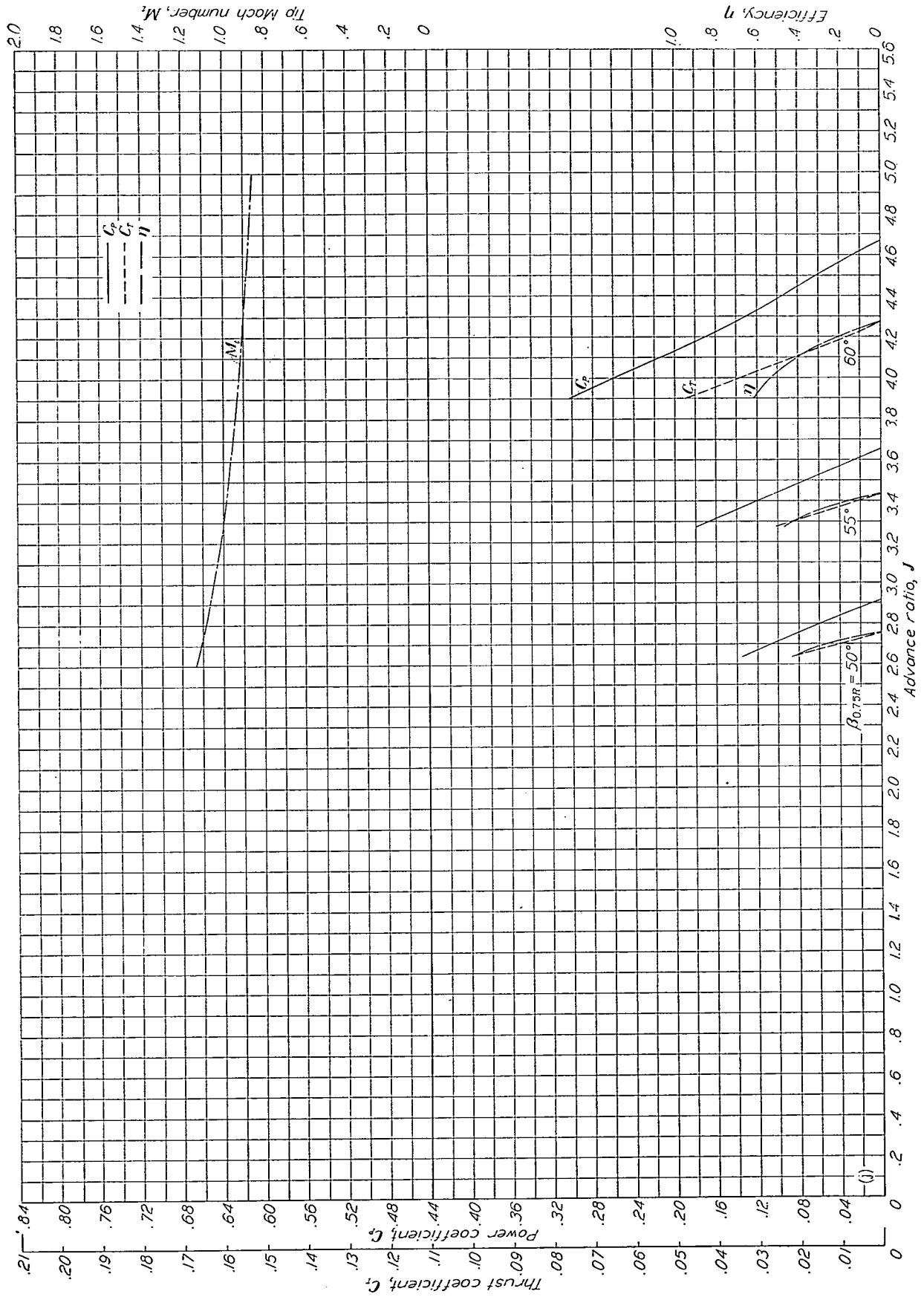
(g) $M = 0.65$.
FIGURE 6.—Continued.



(h) $M=0.575$.
FIGURE 6.—Continued.



(i) $M=0.70$
FIGURE 6.—Continued



(i) $M=0.725$.
FIGURE 6.—Concluded.

Effect of camber on thrust coefficient.—The primary effect of using propeller blades of increased design camber (increased design lift coefficient) is to increase the power absorbed by the blades and consequently to increase the thrust. A typical illustration of the increase in thrust produced by increasing the design camber is shown in figure 7 in which the thrust coefficients for the high-camber, medium-camber, and low-camber propellers for a blade angle of 45° and a forward Mach number of 0.23 are compared. The power-coefficient curves are similar and hence are not shown. For cases in which take-off and climb performances are of prime importance, the increased thrust produced by the blades of high design camber may determine the design; greater thrusts may be produced with no increase in propeller weight. The maximum thrust coefficient and the thrust coefficient for maximum efficiency also increase with an increase in design camber, as shown in figures 8 and 9 for Mach

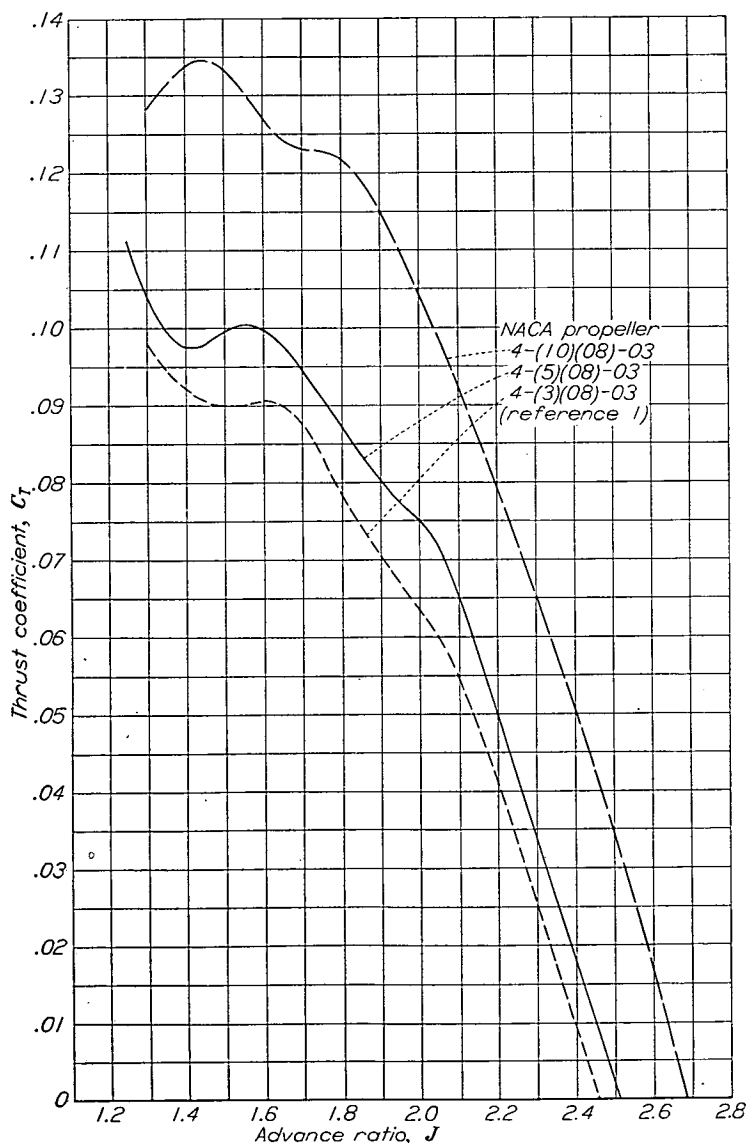


FIGURE 7.—Effect of design camber on thrust coefficient. $M=0.23$; $\beta_{0.75R}=45^\circ$.

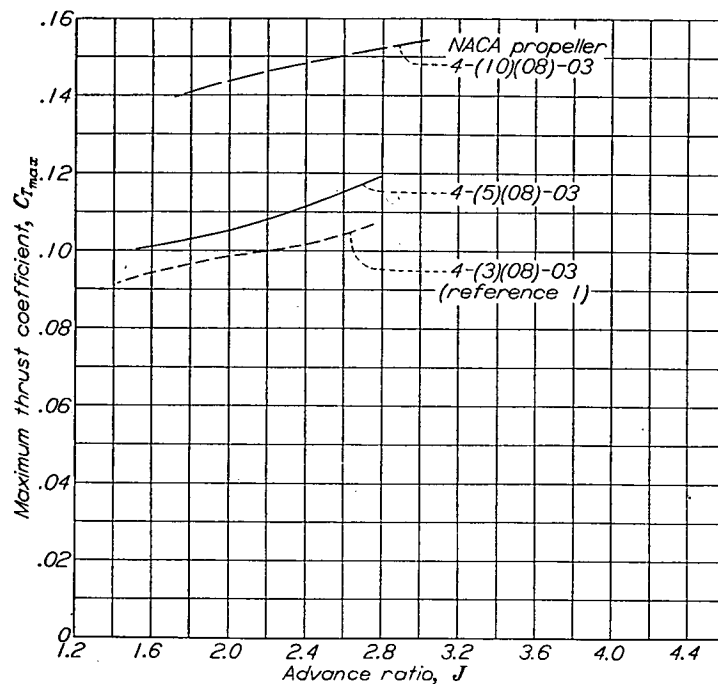


FIGURE 8.—Effect of design camber on maximum thrust coefficient. $M=0.165$.

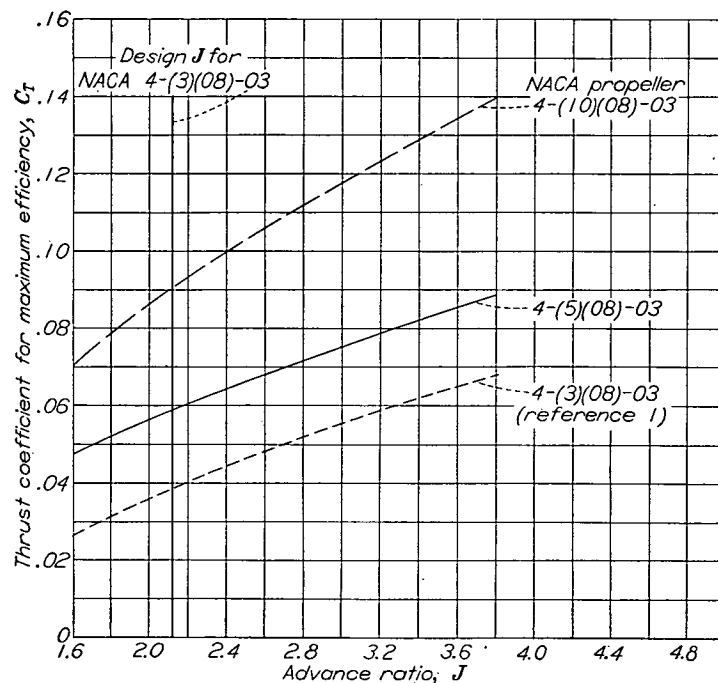


FIGURE 9.—Effect of design camber on thrust coefficient for maximum efficiency. $M=0.35$.

numbers of 0.165 and 0.35, respectively. The percentage of increase in thrust is less than the percentage of increase in corresponding design camber. The maximum thrust coefficients for the medium-camber propeller are 7 to 11 percent greater than for the low-camber propeller. The maximum thrust coefficients for the high-camber propeller are 41 to 46 percent greater than for the low-camber propeller. The increases in thrust coefficient for maximum efficiency are

much greater than the increase in maximum thrust coefficient. The thrust coefficients at maximum efficiency are 30 to 79 percent greater for the medium-camber propeller and 105 to 165 percent greater for the high-camber propeller than for the low-camber propeller.

Operation at high angles of attack may cause the blade sections to become stalled or nearly stalled so that the propeller efficiency is decreased because of increased profile-drag losses. The pressure distribution over these blade sections is therefore far from optimum and has high peaks that have a tendency to cause flow separation or to initiate compressibility shock. The use of blades of high design camber, however, makes it possible for the blade sections to operate at high section lift coefficients, which are obtained at angles of attack much lower than for blades of low design camber; thus, the tendency of the flow to separate is reduced and stalled conditions are largely eliminated. Since the pressure distribution about the sections may closely approximate the design distribution, the profile drag and the tendency for shock to be initiated are reduced.

Effect of camber and power loading on efficiency.—The effect of blade power loading on propeller efficiency for the high-camber, medium-camber, and low-camber propellers is shown in figure 10 for a forward Mach number of 0.165. Values of the power coefficient of about 0.10 for these propellers represent operation at high lift coefficients for values of the advance ratio corresponding to take-off and climb. For this condition, the propeller efficiency decreases very rapidly as the power coefficient is increased because of the increased profile-drag losses and the failure of the lift to increase beyond the maximum section lift coefficient with further increase of angle of attack. The ideal efficiency computed from the momentum theory is also shown in figure 10 for comparison. The divergence of the measured efficiency from the ideal efficiency emphasizes the magnitude of the profile-drag and induced losses.

The effect of design camber for constant values of power coefficient is shown in figure 11 for a forward Mach number of 0.165. At low advance ratios corresponding to take-off and climb, increased camber gives increased efficiency at high power coefficients. At these high power coefficients, the high-camber blade is 15 to 25 percent more efficient than the low-camber and medium-camber blades. At low advance ratios and for low power coefficients, the high-camber blade is approximately 5 percent less efficient than the low-camber and medium-camber blades. These variations emphasize the necessity for choosing the correct blade camber to meet operational requirements. The high-camber blade is generally more efficient than the low-camber and medium-camber blades up to values of the advance ratio approximately 10 times the value of the power coefficient. The medium-camber blade is generally as much as 5 percent more efficient than the low-camber blade for the same operating

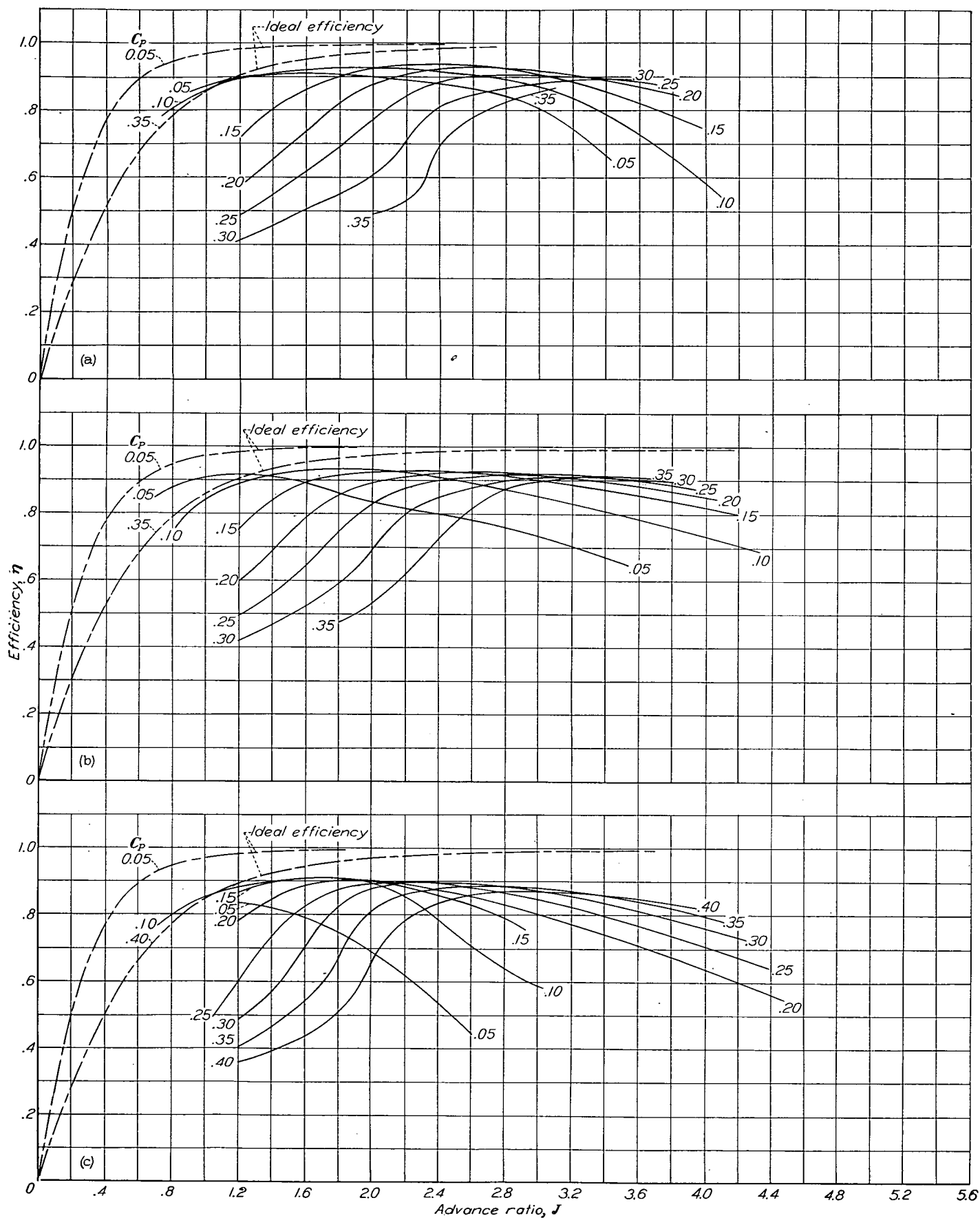
range. These results suggest that a satisfactory compromise propeller may be designed by proper selection of the design camber.

Single-station analysis of camber effects.—In order to show the effect of design camber and operating lift coefficients on propeller section efficiency, the results of tests of the NACA 16-series airfoil sections of 9-percent thickness (reference 2) were chosen as representative of the section at the 0.7-radius station. Since this analysis is not an attempt to explain or present compressibility effects, data at a Mach number of 0.45 are used. It can be shown that the section efficiency is given by

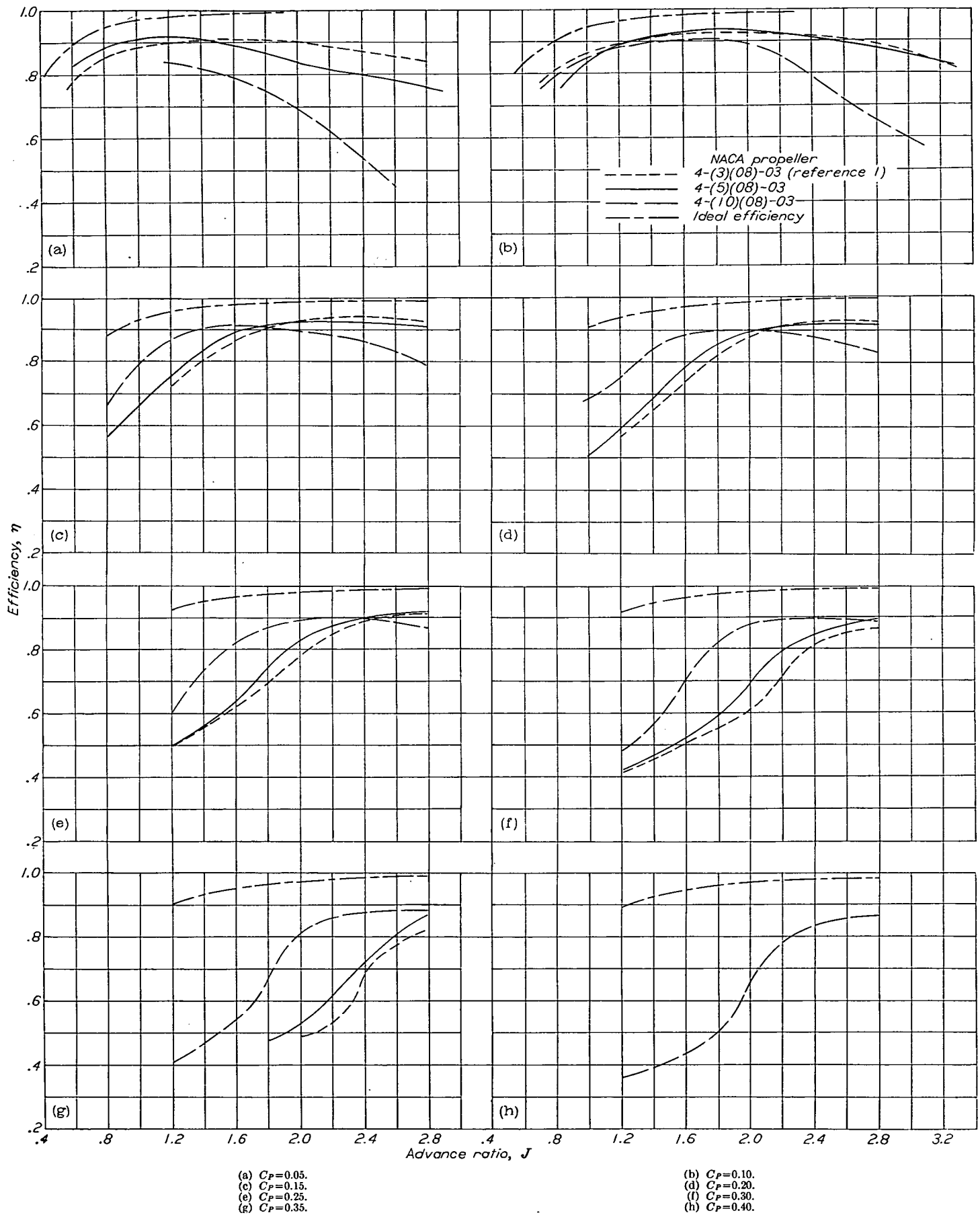
$$\eta = \frac{\tan \phi_0}{\tan (\phi + \gamma)} \quad (1)$$

Figure 12 shows the results obtained by use of equation (1). For a given operating lift coefficient, the induced angle of attack for all the sections is the same; hence, the efficiency shows the effect of lift-drag ratio. The most obvious result is that the sections with design lift coefficients between 0.3 and 0.5 are the most efficient, because these sections have the highest lift-drag ratios at maximum efficiency. For sections with design lift coefficients of 0.3 and lower, the maximum efficiency occurs at operating lift coefficients greater than the design lift coefficients. For sections with design lift coefficients higher than 0.3, the maximum efficiency occurs at operating lift coefficients lower than the design lift coefficient. The maximum attainable efficiency at any blade operating lift coefficient is represented by the envelope of the efficiencies in figure 12. The greatest efficiency attainable for operation at a given lift coefficient occurs when the section has a design lift coefficient equal to the operating lift coefficient.

Effect of compressibility on maximum efficiency.—The envelope efficiencies for the high-camber, medium-camber, and low-camber propellers are presented in figure 13 for forward Mach numbers from 0.23 to 0.70. The values of advance ratio at which propeller tip Mach numbers of 0.9, 1.0, and 1.1 are reached are indicated by vertical dash lines labeled with the value of M_t . The medium-camber propeller gave the highest efficiencies for all advance ratios and for forward Mach numbers up to 0.53. In most cases, the medium-camber propeller was 2 to 5 percent more efficient than the low-camber propeller. The high-camber propeller gave peak efficiencies 3 to approximately 12 percent lower than those for the medium-camber propeller; the higher efficiency losses were due mainly to compressibility effects. At tip Mach numbers greater than approximately 0.90, the low-camber and medium-camber propellers showed compressibility losses. The high-camber propeller, however, showed an appreciable compressibility loss at a tip Mach number considerably below 0.90, but the efficiencies were still above 82 percent.



(a) NACA 4-(3)(08)-03 propeller (reference 1).
 (b) NACA 4-(5)(08)-03 propeller.
 (c) NACA 4-(10)(08)-03 propeller.
 FIGURE 10.—Effect of power loading on efficiency. $M=0.165$


 FIGURE 11.—Effect of design camber and power loading on efficiency. $M=0.165$.

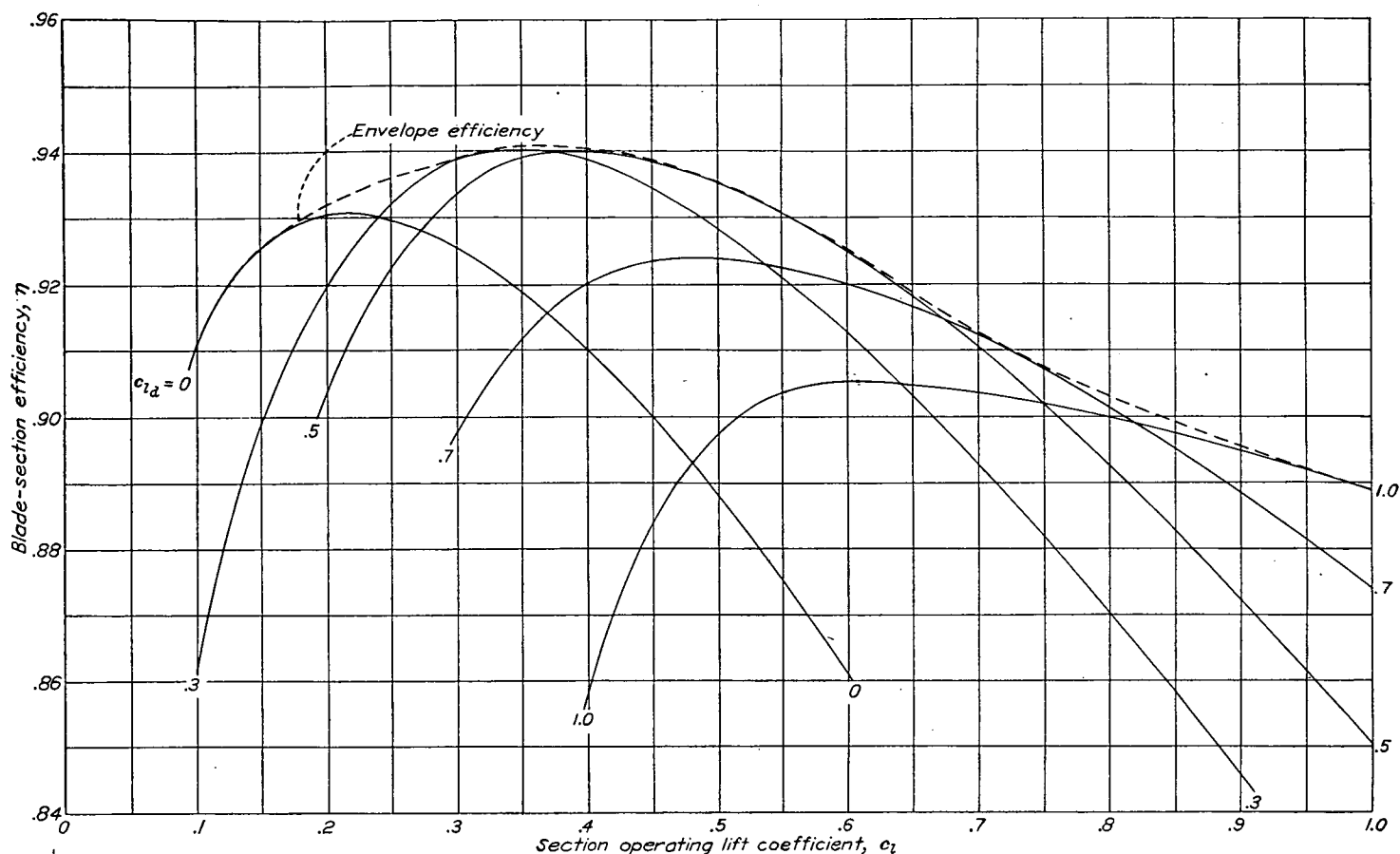


FIGURE 12.—Effect of design camber on blade-section efficiency, from single-station analysis using NACA 16-series sections. $\frac{r}{R}=0.7$; $\frac{h}{b}=0.09$; $\phi=44^\circ$; $M=0.45$.

The effect of compressibility on maximum efficiency is shown in figure 14 for a blade angle of 45° . Maximum efficiency differed very little for the low-camber and medium-camber propellers with critical tip Mach numbers of approximately 0.90 and 0.91, respectively. The high-camber propeller begins to show compressibility losses at a tip Mach number of 0.70, but the rate of loss is less than that for the low-camber and medium-camber propellers. The low critical speed of the propeller with the highest camber obviously excludes the use of this propeller for very efficient high-speed operation. The early compressibility losses for the high-camber propeller are due, in part at least, to the high power absorbed. If the low-camber and medium-camber propellers absorbed the same power as the high-camber propeller, these propellers would have to operate at high angles of attack; this operation would produce high pressure peaks and perhaps earlier and more extensive compressibility losses.

In order for the low-camber and medium-camber propellers to absorb the same power as the high-camber propeller and still operate at high efficiencies, a considerable increase in solidity would be necessary. The large power-absorbing capacity of the high-camber propeller, however, makes it useful for conditions of operation at which large values of thrust are required, even at high speeds. For example, the influence of design camber on maximum efficiency and on the power absorbed at maximum efficiency is presented in

figure 15 for an advance ratio of 2.48 at forward Mach numbers of 0.23 and 0.53 (tip Mach numbers of 0.37 and 0.86, respectively). For these conditions, the high-camber propeller shows a compressibility loss of 8 percent. The high-camber propeller absorbs 55 percent more power than the medium-camber propeller and 75 percent more power than the low-camber propeller at a forward Mach number of 0.53. The corresponding differences in efficiency are reductions of 9 and 10 percent, which result in net thrust increases of 46 and 65 percent, respectively, for the high-camber propeller; in addition, these increases in thrust are obtained with no increase in propeller weight.

Effect of compressibility and power disk loading on maximum efficiency.—The effects of power disk-loading coefficient P_c and compressibility on maximum efficiency are shown in figure 16. The curve for the ideal efficiency obtained from axial-momentum considerations is also shown. The ideal efficiency deviates from 100 percent solely because of the induced loss due to increasing the axial velocity of the air in the slipstream. The additional losses for an actual propeller, however, are due to profile-drag and rotational induced losses. The induced losses for these propellers are small, and the differences shown between the ideal and the measured efficiencies are principally due to blade drag loss. At a given value of the forward Mach number, increased values of P_c correspond to loadings at low values of the advance ratio.

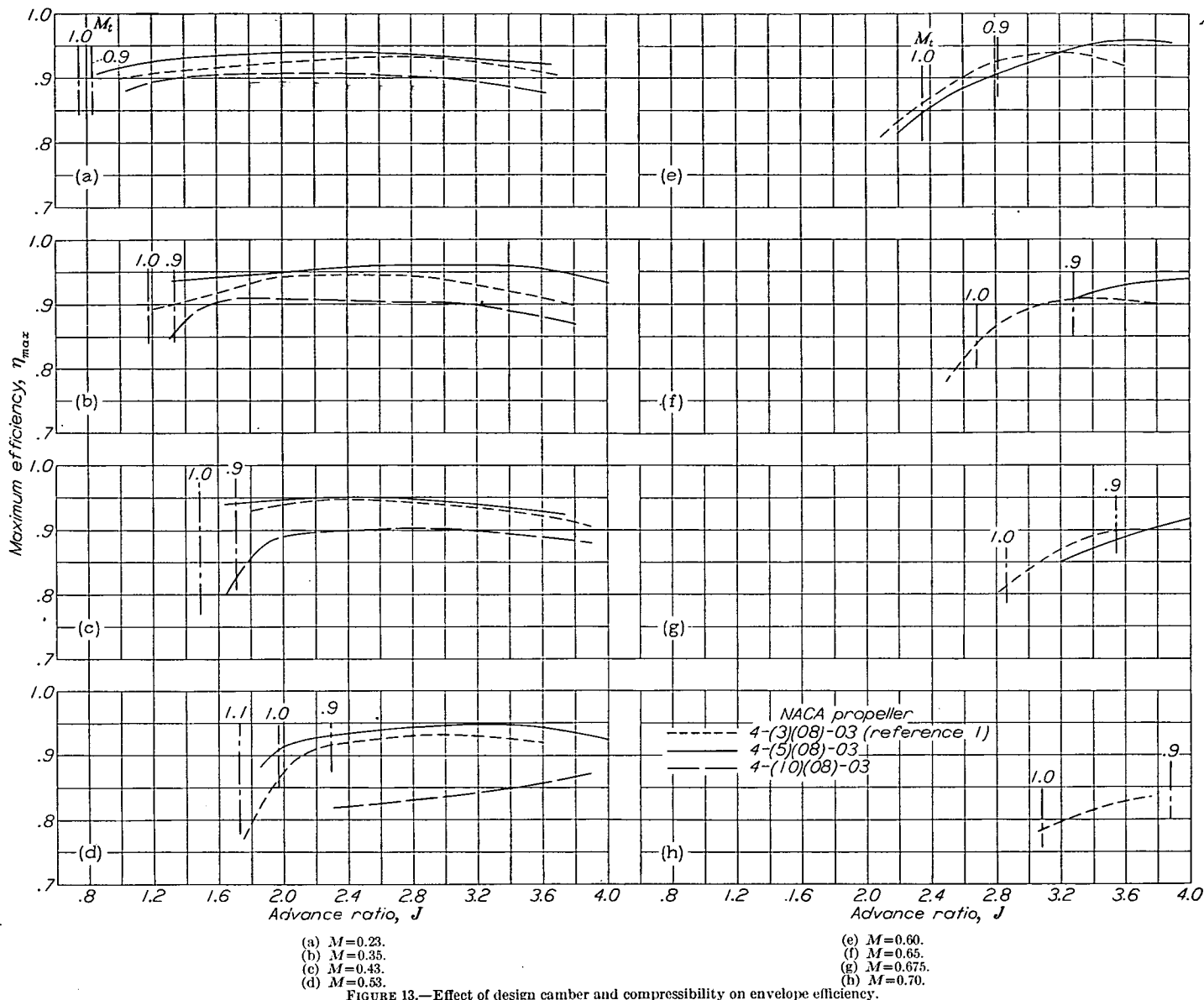
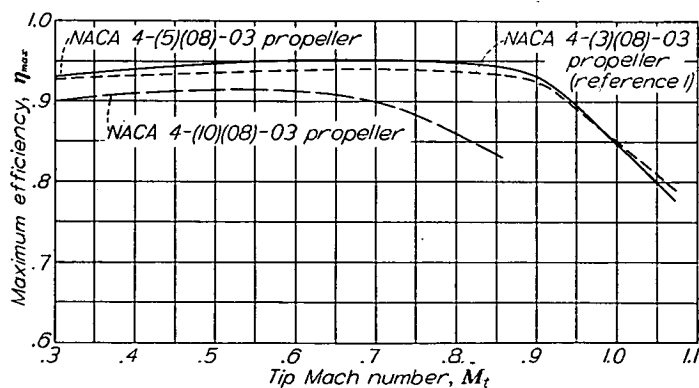
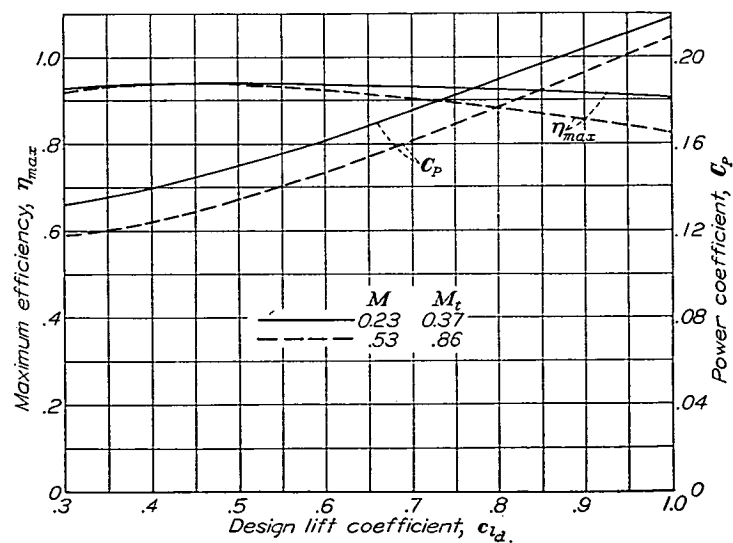


FIGURE 13.—Effect of design camber and compressibility on envelope efficiency.


 FIGURE 14.—Effect of compressibility on maximum efficiency. $\beta_{0.75R} = 45^\circ$.

 FIGURE 15.—Effect of blade-section design lift coefficient on maximum efficiency and power absorbed at maximum efficiency. $J = 2.45$.

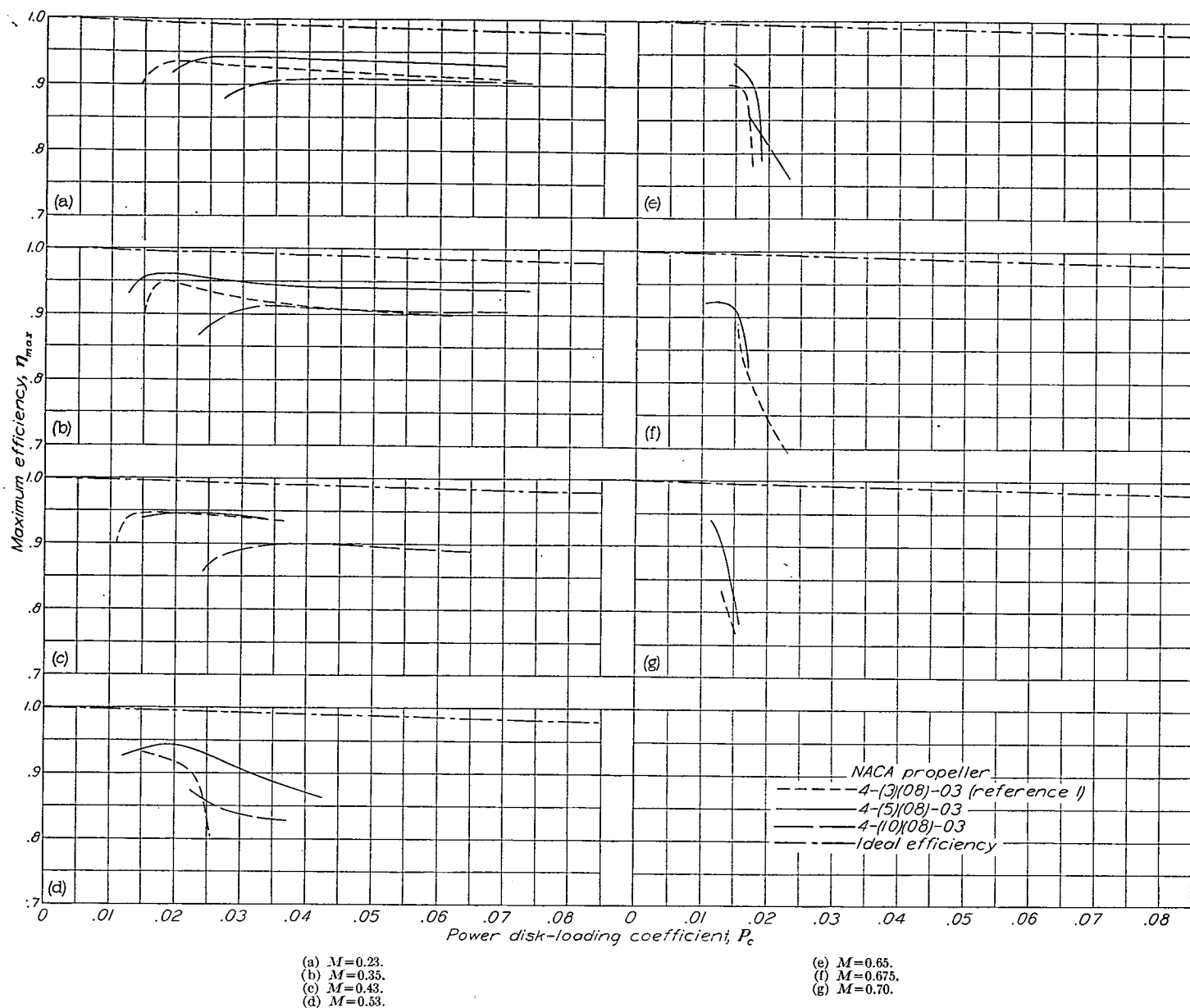


FIGURE 16.—Effect of power disk loading and compressibility on maximum efficiency.

At low values of the forward Mach number (fig. 16 (a)), for example, the maximum-efficiency curves for the three propellers are parallel for high values of P_c and are relatively close to the ideal-efficiency curve. This agreement is expected, because these propellers have approximately the optimum pitch distributions for these values of P_c and because the profile-drag and induced losses are not expected to change very much. Since at a constant forward Mach number the high values of P_c for each curve correspond to operation at the highest tip speeds for that condition, compressibility losses would appear at the high values of the power disk-loading coefficient. No compressibility losses appear for the low forward Mach numbers. For all propellers, large losses begin to appear at a forward Mach number of 0.53 and are most severe for the low-camber propeller and

least severe for the medium-camber propeller. At a Mach number of 0.65, the high-camber propeller appears to be more efficient than the other propellers for operation at the high power disk-loading coefficients. Of particular interest is the greatly reduced range of power disk-loading coefficient for which the maximum efficiencies obtained at low Mach numbers can be maintained at high forward Mach numbers. Similar results are shown in reference 1. At forward Mach numbers of 0.53 or greater, the range of power disk-loading coefficient that gives high efficiencies for the low-camber propeller is greatly reduced because of compressibility losses. Previous results (reference 1) have indicated that the range of power disk-loading coefficient for high efficiency may be selected by change of blade solidity. The results reported herein also indicate that the same effect can be produced by

change of camber. This effect is particularly pronounced for forward Mach numbers of 0.53 and 0.65 (figs. 16 (d) and 16 (e)). The medium-camber and high-camber propellers can operate more efficiently at high values of the power disk-loading coefficient, but compressibility effects have considerably lowered the efficiencies. Changing the design camber thus offers another possibility of operating at high power disk loadings without too much loss in maximum efficiency.

The power disk-loading coefficient for which low-speed efficiencies may be maintained at high speeds can also be increased by using a greater number of similar blades, as was pointed out in reference 1. For operation at very high speeds, particular consideration must therefore be given to the aerodynamic design. The design of a propeller then approaches the design for a specific condition of operation to obtain high efficiencies because of the reduced range of available power disk loading.

CONCLUSIONS

Two-blade propellers designated the NACA 4-(5)(08)-03 (medium camber) and the NACA 4-(10)(08)-03 (high camber) propellers have been investigated in the Langley 8-foot high-speed tunnel through a range of blade angles from 20° to 60° for forward Mach numbers from 0.165 to 0.70 to determine the effect of camber and compressibility on propeller characteristics. The results of these tests and comparisons with results obtained from previous tests of the NACA 4-(3)(08)-03 (low camber) propeller indicated the following conclusions:

1. Propellers of high design camber were more efficient than propellers of low design camber for operation at high power coefficients. The propeller of highest camber gave efficiencies 15 to 25 percent greater than the efficiencies of the low-camber and medium-camber propellers for high power coefficients at advance ratios corresponding to take-off and climb at low Mach numbers.

2. The medium-camber propeller generally gave peak efficiencies 2 to 5 percent higher than the low-camber

propeller and 3 to approximately 12 percent higher than the high-camber propeller. The high-camber propeller was operating at much higher power coefficients, which led to early compressibility effects.

3. The critical tip Mach number for maximum efficiency at the design blade angle of 45° was 0.01 higher for the medium-camber propeller than that for the low-camber propeller, which began to show compressibility losses at a tip Mach number of approximately 0.90. The high-camber propeller, which was operating at higher power coefficients than the other propellers, showed the largest compressibility losses. The compressibility losses for the high-camber propeller began at a tip Mach number as low as 0.70 but efficiencies of more than 82 percent were still maintained.

4. For a forward Mach number of 0.53 and an advance ratio of 2.48 (tip Mach number of 0.86), the high-camber propeller showed a compressibility loss of 8 percent in maximum efficiency but, because of the large power-absorbing capacity of this propeller, produced about 65 percent more thrust than the low-camber propeller and 46 percent more thrust than the medium-camber propeller.

5. The range of power disk-loading coefficient over which high efficiencies could be obtained was greatly reduced at high speeds.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., May 7, 1945.

REFERENCES

1. Stack, John, Draley, Eugene C., Delano, James B., and Feldman, Lewis: Investigation of the NACA 4-(3)(08)-03 and NACA 4-(3)(08)-045 Two-Blade Propellers at Forward Mach Numbers to 0.725 to Determine the Effects of Compressibility and Solidity on Performance. NACA Rep. 999, 1950.
2. Stack, John: Tests of Airfoils Designed to Delay the Compressibility Burble. NACA Rep. 763, 1943.
3. Hartman, Edwin P., and Feldman, Lewis: Aerodynamic Problems in the Design of Efficient Propellers. NACA ACR, Aug. 1942.

REPORT 1013

EFFECTS OF WING FLEXIBILITY AND VARIABLE AIR LIFT UPON WING BENDING MOMENTS DURING LANDING IMPACTS OF A SMALL SEAPLANE ¹

By KENNETH F. MERTEN and EDGAR B. BECK

SUMMARY

A smooth-water-landing investigation was conducted with a small seaplane to obtain experimental wing-bending-moment time histories together with time histories of the various parameters necessary for the prediction of wing bending moments during hydrodynamic impact. The experimental results were compared with calculated results which include inertia-load effects and the effects of air-load variation during impact. The responses of the fundamental mode were calculated with the use of the measured hydrodynamic forcing functions. From these responses, the wing bending moments due to the hydrodynamic load were calculated according to the procedure given in R. & M. No. 2221. This comparison of the time histories of the experimental and calculated wing bending moments showed good agreement both in phase relationship of the oscillations and in numerical values.

The effects of structural flexibility on the wing bending moment were large, the dynamic component of the total moment being as much as 97 percent of the static component. Changes in the wing bending moment due to the variation in air load during impact were of about the same magnitude as the static water-load component.

INTRODUCTION

Recent trends in the design of aircraft have led to an important increase of the stresses produced in wings by landing impacts. Two significant factors contributing to these increased stresses during landing are an increased proportion of the airplane weight in the wings and an increased structural flexibility, since, in most cases, these factors have caused the ratios of the times to peak of the applied landing loads to the period of the fundamental mode to approach a critical value.

Several simplified methods have been developed for determining the inertia loads in wing structures during landing impacts, and studies have been made of the landing-impact inertia loads in simplified structures with the use of the principles of these methods. Although experimental investigations have been made to determine the magnitudes of inertia loads in actual airplane structures, little correlation of theory and experiment has been made concerning the nature and magnitude of inertia loads in airplane wings during actual landing impacts.

Another aspect of the problem of wing loads during landing is the variation of air load due to changes of attitude and flight path during impact. The importance of this change in air load has been the subject of some speculation but little investigation.

In order to evaluate the importance of the various components of the load, including dynamic effects and variation in air load, data were obtained during full-scale landing tests of a small seaplane to provide a comparison of actual wing loads with those predicted by a simplified method (reference 1).

The present report gives a comparison of the theoretical and experimental wing loads, in the form of time histories of the wing bending moments, and discusses the contributions of each of the components of the moment (static water-load moment, dynamic water-load moment, and air-load moment) to the total. The static and dynamic components of the total moment were calculated and combined according to the procedure of reference 1, with the responses of the fundamental mode being calculated from the recorded time histories of the applied forces. The air-load component was calculated by a simplified method which is described in this report.

SYMBOLS

C_L	lift coefficient $\left(\frac{2L}{\rho S V^2}\right)$
g	acceleration due to gravity (32.2 ft/sec ²)
L	lift, pounds
M	bending moment in wing, pound-inches
n	load factor, multiples of g
S	wing surface area, square feet
T	duration of impact, seconds
t_i	time to peak of applied load, seconds
t_n	natural period of fundamental mode, seconds
u	dynamic response factor, ratio of maximum total water-load wing bending moment to maximum static water-load bending moment
V	velocity of seaplane, feet per second
α	angle of attack, degrees
γ	flight-path angle, degrees
Δ	prefix denoting change
ρ	density, slugs per cubic foot
τ	trim angle, degrees

¹ Supersedes NACA TN 2063, "A Comparison of Theoretical and Experimental Wing Bending Moments during Seaplane Landings" by Kenneth F. Merten, José L. Rodríguez, and Edgar B. Beck, 1950.

Subscripts:

<i>c</i>	corrected for air load
<i>h</i>	horizontal
<i>n</i>	normal to keel
<i>0</i>	at time of water contact
<i>p</i>	parallel to keel
<i>r</i>	recorded
<i>T</i>	total
<i>v</i>	vertical
<i>max</i>	maximum

APPARATUS AND INSTRUMENTATION

The airplane used in the present investigation was a small two-engine seaplane (fig. 1). Pertinent information about the seaplane is given in table I, and additional information may be obtained from reference 2. The frequency and shape of the fundamental wing bending mode were found from ground vibration tests and are given in table II and figure 2. The spanwise weight distribution is also given in table II.

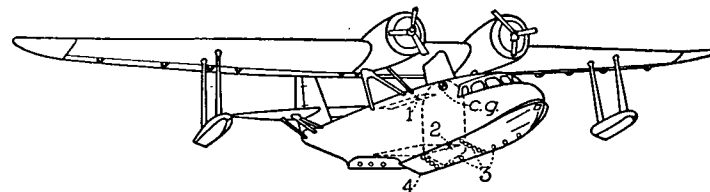
The trim variation was measured with a gyroscopic trim recorder mounted in the cabin floor. The airspeed was measured with an NACA airspeed recorder, pitot-static-tube type, mounted above the cabin. Accelerations of the center of gravity were obtained from an NACA optical-recording three-component accelerometer mounted securely in the fuselage near the center of gravity. The time of contact was determined from a water-contact indicator located on the keel at the main step. The hull immersions were determined from pressure gages installed along the bottom of the hull. The wing bending moments were measured by means of a strain gage mounted on the wing main spar 9 inches from the center line of the seaplane (hereinafter referred to as station 9).

The estimated accuracies of the experimental data based on calibration, instrument, and reading error are as follows:

Horizontal velocity, V_h , feet per second	± 4
Trim angle, τ , degrees	± 0.25
Load factor, n , multiples of g	± 0.2
Initial wing lift, L_0 , multiples of g	± 0.05
Total wing bending moment, M_T , pound-inches	$\pm 0.05 \times 10^6$

TEST PROCEDURE

The landing-impact tests were made in smooth water. During these landing tests, airspeed, trim variation, center-of-gravity accelerations, and wing-spar bending moments were recorded. The landings were made at horizontal velocities ranging from 95.4 to 112.0 feet per second, trim angles ranging from 3.00° to 7.83° , and initial flight-path angles ranging from 2.0° to 4.4° . The resulting maximum center-of-gravity accelerations normal to the keel line ranged from $1.10g$ to $1.96g$ and the duration of the impacts varied from 0.63 to 0.87 second. The times to peak of these normal accelerations ranged from 0.10 to 0.36 second. Values of these parameters and other pertinent information for all the tests are presented in table III.



1 NACA optical-recording three-component accelerometer
2 Gyroscopic trim recorder
3 Pressure gages and water contacts
4 Water-contact indicator

FIGURE 1.—Seaplane and instrumentation.

TABLE I
GENERAL INFORMATION ABOUT SEAPLANE USED IN LANDING-IMPACT INVESTIGATION

Approximate flying weight during tests, lb.	19,200
Stalling speed (flaps down), fps	94
Wing area, sq ft.	780.6
First natural frequency, cps	4.76
Second natural frequency, cps	13.0

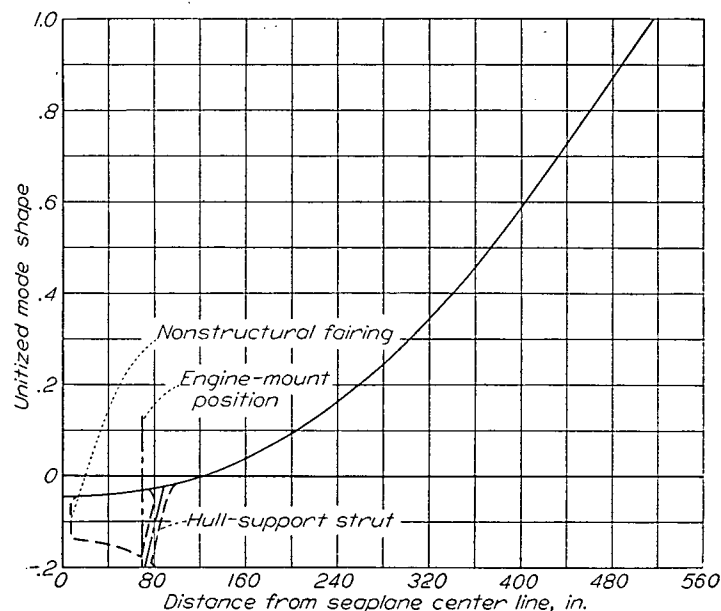


FIGURE 2.—Experimental fundamental bending-mode shape of wing semispan. Frequency, 4.76 cycles per second.

TABLE II
WEIGHT DISTRIBUTION AND UNITIZED MODE SHAPE FOR FUNDAMENTAL BENDING MODE (4.76 cps) OF WING SEMISPAN

Distance from center line (in.)	Associated station weight (lb)	Unitized mode shape
0	---	-0.045
31	881	-.044
75	2,057	-.026
87.7	5,076	-.022
119	881	-.004
170	116	.053
210	102	.110
250	88	.190
290	181	.270
330	64	.370
370	53	.490
410	43	.625
440	18	.730
477.7	40	.860
516	---	1.000
Total 9,600		

TABLE III
FLIGHT AND IMPACT VARIABLES

Run	V_{h_0} (fps)	$\Delta V_{h_{max}}$ (fps)	V_{v_0} (fps)	$\Delta V_{v_{max}}$ (fps)	$n_{n_{max}}$	$n_{n_{max}}$	$n_{p_{max}}$	$n_{v_{max}}$	$n_{h_{max}}$	T (sec)	$\frac{t_i}{L_0/4}$	ΔM_{max} (lb-in.)	u	τ_0 (deg)	$\Delta \tau_{max}$ (deg)	α_0 (deg)	$\Delta \alpha_{max}$ (deg)	$\Delta \gamma_{max}$ (deg)
1	103.0	4.91	5.1	8.37	1.96	2.03	0.264	1.90	0.508	0.69	3.72	0.575×10^5	1.47	7.83	1.30	10.66	5.20	4.7
2	95.4	3.55	3.5	7.29	1.21	1.26	.181	1.19	.259	.87	6.28	.250	1.23	6.01	5.07	8.11	2.10	4.2
3	100.5	3.46	7.2	11.29	1.61	1.69	.209	1.59	.340	.65	3.43	.410	1.53	5.72	5.21	9.82	4.36	6.5
4	112.0	3.83	8.6	13.64	1.90	1.94	.255	1.89	.355	.63	1.90	.483	1.97	3.00	9.31	7.39	4.45	7.0
5	105.9	3.54	4.4	8.10	1.16	1.24	.183	1.14	.285	.77	4.95	.345	1.42	6.07	4.28	8.45	3.29	4.4
6	104.1	3.42	3.6	6.90	1.10	1.19	.208	1.07	.310	.80	6.85	.335	1.13	7.39	3.30	9.34	2.67	3.8

THEORY

WILLIAMS' METHOD

In reference 1 a method was proposed for calculating the dynamic effect of an impulsive load applied to an elastic structure. Basically, the method follows classical normal-mode vibration theory by considering the total response of the elastic structure to a forcing function at any instant to be the summation of the responses of all of its normal modes at that instant. However, a unique feature of the method is that the total response of each mode is separated into a static and a dynamic component, and the stress due to the sum of the static components of the responses of all the modes is found in one calculation by rigid-body analysis. This stress is referred to as the static-load stress. The stress of each mode due to its dynamic component of response is found separately. The total stress is the sum of the static-load stress and the dynamic components of stress for the significant modes. Time histories of stress are found in these calculations and thus phase relationships of the modes are considered when the stresses for each mode are added.

AIR-LOAD VARIATION

In determining the effect of air-load variations on the wing bending moment during impact, the air load was assumed to change instantaneously with change in angle of attack and the rate of change in air load was assumed to be slow enough to neglect excitation of structural dynamic response. Also, the ratio of air-load bending moment to lift was assumed to be constant for all impacts. On the basis of these assumptions, the air-load moment at any time during the impact M was related to the lift L and the ratio of bending moment to lift at time of contact M_0/L_0 by the following equation:

$$M = \left(\frac{M_0}{L_0} \right) L$$

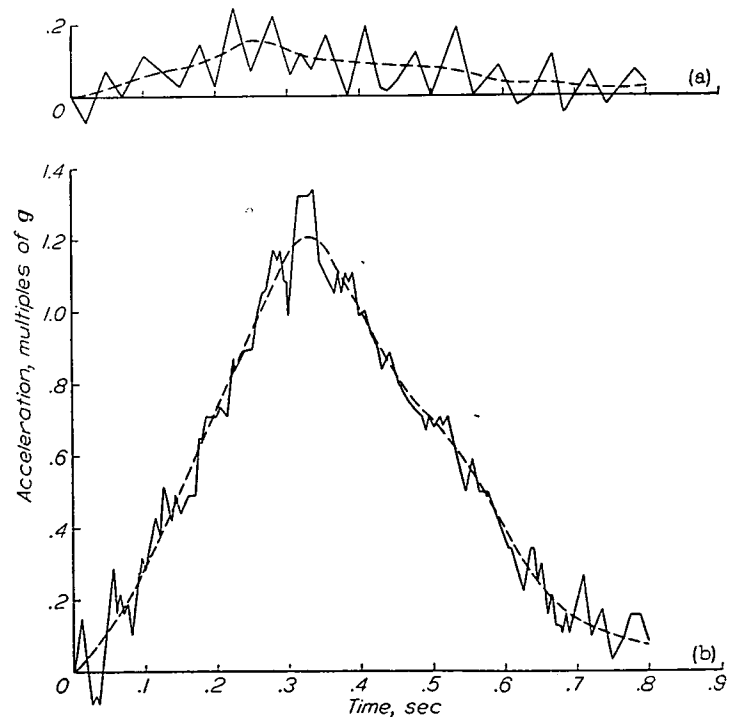
An average value of M_0/L_0 , obtained from the measured values of lift and moment at time of contact, was used in this equation. The variation of lift L was determined from time histories of the angle of attack and velocity in conjunction with a lift curve (C_L plotted against α) determined for the wing with flaps deflected. This lift curve was determined by fairing a line with the slope of the available lift curve without flaps deflected through the values of C_L computed for all the impacts from the measured velocity,

angle of attack, and center-of-gravity acceleration at time of contact. With the variation in total air-load moment thus determined, the change in air-load wing bending moment during impact ΔM was found from the following equation:

$$\Delta M = M_0 - M$$

CALCULATIONS AND RESULTS

The variations of the wing angle of attack and velocity during impact necessary for computing the changes in air-load bending moment were determined for each impact from the recorded data in the following manner. The accelerations normal and parallel to the keel line, obtained from the three-component center-of-gravity accelerometer, were plotted (fig. 3). After the trim-angle variation (fig. 4 (a)) was taken into account, these accelerations were resolved into vertical and horizontal components. Integration of the time histories of these accelerations over the duration of the impact produced time histories of the changes in vertical and horizontal velocities. Since the vertical velocity at the time of contact was not accurately known, the initial



(a) Acceleration parallel to seaplane keel line during impact.
(b) Acceleration normal to seaplane keel line during impact.
FIGURE 3.—Typical accelerometer records and fairings.

velocity was determined so that integration of the time-history curve over the duration of the impact resulted in a final vertical displacement of zero. (The duration of the impact is defined as the interval between the time of contact and the time when the center of gravity again reached its initial height above the mean water line. The instant of contact was found from a water-contact indicator on the step and the time history of the center-of-gravity displacement was determined from the times of immersion and emersion of the hull pressure gages, the fixed location of the center of gravity relative to the step, and trim-angle time history.) Integration of the time-history curve of the corrected vertical velocity from time of contact to the time of zero vertical velocity determined the maximum displacement of the center of gravity. The maximum displacements determined in this manner for all the impacts agreed within experimental error with the maximum displacements calculated from the hull pressure gages. With the use of the corrected vertical-velocity and horizontal-velocity time histories, time histories of the flight-path angle γ and the resultant velocity were computed. From the time histories of trim angle τ (fig. 4 (a)) and flight-path angle γ (fig. 4 (b)), the time history of the angle of attack α was computed (fig. 4 (c)).

With the use of the time histories of angle of attack and resultant velocity, the changes in bending moment in the wing at station 9 due to the changes in air load were determined for each impact in the manner described in the section entitled "Air-Load Variation" and are presented in parts (a) of figures 5 to 10.

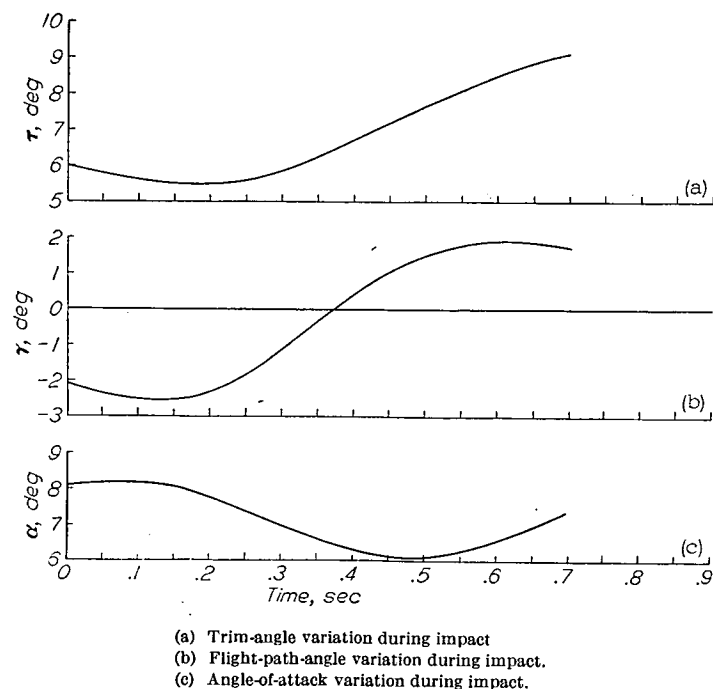


FIGURE 4.—Typical time histories of pitching motion based on recorded and calculated data.

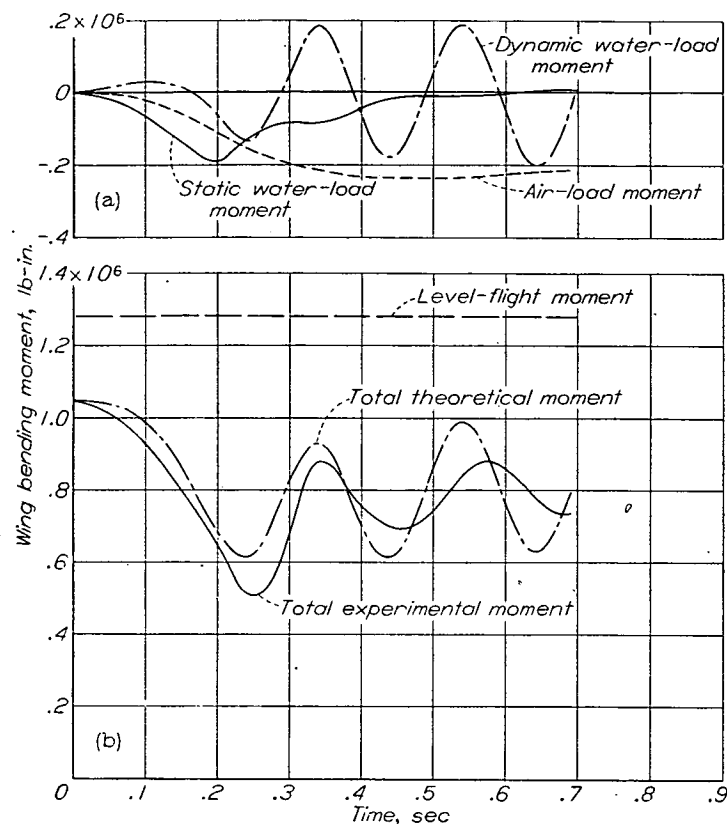


FIGURE 5.—Wing-bending-moment time histories during impact; run 1.

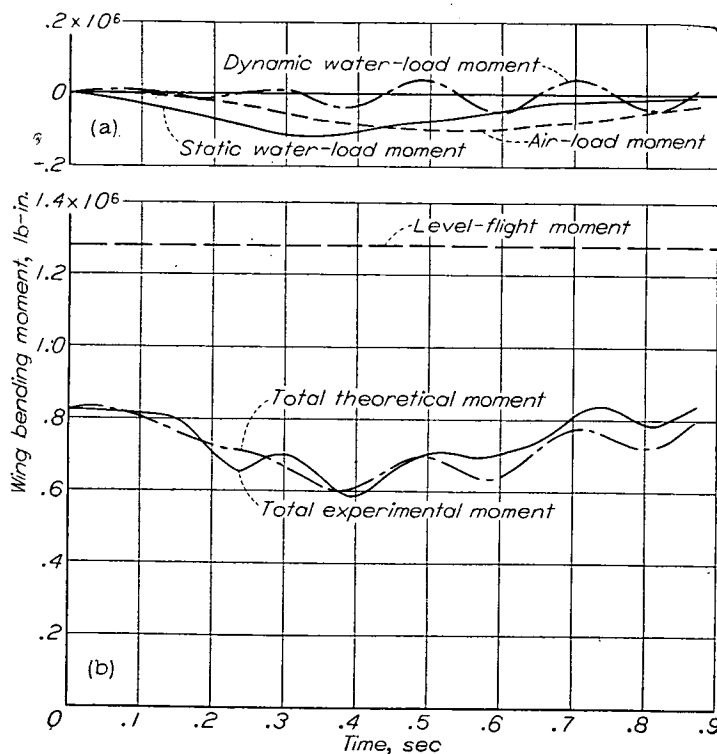
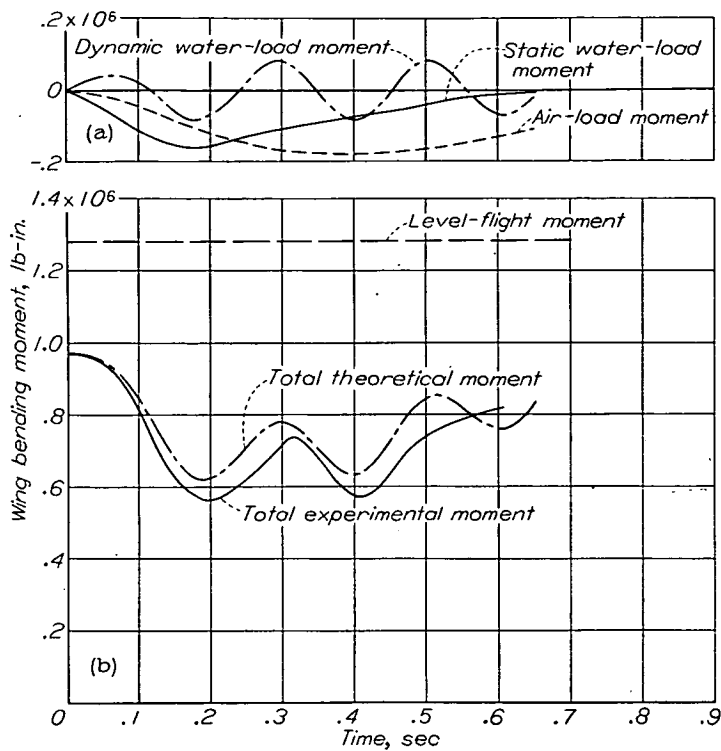
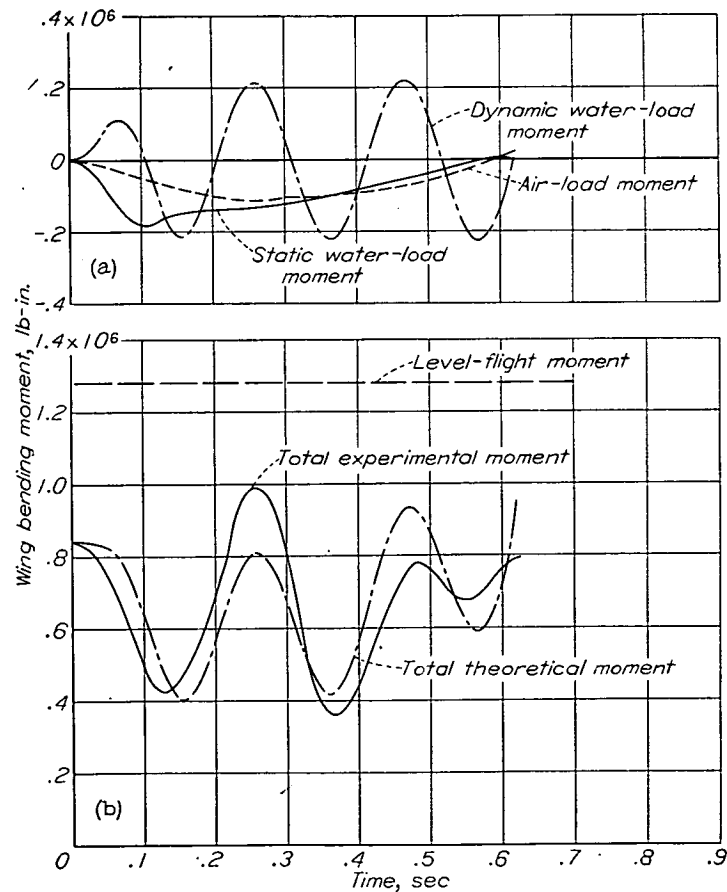


FIGURE 6.—Wing-bending-moment time histories during impact; run 2.



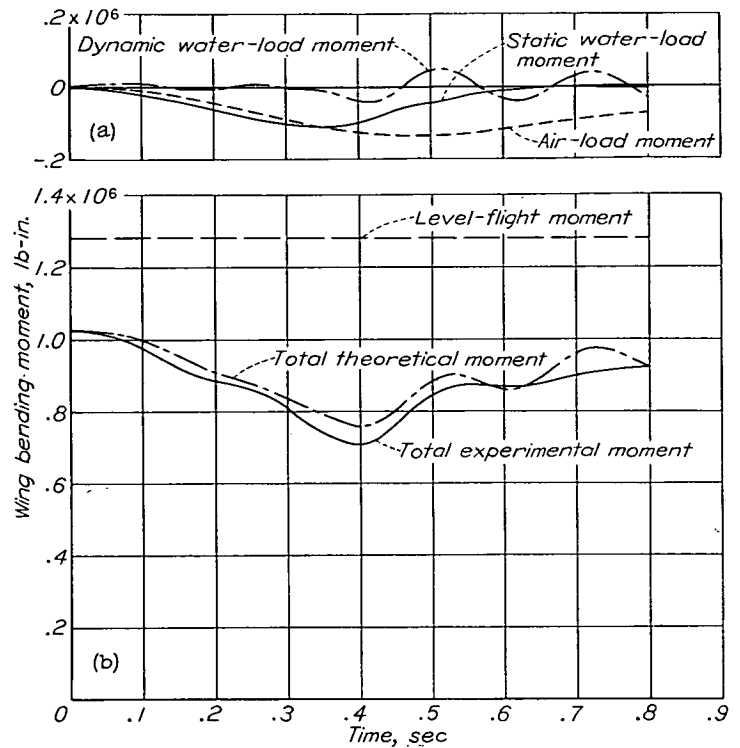
(a) Components of wing-bending-moment changes during impact.
(b) Total wing bending moment during impact.

FIGURE 7.—Wing-bending-moment time histories during impact; run 3.



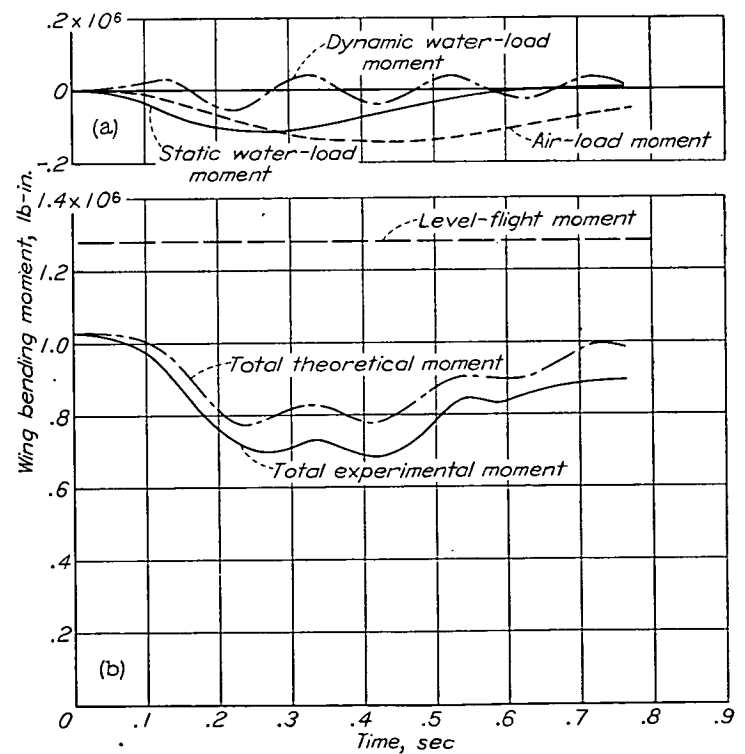
(a) Components of wing-bending-moment changes during impact.
(b) Total wing bending moment during impact.

FIGURE 8.—Wing-bending-moment time histories during impact; run 4.
213637—53—16



(a) Components of wing-bending-moment changes during impact.
(b) Total wing bending moment during impact.

FIGURE 9.—Wing-bending-moment time histories during impact; run 5.



(a) Components of wing-bending-moment changes during impact.
(b) Total wing bending moment during impact.

FIGURE 10.—Wing-bending-moment time histories during impact; run 6.

The procedure of reference 1 was used to compute the bending moments because it provides a convenient means of applying the principles essential to a dynamic-loads analysis which results in time histories of the wing bending moments. The forcing function for each impact was determined from the normal acceleration measured in the hull by an accelerometer located near the seaplane center of gravity. Because of the manner of connection between the wing and hull (fig. 1), the measured acceleration was not appreciably affected by the oscillations of the wing. By including the effect of the varying wing lift on the center-of-gravity acceleration, the acceleration normal to the keel line due to the hydrodynamic force only was determined. From this acceleration and the mass of the seaplane, the hydrodynamic forcing function was calculated. The dynamic responses of the significant modes to this forcing function were computed by a recurrence method developed at the Langley Structures Research Division in which the actual forcing function is used without approximation. Only the dynamic effects of the fundamental bending mode were included in the final results because calculations showed the dynamic effects of the second symmetrical bending mode to be negligible. This observation was borne out by the absence of higher mode effects on the strain-gage records. The calculated time histories of the static water-load and dynamic water-load components of wing bending moment at station 9 are presented in parts (a) of figures 5 to 10. The spanwise bending-moment distribution for the fundamental-mode $1g$ inertia loading calculated as set forth in reference 1, a $1g$ static water loading, and a level-flight loading are plotted in figure 11. The values of the bending moment at station 9 used in the application of the method of reference 1 were obtained from this plot.

These three components of wing-bending-moment changes obtained in this manner for station 9 were combined and added to the wing bending moment existing at the instant of contact. This total theoretical wing bending moment is presented in parts (b) of figures 5 to 10 together with the wing-bending-moment variation measured by the strain gage at station 9.

DISCUSSION

Comparisons between the total theoretical and experimental wing bending moments are presented in parts (b) of figures 5 to 10. The comparisons are made only for wing station 9 because the bending moments in the outer wing section were so small as to be of the same order as the estimated error. Only the dynamic effects of the fundamental bending mode were included in the final results because calculations showed the dynamic effects of the second symmetrical bending mode to be negligible. This negligibility was borne out by the absence of higher-mode effects on the strain-gage records. The comparisons show the predicted values to be in good agreement with the experimental values. As can be seen from the figures, the phase relationships between the theoretical and experimental

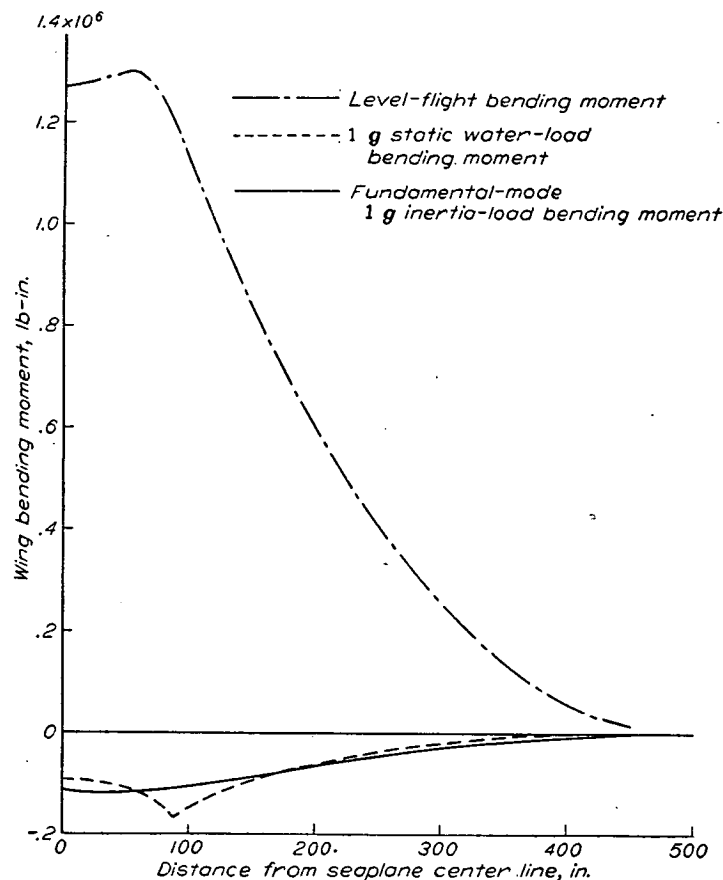


FIGURE 11.—Level-flight and landing-impact bending moments.

values are consistently good, and the maximum changes from initial conditions show a range of error of 5 to 28 percent based on the experimental values of the maximum changes in wing moment. These results indicate that when the three components of moment are included in the theory good agreement is obtained.

A comparison of the level-flight bending moment and the change in bending moment due to the heaviest impact of the tests shows that the maximum change in bending moment accompanying a downward motion of the wing was approximately 50 percent of the level-flight moment (figs. 11 and 5 (b)). This maximum bending-moment change was produced by a $2.03g$ impact. When any differences in the response factor and any change in air-load bending moment are neglected, landings of over $4g$ would be required for the downward motion of the wing merely to start stress reversal in the wing. Similarly, even if the maximum dynamic water-load bending moment (fig. 8 (a)) caused by a $1.94g$ impact were to be twice as large for a $4g$ impact, were to exist after the water load was removed, and were to be superposed on a level-flight moment, the moment produced by the upward motion of the wing would still be less than twice the level-flight moment. Therefore, the change in bending moment due to a landing impact is unimportant in this seaplane insofar as this change will not produce critical stresses at the wing root. This unimportance may

be largely attributed to the fact that the fundamental mode $1g$ inertia loading is relatively small as compared with a level-flight loading (fig. 11).

The effects of structural flexibility on the computed bending moments can be seen by comparing the static and dynamic components of the water-load bending moment in parts (a) of figures 5 to 10 and by observing the dynamic response factors u in table III. The dynamic overstress attributable to structural flexibility is the dynamic component of stress in parts (a) of figures 5 to 10 and is represented in the response factor u by the amount that this factor differs from unity. Since response factors as high as 1.97 are obtained, the dynamic overstress sometimes contributes an increment of stress almost as large as the static water-load stress. This observation is in agreement with the results of other investigators and shows the necessity of using dynamic analyses in landing-load investigations.

The change in wing bending moment due to change in air load on the wing is a function of the changes in velocity and angle of attack. (See the section entitled "Air-Load Variation.") In these tests the changes in velocity during the impacts were small (table III) and the changes in air-load bending moment were therefore almost entirely attributable to the changes in angle of attack. Since the angle of attack is a function of the trim and flight-path angles, large changes in angle of attack will occur when the trim and flight-path angles change to a great degree. For the relatively small changes in trim angle and flight-path angle which occurred in these tests (the maximum values being 9.31° and 7.0° , respectively), air-load changes as large as $0.2g$ were computed. For this airplane, these changes in air-load bending moment were of about the same magnitude as the bending-moment changes due to the static component of loading (parts (a) of figs. 5 to 10) and inclusion of the effects of air-load changes in the calculations was therefore necessary. For other airplanes with structural and mass characteristics conducive to large inertia-load moments, the bending-moment changes accompanying a $0.2g$ change in air load would be small relative to the changes in bending moment caused by inertia loads. However, for more severe changes in flight-path angle, which should be considered in a design analysis, the effects of the change in air load on the bending moment in the wing may still be large enough to warrant consideration in a design analysis. Further investigation is necessary to determine the importance of this air-load variation in design-strength calculations.

In most landing tests the applied forcing functions are not easily obtained from the center-of-gravity accelerations because of the superposed accelerations caused by structural oscillations. But because the fuselage of this airplane is connected to the wing by struts located near the nodal point of the fundamental mode which represented the greatest portion of the wing bending and by a nonstructural fairing which neither transmitted nor interfered with the wing oscillations, the accelerometer in the fuselage was not appreciably affected by the wing oscillations in these tests.

The calculation of the dynamic response of each of the normal modes of the seaplane involves solving for the response of an equivalent simple spring-mass system to the given forcing function. In this calculation it is common practice to approximate the forcing function in order to simplify the computation. However, the responses used in this report were computed from the actual forcing functions because trial calculations showed that errors as large as 20 percent in the total response of the simple spring-mass system could be introduced by use of apparently good simple approximations.

CONCLUSIONS

Experimental wing bending moments obtained from a landing-impact investigation with a small full-scale seaplane were compared with analytical results. The effects of the variation in air load during the impacts were included in the analytical procedure. The responses of the fundamental mode were calculated from the recorded time histories of the applied hydrodynamic forces. For the seaplane tested and the conditions of the impacts encountered, the following conclusions may be drawn:

1. Good agreement between measured and computed wing bending moments was obtained when the three components of the wing moment (static water-load moment, dynamic water-load moment, and air-load moment) were included in the calculations.
2. The effects of structural flexibility on the wing bending moments, represented by the dynamic overstress and understress, were large, the moment due to the dynamic component of the total response being as much as 97 percent of that caused by the static water-load component.
3. Although the changes in seaplane attitude during the landing impacts were small, the variation in the air-load component of the total moment was of about the same magnitude as the static water-load component. Although this comparison of changes is not representative of the relative importance of the air-load variation in seaplanes with structures conducive to large inertia-load components, it indicates the probable significance of the effects of air-load variation since large changes in seaplane attitude must also be considered.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., December 21, 1949.

REFERENCES

1. Williams, D., and Jones, R. P. N.: Dynamic Loads in Aeroplanes under Given Impulsive Loads with Particular Reference to Landing and Gust Loads on a Large Flying Boat. R. & M. No. 2221, British A.R.C., Sept. 1945.
2. Steiner, Margaret F.: Comparison of Over-All Impact Loads Obtained during Seaplane Landing Tests with Loads Predicted by Hydrodynamic Theory. NACA TN 1781, 1949.

REPORT 1014

STUDY OF EFFECTS OF SWEEP ON THE FLUTTER OF CANTILEVER WINGS¹

By J. G. BARMBY, H. J. CUNNINGHAM, and I. E. GARRICK

SUMMARY

An experimental and analytical investigation of the flutter of sweptback cantilever wings is reported. The experiments employed groups of wings swept back by rotating and by shearing. The angle of sweep ranged from 0° to 60° and Mach numbers extended to approximately 0.85. A theoretical analysis of the air forces on an oscillating swept wing of high length-chord ratio is developed, and the approximations inherent in the assumptions are discussed. Comparison with experiment indicates that the analysis developed in the present report is satisfactory for giving the main effects of sweep, at least for nearly uniform cantilever wings of high and moderate length-chord ratios. A separation of the effects of finite span and compressibility in their relation to sweep has not been made experimentally but some combined effects are given. A discussion of some of the experimental and theoretical trends is given with the aid of several tables and figures.

INTRODUCTION

The present report is an outgrowth of the trend toward the use of swept wings for high-speed flight and presents the results of an analysis and of an accompanying exploratory program of research in the Langley 4.5-foot flutter research tunnel on swept cantilever wings. The material was assembled in a memorandum form with a similar title in 1948. The chief purposes of the present report are to provide a more detailed exposition of the analysis and to make the main material more generally available.

Some previous experimental and analytical work on swept wings is mentioned here. A preliminary experimental investigation of the effect of sweep on flutter has been made (reference 1) with a single, simple rigid wing mounted flexibly on a base which could be rotated to various desired sweep angles. This investigation was made at low Mach numbers for two bending-torsion frequency ratios and at several angles of sweepback. Another investigation (data unpublished) in which the density of the test medium was a variable was conducted by D. Benun on the same type of rigid, flexibly mounted wing at higher Mach numbers and at sweep angles of 0° and 45°. Other unpublished work on swept wings exists, but a search of the available information indicates a need for further systematic study.

The experimental work reported herein dealt with models mounted as cantilevers at their roots. These cantilever models differed from the rigid, flexibly mounted wings, which had all bending and torsion flexibility concentrated at the root, and thus were subject to different root effects. In order to facilitate analysis the cantilever models were uniform and untapered. The intent of the experimental program was to establish trends and to indicate orders of magnitude of the various effects of sweep on flutter rather than to isolate precisely the separate effects.

The models were swept back in two basic manners—shearing and rotating. For the case in which the wings were swept back by shearing the cross sections parallel to the air stream, the span and aspect ratio remained constant. For the other case, a series of rectangular-plan-form wings were mounted on a special base which could be rotated to provide any desired angle of sweepback. This rotatory base was also used to examine the critical speed of swept-forward wings.

Tests were conducted also on special models that were of the “rotated” type (sections normal to the leading edge were the same at all sweep angles) with the difference that the bases were aligned parallel to the air stream. Two series of such rotated models having different lengths were tested.

Inasmuch as the location of the center of gravity, the mass-density ratio, and the Mach number have important effects on the flutter characteristics of unswept wings, these parameters were varied for swept wings. In order to investigate possible changes in flutter characteristics which might be due to different flow over the tips, various tip shapes were included in the experiments.

In an analysis of flutter, vibrational characteristics are very significant; accordingly, vibration tests were made on each model. A special study of the change in frequency and mode shape with angle of sweep was made for a simple aluminum-alloy beam and is reported in appendix A.

Theoretical analyses of the effect of sweep on flutter exist only in brief or preliminary forms. In England in 1942, W. J. Duncan estimated by certain dimensional considerations the effect of sweep on the flutter speed of certain specialized wing types. Among other British workers whose names are mentioned in connection with problems

¹ Supersedes NACA TN 2121, “Study of Effects of Sweep on the Flutter of Cantilever Wings” by J. G. Barmby, H. J. Cunningham, and I. E. Garrick, 1950.

of flutter involving sweep are R. McKinnon Wood, A. R. Collar, and I. T. Minhinnick. An account of Minhinnick's work was given by Broadbent in reference 2. In reference 3 a preliminary analysis for the flutter of swept wings in incompressible flow is developed on the basis of a "strip theory" (with the strips taken in the stream direction) and is applied to the experimental results of reference 1. Examination of the limiting case of infinite span discloses that the aerodynamic assumptions employed in reference 3 are not well-grounded. Reference 4 adapts this strip theory to flexible wings and also presents an alternative "velocity component" treatment employing other aerodynamic assumptions which in their end result appear more akin to those employed in the analysis of the present report. No definite choice is made in reference 4 between the two methods although the strip-theory method is favored.

In the present report a theoretical analysis is developed anew and given a general presentation. Application of the analysis has been limited at this time chiefly to those calculations needed for comparison with experimental results. A wider examination of the effect of various parameters and of additional degrees of freedom on the flutter characteristics is desirable.

SYMBOLS

b	half-chord of wing measured perpendicular to elastic axis, feet
b_r	half-chord perpendicular to elastic axis at reference station, feet
l'	effective length of wing, measured along elastic axis, feet
c	wing chord measured perpendicular to elastic axis, inches
l	length of wing measured along midchord line, inches
Λ	angle of sweep, positive for sweepback, degrees
A_g	geometric aspect ratio $\left(\frac{l \cos \Lambda}{lc}\right)^2$
x'	coordinate perpendicular to elastic axis in plane of wing, feet
y'	coordinate along elastic axis, feet
z'	coordinate in direction perpendicular to $x'y'$ -plane, feet
Z	coordinate of wing surface in z' -direction, feet
η	nondimensional coordinate along elastic axis (y'/l')
ξ	coordinate in wind-stream direction
h	bending deflection of elastic axis, positive downward, feet
θ	torsional deflection of elastic axis, positive with leading edge up, radians
σ	local bending slope of elastic axis $\left(\frac{\partial h}{\partial y'}\right)$
τ	local rate of change of twist $\left(\frac{\partial \theta}{\partial y'}\right)$
$f_h(y'), F_h(\eta)$	deflection function of wing in bending
$f_\theta(y'), F_\theta(\eta)$	deflection function of wing in torsion
t	time
ω	angular frequency of vibration, radians per second

ω_h	angular uncoupled bending frequency, radians per second
ω_a	angular uncoupled torsional frequency about elastic axis, radians per second
f_{h_1}	first bending natural frequency, cycles per second
f_{h_2}	second bending natural frequency, cycles per second
f_t	first torsion natural frequency, cycles per second
f_a	uncoupled first torsion frequency relative to elastic axis, cycles per second $\left(f_t \left[1 - \frac{(x_a/r_a)^2}{1 - (f_{h_1}/f_t)^2}\right]^{\frac{1}{2}}\right)$
f_e	experimental flutter frequency, cycles per second
f_R	reference flutter frequency, cycles per second
f_Λ	flutter frequency determined by analysis of present report, cycles per second
v	free-stream velocity, feet per second
v_e	experimental flutter speed, feet per second
v_n	component of air-stream velocity perpendicular to elastic axis, feet per second $(v \cos \Lambda)$
V_e	experimental flutter speed taken parallel to air stream, miles per hour
V_R	reference flutter speed, miles per hour
V_R'	reference flutter speed based on wing elastic axis, miles per hour (defined in appendix B)
V_Λ	flutter speed determined by theory of present report, miles per hour
V_D	theoretical divergence speed, miles per hour
k_n	reduced frequency employing velocity component perpendicular to elastic axis $(\omega b/v_n)$
φ	phase difference between wing bending and wing torsion strains, degrees
ρ	density of testing medium at flutter, slugs per cubic foot
q	dynamic pressure at flutter, pounds per square foot
M	Mach number at flutter
M_{cr}	critical Mach number
x_{cg}	distance of center of gravity behind leading edge taken perpendicular to elastic axis, percent chord
x_{ea}	distance of elastic center of wing cross section behind leading edge taken perpendicular to elastic axis, percent chord
x_{ea}'	distance of elastic axis of wing behind leading edge taken perpendicular to elastic axis, percent chord
a	nondimensional elastic-axis position $\left(\frac{2x_{ea}}{100} - 1\right)$
$a + x_a$	nondimensional center-of-gravity position $\left(\frac{2x_{cg}}{100} - 1\right)$
m	mass of wing per unit length, slugs per foot
κ	wing mass-density ratio at flutter $(\pi \rho b^2/m)$
I_a	mass moment of inertia of wing per unit length about elastic axis, slug-feet ² per foot

r_α	nondimensional radius of gyration of wing about elastic axis $\left(\sqrt{\frac{I_\alpha}{m b^2}}\right)$
EI	bending stiffness, pound-inches ² in tables, pound-feet ² in analysis
GJ	torsional stiffness, pound-inches ² in tables, pound-feet ² in analysis
g_h	structural damping coefficient for bending vibration
g_α	structural damping coefficient for torsional vibration
P	oscillatory lift per unit length, positive downward (defined in equation (6))
M_α	oscillatory moment about elastic axis, positive leading edge up (defined in equation (7))
[]	a special bracket used to identify terms which are due solely to inclusion of the last term in equation (5b)

In order to preserve continuity and to facilitate comparison with previous work on the unswept wing, the subscript α rather than θ is retained with certain quantities to refer to the torsional degree of freedom.

ANALYTICAL INVESTIGATION

GENERAL

Assumptions.—An attempt is first made to point out the main assumptions which seem to be applicable for swept wings of moderate taper and of high or moderate length-chord ratios.

(a) The assumptions, such as small disturbances and potential flow, commonly employed in linearized treatment of unswept wings in an ideal incompressible fluid are made.

(b) The structural behavior is such that over the main part of the wing the elastic axis may be considered straight. The wing is also considered sufficiently stiff at the root so that it behaves as if it were clamped normal to the elastic axis. An effective length l' needed for integration reasons may be defined (for example, as in fig. 1). The angle of sweepback is measured in the plane of the wing from the direction normal to the air stream to the elastic axis. All section parameters such as semichord, locations of elastic axis and center of gravity, radius of gyration, and so forth, are based on sections normal to the elastic axis.

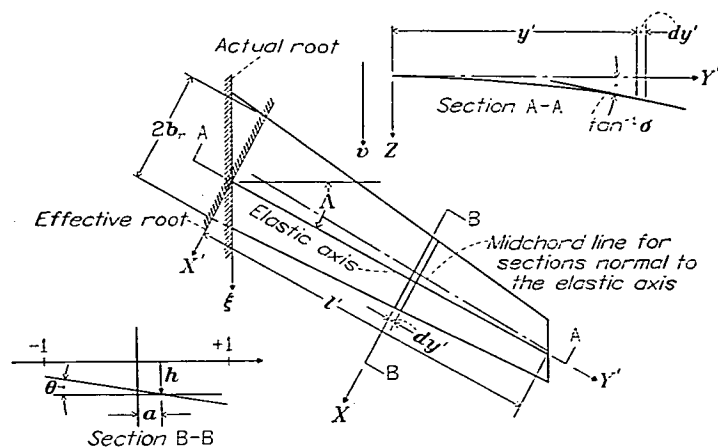


FIGURE 1.—Nonuniform swept wing treated in the present analysis.

(c) The aerodynamic behavior is such that any section dy' of the wing normal to the elastic axis, taken in the direction of the component $v \cos \Lambda$ of the main-stream velocity, generates a velocity potential associated with a uniform infinite swept wing having the same instantaneous distribution over the chord of velocity normal to the wing surface as does the actual section.

Additional remarks on these assumptions are appropriate. With regard to assumption (a), in accordance with linearization of the problem, the boundary conditions are stated and treated with respect to a reference surface, in this case a plane, containing the mean equilibrium position of the wing and the main-stream velocity. Furthermore, incompressible flow is assumed in order to avoid complexity of the analysis, although modifications due to Mach number effects can be added. Such modifications may be based, for example, for wings having large length-chord ratios, on existing theoretical calculations of aerodynamic coefficients for subsonic or supersonic two-dimensional flow appropriate to the component $v \cos \Lambda$. On the other hand the modifications may be partly empirical, especially for "transonic" conditions and for small length-chord ratios. The transonic conditions and the general aerodynamic behavior of swept wings may depend, for large length chord ratios, on the component $v \cos \Lambda$, but the dependence may shift to the stream velocity v for small length-chord ratios.

With respect to assumption (b), results of analyses of and experiment on unswept wings having low ratios of bending frequency to torsion frequency show that small variations of position of the elastic axis are not important. The assumption of a straight elastic axis over the main part of a swept wing, similarly, is not critical for many cases. This assumption is made for convenience, however, and modifications for a curved elastic axis can be made when necessary, for example, for plate-like wings. Small differences in the angle of sweepback of the leading edge, quarter-chord line, elastic axis, and so forth, are neglected. The analysis could be further modified to take into account variation of the angle of sweepback along the length of the wing.

Assumption (c) implies that associated with the action of the wing in pushing air downward there is a noncirculatory potential-type flow similar to that around sections of an infinite flat-plate wing. Furthermore, as in the case of the unswept airfoil, a circulatory potential-type flow is generated in which for the swept airfoil the component $v \cos \Lambda$ is decisive in fixing the circulation. (This assumption differs from that made in the strip theory of references 3 and 4 which employs the main-stream velocity together with sections of the wings parallel to the stream direction.) Effects of the floating of the wake in the stream direction rather than in the direction of $v \cos \Lambda$ and induced effects of variation of the strength of the wake in the wing-length direction are neglected, as are three-dimensional tip effects. For large values of the reduced frequency k_n , a given segment of the wing might be influenced chiefly by the nearby wake and the correction would be small. On the other hand, for small values of k_n a given segment might be influenced by a more widespread portion of the wake; corrections for this condition may possibly be based on knowledge of the static

case (for example, slope of the lift curve). As the angle of sweep approaches 90° , obviously the mechanism for the generation of lift is different from the one postulated here; for example, a tip condition may replace the trailing-edge condition and considerations of very small aspect ratio arise.

Basic considerations.—Consider the configuration shown in figure 1 where the vertical coordinate of the wing surface is denoted by $z' = Z(x', y', t)$ (positive downward). The effect of the position and motion of the wing may be given by the disturbance-velocity distribution to be superposed on the uniform stream in order to represent the condition of tangential flow at the wing surface. This velocity distribution normal to the surface (positive upward) is, for small disturbances,

$$w(x', y', t) = \frac{\partial Z}{\partial t} + v \frac{\partial Z}{\partial \xi} \quad (1a)$$

where ξ is the coordinate in the wind-stream direction. With the use of the relation

$$\begin{aligned} \frac{\partial Z}{\partial \xi} &= \frac{\partial Z}{\partial x'} \frac{\partial x'}{\partial \xi} + \frac{\partial Z}{\partial y'} \frac{\partial y'}{\partial \xi} \\ &= \frac{\partial Z}{\partial x'} \cos \Lambda + \frac{\partial Z}{\partial y'} \sin \Lambda \end{aligned}$$

the vertical velocity at any point is

$$w(x', y', t) = \frac{\partial Z}{\partial t} + v \frac{\partial Z}{\partial x'} \cos \Lambda + v \frac{\partial Z}{\partial y'} \sin \Lambda \quad (1b)$$

Let the wing be bending so that a segment dy' (see fig. 1) is displaced from its equilibrium position by an incremental distance h (positive down) and also let the wing segment be twisting about the elastic axis through an incremental angle θ (positive leading edge up). The position of each point of the segment may be defined, for small deflections, by

$$Z = h + x'\theta \quad (2)$$

The velocity distribution normal to the surface, equation (1b), consequently becomes

$$w = \dot{h} + x'\dot{\theta} + v\theta \cos \Lambda + v(\sigma + x'\tau) \sin \Lambda \quad (3)$$

where $\sigma = \frac{\partial h}{\partial y'}$ is the local bending slope of the elastic axis and is thus analogous to dihedral, and where $\tau = \frac{\partial \theta}{\partial y'}$ is the local change of twist of the elastic axis.

In accordance with assumption (c) the noncirculatory-flow velocity potentials associated with the vertical-velocity distribution are first needed. In equation (3) the terms involving $\dot{h}$, $\dot{\theta}$, and σ are constant across the chord, whereas those

involving θ and τ vary in a linear manner. The noncirculatory velocity potentials as in reference 5 and the new potentials associated with σ and τ are

$$\left. \begin{aligned} \phi_h &= \dot{h} b \sqrt{1-x^2} \\ \phi_\theta &= v_n \theta b \sqrt{1-x^2} \\ \phi_\sigma &= v_n \sigma \tan \Lambda b \sqrt{1-x^2} \\ \phi_\theta &= \theta b^2 \left(\frac{x}{2} - a \right) \sqrt{1-x^2} \\ \phi_\tau &= v_n \tau \tan \Lambda b^2 \left(\frac{x}{2} - a \right) \sqrt{1-x^2} \end{aligned} \right\} \quad (4)$$

where $v_n = v \cos \Lambda$ and x is the nondimensional chordwise coordinate measured from the midchord as in reference 5 and related to the coordinate x' in the manner

$$x = \frac{x'}{b} + a$$

The velocity potential for the circulatory flow associated with the wake may be developed on the basis of assumption (c) and the concepts for the infinite unswept wing introduced in reference 5. (Thus the circulatory-flow pattern for a section dy' of the finite swept wing is to be obtained from the corresponding flow pattern for an infinite uniform yawed wing. This infinite wing is assumed to have undergone harmonic oscillations for a long time; the full wake is established, remains where formed, and consequently is harmonically distributed in space. For the infinite uniform yawed wing, results for the circulatory flow are like those of reference 5 with v replaced by the component v_n and with the addition of terms to take care of σ and τ .) In particular, the strength of the wake acting on each section is determined by the condition of smooth flow (the velocity remaining finite) at the trailing edge. This condition is utilized in the form $\frac{\partial}{\partial x} (\phi_r + \phi_N)$

is equal to a finite quantity at the trailing edge (where ϕ_r is the velocity potential due to the vorticity in the wake, and ϕ_N is the total noncirculatory velocity potential), and this condition leads to a relation analogous to equation (VII) of reference 5 involving the basic quantity

$$Q = \dot{h} + v_n \dot{\theta} + v_n \sigma \tan \Lambda + b \left(\frac{1}{2} - a \right) (\dot{\theta} + v_n \tau \tan \Lambda)$$

which occurs in the terms associated with the wake. The net result of these considerations is that the circulatory-flow velocity potential may be regarded as determined.

The pressure difference between upper and lower surfaces of the wing at a point x is (positive downward)

$$\begin{aligned} p &= -2\rho \left(\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial \xi} \right) \\ &= -2\rho \left(\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x'} \cos \Lambda + v \frac{\partial \phi}{\partial y'} \sin \Lambda \right) \end{aligned} \quad (5a)$$

where ϕ is in general the total potential (the sum of circulatory-flow and noncirculatory-flow potentials). The last term in equation (5a) is the product of the component of main-stream velocity taken along the wing and the lengthwise change in the velocity potential and is often neglected even in steady-flow work. The question of the retention or neglect of this last term seems partly dependent on the order in which the approximations are introduced—specifically, whether velocity potentials for the whole flow pattern are found and then the integrated forces are determined or whether section forces are first determined and then integrated. It seems appropriate to retain at least the noncirculatory part ϕ_N of ϕ in the last term of equation (5a). In view, however, of the nature of the approximate treatment of the circulatory potential and of the inherent shortcomings of a strip analysis, in particular the neglect of lengthwise variations in wake vortex strength, complicating the results by also including ϕ_r in this term does not appear worth while. (This neglect of ϕ_r and retention of ϕ_N is realized to involve some inconsistencies in that account may not be taken of other higher order terms associated with lengthwise variation of the wing wake, which may be of the same order as terms retained.) Thus equation (5a) becomes

$$p = -2\rho \left(\frac{\partial \phi}{\partial t} + v \frac{\partial \phi}{\partial x'} \cos \Lambda + v \frac{\partial \phi_N}{\partial y'} \sin \Lambda \right) \quad (5b)$$

For harmonic motion in each degree of freedom, relations for the pressure may be integrated over the chord to yield expressions for the air forces and moments. For the sake of separating and identifying the terms in force and moment expressions which are due solely to the inclusion of the last term in equation (5b), a special bracket [] is employed. Thus these terms may be readily omitted. Numerical checks among the calculations made for the present report showed the effect of inclusion of the last term in equation (5b) on the calculated results to be quite small, even for 60° of sweepback within the range of other parameters investigated.

The expressions for the aerodynamic lift (positive down) and for the moment about the elastic axis (positive leading edge up), each per unit length of the wing, are as follows:

$$\begin{aligned} P &= -2\pi\rho v_n b C \left[\dot{h} + v_n \dot{\theta} + v_n \dot{\sigma} \tan \Lambda + b \left(\frac{1}{2} - a \right) (\dot{\theta} + v_n \dot{\tau} \tan \Lambda) \right] - \\ &\quad \pi\rho b^2 \left[\ddot{h} + v_n \ddot{\theta} + v_n \ddot{\sigma} \tan \Lambda + \{ v_n \ddot{\sigma} \tan \Lambda + v_n^2 \dot{\tau} \tan \Lambda + v_n^2 \frac{\partial \sigma}{\partial y'} \tan^2 \Lambda \} \right] + \\ &\quad \pi\rho b^3 a \left[\ddot{\theta} + v_n \ddot{\tau} \tan \Lambda + \left\{ v_n \ddot{\tau} \tan \Lambda + v_n^2 \frac{\partial \tau}{\partial y'} \tan^2 \Lambda \right\} \right] \end{aligned} \quad (6)$$

$$\begin{aligned} M_a &= 2\pi\rho v_n b^2 \left(\frac{1}{2} + a \right) C \left[\dot{h} + v_n \dot{\theta} + v_n \dot{\sigma} \tan \Lambda + b \left(\frac{1}{2} - a \right) (\dot{\theta} + v_n \dot{\tau} \tan \Lambda) \right] - \\ &\quad \pi\rho v_n b^3 \left[\left(\frac{1}{2} - a \right) \ddot{\theta} + \frac{1}{2} v_n \ddot{\tau} \tan \Lambda \right] + \pi\rho b^3 a \left[\ddot{h} + v_n \ddot{\sigma} \tan \Lambda + \right. \\ &\quad \left. \left\{ v_n \ddot{\sigma} \tan \Lambda + v_n^2 \dot{\tau} \tan \Lambda + v_n^2 \frac{\partial \sigma}{\partial y'} \tan^2 \Lambda \right\} \right] - \pi\rho b^4 \left(\frac{1}{8} + a^2 \right) \left[\ddot{\theta} + \right. \\ &\quad \left. v_n \ddot{\tau} \tan \Lambda + \left\{ v_n \ddot{\tau} \tan \Lambda + v_n^2 \frac{\partial \tau}{\partial y'} \tan^2 \Lambda \right\} \right] \end{aligned} \quad (7)$$

where

$$C = C(k_n) = F(k_n) + iG(k_n)$$

is the function associated with the wake developed by Theodorsen in reference 5; the reduced frequency parameter k_n is defined by

$$k_n = \frac{\omega b}{v_n} = \frac{\omega b}{v \cos \Lambda} \quad (8)$$

As has already been stated, the foregoing expressions were developed and apply for steady sinusoidal oscillations,

$$\left. \begin{aligned} h &= h_1(y') e^{i\omega t} \\ \theta &= \theta_1(y') e^{i\omega t} \end{aligned} \right\} \quad (9)$$

The amplitude, velocity, and acceleration in each degree of freedom are related as in the degree of freedom h ; that is,

$$\dot{h} = i\omega h$$

$$\ddot{h} = -\omega^2 h$$

Expressions for force and moment.—With the use of such relations, equations (6) and (7) may be put into the form

$$P = -\pi \rho b^3 \omega^2 (h B_{ch} + \theta B_{c\theta}) \quad (10a)$$

$$M_a = -\pi \rho b^4 \omega^2 (h B_{ah} + \theta B_{a\theta}) \quad (11a)$$

where

$$B_{ch} = \frac{1}{b} A_{ch} + \frac{\sigma}{h} \tan \Lambda \left(-i \frac{1}{k_n} \right) ([-1] + A_{ch}) + \left\{ \frac{b}{h} \frac{\partial \sigma}{\partial y'} \tan^2 \Lambda \left(\frac{1}{k_n^2} \right) \right\}$$

$$B_{c\theta} = A_{c\alpha} + \frac{\tau}{\theta} b \tan \Lambda (A_{c\tau}) + \left\{ \frac{b^2}{\theta} \frac{\partial \tau}{\partial y'} \tan^2 \Lambda \left(-\frac{a}{k_n^2} \right) \right\}$$

$$B_{ah} = \frac{1}{b} A_{ah} + \frac{\sigma}{h} \tan \Lambda \left(-i \frac{1}{k_n} \right) ([a] + A_{ah}) + \left\{ \frac{b}{h} \frac{\partial \sigma}{\partial y'} \tan^2 \Lambda \left(-\frac{a}{k_n^2} \right) \right\}$$

$$B_{a\theta} = A_{a\alpha} + \frac{\tau}{\theta} b \tan \Lambda (A_{a\tau}) + \left\{ \frac{b^2}{\theta} \frac{\partial \tau}{\partial y'} \tan^2 \Lambda \left(\frac{1}{8} + a^2 \right) \frac{1}{k_n^2} \right\}$$

in which the four following coefficients:

$$A_{ch} = -1 - \frac{2G}{k_n} + i \frac{2F}{k_n}$$

$$A_{c\alpha} = a + \frac{2F}{k_n^2} - \left(\frac{1}{2} - a \right) \frac{2G}{k_n} + i \left[\frac{1}{k_n} + \frac{2G}{k_n^2} + \left(\frac{1}{2} - a \right) \frac{2F}{k_n} \right]$$

$$\begin{aligned} A_{ah} &= a + \left(\frac{1}{2} + a \right) \frac{2G}{k_n} + i \left(\frac{1}{2} + a \right) \left(-\frac{2F}{k_n} \right) \\ &= -\frac{1}{2} - \left(\frac{1}{2} + a \right) A_{ch} \end{aligned}$$

$$\begin{aligned} A_{a\alpha} &= -\frac{1}{8} - a^2 - \left(\frac{1}{2} + a \right) \frac{2F}{k_n^2} + \left(\frac{1}{4} - a^2 \right) \frac{2G}{k_n} + \\ &+ i \left[\left(\frac{1}{2} - a \right) \frac{1}{k_n} - \left(\frac{1}{4} - a^2 \right) \frac{2F}{k_n} - \left(\frac{1}{2} + a \right) \frac{2G}{k_n^2} \right] \end{aligned}$$

are identical with those used in the case of the unswept wing. Additionally,

$$A_{c\tau} = \left\{ \frac{1}{k_n^2} \right\} - i \frac{1}{k_n} \left[\frac{1}{2} + [a] + \left(\frac{1}{2} - a \right) A_{ch} \right]$$

$$A_{a\tau} = -i \frac{1}{k_n} \left[-\left(\frac{3}{8} + \left\{ \frac{1}{8} + a^2 \right\} \right) + i \frac{1}{k_n} \left(\frac{1}{2} - [a] \right) - \left(\frac{1}{4} - a^2 \right) A_{ch} \right]$$

It is of interest to note that equations (6) and (7) reduce, for the case of the wing in steady flow ($k_n = 0$), to

$$\begin{aligned} P &= -2\pi \rho b v_n^2 \left[\theta + \sigma \tan \Lambda + \tau b \tan \Lambda \left(\left\{ \frac{1}{2} \right\} + \frac{1}{2} - a \right) + \right. \\ &\quad \left. \left\{ \frac{b}{2} \frac{\partial \sigma}{\partial y'} \tan^2 \Lambda - \frac{a}{2} b^2 \frac{\partial \tau}{\partial y'} \tan^2 \Lambda \right\} \right] \quad (10b) \end{aligned}$$

$$\begin{aligned} M_a &= 2\pi \rho b^2 v_n^2 \left[(\theta + \sigma \tan \Lambda) \left(\frac{1}{2} + a \right) + \tau a b \tan \Lambda \left(\left\{ \frac{1}{2} \right\} - a \right) + \right. \\ &\quad \left. \left\{ \frac{a b}{2} \frac{\partial \sigma}{\partial y'} \tan^2 \Lambda - \frac{1}{2} b^2 \left(\frac{1}{8} + a^2 \right) \frac{\partial \tau}{\partial y'} \tan^2 \Lambda \right\} \right] \quad (11b) \end{aligned}$$

per unit length of wing.

Introduction of modes.—Equations (10a) and (11a) give the total aerodynamic force and moment on a segment of a sweptback wing oscillating in a simple harmonic manner. Relations for mechanical equilibrium applicable to a wing segment may be set up, but it is preferable to bring in directly the three-dimensional-mode considerations. (See for example, reference 6.) This end may be readily accomplished by the combined use of Rayleigh type approximations and the classical methods of Lagrange. The vibrations at flutter are assumed to consist of a combination of fixed mode shapes, each mode shape representing a degree of freedom associated with a generalized coordinate. The total mechanical energy, the potential energy, and the work done by applied forces, aerodynamic and structural, are then obtained by the integration of the section characteristics over the span. The Rayleigh type approximation enters in the representation of the potential energy in terms of the uncoupled frequencies.

As is customary, the modes are introduced into the problem as varying sinusoidally with time. For the purpose of simplicity of analysis, one bending degree of freedom and one torsion degree of freedom are carried through in the present development. Actually, any number of degrees of freedom may be added if desired, exactly as with an unswept wing. Let the mode shapes be represented by

$$\left. \begin{aligned} h &= [f_h(y')] \underline{h} \\ \theta &= [f_\theta(y')] \underline{\theta} \end{aligned} \right\} \quad (12)$$

where $h = h_o e^{i\omega t}$ is the generalized coordinate in the bending degree of freedom, and $\theta = \theta_o e^{i\omega t}$ is the generalized coordinate in the torsion degree of freedom. (In a more general treatment the mode shapes must be solved, but in this procedure $f_h(y')$ and $f_\theta(y')$ are chosen, ordinarily as real functions of y' . Complex functions may be used to represent twisted modes.) The constants h_o and θ_o are in general complex and thus signify the phase difference between the two degrees of freedom.

In the subsequent treatment the reader will notice that in some expressions, namely for force and moment, h and θ can conveniently and logically be retained in their complex form. In other expressions, notably for energy, one is forced to utilize h and θ as real quantities. Appropriate statements will be made where necessary.

For each degree of freedom an equation of equilibrium may be obtained from Lagrange's equation

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = Q_i \quad (13)$$

where q_i is a generalized coordinate and Q_i is the corresponding generalized force. The kinetic energy of the mechanical system is

$$T = \frac{1}{2} \dot{h}^2 \int_0^{l'} m [f_h(y')]^2 dy' + \frac{1}{2} \dot{\theta}^2 \int_0^{l'} I_a [f_\theta(y')]^2 dy' + \dot{h} \dot{\theta} \int_0^{l'} m x_a b [f_h(y')] [f_\theta(y')] dy' \quad (14)$$

where h and θ here and in the subsequent equations (15) are to be interpreted as real in order that the energy be always positive (or zero), and for definiteness can be regarded as the real parts of $h_0 e^{i\omega t}$ and $\theta_0 e^{i\omega t}$ respectively; and where

m mass of wing per unit length, slugs per foot
 I_a mass moment of inertia of wing about its elastic axis per unit length, slug-feet² per foot
 $x_a b$ distance of sectional center of gravity from the elastic axis, positive rearward, feet

The potential energy of the mechanical system may be expressed in a form not involving bending-torsion cross-stiffness terms:

$$U = \frac{1}{2} h^2 \int_0^{l'} EI \left(\frac{d^2 f_h}{dy'^2} \right)^2 dy' + \frac{1}{2} \theta^2 \int_0^{l'} GJ \left(\frac{df_\theta}{dy'} \right)^2 dy' \quad (15a)$$

where

EI bending stiffness, pound-feet²
 GJ torsional stiffness, pound-feet²

If Rayleigh type approximations are used to introduce frequency, the expression for the potential energy may be written in a more convenient form:

$$U = \frac{1}{2} \omega_h^2 h^2 \int_0^{l'} m f_h^2 dy' + \frac{1}{2} \omega_a^2 \theta^2 \int_0^{l'} I_a f_\theta^2 dy' \quad (15b)$$

Another expression for the potential energy is

$$U = \frac{1}{2} h^2 \int_0^{l'} C_h f_h^2 dy' + \frac{1}{2} \theta^2 \int_0^{l'} C_a f_\theta^2 dy' \quad (15c)$$

The effective spring constants C_h and C_a correspond to unit length of wing and thus conform to their use in references 5 to 7. The constants are effectively defined by

$$\omega_h^2 = \frac{\int_0^{l'} C_h f_h^2 dy'}{\int_0^{l'} m f_h^2 dy'} \quad \omega_a^2 = \frac{\int_0^{l'} C_a f_\theta^2 dy'}{\int_0^{l'} I_a f_\theta^2 dy'}$$

These effective spring constants are related to the frequencies associated with the chosen modes. For so-called uncoupled

modes the frequencies appropriate to pure modes (obtained by proper constraints) are often used. On the other hand, employment of the normal or natural modes and frequencies appropriate to them, which might be obtained by proper ground test or by calculation, may be preferred. In either case the convenience of not having cross-stiffness terms in the potential-energy expression is noted.

Application is now made to obtain the equation of equilibrium in the bending degree of freedom. Equation (13) becomes

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{h}} \right) - \frac{\partial T}{\partial h} + \frac{\partial U}{\partial h} = Q_h \quad (16)$$

The term Q_h represents all the bending forces not derivable from the potential-energy function and consists of the aerodynamic forces together with the structural damping forces. The virtual work δW done on the wing by these forces as the wing moves through the virtual displacements δh and $\delta \theta$ is

$$\begin{aligned} \delta W &= \int_0^{l'} \left[\left(P - C_h \frac{g_h}{\omega} \dot{h} \right) \delta h + \left(M_a - C_a \frac{g_a}{\omega} \dot{\theta} \right) \delta \theta \right] dy' \\ &= \int_0^{l'} \left(P - m \omega_h^2 \frac{g_h}{\omega} f_h \dot{h} \right) f_h dy' \delta h + \\ &\quad \int_0^{l'} \left(M_a - I_a \omega_a^2 \frac{g_a}{\omega} f_\theta \dot{\theta} \right) f_\theta dy' \delta \theta = Q_h \delta h + Q_\theta \delta \theta \end{aligned} \quad (17)$$

where

g_h structural damping coefficient for bending vibration
 g_a structural damping coefficient for torsional vibration

In this expression the aerodynamic forces appropriate to sinusoidal oscillations are used. The application of the structural damping as in equation (17) (proportional to deflection and in phase with velocity) corresponds to the manner in which it is introduced in reference 7. In accordance with the preceding development, the aerodynamic and structural damping forces and moments in equation (17) are regarded as complex, but the virtual displacements δh and $\delta \theta$ should be considered real. Thus, the physically significant part of the resulting expression for virtual work is the real part. Since the subsequent analysis reverts to expressions for forces and moments, no further qualifications on the use of h and θ in their convenient complex forms are needed.

For the half-wing

$$\begin{aligned} Q_h &= \int_0^{l'} \left(P - m \omega_h^2 \frac{g_h}{\omega} f_h \dot{h} \right) f_h dy' \\ &= -\pi \rho b r^3 \omega^2 \int_0^{l'} \left(\frac{b}{b_r} \right)^3 \left[\frac{h}{b} A_{ch} f_h^2 + \frac{h}{b} \left(-i \frac{1}{k_n} \right) ([-1] + \right. \\ &\quad \left. A_{ch}) (\tan \Lambda) f_h \frac{df_h}{dy'} + \left\{ \frac{h}{b} \frac{1}{k_n^2} b (\tan^2 \Lambda) f_h \frac{d^2 f_h}{dy'^2} \right\} + \right. \\ &\quad \left. \theta A_{ca} f_\theta f_h + \theta A_{ca} b (\tan \Lambda) f_h \frac{df_\theta}{dy'} + \right. \\ &\quad \left. \left\{ \theta \left(-\frac{a}{k_n^2} \right) b^2 (\tan^2 \Lambda) f_h \frac{d^2 f_\theta}{dy'^2} \right\} + \frac{h}{b} \frac{1}{k_n} i \left(\frac{\omega_h}{\omega} \right)^2 g_h f_h^2 \right] dy' \end{aligned} \quad (18)$$

where b_r is the semichord at some reference section. Performance of the operations indicated in equation (16) and collection of terms lead to the equation of equilibrium in the bending degree of freedom

$$\begin{aligned} & \left(\underline{h} \left[1 - \left(\frac{\omega}{\omega_n} \right)^2 (1 + i g_n) \right] \frac{l'}{b_r} \int_0^{l'} \left(\frac{b}{b_r} \right)^2 \frac{1}{\kappa} f_h^2 dy' - \frac{1}{b_r} \int_0^{l'} \left(\frac{b}{b_r} \right)^2 A_{ch} f_h^2 dy' + \int_0^{l'} \left(\frac{b}{b_r} \right)^3 \tan \Lambda \left(i \frac{1}{k_n} \right) ([1] + A_{ch}) f_h \frac{df_h}{dy'} dy' - \left[b_r \int_0^{l'} \left(\frac{b}{b_r} \right)^4 \tan^2 \Lambda \left(\frac{1}{k_n^2} \right) f_h \frac{d^2 f_h}{dy'^2} dy' \right] \right) + \underline{\theta} \left\{ \int_0^{l'} \left(\frac{b}{b_r} \right)^3 \left(\frac{x_\alpha}{\kappa} - A_{ca} \right) f_h f_\theta dy' - b_r \int_0^{l'} \left(\frac{b}{b_r} \right)^4 A_{cr} (\tan \Lambda) f_h \frac{df_\theta}{dy'} dy' + \left[b_r^2 \int_0^{l'} \left(\frac{b}{b_r} \right)^5 \frac{a}{k_n^2} \tan^2 \Lambda f_h \frac{d^2 f_\theta}{dy'^2} dy' \right] \right\} \pi \rho b_r^3 \omega^2 = 0 \end{aligned} \quad (19a)$$

where

$$\frac{1}{\kappa} = \frac{m}{\pi \rho b^2}$$

By a parallel development the equation of equilibrium for the torsional degree of freedom may also be obtained as follows:

$$\begin{aligned} & \left(\underline{h} \left\{ \frac{1}{b_r} \int_0^{l'} \left(\frac{b}{b_r} \right)^3 \left(\frac{x_\alpha}{\kappa} - A_{ah} \right) f_h f_\theta dy' + \int_0^{l'} \left(\frac{b}{b_r} \right)^4 \tan \Lambda \left(i \frac{1}{k_n} \right) ([a] + A_{ah}) f_\theta \frac{df_h}{dy'} dy' + \left[b_r \int_0^{l'} \left(\frac{b}{b_r} \right)^5 \tan^2 \Lambda \left(\frac{a}{k_n^2} \right) f_\theta \frac{d^2 f_h}{dy'^2} dy' \right] \right\} + \underline{\theta} \left\{ \left[1 - \left(\frac{\omega}{\omega_n} \right)^2 (1 + i g_n) \right] \int_0^{l'} \left(\frac{b}{b_r} \right)^4 \frac{r_\alpha^2}{\kappa} f_\theta^2 dy' - \int_0^{l'} \left(\frac{b}{b_r} \right)^4 A_{aa} f_\theta^2 dy' - b_r \int_0^{l'} \left(\frac{b}{b_r} \right)^5 \tan \Lambda (A_{ar}) f_\theta \frac{df_\theta}{dy'} dy' - \left[b_r^2 \int_0^{l'} \left(\frac{b}{b_r} \right)^6 \tan^2 \Lambda \left(\frac{1}{8} + a^2 \right) \frac{1}{k_n^2} f_\theta \frac{d^2 f_\theta}{dy'^2} dy' \right] \right\} \right) \pi \rho b_r^4 \omega^2 = 0 \end{aligned} \quad (20a)$$

where $r_\alpha = \sqrt{I_\alpha / m b^2}$ (radius of gyration of wing about the elastic axis).

Determinantal equation for flutter.—Equations (19a) and (20a) may be rewritten with the use of the nondimensional coordinate $\eta = \frac{y'}{l'}$. They then are in the form

$$(\underline{h} A_2 + \underline{\theta} B_2) \pi \rho b_r^3 \omega^2 = 0 \quad (19b)$$

$$(\underline{h} D_2 + \underline{\theta} E_2) \pi \rho b_r^4 \omega^2 = 0 \quad (20b)$$

where

$$\begin{aligned} A_2 &= \left[1 - \left(\frac{\omega}{\omega_n} \right)^2 (1 + i g_n) \right] \frac{l'}{b_r} \int_0^{1.0} \left(\frac{b}{b_r} \right)^2 \frac{1}{\kappa} [F_h(\eta)]^2 d\eta - \frac{l'}{b_r} \int_0^{1.0} \left(\frac{b}{b_r} \right)^2 A_{ch} [F_h(\eta)]^2 d\eta + \\ & \quad \int_0^{1.0} \left(\frac{b}{b_r} \right)^3 \tan \Lambda \left(i \frac{1}{k_n} \right) ([1] + A_{ch}) [F_h(\eta)] \frac{dF_h}{d\eta} d\eta - \left[\frac{b_r}{l'} \int_0^{1.0} \left(\frac{b}{b_r} \right)^4 \tan^2 \Lambda \left(\frac{1}{k_n^2} \right) [F_h(\eta)] \frac{d^2 F_h}{d\eta^2} d\eta \right] \\ B_2 &= l' \int_0^{1.0} \left(\frac{b}{b_r} \right)^3 \left(\frac{x_\alpha}{\kappa} - A_{ca} \right) [F_h(\eta)] [F_\theta(\eta)] d\eta - b_r \int_0^{1.0} \left(\frac{b}{b_r} \right)^4 \tan \Lambda (A_{cr}) [F_h(\eta)] \frac{dF_\theta}{d\eta} d\eta + \\ & \quad \left[\frac{b_r^2}{l'} \int_0^{1.0} \left(\frac{b}{b_r} \right)^5 \tan^2 \Lambda \left(\frac{a}{k_n^2} \right) [F_h(\eta)] \frac{d^2 F_\theta}{d\eta^2} d\eta \right] \\ D_2 &= \frac{l'}{b_r} \int_0^{1.0} \left(\frac{b}{b_r} \right)^3 \left(\frac{x_\alpha}{\kappa} - A_{ah} \right) [F_\theta(\eta)] [F_h(\eta)] d\eta + \int_0^{1.0} \left(\frac{b}{b_r} \right)^4 \tan \Lambda \left(i \frac{1}{k_n} \right) ([a] + A_{ah}) [F_\theta(\eta)] \frac{dF_h}{d\eta} d\eta + \\ & \quad \left[\frac{b_r}{l'} \int_0^{1.0} \left(\frac{b}{b_r} \right)^5 \tan^2 \Lambda \left(\frac{a}{k_n^2} \right) [F_\theta(\eta)] \frac{d^2 F_h}{d\eta^2} d\eta \right] \\ E_2 &= l' \left[1 - \left(\frac{\omega}{\omega_n} \right)^2 (1 + i g_n) \right] \int_0^{1.0} \left(\frac{b}{b_r} \right)^4 \frac{r_\alpha^2}{\kappa} [F_\theta(\eta)]^2 d\eta - l' \int_0^{1.0} \left(\frac{b}{b_r} \right)^4 A_{aa} [F_\theta(\eta)]^2 d\eta - \\ & \quad b_r \int_0^{1.0} \left(\frac{b}{b_r} \right)^5 \tan \Lambda (A_{ar}) [F_\theta(\eta)] \frac{dF_\theta}{d\eta} d\eta - \left[\frac{b_r^2}{l'} \int_0^{1.0} \left(\frac{b}{b_r} \right)^6 \tan^2 \Lambda \left(\frac{1}{8} + a^2 \right) \frac{1}{k_n^2} [F_\theta(\eta)] \frac{d^2 F_\theta}{d\eta^2} d\eta \right] \end{aligned}$$

in which $F_h(\eta) = f_h(l'\eta)$ and $F_\theta(\eta) = f_\theta(l'\eta)$.

The borderline condition of flutter, separating damped and undamped oscillations, is determined from the nontrivial solution of the simultaneous homogeneous equations (19b) and (20b). Such a solution corresponds to the fact that mechan-

ical equilibrium exists for sinusoidal oscillations at a certain airspeed and with a certain frequency. The flutter condition thus is given by the vanishing of the determinant of the coefficients

$$\begin{vmatrix} A_2 & B_2 \\ D_2 & E_2 \end{vmatrix} = 0$$

Application to the case of uniform, cantilever swept wings is made in the next section.

APPLICATION TO UNIFORM CANTILEVER SWEPT WINGS

The first step in the application of the theory is to assume or develop the deflection functions to be used. For the purpose of applying the analysis to the wing models employed in the experiments it appeared reasonable to use for the deflection functions, $F_h(\eta)$ and $F_\theta(\eta)$, the uncoupled first bending and first torsion mode shapes of an ideal uniform cantilever beam. Although approximations for these mode shapes could be used, the analysis utilized the exact expressions developed from equations (120) and (106d), respectively, of reference 8 by application of appropriate boundary conditions.

The bending-mode shape can be written

$$F_h(\eta) = C_1 \left[\frac{\sinh \beta_1 + \sin \beta_1}{\cosh \beta_1 + \cos \beta_1} (\cos \beta_1 \eta - \cosh \beta_1 \eta) + \sinh \beta_1 \eta - \sin \beta_1 \eta \right]$$

where $\beta_1 = 0.5969\pi$ for first bending. The torsion mode shape can be written

$$F_\theta(\eta) = C_2 \sin \beta_2 \eta$$

where $\beta_2 = \frac{\pi}{2}$ for first torsion and C_1 and C_2 are constants.

The integrals appearing in the determinant elements A_2 , B_2 , D_2 , and E_2 are

$$\begin{aligned} \int_0^{1.0} F_h^2 d\eta &= 1.8554 C_1^2 \\ \int_0^{1.0} F_\theta^2 d\eta &= 0.5000 C_2^2 \\ \int_0^{1.0} F_h \frac{dF_h}{d\eta} d\eta &= 3.7110 C_1^2 \\ \int_0^{1.0} F_\theta \frac{dF_\theta}{d\eta} d\eta &= 0.3183 C_2^2 \\ \int_0^{1.0} F_h \frac{d^2 F_h}{d\eta^2} d\eta &= 1.5926 C_1^2 \\ \int_0^{1.0} F_\theta \frac{d^2 F_\theta}{d\eta^2} d\eta &= -1.2337 C_2^2 \\ \int_0^{1.0} F_h F_\theta d\eta &= \int_0^{1.0} F_\theta F_h d\eta = -0.9233 C_1 C_2 \\ \int_0^{1.0} F_h \frac{dF_\theta}{d\eta} d\eta &= -1.4040 C_1 C_2 \\ \int_0^{1.0} F_\theta \frac{dF_h}{d\eta} d\eta &= -2.0669 C_1 C_2 \\ \int_0^{1.0} F_h \frac{d^2 F_\theta}{d\eta^2} d\eta &= 2.2782 C_1 C_2 \\ \int_0^{1.0} F_\theta \frac{d^2 F_h}{d\eta^2} d\eta &= -1.4722 C_1 C_2 \end{aligned}$$

The flutter determinant becomes

$$\begin{vmatrix} 1.8554C_1^2 \frac{V'}{b_r} A + 3.7110C_1^2 \left(i \frac{1}{k_n}\right) \left\{(-1) + A_{ea}\right\} \tan \Lambda - \left\{1.5926C_1^2 \frac{\tan^2 \Lambda}{V'/b_r} \frac{1}{k_n^2}\right\} & -0.9233C_1C_2 \frac{V'}{b_r} D - 2.0669C_1C_2 \left(i \frac{1}{k_n}\right) \left\{(a) + A_{ea}\right\} \tan \Lambda + \left\{-1.4722C_1C_2 \frac{\tan^2 \Lambda}{V'/b_r} \frac{a}{k_n^2}\right\} & -0.9233C_1C_2 \frac{V'}{b_r} B - (-1.4040C_1C_2) b_r A_{ea} \tan \Lambda + \left\{2.2782C_1C_2 b_r \frac{\tan^2 \Lambda}{V'/b_r} \frac{a}{k_n^2}\right\} \\ -0.9233C_1C_2 \frac{V'}{b_r} D - 2.0669C_1C_2 \left(i \frac{1}{k_n}\right) \left\{(a) + A_{ea}\right\} \tan \Lambda + \left\{-1.4722C_1C_2 \frac{\tan^2 \Lambda}{V'/b_r} \frac{a}{k_n^2}\right\} & A + 2.0000 \frac{\tan \Lambda}{V'/b_r} \left(i \frac{1}{k_n}\right) \left\{(-1) + A_{ea}\right\} - \left\{0.85837 \left(\frac{\tan \Lambda}{V'/b_r}\right)^2 \frac{1}{k_n^2}\right\} & 0.5000C_2^2 \frac{V'}{b_r} E - 0.3183C_2^2 b_r A_{ea} \tan \Lambda - \left\{-1.2337C_2^2 b_r \frac{\tan^2 \Lambda}{V'/b_r} \left(\frac{1}{8} + a^2\right) \frac{1}{k_n^2}\right\} \end{vmatrix} = 0$$

or more conveniently, when columns and rows of the determinant are divided by appropriate terms

$$\begin{vmatrix} A + 2.0000 \frac{\tan \Lambda}{V'/b_r} \left(i \frac{1}{k_n}\right) \left\{(-1) + A_{ea}\right\} - \left\{0.85837 \left(\frac{\tan \Lambda}{V'/b_r}\right)^2 \frac{1}{k_n^2}\right\} & B - 1.5206 \frac{\tan \Lambda}{V'/b_r} A_{ea} - \left\{2.4675 \left(\frac{\tan \Lambda}{V'/b_r}\right)^2 \frac{a}{k_n^2}\right\} \\ 0.9189D + 2.0571 \frac{\tan \Lambda}{V'/b_r} \left(i \frac{1}{k_n}\right) \left\{(a) + A_{ea}\right\} + \left\{1.4652 \left(\frac{\tan \Lambda}{V'/b_r}\right)^2 \frac{a}{k_n^2}\right\} & E - 0.63660 \frac{\tan \Lambda}{V'/b_r} A_{ea} + \left\{2.4675 \left(\frac{\tan \Lambda}{V'/b_r}\right)^2 \left(\frac{1}{8} + a^2\right) \frac{1}{k_n^2}\right\} \end{vmatrix} = 0$$

where

$$A = \frac{1}{K} \left[1 - \left(\frac{\omega_n}{\omega}\right)^2 (1 + i g_n) \right] - A_{ea}$$

$$B = \frac{x_{\alpha}}{K} - A_{ea}$$

$$D = \frac{x_{\alpha}}{K} - A_{ea}$$

$$E = \frac{r_{\alpha}^2}{K} \left[1 - \left(\frac{\omega_{\alpha}}{\omega}\right)^2 (1 + i g_{\alpha}) \right] - A_{ea}$$

It is interesting to note that the parameters Λ and V'/b_r appear only in the combination $\frac{\tan \Lambda}{V'/b_r}$ in the immediately preceding determinant. The solution of the determinant results in the flutter condition.

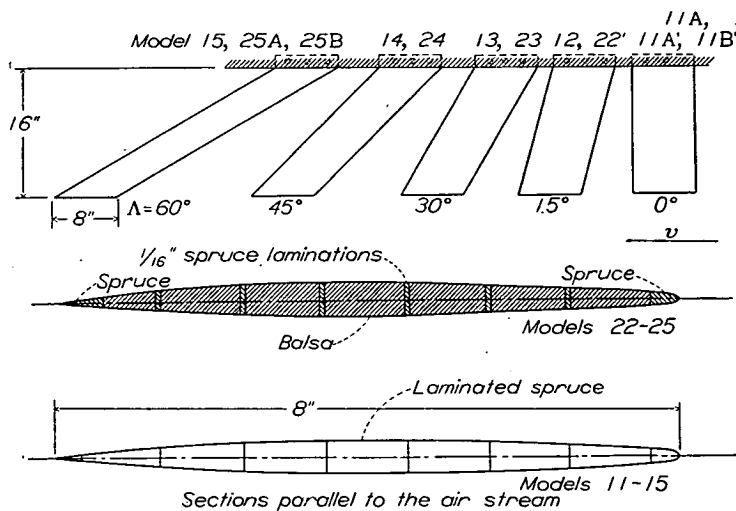
EXPERIMENTAL INVESTIGATION

APPARATUS

Wind tunnel.—The tests were conducted in the Langley 4.5-foot flutter research tunnel which is of the closed-throat, single-return type employing either air or Freon-12 as a testing medium at pressures varying from 4 inches of mercury to 30 inches of mercury. In Freon-12, the speed of sound is 324 miles per hour and the density is 0.0106 slug per cubic foot at standard pressure and temperature. The maximum choking Mach number for these tests was approximately 0.92. The Reynolds number range was from 0.26×10^6 to 2.6×10^6 with most of the tests at Reynolds numbers of the order of 1.0×10^6 .

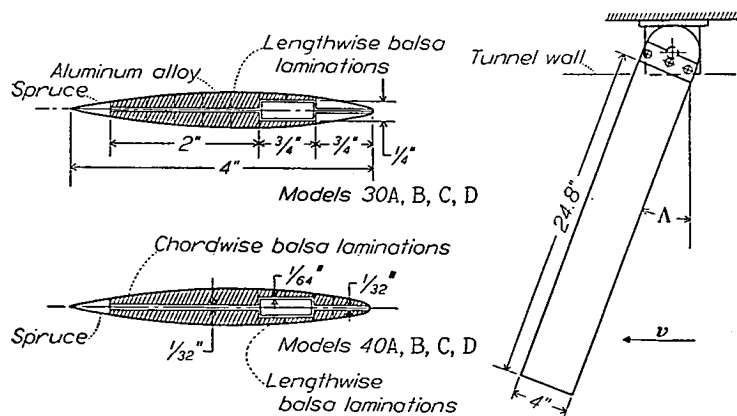
Models.—In order to obtain structural parameters required for the flutter studies, different types of construction were used for the models. Some models were solid spruce, others were solid balsa, and many were combinations of balsa with various aluminum-alloy inserts. Seven series of models were investigated, for which the cross sections and plan forms are shown in figure 2.

Figure 2 (a) shows the series of models which were swept back by shearing the cross sections parallel to the air stream. In order to obtain flutter with these low-aspect-ratio models, thin sections and relatively light and weak wood construction were employed.



(a) Sheared swept models with a constant geometric aspect ratio of 2. Series I.

FIGURE 2.—Model plan form and cross-sectional construction.



(b) Models swept back by use of a rotating mount. Series II.

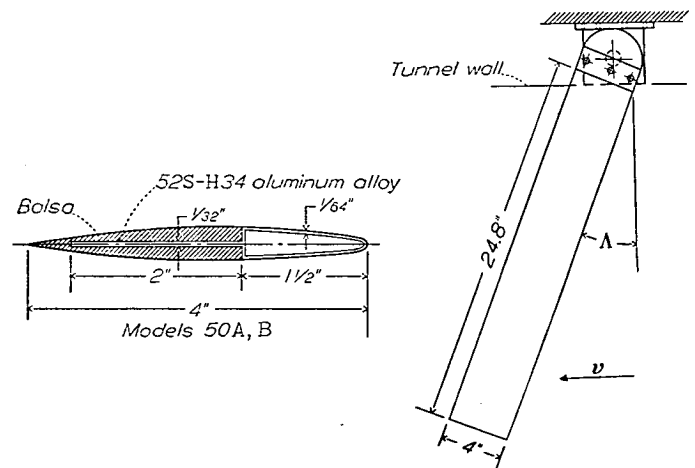
FIGURE 2.—Continued.

The series of rectangular-plan-form models shown in figure 2 (b) were swept back by using a base mount that could be rotated to give the desired sweep angle. The same base mount was used for testing models at forward sweep angles. It is known that for forward sweep angles divergence is critical. In an attempt to separate the divergence and flutter speeds in the sweepforward tests, a D-spar cross-sectional construction was used to get the elastic axis relatively far forward (fig. 2 (c)).

Two series of wings (figs. 2 (d) and 2 (e)) were swept back with the length-chord ratios kept constant. In these series of models, the chord perpendicular to the leading edge was kept constant and the bases were aligned parallel to the air stream. The wings of length-chord ratio 8.5 (fig. 2 (d)) were cut down to get the wings of length-chord ratio 6.5 (fig. 2 (e)).

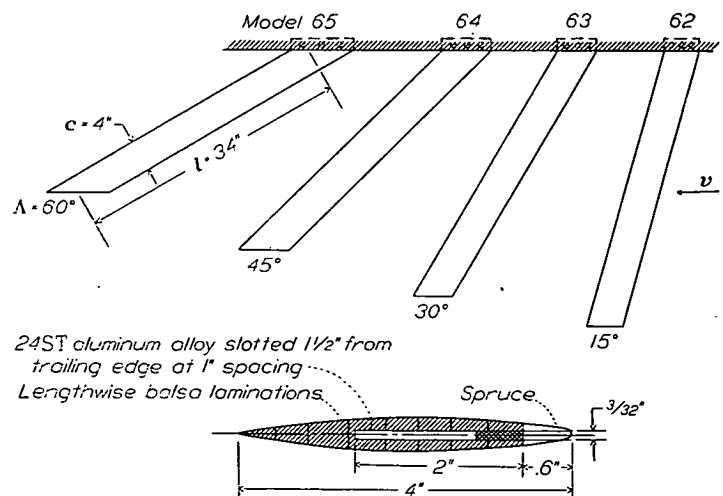
Another series of models obtained by using this same manner of sweep (fig. 2 (f)) was used for investigating some effects of tip shape.

Spanwise strips of lead were fastened to the models shown in figure 2 (e) and a series of tests were conducted with these weighted models to determine the effect of center-of-gravity shift on the flutter speed of swept wings. The method of



(c) Models in which a rotating mount is used to determine the effect of sweepback and sweepforward on the critical velocity. Series III.

FIGURE 2.—Continued.



(d) Swept models having a length-chord ratio of 8.5. Series IV.

FIGURE 2.—Continued.

varying the center of gravity is shown in figure 2 (g). In order to obtain data at zero sweep angle it was necessary, because of the proximity of flutter speed to wing-divergence speed, to use three different wings. These zero-sweep-angle wings, of 8-inch chord and 48-inch length, had an internal weight system.

The models were mounted from the top of the tunnel as cantilever beams with rigid bases (fig. 3). Near the root of each model two sets of strain gages were fastened, one set for recording principally bending deformations and the other set for recording principally torsional deflections.

METHODS

Determination of model parameters.—Pertinent geometric and structural properties of the model are given in tables I to VII. Some parameters of interest are discussed in the following paragraphs.

As an indication of the nearness to sonic-flow conditions, the critical Mach number is listed. This Mach number is determined by the Kármán-Tsien method for a wing section normal to the leading edge at zero lift.

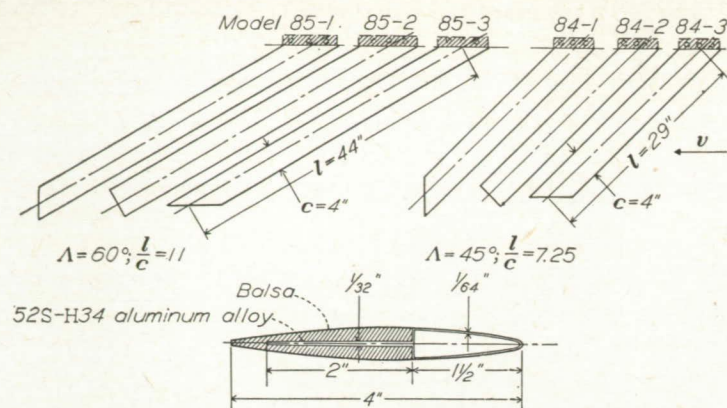
The geometric aspect ratio of a wing is here defined as

$$A_g = \frac{\text{Semispan}^2}{\text{Plan-form area}} = \frac{(l \cos \Lambda)^2}{lc} = \frac{l}{c} \cos^2 \Lambda = \frac{A}{2}$$

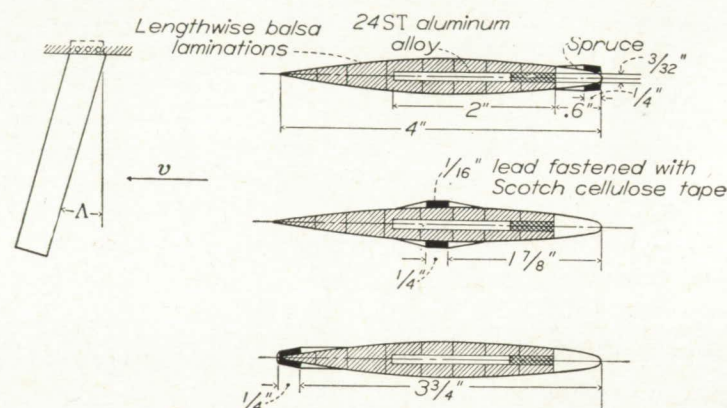
The geometric aspect ratio A_g is used in place of the conventional aspect ratio A because the models were only semispan wings. For sheared swept wings, obtained from a given unswept wing, the geometric aspect ratio is constant, whereas for the wings of constant length-chord ratio the geometric aspect ratio decreases with $\cos^2 \Lambda$ as the angle of sweep is increased.

The weight, center-of-gravity position, and polar moment of inertia of the models were determined by usual means. The models were statically loaded at the tip to obtain the rigidities in torsion and bending GJ and EI .

A parameter occurring in the methods of analysis of this report is the position of the elastic axis. A "section" elastic axis located at x_{ea} was obtained for wings from each series of models as follows: The wings were clamped at the root normal to the leading edge and at a chosen spanwise station



(f) Models used to investigate the effect of tip shape on the flutter velocity. Series VI
FIGURE 2.—Continued.

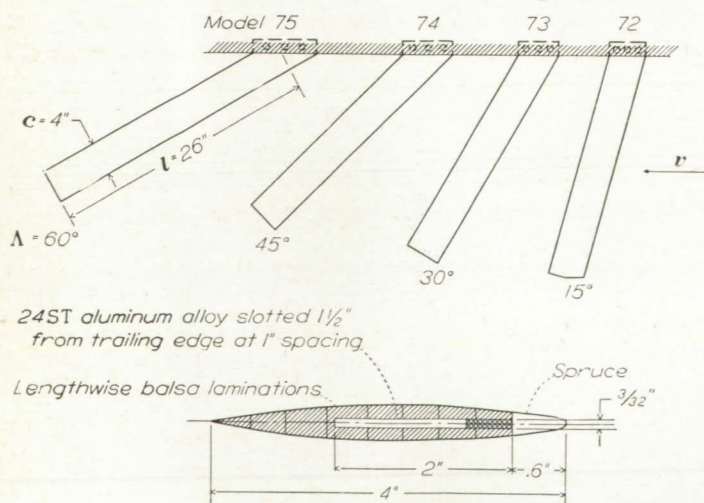


Model	Λ (deg)
91-1, 91-2, 91-3 *	0
92-1, 92-2, 92-3	15
93-1, 93-2, 93-3	30
94-1, 94-2, 94-3	45
95-1, 95-2, 95-3	60

* Chord=8", lead inside balsa.

(g) Models used to determine the effect of center-of-gravity shift on the flutter velocity of swept wings. Series VII.

FIGURE 2.—Concluded.



(e) Swept models having a length-chord ratio of 6.5. Series V.

FIGURE 2.—Continued.

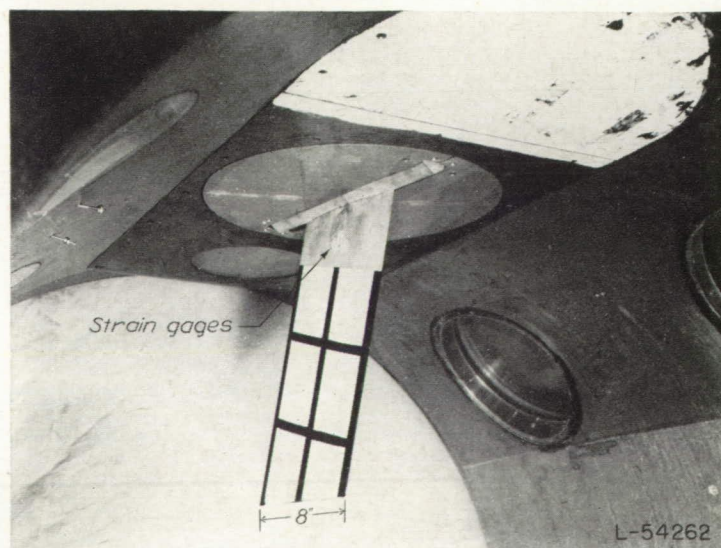


FIGURE 3.—Model 12 in the tunnel test section.

were loaded at points lying in the chordwise direction. The point for which pure bending deflection occurred, with no twist in the plane normal to the leading edge, was determined. The same procedure was used for those wings which were clamped at the root, not normal, but at an angle to the leading edge. A different elastic axis designated the "wing" elastic axis and located at x_{ea}' was thus determined.

For these uniform, swept wings with fairly large length-chord ratios, the wing elastic axis was reasonably straight and remained essentially parallel to the section elastic axis, although it was found to move farther behind the section elastic axis as the angle of sweep was increased. It is realized that in general for nonuniform wings—for example, wings with cut-outs or skewed clamping—a certain degree of cross stiffness exists and the concept of an elastic axis is an oversimplification. More general concepts such as those involving influence coefficients may be required. These more strict considerations, however, are not required here since the elastic-axis parameter is of fairly secondary importance.

The wing mass-density ratio κ is the ratio of the mass of a cylinder of testing medium, of a diameter equal to the chord of the wing, to the mass of the wing, both taken for unit length along the wing. The density of the testing medium when flutter occurred was used in the evaluation of κ .

Determination of the reference flutter speed.—It is convenient in presenting and comparing data of swept and unswept wings to employ a certain reference flutter speed. This reference flutter speed will serve to reduce variations in flutter characteristics which arise from changes in the various model parameters such as density and section properties not pertinent to the investigation. It thus aids in systematizing the data and emphasizing the desired effects of sweep including effects of aspect ratio and Mach number.

This reference flutter speed V_R may be obtained in the following way. Suppose the wing to be rotated about the intersection of the elastic axis with the root to a position of zero sweep. In this position the reference flutter speed is calculated by the method of reference 7, which assumes an idealized, uniform, infinite wing mounted on springs in an incompressible medium. For nonuniform wings, a reference section taken at a representative spanwise position, or some integrated value, may be used. Since the wings used were uniform, any reference section will serve. The reference flutter speed may thus be considered a "section" reference flutter speed and parameters of a section normal to the leading edge are used in its calculation. This calculation also employs the uncoupled first bending and torsion frequencies of the wing (obtained from the measured frequencies) and the measured density of the testing medium at time of flutter. The calculation yields a corresponding reference flutter frequency which is useful in comparing the frequency data. For the sake of completeness a further discussion of the reference flutter speed is given in appendix B.

Test procedure and records.—Since flutter is often a sudden and destructive phenomenon, coordinated test procedures were required. During each test, the tunnel speed was slowly raised until a speed was reached for which the

amplitudes of oscillation of the model in bending and torsion increased rapidly while the frequencies in bending and torsion, as observed on the screen of the recording oscillograph, merged to the same value. At this instant, the tunnel conditions were recorded and an oscillograph record of the model deflections was taken. The tunnel speed was immediately reduced in an effort to prevent destruction of the model.

From the tunnel data, the experimental flutter speed V_e , the density of the testing medium ρ , and the Mach number M were determined. No blocking or wake corrections to the measured tunnel velocity were applied.

From the oscillogram the experimental flutter frequency f_e and the phase difference φ (or the phase difference $\pm 180^\circ$) between the bending and torsion deflections near the root were read. A reproduction of a typical oscillograph flutter record, which indicated the flutter to be a coupling of the wing bending and torsion degrees of freedom, is shown as figure 4. Since semispan wings mounted rigidly at the base were used, the flutter mode may be considered to correspond to the flutter of a complete wing having a very heavy fuselage at midspan—that is, to the symmetrical type.

The natural frequencies of the models in bending and torsion at zero airspeed were recorded before and after each test in order to ascertain possible changes in structural characteristics. In most cases there were no appreciable changes in frequencies but there were some reductions in stiffnesses for models which had been weakened by fluttering violently. Analysis of the decay records of the natural frequencies indicated that the wing damping coefficients g_b and g_a (reference 7) were about 0.02 in the first bending mode and 0.03 in the torsion mode.

RESULTS AND DISCUSSION

EXPERIMENTAL INVESTIGATION

Presentation of experimental data.—Results of the experimental investigation are listed in detail in tables I to VII, and some significant experimental trends are illustrated

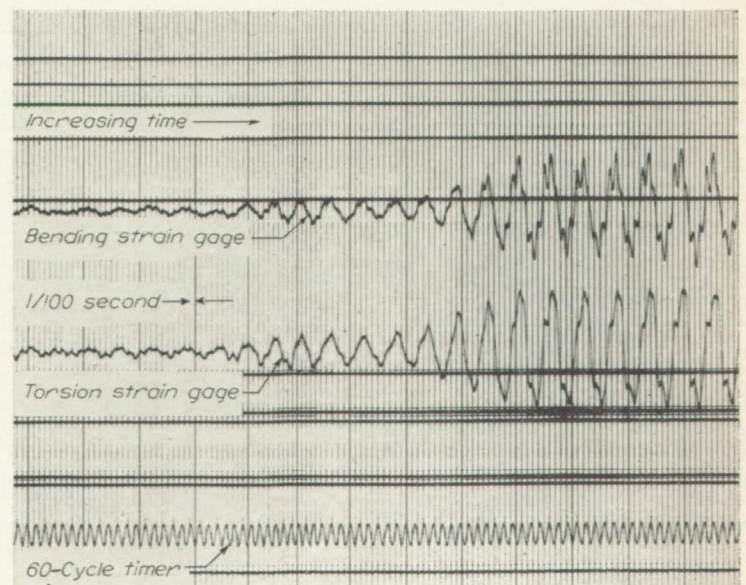


FIGURE 4.—Oscillograph record of model at flutter.

in figures 5 to 10. As a basis for presenting and comparing the test results, the ratio of experimental tunnel stream conditions to the reference flutter conditions is employed so that the data indicate more clearly combined effects of aspect ratio, sweep, and Mach number. As previously mentioned, use of the reference flutter speed V_R serves to reduce variations in flutter characteristics which arise from changes in other parameters, such as density and section properties, which are not pertinent to this investigation. (See appendix B.)

Some effects on flutter speed.—A typical plot showing the effect of compressibility on the flutter speed of wings at various angles of sweepback is shown in figure 5. These data are from tests of the rectangular-plan-form models (type 30) that were swept back by use of the rotating mount, for which arrangement the reference flutter speed does not vary with either Mach number or sweep angle. Observe the large increase in speed ratio at the high sweep angles.

The data of reference 1 from tests of a rigid, flexibly mounted rectangular model having a rotating base are also plotted in figure 5. It can be seen that the data from the cantilever models of the present report which had a similar method of sweep are in conformity with the data from the flexibly mounted model. This indicates that, for uniform wings having the range of parameters involved in these tests, the differences due to mode shape are not very great.

Figure 6 is a cross plot of the data from figure 5 plotted against Λ at a Mach number approximately equal to 0.65. The data of the swept wings of constant length-chord ratio and of the sheared swept wings are also included for comparison. The velocity ratio V_e/V_R is relatively constant at small sweep angles but rises noticeably at the large sweep angles. It is pointed out that the reference flutter speed V_R may be considered to correspond to a horizontal line at $\frac{V_e}{V_R}=1$ for the rotated and constant-length-chord-ratio wings, but for the sheared wings this reference speed corresponds to a curve decreasing somewhat less rapidly than $\sqrt{\cos \Lambda}$ as Λ increases. (See appendix B.)

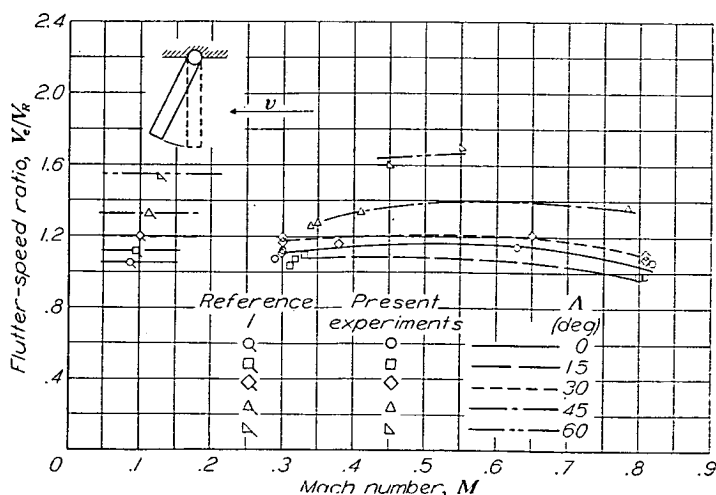


FIGURE 5.—Ratio of experimental to reference flutter speed as a function of Mach number for various sweep angles for series II models (fig. 2 (b)) on the rotating mount.

The order of magnitude of some three-dimensional effects may be noted from the fact that the shorter wings ($\frac{l}{c}=6.5$, fig. 6, series V) have higher velocity ratios than the longer wings ($\frac{l}{c}=8.5$, series IV). This increase may be due partly to differences in flutter modes as well as aerodynamic effects.

Some effects on flutter frequency.—Figure 7 is a representative plot of the flutter-frequency data given in table II. The figure shows the variation in flutter-frequency ratio with Mach number for different values of sweep angle for the models rotated back on the special mount. The ordinate

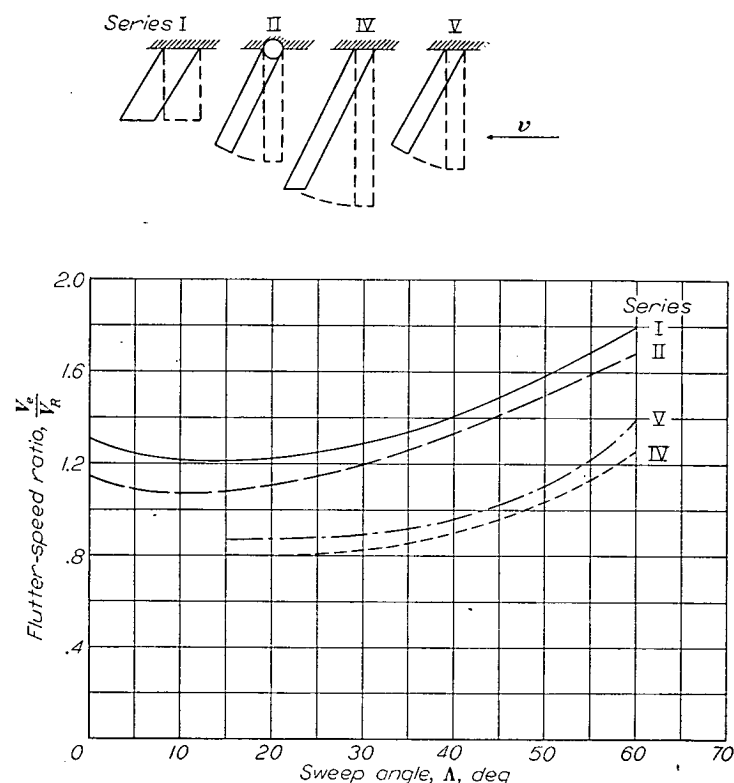


FIGURE 6.—Cross plot of ratio of experimental to reference flutter speed as a function of sweep angle for various wings. Mach number is approximately 0.65.

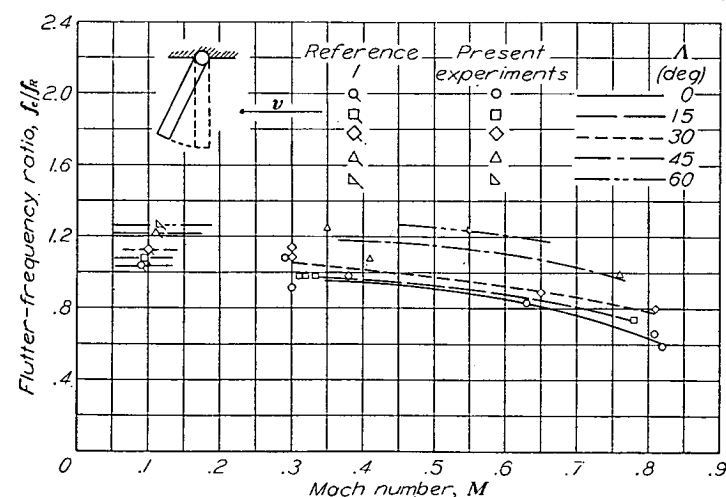


FIGURE 7.—Ratio of experimental to reference flutter frequency as a function of Mach number for various sweep angles for series II models (fig. 2 (b)) on the rotating mount.

is the ratio of the experimental flutter frequency to the reference flutter frequency f_e/f_R . It appears that there is a reduction in flutter frequency with increase in Mach number and also an increase in flutter frequency with increase in sweep. The data from reference 1 show the same trend with increase in sweep. Considerably more scatter may be noted in the frequency data than in the speed data (fig. 5) from the same tests.

The results of the tests for rotated wings with chordwise laminations (models 40A, B, C, D) are given in table II. At sweep angles up to 30° the values of the speed ratio V_e/V_R for wings of this construction were low (in the neighborhood of 0.9), and the flutter frequency ratios f_e/f_R were high (of the order of 1.4). As these results indicate and as visual observation showed, these models fluttered in a mode that apparently involved an appreciable proportion of the second bending mode. The models with spanwise laminations (models 30A, B, C, D) also showed indications of this higher flutter mode at low sweep angles; however, these models were able to pass through the small speed range of higher mode flutter without sufficiently violent oscillations to cause failure. At a still higher speed these models with spanwise laminations fluttered in a lower mode resembling a coupling of the torsion and first bending modes. This lower mode type of flutter characterized the flutter of both the sheared and constant-length-chord-ratio models.

For those wing models having the sheared type of balsa construction (models 22', 23, 24, and 25), the results are more difficult to compare with those of the other models. This difficulty arises chiefly because the lightness of the wood produced relatively high mass-density ratios κ and partly because of the nonhomogeneity of the mixed wood construction. For high values of κ the flutter-speed coefficient changes rather abruptly even for the unswept models (reference 7). The data are nevertheless included in table I.

Effect of shift in center-of-gravity position on the flutter speed of swept wings.—Results of the investigation of the effects of center-of-gravity shift on the flutter speed of swept wings are illustrated in figure 8. This figure is a cross plot of the experimental indicated air speeds as a function of sweep angle for various center-of-gravity positions. The ordinate is the experimental indicated air

speed $V_e \sqrt{\frac{\rho}{0.00238}}$, which serves to reduce the scatter resulting from flutter tests at different densities of testing medium. The data were taken in the Mach number range between 0.14 and 0.44, so that compressibility effects are presumably negligible. As in the case of unswept wings, forward movement of the center of gravity increases the flutter speed. Again, the flutter speed increases with increase in the angle of sweep.

The models tested at zero sweep angle (models 91-1, 91-2, 91-3) were of different construction from and of larger size than the models tested at the higher sweep angles. Because of the manner of plotting the results, namely as experimental indicated airspeed (fig. 8), a comparison of the results of tests at $\Lambda=0^\circ$ with the results of the tests of swept models

is not particularly significant. The points at zero sweep angle are included, however, to show that the increase in flutter speed due to a shift in the center-of-gravity position for the swept models is of the same order of magnitude as for the unswept models. For the unswept models, the divergence speed V_D and the reference flutter speed V_R are fairly near each other, and although the models appeared to flutter, the proximity of the flutter speed to the divergence speed may have influenced the value of the critical speed.

The method used to vary the center of gravity (see fig. 2 (g)) produced two bumps on the airfoil surface. At the low Mach numbers of these tests, however, the effect of this roughness on the flutter speed is considered negligible. For proper interpretation of figure 8, the fact must be kept in mind that the method of varying the location of the center of gravity changed the radius of gyration r_a and the torsional frequency f_a .

The effect of sweepforward on the critical speed.—An attempt was made to determine the variation in flutter speed with angle of sweepforward by testing wings on the mount that could be rotated both backward and forward. As expected, however, the model tended to diverge at forward sweep angles in spite of the relatively forward position of the elastic axis in this D-spar wing.

Figure 9 shows a plot of the ratio of critical speed to the reference flutter speed V_R against sweep angle Λ . Note the different curves for the sweptback and for the sweptforward conditions and the sharp reduction in critical speed as the

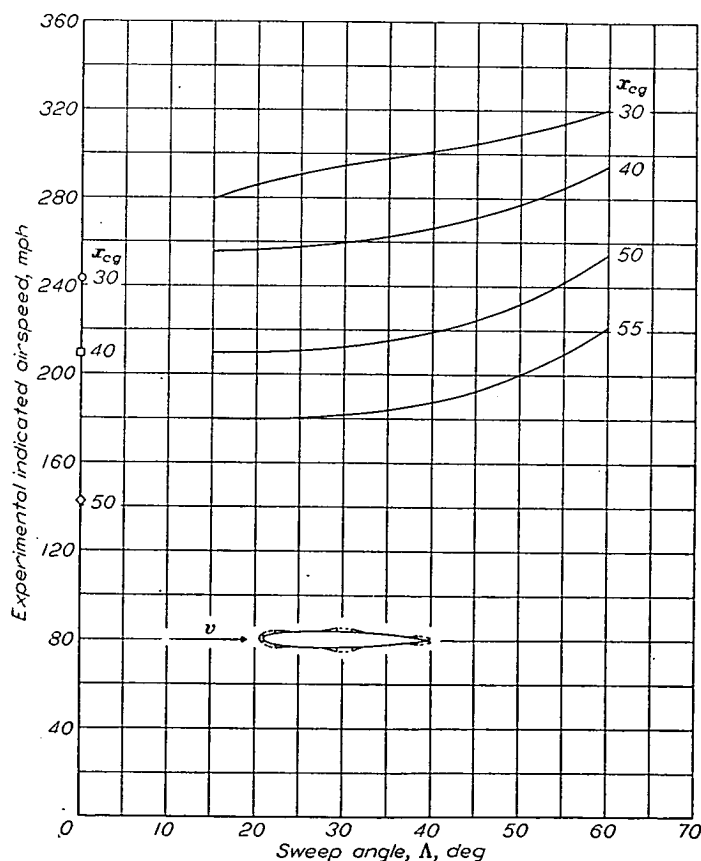


FIGURE 8.—Cross plot of flutter speed as a function of sweep angle for several center-of-gravity positions. Series VII models (fig. 2 (g)). Length-chord ratio is approximately 6.

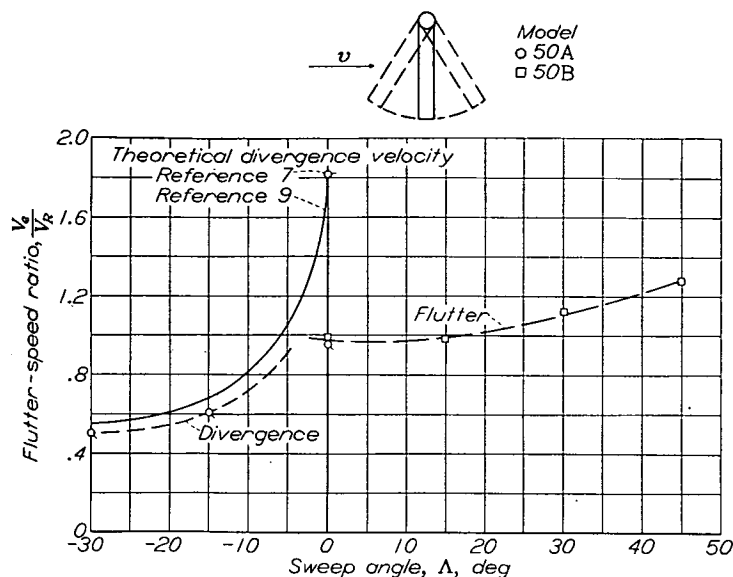


FIGURE 9.—Comparison of sweepforward and sweepback tests on wings tested on a rotating mount. Series III models (fig. 2 (c)).

angle of sweepforward is increased. The different curves result from two different phenomena. When the wing was swept back, it fluttered; whereas at forward sweep angles it diverged before the flutter speed was reached. Superposed on this plot for the negative values of sweep are the results of calculations based on an analytical study of divergence (reference 9). Reasonable agreement exists between theory and experiment at forward sweep angles. The small difference between the theoretical and experimental results may perhaps be due to an inaccuracy in determining either the position of the elastic axis of the model or the required slope of the lift curve or both.

The divergence speed V_D for the wing at zero sweep angle, as calculated by the simplified theory of reference 7, is also plotted in figure 9. This calculation is based on the assumption of a two-dimensional unswept wing in an incompressible medium. The values of the uncoupled torsion frequency and the density of the testing medium at time of flutter or divergence are employed. Reference 9 shows that a relatively small amount of sweepback raises the divergence speed sharply. For convenience, however, the numerical quantity V_D (based on the wing at zero sweep) is listed in table I for all the tests.

Effect of tip modifications.—Tests to investigate some of the over-all effects of tip shape were conducted and some results are shown in figure 10. Two sweep angles and two length-chord ratios were used in the experiments conducted at two Mach numbers. It is seen that, of the three tip shapes used, namely, tips perpendicular to the air stream, perpendicular to the wing leading edge, and parallel to the air stream, the wings with tips parallel to the air stream gave the highest flutter speeds.

DISCUSSION AND COMPARISON OF ANALYTICAL AND EXPERIMENTAL RESULTS

Correlation of analytical and experimental results has been made for wings swept back in the two different manners; that is, (1) sheared back with a constant value of A_g , and (2) rotated back. The two types of sheared wings (series I) and two rotated wings (models 30B and 30D) have been analyzed.

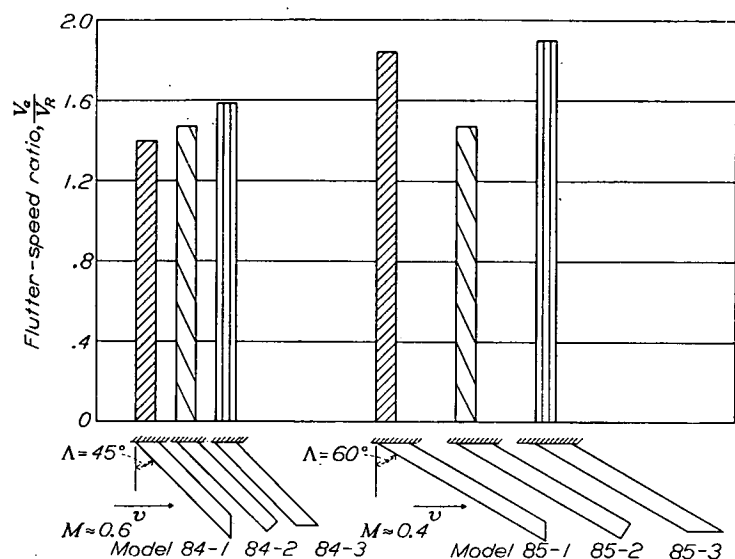


FIGURE 10.—Effect of tip shape on the flutter speed of swept wings. Wings of length-chord ratios of 7.25 and 11 (fig. 2 (f)). Series VI models.

Results of some solutions of the flutter determinant for a wing (model 30B) on a rotating base at several angles of sweepback are shown in figures 11 and 12. Figure 11 shows the flutter-speed coefficient as a function of the bending-to-torsion frequency ratio, and figure 12 shows the flutter frequency ratio as a function of the bending-to-torsion frequency ratio.

The calculated results (for those wings investigated analytically) are included in tables I and II. The ratios of experimental to analytical flutter speeds and flutter frequencies have been plotted against the angle of sweep in figures 13 to 16. If an experimental value coincides with the corresponding analytical predicted value, the ratio will fall at a value of 1.0 on the figures. Deviations of experimental results above or below the analytical results appear on the figures as ratios greater than or less than 1.0, respectively. The flutter-speed ratios plotted in figure 13 for the two rotated wings show very good agreement between analysis and experiment over the range of sweep angle, 0° to 60° . Such good agreement in both the trends and in the numerical quantities is gratifying but probably should not be expected in general. In view of the discussion of the last term in equation (5b) it may be of interest to mention that failure to include the terms arising from the last term of equation (5b) in the calculations for model 30B would decrease the ratio V_e/V_a corresponding to $\Lambda = 60^\circ$ by about 3 percent. The flutter frequency ratios of figure 14 obtained from the same two rotated wings are in good agreement.

The flutter-speed ratios plotted in figure 15 for the two types of sheared wings do not show such good conformity at the low angles of sweep, whereas for sweep angles beyond 45° the ratios are considerably nearer to 1.0. The sheared wings are again observed to have a constant value of A_g of 2.0 (aspect ratio for the whole wing would be 4.0). For this small value of aspect ratio the finite-span correction is appreciable at zero angle of sweep and, if made, would bring better agreement at that point. Analysis of the corrections for finite-span effects on swept wings requires further consideration.

Figures 13 and 15 also afford a comparison of the behavior of wings swept back in two manners: (1) rotated back with

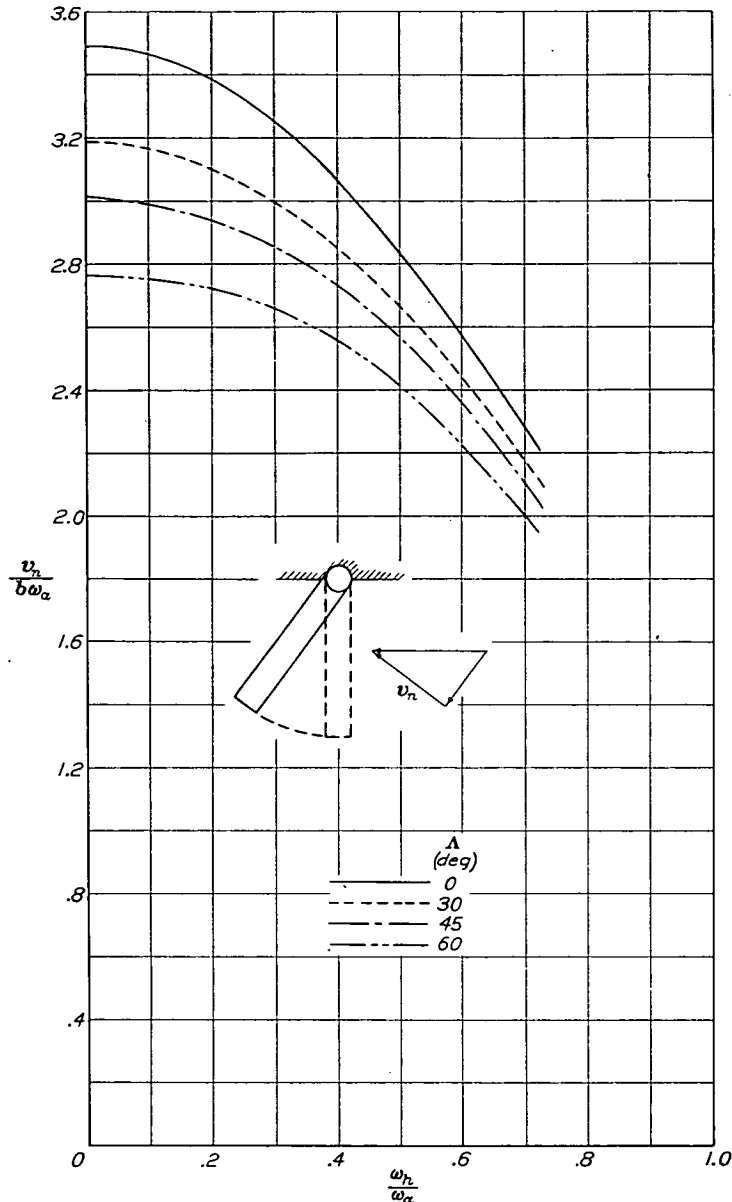


FIGURE 11.—Theoretical flutter-speed coefficient as a function of the ratio of bending to torsion frequency for the rotated model 30B at four angles of sweep and with a constant mass-density ratio ($\frac{1}{\kappa}=37.8$).

constant length-chord ratio but decreasing aspect ratio (fig. 13), and (2) sheared back with constant aspect ratio and increasing length-chord ratio (fig. 15). A study of these two figures suggests that the length-chord ratio rather than the aspect ratio ($\frac{\text{Span}^2}{\text{Area}}$) may be the relevant parameter in determining corrections for finite swept wings. (Admittedly, effects of tip shape and root condition are also involved and have not been precisely separated.)

Figure 16, which refers to the same sheared wings as figure 15, shows the ratios of experimental to predicted flutter frequencies. The trend is for the ratio to decrease as the angle of sweep increases. Table I shows that the flutter frequency f_R obtained with V_R and used as a reference in a previous section of the report is not significantly different from the frequency f_Λ predicted by the present analysis.

A few remarks can be made on estimates of over-all trends of the flutter speed of swept wings. As a first consideration the conclusion may be made that, if a rigid infinite yawed

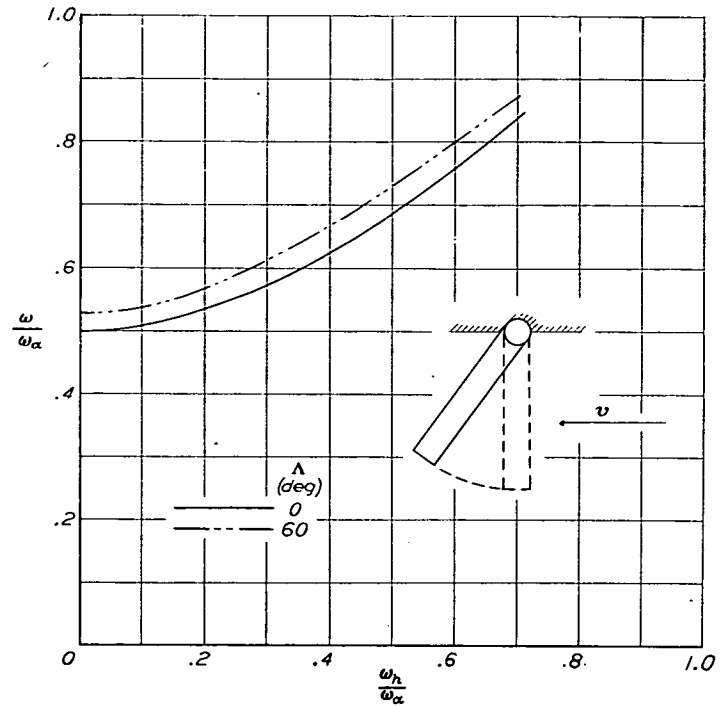


FIGURE 12.—Ratio of theoretical flutter frequency to torsional frequency as a function of the ratio of bending to torsion frequency for the rotated model 30B at two angles of sweep and with a constant mass-density ratio ($\frac{1}{\kappa}=37.8$).

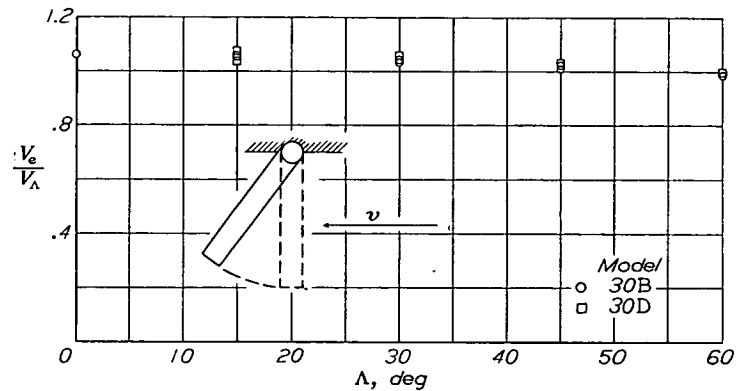


FIGURE 13.—Ratio of experimental to theoretically predicted flutter speed as a function of sweep angle for two rotated models.

wing were mounted on springs which permitted it to move vertically as a unit and to rotate about an elastic axis, the flutter speed would be proportional to $1/\cos \Lambda$. A finite yawed wing mounted on similar springs would be expected to have a flutter speed lying above the curve of $1/\cos \Lambda$ because of finite-span effects. For a finite sweptback wing clamped at its root, however, the greater degree of coupling between bending and torsion adversely affects the flutter speed so as to bring the speed below the curve of $1/\cos \Lambda$ for an infinite wing. This statement is illustrated in figure 17 which refers to a wing (model 30B) on a rotating base. The ordinate is the ratio of flutter speed at a given angle of sweep to the flutter speed calculated at zero angle of sweep. A theoretical curve is shown, together with experimentally determined points. Curves of $1/\cos \Lambda$ and $1/\sqrt{\cos \Lambda}$ are shown for convenience of comparison. The curve for model 30D (not shown in figure 17) also followed this trend quite closely. The foregoing remarks should prove useful for making estimates and discussing trends but are not intended to replace more complete calculation. In particular, mention may be made, for example, that a far-forward location

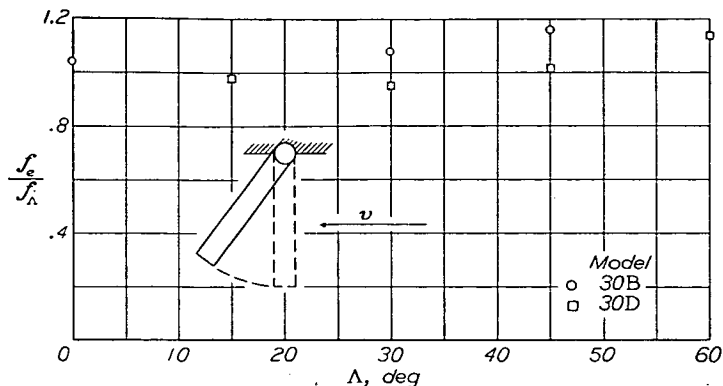


FIGURE 14.—Ratio of experimental to theoretically predicted flutter frequency as a function of sweep angle for two rotated models.

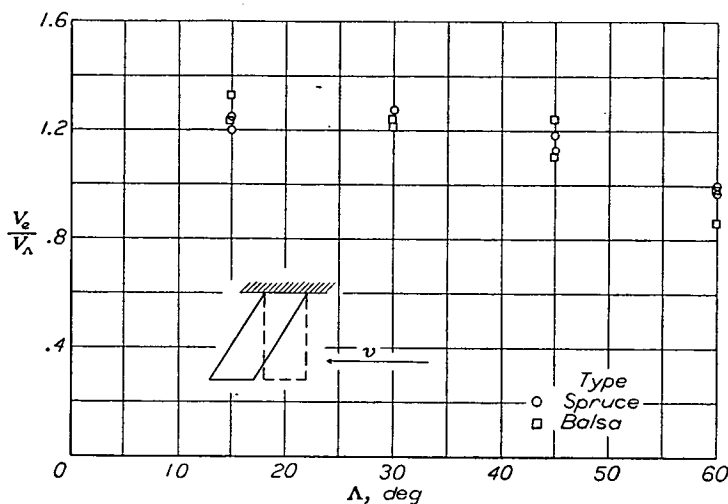


FIGURE 15.—Ratio of experimental to theoretically predicted flutter speed as a function of sweep angle for two types of sheared models.

of section center of gravity would lead to an entirely different trend. Moreover, as is apparent from the analysis, the bending stiffness can play an increasingly significant role with increase in the angle of sweep.

The experiments and calculations deal in general with wings having low ratios of natural first bending to first torsion frequencies. At high values of the ratio of bending frequency to torsion frequency, the position of the elastic axis becomes relatively more significant. Additional calculations to develop the theoretical trends are desirable.

CONCLUSIONS

In a discussion and comparison of the results of an investigation of the flutter of a group of swept wings, the manner of sweep is significant. This report deals with two main groups of uniform, swept wings: rotated wings and sheared wings. In presenting the data, employment of a certain reference flutter speed was found convenient. The following conclusions seem to apply:

1. Comparison with experiment indicates that the analysis presented is satisfactory for giving the main effects of sweep, at least for nearly uniform cantilever wings of moderate length-chord ratios. Additional calculations are desirable to investigate various theoretical trends.

2. The coupling between bending and torsion adversely affects the flutter speed. The fact, however, that only a part of the forward velocity is aerodynamically effective

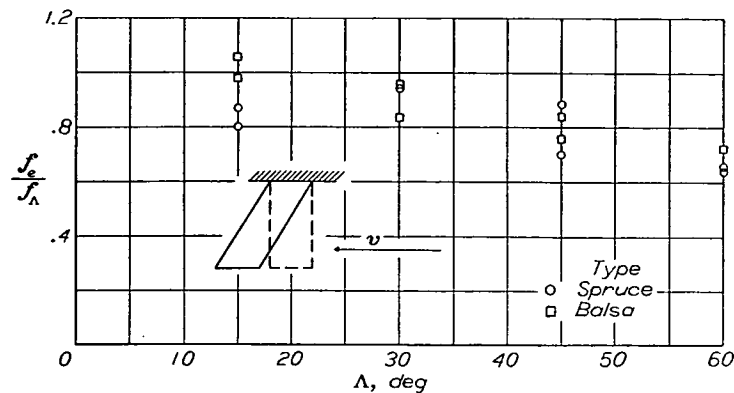


FIGURE 16.—Ratio of experimental to theoretically predicted flutter frequency as a function of sweep angle for two types of sheared wings.

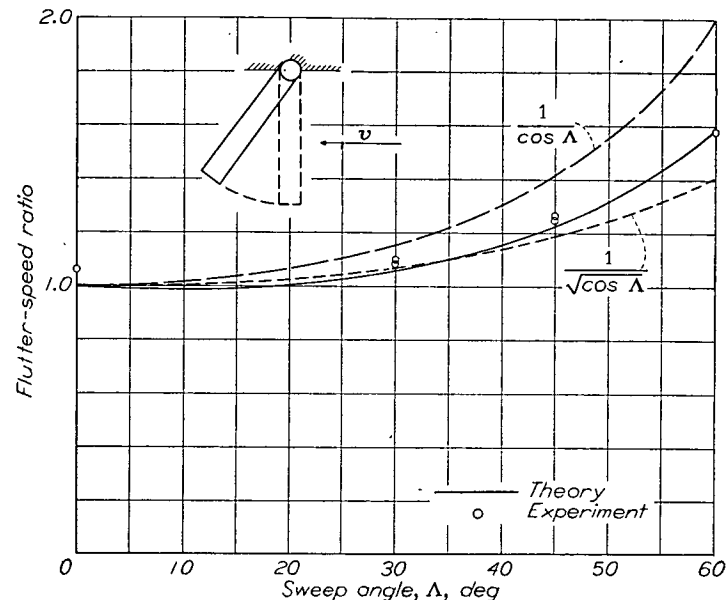


FIGURE 17.—Flutter-speed ratio as a function of sweep angle for model 30B at a constant mass-density ratio ($\frac{1}{\kappa}=37.8$), showing analytical and experimental results.

increases the flutter speed. Certain approximate relations can be used to estimate some of the trends.

3. Although a precise separation of the effects of Mach number, aspect ratio, tip shape, and center-of-gravity position has not been accomplished, the order of magnitude of some of these combined effects has been experimentally determined. Experimental results indicated are

(a) The location of the section center of gravity is an important parameter and produces effects for swept wings similar to those for unswept wings over the range (30 percent to 70 percent chord) of locations tested.

(b) Appreciable differences in flutter speed have been found to be due to tip shape.

(c) The length-chord ratio of swept wings is a more relevant finite-span parameter than is the aspect ratio.

(d) Compressibility effects attributable to Mach number are fairly small, at least up to a Mach number of 0.8.

(e) The sweptforward wings could not be made to flutter but diverged before the flutter speed was reached.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., September 9, 1948.

APPENDIX A

THE EFFECT OF SWEEP ON THE FREQUENCIES OF A CANTILEVER BEAM

Early in the investigation it was decided to make an experimental vibration study of a simple beam at various sweep angles. The uniform, plate-like aluminum-alloy beam shown in figure 18 was used to make the study amenable to analysis. Length-chord ratios of 6, 3, and 1.5 were tested, the length l being defined as the length along the midchord. A single 60-inch beam was used throughout the investigation, the desired length and sweep angle being obtained by clamping the beam in the proper position with a 1½- by 1½- by 14-inch aluminum-alloy crossbar.

Figures 18 and 19 show the variation in modes and frequencies with sweep angle. In most cases, an increase in sweep angle increased the natural vibrational frequencies. As expected, the effect of sweep was more pronounced at the

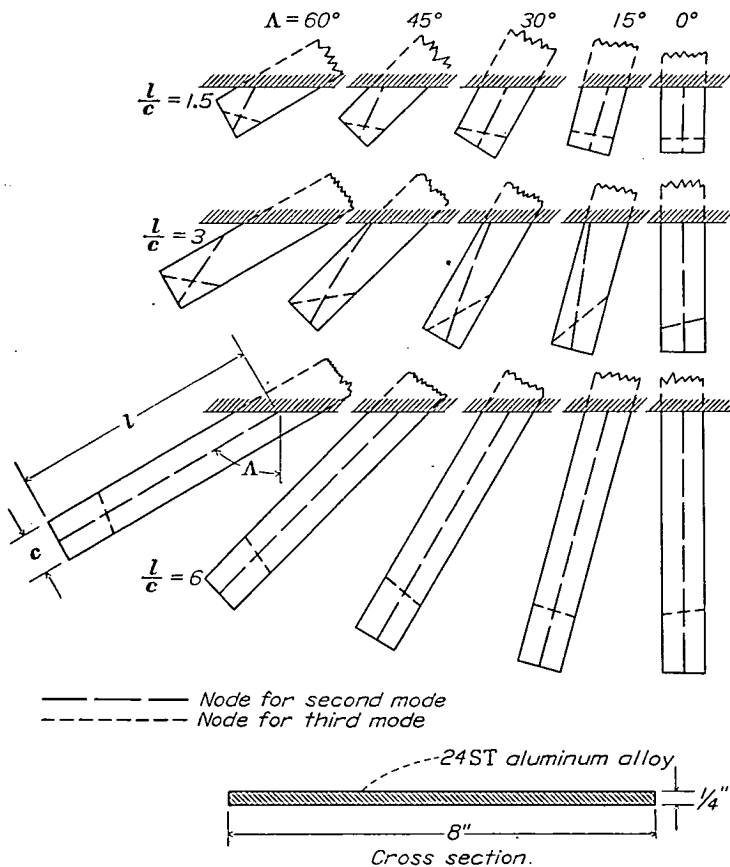


FIGURE 18.—Change in nodal lines with sweep and length-chord ratio for the vibration of an aluminum-alloy beam.

smaller values of length-chord ratio. The fundamental mode was found by striking the beam and measuring the frequency with a self-generating vibration pick-up and paper recorder. The second and third modes were excited by light-weight electromagnetic shakers clamped to the beam. These shakers were attached as close to the root as possible to give a node either predominantly spanwise or chordwise. The mode with the spanwise node, designated second mode, was primarily torsional vibration, whereas the mode with the chordwise node, designated third mode, was primarily a second bending vibration.

The first two bending frequencies and the lowest torsion frequency, determined analytically for a straight uniform unswept beam, are plotted in figure 19. Good agreement exists with the experimental results for the length-chord ratios of 6 and 3, but for a ratio of 1.5 (length equal to 12 inches and chord equal to 8 inches) less favorable agreement exists. This discrepancy may be attributed to the fact that the beam at the short length-chord ratio of 1.5 resembled more a plate than a beam and did not meet the theoretical assumptions of a perfectly rigid base and of simple-beam stress distributions. The data are valid for use in comparing the experimental frequencies of the beam when swept with the frequencies at zero sweep, which was the purpose of the test.

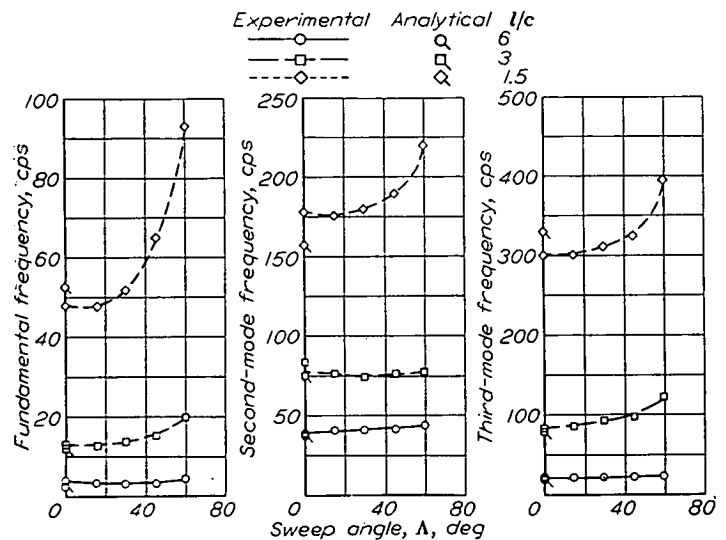


FIGURE 19.—Variation of frequencies with sweep and length-chord ratio for the vibration of an aluminum-alloy beam.

APPENDIX B

DISCUSSION OF THE REFERENCE FLUTTER SPEED

For use in comparing data of swept and unswept wings, a reference flutter speed V_R is convenient. This reference flutter speed is the flutter speed determined from the simplified theory of reference 7. This theory deals with two-dimensional unswept wings in incompressible flow and depends upon a number of wing parameters. The calculations in this report utilize parameters of sections perpendicular to the leading edge, first bending frequency, uncoupled torsion frequency, density of testing medium at time of flutter, and zero damping. Symbolically,

$$V_R = b \omega_\alpha f \left(\kappa, x_{cg}, x_{ea}, r_\alpha^2, \frac{f_h}{f_\alpha} \right)$$

Variation in reference flutter speed with sweep angle for sheared swept wings.—The reference flutter speed is independent of sweep angle for a homogeneous rotated wing and for homogeneous wings swept back by keeping the length-chord ratio constant. For a series of homogeneous wings swept back by the method of shearing, however, a definite variation in reference flutter speed with sweep angle exists since sweeping a wing by shearing causes a reduction in chord perpendicular to the wing leading edge and an increase in length along the midchord as the angle of sweep is increased. The resulting reduction in the mass-density-ratio parameter and first bending frequency tends to raise the reference flutter speed, whereas the reduction in semichord tends to lower the reference flutter speed as the angle of sweep is increased. The final effect upon the reference flutter speed depends on the other properties of the wing. The purpose of this section is to show the effect of these changes on the magnitude of the reference flutter speed for a series of homogeneous sheared wings having properties similar to those of the sheared swept models used in this report.

Let the subscript 0 refer to properties of the wing at zero sweep angle. The following parameters are then functions of the sweep angle:

$$b = b_0 \cos \Lambda$$

$$l = \frac{l_0}{\cos \Lambda}$$

Since m is proportional to b ,

$$\kappa = \frac{\pi \rho b^2}{m} = \kappa_0 \cos \Lambda$$

Similarly, since I is proportional to b ,

$$f_{h1} = \frac{0.56}{l^2} \sqrt{\frac{EI}{m}} = (f_{h1})_0 (\cos \Lambda)^2$$

Also, because f_α is independent of Λ ,

$$\frac{f_{h1}}{f_\alpha} = \left(\frac{f_{h1}}{f_\alpha} \right)_0 (\cos \Lambda)^2$$

An estimate of the effect on the flutter speed of these changes in semichord and mass parameter with sweep angle may be obtained from the approximate formula given in reference 7,

$$V_R \approx b \omega_\alpha \sqrt{\frac{r_\alpha^2}{\kappa} \frac{0.5}{0.5 + a + x_\alpha}} = V_{R_0} \sqrt{\cos \Lambda}$$

This approximate analysis of the effect on the reference flutter speed does not depend upon the first bending frequency but assumes f_h/f_α to be small.

In order to include the effect of changes in bending-torsion frequency ratio, a more complete analysis must be carried out. Figure 20 presents some results of a numerical analysis based on a homogeneous wing with properties at zero sweep angle as follows:

$$\begin{array}{ll} x_{cg} = 50 & b_0 = 0.333 \\ x_{ea} = 45 & \left(\frac{1}{\kappa} \right)_0 = 10 \\ r_\alpha^2 = 0.25 & \\ f_\alpha = 100 & \left(\frac{f_{h1}}{f_\alpha} \right)_0 = 0.4 \end{array}$$

In figure 20 the curve showing the decrease in V_R with Λ is slightly above the $\sqrt{\cos \Lambda}$ factor indicated by the approximate formula.

Effect of elastic-axis position on reference flutter speed.—As pointed out in the definition of elastic axis, the measured

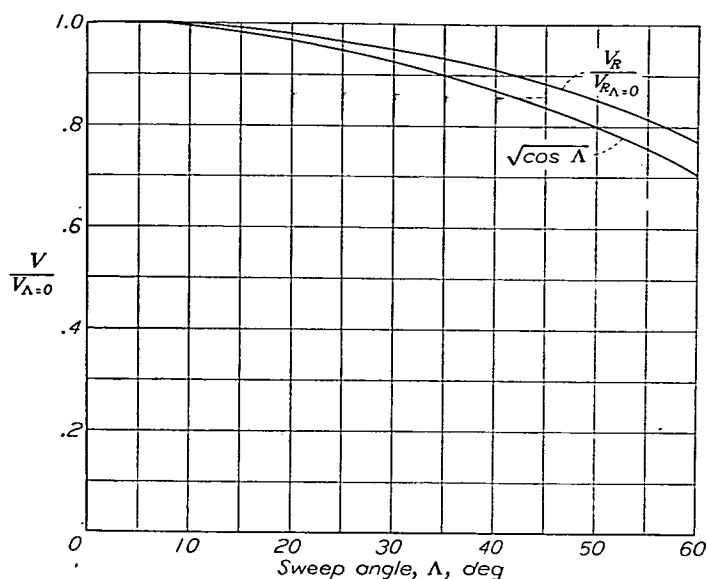


FIGURE 20.—Variation in reference flutter speed with sweep for sheared wings.

locus of elastic centers x_{ea}' fell behind the section elastic axis for the swept models with bases parallel to the air stream. In order to get an idea of the effect of elastic-axis position on the chosen reference flutter speed, computations were made both of V_R and a second reference flutter speed V_R' similar to V_R except that x_{ea}' was used in place of x_{ea} . The maximum difference between these two values of reference flutter speed was of the order of 7 percent. This difference occurred at a sweep angle of 60° when the wing elastic axis was farthest behind the section elastic axis. Thus, for wings of this type,

the reference flutter speed is not very sensitive to elastic-axis position. The reference flutter frequency f_R' was found in conjunction with V_R' . The maximum difference between f_R and f_R' was less than 10 percent. Thus, the convenient use of the reference flutter speed and reference frequency is not altered by these elastic-axis considerations.

REFERENCES

1. Kramer, Edward H.: The Effect of Sweepback on the Critical Flutter Speed of Wings. MR No. TSEAC5-4595-2-5, Eng. Div., Air Technical Service Command, Army Air Forces, March 15, 1946.
2. Broadbent, E. G.: Some Considerations of the Flutter Problems of High-Speed Aircraft. Sec. International Aero. Conference, N. Y., 1949, pp. 556-581.
3. Fettis, Henry E.: Calculations of the Flutter Characteristics of Swept Wings at Subsonic Speeds. MR No. TSEAC5-4595-2-9, Air Materiel Command, Army Air Forces, May 13, 1946.
4. Spielberg, Irvin, Fettis, H. E., and Toney, H. S.: Methods for Calculating the Flutter and Vibration Characteristics of Swept Wings. MR No. MCREXA5-4595-8-4, Air Materiel Command, U. S. Air Force, Aug. 3, 1948.
5. Theodorsen, Theodore: General Theory of Aerodynamic Instability and the Mechanism of Flutter. NACA Rep. 496, 1935.
6. Smilg, Benjamin, and Wasserman, Lee S.: Application of Three-Dimensional Flutter Theory to Aircraft Structures. ACTR No. 4798, Materiel Div., Army Air Corps, July 9, 1942.
7. Theodorsen, Theodore, and Garrick, I. E.: Mechanism of Flutter—A Theoretical and Experimental Investigation of the Flutter Problem. NACA Rep. 685, 1940.
8. Timoshenko, S.: Vibration Problems in Engineering. D. Van Nostrand Co., Inc., 1928.
9. Diederich, Franklin W., and Budiansky, Bernard: Divergence of Swept Wings. NACA TN 1680, 1948.

TABLE I.—DATA FOR SHEARED

Model	Λ (deg)	A_s	f_{h_1} (cps)	f_{h_2} (cps)	f_t (cps)	f_a (cps)	GJ (lb-in. ²)	EI (lb-in. ²)	NACA airfoil section	M_{cr}	l (in.)	c (in.)	b (ft)	x_{cg} (percent chord)	x_{ca} (percent chord)	x_{ca}' (percent chord)	$a+x_a$	a	r_a^2	$\frac{1}{\kappa}$	ρ (slugs cu ft)	Percent Freon
Spruce wings																						
11A	0	2	45	272	108	107	15,000	25,100	16-005	0.89	16.0	8.0	0.333	48.4	45	45	-0.032	-0.10	0.232	13.3	0.00287	95
11A'	0	2	26	-----	59	37	-----	-----	16-005	.89	16.0	8.0	.333	48.4	26.6	26.6	-.032	-.468	.396	17.6	.00217	0
11B'	0	2	29	-----	61	43	-----	-----	16-005	.89	16.0	8.0	.333	48.4	29.7	29.7	-.032	-.406	.371	40.5	.000943	88
12	15	2	43	-----	103	103	14,400	54,700	16-005.2	.88	16.6	7.72	.321	48.5	46.3	46	-.03	-.074	.23	5.69	.00725	96
12	15	2	42	-----	105	105	14,400	54,700	16-005.2	.88	16.6	7.72	.321	48.5	46.3	46	-.03	-.074	.23	8.47	.00486	98
12	15	2	42	-----	103	102	14,400	54,700	16-005.2	.88	16.6	7.72	.321	48.5	46.3	46	-.03	-.074	.23	11.2	.00367	97
13	30	2	33	196	94	93	11,100	53,500	16-005.8	.87	18.2	6.87	.284	48.8	46.0	49	-.024	-.080	.23	7.15	.00746	99
14	45	2	22	139	93	92	9,240	33,000	16-007.1	.85	22.6	5.62	.234	48.8	46.0	60	-.024	-.080	.23	7.78	.00720	85
14	45	2	21	136	92	91	9,240	33,000	16-007.1	.85	22.6	5.62	.234	48.8	46.0	60	-.024	-.080	.23	19.8	.00285	94
15	60	2	12	68	93	93	4,520	19,100	16-010	.81	32.0	4.0	.167	48.8	46.0	65	-.024	-.080	.23	9.10	.00757	92
15	60	2	12	67	93	93	4,520	19,100	16-010	.81	32.0	4.0	.167	48.8	46.0	65	-.024	-.080	.23	14.0	.00493	90
Balsa wings																						
22'	15	2	31	155	63	61	-----	-----	16-005.2	0.88	16.6	7.72	0.321	48.8	42.4	42.4	-0.024	-0.152	0.292	2.19	0.00854	98
22'	15	2	31	154	64	62	-----	-----	16-005.2	.88	16.6	7.72	.321	48.8	42.4	42.4	-.024	-.152	.292	3.82	.00488	93
22'	15	2	31	154	64	62	-----	-----	16-005.2	.88	16.6	7.72	.321	48.8	42.4	42.4	-.024	-.152	.292	18.7	.00100	92
23	30	2	35	219	89	89	6,230	27,900	16-005.8	.87	18.2	6.87	.284	48.0	48.0	52	-.04	-.04	.304	3.18	.00864	91
23	30	2	34	216	89	89	6,230	27,900	16-005.8	.87	18.2	6.87	.284	48.0	48.0	52	-.04	-.04	.304	8.54	.00321	91
23	30	2	34	220	91	91	6,230	27,900	16-005.8	.87	18.2	6.87	.284	48.0	48.0	52	-.04	-.04	.304	9.15	.00300	89
23	30	2	34	216	89	89	6,230	27,900	16-005.8	.87	18.2	6.87	.284	48.0	48.0	52	-.04	-.04	.304	14.9	.00184	90
24	45	2	19	123	73	73	2,810	10,800	16-007.1	.85	21.8	5.66	.236	47.0	49.0	57	-.06	-.02	.311	8.40	.00784	85
24	45	2	19	122	75	75	2,810	10,800	16-007.1	.85	21.8	5.66	.236	47.0	49.0	57	-.06	-.02	.311	13.2	.00216	93
24	45	2	19	122	75	75	2,810	10,800	16-007.1	.85	21.8	5.66	.236	47.0	49.0	57	-.06	-.02	.311	29.4	.000970	74
24	45	2	19	120	74	74	2,810	10,800	16-007.1	.85	21.8	5.66	.236	47.0	49.0	57	-.06	-.02	.311	30.6	.000933	89
24	45	2	19	120	73	73	2,810	10,800	16-007.1	.85	21.8	5.66	.236	47.0	49.0	57	-.06	-.02	.311	30.6	.000954	88
25A	60	2	8.6	54	66	65	1,950	6,470	16-010	.81	32.0	4.0	.167	46.9	40.0	71	-.062	-.20	.359	34.6	.000954	88
25B	60	2	8.6	48	70	68	-----	5,500	16-010	.81	32.0	4.0	.167	46.9	40.0	71	-.062	-.20	.359	9.36	.00353	91

TABLE II.—ROTATED

Model	Λ (deg)	A_s	f_{h_1} (cps)	f_{h_2} (cps)	f_t (cps)	f_a (cps)	GJ (lb-in. ²)	EI (lb-in. ²)	NACA airfoil section	M_{cr}	l (in.)	c (in.)	b (ft)	x_{cg} (percent chord)	x_{ca} (percent chord)	$x_{ca'}$ (percent chord)	$a+x_a$	a	r_a^2	$\frac{1}{\kappa}$	$\left(\frac{\rho}{\text{cu ft}}\right)$	Percent Freon
Lengthwise laminations																						
30A	0	6.20	11.9	76.0	90.4	83.0	3,760	-----	16-010	0.81	24.8	4	0.167	46.0	35	35	-0.08	-0.30	0.311	36.8	0.00220	0
30B	0	6.20	12.0	72.6	90.0	88.0	3,760	6,920	16-010	.81	24.8	4	.167	46.0	40	40	-.08	-.20	.277	37.8	.00214	0
30B	30	4.65	12.1	73.0	91.0	88.8	3,760	6,920	16-010	.81	24.8	4	.167	46.0	40	40	-.08	-.20	.277	37.7	.00215	0
30B	30	4.65	12.0	73.0	90.0	88.0	3,760	6,920	16-010	.81	24.8	4	.167	46.0	40	40	-.08	-.20	.277	37.8	.00214	0
30B	45	3.10	12.1	73.0	91.0	88.8	3,760	6,920	16-010	.81	24.8	4	.167	46.0	40	40	-.08	-.20	.277	37.8	.00214	0
30B	45	3.10	12.2	73.0	90.0	88.0	3,760	6,920	16-010	.81	24.8	4	.167	46.0	40	40	-.08	-.20	.277	37.8	.00214	0
30B	60	1.55	12.0	72.5	90.0	88.0	3,760	6,920	16-010	.81	24.8	4	.167	46.0	40	40	-.08	-.20	.277	39.8	.00204	0
30C	0	6.20	12.2	69.0	86.0	75.8	4,000	6,950	16-010	.81	24.8	4	.167	48.5	39	39	-.03	-.22	.292	40.5	.00200	89
30C	0	6.20	12.2	69.0	86.0	75.8	4,000	6,950	16-010	.81	24.8	4	.167	48.5	39	39	-.03	-.22	.292	98.9	.000820	86
30C	0	6.20	13.3	70.0	84.0	74.2	4,000	6,950	16-010	.81	24.8	4	.167	48.5	39	39	-.03	-.22	.292	92.6	.000876	83
30C	15	5.78	12.2	69.0	86.0	75.8	4,000	6,950	16-010	.81	24.8	4	.167	48.5	39	39	-.03	-.22	.292	92.6	.000870	81
30C	30	4.65	12.2	69.0	86.0	75.8	4,000	6,950	16-010	.81	24.8	4	.167	48.5	39	39	-.03	-.22	.292	40.0	.00202	89
30C	30	4.65	12.2	70.0	86.5	76.2	4,000	9,950	16-010	.81	24.8	4	.167	48.5	39	39	-.03	-.22	.292	81.4	.000995	86
30C	30	4.65	12.2	70.0	86.5	76.2	4,000	6,950	16-010	.81	24.8	4	.167	48.5	39	39	-.03	-.22	.292	80.0	.00100	85
30C	45	3.10	12.2	70.0	86.5	76.2	4,000	6,950	16-010	.81	24.8	4	.167	48.5	39	39	-.03	-.22	.292	45.2	.00179	87
30D	15	5.78	13.2	80.2	87.1	82.4	4,350	-----	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	8.70	.00933	99
30D	15	5.78	13.2	80.2	87.1	82.4	4,350	-----	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	8.72	.00930	99
30D	15	5.78	13.2	80.2	87.1	82.4	4,350	-----	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	8.76	.00927	99
30D	30	4.65	13.5	81.7	92.5	87.4	4,350	-----	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	8.90	.00910	99
30D	45	3.10	13.3	81.7	88.2	83.4	4,350	-----	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	8.85	.00905	99
30D	60	1.55	13.5	82.0	90.5	85.5	4,350	-----	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	9.54	.00852	99
Chordwise laminations																						
40A	0	6.20	9.4	57.4	90.0	88.4	3,540	5,250	16-010	0.81	24.8	4	0.167	46	40	40	-0.08	-0.20	0.277	36.5	0.00222	0
40A	0	6.20	9.6	57.1	91.0	88.5	3,540	5,250	16-010	.81	24.8	4	.167	46	40	40	-.08	-.20	.277	24.2	.00334	90
40A	0	6.20	9.6	57.1	91.0	88.5	3,540	5,250	16-010	.81	24.8	4	.167	46	40	40	-.08	-.20	.277	37.7	.00215	89
40A	0	6.20	9.6	57.1	91.0	88.5	3,540	5,250	16-010	.81	24.8	4	.167	46	40	40	-.08	-.20	.277	75.0	.00108	82
40A	15	5.78	9.3	55.8	90.6	88.2	3,540	5,250	16-010	.81	24.8	4	.167	46	40	40	-.08	-.20	.277	35.1	.00231	0
40A	30	4.65	9.3	55.8	90.6	88.2	3,540	5,250	16-010	.81	24.8	4	.167	46	40	40	-.08	-.20	.277	37.5	.00216	0
40B	0	6.20	9.5	55.0	90.5	85.5	3,710	5,020	16-010	.81	24.8	4	.167	49	40	40	-.02	-.20	.297	35.5	.00228	0
40C	0	6.20	9.0	54.4	61.0	58.2	2,280	4,350	16-010	.81	24.8	4	.167	46	38.5	38.5	-.08	-.23	.287	8.74	.00928	100
40D	0	6.20	9.4	58.0	88.9	84.0	3,330	5,050	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	79.0	.000969	84
40D	15	5.78	9.6	58.3	88.9	84.0	3,330	5,050	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	36.2	.00212	89
40D	15	5.78	9.5	57.9	87.5	82.6	3,330	5,050	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	80.0	.000956	87
40D	30	4.65	9.5	57.5	89.0	84.1	3,330	5,050	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	88.2	.000867	85
40D	45	3.10	9.6	58.3	88.9	84.0	3,330	5,050	16-010	.81	24.8	4	.167	48	39.5	39.5	-.04	-.21	.280	39.1	.00196	86

SWEPT MODELS—SERIES I

f_s (cps)	f_R (cps)	f_A (cps)	$\frac{f_s}{f_a}$	$\frac{f_s}{f_R}$	$\frac{f_s}{f_A}$	ϕ (deg)	$\left(\frac{q}{\text{lb}}\right)$ (sq ft)	M	V_s (mph)	V_R (mph)	V_R' (mph)	V_A (mph)	$\frac{v_s}{b\omega_\alpha}$	$\frac{V_s}{V_R}$	$\frac{V_s}{V_A}$	V_D (mph)	Remarks
Spruce wings																	
66	70	-----	0.62	0.93	----	50	235	0.82	274	260	260	----	1.80	1.05	----	314	Tunnel excitation frequency, 67 cps. Model failed. { Slotted 4¾ inches from trailing edge. Slots uncovered.
42	40	-----	1.12	1.03	----	80	85.0	.24	191	129	129	----	3.58	1.48	----	583	
38	42	-----	.87	.91	----	170	70.5	.74	262	197	197	----	4.22	1.33	----	183	
64	70	-----	.63	.92	----	70	375	.64	218	176	----	----	1.54	1.24	----	175	Tunnel excitation frequency, 61 cps.
62	71	71	.59	.87	0.87	50	320	.71	245	206	----	205	1.70	1.19	1.20	217	
55	69	69	.54	.80	.80	50	307	.79	276	225	----	220	1.95	1.23	1.25	245	
61	60	65	.66	1.01	.94	70	334	.62	202	154	----	159	1.77	1.31	1.27	149	Model failed.
54	56	61	.59	.97	.88	60	300	.56	196	134	----	166	2.11	1.46	1.18	119	
37	51	53	.41	.72	.70	40	234	.81	275	191	----	245	2.99	1.44	1.12	187	
37	53	58	.40	.70	.64	40	265	.51	179	103	107	184	2.71	1.73	.97	105	Model failed.
36	51	55	.39	.71	.65	30	264	.62	222	124	127	222	3.35	-1.79	1.00	122	
Balsa wings																	
50	46	---	0.82	1.07	-----	70	101	0.30	104	97.3	-----	-----	1.25	1.07	-----	79.9	Tunnel excitation frequency, 49 cps. Slotted 2¾ inches from trailing edge.
51	48	48	.83	1.07	1.06	50	74.7	.34	119	95.0	-----	96	1.41	1.25	1.24	107	
45	46	46	.72	.96	.98	50	54.2	.64	224	167	-----	168	2.64	1.34	1.33	238	
60	62	---	.68	.96	-----	130	189	.42	142	137	-----	-----	1.31	1.04	-----	110	Tunnel excitation frequency, 61 cps. Tunnel excitation frequency, 61 cps.
62	62	65	.70	1.01	.95	70	152	.62	212	176	-----	175	1.95	1.21	1.21	180	
60	63	---	.67	.96	-----	60	171	.66	229	185	-----	-----	2.07	1.24	-----	190	
53	60	65	.59	.87	.82	90	152	.81	275	221	-----	224	2.53	1.24	1.23	237	Model failed.
51	49	---	.71	1.06	-----	90	125	.34	121	97.1	-----	-----	1.63	1.25	-----	80.1	
49	49	58	.65	1.00	.84	40	120	.54	180	132	-----	145	2.35	1.37	1.24	127	
45	48	---	.60	.95	-----	40	108	.64	215	160	-----	-----	2.82	1.35	-----	159	Model failed.
---	44	---	---	---	---	---	83.5	.76	281	226	-----	-----	3.76	1.25	-----	232	
34	43	45	.47	.79	.75	60	78.0	.81	277	226	-----	252	3.77	1.22	1.10	232	
29	37	40	.44	.75	.72	10	76.8	.79	272	161	169	278	5.90	1.69	.98	210	Model failed.
---	45	48	---	---	---	---	73.6	.41	139	93.5	97.5	161	2.85	1.49	.86	115	

WINGS—SERIES II

f_s (cps)	f_R (cps)	f_A (cps)	$\frac{f_s}{f_a}$	$\frac{f_s}{f_R}$	$\frac{f_s}{f_A}$	φ (deg)	$\left(\frac{q}{\text{sq ft}}\right)$	M	V_s (mph)	V_R (mph)	V_R' (mph)	V_A (mph)	$\frac{v_s}{b\omega_\alpha}$	$\frac{V_s}{V_R}$	$\frac{V_s}{V_A}$	V_D (mph)	Remarks
Lengthwise laminations																	
42	45	---	0.51	0.91	---	70	127	0.30	232	209	209	---	3.91	1.11	---	318	Wing failed; tunnel excitation frequency, 40.7 cps.
48	44	46	.54	1.08	1.04	60	121	.29	229	212	212	215	3.64	1.08	1.06	263	
51	47	47	.57	1.08	1.08	60	126	.30	235	214	214	229	3.74	1.10	1.03	266	
50	44	47	.57	1.14	1.08	40	129	.30	237	212	212	229	3.77	1.12	1.04	263	
---	44	---	---	---	---	---	166	.34	269	214	214	265	4.28	1.26	1.01	266	Wing failed.
55	44	47	.62	1.25	1.16	50	169	.35	272	212	212	265	4.32	1.28	1.02	263	
---	46	48	---	---	---	---	275	.45	350	219	219	353	5.59	1.60	.99	265	
34	41	---	.45	.83	---	30	104	.63	219	189	189	---	4.05	1.16	---	249	
24	37	---	.32	.66	---	30	74.4	.81	286	290	290	---	5.29	.986	---	393	Wing failed.
21	36	---	.29	.59	---	30	79.6	.82	288	270	270	---	5.43	1.07	---	369	
27	36	---	.36	.74	---	30	72.5	.78	278	282	282	---	5.13	.986	---	376	
37	41	---	.48	.89	---	50	113	.65	226	187	187	---	4.18	1.21	---	248	
---	41	---	---	---	---	---	88.1	.81	284	263	263	---	5.22	1.08	---	355	Wing failed.
31	38	---	.40	.80	---	30	88.6	.81	289	260	260	---	5.32	1.11	---	352	
40	41	---	.53	.98	---	40	147	.76	273	199	199	---	5.02	1.37	---	265	
50	51	51	.61	.98	.98	50	110	.31	104	100	100	100	1.77	1.05	1.04	119	
51	52	52	.61	.98	.98	50	115	.32	107	100	100	101	1.82	1.08	1.06	119	Tunnel excitation frequency, 57 cps.
51	52	52	.61	.98	.98	50	121	.33	109	100	100	101	1.85	1.10	1.08	119	
53	54	56	.61	.98	.95	40	150	.38	123	106	106	116	1.97	1.16	1.06	129	
56	52	55	.67	1.08	1.02	60	178	.41	135	101	101	130	2.26	1.34	1.04	122	
65	53	57	.77	1.24	1.14	90	307	.55	182	107	107	182	2.98	1.70	1.00	130	
Chordwise laminations																	
62	47	---	0.70	1.33	---	140	82.0	0.24	188	211	211	---	2.98	0.892	---	260	Tunnel excitation frequency, 57 cps.
56	49	---	.63	1.15	---	60	86.7	.45	155	184	184	---	2.45	.843	---	212	
61	46	---	.69	1.33	---	70	69.2	.50	172	215	215	---	2.72	.800	---	265	
61	43	---	.69	1.44	---	70	63.6	.65	234	299	299	---	3.70	.784	---	373	
61	46	---	.68	1.30	---	90	93.9	.26	201	208	208	---	3.19	.967	---	254	Wing failed.
---	46	---	---	---	---	---	127	.30	235	213	213	---	3.73	1.10	---	263	
61	45	---	.71	1.37	---	10	77.7	.23	178	191	191	---	2.91	.932	---	247	
29	36	---	.51	.83	---	80	57.6	.23	75.3	74.5	74.5	---	1.81	1.01	---	90.4	
62	40	---	.73	1.54	---	30	52.3	.62	221	281	281	---	3.69	.787	---	370	Tunnel excitation frequency, 61 cps.
62	44	---	.74	1.41	---	70	72.7	.51	177	194	194	---	2.95	.913	---	251	
61	40	---	.74	1.54	---	50	57.9	.67	236	279	279	---	3.99	.846	---	367	
65	40	---	.77	1.63	---	60	79.4	.82	290	298	298	---	4.83	.973	---	392	
32	44	---	.38	.73	---	80	138	.73	254	200	200	---	4.24	1.27	---	261	Wing failed.

SWEEP FORWARD TESTS—SERIES III

f_s (cps)	f_R (cps)	f_A (cps)	$\frac{f_s}{f_a}$	$\frac{f_s}{f_R}$	$\frac{f_s}{f_A}$	ϕ (deg)	$\left(\frac{q}{\text{sq ft}}\right)$	M	V_s (mph)	V_R (mph)	V_R' (mph)	V_A (mph)	$\frac{v_s}{b\omega_\alpha}$	$\frac{V_s}{V_R}$	$\frac{V_s}{V_A}$	V_D (mph)	Remarks
---	98	---	---	---	---	---	73.4	0.26	86.9	174	174	-----	0.888	0.498	-----	294	Model diverged.
---	98	---	---	---	---	---	107	.31	105	174	174	-----	1.075	.603	-----	294	
102	79	---	0.77	1.29	---	40	211	.40	303	319	319	-----	3.18	.949	-----	579	
91	94	---	.78	.97	---	100	260	.52	170	172	172	-----	2.05	.989	-----	704	Model failed.
84	94	---	.72	.90	---	70	257	.51	169	172	172	-----	2.04	.982	-----	700	
74	93	---	.63	.80	---	180	352	.61	202	179	179	-----	2.44	1.125	-----	720	
98	93	---	.84	1.05	---	100	423	.68	226	179	179	-----	2.73	1.265	-----	736	

TABLE IV.—SWEEPED MODELS OF A CONSTANT

Model	Λ (deg)	A_s	f_{h_1} (cps)	f_{h_2} (cps)	f_t (cps)	f_a (cps)	GJ (lb-in. ²)	EI (lb-in. ²)	NACA airfoil section	M_{cr}	l (in.)	c (in.)	b (ft)	x_{cg} (percent chord)	x_{sa} (percent chord)	$x_{sa'}$ (percent chord)	$a + x_a$	a	r_a^2
62	15	7.95	4.9	29.1	72.5	71.8	3,730	7,820	16-010	0.81	34	4	0.167	41	44	46	-0.18	-0.12	0.175
62	15	7.95	4.9	29.1	73.4	72.5	3,730	7,820	16-010	.81	34	4	.167	41	44	46	-.18	-.12	.175
62	15	7.95	4.9	29.1	73.4	72.5	3,730	7,820	16-010	.81	34	4	.167	41	44	46	-.18	-.12	.175
62	15	7.95	4.9	29.6	73.5	72.7	3,730	7,820	16-010	.81	34	4	.167	41	44	46	-.18	-.12	.175
63	30	6.38	4.6	25.8	73.5	73.0	5,450	5,870	16-010	.81	34	4	.167	41	44	47	-.18	-.12	.175
63	30	6.38	3.9	24.0	73.0	72.4	5,450	5,870	16-010	.81	34	4	.167	41	44	47	-.18	-.12	.175
63	30	6.38	4.6	25.8	73.5	73.0	5,450	5,870	16-010	.81	34	4	.167	41	44	47	-.18	-.12	.175
63	30	6.38	4.0	24.0	73.0	72.4	5,450	5,870	16-010	.81	34	4	.167	41	44	47	-.18	-.12	.175
63	30	6.38	4.0	24.0	73.0	72.4	5,450	5,870	16-010	.81	34	4	.167	41	44	47	-.18	-.12	.175
64	45	4.75	4.4	29.0	66.0	65.5	3,500	6,080	16-010	.81	34	4	.167	41	44	57	-.18	-.12	.175
64	45	4.75	4.2	27.0	66.0	65.5	3,500	6,080	16-010	.81	34	4	.167	41	44	57	-.18	-.12	.175
64	45	4.75	4.2	27.0	66.0	65.5	3,500	6,080	16-010	.81	34	4	.167	41	44	57	-.18	-.12	.175
64	45	4.75	4.1	27.0	65.0	64.4	3,500	6,080	16-010	.81	34	4	.167	41	44	57	-.18	-.12	.175
64	45	4.75	4.1	27.0	65.0	64.4	3,500	6,080	16-010	.81	34	4	.167	41	44	57	-.18	-.12	.175
65	60	2.12	5.7	33.4	77.0	76.2	4,650	11,980	16-010	.81	34	4	.167	41	44	71	-.18	-.12	.175

TABLE V.—DATA FOR SWEEPED MODELS OF A

Model	Λ (deg)	A_s	f_{h_1} (cps)	f_{h_2} (cps)	f_t (cps)	f_a (cps)	GJ (lb-in. ²)	EI (lb-in. ²)	NACA airfoil section	M_{cr}	l (in.)	c (in.)	b (ft)	x_{cg} (percent chord)	x_{sa} (percent chord)	$x_{sa'}$ (percent chord)	$a + x_a$	a	r_a^2
72	15	6.09	7.6	54	97.3	96.3	3,730	7,820	16-010	0.81	26	4	0.167	41	44	46	-0.18	-0.12	0.175
72	15	6.09	7.6	54	97.3	96.3	3,730	7,820	16-010	.81	26	4	.167	41	44	46	-.18	-.12	.175
73	30	4.88	6.4	40.0	98.0	97.0	5,450	5,870	16-010	.81	26	4	.167	41	44	47	-.18	-.12	.175
73	30	4.88	6.4	40.0	98.0	97.0	5,450	5,870	16-010	.81	26	4	.167	41	44	47	-.18	-.12	.175
73	30	4.88	6.4	40.0	98.0	97.0	5,450	5,870	16-010	.81	26	4	.167	41	44	47	-.18	-.12	.175
74	45	3.25	6.5	40.0	79.0	78.2	3,500	6,080	16-010	.81	26	4	.167	41	44	57	-.18	-.12	.175
74	45	3.25	6.7	39.5	78.5	77.7	3,500	6,080	16-010	.81	26	4	.167	41	44	57	-.18	-.12	.175
74	45	3.25	6.7	39.5	78.5	77.7	3,500	6,080	16-010	.81	26	4	.167	41	44	57	-.18	-.12	.175
75	60	1.65	7.2	51.8	82.4	81.6	4,650	11,980	16-010	.81	26	4	.167	41	44	71	-.18	-.12	.175
75	60	1.65	7.2	51.8	84.6	83.8	4,650	11,980	16-010	.81	26	4	.167	41	44	71	-.18	-.12	.175

TABLE VI.—DATA FOR TIP-

Model	Λ (deg)	A_s	f_{h_1} (cps)	f_{h_2} (cps)	f_t (cps)	f_a (cps)	GJ (lb-in. ²)	EI (lb-in. ²)	NACA airfoil section	M_{cr}	l (in.)	c (in.)	b (ft)	x_{cg} (percent chord)	x_{sa} (percent chord)	$x_{sa'}$ (percent chord)	$a + x_a$	a	r_a
84-1	45	3.63	10	60	133	104	-----	-----	16-010	0.81	29	4	0.167	51	32	44	0.02	-0.36	0.378
84-2	45	3.63	10	61	135	107	-----	-----	16-010	.81	29	4	.167	51	32	44	.02	-.36	.378
84-3	45	3.63	9.6	58	118	93	-----	-----	16-010	.81	29	4	.167	51.5	32	44	.03	-.36	.378
85-1	60	2.75	5.0	32	92	72	10,800	13,400	16-010	.81	44	4	.167	50	32	58	.0	-.36	.378
85-2	60	2.75	5.0	31	95	75	9,850	12,400	16-010	.81	44	4	.167	50	32	58	.0	-.36	.378
85-3	60	2.75	5.0	30	80	63	11,200	16,600	16-010	.81	44	4	.167	51	32	58	.02	-.36	.378

TABLE VII.—DATA FOR MODELS USED TO DETERMINE

Model	Λ (deg)	A_s	f_{h_1} (cps)	f_{h_2} (cps)	f_t (cps)	f_a (cps)	GJ (lb-in. ²)	EI (lb-in. ²)	NACA airfoil section	l (in.)	c (in.)	b (ft)	x_{cg} (percent chord)	x_{sa} (percent chord)	$x_{sa'}$ (percent chord)	$a + x_a$	a	r_a^2	$\frac{1}{\kappa}$
91-1	0	6	4.2	24	31	23	34,100	128,000	16-010	48	8	0.333	29.9	48	48	-0.402	-0.04	0.307	17.3
91-2	0	6	5.5	36	43	43	41,200	108,300	16-010	48	8	.333	41.0	43.8	43.8	-.18	-.124	.179	41.7
91-2	0	6	5.5	36	43	43	41,200	108,300	16-010	48	8	.333	41.0	43.8	43.8	-.18	-.124	.179	56.4
91-2	0	6	5.3	33	42	42	41,200	108,300	16-010	48	8	.333	41.0	43.8	43.8	-.18	-.124	.179	12.8
91-2	0	6	5.5	36	43	43	41,200	108,300	16-010	48	8	.333	41.0	43.8	43.8	-.18	-.124	.179	95.5
91-3	0	6	5.0	30	40	40	28,500	83,700	16-010	48	8	.333	49.0	48.4	48.4	-.02	-.032	.160	44.3
91-3	0	6	4.7	29	39	39	28,500	83,700	16-010	48	8	.333	49.0	48.4	48.4	-.02	-.032	.160	36.4
91-3	0	6	4.7	29	39	39	28,500	83,700	16-010	48	8	.333	49.0	48.4	48.4	-.02	-.032	.160	48.4
92-1	15	6.09	8.3	48	70	62	3,730	7,820	Modified 16-010	26	4	.167	31.2	44	46	-.376	-.12	.298	77.9
92-2	15	6.09	8.3	49	95	95	3,730	7,820	Modified 16-010	26	4	.167	42.9	44	46	-.142	-.12	.136	76.0
92-3	15	6.09	8.1	47	55	52	3,730	7,820	Modified 16-010	26	4	.167	54.5	44	46	.090	-.12	.411	74.5
93-1	30	4.42	6.3	40	78	68	5,450	5,870	Modified 16-010	23.6	4	.167	30	44	47	-.40	-.12	.310	78.0
93-2	30	4.42	6.8	44	99	99	5,450	5,870	Modified 16-010	23.6	4	.167	43	44	47	-.16	-.12	.134	74.0
93-3	30	4.42	6.3	51	54	50	5,450	5,870	Modified 16-010	23.6	4	.167	56	44	47	.12	-.12	.428	73.2
94-1	-(-45)	3.81	4.5	26	38	35	2,120	4,520	Modified 16-010	30.5	4	.167	44.5	56	-----	-.11	.12	.427	68.2
94-2	-(-45)	3.81	4.8	28	70	70	2,120	4,520	Modified 16-010	30.5	4	.167	57.0	56	-----	.14	.12	.134	68.2
94-3	-(-45)	3.81	4.6	28	40	38	2,120	4,520	Modified 16-010	30.5	4	.167	69.3	56	-----	.386	.12	.307	68.2
95'-1	60	1.65	5.6	-----	54	50	1,900	4,560	Modified 16-010	26.4	4	.167	31.4	22	41	-.372	-.56	.267	75.8
95'-2	60	1.65	5.9	-----	71	47	1,900	4,560	Modified 16-010	26.4	4	.167	42.8	22	41	-.144	-.56	.308	73.0
95'-3	60	1.65	5.8	35	40	27	1,900	4,560	Modified 16-010	26.4	4	.167	54.3	22	41	.086	-.56	.779	69.0

LENGTH-CHORD RATIO OF 8.5—SERIES IV

$\frac{1}{\kappa}$	$\left(\frac{\rho}{\text{slugs}}\right)$ (cu ft)	Percent Freon	f_s (cps)	f_R (cps)	$\frac{f_s}{f_a}$	$\frac{f_s}{f_R}$	φ (deg)	$\left(\frac{q}{\text{lb}}\right)$ (sq ft)	M	V_s (mph)	V_R (mph)	V_R' (mph)	$\frac{v_s}{b\omega_a}$	$\frac{V_s}{V_R}$	V_D (mph)	Remarks
13.5	0.00925	99	22	35	0.28	0.59	30	91.8	0.29	95.4	105	104	1.85	0.905	91.6	No record. Record shown in figure 4.
37.6	.00333	88	20	32	.28	.64	20	73.7	.41	143	167	171	2.76	.856	153	
59.5	.00210	87	19	31	.26	.60	20	69.7	.49	175	206	---	3.37	.850	192	
130.0	.000964	85	16	29	.22	.55	20	57.5	.66	234	300	---	4.50	.780	284	
15.2	.00745	73	19	35	.27	.56	180	98.8	.29	111	111	---	2.12	1.000	97.6	
26.8	.00424	98	18	33	.25	.56	110	78.0	.38	129	142	---	2.49	.908	128	
46.0	.00246	50	22	32	.30	.69	180	82.1	.40	176	183	---	3.37	.962	170	
53.0	.00214	94	19	31	.26	.61	140	74.0	.52	179	195	---	3.46	.918	180	
98.2	.00116	92	15	29	.20	.50	120	62.2	.64	222	262	---	4.30	.848	246	
50.9	.00217	0	19	28	.29	.67	30	69.6	.22	173	174	176	3.69	.995	166	
12.1	.00914	97	---	32	---	---	---	70.6	.24	83.9	91	90	1.80	.923	111	
41.9	.00263	54	18	29	.27	.61	0	68.3	.36	155	160	160	3.31	.968	132	
51.3	.00215	92	17	27	.26	.62	30	63.5	.47	165	172	171	3.59	.960	173	
116.0	.000953	86	16	25	.25	.65	0	57.5	.66	235	248	---	5.10	.948	260	
44.1	.00297	94	17	33	.22	.51	0	172	.67	234	186	---	4.29	1.258	176	

CONSTANT LENGTH-CHORD RATIO OF 6.5—SERIES V

$\frac{1}{\kappa}$	$\left(\frac{\rho}{\text{slugs}}\right)$ (cu ft)	Percent Freon	f_s (cps)	f_R (cps)	$\frac{f_s}{f_a}$	$\frac{f_s}{f_R}$	φ (deg)	$\left(\frac{q}{\text{lb}}\right)$ (sq ft)	M	V_s (mph)	V_R (mph)	V_R' (mph)	$\frac{v_s}{b\omega_a}$	$\frac{V_s}{V_R}$	V_D (mph)	Remarks
37.2	0.00336	94	30	43	0.31	0.71	10	143	0.59	197	220	221	2.88	0.895	201	Wing failed. Model damaged at root. Rear half separated from base.
81.5	.00153	89	22	40	.23	.55	0	109	.74	255	318	319	3.73	.804	297	
34.7	.00327	96	29	43	.30	.67	---	133	.57	193	216	214	2.78	.893	196	
57.4	.00188	95	24	41	.24	.57	80	118	.69	234	273	---	3.38	.853	252	
108	.00105	93	22	39	.22	.55	---	90.8	.82	280	363	---	4.05	.770	345	
14.2	.00779	98	29	37	.37	.77	0	118	.35	118	115	---	2.11	1.025	111	
56.0	.00197	93	26	33	.33	.77	0	104	.64	219	214	---	3.95	1.023	218	
120	.000923	90	21	31	.28	.69	0	85.5	.83	291	308	---	5.24	.945	320	
15.8	.00829	95	39	39	.47	.99	30	294	.54	181	127	128	3.11	1.425	113	
16.7	.00783	100	39	38	.46	.97	0	295	.56	186	134	136	3.05	1.386	122	

EFFECT MODELS—SERIES VI

$\frac{1}{\kappa}$	$\left(\frac{\rho}{\text{slugs}}\right)$ (cu ft)	Percent Freon	f_s (cps)	f_R (cps)	$\frac{f_s}{f_a}$	$\frac{f_s}{f_R}$	φ (deg)	$\left(\frac{q}{\text{lb}}\right)$ (sq ft)	M	V_s (mph)	V_R (mph)	V_R' (mph)	$\frac{v_s}{b\omega_a}$	$\frac{V_s}{V_R}$	V_D (mph)	Remarks
9.15	0.00781	99	75	76	0.65	0.89	50	339	0.60	199	142	---	2.66	1.40	253	Tip perpendicular to air stream. Model failed. Tip perpendicular to leading edge. Model failed. Tip parallel to air stream. Model failed. Tip perpendicular to air stream. Model failed. Tip perpendicular to leading edge. Model failed. Tip parallel to air stream. Model failed.
9.25	.00764	99	60	78	.51	.70	0	382	.63	213	146	---	2.80	1.47	259	
9.55	.00778	99	---	68	---	---	---	346	.60	201	127	---	3.02	1.58	229	
34.6	.00205	0	35	43	.44	.72	---	225	.41	322	185	189	6.24	1.74	341	
34.1	.00208	0	27	46	.33	.54	---	173	.35	278	189	196	5.21	1.47	348	
34.5	.00207	0	22	38	.32	.53	0	203	.39	304	159	159	6.77	1.91	295	

EFFECT OF CENTER-OF-GRAVITY SHIFT—SERIES VII

$\left(\frac{\rho}{\text{slugs}}\right)$ (cu ft)	Percent Freon	f_s (cps)	f_R (cps)	$\frac{f_s}{f_a}$	$\frac{f_s}{f_R}$	φ (deg)	$\left(\frac{q}{\text{lb}}\right)$ (sq ft)	M	V_s (mph)	V_R (mph)	V_R' (mph)	$\frac{v_s}{b\omega_a}$	$\frac{V_s}{V_R}$	V_D (mph)	Remarks
0.00871	95	12.5	15	0.54	0.82	---	153	0.37	127	231	231	3.83	0.548	79.9	Model failed.
.00239	0	16	19	.37	.81	40	109	.28	208	207	207	3.40	1.000	192	
.00177	0	16	19	.38	.86	20	105	.32	239	239	239	3.93	1.000	224	
.00783	81	20	21	.47	.94	40	128	.33	122	120	120	2.05	1.02	104	
.00105	0	15	18	.35	.83	30	106	.40	303	308	308	4.97	.985	291	
.00226	0	18	17	.45	1.09	100	61.5	.20	159	158	158	2.78	1.01	157	
.00274	76	15	17	.39	.91	10	58.4	.39	142	141	141	2.54	1.01	139	
.00207	75	14	16	.37	.89	0	57.2	.44	163	161	161	2.92	1.01	161	
.00214	0	26	36	.42	.72	0	195	.38	293	415	422	6.60	.706	245	
.00219	0	22	36	.23	.66	20	151	.33	255	258	257	3.76	.990	251	
.00224	0	26	28	.49	.93	20	87.5	.25	191	176	177	5.12	1.09	237	
.00199	0	26	26	.39	.65	---	225	.41	324	503	---	6.73	.645	267	
.00210	0	23	37	.23	.64	70	156	.34	264	265	---	3.72	.997	257	
.00212	0	23	27	.45	.85	20	77.2	.23	185	170	---	5.15	1.09	231	
.00223	0	18	20	.51	.88	20	61.0	.20	160	160	---	6.38	1.00	122	
.00223	0	18	23	.26	.78	---	62.2	.21	162	139	---	3.24	1.17	136	
.00223	0	17	16	.44	1.04	40	39.5	.17	129	93.2	---	4.78	1.39	110	
.00201	0	24	27	.49	.89	30	258	.44	345	279	300	5.20	1.24	∞	Section reversed. Slotted 2 1/4 inches from trailing edge.
.00209	0	23	26	.48	.86	20	212	.40	307	186	189	9.15	1.66	∞	
.00218	0	23	20	.84	1.03	30	125	.30	234	121	123	12.1	1.94	∞	

REPORT 1015

ANALYSIS OF TURBULENT FREE-CONVECTION BOUNDARY LAYER ON FLAT PLATE¹

By E. R. G. ECKERT AND THOMAS W. JACKSON

SUMMARY

With the use of Kármán's integrated momentum equation for the boundary layer and data on the wall-shearing stress and heat transfer in forced-convection flow, a calculation was carried out for the flow and heat transfer in the turbulent free-convection boundary layer on a vertical flat plate. The calculation is for a fluid with a Prandtl number that is close to 1.

A formula was derived for the heat-transfer coefficient that was in good agreement with experimental data in the range of Grashof numbers from 10^{10} to 10^{12} . Because of the good agreement between the theoretical formula and the experimental data, the formula may be used to obtain data for higher Grashof numbers. The calculation also yielded formulas for the maximum velocity in the boundary layer and for the boundary-layer thickness.

INTRODUCTION

Recent developments in the field of gas turbines have revealed the need for data on heat transfer in turbulent free-convection flow at very high Grashof numbers (10^{12} to 10^{15}). For example, in using the method of free-convection cooling for turbine blades, the centrifugal forces generate a free-convection flow of the cooling fluid in the blade passages that is within the preceding range of Grashof numbers. Free-convection flow is also superimposed on the flow of the cooling air in the hollow blades of air-cooled gas turbines and may influence considerably the heat transfer under certain conditions. The radial flow present in the cooled boundary layers on the outside of cooled turbine blades is also of a type similar to the free-convection flow in the blade coolant passages.

In order to understand and to evaluate such cooling processes, information on the heat transfer, the boundary-layer thickness, and the velocities connected with free-convection flow is necessary. The knowledge of turbulent free-convection flow, however, is limited. Experiments on plane vertical surfaces give heat-transfer data up to Grashof times Prandtl numbers of 10^{12} (summarized in references 1 and 2). Griffiths and Davis (reference 3) determined, in addition, temperature and velocity profiles. Watzinger and Johnson (reference 4) measured heat transfer and temperature and velocity profiles in a superimposed forced- and free-convection flow in a vertical tube. Brown and Marko (reference 5) show by theoretical considerations that a gen-

eral relation exists between the Grashof number that characterizes free-convection flow and the Reynolds number that characterizes forced flow. Colburn and Hougen (reference 6) derive a formula for the heat transfer in turbulent free-convection flow on a vertical plate. However, only the laminar sublayer in the immediate neighborhood of the wall was investigated in reference 6 and the thickness of this sublayer, made dimensionless by the shearing stress velocity and the kinematic viscosity, was assumed to be the same as in forced flow. This analysis therefore gives no information on the whole boundary-layer thickness and on the velocities.

The problem of turbulent free-convection flow is approached herein using another method. Certain shapes are assumed for the temperature and velocity profiles in the free-convection boundary layer. In addition, an empirical relation for the shearing stress on the wall and a heat-transfer coefficient derived from forced-convection flow are used to estimate the boundary-layer thickness, the maximum velocity within the boundary layer, and a heat-transfer coefficient for free-convection flow. No experimental free-convection measurements are used in the calculations.

SHEARING STRESS AND HEAT FLOW IN FORCED-CONVECTION BOUNDARY LAYER

The relations for forced-convection flow that are needed in the theoretical free-convection calculations are compiled in this section. It is known that for Reynolds numbers that are not too high, the velocity profile in a turbulent boundary layer on a flat plate can be represented by the equation

$$u = u_1 \left(\frac{y}{\delta} \right)^{1/7} \quad (1)$$

(All symbols are defined in the appendix.) The shearing stress on the plate surface in such a flow is given by the equation (reference 7,)

$$\tau_w = 0.0225 \rho u^2 \left(\frac{\nu}{uy} \right)^{1/4} \quad (2)$$

This equation is the relation for the shearing stress on the wall that is used in the derivation of the boundary-layer equations for the turbulent free-convection boundary layer. By introducing equation (1), the velocity profile, the shear-

¹ Supersedes NACA TN 2207, "Analysis of Turbulent Free-Convection Boundary Layer on Flat Plate" by E. R. G. Eckert and Thomas W. Jackson, 1950.

ing stress can also be expressed by the velocity u_1 at the outer edge of the boundary layer and by the boundary-layer thickness δ :

$$\tau_w = 0.0225 \rho u_1^2 \left(\frac{\nu}{u_1 \delta} \right)^{1/4} \quad (3)$$

For fluids or gases with a Prandtl number equal to 1, the temperature profile is similar in shape to the velocity profile. When the temperatures within the boundary layer are measured with the temperature outside the boundary layer as reference, the expression for the temperature profile is

$$\theta = \theta_w \left[1 - \left(\frac{y}{\delta} \right)^{1/7} \right] \quad (4)$$

Equation (4) is valid for values of y less than δ . For larger values of y , $\theta = 0$.

Reynolds analogy between the turbulent exchange of momentum and heat gives the relation (reference 8)

$$\frac{q_w}{\tau_w} = g c_p \frac{dt}{du} \quad (5)$$

Because the temperature and velocity profiles are similar in shape for a Prandtl number equal to 1, the finite temperature and velocity differences at two arbitrary points in the flow can be introduced into the last equation instead of the differentials. Using the differences between the wall and the flow outside the boundary layer and specifying the heat flow and the shearing stress for the position at the wall give the following equation:

$$\frac{q_w}{\tau_w} = g c_p \frac{\theta_w}{u_1} \quad (6)$$

The introduction of the law (equation (3)) for the shearing stress into this expression yields the following equation:

$$q_w = 0.0225 g \rho c_p u_1 \theta_w \left(\frac{\nu}{u_1 \delta} \right)^{1/4} \quad (7)$$

Heat transfer is usually calculated with dimensionless moduli. Such a modulus is the Stanton number

$$St = \frac{q_w}{g \rho c_p u_1 \theta_w} = 0.0225 \left(\frac{\nu}{u_1 \delta} \right)^{1/4}$$

Experimental investigations show that for fluids with Prandtl numbers from approximately 0.5 to 50 (reference 1, p. 520) the same relation holds when it is multiplied by the factor $Pr^{-2/3}$.

$$St = \frac{q_w}{g \rho c_p u_1 \theta_w} = 0.0225 \left(\frac{\nu}{u_1 \delta} \right)^{1/4} (Pr)^{-2/3} \quad (8)$$

This relation for the heat flow q_w is used in the derivation of the boundary-layer equations for the turbulent free-convection boundary layer.

Replacing the boundary-layer thickness δ by

$$\delta = 0.366 x \left(\frac{\nu}{u_1 x} \right)^{1/5}$$

which is the boundary-layer thickness on a flat plate in turbulent forced-convection flow (reference 1, p. 481), transforms equation (8) into the widely accepted formula for heat transfer on a flat plate in the turbulent range

$$Nu = \frac{q_w x}{k \theta_w} = 0.029 \left(\frac{u_1 x}{\nu} \right)^{0.8} (Pr)^{1/3} \quad (9)$$

DETERMINATION OF TURBULENT FREE-CONVECTION BOUNDARY LAYER

DERIVATION OF BOUNDARY-LAYER EQUATIONS

If a stationary plane vertical wall is heated to a temperature higher than the surroundings, the layer of fluid adjacent to the wall is heated by conduction from the wall. In this way, buoyant forces are generated that cause this layer to flow in an upward direction. This layer of fluid adjacent to the wall to which the vertical motion is confined is called the free-convection boundary layer. The boundary layer begins with zero thickness at the lower end of the vertical wall and increases in thickness in the upward direction. At a certain distance from the lower end of the wall the boundary layer becomes turbulent, depending on the critical Grashof number. The distance measured vertically from the lower end of the wall is called x and the distance normal to the wall y . In order to determine the boundary-layer thickness for steady state, a small stationary volume element in the turbulent region of the boundary layer is considered. Figure 1 shows this volume element. The dimensions of the element are dx along the wall and l normal to the wall. The length l should be larger than the boundary-layer thickness δ . For two-dimensional flow, the dimension of the volume element normal to the plane of figure 1 may be considered to be 1. The upward velocity of the fluid in plane 1-1 at a distance y from the surface of the wall is u . Then the mass flow through a small area with a width dy is $\rho u dy$ and the flow of momentum in the x -direction is $\rho u^2 dy$. The momentum flow in the x -direction entering the volume element through plane 1-1 is

$$\int_0^l \rho u^2 dy$$

In progressing to plane 2-2 the momentum flow changes by

$$\frac{d}{dx} \left(\int_0^l \rho u^2 dy \right) dx$$

The mass flow entering the volume element through plane 1-1 is generally different from the mass flow leaving the element through plane 2-2. Therefore fluid enters or leaves the volume element through the plane parallel to the wall at a distance l . Because it is assumed that the velocity in the x -direction outside the boundary layer is so small that it can be neglected, no momentum in the x -direction is carried through the plane.

The rate of change of momentum must be in equilibrium with the forces acting in the x -direction on the fluid within or on the surface of the volume element considered. A shearing stress τ_w acts on the wall. The force connected with

this stress is $\tau_w dx$. No shearing stress occurs on the surface of the volume element that is parallel to the wall and at a distance l from the wall because outside the boundary layer the velocity in the x -direction is zero.

According to the boundary-layer theory, the pressure change can be neglected along any normal to the surface. A constant pressure difference dp therefore exists between planes 1-1 and 2-2. This pressure difference gives a force on the volume element of magnitude $l dp$. In addition to the forces on the surfaces, there is a force due to the weight of the fluid within the volume element

$$\left(\int_0^l \rho g dy \right) dx$$

The process of summing up all the forces and equating them to the change in momentum flow gives the momentum equation

$$\frac{d}{dx} \left(\int_0^l \rho u^2 dy \right) dx = l dp - \left(\int_0^l \rho g dy \right) dx - \tau_w dx$$

Because no flow exists outside the boundary layer, the pressure difference between planes 1-1 and 2-2 is balanced by the weight of the fluid layer between the planes

$$dp = \rho_s g dx$$

When both sides of this equation are multiplied by l and the right side of the equation is changed to the integral form, the following equation is obtained

$$l dp = \left(\int_0^l \rho_s g dy \right) dx$$

Introducing the preceding equation into the momentum equation gives

$$\frac{d}{dx} \left(\int_0^l \rho u^2 dy \right) dx = g \left[\int_0^l (\rho_s - \rho) dy \right] dx - \tau_w dx$$

Introducing the expansion coefficient defined by the equation

$$\beta = \frac{v - v_s}{v_s} \left(\frac{1}{t - t_s} \right)$$

the first term on the right side of the preceding equation can be transformed into

$$g dx \int_0^l \beta \rho (t - t_s) dy$$

Designating the difference between the temperature t at the distance y and the temperature t_s outside the boundary layer by θ , the following expression is obtained:

$$g dx \int_0^l \beta \rho \theta dy$$

In the applications considered, the essential influence of the density changes on the flow is taken into account by the introduction of the expansion coefficient β . The other influences of variable density on the flow and the variation of

the expansion coefficient β with small temperature differences are negligible. Therefore, β and ρ can be assumed constant in the preceding expressions. The momentum equation for the free-convection boundary layer therefore becomes

$$\frac{d}{dx} \int_0^l u^2 dy = g \beta \int_0^l \theta dy - \frac{\tau_w}{\rho} \quad (10)$$

A similar equation is set up for the heat flow through the volume element in figure 1. The heat carried with the fluid through plane 1-1 is

$$g \rho c_p \int_0^l u \theta dy$$

where the enthalpies $c_p \theta$ are measured from the temperature outside the boundary layer. The specific heat and density are considered constant.

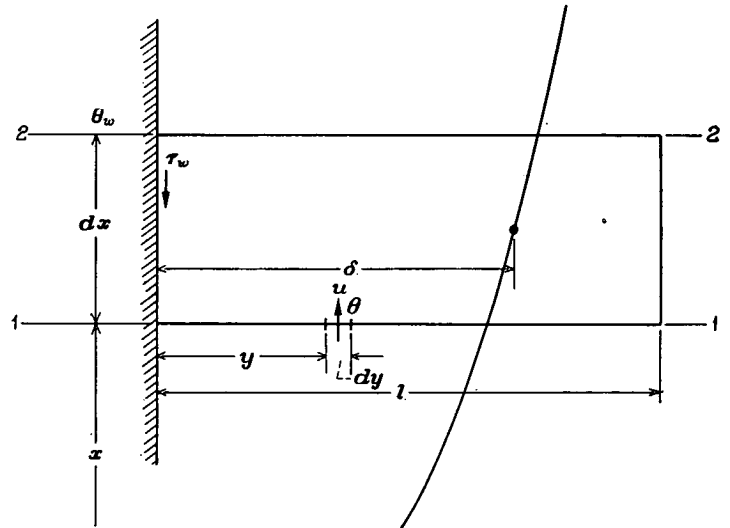


FIGURE 1.—Nomenclature in deriving equations for free-convection boundary layer.

The heat carried out of plane 2-2 differs from that carried into plane 1-1 by

$$g \rho c_p \frac{d}{dx} \left(\int_0^l u \theta dy \right) dx$$

The difference in the heat flow through planes 1-1 and 2-2 must come from the surface and the heat flow leaving the plate per unit time and area is therefore

$$q_w = g \rho c_p \frac{d}{dx} \int_0^l u \theta dy \quad (11)$$

Equation (11) is the heat-flow equation for the free-convection boundary layer.

Equations for the laminar free-convection boundary layer that are analogous to equations (10) and (11) were derived and solved by Squire. (See reference 9.)

SOLUTION OF BOUNDARY-LAYER EQUATIONS

Equations (10) and (11) are sufficient to calculate the boundary-layer thickness δ and the velocity u when the shape of the velocity and temperature profiles within the boundary layer and the laws for the shearing stress and the

heat flow on the wall are known. It was suggested by von Kármán that an approximate solution may be obtained by introducing approximate shapes for both profiles (reference 7). The accuracy of the solution is better the more closely the assumed shapes correspond to the real profiles. The method, however, proved comparatively insensitive to changes in the profile shape.

As mentioned in the section "INTRODUCTION" some information on the shape of the velocity and temperature profiles in turbulent free-convection flow can be obtained from reference 3. In figure 2, the measured values of both profiles in the turbulent range are presented with the ratios θ/θ_w and u/u_{max} plotted against the distance from the wall in an arbitrary scale. The value u_{max} was determined by drawing curves through the measured velocity values given in table VI of reference 3.

The distance from the wall where the curves u/u_{max} have the value 0.5 was arbitrarily assigned the value 1. The lines in figure 2 are the curves by which the temperature and velocity profiles are approximated herein. These profiles are presented in figure 2 in such a way that the velocity profile has the value 0.5 at a distance 1 and the temperature profile the value 0.2 at a distance 0.5 from the wall. An indication of the shape of turbulent free-convection flow profiles can also be obtained from the measurements by Watzinger and Johnson (reference 4) on mixed forced- and free-convection flow in a vertical tube. Some of these measured profiles are presented in figure 3.

Equation (4), which describes the forced-convection temperature profile, is used for the temperature profile in turbulent free-convection flow. The profiles of figure 2 indicate that this assumption is reasonable. The velocity profile in free-convection flow, however, differs from the forced-flow profile by having a velocity of zero outside the boundary layer. Accordingly, equation (1) is multiplied by a function of y/δ , which brings the velocities back to the value zero at the outer edge of the boundary layer where $y=\delta$. Two simple expressions that fulfill this condition and that agree quite well with the measured values are plotted in figure 2.

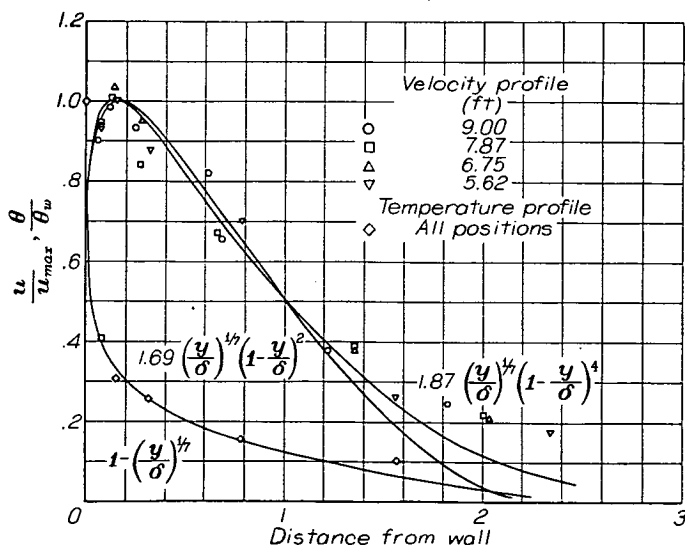
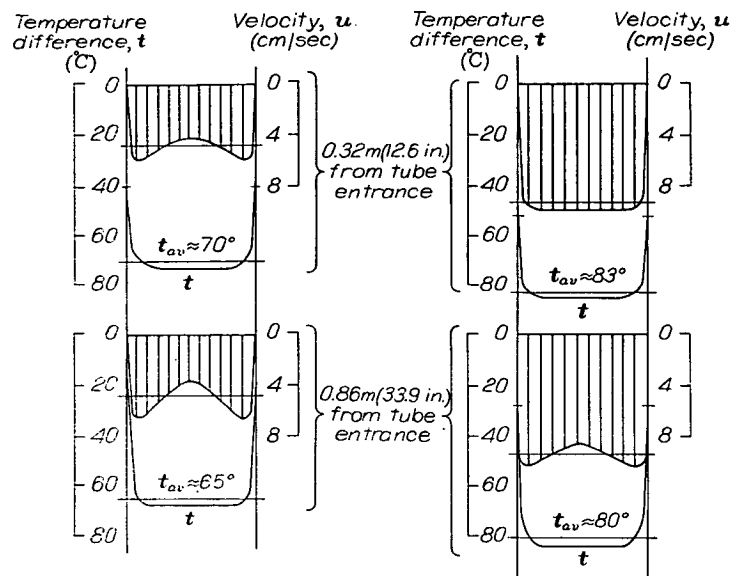


FIGURE 2.—Experimental and theoretical velocity and temperature profiles for turbulent free-convection flow on a flat plate. Experimental data from reference 3.



(a) Reynolds number of 6340 (corresponding to average velocity of 4.7 cm/sec). (b) Reynolds number of 14,200 (corresponding to average velocity of 9.4 cm/sec).

FIGURE 3.—Temperature and velocity profiles in mixed forced- and free-convection flow of water in vertical tube of 50.3 millimeter (1.98 in.) diameter. Data from figure 10 of reference 4.

The velocity profile that represents the measured points somewhat better is used in the following calculations:

$$u = u_1 \left(\frac{y}{\delta} \right)^{1/7} \left(1 - \frac{y}{\delta} \right)^4 \quad (12)$$

Equation (12) is valid for values of y less than δ . For larger values of y , $u=0$. The maximum velocity u_{max} of this profile can be found by differentiating equation (12), giving

$$u_{max} = 0.537 u_1 \quad (13)$$

By use of equations (4) and (12) and the fact that $u=0$ and $\theta=0$ for distances greater than δ from the wall, the integrals in the momentum and heat-flow equations (equations (10) and (11)) become

$$\int_0^\delta u^2 dy = 0.0523 \delta u_1^2$$

$$\int_0^\delta \theta dy = 0.125 \delta \theta_w$$

$$\int_0^\delta u \theta dy = 0.0366 \delta u_1 \theta_w$$

The same value of δ was used for the velocity and temperature profiles. Calculations on forced-convection heat transfer indicate that the same value of δ for the velocity and temperature profiles can be used for fluids that have a Prandtl number close to 1 (reference 8). For very large or small Prandtl numbers in forced flow, the thickness of the temperature boundary layer is considerably smaller or larger than the thickness of the velocity boundary layer. This condition is probably true for free-convection flow also.

In order to solve boundary-layer equations (10) and (11), the laws for the shearing stress on the wall and the heat flow must be introduced. It is assumed that, in the layers very

near the wall, the conditions are similar for free- and forced-convection flow and that the same laws for the shearing stress and the heat flow that are used in forced-convection flow can therefore be used in free-convection flow. A similar assumption for the shearing stress was used by von Kármán to calculate the radial flow on a rotating disk, with satisfactory results (reference 7).

As a consequence of the preceding assumption, equations (2) and (8) will be used for free-convection flow. Very near the wall the second term in equation (12) for the free-convection-velocity profile is 1 and, therefore, the same transformation that led in forced flow from equation (2) to equation (3) can be made for free-convection flow. By introducing the evaluated integrals and equations (3) and (8), the momentum and heat-flow equations become

$$0.0523 \frac{d}{dx} (u_1^2 \delta) = 0.125 g \beta \theta_w \delta - 0.0225 u_1^2 \left(\frac{\nu}{u_1 \delta} \right)^{1/4} \quad (14)$$

$$0.0366 \frac{d}{dx} (u_1 \delta) = 0.0225 u_1 \left(\frac{\nu}{u_1 \delta} \right)^{1/4} (Pr)^{-2/3} \quad (15)$$

Equations (14) and (15) are total differential equations from which the two unknown values δ and u_1 may be determined as functions of x .

Equations (14) and (15) can be solved by introducing

$$u_1 = C_u x^m \quad (16)$$

$$\delta = C_\delta x^n \quad (17)$$

The introduction of these values transforms equations (14) and (15) into

$$0.0523 (2m+n) C_u^2 C_\delta x^{2m+n-1} = 0.125 g \beta \theta_w C_\delta x^n - 0.0225 C_u^{7/4} C_\delta^{-1/4} \nu^{1/4} x^{(7m/4)-(n/4)} \quad (18)$$

$$0.0366 (m+n) C_u C_\delta x^{m+n-1} = 0.0225 (Pr)^{-2/3} \nu^{1/4} C_u^{3/4} C_\delta^{-1/4} x^{(3m/4)-(n/4)} \quad (19)$$

Because equations (18) and (19) must be valid for any value of x , the exponents of x must be identical, that is,

$$2m+n-1 = n = \frac{7m}{4} - \frac{n}{4}$$

$$m+n-1 = \frac{3m}{4} - \frac{n}{4}$$

It can be seen that this condition is fulfilled if

$$m = \frac{1}{2} \text{ and } n = \frac{7}{10} \quad (20)$$

With these values of m and n , equation (19) can be solved for the constant C_u

$$C_u = 0.0689 \nu C_\delta^{-5} (Pr)^{-8/3} \quad (21)$$

Introducing this constant into equation (18) gives

$$C_\delta^{10} = 0.00338 \frac{\nu^2}{g \beta \theta_w} [1 + 0.494 (Pr)^{2/3}] (Pr)^{-16/3} \quad (22)$$

Introducing the Grashof number $Gr = \frac{g \beta \theta_w x^3}{\nu^2}$ gives for the velocity

$$u_1 = 1.185 \frac{\nu}{x} (Gr)^{1/2} [1 + 0.494 (Pr)^{2/3}]^{-1/2} \quad (23)$$

and for the boundary-layer thickness

$$\delta = 0.565 x (Gr)^{-1/10} (Pr)^{-8/15} [1 + 0.494 (Pr)^{2/3}]^{1/10} \quad (24)$$

More significant than the value u_1 is the maximum velocity u_{max} within the boundary layer. (See equation (13).) A Reynolds number is obtained to make this velocity dimensionless:

$$Re_{max} = \frac{u_{max} x}{\nu} = 0.636 (Gr)^{1/2} [1 + 0.494 (Pr)^{2/3}]^{-1/2} \quad (25)$$

A more significant value characterizing the thickness of the boundary layer than the value δ used up to now is the displacement thickness δ^* , which is defined for free-convection flow by the equation

$$\delta^* = \int_0^\infty \frac{u}{u_{max}} dy$$

With the use of equations (12) and (13) it is found that

$$\delta^* = 0.272 \delta$$

The displacement thickness made dimensionless by the distance x from the leading edge of the plate is therefore

$$\frac{\delta^*}{x} = 0.154 (Gr)^{-1/10} (Pr)^{-8/15} [1 + 0.494 (Pr)^{2/3}]^{1/10}$$

The heat-transfer coefficient $h = q_w / \theta_w$ at the point x on the wall is found by introducing the boundary-layer thickness δ and the velocity u_1 into equation (8). Changing to the usually presented Nusselt number $Nu = hx/k$ gives

$$Nu = 0.0295 (Gr)^{2/5} (Pr)^{7/15} [1 + 0.494 (Pr)^{2/3}]^{-2/5} \quad (26)$$

In order to compare this equation with experimental results, it is necessary to change from the local heat-transfer coefficient to the average value along the plate. By introducing the expression for the Grashof number into equation (26), it can be seen that the local heat-transfer coefficient is proportional to the power 0.2 of the distance x ($h = C_h x^{0.2}$). With the assumption that the boundary layer is turbulent from the leading edge, the average heat-transfer coefficient becomes

$$h_{av} = \frac{1}{x} \int_0^x h dx = \frac{C_h}{x} \int_0^x x^{0.2} dx = \frac{C_h}{1.2} x^{0.2} = \frac{h}{1.2} \quad (27)$$

In reality the boundary layer is first laminar and only at a certain distance from the lower edge of the plate does it become turbulent. The preceding expression for the average heat-transfer coefficient can therefore be expected to represent the true value only at Grashof numbers so high that the extent of the laminar boundary layer is small compared

with the total length x . This limit for the Grashof number seems to be near 10^{10} . For Grashof numbers higher than this value, the average Nusselt number can be calculated with equation (27); therefore,

$$Nu_{av} = 0.0246(Gr)^{2/5}(Pr)^{1/5}[1 + 0.494(Pr)^{2/3}]^{-2/5} \quad (28)$$

In order to determine to what extent the shape of the velocity profile influences the results, a calculation was made with the second velocity distribution shown in figure 2:

$$u = u_1 \left(\frac{y}{\delta} \right)^{1/7} \left(1 - \frac{y}{\delta} \right)^2 \quad (29)$$

Only the numerical constants are influenced by the change in the profile shape. The constant preceding the value $(Pr)^{2/3}$ changes from 0.494 to 0.342 in all equations. In addition, the value 0.636 changes to 0.487 in the equation for Re_{max} ; in the equation for δ^*/x the value 0.154 changes to 0.200 and in the equation for Nu_{av} the constant 0.0246 changes to 0.0198.

COMPARISON WITH EXPERIMENTS

In figure 4 the results of experiments carried out in different investigations (references 1 and 2) are plotted as the average Nusselt number Nu_{av} against the product $Gr Pr$. For the

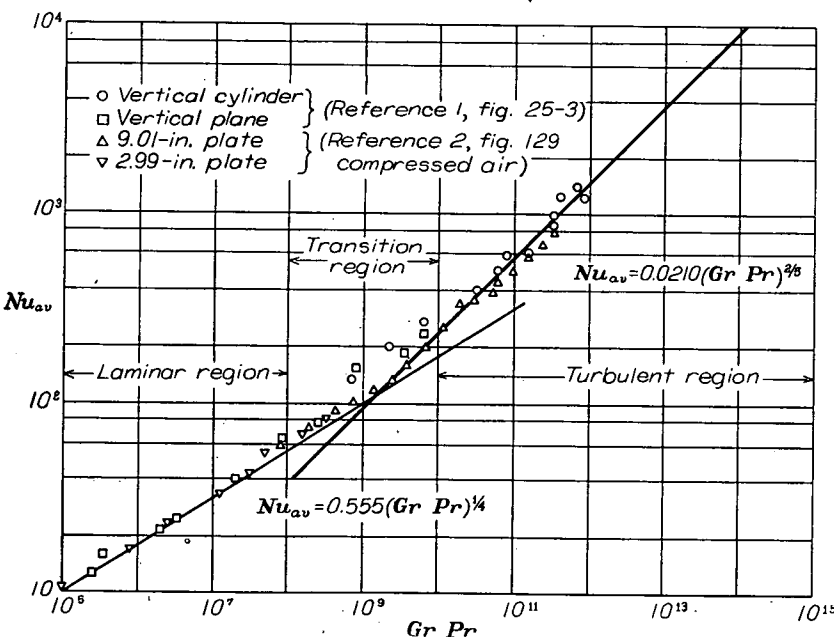


FIGURE 4.—Average Nusselt number for free-convection flows on a vertical plate. Experimental data from references 1 and 2.

lower values of $Gr Pr$ the experimental results quite accurately fit the equation

$$Nu_{av} = 0.555(Gr Pr)^{1/4}$$

which was, with a slight adjustment of the constant, theoretically derived for laminar free-convection flow. The experimental results in the turbulent range ($Gr Pr = 10^{10}$ to 10^{12}) may be represented by the equation

$$Nu_{av} = 0.0210(Gr Pr)^{2/5} \quad (30)$$

The exponent of the Grashof number in this equation is the same as that of equation (28) derived in the previous section. When equation (28) is transformed into the form of equation (30) in such a way that the values of both equations are the same for $Pr = 0.72$, the constant of equation (28) becomes 0.0210. The heat-transfer coefficient derived with equation (12) for the velocity profile is in perfect agreement with the experimental results. Such agreement is probably a coincidence. The profile given by equation (29), which does not fit the velocity distributions measured by Griffiths and Davis in reference 3 as well as does the first profile, gives heat-transfer coefficients that are 17 percent lower than the measured values.

The values for the maximum velocity within the boundary layer and the boundary-layer-displacement thickness agree poorly with the values measured in reference 3 in the turbulent range. Heat-transfer coefficients were not measured therein on the experimental apparatus on which velocity and temperature profiles were obtained.

Whereas the velocity and temperature profile shapes as measured in reference 3 are typical of turbulent free-convection flow, the order of magnitude of these profiles appears to be in error. It can be shown that there is disagreement within the measured values themselves. The heat given off by the plate to the air stream must be carried away within the boundary layer. The measured temperature and velocity profiles as well as the measured maximum velocity and the boundary-layer thickness vary little along the plate in the turbulent range, which means that only a small part of the heat given off by the wall is found in the boundary layer. The horizontal dimension of the plate may have been too small compared with the vertical length to make the flow in the center part two-dimensional and air may have flowed into the boundary layer from the sides.

SUMMARY OF RESULTS

With the use of Kármán's approximate method, a calculation was carried out for the flow and heat transfer in the turbulent free-convection boundary layer on a vertical flat plate. The calculation used relations for the heat flow and shearing stress on the wall developed for forced turbulent flow and velocity and temperature profiles that approximate well the shapes of profiles measured by Griffiths and Davis.

A formula was derived for the heat-transfer coefficient that was in good agreement with measured values in the range of Grashof numbers from 10^{10} to 10^{12} and that can be used to extrapolate the values into the range of higher Grashof numbers. The formula is valid for Prandtl numbers that are close to 1.

The calculation also yielded formulas for the maximum velocity in the boundary layer and for the boundary-layer thickness.

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, July 12, 1950.

APPENDIX

SYMBOLS

The following symbols are used in this report:

a	thermal diffusivity, sq ft/sec
C_h	constant for variation of heat-transfer coefficient with $x^{0.2}$
C_u	constant for variation of velocity u_1 in boundary layer with x^m
C_δ	constant for variation of boundary-layer thickness with x^n
c_p	specific heat at constant pressure, Btu/(lb) (°F)
Gr	Grashof number, $\frac{g\beta\theta_w x^3}{\nu^2}$
g	acceleration due to gravity, ft/sec ²
h	heat-transfer coefficient, Btu/(sq ft) (sec) (°F)
k	heat conductivity, Btu/(ft) (sec) (°F)
l	length (fig. 1), ft
m	exponent
Nu	Nusselt number, $\frac{hx}{k}$
n	exponent
Pr	Prandtl number, $\frac{\nu}{a} = \frac{gc_p\mu}{k}$
p	pressure, lb/sq ft
q	specific heat flow, Btu/(sec) (sq ft)
Re_{max}	Reynolds number based on maximum velocity u_{max} , $\frac{u_{max}x}{\nu}$

St	Stanton number, $\frac{q_w}{g\rho c_p u_1 \theta_w}$
t	temperature, °F
u	velocity component in x -direction, ft/sec
u_1	velocity outside boundary layer of comparable forced-convection flow, ft/sec
v	specific volume, cu ft/lb
x	coordinate (distance along plate from starting point of boundary layer), ft
y	coordinate (distance from wall), ft
β	expansion coefficient, 1/°F
δ	boundary-layer thickness, ft
δ^*	displacement thickness of boundary layer, ft
θ	temperature difference, °F
θ_w	temperature difference between wall and fluid outside of boundary layer, °F
μ	absolute viscosity, $\rho\nu$, lb-sec/sq ft
ν	kinematic viscosity, sq ft/sec
ρ	mass density, (lb) (sec ²)/ft ⁴
τ	shearing stress, lb/sq ft
Subscripts:	
av	average value
max	maximum value
w	on wall
δ	at outer edge of boundary layer

REFERENCES

1. Jakob, Max: Heat Transfer. John Wiley & Sons, Inc., 1949.
2. McAdams, William H.: Heat Transmission. McGraw-Hill Book Co., Inc., 2d ed., 1942, p. 248.
3. Griffiths, Ezer, and Davis, A. H.: The Transmission of Heat by Radiation and Convection. Special Rep. No. 9, Food Investigation Board, British Dept. Sci. and Ind. Res., 1922.
4. Watzinger, A., und Johnson, Dag G.: Wärmeübertragung von Wasser an Rohrwand bei senkrechter Strömung im Übergangsbereich zwischen laminarer und turbulenter Strömung. Forschung Ingenieurwesen, Bd. 10, Heft 4, Juli/Aug. 1939, S. 182-196.
5. Brown, Aubrey I., and Marko, Salvatore M.: Introduction to Heat Transfer. McGraw-Hill Book Co., Inc., 1942, p. 111.
6. Boelter, L. M. K., Cherry, V. H., Johnson, H. A., and Martinelli, R. C.: Heat Transfer Notes. Univ. Calif. Press (Berkeley), 1948, p. XII-35.
7. von Kármán, Th.: On Laminar and Turbulent Friction. NACA TM 1092, 1946.
8. Eckert, E. R. G.: Introduction to the Transfer of Heat and Mass. McGraw-Hill Book Co., Inc., 1950, p. 107.
9. Goldstein, Sidney: Modern Developments in Fluid Dynamics. Vol. II. Clarendon Press (Oxford), 1938, p. 641.

REPORT 1016

EFFECT OF TUNNEL CONFIGURATION AND TESTING TECHNIQUE ON CASCADE PERFORMANCE¹

By JOHN R. ERWIN and JAMES C. EMERY

SUMMARY

An investigation has been conducted to determine the influence of aspect ratio, boundary-layer control by means of slots and porous surfaces, Reynolds number, and tunnel end-wall condition upon the performance of airfoils in cascades. A representative compressor-blade section (the NACA 65-(12)10) of aspect ratios of 1, 2, and 4 has been tested at low speeds in cascades with solid and with porous side walls. Two-dimensional flow was established in porous-wall cascades of each of the three aspect ratios tested; the flow was not two-dimensional in any of the solid-wall cascades.

Turbine-blade sections of aspect ratio 0.83 were tested in cascades with solid and porous side walls and blade sections of aspect ratio 3.33 were tested in cascades with solid walls. No particular advantage was observed in the use of porous walls for the turbine cascades tested.

INTRODUCTION

Airfoils are tested in cascades to provide fundamental information for the design of compressors and turbines. This information can be applied directly as basic data in many designs. The advantage of cascade testing lies in the relative ease and rapidity with which tests can be made, in the elimination of three-dimensional and boundary-layer effects not related to section performance, and in the much more detailed information concerning section performance which can be obtained in comparison with that obtainable from tests of rotating compressors and turbines. Cascade results have always contained inherent discrepancies, however, because the flow could not be made truly two-dimensional. These discrepancies arose from the interference and interaction of the boundary layers on the side walls with the flow about the test airfoils because of the finite aspect ratios necessarily used.

The data reported in references 1 and 2 are in some ways inconsistent and, in cross-plotting these data for design studies, irregularities appear such that the design would be indeterminant within the limits of required accuracy. These irregularities have caused much difficulty to persons attempting to interpolate or extrapolate the data for particular applications. Inconsistencies also arose from the fact that data from the Langley 5-inch cascade tunnel had always been

subject to operating technique; for example, much skill and experience were necessary in adjusting the flexible floors correctly and data that were repeatable were difficult to obtain.

The specific difficulties that have led to distrust of previously obtained cascade data are:

(1) As was pointed out in reference 3, the lift coefficient obtained by integrating the pressure-distribution plots of reference 1 did not agree with that calculated from the measured turning angle, when two-dimensional flow was assumed.

(2) The fact that the pressure rise expected to result from the measured turning angle was not obtained obviously should have an effect on the magnitude of the turning angle; therefore, some question arises as to the validity of the data in references 1 and 2. The pressure distribution is also affected in magnitude and in shape by the failure to obtain the calculated pressure rise.

(3) Compressor-blade sections of higher camber than could be satisfactorily tested in the original 5-inch cascade have been successfully used in a single-stage test blower (reference 4).

These effects are believed to be caused by the interaction of the tunnel-wall and test-blade-surface boundary layers since premature separation occurs at the juncture of the side walls and test blades and produces a large low-energy region at the exit from the cascade. This large wake acts as a restriction on the flow, higher average exit velocities result, and the flow is not two-dimensional.

Two-dimensional flow is believed to exist when the following criteria are satisfied:

(1) Equal pressures, velocities, and directions exist at different spanwise locations.

(2) The static-pressure rise across the cascade equals the value associated with the measured turning angle and wake.

(3) No regions of low-energy flow other than blade wakes exist. The blade wakes are constant in the spanwise direction.

(4) The measured force on the blades equals that associated with the measured momentum and pressure change across the cascade.

(5) The various performance values do not change with aspect ratio, number of blades, or other physical factors of the tunnel configuration.

¹ Supersedes NACA TN 2028, "Effect of Tunnel Configuration and Testing Technique on Cascade Performance" by John R. Erwin and James C. Emery, 1950.

Past attempts to establish two-dimensional flows in cascades have utilized boundary-layer removal slots on two or all of the tunnel walls upstream of the cascade, and cascades using blades of aspect ratio 4 or higher have been constructed. The present investigation was intended to determine the value of these methods and that of a method believed to be new—the use of continuous boundary-layer removal through porous surfaces from the cascade side and end walls. A representative compressor-blade section of aspect ratios of 1, 2, and 4 was tested at low speeds in cascades with solid and with porous side walls. This section was tested over a range of Reynolds number for each of these conditions. For comparison, turbine-blade sections in which the flow is characterized by a pressure drop through the test section were tested in cascades with solid and with porous side walls.

When schlieren or shadow photographs of flows through cascades are desired, the use of porous side walls would be difficult; therefore, several methods of correcting solid-wall-cascade results to the two-dimensional case have been compared and their accuracy discussed.

SYMBOLS

A	aspect ratio, span of blades divided by chord of blades
C_w	wake coefficient, coefficient of momentum difference between wake and free stream, based on entering velocity
C_N	blade normal-force coefficient based on entering velocity
$(C_N)_P$	blade normal-force coefficient obtained by integration of blade pressure distribution
$(C_N)_M$	blade normal-force coefficient calculated from measured momentum and pressure changes
q	dynamic pressure, pounds per square foot
R	Reynolds number, based on blade chord and entering air velocity
V	velocity, feet per second
α	angle of attack, angle between entering air and chord line of blade, degrees
β	inlet air angle, angle between entering air and axis, degrees
θ	turning angle, angle through which air is turned by blades, degrees
σ	solidity, chord of blade divided by gap between blades

Subscripts:

1	upstream of cascade
2	downstream of cascade
l	local
c	corrected
a	axial
m	mean value
t	tangential

DESCRIPTION OF TEST EQUIPMENT

The test facilities used in this investigation were the Langley 5-inch and 20-inch cascade tunnels. The 5-inch cascade test section proper uses the same design and, to a considerable extent, the same parts as the one described in reference 1. A larger settling chamber having an area of

about 25 square feet is used, however, and provides a ratio of settling-chamber area to test-section area of about 40:1. A similar settling chamber is used on the 20-inch cascade with an area ratio of only 10:1. It is believed that, if a larger area ratio were used, this cascade would yield satisfactory entrance flows and less attention to the flexible-wall curvatures and suction pressure on the upstream slots would be needed. A sketch of a vertical cross section of either tunnel is shown as figure 1. Photographs of the two tunnels are presented as figures 2 and 3.

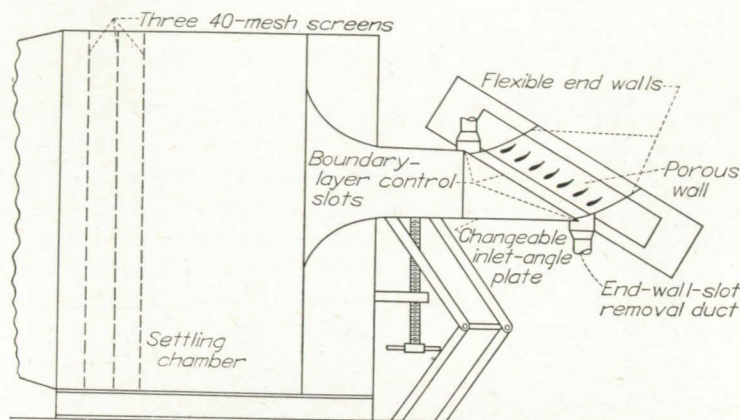


FIGURE 1.—Vertical cross section of two-dimensional low-speed cascade tunnels.

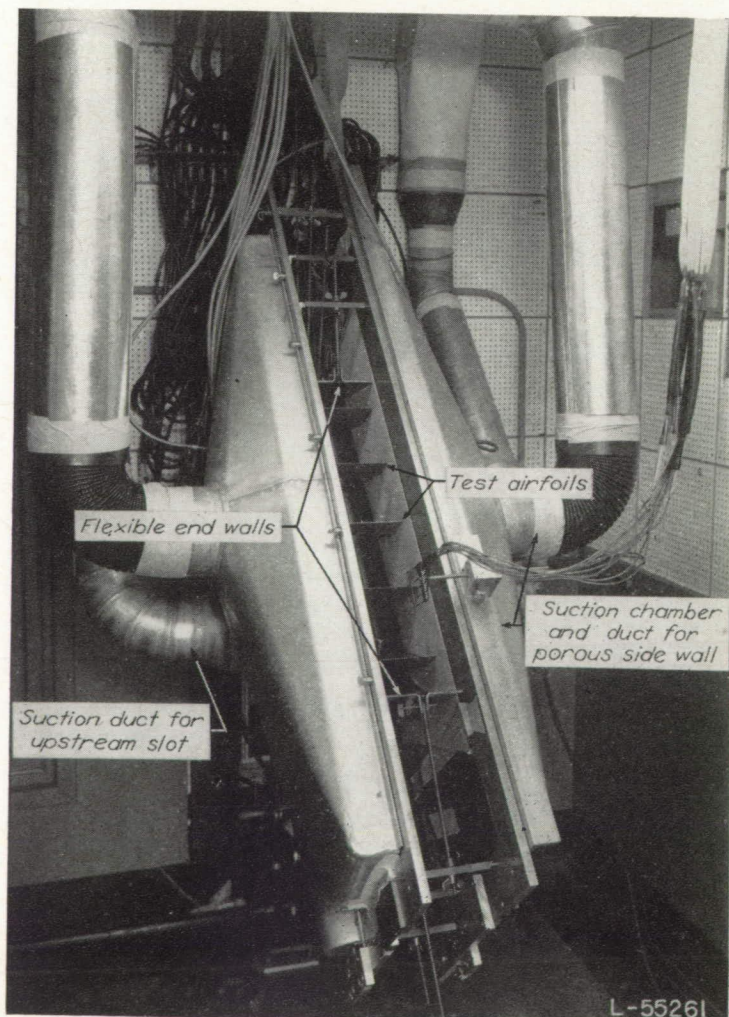


FIGURE 2.—Langley 5-inch cascade tunnel equipped with porous walls.

The porous surface is supported by a 120-mesh screen, which is in turn supported by a sheet of perforated metal having $\frac{1}{8}$ -inch-diameter holes with $\frac{1}{4}$ inch between their centers in all directions. (See fig. 4.) This perforated sheet metal is available from steel suppliers as a standard item. A rigid frame of cellular construction is employed to carry the test airfoils and to minimize the bowing of the side walls due to the suction pressure. A number of different materials were investigated for use as porous surfaces; the results of this study are presented in a subsequent section.

The consideration that determined the particular cascade selected for detailed study was the desire to have as direct a comparison as possible with the present blading in the rotor of the 42-inch test compressor. The NACA 65-(12)10 compressor-blade section as used in this compressor has an inlet angle of 60° at the mean diameter and a solidity of 1.182 and is fairly typical of axial-flow compressor rotors. Because of the geometry of the cascade tunnel used, the sections were tested at a solidity of 1. This cascade has a relatively high static-pressure rise and high blade normal-force coefficient, conditions which make cascade testing difficult.

In addition, some data obtained at the less severe condition of 45° inlet angle are included to provide a more complete picture of the problem and of the results obtained.

The cascade of airfoils selected for detailed study was tested at low speed in the following tunnel configurations:

- (1) 5-inch, blade aspect ratio 1, solid-wall cascade
- (2) 5-inch, blade aspect ratio 1, porous-wall cascade
- (3) 5-inch, blade aspect ratio 2, solid-wall cascade
- (4) 5-inch, blade aspect ratio 2, porous-wall cascade
- (5) 20-inch, blade aspect ratio 4, solid-wall cascade
- (6) 20-inch, blade aspect ratio 4, porous-wall cascade

Turning-angle, pressure-rise, pressure-distribution, and wake-survey measurements were taken. For comparison purposes, results obtained by using a 42-inch-tip-diameter test compressor and a 28-inch-tip-diameter test compressor are presented. All but wake-survey measurements were taken in the 42-inch test compressor described in reference 5. Turning-angle and pressure-rise measurements were obtained with the 28-inch test compressor (reference 4).

The testing procedures used in this investigation were the same as those reported in reference 1. Most of the tests were made with a fixed entrance velocity of about 95 feet per second and with blades of 5-inch chord. A few tests at other speeds were made to vary the Reynolds number. Tests on blades of aspect ratio 2 used airfoils of $2\frac{1}{2}$ -inch chords. With the exception of tests run to determine proper end conditions, cascades of seven blades were used throughout.

In order to expedite plotting of test pressure-distribution results, a constant entrance dynamic pressure is normally used in cascade testing at the Langley Laboratory. Manometers scaled with values of q_1/q_2 are used so that no computing is required to make these plots. It is therefore convenient to obtain force coefficients on the basis of the entering dynamic pressures. All coefficients presented herein are so calculated.

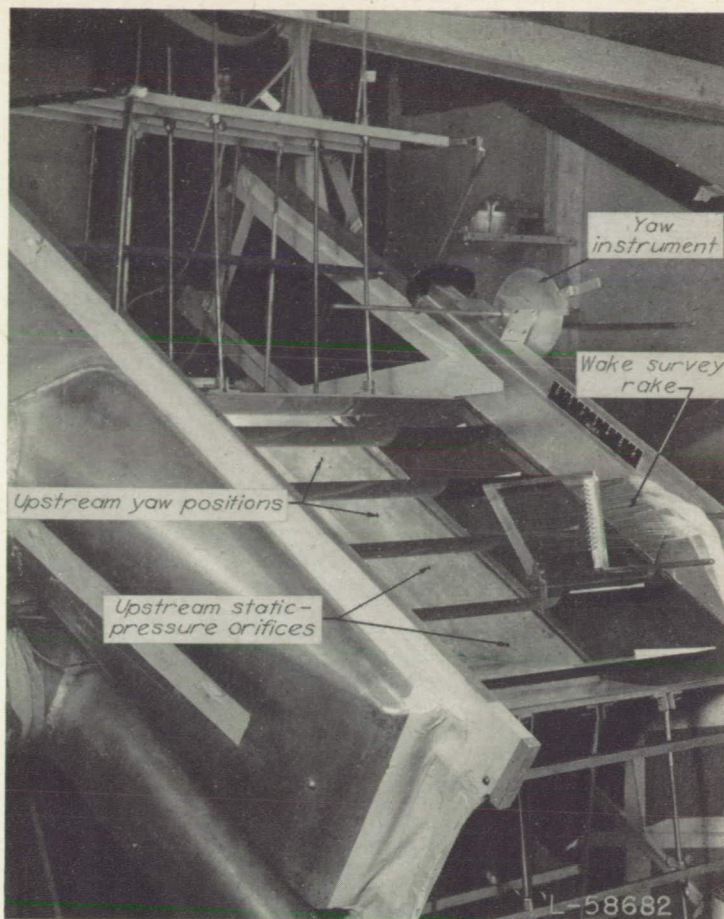


FIGURE 3.—Langley 20-inch cascade tunnel equipped with porous walls.

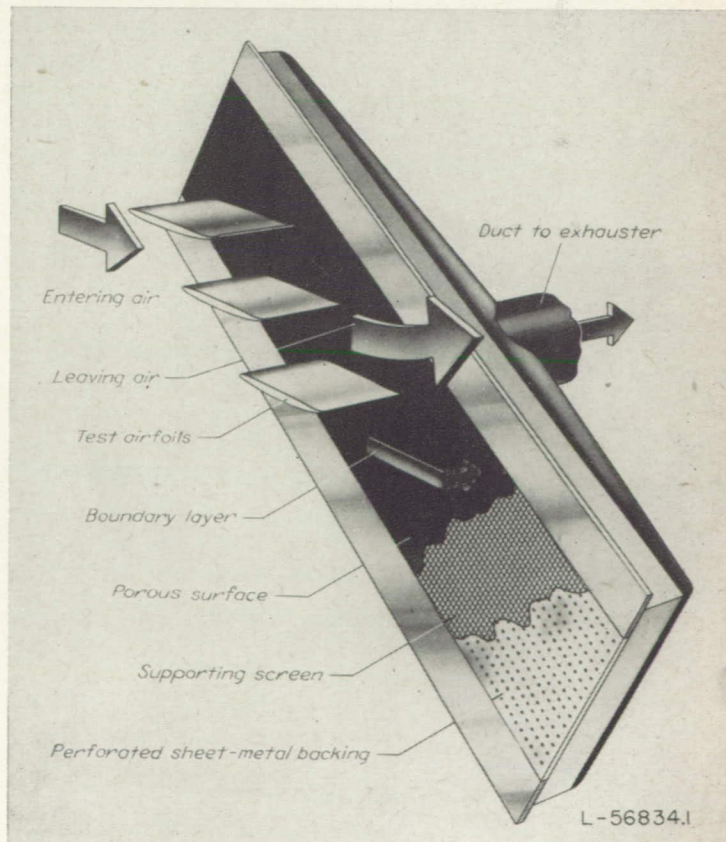


FIGURE 4.—Sketch showing construction of porous wall.

The static-pressure rise across the test section can be controlled to a considerable extent by the quantity of flow drawn through the porous walls. The pressure rise corresponding to two-dimensional flow through the observed turning angle, when allowance for effective passage-area reduction due to the observed wake is made, could be established by this means. A chart of pressure rise or, more specifically, of q_2/q_1 , was prepared for a range of turning angle and wake width for the test inlet air angle and solidity (fig. 5). The effect of the blade wake on the exit velocity was taken into account by measuring the exit velocities and averaging the values for a few test points. A factor, dependent upon the wake width, was so obtained and was applied in computing values of q_2/q_1 for other conditions. This factor assumes that wakes of similar width will restrict the exit flow similarly. The validity of the curves was later confirmed by detailed velocity calculations for many tests. For all tests in which continuous boundary-layer removal was used, the pressure rise across the cascade was set by reference to the chart. Of course, this method was not applied to the series in which the pressure rise was intentionally made different from the two-dimensional value in order to examine the resulting effects.

RESULTS AND DISCUSSION

GENERAL

The investigations reported in references 1 and 2 used blades of aspect ratio 1. The results obtained, although providing useful information, failed to satisfy the criteria of two-dimensional flow. The Langley 20-inch cascade tunnel was constructed to permit blade models of aspect ratio 4 to be

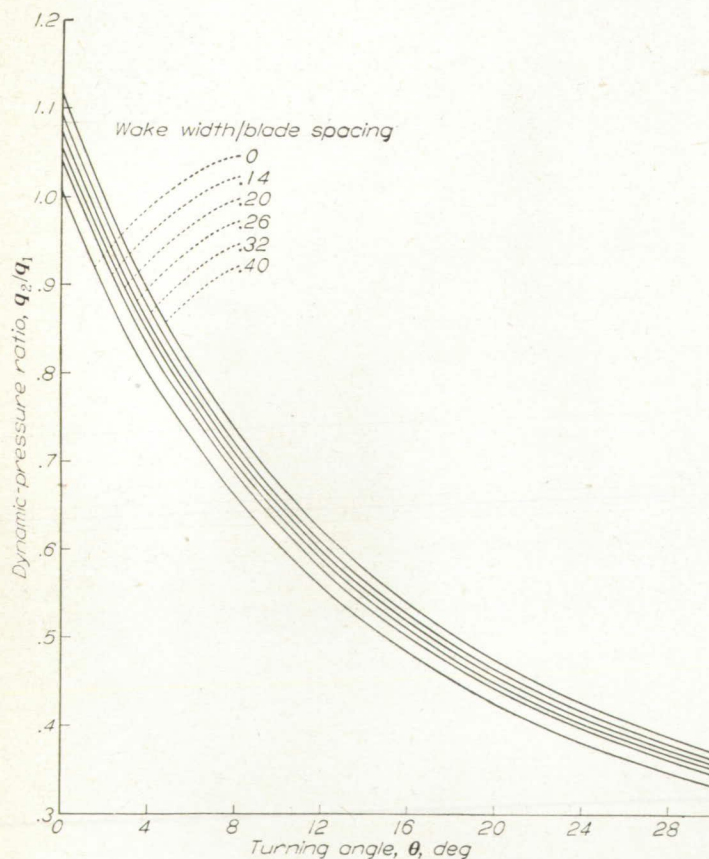


FIGURE 5.—Relation between dynamic-pressure ratio across cascade to turning angle. $\beta=60^\circ$.

tested without reduction in blade chord or test Reynolds number. The results for blades of aspect ratio 4 indicated that the static-pressure rise obtained did not agree with that calculated by using the measured values of turning angle and by assuming two-dimensional flow, even though correction was made for the blade wake measured at midspan. Further, the turning angles observed were quite different from the turning-angle values obtained in the test compressors. Correcting the data to account for the constriction due to separation near the walls failed to produce corrected turning angles in agreement with the values measured in the test compressors. When the data for the 5-inch blades of aspect ratio 1, which agree with the data for the test compressors, were so corrected, however, the resultant values were lower than test-compressor results. The values for the normal-force coefficient for the tests on blades of aspect ratio 4 appear to be in closer agreement with the values calculated on a basis of two-dimensional momentum and pressure changes when they are plotted with corrected turning angles than when the measured turning-angle values are used, but since the slopes of C_N against θ_c differ, the results are in question.

Thus it appears that cascades of aspect ratio 4 do not yield directly usable information even when axial-velocity corrections are applied. Because the blades are not operating under pressure-rise, turning-angle, and angle-of-attack conditions that represent the two-dimensional case, cascades of neither $A=1$ nor $A=4$ provide the conditions sought. Furthermore, large air supplies are required for high-aspect-ratio cascades, particularly for high-speed tests. For these reasons, attempts were made to devise methods that would provide two-dimensional flows in cascades using blades of low aspect ratio.

The interference and interaction of the tunnel-side-wall boundary layers with the boundary layers of the test airfoils were believed to be the reason for the flow not being two-dimensional. If the side-wall boundary layers could be removed continuously through the side walls, two-dimensional flow conditions could perhaps be established. In order to examine this possibility, the Langley 5-inch and 20-inch cascade tunnels were modified by replacing the test-section side walls with a frame supporting a perforated metal sheet and a porous material was attached to this perforated sheet. As illustrated in figure 2, a suction chamber connected to a blower was provided on the outside of the test section, and a part of the flow was drawn through the side wall. First tests using this system indicated considerable promise but also indicated that the material used for the porous surface affected the quantity of flow that had to be removed to establish two-dimensional conditions. For this reason, a search for suitable porous materials was initiated, and several materials were tested in the 5-inch cascade.

Other elements of the tunnel configuration that affect the character of the flow into the cascade are the entrance-cone shape, the type, location, and number of the wall slots, and the treatment of the end wall. Only one modification of the circular-arc entrance-cone shape used on both tunnels was tried: The 20-inch-tunnel fairings were replaced with fairings of parabolic curvature in an attempt to eliminate the unfavorable pressure gradient that existed with the circular-arc

cone. No measurable reduction of end-wall boundary-layer thickness was observed, however.

TEST-SECTION CONDITIONS FOR TWO-DIMENSIONAL FLOWS

Porous materials.—Ideally, it would be desirable to employ permeable surfaces having varying porosity in order to distribute the boundary-layer removal in a manner related to the local boundary-layer thickness. No scheme that would be practical for a series of tests was evolved to accomplish this result. In a general way, a surface of uniform porosity accomplishes the desired result, for, where the pressure recovery has been the greatest, the quantity flowing through the surface will be greatest. The regions of highest pressure are unfortunately not necessarily regions where the boundary layer is thickest or most likely to separate; therefore, a surface of uniform porosity is not ideal for compressor-cascade side walls. The surface porosity for these low-speed compressor-blade tests was determined by the requirement that the suction-chamber pressure be lower than the lowest pressure on the test airfoils. In this condition, the flow through the wall surface was everywhere outward from the test section to the suction chamber.

The first tunnel configuration using porous walls had a very heavy canvas as the surface. Because of the appreciable roughness of this material, a very large quantity of the main flow had to be removed through the canvas in order to obtain the pressure rise associated with the measured turning angle. Although a direct quantity measurement was not made, an estimate based on the suction pressure used would be that an air quantity of from 20 to 25 percent of the entering flow was removed through the walls. A medium-weight broadcloth material having a fairly smooth surface on one side was next tried. The quantity of air removed was 10.3 percent of the entering flow; this amount includes some leakage flow. Broadcloth surfaces were used for most of the tests reported herein.

One test using 2-ounce nylon sailcloth was run. This material appears to have desirable smoothness, strength, and abrasion-resistance properties, but the sample used was too closely woven for the conditions of the test. A flow volume sufficiently large to produce the required pressure rise through the cascade could not be removed with the existing suction blower. A very light parachute silk was also tried, but with the opposite result—the porosity was too great, and several layers would have been required to obtain a satisfactory permeable surface. This arrangement was considered impractical for routine operations requiring many tunnel changes.

Several tests were run with a metal screen, made by electroplating, as the porous surface. This material, a copper-nickel alloy, has a very smooth, uniform surface, which is much smoother than wire screen of similar mesh (100), and is without the fuzz of cloth. The sample used had 16-percent open area. One or two layers of broadcloth backing were employed to produce the necessary resistance. A significant reduction in the porous-wall air removal was measured in the tests; the air removal dropped from the previous value of 10.3 to 4.5 percent of the main flow. Better agreement resulted between normal-force coefficients

calculated from measured blade pressure distributions and those calculated by using the momentum equation. The electroplated metal screen was rather expensive, however. A porous surface formed by hammering copper or bronze screen was found to be very practical and quickly made. This technique is attributed to Mr. P. K. Pierpont of the Langley Full-Scale Research Division. By varying the number of hammering operations (each followed by torch annealing), a material of required porosity could be obtained. A screen successfully used over a wide range of test conditions had initial wire diameter of 0.014 inch, 30 mesh, and was hammered to a sheet thickness of 0.009 inch. The normal flow velocity plotted against pressure drop for this material is presented in figure 6.

A commercially available material, twill dutch double weave filter cloth, usually woven of monel wire, having 250 fill wires of 0.008-inch diameter per inch and 30 warp wires of 0.010-inch diameter per inch was found to be very satisfactory when commercially calendered from the as-woven thickness of 0.027 inch to 0.018 inch. This filter cloth permitted normal velocities at a given pressure drop similar to those presented in figure 6 for the hammered screen.

No comparative tests were run in the 20-inch tunnel, but it is believed that a similar reduction in porous-wall air removal could have been achieved; however, a typical value for this removal in the 20-inch tunnel is 2.9 percent when broadcloth is used. This percentage was so small that attempts to reduce the amount seemed unnecessary.

Slot configurations.—A series of runs was made with several combinations of tunnel side-wall and end-wall slots in order to produce uniform entering flows and to reduce porous-wall suction-flow quantity. The possibility of using flush side-wall slots was considered, since flush side slots would eliminate the width change in the end plates and would increase the ease and decrease the time of operation. The tests were made by using existing sharp-edge protruding slots with and without fairings in order to vary the configuration and therefore these slots do not represent optimum slot shapes. The side-wall slots were closed by fairings extending

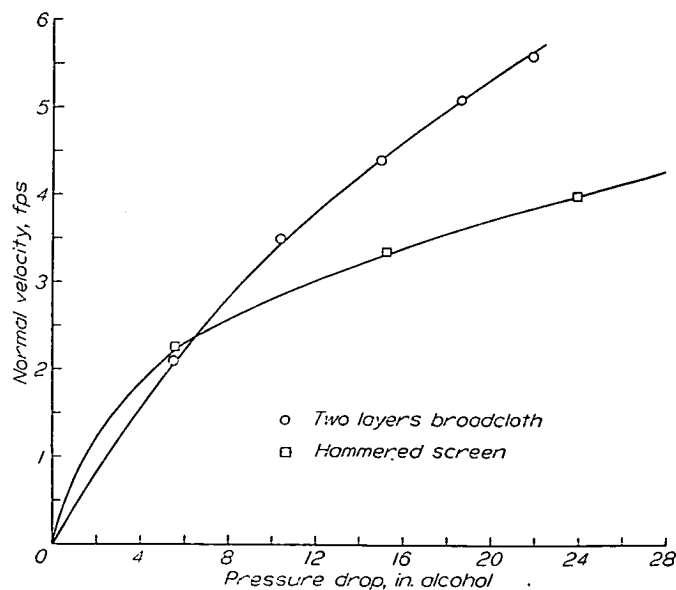


FIGURE 6.—Velocity through two porous materials as a function of the pressure drop.

from the entrance cone to the slot; these fairings reduced the tunnel width to a constant 5 inches. An increase of 2 percent of the main flow or 21 percent of the permeable-wall flow was required to establish the desired cascade pressure rise.

The fairing was then moved $\frac{1}{4}$ inch upstream to produce a flush slot, and 10.5 percent of the main flow was drawn through the porous wall when 8.8 percent was removed through the flush slot. An increase in slot suction pressure, which raised the slot flow to 11.4 percent, decreased the required porous-wall flow only slightly to 10.3 percent of the main flow. Increasing the flush-slot width to $\frac{1}{4}$ inch had a detrimental effect on the required flow through the cloth wall. A different slot shape similar to that recommended in reference 6 might have improved the performance, but this possibility has not yet been investigated. Flush slots parallel to the cascade might be expected to deflect the entering flow at inlet air angles other than zero. In these tests, however, this effect was not measurable when the static-pressure drop through the slot was less than one-half the dynamic pressure entering the cascade. For the usual range of cascade inlet angles, no greater static-pressure drop than one-half the entering dynamic pressure has been found necessary.

The few tests run to determine the end-wall slot configuration indicated the same trends as the side slots. The protruding slots were slightly more effective than flush slots, but only an insignificant reduction in air removal was gained over the sealed or no-slot condition. However, when seven blades were used with the end blades serving as the cascade boundary, boundary-layer removal was necessary. Without the end slots, separated turbulent flow occurred over the end blades, and uniform entering conditions for the cascade apparently could not be established.

Figure 7 shows the boundary-layer profile measured at several stations about 1 chord upstream of the cascade. The origin of each of the small plots is placed at the survey point. It is evident that the protruding slot produces a clean flow at these locations.

For conditions of large inlet air angles, the side-wall length from the entrance cone to the slot varies considerably from one end of the cascade to the other (fig. 1), and the boundary thickness varies in a similar manner. With a fixed slot width and with essentially constant slot-suction-chamber

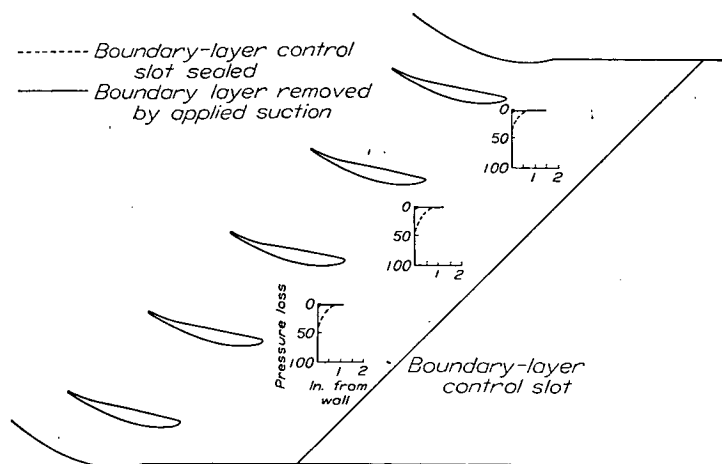
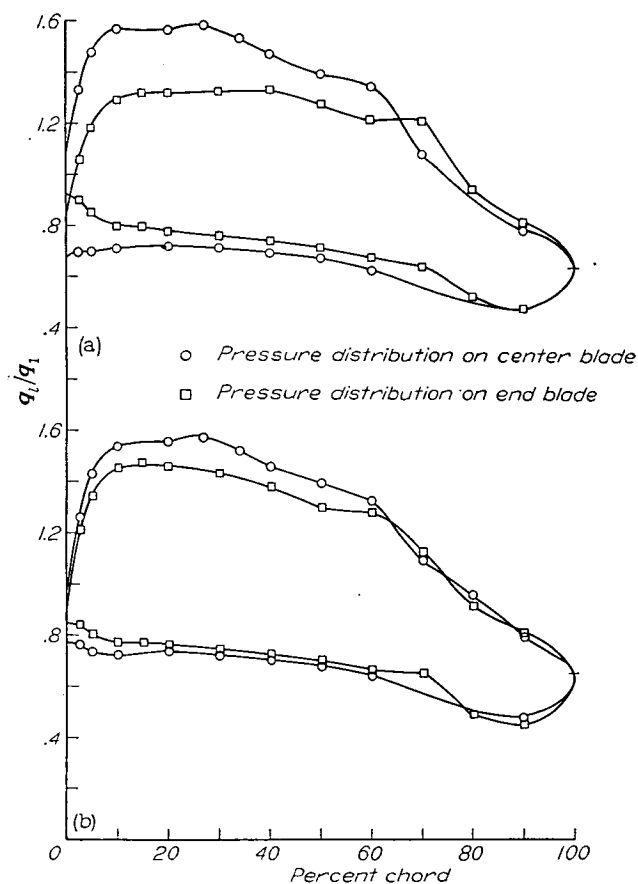


FIGURE 7.—Total-pressure loss in percent of entering dynamic pressure, with and without boundary-layer slot. Survey stations located at origins of small coordinate axes.

pressure, asymmetrical flow in the test section could result from the difference in boundary-layer thickness. No difficulties encountered during the investigation reported herein were traced to this source, however. When testing at $\beta=70^\circ$, uniform flow entering the test section could not be satisfactorily obtained with the slot configurations previously described. Inlet-angle plates having a construction similar to the porous-wall test section were installed in place of the solid plates normally used. When a small quantity of flow was drawn through the porous surface of the inlet-angle plates, uniform flow entering the test section was readily established at $\beta=70^\circ$.

End conditions.—In order to study the effect of the end gap on the behavior of the main flow, a few tests using five blades were run with varying end gaps. A blade equipped with orifices was placed nearest the end walls. As indicated in figure 8, in the case of the half-gap end spacing, the last blade did not carry lift equal to that of the central blade. When the end spacing was increased to a whole gap, the lift on the end blade increased considerably, although not up to the value of lift on the central airfoil. The important result, however, is that the change of end gap had no measurable effect on the performance of the central blade. The pressure distribution, turning angle, and drag coefficient remained as before.

Other investigators have used the end blades in the cascade as the boundary (reference 7). In order to examine the possibilities of this method, two extra blades were installed in the 5-inch cascade tunnel. Runs were first made with the end walls sealed to the end blades approximately at their stagna-



(a) Approximate half-gap end spacing.
(b) Approximate whole-gap end spacing.

FIGURE 8.—Effect of end gap on the pressure distribution measured on the end blade.

tion points. This system provided too few variables, and proper entering conditions could not be established; in addition, the end passages exhibited stalling and fluctuating flows. The end walls were moved outward about $\frac{1}{2}$ inch for the subsequent test, and a gap between the flexible end walls and the end blades resulted. Attempts were made to control the entering static pressure and direction of flow by bending the end walls. Considerable stalling continued to occur, however, apparently because of the thick end-wall boundary layer. To investigate this possibility a $\frac{3}{16}$ -inch protruding slot was built into the upper and lower end walls 1 chord ahead of the test blades. These slots were connected by ducts to an exhaustor. Satisfactory control of entering conditions was possible by bending the flexible surfaces. Some operating time is eliminated with this system since it is not necessary to set the end wall exactly in the direction of the exit flow, and less skill and time are required in adjusting the entering conditions. However, the flow about the central blade appeared to be the same as that for the other two satisfactory tunnel configurations.

The number of blades required in a cascade appears to be an inverse function of the amount of time and care employed in establishing the end conditions. With the flexible end walls used in the cascades described herein, little difference in performance was noticed between similar configurations of five or seven blades. In a setup where end-wall adjustments are not readily made or where running time is limited to a few minutes, significant differences in the performance might be observed.

In tests of airfoils more highly cambered than the NACA 65-(12)10 section reported herein, continuous boundary-layer removal on the convex end wall has been necessary in order to prevent separation from this wall and to permit uniform exit flow from the cascade. A porous surface beginning at the end-wall slot, about 1 chord upstream of the blades, was extended through the cascade to about 1 chord downstream of the blades. Provision was made for flexing this end wall to the desired curvatures. Although continuous boundary-layer removal might also be desirable for the concave end wall, the additional complexity is undesirable. A thin sheet-brass end wall has been found to provide satisfactory flows at an inlet angle of 60° and at a turning angle of 30° , which is a rather severe case.

COMPARISON OF CASCADE AND TEST-BLOWER RESULTS

Turning angle and static-pressure rise.—Results for the several cascades and test blowers investigated are given in figures 9 and 10 as plots of turning angle against angle of attack. All test airfoils are the NACA 65-(12)10 section. All inlet angles are 60° and all solidities are unity, with the single exception of the 42-inch-diameter test compressor, in which a mean diameter solidity of 1.182 existed and in which the inlet angle increased with angle of attack so that $\beta=60^\circ$ occurred at $\alpha=14^\circ$. Turning-angle corrections obtained from reference 2 for varying inlet angle in the 42-inch compressor are presented. Because the rotor-blade solidity was 1.182, the turning angles measured in the 42-inch compressor were predicted to be about 0.4° higher than in the 28-inch test blower or cascades of solidity 1.

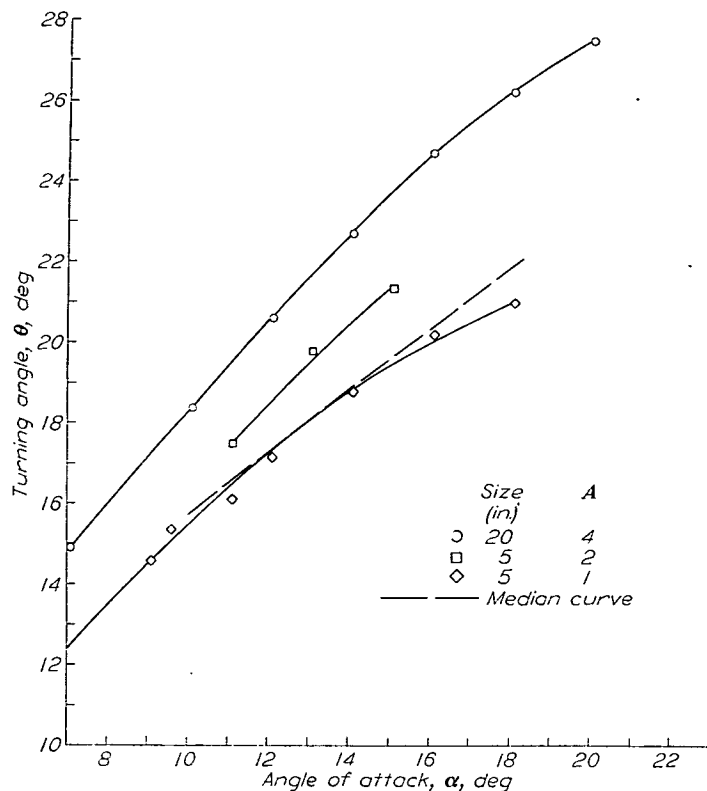


FIGURE 9.—Effect of aspect ratio on turning angle for solid-wall cascades. A median curve (dashed line) obtained from figure 10 is presented for comparison.

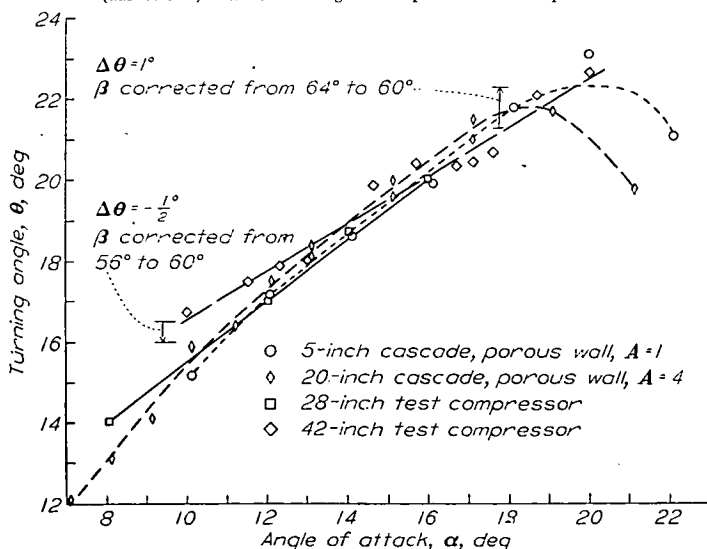


FIGURE 10.—Effect of aspect ratio on turning angle for porous-wall cascades. Results from two test compressors are also given. Turning-angle corrections for varying inlet air angles of the 42-inch compressor are indicated. For the 42-inch compressor, $\beta=56^\circ$ at $\alpha=10^\circ$ and $\beta=64^\circ$ at $\alpha=18^\circ$.

If the results of the solid-wall cascade tests are compared, an increasing turning angle with increasing aspect ratio is very evident. The results from porous cascades and test blowers agree with the data from the solid-wall cascade of aspect ratio 1. This behavior is believed to be due to counteracting effects. The velocities induced by the trailing vortices due to reduced lift in the tunnel-wall boundary layers and the movement of the wall boundary layer onto the suction surface of the airfoils have an over-all result of reducing the blade lift and hence the net turning angle. A factor having an opposite effect is the increase of axial velocity resulting from thickened wall boundary layers (see

reference 3). That this effect exists is evident during operation of the porous-wall cascades: As the porous-wall flow is increased and the pressure rise across the cascade is increased, a measurable reduction in turning angle takes place. Figure 11 shows this effect for the cascade with blades of aspect ratio 4.

In a recent paper (reference 8), Hausmann has presented an analysis of the simplified trailing-vortex problem. In the rather severe case calculated, the indication is that the turning angle would be increased because of the downwash associated with the trailing vortices. The opposite effect is predicted by Carter and Cohen in reference 9. A discussion of these counteracting effects is presented in a subsequent section, with the conclusion that no method is now available to explain satisfactorily the flows observed in the solid-wall cascades.

The solid-wall cascades studied failed to produce the pressure rise associated with the measured turning angles (fig. 12). At low aspect ratio the three-dimensional effects predominated, and low turning angles were observed. With higher aspect ratios, the induced effects were of smaller magnitude, and more turning of the flow resulted. To approach two-dimensional flows in solid-wall cascades, aspect ratios much greater than 4 must be used.

Satisfactory agreement of θ with α and q_2/q_1 with θ was obtained between porous-wall cascades of aspect ratios of

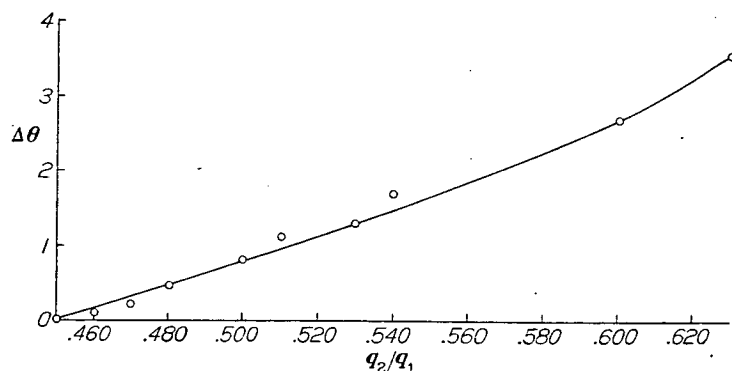


FIGURE 11.—Variation of turning angle with variation in pressure rise as obtained in the porous-wall cascade of aspect ratio 4. The calculated value for the measured turning angle of 20° ($\Delta\theta = 0^\circ$) is $\frac{q_2}{q_1} = 0.450$; the value measured with no flow through the porous walls was $\frac{q_2}{q_1} = 0.630$. $\alpha = 15.1^\circ$.

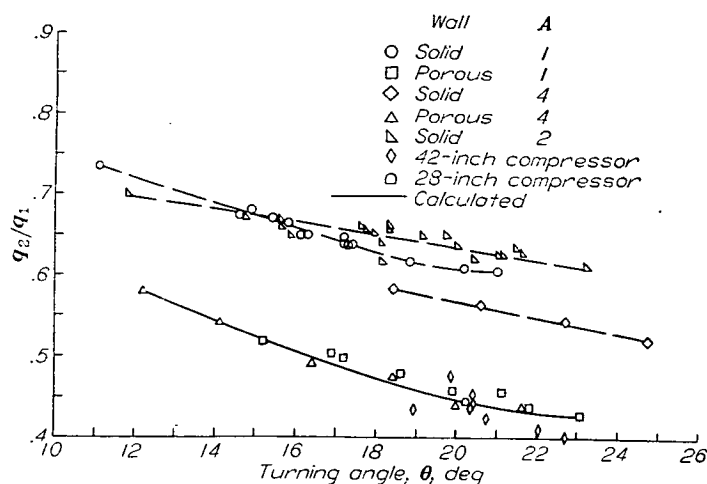


FIGURE 12.—Dynamic-pressure ratio as measured in the various tunnel configurations tested. The solid-line curve indicates the calculated value of q_2/q_1 .

1, 2, and 4 and two test blowers at their mean diameters. The porous-wall cascades appear to satisfy these criteria of two-dimensionality.

Passage wake surveys.—An examination of the flow downstream of the cascade was made in order to determine how much of the flow was homogeneous. Total-pressure surveys of the test cascade were made in the solid-wall and porous-wall configurations with blades of aspect ratio 1 at an angle of attack of 15.1° . The results are presented in figure 13. In the solid-wall case, a large region of low-energy air, which had its core originating at the junction between the convex surfaces of the test airfoil and the tunnel wall, was observed. In reference 9, a similar plot is presented for a cascade of aspect ratio 2 in which similar results are indicated. The percentage of the flow area affected is less, however, and a short region of uniform flow appears in the center of the tunnel.

The porous-wall configuration had only a relatively small region of loss greater than that of the wake of the test blades. The portion of the wake unaffected by the spread near the wall was 80 percent of the tunnel width. The total-pressure-loss values in this "two-dimensional" wake were lower and of smaller extent than in the solid-wall test.

A comparison of these plots illustrates clearly how continuous removal of the wall boundary layers alters the static-pressure rise across a compressor cascade. With solid walls, the large low-energy regions act to constrict the main flow and so reduce the effective flow area, so that an increase in the main exit flow velocity results. With permeable walls, since the effective exit flow area is much larger, the exit velocity is smaller and the static-pressure rise is higher.

Comparison of normal-force coefficients.—One of the important criteria for two-dimensional flow is that the value for the force on the blades determined by calculation of the momentum and pressure changes associated with the meas-

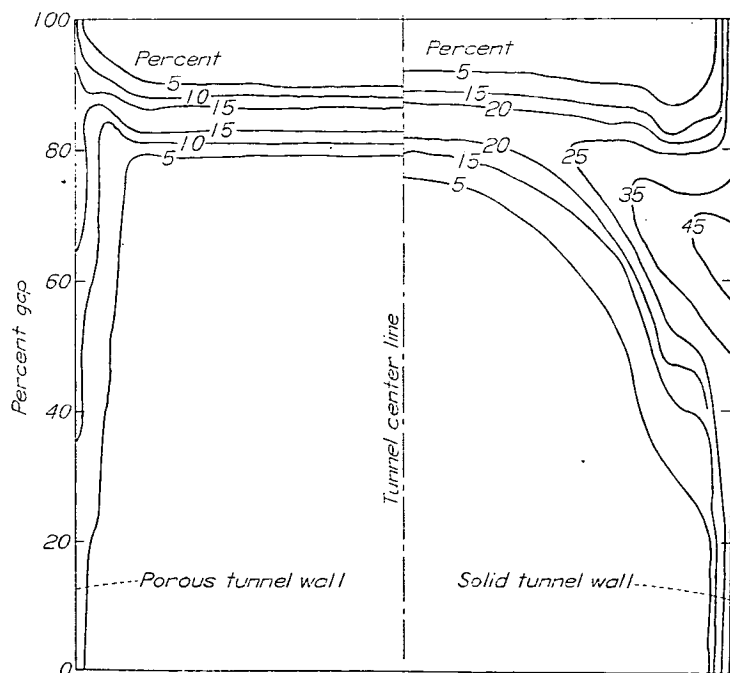


FIGURE 13.—Comparison of downstream total-pressure-loss contours obtained in the Langley 5-inch cascade tunnel with solid walls and with porous walls. $A=1$; $\sigma=1$; $\alpha=15.1^\circ$.

ured turning angle be equal to the force as determined by integration of the surface pressures. The ability of the configurations used to satisfy this criterion is illustrated in figure 14. The solid-wall cascade of aspect ratio 1 fails badly. The solid-wall cascade of aspect ratio 4 is better but the normal-force coefficient obtained from pressure distributions is from 7 to 15 percent low. The porous-wall cascades are clearly superior on this count, because reasonable correlation between $(C_N)_P$ and $(C_N)_M$ is indicated over the usual operating range.

Figure 15 shows that good agreement was obtained between $(C_N)_P$ and $(C_N)_M$ with the porous-wall cascade of aspect ratio 1 at the lower-pressure-rise case of $\beta=45^\circ$. No tests at lower than 45° inlet angle were included in this investigation, but it is believed that satisfaction of all the criteria of two-dimensionality would be obtained.

The pressure rise across the cascade has an effect on the force normal to the blade. In order to calculate this force, the pressure rise must be known or assumed. The calculated normal-force-coefficient curves of figures 14 and 15 were prepared by using the pressure rise measured in the porous-wall cascades. As previously noted, the values of q_2/q_1 are controllable with continuous boundary-layer removal and were adjusted to correspond to the observed turning angle with allowance for the observed wake. The calculated values of C_N presented are therefore believed to be representative of two-dimensional flow and the most logical basis for comparison.

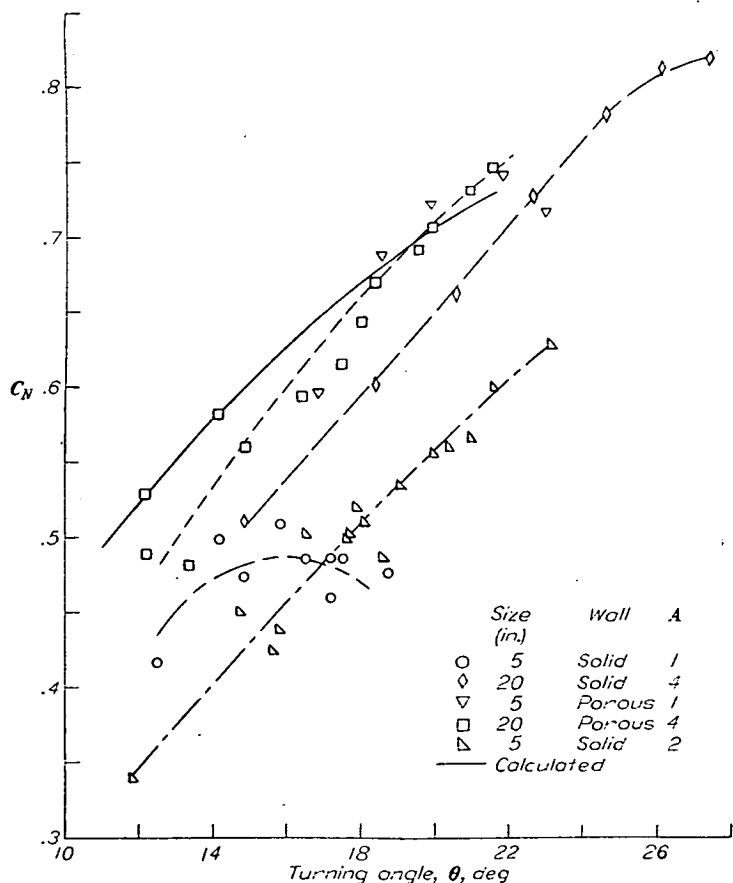


FIGURE 14.—Comparison of normal-force coefficient obtained in various cascades. Solid-line curve indicates calculated values of C_N .

An interesting comparison is that of the pressure-distribution normal-force coefficients obtained with the 42-inch-diameter test compressor, the values calculated for the observed turning angles and the values obtained in solid-wall cascade of aspect ratio 1. This cascade might be expected to produce effects similar to the blower, if the physical similarity were the determining factor, since the test compressor has blades of aspect ratio of about 1, solid "walls," and a solidity of 1.182. The values of σC_N presented in figure 16 illustrate that the effects are not similar. At the mean diameter, the σC_N values of the test compressor match the values calculated for the measured turning angles within the limits of the measuring accuracy, but the values for the solid-wall cascade of aspect ratio 1 are greatly different.

The excellent agreement between the test compressors and the porous-wall cascades on turning-angle (θ against α), normal-force-coefficient ($(C_N)_P$ and $(C_N)_M$ against θ), and pressure-rise (q_2/q_1 against θ) relations probably results from the relatively greater energy of the boundary layers in the compressors compared with the energy of the boundary layer in the cascade tunnels. This energy difference, which is due to greater relative velocities between the boundary layer and the running blades in the compressors as compared with the stationary blades in the cascade, is discussed in reference 3. The excellent agreement obtained illustrates that two-dimensional cascade data can be used directly in the design of axial-flow compressors.

Spanwise pressures.—One of the characteristics of a flow constant in the spanwise direction is that equal pressures exist at one chordwise position at different spanwise locations. In order to examine the ability of the various setups to meet this requirement, airfoils having static-pressure orifices located $\frac{1}{2}$ inch from the tunnel wall as well as at the usual center-line location were constructed. NACA

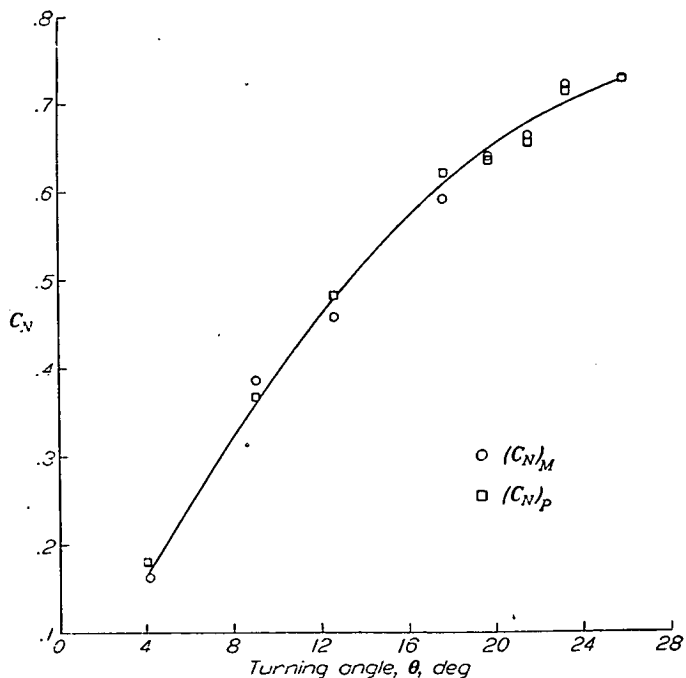


FIGURE 15.—Comparison of normal-force coefficient obtained by integration of blade pressure distribution and by momentum-change calculations. $\beta=45^\circ$; $\sigma=1$; porous walls.

65-(12)10 compressor blades of aspect ratio 1 were tested at $\beta=60^\circ$, with a solidity of 1, with solid and with porous walls. In the solid-wall case (fig. 17) the pressures near the wall are quite different from those along the tunnel center at the various chordwise stations. In addition, the normal-force coefficient obtained by integrating the center-line diagram is significantly lower than that calculated from the measured turning angle. This fact, coupled with the dif-

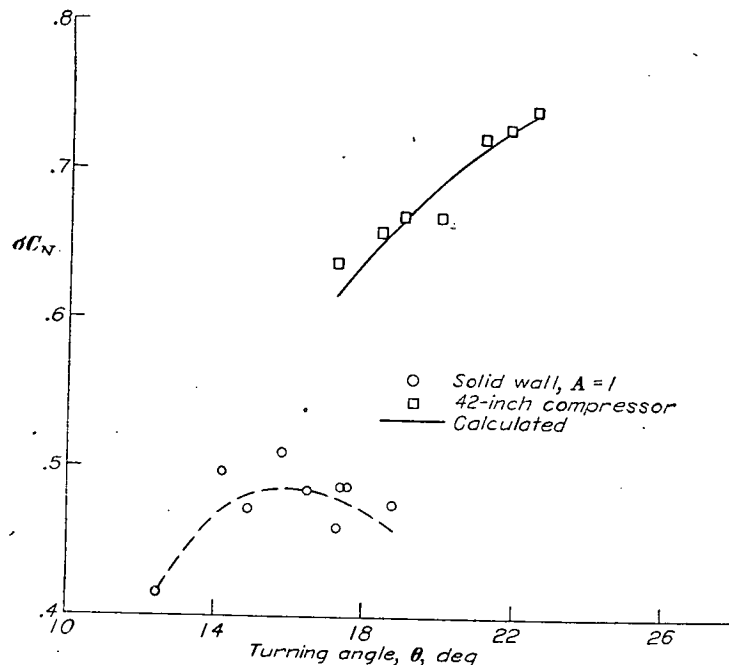


FIGURE 16.—Comparison of normal-force coefficient measured in the 42-inch-diameter test blower with that measured in the solid-wall cascade of $A=1$. The solid curve is calculated from measured turning angle, the appropriate inlet angle being considered.

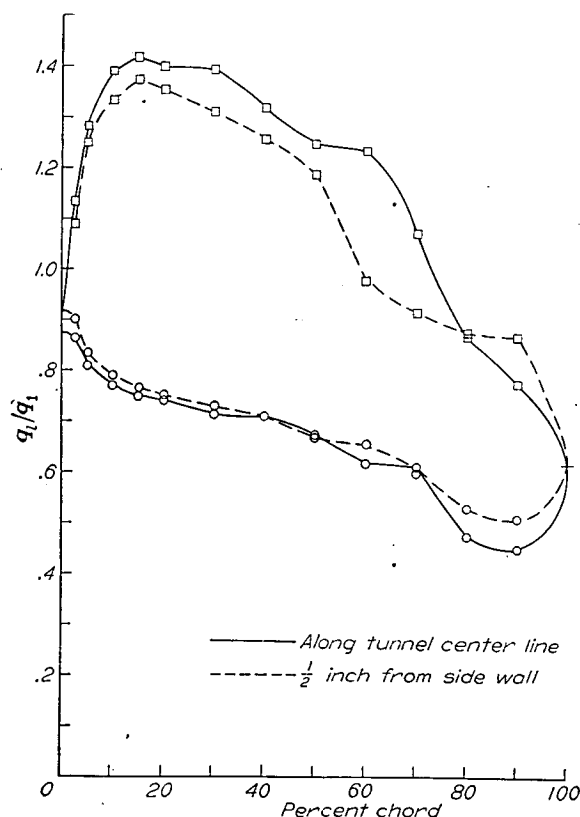


FIGURE 17.—Pressure distributions at two spanwise locations in the 5-inch solid-wall cascade of aspect ratio 1. $\alpha=14.1^\circ$.

ference between the measured and calculated exit velocities, suggests that significant differences in the pressure distribution would occur either in the ideal two-dimensional case or in a compressor (the calculated pressure rise has been obtained in several test compressors). This possibility was discussed in reference 3.

Pressure distributions similar to those for the solid-wall condition are presented in figure 18 for the porous-wall condition. Excellent agreement is obtained between the two diagrams. The normal-force coefficients obtained by integration and by calculation agree within the limits of experimental error.

For purposes of comparison, the center-line pressure distributions of figures 17 and 18 have been replotted in figure 19. Although the angle of attack and turning angle were similar in the solid-wall test, it is clear that the total area is considerably less and that the pressure distributions differ from each other in such a manner that it would be difficult, if not impossible, to devise a method for correcting the solid-wall test results to agree with those which would result in a two-dimensional flow.

Similar agreement was obtained in porous-wall tests at an inlet air angle of 60° with a cascade of aspect ratio 4 (fig. 20) over the test range of angle of attack. It is therefore believed that a reasonably close approach to two-dimensional flow was obtained with continuous removal of the tunnel-wall boundary layers.

EFFECT OF REYNOLDS NUMBER

A set of 2.5-inch-chord airfoils was constructed to permit cascade tests with blades of aspect ratio 2. At the usual test velocity of about 95 feet per second, the Reynolds number of this cascade would be 123,000, a value presumably well above the critical range at which scale effects occur (usually taken to be about 100,000 for typical axial-flow compressors). Because of the low turbulence level of the Langley 5-inch cascade tunnel, laminar separation existed, and very poor performance was observed. (Velocity variation was 0.0004 of stream velocity; however, in both tunnels the turbulence factor varied somewhat with the air velocity.) A series of tests was therefore run in an attempt to define the critical Reynolds number for the test cascade in several tunnel configurations.

The 2.5-inch-chord blades in the original smooth condition were tested at several values of R by increasing the test entrance velocity. In figure 21, a leveling of the turning-angle and drag-coefficient curves around values of $R=250,000$ is indicated for these tests. Because the tunnel motor and blower appeared to be operating at speeds above a safe limit, attempts were made to simulate higher effective values of R by introducing turbulence into the air stream and by using airfoils roughened by a strip of masking tape at the leading edge. Figure 22 indicates that a turning-angle performance indicative of higher effective Reynolds number can be obtained by the use of roughness. Some difficulty was encountered in introducing turbulence to the test air stream and still maintaining uniform entering-flow conditions; therefore, the results are not presented herein. However, qualitatively, the expected effect was observed.

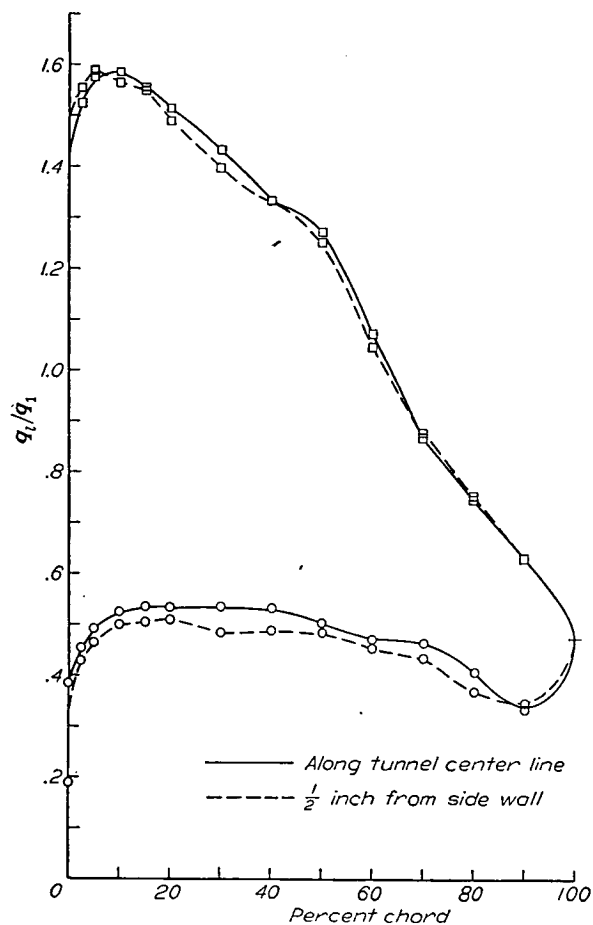


FIGURE 18.—Pressure distributions at two spanwise locations in the 5-inch porous-wall cascade of aspect ratio 1. $\alpha=14.1^\circ$.

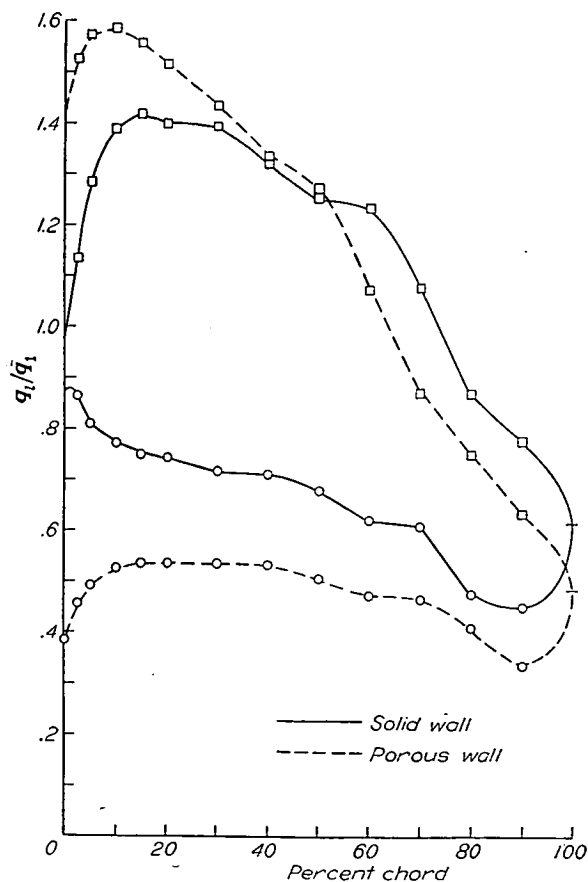


FIGURE 19.—Comparison of test-airfoil surface pressure distributions measured along tunnel center line with solid and with porous walls. $\alpha=14.1^\circ$.

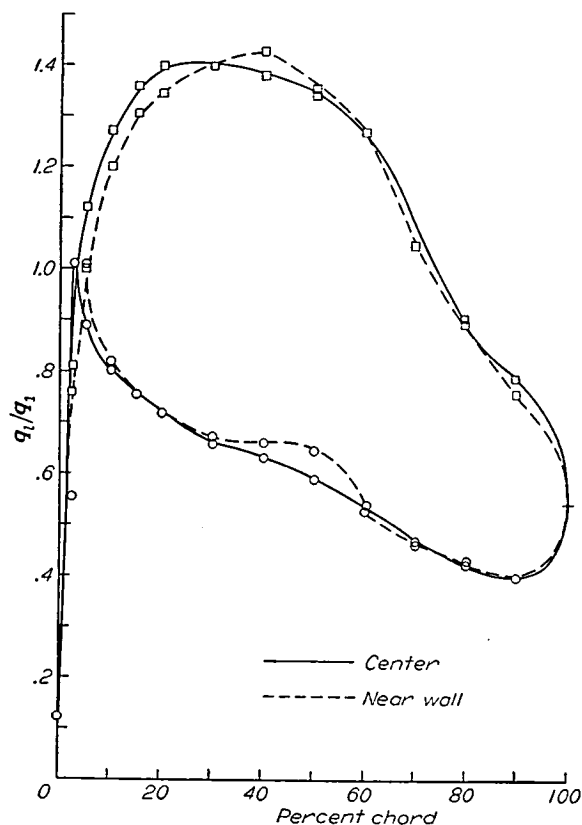


FIGURE 20.—Comparison of pressure distributions at two spanwise locations taken in the porous-wall cascade of aspect ratio 4, at 60° inlet angle, and with a solidity of 1. $\alpha=9^\circ$.

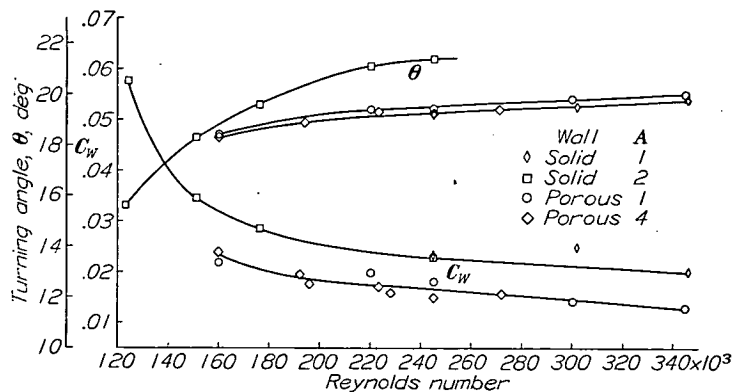


FIGURE 21.—Effect of Reynolds number on cascade performance. $\alpha=15.1^\circ$.

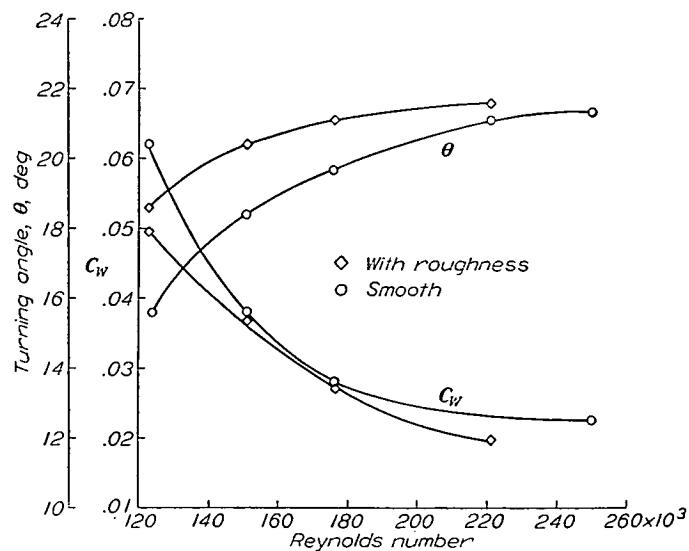


FIGURE 22.—Effect of roughness on cascade performance over a range of Reynolds numbers. Tests were made in 5-inch solid-wall cascade of aspect ratio 2.

To make certain that values of R above 250,000 would produce only insignificant changes in performance, a set of 5-inch-chord airfoils were rerun in the 5-inch cascade with solid walls at the usual test velocities and at higher values. These results are included in figure 21. Since the aspect ratio of the blades used in these tests was unity, the turning-angle values are not directly comparable with the results of the 2.5-inch-chord tests. The indication is that the critical Reynolds number is about 250,000.

Hot-wire anemometer measurements in the 20-inch tunnel indicated that the turbulence factor of this tunnel was about ten times that of the 5-inch tunnel. To obtain further information on Reynolds number effects, the test cascade was rerun in the 20-inch tunnel. Entering speeds above and below the usual value were used. These results are also included in figure 21. In the Reynolds number range between 160,000 and 250,000 the turning-angle values for the porous-wall cascades of $A=1$ and $A=4$ vary less with R than do the values for solid-wall cascades of $A=2$. Apparently, continuous removal of the side-wall boundary layer makes a compressor cascade less sensitive to Reynolds number changes.

A large variation in the value of the measured wake coefficient is noted at $R=250,000$. The section drag for the solid-wall cascade of $A=1$ is influenced by the blade-wall junction losses, as was shown in figure 13. The wake coefficients of the test airfoils of $A=2$ in the solid-wall cascade and $A=1$ in the porous-wall cascade are believed to be greater than those of the airfoils with $A=4$ because of the greater turbulence and higher effective Reynolds number in the 20-inch tunnel. This trend is evident in figure 23, in which a plot of C_w against α is presented for several tunnel configurations. The scatter of the C_w values suggests the order of accuracy of this measurement.

Data taken at angles of attack other than 15.1° indicate a large change in the values of $d\theta/d\alpha$ with R (fig. 24). Most of these data were obtained with the solid-wall cascade with blades of $A=2$ over only a 4° range of α ; however, a more complete α range was run in a porous-wall cascade with blades of $A=2$ at $R=190,000$ and $176,000$. The points presented in figure 24 are not to be considered final, but they do provide an indication of the trend to be expected.

The decrease of $d\theta/d\alpha$ with R is believed to be related to the pressure gradients near the leading edge of the convex surface of the test airfoils and to the extent of the laminar separation. At lower angles of attack, the pressure recovery in the forward 35 to 40 percent chord is small (fig. 20) and laminar flow can exist for some distance so that a relatively thick laminar boundary layer results. Farther downstream a rapid pressure rise occurs and, at low Reynolds numbers, a severe laminar separation takes place and low lift and low turning angles result. The fact that severe separation occurs is indicated by the high C_w values measured at low Reynolds numbers (fig. 21). At higher angles of attack, the region favorable to a laminar boundary layer is reduced, so that when laminar separation takes place the separation is of lesser extent. (In some cases, particularly at higher Reynolds numbers, the flow may reattach to the surface.) Thus,

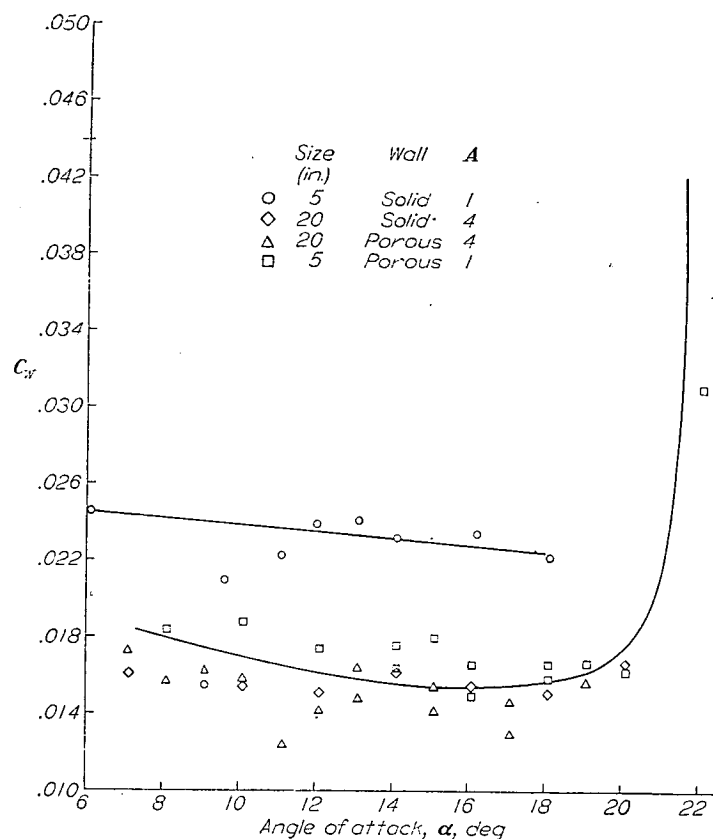


FIGURE 23.—Wake coefficient of the NACA 65-(12)10 airfoils at $\beta=60^\circ$ and $\sigma=1$ for various angles of attack.

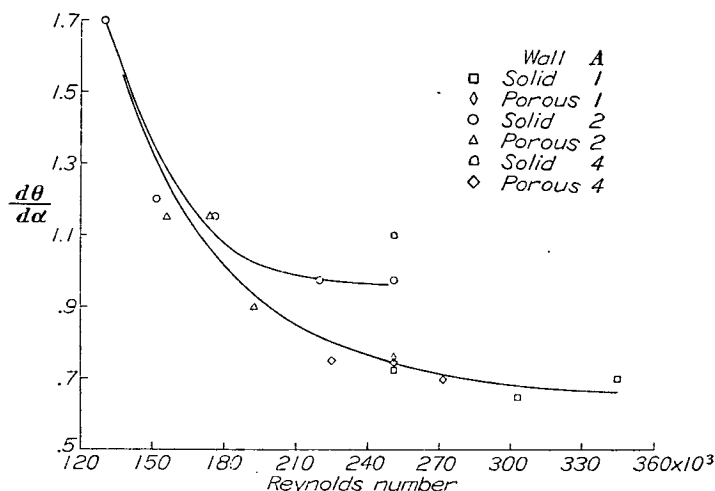


FIGURE 24.—Effect of Reynolds number on curve of $\frac{d\theta}{d\alpha}$.

as the angle of attack is increased, a disproportionate increase in turning angle occurs at low Reynolds numbers. At higher Reynolds numbers, laminar separation is less likely to occur so that these effects are less pronounced and a lower value of $d\theta/d\alpha$ results.

Attempts were made to verify the apparently reduced effect of Reynolds number on porous-wall cascades of aspect ratios 1 and 4 by testing the 2.5-inch-chord airfoils of $A=2$ with permeable walls. With the exhausting equipment available, it was not possible to establish the desired flow conditions at Reynolds numbers above 190,000. At this value, a turning angle of 19.2° was measured at 15.1° angle of attack, and $d\theta/d\alpha$ equalled 0.90. These results agree with the previously recorded data.

COMPARISON OF SEVERAL METHODS OF CORRECTING TURNING ANGLES TO THE TWO-DIMENSIONAL CASE

There are many situations in which it would be desirable to use solid-wall cascades if suitable means for correcting the results were available, as, for example, when schlieren photographs of the flow are desired. Several methods for converting cascade and compressor results that are not two-dimensional to the two-dimensional case were applied to the data from the solid-wall cascade with blades of $A=4$. One method is based on the discussion in reference 3, in which the exit axial velocity is made equal to the inlet axial velocity. The entrance inlet angle and angle of attack are assumed to be unaffected by the exit flow (fig. 25(a)). A second

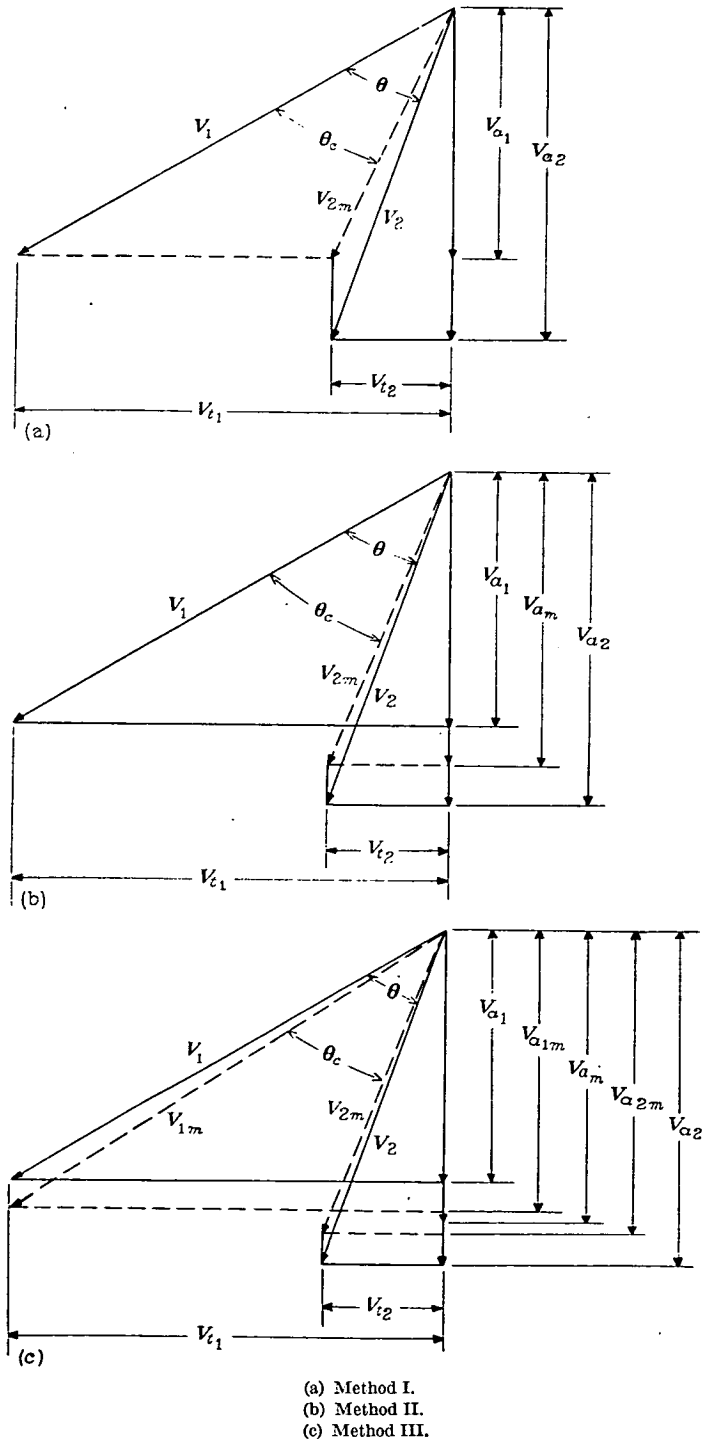
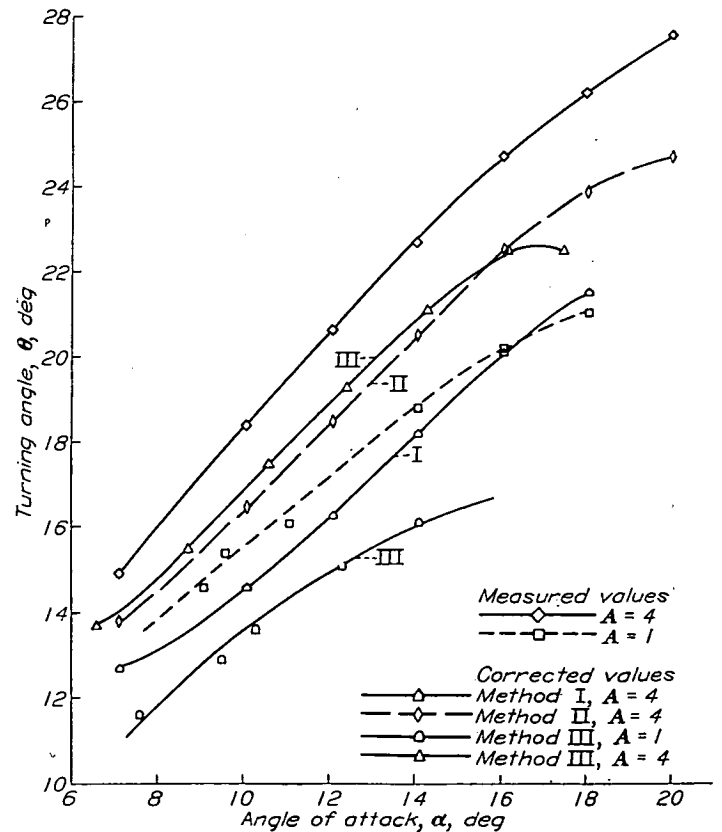


FIGURE 25.—Velocity diagrams illustrating several correction methods.

method, based on a discussion in reference 10, suggests that the exit axial velocity be adjusted by one-half the difference between inlet and exit values and the corrected turning angle be the difference between the actual inlet vector and the adjusted exit vector (fig. 25(b)). A third method is to average the inlet and exit axial velocities and then to reduce the averaged axial velocity by one-half the axial-velocity increment due to the wake to obtain the inlet axial velocity. The exit axial velocity would be greater than the average value by one-half the wake increment (fig. 25(c)). In all these methods, the change in the tangential velocity is assumed to be unaffected by the change in axial velocity. This assumption may not be valid.

The turning angles measured in the solid-wall cascade of $A=4$ have been corrected by these three methods, and the results are presented in figure 26. The turning angles measured in the solid-wall cascade of $A=1$, which were in good agreement with test-blower and porous-wall-cascade results as indicated in figure 9, are included in figure 26. Corrected values obtained by the wake-allowance method (method III), which gave the smallest change for a given axial-velocity increase, are also included for the $A=1$ solid-wall case. The turning angles for the $A=4$ case as measured were 2° to 4° higher than the median values of figure 10. The second and third methods reduced these differences to about one-half. The first method yielded results in better agreement with median curve, but the corrected values tended to be low, with a maximum difference of about 1° .

The wake-allowance method was applied to the data from the solid-wall cascade with blades of $A=1$. The corrected turning angles appear 2° to 3° below the measured data.

FIGURE 26.—Comparison of turning angles measured in the solid-wall cascades of $A=1$ and $A=4$ with values corrected by several methods.

(or the median curve) if this method is used and would be even lower if the other methods were used. Since the corrected turning angles for blades of $A=4$ tend to be too high and the values for blades of $A=1$ too low, the conclusion might be drawn that some combination of test aspect ratio and correction method could be obtained that would make possible the adjustment of cascade data that are not two-dimensional to the two-dimensional case. That such a combination would yield satisfactory results over the useful range of inlet angles, solidities, and cambers seems unlikely. Consideration of correction attempts for aspect ratios 1 and 4 indicates that a single simple method of adjusting measured values for observed axial-velocity changes is not likely to be found. Apparently factors other than the increase or decrease of axial velocity through the cascade must be taken into account. Undoubtedly one of these other factors is the effect of the flow induced by the trailing vortices. The method of reference 9 was used to estimate the magnitude of these effects in the solid-wall cascades of $A=1$ and $A=4$, for a measured turning angle of 20° . The turning angle in the cascade of $A=1$ is decreased about 2° by the induced flow, whereas the turning angle for the blade of $A=4$ is reduced by 1.7° . Adding 2° to the corrected turning angles in figure 26 would bring the $A=1$ results into closer agreement with the median curve. Adding 1.7° to the $A=4$ results, however, would cause greater disagreement between the values corrected by any of the three methods used and the median curve of figure 10.

The values of normal-force coefficient obtained by integrating the pressure distribution about the test airfoils have been replotted with the turning angles corrected by three methods (fig. 27). The corrected $(C_N)_P$ curves intersect the theoretical curve within the test range. The $(C_N)_P$ curve plotted against measured values of θ did not intersect the theoretical curve (fig. 14). Thus the axial-velocity adjustment methods used might be considered to improve the agreement between measured and calculated normal-force coefficients, but, because the slopes are quite different, only little improvement is gained. If the induced flows were taken into account by the relations of reference 9, the various values of $dC_N/d\theta_c$ would be in less agreement with the theoretical values, because the reduction of turning angle due to the trailing vortices increases with C_N . Thus no methods are at hand to explain the observed flows in the solid-wall cascades.

TURBINE-BLADE TESTS

A turbine blade designed to operate at $\beta=30^\circ$ with about 85° turning angle, a pressure-drop, reaction condition, was tested in the 5-inch tunnel in cascades of $A=0.83$ and in the 20-inch tunnel in cascades of $A=3.33$; solid walls were used. The curves of α against θ are presented in figure 28. Little difference other than experimental scatter is noted. Blade-surface static-pressure measurements taken along the tunnel center line are shown in figure 29. Greater differences than can be accounted for by inaccuracies of measurement existed, but general agreement was obtained. Tuft surveys indicated a collecting of low-energy flow at the juncture of

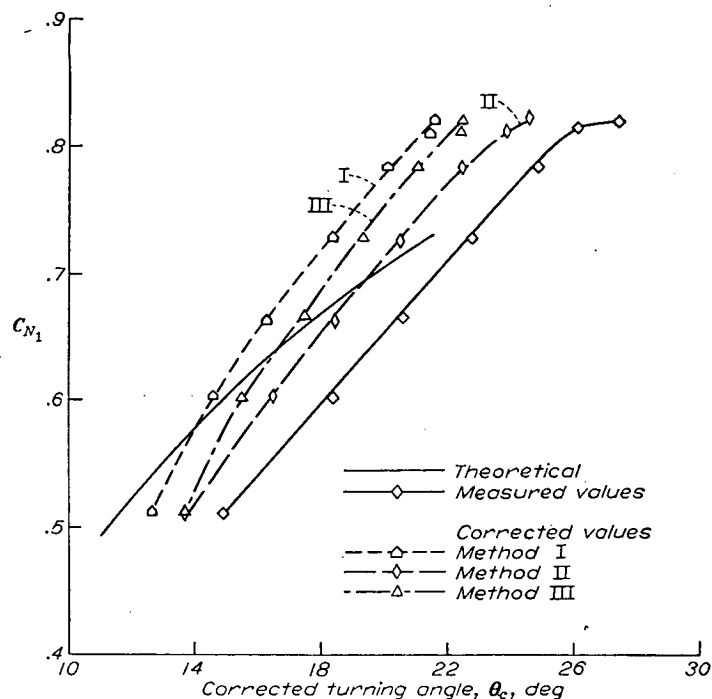


FIGURE 27.—Comparison of normal-force coefficients obtained in the solid-wall cascade of $A=4$ when plotted against the measured turning angles and the turning angles corrected by three methods.

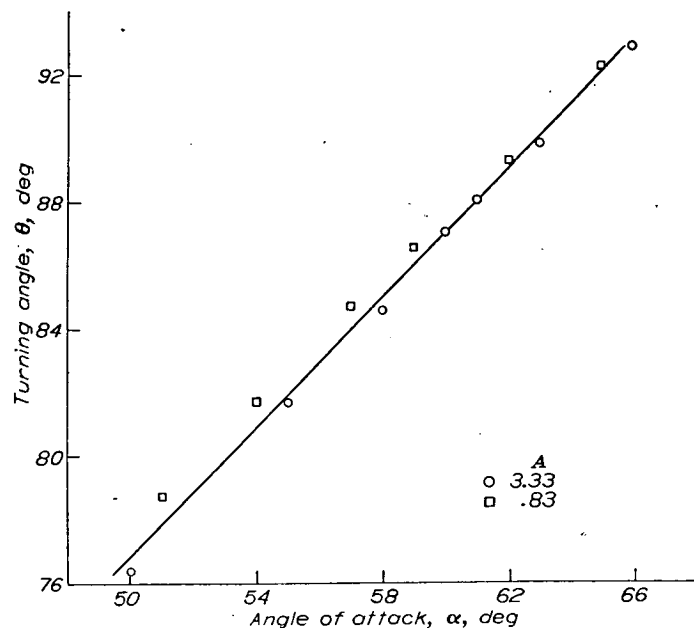


FIGURE 28.—Variation of turning angle with angle of attack for one turbine-blade section tested at aspect ratios of 0.83 and 3.33.

the convex surface of the test blades and the tunnel side walls. In these tests, even at the lower aspect ratio, this condition seemed to have little effect on the test results. It is therefore believed that, with usual pressure-drop cascades, as with reaction turbine blades, nozzles, and guide vanes, useful results can be obtained with cascades of low aspect ratio.

Another reaction turbine section designed for $\theta=105^\circ$ at $\beta=45^\circ$ was tested in the 5-inch tunnel in cascades of $A=0.83$ with solid and with porous walls. Since the dynamic pressure

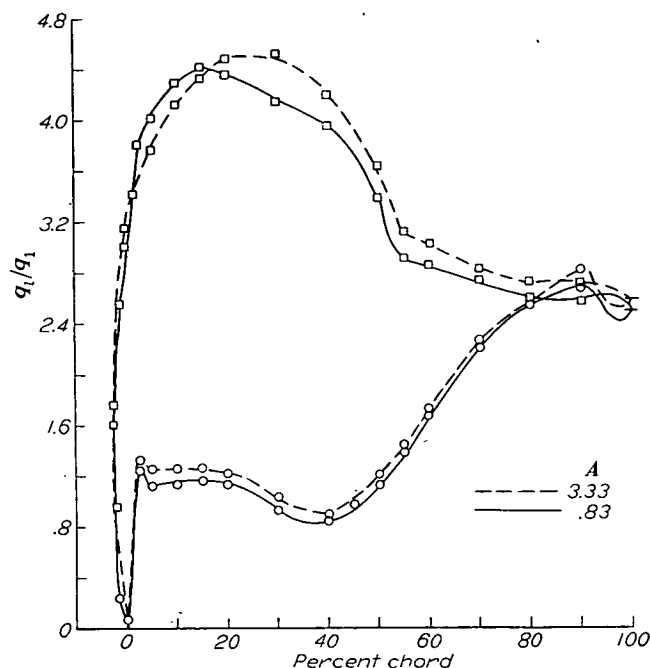


FIGURE 29.—Comparison of surface pressure distributions of a turbine blade of aspect ratio 0.83 and aspect ratio 3.33 in solid-wall tunnels.

at the exit from the solid-wall cascade was not much greater than that calculated for two-dimensional flow, very little air could be drawn through porous walls without lowering the exit dynamic pressure below the desired value. With this low porous-wall flow rate, the pressure drop across the porous walls in use with this test was not sufficiently great to produce an inflow through the entire wall. Therefore, the flow through the cascade could not be considered valid. Nevertheless, a turning angle only 1° different (in 105°) was measured. This test is at best only indicative of the effects of using porous walls with turbine cascades, but it seems clear that with pressure-drop cascades little benefit is to be gained through the use of permeable side walls.

CONCLUSIONS

The results of an investigation to determine the influence of aspect ratio, boundary-layer control by means of slots and porous surfaces, Reynolds number, and tunnel end-wall condition upon the performance of airfoils in cascades have led to the following conclusions:

1. Conventional cascades of compressor blades with aspect ratio 4 do not simulate the two-dimensional case. The indication is that very large aspect ratios would be necessary to satisfy all criteria of two-dimensionality.

2. Because of opposing effects, the solid-wall cascade of aspect ratio 1 produced a relation between turning angle and angle of attack for the NACA 65-(12)10 compressor blade at a solidity of 1 and an inlet air angle of 60° more nearly like that of two test blowers than that of the solid-wall cascade of aspect ratio 4. However, the normal-force coefficients obtained from the measured pressure distributions were significantly less than those calculated from momentum and pressure changes, and the measured pressure rise across

the cascade was significantly less than that associated with the measured turning angle.

3. With continuous boundary-layer removal, cascades of aspect ratios 1 and 4 produced curves of turning angle against angle of attack for the NACA 65-(12)10 airfoil very similar to those of two test blowers, when the pressure rise associated with the measured turning angle was established.

4. Much better agreement was achieved with the porous-wall cascades between the normal-force coefficients calculated from momentum and pressure changes and those obtained by integration of the measured pressure distributions than was obtained with the solid-wall cascades.

5. With cascades having a decrease in static pressure in the downstream direction, as with guide vanes and turbine blades, the use of permeable walls appears to be of negligible advantage. With such cascades, solid-wall tunnel configurations having blades of aspect ratios of 0.83 and 3.33 produced similar results.

6. For typical compressor-blade cascades in tunnels of low turbulence level, scale effects seem to be negligible above Reynolds numbers of 250,000 but may be appreciable below this value.

7. The porous-wall technique as applied to cascades of compressor blades makes practical the attainment of two-dimensional flows.

LANGLEY AERONAUTICAL LABORATORY,¹

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., November 30, 1949.

REFERENCES

1. Bogdonoff, Seymour M., and Bogdonoff, Harriet E.: Blade Design Data for Axial-Flow Fans and Compressors. NACA ACR L5F07a, 1945.
2. Bogdonoff, Seymour, M., and Hess, Eugene E.: Axial-Flow Fan and Compressor Blade Design Data at 52.5° Stagger and Further Verification of Cascade Data by Rotor Tests. NACA TN 1271, 1947.
3. Katzoff, S., Bogdonoff, Harriet E., and Boyet, Howard: Comparisons of Theoretical and Experimental Lift and Pressure Distributions on Airfoils in Cascade. NACA TN 1376, 1947.
4. Herrig, L. Joseph, and Bogdonoff, Seymour M.: Performance of an Axial-Flow Compressor Rotor Designed for a Pitch-Section Lift Coefficient of 1.20. NACA TN 1388, 1947.
5. Runckel, Jack F., and Davey, Richard S.: Pressure-Distribution Measurements on the Rotating Blades of a Single-Stage Axial-Flow Compressor. NACA TN 1189, 1947.
6. Pierpont, P. Kenneth: Investigation of Suction-Slot Shapes for Controlling a Turbulent Boundary Layer. NACA TN 1292, 1947.
7. Howell, A. R.: Fluid Dynamics of Axial Compressors. Proc. Inst. Mech. Eng. (London), vol. 153, 1945, pp. 441-452. Reprinted in U. S. by A.S.M.E.
8. Hausmann, George F.: The Theoretical Induced Deflection Angle in Cascades Having Wall Boundary Layers. Jour. Aero. Sci., vol. 15, no. 11, Nov. 1948, pp. 686-690.
9. Carter, A. D. S., and Cohen, Elizabeth M.: Preliminary Investigation into the Three-Dimensional Flow through a Cascade of Aerofoils. R. & M. No. 2339, British A.R.C., 1946.
10. Hawthorne, W. R.: Induced Deflection Angle in Cascades. Jour. Aero. Sci. (Readers' Forum), vol. 16, no. 4, April 1949, p. 252.

REPORT 1017

INVESTIGATION OF FREQUENCY-RESPONSE CHARACTERISTICS OF ENGINE SPEED FOR A TYPICAL TURBINE-PROPELLER ENGINE¹

By BURT L. TAYLOR, III, and FRANK L. OPPENHEIMER

SUMMARY

Experimental frequency-response characteristics of engine speed for a typical turbine-propeller engine are presented. These data were obtained by subjecting the engine to sinusoidal variations of fuel flow and propeller-blade-angle inputs. Correlation is made between these experimental data and analytical frequency-response characteristics obtained from a linear differential equation derived from steady-state torque-speed relations.

The results of this investigation indicate that engine speed is a linear function of fuel flow and propeller-blade angle for limited variations of these parameters. Algebraic expressions approximately describing the frequency-response characteristics of the engine tested are primarily first order. Approximate frequency-response characteristics limited to first-order effects may be calculated from solutions of a linear differential equation and equilibrium torque characteristics of the engine and the propeller.

INTRODUCTION

In order to design controls that will provide desirable transient operation, the dynamics of the various components of the controlled system should be known. The most general description of the dynamics of any system is the differential equations of motion, but use of the detailed and precise aerodynamic and thermodynamic equations for analysis of dynamic behavior of a gas-turbine engine and control system presents insurmountable problems. On the other hand, any description whereby linear behavior of the system components is assumed can be placed in a form that is readily handled and productive of results. The advantages of the linear assumption are so far-reaching that its use is justified, even where the system is of known nonlinearity but where high precision of results is not required, such as in the investigation of controls for gas-turbine engines.

In the linear field, several descriptive forms can be used. The most general of these is the frequency response, whereby the attenuation and the phase lag of steady sinusoidal inputs are given for the continuous frequency spectrum. When these data are available for each component of a closed-loop control system, the time response of the system can be calculated. By employing techniques of analysis and synthesis described in reference 1, system behavior can be altered to obtain desired frequency-response characteristics and the related time response.

A general investigation is in progress at the NACA Lewis laboratory to determine the nature of the dynamics of gas-turbine power plants and to devise and to evaluate methods for describing this behavior in a form applicable to control

synthesis and analysis. Special emphasis has been placed on the determination of the behavior of rotative speed because of its importance in this type of engine for safety and optimum operation. The results of an investigation of the frequency-response characteristics of a turbine-propeller engine are presented herein. Experimental frequency-response data were analyzed to obtain the general form and nature of the frequency-response and dynamic characteristics of such engines, to verify the linear assumption, and to obtain a basis upon which other descriptive forms of the dynamic behavior may be compared. One such comparison is made between the linear differential equation obtained from equilibrium-torque curves and experimental frequency-response characteristics.

For purposes of this investigation, the dynamic response of engine speed to disturbances in fuel flow and propeller-blade angle for a typical turbine-propeller engine is used.

ANALYSIS

DEVELOPMENT OF LINEAR DIFFERENTIAL EQUATIONS

With the hypothesis that quasi-static conditions exist in an engine during transient conditions of operation, a differential equation may be derived to express the behavior of engine speed as a function of fuel flow (reference 2). An assumption is made that the torque output of the engine is some function of only two variables, engine speed and fuel flow, for a given condition of altitude and ram pressure ratio. This function may be expanded and linearized so that the following equation approximates the behavior of engine speed for operation in a small region

$$\left. \begin{aligned} Q_e &= f_1(W_f, N_e) \\ \Delta Q_e &= a \Delta W_f - b \Delta N_e \end{aligned} \right\} \quad (1)$$

where a is the partial derivative of engine torque with respect to fuel flow and b is the absolute value of the partial derivative of engine torque with respect to engine speed. A delta prefix before a variable signifies deviation from the initial operating point. (All symbols used in this report are defined in the appendix.)

In a similar manner, the torque absorbed by a propeller may be expressed as some function of rotational speed and blade angle

$$\left. \begin{aligned} Q_p &= f_2(\beta, N_p) \\ \Delta Q_p &= c \Delta \beta + d \Delta N_p \\ \Delta Q_p &= c \Delta \beta + \frac{d}{R} \Delta N_e \end{aligned} \right\} \quad (2)$$

¹ Supersedes NACA TN 2184, "Investigation of Frequency-Response Characteristics of Engine Speed for a Typical Turbine-Propeller Engine" by Burt L. Taylor, III, and Frank L. Oppenheimer, 1950.

where c is the partial derivative of propeller torque with respect to blade angle and d is the partial derivative of propeller torque with respect to propeller speed.

Any unbalance between torque developed by the engine and torque absorbed by the propeller results in an acceleration. If the effective moment of inertia is assumed to be comprised of the propeller and the rotating parts of the engine, the unbalanced torque may be expressed in terms of engine speed and the moments of inertia by the following equation:

$$\Delta Q_e - \frac{\Delta Q_p}{R} = \left(I_e + \frac{I_p}{R^2} \right) \Delta \dot{N}_e \quad (3)$$

If equation (2) is divided by the gear ratio R and subtracted from equation (1), the following equation results:

$$\left. \begin{aligned} & \frac{\left(I_e + \frac{I_p}{R^2} \right) \Delta N_e}{\left(I_e + \frac{I_p}{R^2} \right) N_{e(max)}} + \left\{ \left[\frac{\partial Q_e}{\partial N_e} \right] \left[\frac{1}{N_{e(max)}} \right] \left(\frac{1}{I_e + \frac{I_p}{R^2}} \right) + \left[\frac{\partial Q_p}{\partial N_p} \right] \left(\frac{1}{R^2} \right) \left[\frac{1}{N_{p(max)}} \right] \left(\frac{1}{I_e + \frac{I_p}{R^2}} \right) \right\} \frac{\Delta N_e}{N_{e(max)}} \\ & = \left[\frac{\partial Q_e}{\partial W_f} \right] \left[\frac{1}{W_{f(max)}} \right] \left[\frac{1}{N_{e(max)}} \right] \left(\frac{1}{I_e + \frac{I_p}{R^2}} \right) \Delta W_f - \left(\frac{\partial Q_p}{\partial \beta} \right) \left(\frac{1}{R} \right) \left[\frac{1}{N_{e(max)}} \right] \left(\frac{1}{I_e + \frac{I_p}{R^2}} \right) \Delta \beta \\ & \quad + \left\{ \frac{\partial \left[\frac{Q_e}{N_{e(max)} I_e \left(1 + \frac{I_p/R^2}{I_e} \right)} \right]}{\partial \left[\frac{N_e}{N_{e(max)}} \right]} + \frac{\partial \left[\frac{Q_p}{N_{p(max)} I_p \left(1 + \frac{I_e}{I_p/R^2} \right)} \right]}{\partial \left[\frac{N_p}{N_{p(max)}} \right]} \right\} \frac{\Delta N_e}{N_{e(max)}} \\ & = \frac{\partial \left[\frac{Q_e}{N_{e(max)} I_e \left(1 + \frac{I_p/R^2}{I_e} \right)} \right]}{\partial \left[\frac{W_f}{W_{f(max)}} \right]} \frac{\Delta W_f}{W_{f(max)}} - \frac{\partial \left[\frac{Q_p}{N_{p(max)} I_p \left(1 + \frac{I_e}{I_p/R^2} \right)} \right]}{\partial \beta} \Delta \beta \\ & \quad \frac{\Delta \dot{N}_e}{N_{e(max)}} + (b_1 + d_1) \frac{\Delta N_e}{N_{e(max)}} = a_1 \frac{\Delta W_f}{W_{f(max)}} - c_1 \Delta \beta \end{aligned} \right\} \quad (6)$$

The form of equation (6) is that of a first-order lag, which can be written as follows:

$$\tau \frac{\Delta \dot{N}_e}{N_{e(max)}} + \frac{\Delta N_e}{N_{e(max)}} = K_1 \frac{\Delta W_f}{W_{f(max)}} - K_2 \Delta \beta \quad (7)$$

where the dynamic characteristics are entirely defined by the time constant τ and the equilibrium condition is defined by K_1 and K_2 . Thus,

$$\left. \begin{aligned} \tau &= \frac{1}{b_1 + d_1} \\ K_1 &= \frac{a_1}{b_1 + d_1} \\ K_2 &= \frac{c_1}{b_1 + d_1} \end{aligned} \right\} \quad (8)$$

$$\Delta Q_e - \frac{\Delta Q_p}{R} = a \Delta W_f - \frac{c}{R} \Delta \beta - \left(b + \frac{d}{R^2} \right) \Delta N_e \quad (4)$$

Equations (3) and (4) may be combined, yielding the following differential equation relating engine speed to fuel flow and blade angle:

$$\left(I_e + \frac{I_p}{R^2} \right) \Delta \dot{N}_e + \left(b + \frac{d}{R^2} \right) \Delta N_e = a \Delta W_f - \frac{c}{R} \Delta \beta \quad (5)$$

If equation (5) is divided through by $N_{e(max)}$ and $\left(I_e + \frac{I_p}{R^2} \right)$ and the appropriate terms are divided by $\frac{N_{e(max)}}{N_{e(max)}}$, $\frac{N_{p(max)}}{N_{p(max)}}$, or $\frac{W_{f(max)}}{W_{f(max)}}$, the following dimensionless equation expressing speed and fuel flow is obtained:

FREQUENCY RESPONSE

Frequency-response characteristics may be calculated by solving the differential equation describing a system subjected to sinusoidal forcing functions of various frequencies. The same results may be obtained by applying the operational calculus to the differential equation. Thus, the response of engine speed to fuel flow at constant blade angle can be expressed in terms of the operator $i\omega$ as follows:

$$\frac{\Delta N_e}{N_{e(max)}} \frac{1}{\Delta W_f} (i\omega) = \frac{K_1}{(1 + i\tau\omega)} \quad (9)$$

which is obtained from equation (7) by replacing the first derivative with respect to time by $(i\omega)$ and treating the

results algebraically. The complex operator represents both the ratio of the amplitudes of the engine-speed and fuel-flow sine waves and the relative phase angles. The amplitude ratio is the absolute value of the complex function, as follows:

$$\left| \frac{\Delta N_e}{N_{e(max)} \frac{\Delta W_f}{W_{f(max)}}} \right| = \frac{K_1}{\sqrt{1 + \tau^2 \omega^2}} \quad (10)$$

The angle by which the engine-speed sine wave lags the fuel-flow sine wave is as follows:

$$\text{Ang} \frac{\Delta N_e}{N_{e(max)} \frac{\Delta W_f}{W_{f(max)}}} = \tan^{-1}(\omega \tau) \quad (11)$$

The frequency response of engine speed to blade angle at constant fuel flow may be obtained by the same procedure,

$$\frac{\Delta N_e}{N_{e(max)} \Delta \beta} (i\omega) = -\frac{K_2}{(1 + i\tau\omega)} \quad (12)$$

$$\left| \frac{\Delta N_e}{N_{e(max)} \Delta \beta} \right| = \frac{K_2}{\sqrt{1 + \tau^2 \omega^2}} \quad (13)$$

$$\text{Ang} \frac{\Delta N_e}{N_{e(max)} \Delta \beta} = \tan^{-1}(\omega \tau) \quad (14)$$

APPARATUS AND PROCEDURE

ENGINE INSTALLATION

Engine.—The principal components of the engine used for the investigation are an axial-flow compressor, reverse-flow combustion chambers, and a single-stage direct-coupled turbine. A two-stage planetary gear system provides a speed reduction between the turbine and the propeller.

A 12-foot-1-inch-diameter, four-bladed propeller was installed on the engine. The pitch-changing mechanism is a self-contained hydraulic unit located in the propeller hub. A lever, or beta arm, on the stationary portion of the hub operates the hydraulic control valves in the rotating part so that a linear relation exists between the beta arm and the blade angle. Each position of the beta arm corresponds to a unique blade angle during steady-state operation.

Harmonic-motion generator.—An approximation of simple harmonic motion by the beta arm was obtained by connecting it to a rotating crank driven by a direct-current motor through a worm gear reduction unit. A minimum ratio of 100:1 between the connecting rod and the crank length made the motion of the beta arm vary less than 1 percent from a true sine wave. By controlling the armature current and

substituting various speed reducers in the harmonic-motion generator, a range of frequencies could be obtained. The center of amplitude of the sine wave could be varied by shifting the position of the harmonic-motion generator, and various amplitudes could be obtained by changing the crank length.

In order to vary fuel flow sinusoidally, the fuel system of the engine was disconnected from the manifold and replaced by an external system, as shown in figure 1. A line was connected in parallel with the main throttle in order to obtain a small variation of fuel input about a center of amplitude set by the main throttle. The harmonic-motion generator used in the blade-angle runs varied the flow in the parallel line by means of a linear valve. Changes in pressure drop across the valve were too small to have an effect upon the linear variation of flow during transient operation.

INSTRUMENTATION

Steady-state measurements were taken of propeller-blade angle, fuel flow, engine torque, and engine speed. Provisions were made to measure all of these variables except the torque during transient conditions of operation.

Blade-angle position was measured by the use of a potentiometer attached to the propeller blade that varied the flow of current in an electric circuit in direct proportion to the blade position. For steady-state measurements, the current was measured by a milliammeter; during transient conditions the circuit was switched to a recording oscillograph element. A conventional slip-ring arrangement was used to complete the circuit between the potentiometer on the propeller hub and the recording device. Beta-arm position was recorded in a similar manner.

In order to measure fuel flow, an A.S.M.E. orifice was installed in the main fuel line immediately upstream of the engine fuel manifold. A bellows-type differential-pressure gage measured the pressure drop across the orifice and actuated a small potentiometer that varied the current in a circuit similar to that used for blade-angle measurement. Inasmuch as flow variations were small in comparison with mean flow, the nonlinear relation between pressure drop and actual flow was insignificant.

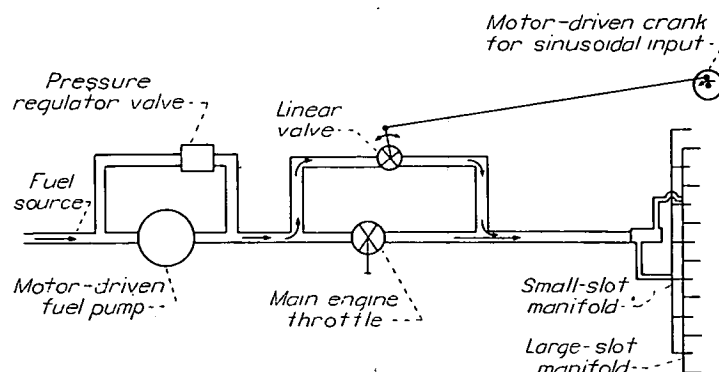


FIGURE 1.—System for providing sinusoidal fuel flow.

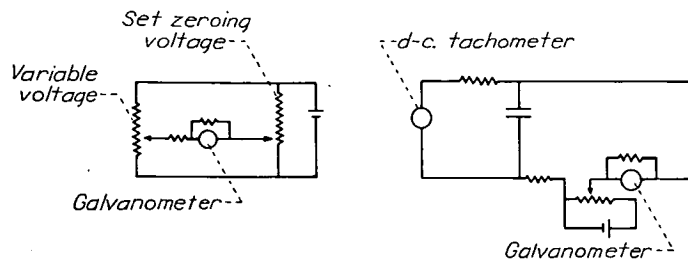


FIGURE 2.—Schematic diagrams of transient response instrumentation.

The ring gear of the planetary reduction unit of the engine is restrained by a self-balancing hydraulic system. The torque output of the engine was indicated by the pressure required to act on the hydraulic piston to maintain balance. This pressure was measured by a Bourdon gage.

Steady-state measurement of engine speed was accomplished by means of a three-phase tachometer generator and indicator. For recording transients in engine speed on the oscillograph, a direct-current tachometer generator was utilized in a current-measuring circuit.

A 10-cycle-per-second timing signal generated by an audio oscillator was recorded on the oscillograph film to show the time variation of the measured variables. The signal was frequently compared with a standard 60-cycle-per-second current on a cathode-ray oscilloscope for calibration purposes.

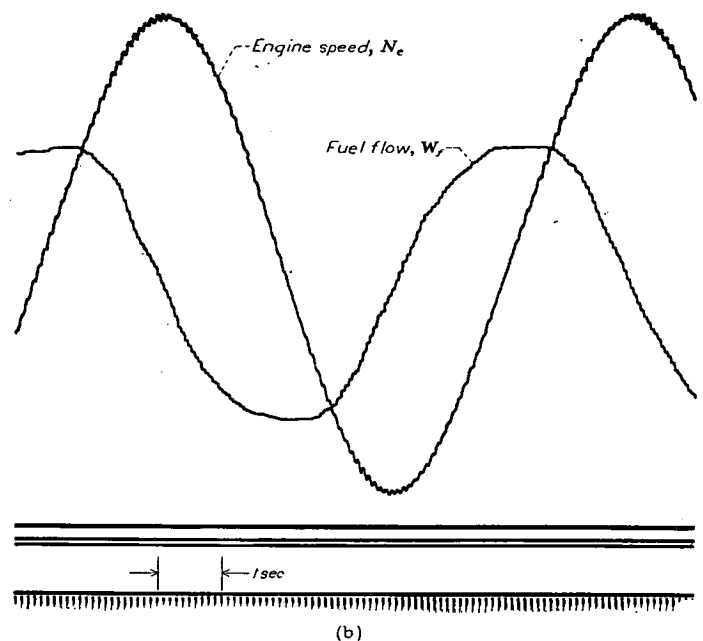
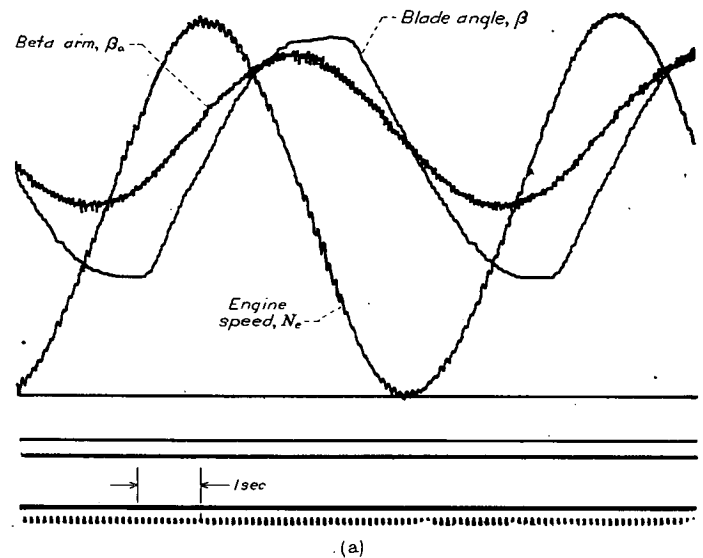
Schematic diagrams of the transient measurement circuits are shown in figure 2. Table I indicates the steady-state and transient characteristics of the instruments used. The oscillograph was used to record only increments of deviation from the initial operating point. Conventional steady-state instruments were used to measure the level at which the transient occurred and to evaluate the increments. The oscillograph was calibrated frequently during the runs for known values of deviation as measured by the steady-state instruments.

PROCEDURE

Frequency-response runs.—In order to determine the frequency response of engine speed, sinusoidal variations were made in propeller-blade angle at constant fuel flow and in fuel flow at constant blade angle. The frequency range to be investigated was determined by the following considerations: It is shown in reference 1 that for a first-order system the break frequency is inversely proportional to the system time constant and that an adequate frequency range to describe the unit is 0.1 to 10 times this value. Therefore, by assuming the time constant to be 5 seconds and considering the engine primarily of first order, the range of descriptive frequencies was defined to be 0.02 to 2 radians per second.

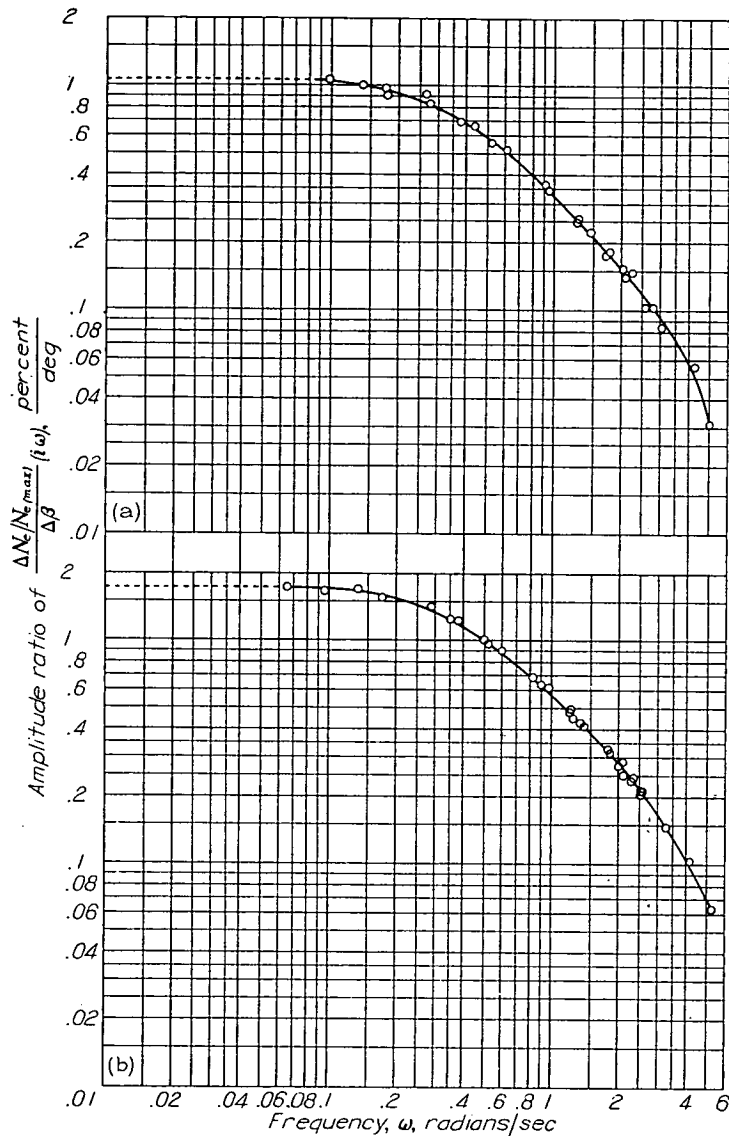
Dynamic behavior about two equilibrium running conditions of the engine was investigated. The first condition was 65 percent of maximum rated fuel flow and a blade angle of 18° (97.5 percent of maximum engine speed). The other condition was 81 percent of maximum rated fuel flow and a blade angle of 28° (92.5 percent of maximum engine speed).

The amplitude of the forcing function was the same at each operating condition; namely, a blade angle of 2° and 3.9 percent of maximum rated fuel flow. For each run, oscillographic records were made of the time variation of pertinent variables during steady-state sinusoidal operation. Typical records of these data are reproduced in figure 3. The high frequency variation of the fundamental waves in these figures resulted from electrical and mechanical disturbances to the sensing elements.

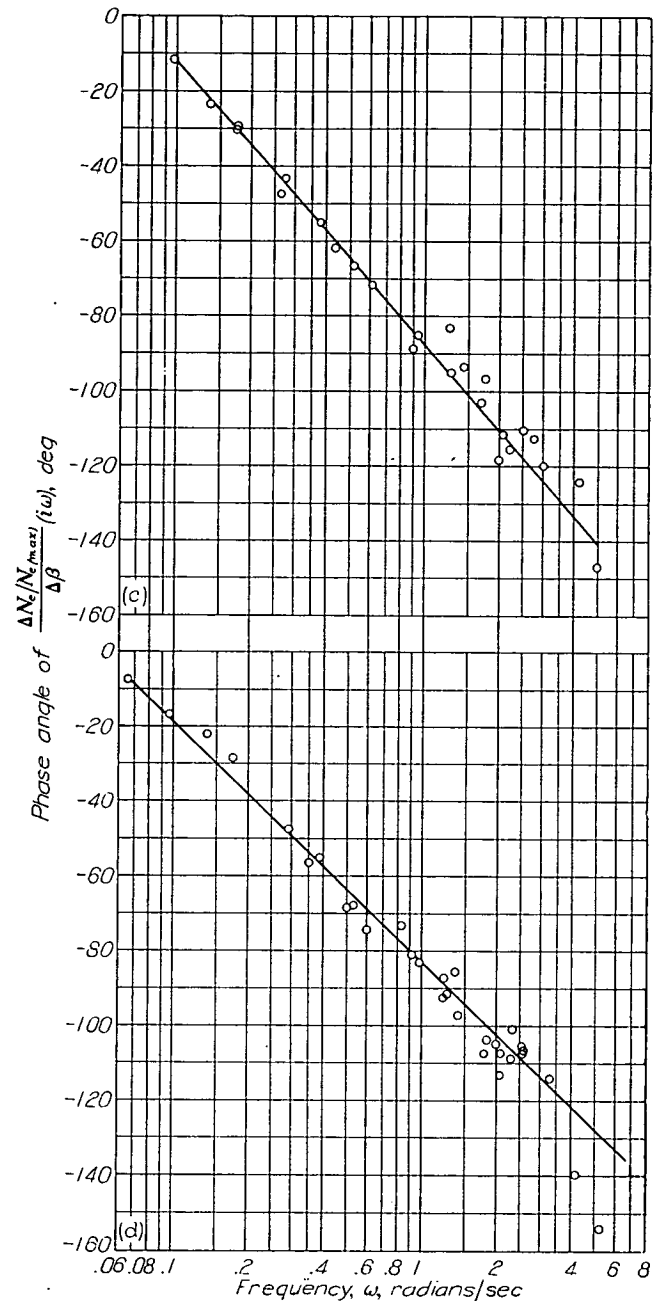


- (a) Typical data showing steady-state sinusoidal response of engine speed to blade angle. Fuel flow, 81 percent of maximum rated; mean engine speed, 92.5 percent of maximum; mean blade angle, 28° ; frequency, 0.986 radian per second.
- (b) Typical data showing steady-state sinusoidal response of engine speed to fuel flow. Blade angle, 18° ; mean engine speed, 97.5 percent of maximum; mean fuel flow, 65 percent of maximum rated; frequency, 0.873 radian per second.

FIGURE 3.—Typical steady-state sinusoidal data for turbine-propeller engine.



(a) Amplitude-ratio relation. Fuel flow, 65 percent of maximum rated; mean engine speed, 97.5 percent of maximum; mean blade angle, 18°; blade-angle amplitude, 2°.
 (b) Amplitude-ratio relation. Fuel flow, 81 percent of maximum rated; mean engine speed, 92.5 percent of maximum; mean blade angle, 28°; blade-angle amplitude, 2°



(c) Phase-angle relation. Fuel flow, 65 percent of maximum rated; mean engine speed, 97.5 percent of maximum; mean blade angle, 18°; blade-angle amplitude, 2°.
 (d) Phase-angle relation. Fuel flow, 81 percent of maximum rated; mean engine speed, 92.5 percent of maximum; mean blade angle, 28°; blade-angle amplitude, 2°.

FIGURE 4.—Propeller-blade angle—engine-speed relations at constant fuel flow obtained from experimental sinusoidal response data for typical turbine-propeller engine.

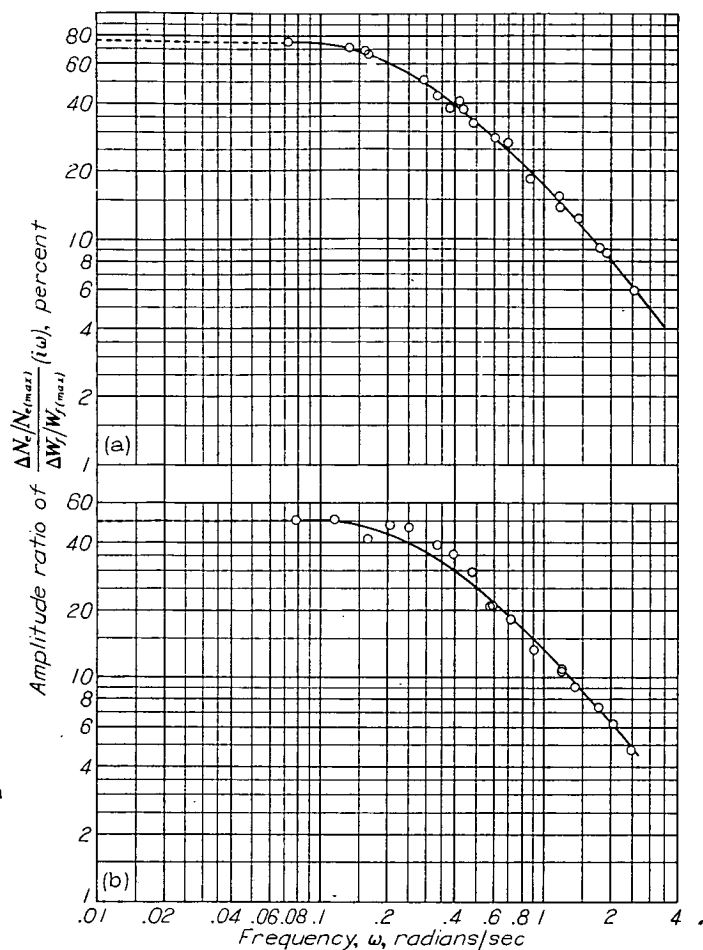
Equilibrium runs.—Equilibrium data required to evaluate the constants of equation (6) were obtained by the following procedure: For a set value of fuel flow, propeller-blade angle was adjusted to obtain nine operating points between 85 and 100 percent of maximum engine speed. These speed points were used for fuel flows of 50.0, 57.6, 65.4, 69.1, 77.0, and 84.5 percent of maximum fuel flow except when torque or turbine-outlet-temperature limitations prohibited operation throughout the complete range. Engine torque, engine speed, fuel flow, and blade-angle measurements were taken at each operating point.

RESULTS AND DISCUSSION

EXPERIMENTALLY DETERMINED FREQUENCY-RESPONSE CHARACTERISTICS

Results of sinusoidal inputs.—Results of the sinusoidal blade-angle and fuel-flow inputs as obtained from the oscillographic traces are shown in figures 4 and 5. Amplitude ratio of output to input is presented on log-log coordinates as a function of frequency. Phase-angle lag of the output relative to input is presented on semilog coordinates as a function of frequency.

The form of these frequency-response curves indicates the nature of the dynamic characteristics of this power plant.



(a) Amplitude-ratio relation. Mean fuel flow, 65 percent of maximum rated; mean engine speed, 97.5 percent of maximum; blade angle, 18°; fuel-flow amplitude, 3.9 percent of maximum.
 (b) Amplitude-ratio relation. Mean fuel flow, 81 percent of maximum rated; mean engine speed, 92.5 percent of maximum; blade angle, 28°; fuel-flow amplitude, 3.9 percent of maximum.

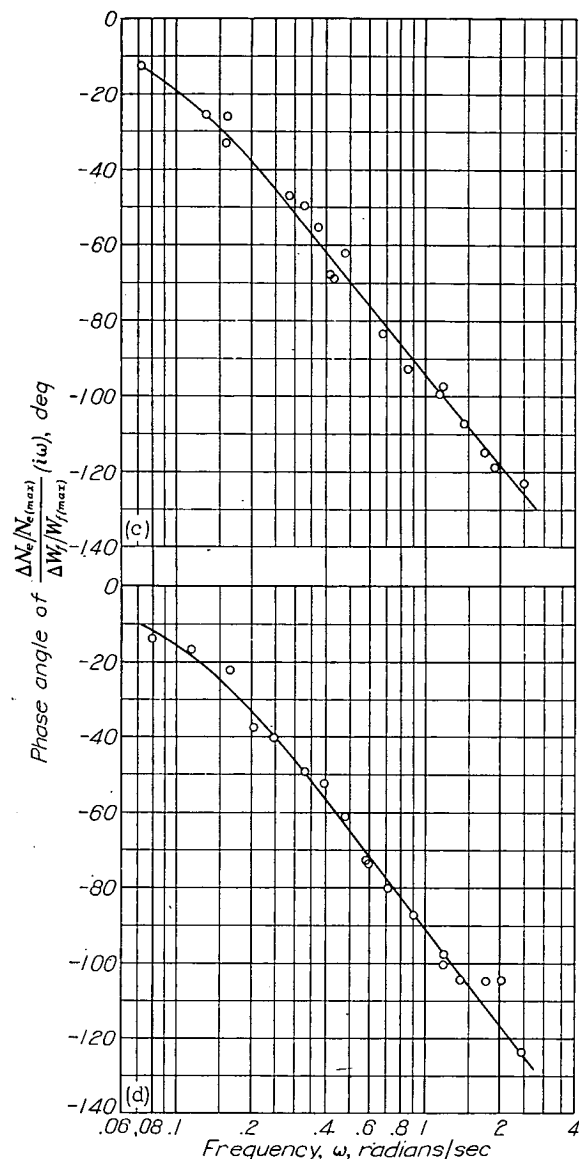
FIGURE 5.—Fuel-flow—engine-speed relations at constant blade angle obtained from experimental sinusoidal response data for typical turbine-propeller engine.

The regularity of data points indicates that the system follows these predicted characteristics better than might be anticipated. Furthermore, in support of the assumptions made for the differential-equation analysis, the system appears to be primarily a first-order lag system. Despite the complexity of the physical system and the processes performed in the turbine cycle, a fairly simple mathematical form approximates the actual behavior of the engine.

Algebraic form of data.—As shown in figures 4 and 5, although the system is primarily first order, higher order effects are present. Increased attenuation of the output amplitude at phase shifts approaching 180° indicates that a second-order approximation may more exactly describe the data. An equation of the following form has therefore been fitted to the experimental data:

$$\frac{\text{output}}{\text{input}} = \frac{K}{(1 + i\tau_1\omega)(1 + i\tau_2\omega)}$$

The equation was matched to the experimental data at the steady-state point and at phase shifts of 45° and 90°, thereby determining the three variables of the equation, K , τ_1 , and τ_2 . Results of these computations are presented in column 3 of table II.

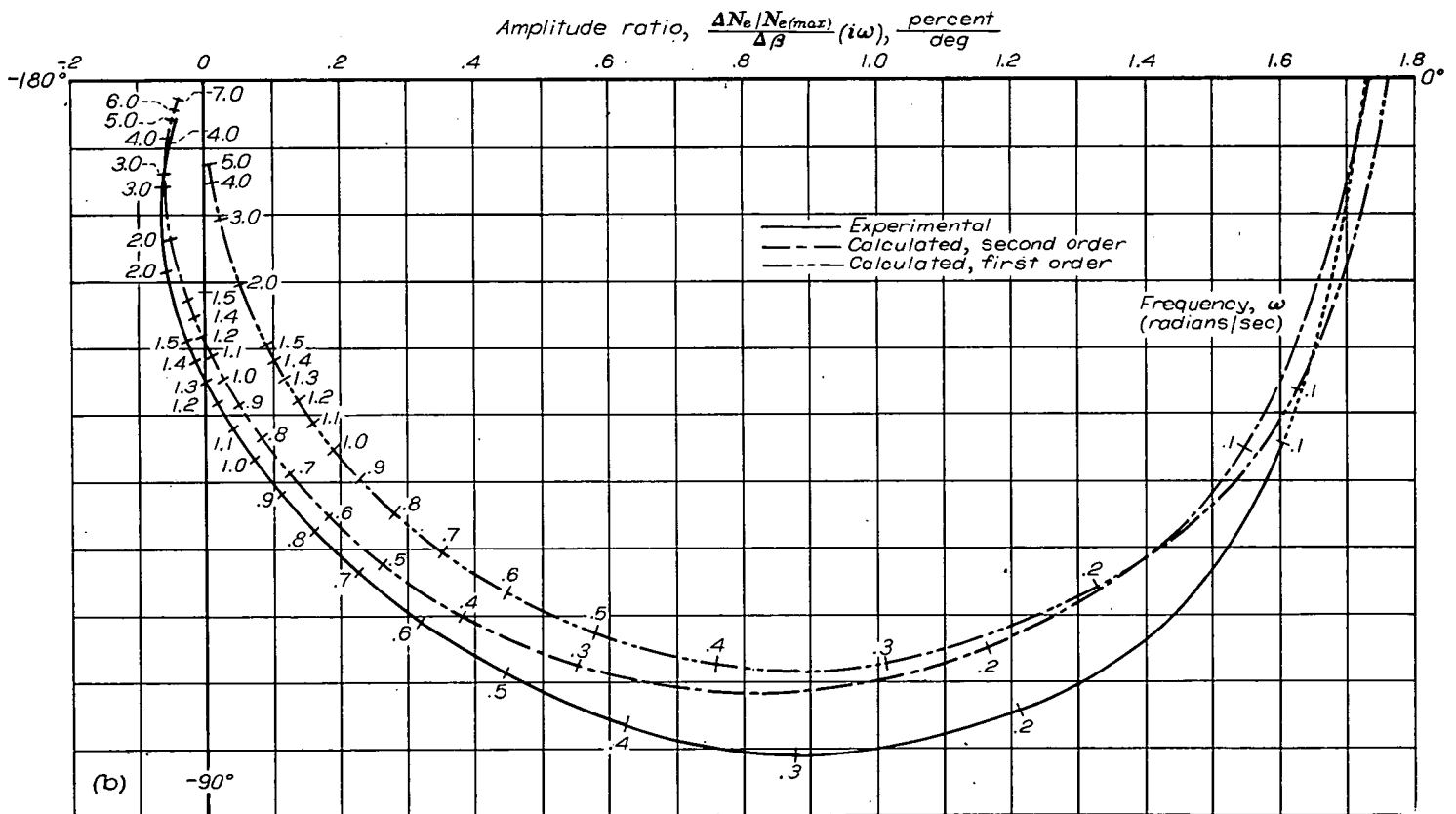
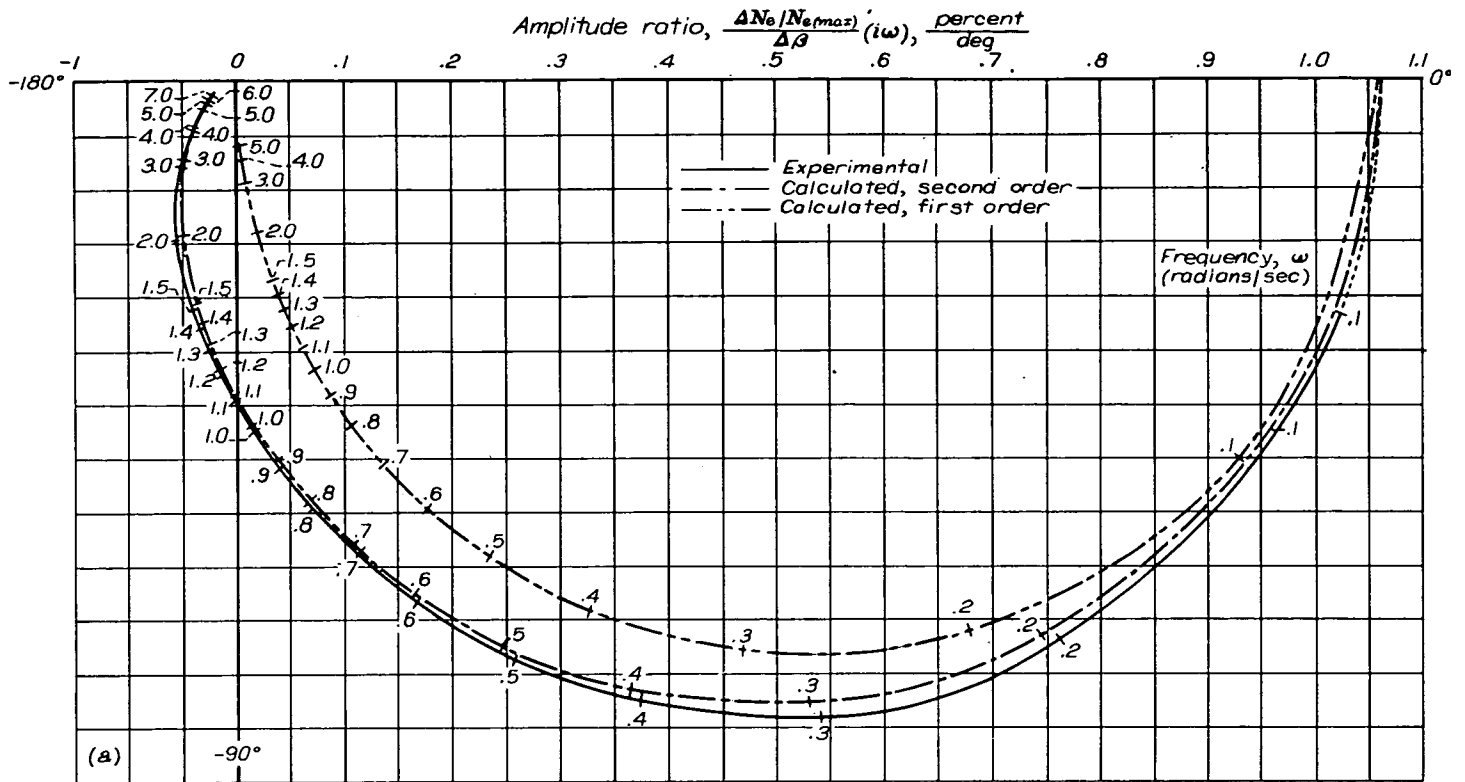


(c) Phase-angle relation. Mean fuel flow, 65 percent of maximum rated; mean engine speed, 97.5 percent of maximum; blade angle, 18°; fuel-flow amplitude, 3.9 percent of maximum.
 (d) Phase-angle relation. Mean fuel flow, 81 percent of maximum rated; mean engine speed, 92.5 percent of maximum; blade angle, 28°; fuel-flow amplitude, 3.9 percent of maximum.

FIGURE 5.—Concluded. Fuel-flow—engine-speed relations at constant blade angle obtained from experimental sinusoidal response data for typical turbine-propeller engine.

A maximum error in the value of the second time constant τ_2 of 0.1 second could be attributed to instrument lags; the numerical values for this time constant τ_2 are therefore presented only to indicate the order of magnitude of this second-order effect. Because the fuel-measurement device was slower than the blade-angle measurement circuit, instrumentation error would reduce the value of τ_2 in the functions of engine speed to fuel flow input. The opposite trend of experimental results indicates that the second-order effect is an engine phenomenon. The fact that a ratio in the order of 10:1 exists between the two time constants τ_1 and τ_2 makes the second-order effect negligible for many problems of control analysis.

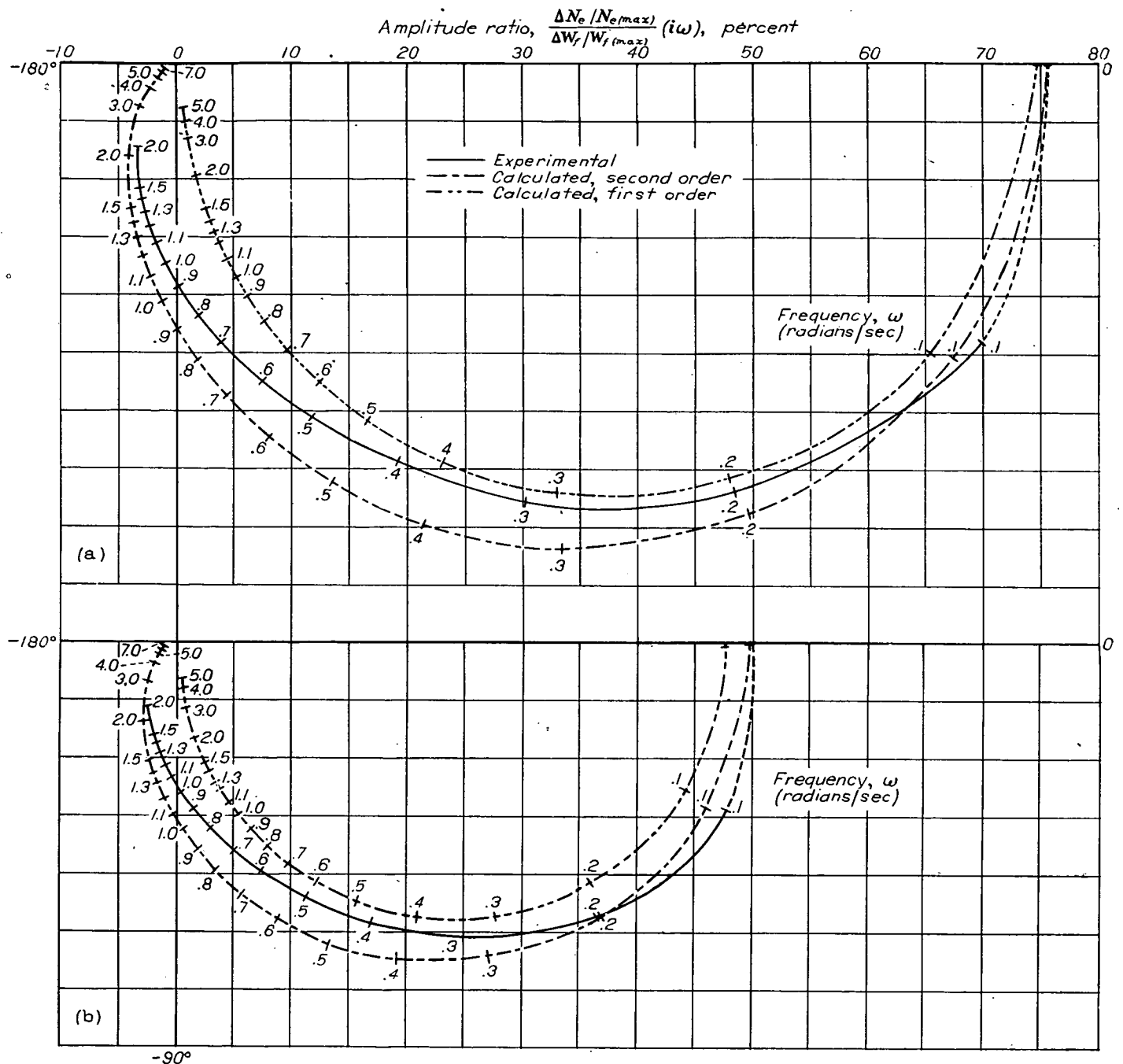
In order to compare the form of the algebraic approximations with the actual data, cross plots of figures 4 and 5 are presented on figures 6 and 7, respectively, together with curves of the algebraic expressions. The calculated first-order curves of figures 6 and 7 will be subsequently discussed.



(a) Fuel flow, 65 percent of maximum rated; mean engine speed, 97.5 percent of maximum; mean blade angle, 18°.

(b) Fuel flow, 81 percent of maximum rated; mean engine speed, 92.5 percent of maximum; mean blade angle, 28°.

FIGURE 6.—Frequency response of engine speed to variations in propeller-blade angle obtained from experimental data and theoretical calculations for typical turbine-propeller engine.



(a) Mean fuel flow, 65 percent of maximum rated; mean engine speed, 97.5 percent of maximum; blade angle, 18°.
 (b) Mean fuel flow, 81 percent of maximum rated; mean engine speed, 92.5 percent of maximum; blade angle, 28°.

FIGURE 7.—Frequency response of engine speed to variations in fuel flow obtained from experimental data and theoretical calculations for typical turbine-propeller engine.

Comparison of frequency-response functions.—Variation of the first-order time constant with operating conditions may be of more importance than the second-order effect. A comparison of first-order time constants in table II shows that these constants not only vary with the engine-power points but also vary with the forcing function at the same power point. For an exact description of dynamic behavior of engine speed, extensive data may be required to determine the trend of transient characteristics.

Linearity of dynamic behavior.—For purposes of analysis, an assumption was made that engine speed was a linear function of fuel flow and blade angle for small transients. In

order to evaluate this assumption from the results of experimental data, the form of the outputs recorded on oscillograph film was compared with an analytical sine wave. For inputs resulting in engine-speed changes of as much as 6 percent of maximum speed the form of the speed sine wave varied negligibly from an exact sine wave. Greater input amplitudes resulting in larger speed oscillations caused appreciable distortion, the effect being greater at low-power points. An assumption of linearity is therefore justified only for restricted deviations on the order of 3 or 4 percent from an equilibrium point.

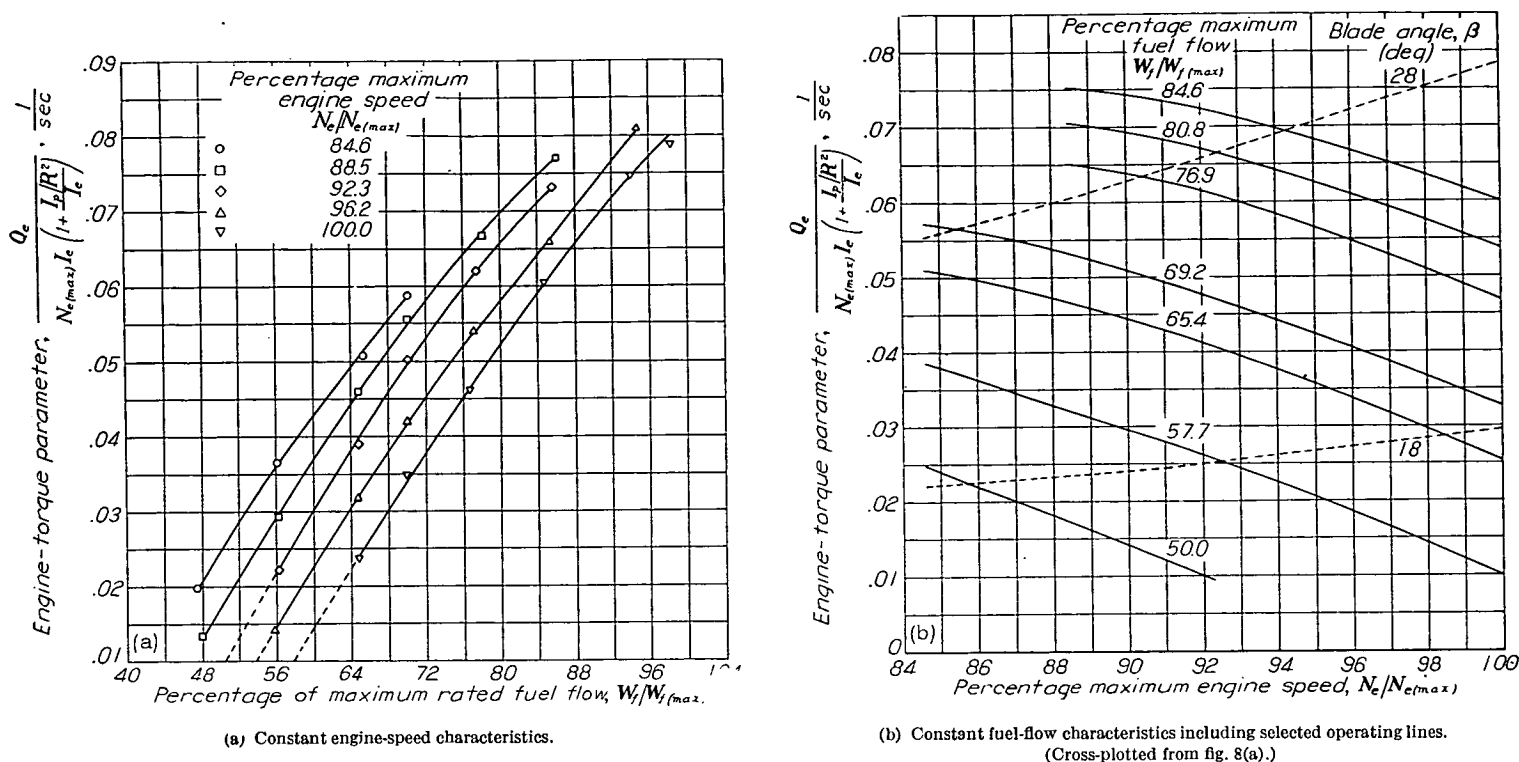


FIGURE 8.—Relation between engine-torque parameter, percentage of maximum engine speed, and percentage of maximum rated fuel flow as obtained from experimental data for typical turbine-propeller engine.

ANALYTICALLY DETERMINED FREQUENCY-RESPONSE CHARACTERISTICS

Results of equilibrium runs.—The relation between steady-state values of engine torque and fuel flow is plotted in figure 8 (a) for several values of constant speed. Cross plots for lines of constant fuel flow are shown in figure 8 (b). Figure 9 (a) is a plot of propeller torque against blade angle for various values of constant speed and cross plots for lines of constant blade angle are shown in figure 9 (b). In order to show the equilibrium running conditions about which sinusoidal data were taken, two lines of constant blade angle, 18° and 28°, are superimposed on figure 8 (b).

Losses to the accessories and the gear reduction unit were neglected in calculations of propeller torque.

Range of linearity.—In the derivation of equations (1) and (2), an assumption was made that the relations presented in figures 8 and 9 may be considered as straight lines for small deviations from a given steady-state point. The actual data indicate that this assumption is justified for the engine characteristics over rather large regions, but the propeller characteristics should be considered linear only in very small regions. Furthermore, these engine data as obtained under steady-state conditions of operation may not be accurate under transient conditions if the turbine is required to operate at a point far from the design condition (reference 2).

Analytical sinusoidal response.—Frequency-response characteristics of the engine were calculated for operating points at which experimental sinusoidal data were taken. Values of the constants a_1 , b_1 , c_1 , and d_1 for an operating condition were obtained from the slopes of the curves of figures 8 (a), 8 (b), 9 (a), and 9 (b), respectively. With values so determined, equations (10), (11), (13), and (14) were evaluated

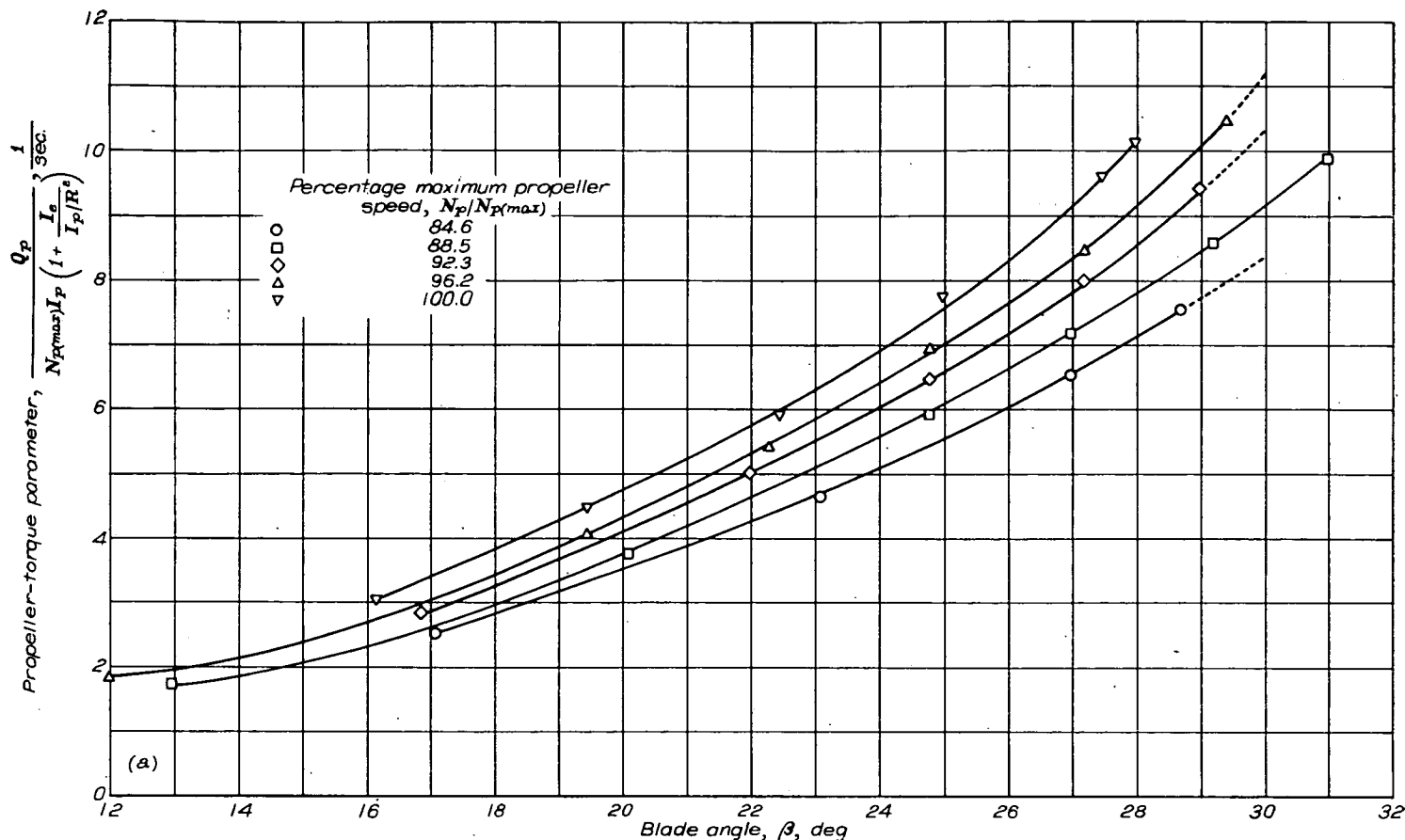
for a range of frequencies coinciding with that used in the experimental runs. Results of these calculations are included in figures 6 and 7.

Algebraic form of analytical results.—The algebraic form of the first-order curves presented in figures 6 and 7 is shown in equations (9) and (12). These equations were evaluated for the selected operating points and the results are presented in column 4 of table II.

For the assumptions made in the analysis, the engine time constant is independent of the forcing function, as shown in equation (7). Variation of time constants for the two power points results from the nonlinearity of torque-speed characteristics of the engine and the propeller. The magnitude of this variation is sufficient to limit the utility of an analysis in which linear approximations are assumed. For studies of behavior within restricted regions, the analysis is satisfactory.

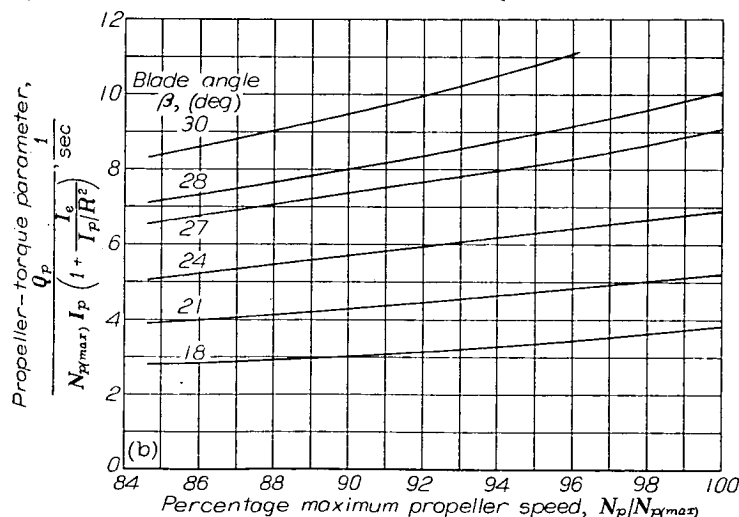
COMPARISON OF DIFFERENTIAL-EQUATION SOLUTION WITH EXPERIMENTAL RESULTS

Comparison of form.—Calculated frequency-response characteristics are similar to the experimental results at low-frequency values, as shown by figures 6 and 7. Because of higher-order effects exhibited by the experimental data, deviation between the two sets of curves increases at higher frequencies. This effect is particularly evident in the phase shift occurring at high frequencies. Presence of phase shifts greater than 90° in the engine system may limit the over-all loop gain for stability in a combined engine and control system. For the usual case in which the control contributes at least 90° additional phase shift to the engine polar diagram, the gain at the 180° phase shift point has a definite limit according to the Nyquist criterion (reference 1).



(a) Constant propeller-speed characteristics.

FIGURE 9.—Relations between propeller-torque parameter, percentage of maximum propeller speed, and blade angle for 12-foot-1-inch-diameter, four-bladed propeller.



(b) Constant blade-angle characteristics. (Cross-plotted from fig. 9(a).)

FIGURE 9.—Concluded. Relations between propeller-torque parameter, percentage of maximum propeller speed, and blade angle for 12-foot-1-inch-diameter, four-bladed propeller.

Comparison of numerical values.—In general, the value of the experimental time constant is lower than the value of the analytical time constant for a given power operating point. The significance of this variation depends on the particular application of the data; for some problems of control analysis

a range of variation of 2 to 1 in the engine time constant causes little change in the combined engine-control system. Discrepancies between steady-state gain values, analytical and experimental, can be accounted for by the fact that the analytical values are calculated from slopes of equilibrium-torque curves.

CONCLUSIONS

From a sea-level static investigation of the frequency-response characteristics of a turbine-propeller engine, the following conclusions may be drawn:

1. Engine speed is a linear function of fuel flow and propeller-blade angle during transient operation, if deviations from the equilibrium operating point are of the order of 3 or 4 percent.

2. The frequency-response functions of engine speed to fuel flow and engine speed to blade angle are primarily first order, but higher-order effects exist.

3. A differential equation that closely describes engine speed as a function of fuel flow and blade angle can be derived from steady-state data.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, March 24, 1950

APPENDIX—SYMBOLS

The following symbols are used in this report:

- a partial derivative of engine torque with respect to fuel flow, $\frac{\partial Q_e}{\partial W_f}$, lb-ft/lb/hr
- a_1 partial derivative of engine-torque parameter with respect to percentage maximum fuel flow,
- $$\frac{\partial \left[\frac{Q_e}{N_{e(max)} I_e \left(1 + \frac{I_p/R^2}{I_e} \right)} \right]}{\partial \left[\frac{W_f}{W_{f(max)}} \right]}, \frac{1}{\text{sec}}$$
- b absolute value of partial derivative of engine torque with respect to engine speed, $\frac{\partial Q_e}{\partial N_e}$, lb-ft/rpm
- b_1 partial derivative of engine-torque parameter with respect to percentage maximum engine speed,
- $$\frac{\partial \left[\frac{Q_e}{N_{e(max)} I_e \left(1 + \frac{I_p/R^2}{I_e} \right)} \right]}{\partial \left[\frac{N_e}{N_{e(max)}} \right]}, \frac{1}{\text{sec}}$$
- c partial derivative of propeller torque with respect to blade angle, $\frac{\partial Q_p}{\partial \beta}$, lb-ft/deg
- c_1 partial derivative of propeller-torque parameter with respect to blade angle,
- $$\frac{\partial \left[\frac{Q_p}{N_{p(max)} I_p \left(1 + \frac{I_e}{I_p/R^2} \right)} \right]}{\beta}, \frac{1}{(\text{sec})(\text{deg})}$$
- d partial derivative of propeller torque with respect to propeller speed, $\frac{\partial Q_p}{\partial N_p}$, lb-ft/rpm
- d_1 partial derivative of propeller-torque parameter with respect to percentage maximum propeller speed,
- $$\frac{\partial \left[\frac{Q_p}{N_{p(max)} I_p \left(1 + \frac{I_e}{I_p/R^2} \right)} \right]}{\partial \left[\frac{N_p}{N_{p(max)}} \right]}, \frac{1}{\text{sec}}$$
- f_1 function of W_f, N_e
- f_2 function of β, N_p

- i imaginary number, $\sqrt{-1}$
- I_e polar moment of inertia of rotating parts of engine, (lb)(ft)(sec)(rad)(min)/revolution
- I_p polar moment of inertia of propeller, (lb)(ft)(sec)(rad)(min)/revolution
- K arbitrary constant
- K_1 constant defining equilibrium condition,
- $$K_1 = \frac{a_1}{b_1 + d_1}, \frac{\text{rpm}}{\text{lb/hr}}$$
- K_2 constant defining equilibrium condition,
- $$K_2 = \frac{c_1}{b_1 + d_1}, \frac{\text{rpm}}{\text{deg}}$$
- N_e engine speed, rpm
- N_p propeller speed, rpm
- Q_e engine torque, lb-ft
- Q_p propeller torque, lb-ft
- R gear ratio of engine speed to propeller speed
- W_f fuel flow, lb/hr
- β propeller-blade angle, deg
- β_a beta arm, used to denote lever input to mechanism for changing propeller-blade angle on engine used to run data
- Δ deviation from initial operating point
- τ engine time constant, sec
- ω frequency, radians/sec

Subscript:

max maximum

REFERENCES

1. Brown, Gordon S., and Campbell, Donald P.: Principles of Servomechanisms. John Wiley and Sons, Inc., 1948.
2. Otto, Edward W., and Taylor, Burt L., III: Dynamics of a Turbojet Engine Considered as a Quasi-Static System. NACA Rep. 1011, 1951. (Formerly NACA TN 2091.)

TABLE I—STEADY-STATE AND TRANSIENT CHARACTERISTICS OF INSTRUMENTS

Measured variable	Steady-state accuracy	Transient characteristics
Blade angle, β	$\frac{1}{2}^\circ$	Limited by oscillograph element (40 cps)
Beta arm, β_a	1 percent of full travel.....	Limited by oscillograph element (40 cps)
Fuel flow, W_f	± 1 percent for full flow.....	10-15 cps
Torque, Q	± 1 percent for full scale.....	
Engine speed, N_e	$\frac{1}{3}$ of 1 percent for full scale.....	Limited by filter circuit (1.59 cps)

TABLE II—FREQUENCY-RESPONSE FUNCTIONS

1	2	3	4
Frequency-response function	Operating point	Algebraic form of frequency-response function from experimental data	Algebraic form of frequency-response function from calculated equation
$\frac{\Delta N_e/N_{e(max)}}{\Delta \beta} (i\omega)$ (percent/deg)	General.....	$\frac{K}{(1+i\tau_1\omega)(1+i\tau_2\omega)}$	$\frac{K_2}{(1+i\tau\omega)}$
	$W_f=65$ percent of maximum rated.	$\frac{-1.06}{(1+i 3.05 \omega)(1+i 0.283 \omega)}$	$\frac{-1.06}{(1+i 3.75 \omega)}$
	$W_f=81$ percent of maximum rated.	$\frac{-1.73}{(1+i 3.35 \omega)(1+i 0.185 \omega)}$	$\frac{-1.76}{(1+i 2.86 \omega)}$
$\frac{\Delta N_e/N_{e(max)}}{\Delta W_f/W_{f(max)}} (i\omega)$ (percent)	General.....	$\frac{K}{(1+i\tau_1\omega)(1+i\tau_2\omega)}$	$\frac{K_1}{(1+i\tau\omega)}$
	$\beta=18^\circ$	$\frac{76}{(1+i 3.32 \omega)(1+i 0.375 \omega)}$	$\frac{75}{(1+i 3.75 \omega)}$
	$\beta=28^\circ$	$\frac{50}{(1+i 2.73 \omega)(1+i 0.332 \omega)}$	$\frac{48}{(1+i 2.86 \omega)}$

REPORT 1018

A THEORETICAL ANALYSIS OF THE EFFECT OF TIME LAG IN AN AUTOMATIC STABILIZATION SYSTEM ON THE LATERAL OSCILLATORY STABILITY OF AN AIRPLANE¹

By LEONARD STERNFIELD and ORDWAY B. GATES, Jr.

SUMMARY

A method is presented for determining the effect of time lag in an automatic stabilization system on the lateral oscillatory stability of an airplane. The method is based on an analytical-graphical procedure. The critical time lag of the airplane-autopilot system is readily determined from the frequency-response analysis.

The method is applied to a typical present-day airplane equipped with an automatic pilot sensitive to yawing acceleration and geared to the rudder so that rudder control is applied in proportion to the yawing acceleration. The results calculated for this airplane-autopilot system by this method are compared with the airplane motions calculated by a step-by-step procedure.

INTRODUCTION

Recent calculations and flight tests of several airplanes designed for operation in the transonic speed range have indicated unsatisfactory damping of the lateral oscillation. The results presented in reference 1 show that the oscillatory stability can be improved by the use of an automatic pilot. The calculations of reference 1, however, were made on the assumption of an idealized control system without lag. Reference 2 points out that lag of the type in which the amount of control applied at a given instant is assumed to be proportional to a deviation which existed at a fixed time previous to the given instant can be represented mathematically by use of the lag operator $e^{-\tau_s D_b}$, where τ_s is nondimensional time lag based on the span and D_b is the differential operator. Lag of this type is generally referred to as time lag. Reference 2 suggests that for purposes of calculation the lag operator may be approximated by three terms of the Taylor's series for $e^{-\tau_s D_b}$. This approximation was used in some recent calculations (reference 3) and the results were found to be erroneous and misleading. The purpose of this report is to present a satisfactory method for determining the effect of time lag on the lateral oscillatory stability based on the exact expression of the lag operator rather than any approximation. Some recent analyses on the same problem, unknown to the authors at the time this problem was being analyzed, are presented in references 4 and 5.

SYMBOLS AND COEFFICIENTS

ϕ	angle of roll, radians
ψ	angle of yaw, radians
β	angle of sideslip, radians (v/V)
r	yawing angular velocity, radians per second ($d\psi/dt$)
$\ddot{\psi}$	yawing angular acceleration, radians per second per second ($d^2\psi/dt^2$)
p	rolling angular velocity, radians per second ($d\phi/dt$)
v	sideslip velocity along lateral axis, feet per second
V	airspeed, feet per second
ρ	mass density of air, slugs per cubic foot
q	dynamic pressure, pounds per square foot ($\frac{1}{2} \rho V^2$)
b	wing span, feet
S	wing area, square feet
W	weight of airplane, pounds
m	mass of airplane, slugs (W/g)
g	acceleration due to gravity, feet per second per second
μ_b	relative-density factor ($m/\rho Sb$)
η	inclination of principal longitudinal axis of airplane with respect to flight path, positive when principal axis is above flight path at nose, degrees
γ	angle of flight path to horizontal axis, positive in a climb, degrees
k_{x_0}	radius of gyration in roll about principal longitudinal axis, feet
k_{z_0}	radius of gyration in yaw about principal vertical axis, feet
K_{x_0}	nondimensional radius of gyration in roll about principal longitudinal axis (k_{x_0}/b)
K_{z_0}	nondimensional radius of gyration in yaw about principal vertical axis (k_{z_0}/b)
K_x	nondimensional radius of gyration in roll about longitudinal stability axis ($\sqrt{K_{x_0}^2 \cos^2 \eta + K_{z_0}^2 \sin^2 \eta}$)

¹ Supersedes NACA TN 2005, "A Theoretical Analysis of the Effect of Time Lag in an Automatic Stabilization System on the Lateral Oscillatory Stability of an Airplane" by Leonard Sternfield and Ordway B. Gates, Jr., 1950.

K_z	nondimensional radius of gyration in yaw about vertical stability axis $(\sqrt{K_{z_0}^2 \cos^2 \eta + K_{x_0}^2 \sin^2 \eta})$
K_{xz}	nondimensional product-of-inertia parameter $((K_{z_0}^2 - K_{x_0}^2) \sin \eta \cos \eta)$
C_L	trim lift coefficient $(\frac{W \cos \gamma}{qS})$
C_l	rolling-moment coefficient $(\frac{\text{Rolling moment}}{qSb})$
C_N	yawing-moment coefficient $(\frac{\text{Yawing moment}}{qSb})$
C_Y	lateral-force coefficient $(\frac{\text{Lateral force}}{qS})$
$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$	
$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$	
$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$	
$C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2V}}$	
$C_{n_p} = \frac{\partial C_n}{\partial \frac{pb}{2V}}$	
$C_{l_p} = \frac{\partial C_l}{\partial \frac{pb}{2V}}$	
$C_{Y_p} = \frac{\partial C_Y}{\partial \frac{pb}{2V}}$	
$C_{Y_r} = \frac{\partial C_Y}{\partial \frac{rb}{2V}}$	
$C_{l_r} = \frac{\partial C_l}{\partial \frac{rb}{2V}}$	
$C_{n_{\delta_r}} = \frac{\partial C_n}{\partial \delta_r}$	
t	time, seconds
s_b	nondimensional time parameter based on span (Vt/b)
D_b	differential operator $(\frac{d}{ds_b})$
P	period of oscillation, seconds
$T_{1/2}$	time for amplitude of oscillation to damp to one-half its original value
δ_r	deflection of control, radians
a	real part of complex root of characteristic stability equation
ω	angular frequency, radians per second
$\omega_s = \frac{b}{V} \omega$	
$\lambda = a + i\omega_s$	
τ	time lag between signal for control and its actual motion, seconds
$\tau_s = \frac{V}{b} \tau$	

$(\tau)_c$	critical time lag
K_A	maximum amplitude of acceleration in yaw produced by control deflection of unit amplitude $(\left \frac{\ddot{\psi}}{\delta_r} \right _{\text{airplane}})$
$(K_A)_s = \frac{b^2}{V^2} K_A$	
k	amplitude of control-surface oscillation produced by autopilot in response to oscillation of airplane acceleration $(\left \frac{\delta_r}{\ddot{\psi}} \right _{\text{autopilot}})$
$k_s = \frac{V^2}{b^2} k$	
$K_c = \frac{1}{k}$	
$(K_c)_s = \frac{1}{k_s}$	
θ	phase angle, radians
θ_A	phase angle of lag of δ_r behind $\ddot{\psi}$ when oscillating control surface forces airplane to oscillate, radians
θ_C	phase angle obtained from frequency response of autopilot

ANALYSIS

The investigation of the effect of time lag on the lateral oscillation may be conveniently divided into two parts: (a) determination of the smallest time lag which would result in a neutrally stable oscillation, referred to as critical time lag, and (b) the effect of a given time lag on the lateral oscillatory stability. It is important to know some of the results of the analysis obtained in part (a) in order to facilitate the analysis presented in part (b). Part (a) is based on the frequency-response method of analysis (references 6 and 7). This method affords a relatively simple means of determining the critical time lag of an automatic stabilization system and thereby of establishing the range of time lags for which the airplane motion is stable. Part (b) treats the solution of a transcendental equation by means of an analytical-graphical procedure. The analysis is presented for an airplane equipped with an automatic pilot sensitive to yawing acceleration and geared to the rudder so that rudder control is applied in proportion to the yawing acceleration, as suggested in reference 3. A similar analysis is applicable, however, to any automatic stabilization system with time lag.

The equations of motion used are expressed in terms of the nondimensional time parameter based on the span of the airplane $s_b = \frac{Vt}{b}$, but the results of the calculations obtained in terms of s_b have been converted from the nondimensional time s_b to time t in seconds. Thus the discussion of the results and the figures included in the report are given in terms of t .

DETERMINATION OF CRITICAL TIME LAG

The critical time lag of a system is defined as the time lag that results in a neutrally stable or steady-state oscillation. The motion of the control δ_r and the airplane acceleration $\ddot{\psi}$ when a critical time lag exists are shown in figure 1. This

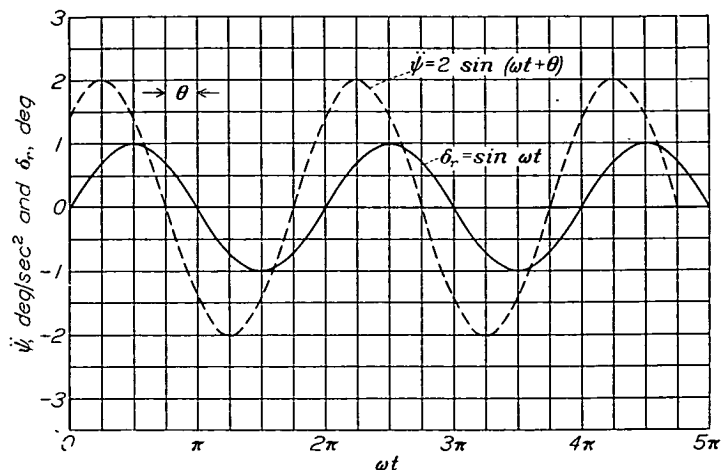


FIGURE 1.—The relationship between the airplane acceleration and the control motion for the case of critical time lag. $\theta = \omega \Delta t$; $\Delta t = \tau = \frac{\theta}{\omega}$.

figure indicates that the relationship between the time lag τ and the phase angle θ between the motion of the control and the airplane acceleration can be expressed as $\tau = \frac{\theta}{\omega}$. Since τ can be expressed as a function of θ , the frequency-response

method can be utilized to determine the critical time lag for a combined airplane-autopilot system. This method requires that the frequency-response curves be calculated for the airplane and autopilot separately and the results analyzed to ascertain the conditions for neutral oscillatory stability of the airplane-autopilot system.

Frequency-response curves for airplane.—The frequency-response curves for the airplane are obtained from the calculation of the steady-state motion of the airplane in response to a sinusoidal forcing function of unit amplitude (see reference 8). Thus if

$$\delta_r = \sin \omega_s s_b$$

the acceleration of the airplane is

$$D_b^2 \psi = (K_A)_s \sin (\omega_s s_b + \theta_A)$$

The values of $(K_A)_s$, known as the amplitude ratio, and θ_A are obtained over the desired range of angular frequencies by substituting $i\omega_s$ for D_b in the expression for $D_b^2 \psi / \delta_r$ which is derived from the lateral equations of motion. The non-dimensional lateral equations of motion, referred to stability axes, for a given control deflection are

$$\left. \begin{aligned} \left(2\mu_b K_X^2 D_b^2 - \frac{1}{2} C_{l_p} D_b \right) \phi + \left(2\mu_b K_{XZ} D_b^2 - \frac{1}{2} C_{l_r} D_b \right) \psi - C_{l_\beta} \beta &= 0 \\ \left(2\mu_b K_{XZ} D_b^2 - \frac{1}{2} C_{n_p} D_b \right) \phi + \left(2\mu_b K_Z^2 D_b^2 - \frac{1}{2} C_{n_r} D_b \right) \psi - C_{n_\beta} \beta &= C_{n_{\delta_r}} \delta_r \\ \left(-\frac{1}{2} C_{Y_p} D_b - C_L \right) \phi + \left(2\mu_b D_b - \frac{1}{2} C_{Y_r} D_b - C_L \tan \gamma \right) \psi + (2\mu_b D_b - C_{Y_\beta}) \beta &= 0 \end{aligned} \right\} \quad (1)$$

The derivatives $C_{l_{\delta_r}} = \frac{\partial C_l}{\partial \delta_r}$ and $C_{Y_{\delta_r}} = \frac{\partial C_Y}{\partial \delta_r}$ are usually very small and therefore have been neglected in equations (1).

Hence,

$$\frac{D_b^2 \psi}{\delta_r} = \frac{\begin{vmatrix} 2\mu_b K_X^2 D_b^2 - \frac{1}{2} C_{l_p} D_b & 0 & -C_{l_\beta} \\ 2\mu_b K_{XZ} D_b^2 - \frac{1}{2} C_{n_p} D_b & C_{n_{\delta_r}} & -C_{n_\beta} \\ -\frac{1}{2} C_{Y_p} D_b - C_L & 0 & 2\mu_b D_b - C_{Y_\beta} \end{vmatrix}}{\begin{vmatrix} 2\mu_b K_X^2 D_b^2 - \frac{1}{2} C_{l_p} D_b & 2\mu_b K_{XZ} D_b^2 - \frac{1}{2} C_{l_r} D_b & -C_{l_\beta} \\ 2\mu_b K_{XZ} D_b^2 - \frac{1}{2} C_{n_p} D_b & 2\mu_b K_Z^2 D_b^2 - \frac{1}{2} C_{n_r} D_b & -C_{n_\beta} \\ -\frac{1}{2} C_{Y_p} D_b - C_L & \left(2\mu_b - \frac{1}{2} C_{Y_r} \right) D_b - C_L \tan \gamma & 2\mu_b D_b - C_{Y_\beta} \end{vmatrix}} \quad (2)$$

After the numerator and denominator are expanded by the method of determinants, the expression for $D_b^2 \psi / \delta_r$ results in the ratio of two polynomials in D_b . The substitution of $i\omega_s$ for D_b in equation (2) gives a complex number $A + iB$ which may be expressed as $(K_A)_s e^{i\theta_A}$. The amplitude ratio

$(K_A)_s$, which is equal to $K_A \frac{b^2}{V^2}$, can be determined from the relation $(K_A)_s = \sqrt{A^2 + B^2}$. The phase angle θ_A can be determined from the relation $\theta_A = \tan^{-1} \frac{B}{A}$.

Frequency response of autopilot.—The frequency response of the autopilot is obtained from the equation for the control motion with time lag taken into account

$$\delta_r = k_s D_b^2 \psi(s_b - \tau_s) \quad (3a)$$

where the term $D_b^2 \psi(s_b - \tau_s)$ signifies the fact that the amount of control applied at a given instant is proportional to the acceleration at a fixed time τ_s previous to the given instant. This time-lag effect can be expressed by the so-called lag operator $e^{-\tau_s D_b}$. Thus equation (3a) becomes

$$\delta_r = k_s D_b^2 \psi e^{-\tau_s D_b} \quad (3b)$$

Solving equation (3b) for $D_b^2 \psi / \delta_r$ gives

$$\frac{D_b^2 \psi}{\delta_r} = \frac{1}{k_s} e^{\tau_s D_b} = (K_C)_s e^{\tau_s D_b} \quad (4)$$

where

$$(K_C)_s = \frac{b^2}{V^2} K_C = \frac{b^2}{V^2} \left| \frac{\ddot{\psi}}{\delta_r} \right|_{\text{autopilot}}$$

Substituting $i\omega_s$ for D_b in equation (4) results in an amplitude ratio $(K_C)_s = \frac{1}{k_s}$, which is independent of frequency, and a phase angle $\theta_C = \tau_s \omega_s$.

Conditions for stability.—The necessary and sufficient conditions for any one of the oscillatory modes to be neutrally stable are that, at a particular frequency, the phase angles and amplitude ratios of the airplane and autopilot must be equal—that is, $\theta_A = \theta_C$ and $(K_A)_s = (K_C)_s$. Even though these conditions are satisfied, the resultant motion of the airplane which is composed of all the individual modes of motion may be unstable since, as is pointed out subsequently, an additional unstable oscillatory mode may be present. In general, the condition which must be satisfied in order that all the oscillatory modes be stable is that, at each angular frequency where $\theta_A = \theta_C$, the ratio $\frac{(K_A)_s}{(K_C)_s}$ must be less than 1 at that frequency. The mathematical proof of this statement is given in reference 9.

Illustrative example.—The foregoing method is applied to a typical present-day high-speed airplane having the characteristics presented in table I. The value of the control-gearing ratio k is arbitrarily assumed to be 0.0427. (This value of $k=0.0427$ corresponds to a rudder deflection of 1° for a yawing acceleration of $23.4^\circ/\text{sec}/\text{sec}$.) The amplitude-ratio and phase-angle curves for the airplane, plotted as a function of angular frequency ω , where $\omega = \omega_s \frac{V}{b}$, are presented as solid-line curves in figures 2 (a) and 2 (b), respectively. As the frequency increases to infinity, K_A approaches a value of 15.98 and θ_A approaches π . The dashed line in figure 2 (a), which is independent of frequency, is the amplitude ratio of the autopilot K_C . The phase-angle curves of the autopilot are straight lines with slopes equal to τ , where

$\tau = \frac{b}{V} \tau_s$, and are shown as dashed lines in figure 2 (b) for several values of τ . Since the phase angle remains between the range of 0 to 2π , $\theta_C = \tau\omega$ continues to repeat itself whenever $\tau\omega \geq 2\pi$. To take account of this fact, θ_C is plotted as a series of parallel lines for each value of τ . Figure 2 (a) indicates that $K_A = K_C$ at $\omega=3.8$ and $\omega=8.5$. The corresponding values of τ where $\theta_A = \theta_C$ at $\omega=3.8$ and $\omega=8.5$ are $\tau=1.63$ and $\tau=0.38$, respectively. One of the oscillatory modes of motion is thus neutrally stable when $\tau=1.63$ and $\tau=0.38$. However, as mentioned previously, the motion of the airplane is neutrally stable only if all other oscillatory modes present are stable. This condition is satisfied for $\tau=0.38$, since for each value of ω where $\theta_A = \theta_C$ the ratio $\frac{K_A}{K_C} < 1$. When $\tau=1.63$, one of the oscillatory modes is neutrally stable but the system is unstable, because at $\omega=6$, $\theta_A = \theta_C$ but $\frac{K_A}{K_C} > 1$. An analysis indicates that for values

of $K_C < 15.98$, which is the limiting value of K_A , the system will be unstable for any infinitesimal time lag. The reason for the instability is that, for any infinitesimal time lag, the value of θ_A is equal to θ_C at some high frequency where it can be shown that $\frac{K_A}{K_C} > 1$ since at the very high frequencies

$$\frac{K_A}{K_C} = \frac{(K_A)_{\omega \rightarrow \infty} + \Delta K_A}{(K_A)_{\omega \rightarrow \infty} - \Delta K_C}$$

where ΔK_A and ΔK_C are small incremental values.

TABLE I.—STABILITY DERIVATIVES AND MASS CHARACTERISTICS OF A TYPICAL PRESENT-DAY AIRPLANE

W/S , lb/ft ² -----	65
S , ft ² -----	130
b , ft-----	28
ρ , slugs/ft ³ -----	0.00089
V , ft/sec-----	797
γ , deg-----	0
C_L -----	0.23
μ_b -----	80.7
K_{x^2} -----	0.00967
K_{z^2} -----	0.0513
K_{xz} -----	-0.00145
η , deg-----	-2.0
C_{l_p} , per radian-----	-0.40
C_{l_r} , per radian-----	0.08
C_{n_p} , per radian-----	-0.0155
C_{n_r} , per radian-----	-0.40
C_{y_p} , per radian-----	0
C_{y_r} , per radian-----	0
$C_{y_{\dot{\beta}}}$, per radian-----	-1.0
$C_{n_{\dot{\beta}}}$, per radian-----	0.25
$C_{l_{\dot{\beta}}}$, per radian-----	-0.126
$C_{n_{\dot{\delta}_r}}$, per radian-----	-0.163

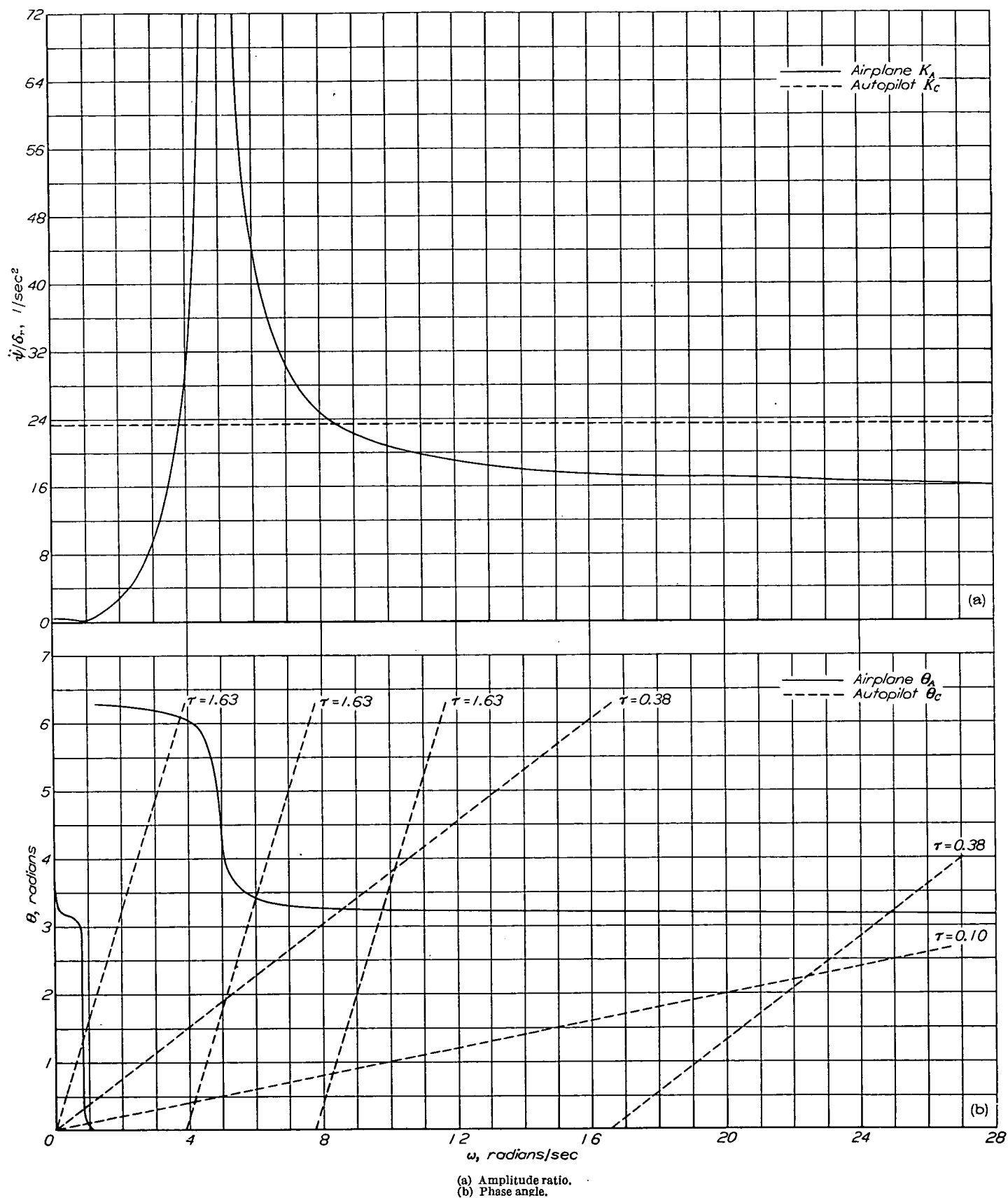


FIGURE 2.—Frequency-response curves of the typical present-day airplane and the assumed autopilot.

In order to verify the results predicted by the preceding analysis, motions of the airplane were calculated subsequent to an initial disturbance of 5° in sideslip. The calculations involved a step-by-step procedure based on the Kutta 3/8 method (reference 10). The results of these calculations for $\tau=0.38$ and $\tau=1.63$ are presented in figures 3 and 4, respectively. The solid-line curve in figure 3 was obtained by using a time increment of 0.095 and the motion is seen to be slightly unstable, whereas neutral stability is predicted by frequency-response analysis for $\tau=0.38$. An additional calculation was made by using a time increment of 0.0475, represented by the dashed-line curve in figure 3, and although the motion was still slightly unstable, the trend indicated by reducing the time increment was such as to make the oscillation more nearly neutrally stable. The airplane motion for $\tau=1.63$ is presented in figure 4 and, as was predicted, the motion is unstable. A neutrally stable oscillation was also predicted for this value of time lag but it is apparent

from figure 4 that the unstable mode influences the airplane motion more than the neutrally stable mode.

EFFECT OF TIME LAG ON LATERAL OSCILLATORY STABILITY

Derivation of equations.—The nondimensional equations of motion, referred to the stability axes, which include the effect of an autopilot applying rudder control in proportion to the yawing acceleration at time $s_b - \tau_s$, are obtained by combining equation (3b) with equations (1). When $\phi_0 e^{\lambda s_b}$ is substituted for ϕ , $\psi_0 e^{\lambda s_b}$ for ψ , and $\beta_0 e^{\lambda s_b}$ for β in the resultant equation written in determinant form, λ must be a root of the characteristic stability equation

$$A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E + k_s e^{-\tau_s \lambda} (A'\lambda^4 + B'\lambda^3 + C'\lambda^2 + D'\lambda) = 0 \quad (5)$$

where $A, B, C, D, E, A', B', C',$ and D' are functions of the mass and aerodynamic parameters of the airplane. The

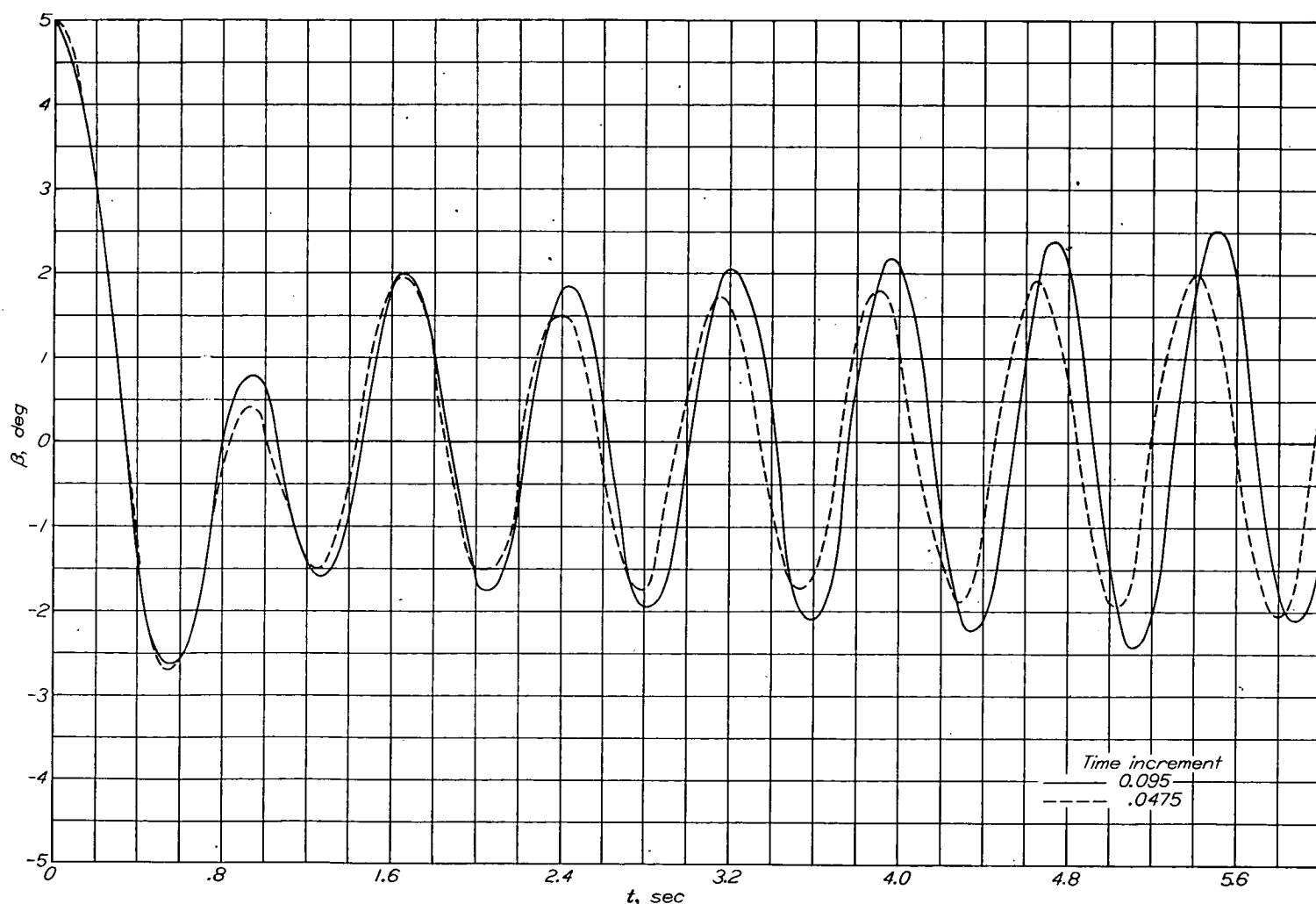


FIGURE 3.—Effect of $\tau=0.38$ on the motion of the airplane.

expressions for A , B , C , D , and E are given in reference 1 and

$$A' = -4\mu_b^2 K_x^2 C_{n_s},$$

$$B' = (2\mu_b K_x^2 C_{Y_\beta} + \mu_b C_{i_p}) C_{n_s},$$

$$C' = \left(-\frac{1}{2} C_{i_p} C_{Y_\beta} + \frac{1}{2} C_{Y_p} C_{i_\beta} \right) C_{n_s},$$

$$D' = C_L C_{i_\beta} C_{n_s},$$

The damping and period of the lateral oscillation in seconds are given by the expressions

$$\left. \begin{aligned} T_{\frac{1}{2}} &= \frac{-0.693}{a} \frac{b}{V} \\ P &= \frac{2\pi}{\omega_s} \frac{b}{V} \end{aligned} \right\} \quad (6)$$

where a and ω_s are the real and imaginary parts of a complex root of equation (5).

Determination of roots of transcendental equation.—The characteristic stability equation of this system (equation (5)) is seen to be a transcendental equation because a constant time lag τ_s in the automatic stabilization system is represented by the so-called lag operator $e^{-\tau_s D}$. It is apparent that the complex roots of such an equation cannot be determined by conventional methods. A method of obtaining the complex roots of this transcendental equation for a particular value of τ_s is therefore presented.

If $a + i\omega_s$ is substituted for λ in equation (5) and the real and imaginary quantities are separated, two equations in a and ω_s result:

$$e^{-\tau_s a} = F_1(a, \omega_s) \quad (7a)$$

$$e^{-\tau_s \omega_s} = F_2(a, \omega_s) \quad (7b)$$

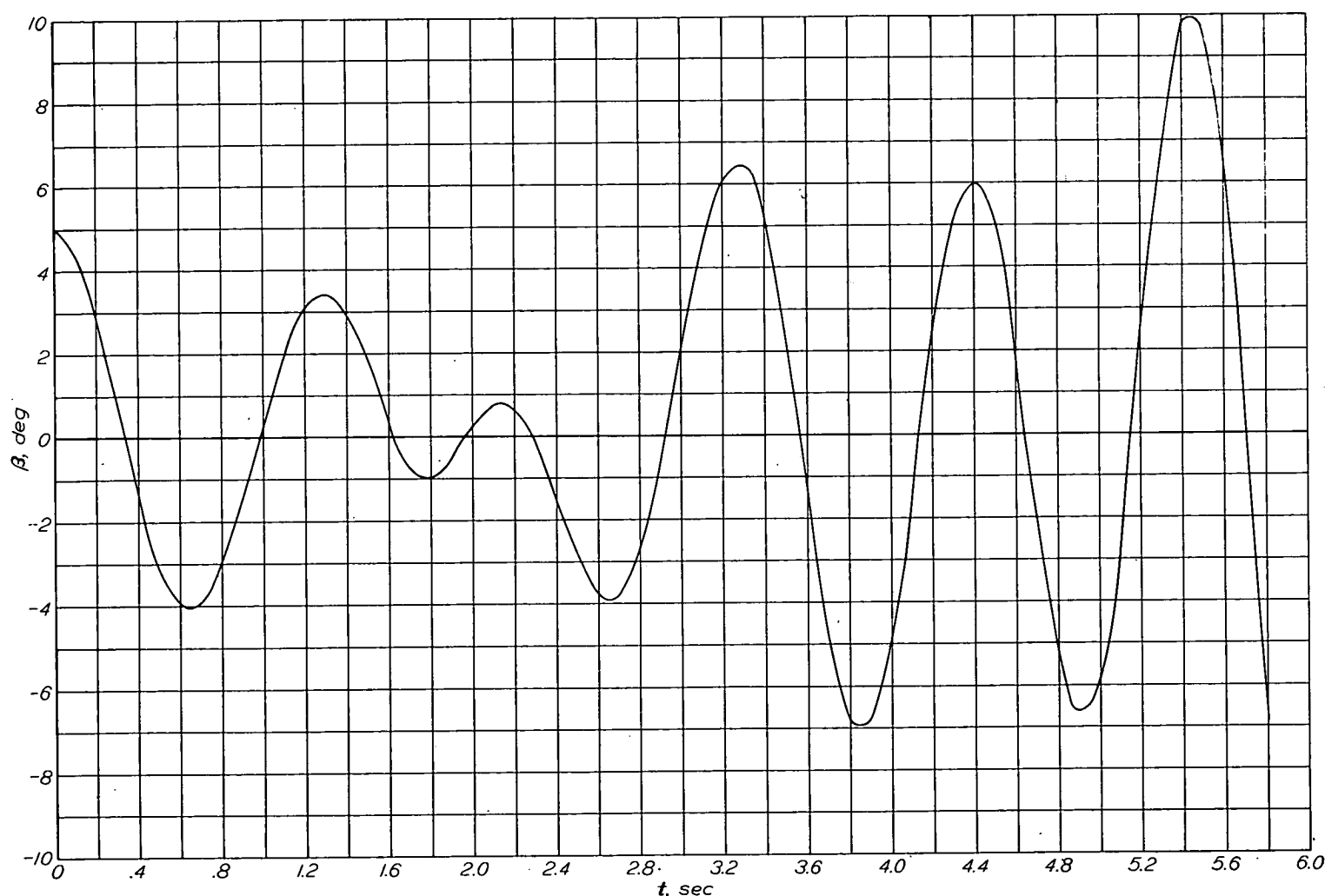


FIGURE 4.—Effect of $\tau = 1.63$ on the motion of the airplane.

A simultaneous solution of equations (7a) and (7b) gives the desired values of a and ω_s which satisfy equation (5). The method used to solve equations (7a) and (7b) simultaneously is basically a graphical procedure. For a series of values of ω_s , the right-hand sides of equations (7a) and (7b) are plotted against a as illustrated in figure 5. The solid lines and dashed lines correspond to the functions $F_1(a, \omega_s)$ and $F_2(a, \omega_s)$, respectively. The variation of $e^{-\tau_s a}$ with a is also plotted in figure 5. The exact values of a and ω_s for which $F_1(a, \omega_s) = F_2(a, \omega_s) = e^{-\tau_s a}$ are determined from a cross plot of the results of figure 5 as shown in figure 6. In this figure, with a as the abscissa and ω_s as the ordinate, are plotted the values of a and ω_s which correspond to the intersection of the $e^{-\tau_s a}$ curve with the functions $F_1(a, \omega_s)$ and $F_2(a, \omega_s)$. The solid curve in figure 6 thus satisfies the equation $e^{-\tau_s a} = F_1(a, \omega_s)$ and the dashed curve satisfies the equation $e^{-\tau_s a} = F_2(a, \omega_s)$. The values of a and ω_s at the intersection of these two curves therefore determine a root of the characteristic stability equation (equation (5)). The period and damping of the lateral oscillation are determined from equations (6) by using this root.

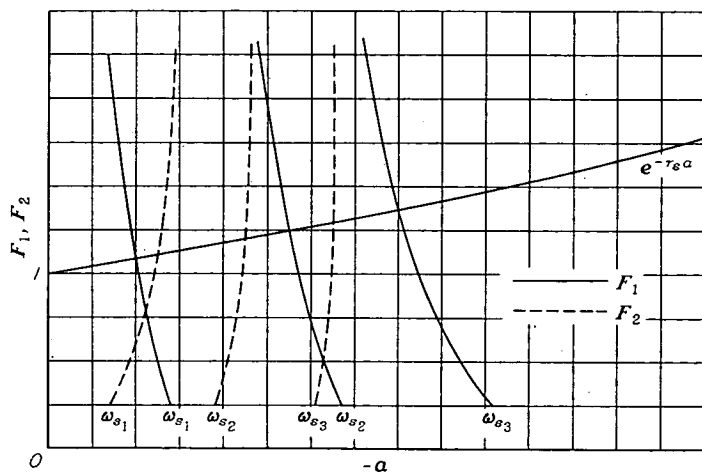


FIGURE 5.—The effect of a and ω_s on the functions $F_1(a, \omega_s)$, $F_2(a, \omega_s)$, and $e^{-\tau_s a}$.

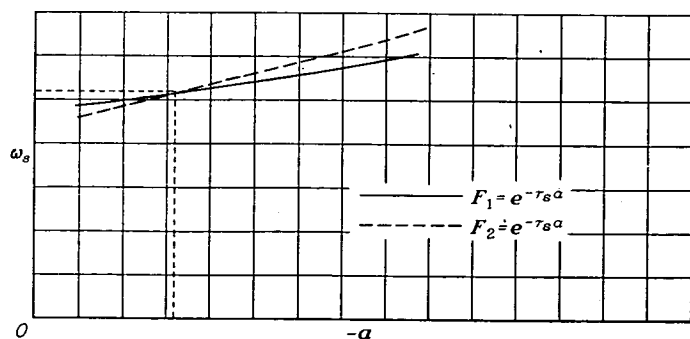


FIGURE 6.—A cross plot of the points of intersection of $F_1(a, \omega_s)$ and $F_2(a, \omega_s)$ with $e^{-\tau_s a}$ as determined from figure 5.

An alternative method for the simultaneous solution of equations (7a) and (7b) is to substitute $1 - \tau_s a + \frac{\tau_s^2 a^2}{2} - \frac{\tau_s^3 a^3}{6}$ for $e^{-\tau_s a}$ in each of these equations. The calculations involved become considerably less laborious when this approximation is made and results obtained by using this alternative method have been found to be in excellent agreement with results obtained from the method described previously. Figure 7 shows that for $|\tau_s a| = 1$ very close agreement is obtained between the exact value of $e^{-\tau_s a}$ and its approximation by three and four terms of the series for $e^{-\tau_s a}$. In actual practice the product $\tau_s a$ will almost invariably be much less than 1. When this substitution is made, both equations (7a) and (7b) become seventh degree in a for a given value of ω_s . Although these equations are of high degree, no serious problem is presented since the values of a desired must be real and, in general, small; that is, only one or two of the roots of these high-degree equations are of interest. In order to calculate the complex roots of equation (5) for a particular value of τ_s , the following procedure should be used. For a sequence of values of ω_s , compute a from equations (7a) and (7b). The results obtained from each equation may then be plotted in a figure similar to figure 6 and the complex root of the characteristic stability equation determined from the intersection of the two resulting curves. If the value of the approximate series for $e^{-\tau_s a}$ is in good agreement with the exact value of $e^{-\tau_s a}$ for the a determined from the intersection of the two curves, then the complex root obtained is valid. A point of intersection for a value of a which would not give satisfactory agreement between the approximate series and $e^{-\tau_s a}$ might exist, however. If such be the case, the accurate point of intersection is readily ascertained. The estimated values of a and ω_s represent the point of intersection of $F_1(a, \omega_s)$ or $F_2(a, \omega_s)$ with the series approximating $e^{-\tau_s a}$. The desired point is the point of intersection of $F_1(a, \omega_s)$ or $F_2(a, \omega_s)$ with $e^{-\tau_s a}$. The first step is to evaluate the term $e^{-\tau_s a}$ for several values of a in the vicinity of the estimated point of intersection. The expressions $F_1(a, \omega_s)$ and $F_2(a, \omega_s)$ are then evaluated for values of ω_s slightly less than and greater than the estimated value of ω_s for several values of a in the range of the estimated point of intersection. Thus, the corrected curves of equations (7a) and (7b) are obtained and at their point of intersection, the accurate values of a and ω_s are determined.

A method for constructing curves of constant period and damping as a function of τ_s and k_s is presented in the appendix.

Range of ω_s and a to be used in the determination of the complex roots of the characteristic stability equation.—In

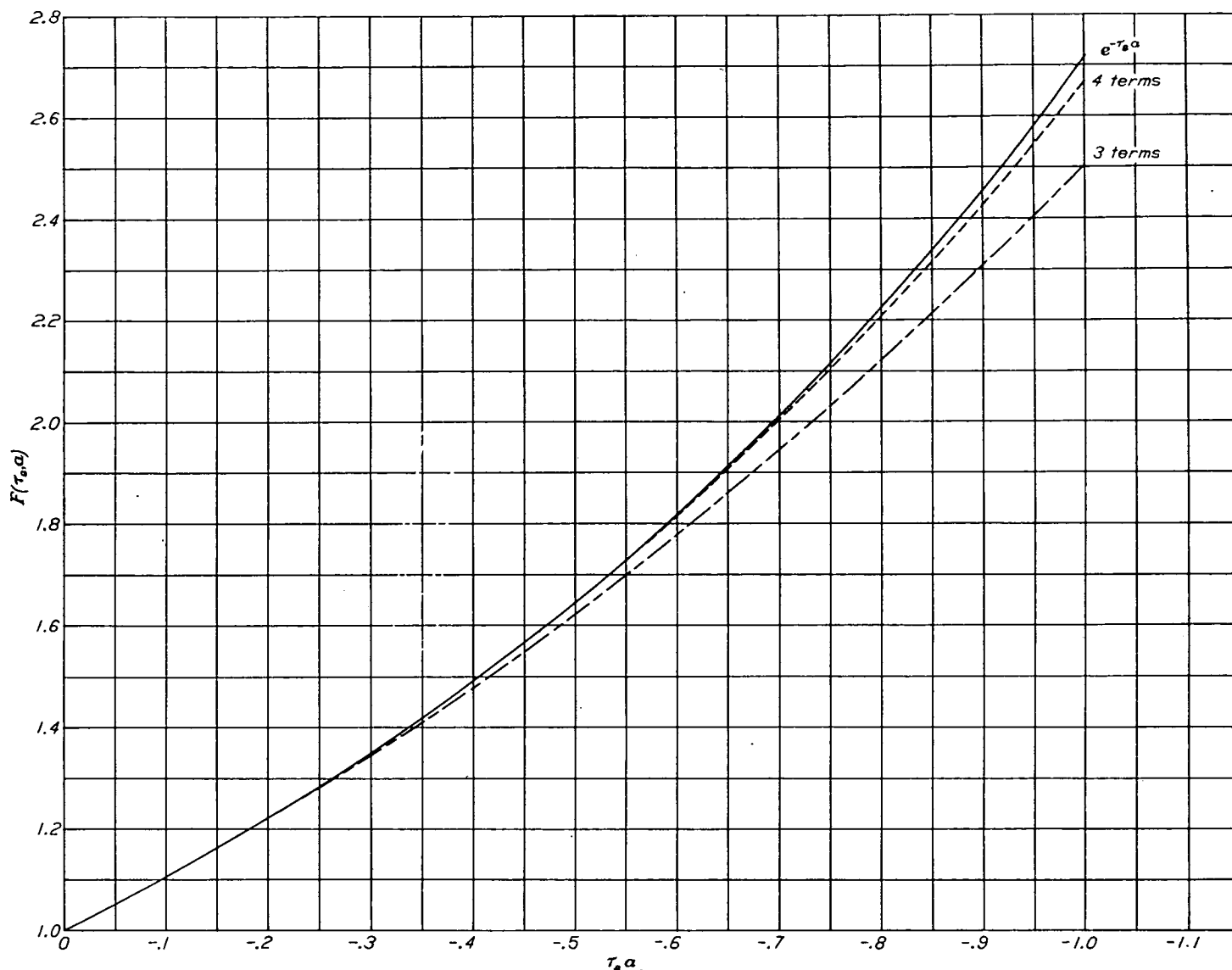


FIGURE 7.—A comparison between $e^{-\tau_s a}$ and its approximation by three and four terms of the series for $e^{-\tau_s a}$.

general, the analysis required to determine the damping of the airplane-autopilot system would be carried out for those values of τ_s that result in a stable system, since the purpose of equipping an airplane with an autopilot is to stabilize or increase the damping of the airplane motions. The range of τ_s for which the system is stable is determined from the frequency-response curves of the airplane and autopilot as indicated in the section entitled "Determination of Critical Time Lag." Thus, if the system is stable, only negative values of a need be investigated. Also, if two stable oscillatory modes of motion exist, the least stable one is of greatest interest—that is, the complex root with the smallest real part is the one of most importance.

The estimated range of values for ω_s that should be used

in the analysis to determine the effect of some particular value of time lag, located between $\tau_s=0$ and the critical time lag $(\tau_s)_c$, on the damping of the oscillation is obtained from the known values of the frequency of the oscillation for the cases where $\tau_s=0$ and $\tau_s=(\tau_s)_c$. The imaginary part of the complex root of equation (5), which becomes a quartic equation when $\tau_s=0$, gives the value of ω_s for the case of $\tau_s=0$. The value of ω_s for $\tau_s=(\tau_s)_c$ is determined from the analysis presented in the section entitled "Determination of Critical Time Lag." As mentioned in reference 11, the frequency of the oscillation decreases as time lag increases; thus for a value of $0 < \tau_s < (\tau_s)_c$, the estimated values of ω_s should include frequencies greater than the value of ω_s at $\tau_s=(\tau_s)_c$ and less than the value of ω_s at $\tau_s=0$.

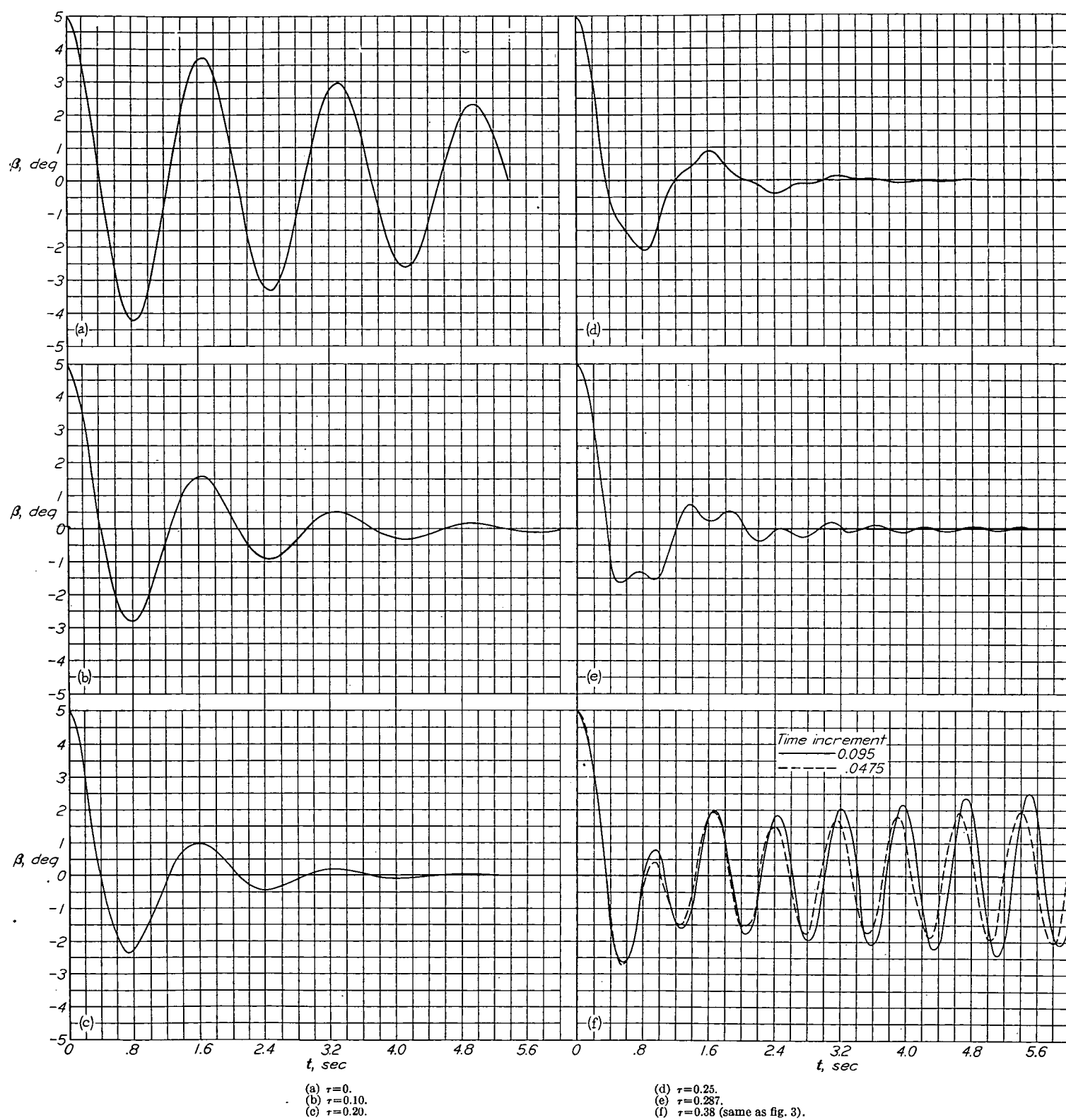


FIGURE 8.—Effect of time lag on the airplane motion in sideslip.

Airplane motions for various values of τ .—For purposes of comparison, the motion of the airplane in sideslip subsequent to an initial displacement in sideslip of 5° was calculated, with the use of a step-by-step procedure, for $\tau=0, 0.10, 0.20, 0.25, 0.287$, and 0.38 . The results are presented in figures 8 (a) to 8 (f). These figures indicate that as τ increases from 0 to 0.2 the period increases slightly, whereas the damping is markedly improved. The frequency of the oscillation is

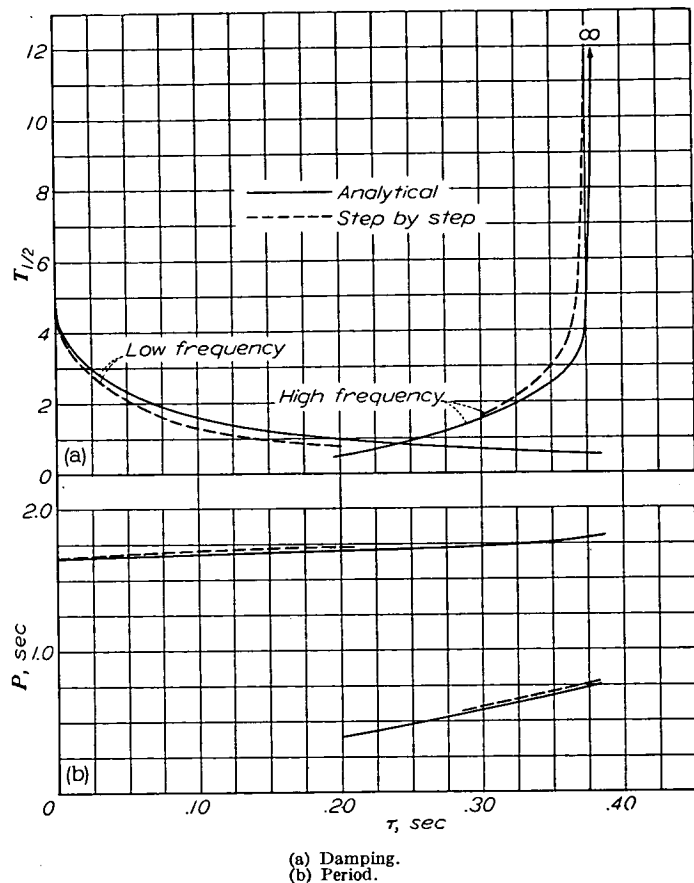


FIGURE 9.—Effect of time lag on the period and damping of the lateral oscillation.

about $\omega=3.7$. However, as τ continues to increase, the presence of a high-frequency oscillation is noted in the motion since this oscillation becomes less damped and the low-frequency oscillation becomes more heavily damped. (See figs. 8 (d) and 8 (e).) For $\tau=0.38$, the motion is neutrally stable at the high frequency of $\omega=8.5$ but the low-frequency oscillation does not appear in this motion since it is very well damped. In figures 9 (a) and 9 (b), the period and damping of the lateral oscillation for several values of τ , calculated by the method discussed in this report, are compared with the period and damping readily obtained from the motion calculations shown in figures 8 (a) to 8 (f). The very good agreement between the results presented in figures 9 (a) and 9 (b) would probably be improved if the increment selected for the step-by-step calculations were reduced. The trends indicated by these results, however, are applicable only to the particular airplane-autopilot system considered in this example and may not be generalized to any other arbitrary airplane-autopilot system.

CONCLUDING REMARKS

A method is presented for determining the effect of time lag in an automatic stabilization system on the lateral oscillatory stability of an airplane. The method is applied to a typical present-day airplane equipped with an automatic pilot sensitive to yawing acceleration and geared to the rudder so that rudder control is applied in proportion to the yawing acceleration. The results calculated for an airplane-autopilot system by the method described are in good agreement with the airplane motions calculated by a step-by-step procedure.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., October 28, 1949.

APPENDIX

A METHOD FOR CONSTRUCTING CURVES OF CONSTANT PERIOD AND DAMPING AS A FUNCTION OF τ , AND h ,

If an over-all picture is desired of the effect of time lag for a given value of k_s or the effect of varying k_s for a given time lag, construction of curves of constant period and damping with the use of the following method is recommended.

The characteristic stability equation of the airplane-autopilot system (equation (5)) may be rewritten in the form:

$$\frac{A\lambda^4 + B\lambda^3 + C\lambda^2 + D\lambda + E}{-(A'\lambda^4 + B'\lambda^3 + C'\lambda^2 + D'\lambda)} = k_s e^{-\tau_s \lambda} \quad (A1)$$

If $\lambda = a + i\omega_s$ is substituted in each side of equation (A1), the condition that λ be a root of the characteristic equation is that the complex number $A_1 + iB_1$ obtained from the left-hand side must be equal to the one obtained from the right-hand side $A_2 + iB_2$. The quantities $A_1 + iB_1$ and $A_2 + iB_2$ may be represented by the expressions $R_1 e^{i\theta_1}$ and $R_2 e^{i\theta_2}$, respectively. Therefore, this requirement is equivalent to saying that $R_1 = R_2$ and $\theta_1 = \theta_2$ if λ is to be a root of equation (A1). If $\lambda = a + i\omega_s$ is substituted in the right-hand side, the following expressions result:

$$k_s e^{-\tau_s a} (\cos \tau_s \omega_s - i \sin \tau_s \omega_s) = A_2 + iB_2 = R_2 e^{i\theta_2}$$

where

$$R_2 = k_s e^{-\tau_s a}$$

and

$$\theta_2 = \tan^{-1} \frac{-\sin \tau_s \omega_s}{\cos \tau_s \omega_s} = -\tau_s \omega_s = 2\pi - \tau_s \omega_s$$

Therefore, if $\lambda = a + i\omega_s$ is substituted in the left-hand side and $A_1 + iB_1 = R_1 e^{i\theta_1}$ is obtained, the value of τ required to make $\theta_1 = \theta_2$ can be determined. Since τ_s is therefore determined and a is fixed, the value of k_s necessary to make $R_1 = R_2$ can be calculated. Thus for these values of k_s and τ_s ,

$$\lambda = a + i\omega_s$$

is a root of the transcendental stability equation (equation (5)).

For a given value of a , the analysis may be made throughout the range of ω_s and the corresponding values of k_s and τ_s determined. A curve representing this value of a may then be plotted as a function of k_s and τ_s . This procedure is repeated for a sequence of values of a and the corresponding curves in the k_s, τ_s plane are plotted. Each point on a curve of constant a represents a particular value of ω_s . Curves of constant frequency may therefore be plotted by drawing a curve through the given value of ω_s on each one of the a curves. The values of a and ω_s are converted to $T_{1/2}$ and P by equations (6). The final result would consist of curves of constant damping $T_{1/2}$ and curves of constant period P in the k_s, τ_s plane. Thus the effect of time lag on the lateral oscillations for any value of the gearing ratio k_s , or the effect of varying k_s for any value of time lag, may be ascertained. For purposes of illustration several lines of constant $T_{1/2}$ and P in the k, τ plane were calculated for the typical present-day airplane described in table I and are shown in figure 10.

A curve can be plotted in the k_s, τ_s plane which divides the quadrant into a satisfactory and an unsatisfactory region

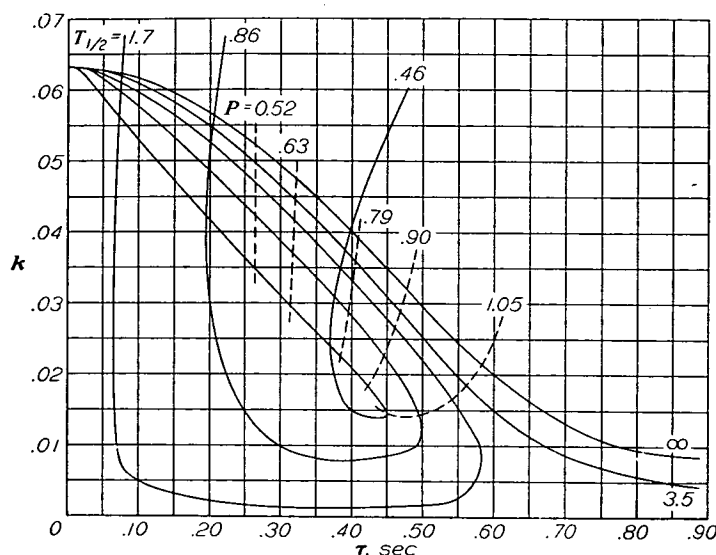


FIGURE 10.—Several curves of constant period and constant damping for the typical present-day airplane described in table I.

according to any prescribed relationship between the period and damping of the lateral oscillation. In order to calculate this curve, several values of a and ω_s that exactly satisfy the criterion should be selected and substituted in equation (A1). The combination of k_s and τ_s is then obtained for each set of values of a and ω_s and the desired curve is plotted in the k_s, τ_s plane.

REFERENCES

1. Sternfield, Leonard: Effect of Automatic Stabilization on the Lateral Oscillatory Stability of a Hypothetical Airplane at Supersonic Speeds. NACA TN 1818, 1949.
2. Imlay, Frederick H.: A Theoretical Study of Lateral Stability with an Automatic Pilot. NACA Rep. 693, 1940.
3. Beckhardt, Arnold R.: A Theoretical Investigation of the Effect on the Lateral Oscillations of an Airplane of an Automatic Control Sensitive to Yawing Accelerations. NACA TN 2006, 1950.
4. Ansoff, H. I.: Stability of Linear Oscillating Systems with Constant Time Lag. Jour. Appl. Mech., vol. 16, no. 2, June 1949, pp. 158-164.
5. Schiff, Leonard I., and Gimprich, Marvin: Automatic Steering of Ships by Proportional Control. Trans. Soc. Naval Arch. and Marine Eng., vol. 57, 1949.
6. Greenberg, Harry: Frequency-Response Method for Determination of Dynamic Stability Characteristics of Airplanes with Automatic Controls. NACA Rep. 882, 1947.
7. Jones, Robert T., and Sternfield, Leonard: A Method for Predicting the Stability in Roll of Automatically Controlled Aircraft Based on the Experimental Determination of the Characteristics of an Automatic Pilot. NACA TN 1901, 1949.
8. Jones, Robert T.: A Simplified Application of the Method of Operators to the Calculation of Disturbed Motions of an Airplane. NACA Rep. 560, 1936.
9. Brown, Gordon S., and Campbell, Donald P.: Principles of Servomechanisms. John Wiley & Sons, Inc., 1948, pp. 170-174.
10. Levy, H., and Baggott, E. A.: Numerical Studies in Differential Equations. Vol. I, Watts & Co. (London), 1934, p. 104.
11. Minorsky, Nicholas: Self-Excited Oscillations in Dynamical Systems Possessing Retarded Actions. Jour. Appl. Mech., vol. 9, no. 2, June 1942, pp. A-65-A-71.

REPORT 1019

CONTENTS

	Page		Page
SUMMARY.....	305	EXPERIMENTAL RESULTS IN PUBLISHED LITERATURE.....	320
INTRODUCTION.....	305	Capacitance Sparks.....	320
I—IGNITION BY HEATED SURFACES.....	305	Effect of mixture composition.....	320
TERMINOLOGY.....	305	Effect of electrode spacing.....	320
Ignition Temperatures.....	305	Effect of electrode material and configuration.....	321
Limits of Inflammability.....	307	Effect of spark potential.....	321
EXPERIMENTAL RESULTS IN PUBLISHED LITERATURE.....	308	Effect of mixture velocity and spark duration.....	322
Effect of Mixture Composition.....	308	Electrostatic sparks.....	322
Ignition temperatures.....	308	Inductance Sparks.....	322
Zones of ignition and nonignition.....	308	Effect of mixture composition.....	322
Effect of Varying Ignition Lag.....	310	Effect of electrode spacing.....	322
Ignition lag of quiescent mixtures.....	310	Effect of electrode material.....	323
Ignition lag of mixtures ignited by heated spheres		Effect of circuit inductance and potential.....	323
and rods.....	311	Effect of type of current in circuit.....	324
Effect of turbulence.....	312	Effect of Gas Pressures and Temperatures.....	324
Effect of Surface Condition and Composition.....	313	Varying pressure.....	324
Open plates in air.....	313	Varying temperature.....	326
Varying surface compositions.....	313	Effect of Diluents.....	328
Varying surface conditions.....	314	III—IGNITION BY FLAMES OR HOT GASES.....	329
Effect of Varying Ignition Surface Area.....	315	EFFECT OF VARYING MIXTURE COMPOSITION.....	329
Heated plates.....	315	LIMITING SIZE OF OPENINGS FOR FLAME PROPAGATION.....	329
Heated wires.....	315	EFFECT OF GAS PRESSURES AND TEMPERATURES.....	330
Heated particles.....	315	EFFECT OF DILUENTS.....	332
Effect of Varying Fuel Composition.....	316	IV—RELATION OF PUBLISHED DATA TO AIRCRAFT-	
Effect of Varying Pressure.....	317	FIRE PROBLEM.....	332
Effect of Diluents.....	317	IGNITION BY HEATED SURFACES.....	332
II—IGNITION BY ELECTRIC SPARKS AND ARCS.....	318	IGNITION BY ELECTRIC SPARKS AND ARCS.....	334
TERMINOLOGY.....	318	IGNITION BY FLAMES OR HOT GASES.....	335
Capacitance Sparks.....	318	CONCLUSIONS.....	335
Inductance Sparks.....	318	REFERENCES.....	336
Fusion Sparks.....	319	BIBLIOGRAPHY.....	338
Minimum Spark-Ignition Energy.....	319	TABLES.....	340
Quenching Distance.....	319		

REPORT 1019

RELATION BETWEEN INFLAMMABLES AND IGNITION SOURCES IN AIRCRAFT ENVIRONMENTS¹

By WILFRED E. SCULL

SUMMARY

A literature survey was conducted to determine the relation between aircraft ignition sources and inflammables. Available literature applicable to the problem of aircraft fire hazards is analyzed and discussed herein. Data pertaining to the effect of many variables on ignition temperatures, minimum ignition pressures, minimum spark-ignition energies of inflammables, quenching distances of electrode configurations, and size of openings through which flame will not propagate are presented and discussed. Ignition temperatures and limits of inflammability of gasoline in air in different test environments, and the minimum ignition pressures and minimum size of openings for flame propagation in gasoline-air mixtures are included; inerting of gasoline-air mixtures is discussed.

The results of the survey indicate the possibility of reducing aircraft-fire hazards by:

- (1) Preventing contact of fuels or fuel vapors with the hot exhaust surfaces
- (2) Releasing an extinguishing agent inside of and around the exhaust system if a crash is imminent
- (3) Reducing the temperature of the exhaust duct or gases below the surface-ignition temperatures of gasoline and lubricating oil in the event of a crash
- (4) Increasing the surface-ignition temperatures of gasoline and lubricating oil
- (5) Using fuels of reduced volatility
- (6) Eliminating the electrical generating system as an ignition hazard in the event of a crash
- (7) Inerting the engine nacelles and wing compartments.

INTRODUCTION

As a part of the aircraft-fire-research program, a literature survey was conducted at the NACA Lewis laboratory during 1949-50 to determine the relation between inflammables and ignition sources. An effort has been made to correlate the limited data available with the problem of fire hazards in aircraft environments. The results of the survey are presented in three parts corresponding to the main sources of ignition in aircraft environments; that is, heated surfaces, electric sparks and arcs, and flames or hot gases.

The following variables have been considered in the ignition of fuels or fuel vapors by the different ignition sources:

Heated surfaces	Electric sparks and arcs	Flames or hot gases
Ignition temperature Mixture composition Ignition lag Mixture velocity Mixture turbulence Condition and composition of the ignition surface Ignition surface area Fuel composition Mixture pressure Effect of diluents	Type of spark Minimum spark-ignition energy Mixture composition Electrode spacing Electrode material and configuration Spark potential Spark duration Mixture velocity Mixture pressure Mixture temperature Effect of diluents	Mixture composition Mixture exposure duration Limiting size of openings for flame propagation Size, shape, and material of openings for flame propagation Mixture pressure Mixture temperature Effect of diluents

Although quantitative agreement among the large quantities of data in the literature is not to be expected because of the many different experimental techniques used, it is believed that comparison of the effects of the different variables as compiled herein will provide valuable information toward reducing aircraft-fire hazards.

I—IGNITION BY HEATED SURFACES

Although the data pertaining to the ignition of inflammable mixtures by heated surfaces are quite extensive, little agreement exists in technique, terminology, or results of various investigators. In the following discussions, an effort has been made to allow for different experimental techniques and to use a system of nomenclature presently acceptable in the combustion field.

TERMINOLOGY

IGNITION TEMPERATURES

The ignition temperature of an inflammable mixture is the minimum temperature to which the mixture must be raised in order that the rate of heat loss from the mixture is more than balanced by the rate at which heat is evolved from the mixture by exothermic chemical reaction. At this temperature, the reaction becomes self-accelerating and the temperature increases until self-propagating inflammation of the mixture occurs. A definite lag during which the reaction is self-accelerating thus exists, the magnitude of which is determined by the rapidity of acceleration of the reaction velocity. Defined in this manner, the ignition temperature of an inflammable is a mixture temperature and does not necessarily correspond to and may be considerably lower than the inflammation temperature or mixture temperature at the instantaneous appearance of flame. The ignition temperature of an inflammable, therefore, cannot be regarded

¹ Supersedes NACA TN 2227, "Relation Between Inflammables and Ignition Sources in Aircraft Environments" by Wilfred E. Scull, 1950.

as a physical constant, but must depend on the experimental conditions under which it is determined. Mixture composition, pressure, and temperature and the presence of catalysts or inhibitors, which accelerate or delay the reaction, are some of the variables affecting the lag before inflammation. The lowest temperature at which any quantity of inflammable will ignite is known as the minimum ignition temperature of the inflammable for the experimental method employed.

Efforts of various investigators to determine the ignition temperatures of inflammables have resulted in different methods of attaining the mixture temperature at which self-accelerating reaction begins. As a result of these experimental methods and the difficulties of determining the defined ignition temperature, many of the ignition temperatures reported in the literature are the temperatures of heated elements, which raise the mixture temperature to a level of self-accelerating reaction. These heated element temperatures are not necessarily the same as the ignition temperature of the mixture. In general, the temperatures reported hereinafter are specifically designated as to the temperature actually measured.

The most important methods employed to determine the ignition temperatures are as follows:

- (1) Crucible methods—static and dynamic
- (2) Dynamic heated-tube method
- (3) Adiabatic-compression method
- (4) Bomb method.

In all these methods, various criterions have been chosen as indications of ignition and inflammation. A sudden rapid pressure rise, an increase in temperature, or both, have been used as indications of ignition, whereas the appearance of flame or explosive sounds have been taken to indicate mixture inflammation.

The crucible methods (static or dynamic, depending on whether the supporting atmosphere in the crucible is quiescent or flowing) appear to be the most popular methods of determining the ignition temperatures of liquid inflammables. Also known as the oil-drop methods, the crucible methods are variations of a method described in references 1 and 2. Essentially, the method consists in dropping a drop of the inflammable liquid through an opening into a crucible fitted into a heated iron or steel block containing either quiescent or flowing air or oxygen. The crucible temperature and ignition lag are measured when the droplet bursts into flame. A precise knowledge of the exact mixture composition at the instant of ignition is unobtainable. Furthermore, the latent heat of vaporization of the droplet must be supplied by the system. In these methods, experimentally measured ignition temperatures are the temperatures of the heated crucible surfaces causing ignition.

The dynamic heated-tube method, which attempts to nullify the effect of enclosing surfaces, is described in reference 3. In this method, the flowing inflammable vapor and the supporting atmosphere are separately heated in concentric quartz or pyrex tubes. The inflammable vapor is metered into the supporting atmosphere in the large tube through a small orifice in the end of the small tube. In this case, the experimentally measured ignition temperature is the mixture

temperature at which inflammation occurs after a measured time lag. Such a method permits easy variation of the inflammable over-all mixture composition.

A description of the adiabatic-compression method of determining ignition temperatures is given in reference 4. In this method, the inflammable mixture of vapor and air or vapor and oxygen is suddenly compressed. Ignition occurs after a lag depending on the temperature reached by the compression process. The ignition temperature is computed from the experimentally measured compression ratio. This method is advantageous in that the mixture composition is known, accurate ignition-lag measurements are possible, and the relatively cold walls of the compression cylinder preclude any catalytic influence. Disadvantages of such a method are the large cooling losses, the difficulty of determining the exact value of the compression exponent γ , and the fact that the experimentally determined ignition temperature pertains to the compression pressure.

The bomb or heated-chamber method of determining ignition temperatures is described in reference 5. Essentially, the method consists in introducing an inflammable mixture into an evacuated chamber heated to a known temperature. The experimentally measured ignition temperature is taken as the chamber temperature that causes ignition after a measured ignition lag. Disadvantages of the method are the possible catalytic effects of the chamber surface and the heat required to increase the mixture temperature to the chamber temperature. Advantages of the method are the knowledge of the mixture composition and the ease with which normal-length ignition lags can be measured. Very small ignition lags are difficult to measure because of the time interval required to admit the mixture to the chamber.

The results of all these methods are affected by the experimental conditions and characteristics of the inflammables investigated. Some of the factors significantly affecting the experimentally determined ignition temperatures of inflammable substances are:

- (1) Fuel composition
- (2) Fuel-air or fuel-oxygen ratio
- (3) State of inflammable—vapor or liquid
- (4) Size of inflammable drop
- (5) Mixture pressure and temperature
- (6) Relative stagnation of mixture—turbulent or quiescent
- (7) Thermal conductivity of mixture
- (8) Ignition-lag period
- (9) Surface composition and physical character of ignition element
- (10) Surface area of ignition element
- (11) Surface-volume ratio of ignition chamber
- (12) Ignition-element temperature
- (13) Approach to ignition temperature—from mixture temperatures above or below ignition temperature
- (14) Method of introduction of inflammable to ignition element.

As a result of the effects of these variables on ignition temperature, the experimentally determined ignition temperature of an inflammable may be different in various sources of

information. Data from the same source, however, show the trends and the relative effects of different variables.

LIMITS OF INFLAMMABILITY

The limits of inflammability of a mixture, also known as the inflammation or explosive limits, are those proportions, generally volumetric, of fuel in air or oxygen that are just capable of self-propagation of flame throughout the entire mixture once inflammation has been initiated in a portion of the mixture. These limits are generally determined for upward, continuous self-propagation of a flame in a vertical tube of sufficient diameter such that the wall cooling effects are negligible. Mixtures just below the lower limits or just above the upper limits of inflammability may burn around the ignition source, but are incapable of propagating flame throughout the entire mixtures. In addition, the mixture volume must be large enough that the flame may still propagate even after the energy input required for the initial inflammation has been dissipated. For any particular inflammable, the limits of inflammability are affected by the direction of flame propagation, type of test apparatus, the amount of water vapor present in the mixture, and the mixture pressure and temperature.

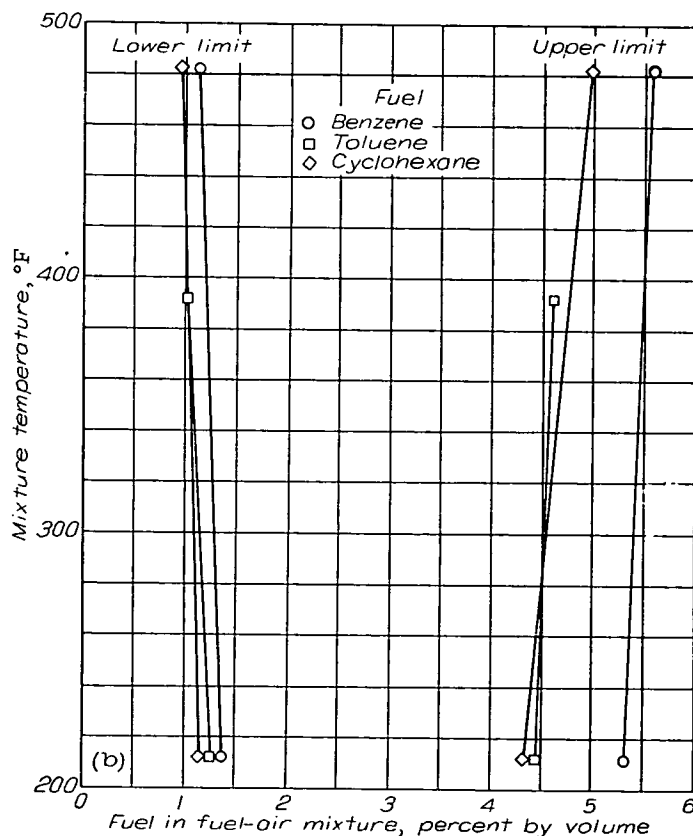
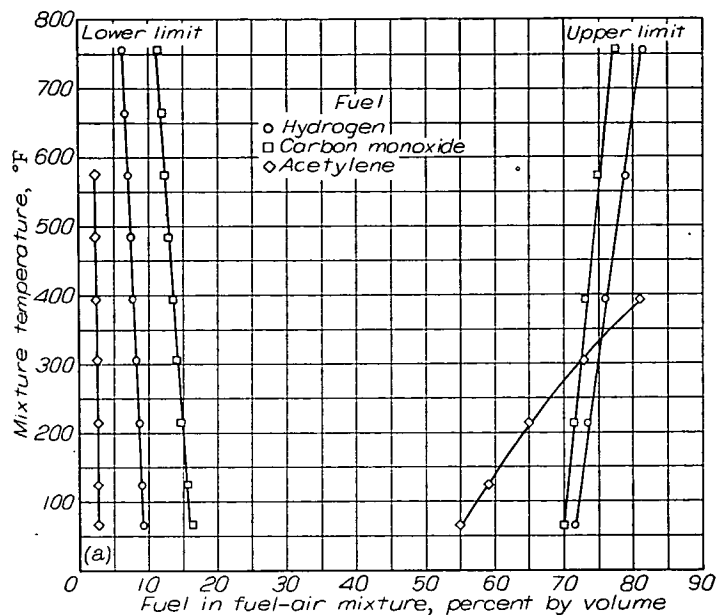
Flame progress may be assisted or retarded by convection depending on whether the flame propagation is in an upward, horizontal, or downward direction; therefore a statement of the method of observation of the limits of inflammability is necessary. Generally, wider limits are found for upward than for downward flame propagation at atmospheric pressure and temperature. Data from reference 6 indicate that the lower and upper limits of inflammability of methane in air at atmospheric conditions, as determined for upward flame propagation, are increased and decreased, respectively, approximately 10 percent for downward flame propagation. Values reported herein are for upward flame propagation unless specified otherwise.

The effect of the type of test apparatus and the method of ignition used in the determination of the inflammability limits of various mixtures are minor. The inflammability range of several mixtures increased with increasing tube diameter up to a tube diameter of 2 inches, because of the decreased cooling effect of the walls. At atmospheric pressure, tube diameters greater than 2 inches have very little effect on the limits. In addition, literature indicates (reference 7) that inflammability limits are unaffected by the method of ignition—spark, flame, or fusion of a platinum wire.

The lower limit of inflammability of a mixture is only slightly affected by the amount of water vapor present in the normal atmosphere. Water-vapor presence reduces the upper limit because of displacement of oxygen by the water vapor. Because the amount of oxygen is important in an upper-limit mixture, the amount of inflammable capable of being ignited decreases with the decreased oxygen content, resulting in an over-all reduction of the upper inflammability limit.

The effect of varying mixture temperature on the downward propagation limits of inflammability of several mix-

tures is expressed in figure 1. The upper and lower limits of inflammability of the various mixtures decrease and increase, respectively, approximately linearly with decreasing mixture temperatures.



(a) Data from reference 8.

(b) Data from references 9 and 10.

FIGURE 1.—Effect of mixture temperature on downward propagation limits of inflammability of combustible fuel-air mixtures at atmospheric pressure.

EXPERIMENTAL RESULTS IN PUBLISHED LITERATURE

Knowledge of the approximate temperature of heated surfaces capable of causing ignition is desirable because ignition of fire in aircraft may be caused by the exposure of inflammables to heated surfaces. Published experimental results for heated-surface ignition indicate the effects of a number of different variable experimental conditions. As far as possible, the results presented here are accompanied by data specifying the experimental conditions.

EFFECT OF MIXTURE COMPOSITION

Ignition temperatures.—The relative effect of mixture composition on the surface ignition temperatures of quiescent natural-gas-air mixtures ignited by single electrically heated nickel strips is shown in figure 2. The nickel strips of varying widths were mounted in the center of a chamber of 4½-cubic-foot capacity. The surface temperatures of the heated strips were measured by peened-in thermocouples. For the various widths of the heated ignition surfaces, the surface ignition temperature of the mixture increases approximately linearly with increasing proportions of natural gas in the mixture between the limits of inflammability. A few comparative experiments with methane in place of natural gas indicated that these mixtures ignited at approximately 30° F greater surface ignition temperatures than the natural-gas-air mixtures. Other investigations (reference 5, for example) indicated that the ignition temperatures of methane-air and natural-gas-air mixtures are lower than those of figure 2. This discrepancy is explained in reference 11 by the fact that the temperature of a heated ignition surface is greater than the true ignition temperature of the mixture. Although the ignition temperatures of methane-air and natural-gas-air mixtures increase with increasing proportions of fuel; the ignition temperatures decrease with increasing proportions of fuel for the higher hydrocarbons of the paraffin series. The

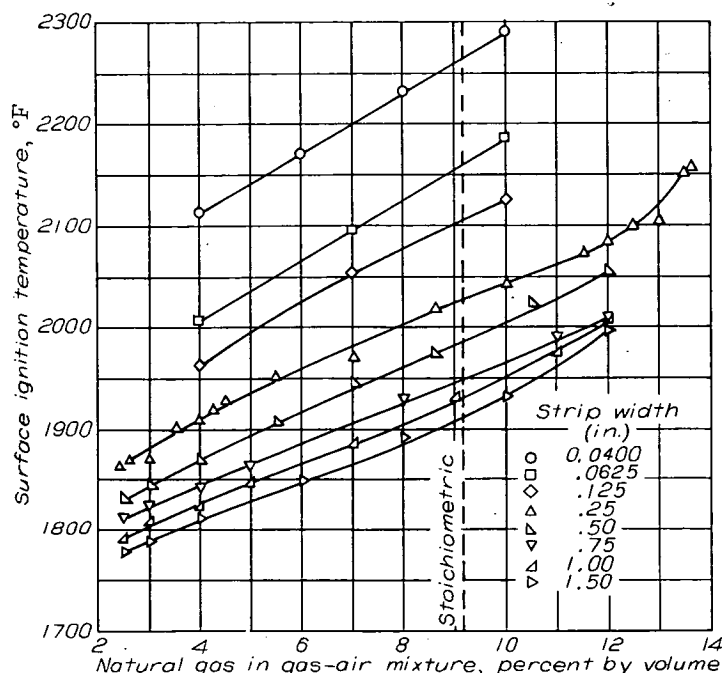


FIGURE 2.—Effect of mixture composition on surface ignition temperature of various quiescent natural-gas-air mixtures. Mixtures ignited by electrically heated nickel strips cut from same sheet of No. 18 B and S gage commercial nickel; length of strips, 4¼ inches. (Data from reference 11.)

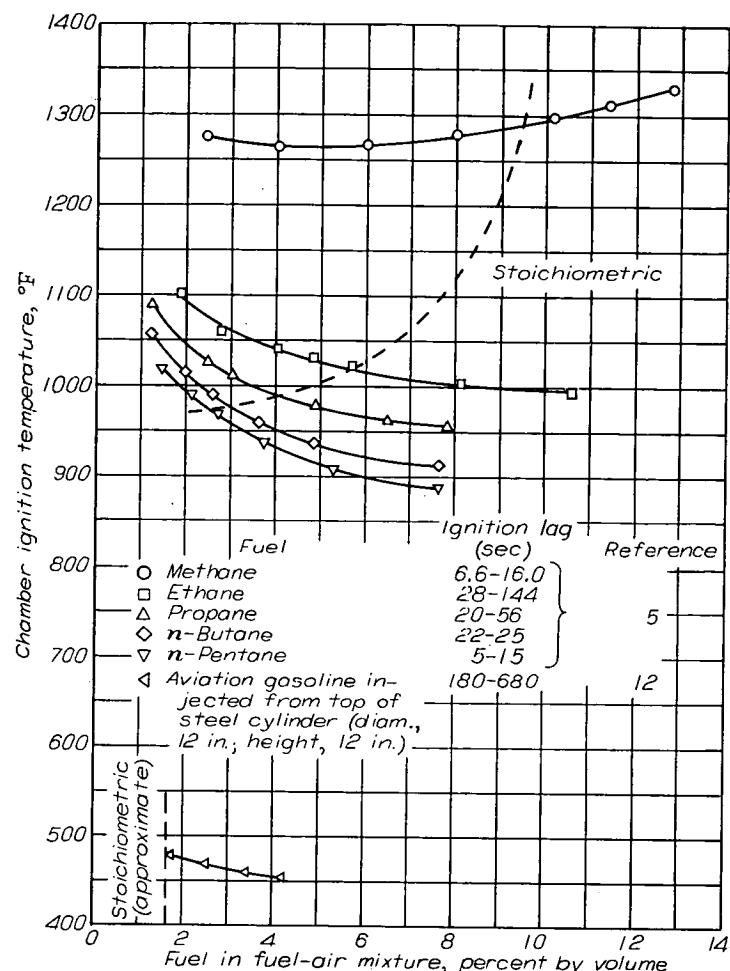


FIGURE 3.—Effect of mixture composition on chamber ignition temperatures of various fuels at atmospheric pressure as determined by bomb ignition method except as shown.

chamber ignition temperatures of mixtures of the higher paraffin hydrocarbons and air decrease with increasing proportions of fuel within the range investigated (fig. 3). It might be expected from figure 3 that the ignition temperatures of mixtures of hydrocarbons and air might again increase with increasing richness of the mixtures. Although data from references 13 to 15 are erratic, it has been concluded by the authors that the ignition temperatures of individual mixtures of coal gas, benzene, toluene, methyl alcohol, S-1 fuel, and air, which are approximately constant over a wide range of fuel-air ratios, do increase eventually with increasing richness of the mixtures.

The ignition temperatures of gasoline mixtures, like the ignition temperatures of the higher paraffin hydrocarbons, decrease with increasing proportions of fuel within the range of data available (figs. 4 and 5). The effect of the proportions of fuel, oxygen, and nitrogen on the ignition temperatures of gasoline mixtures as determined by the dynamic heated-tube method is shown in these figures. Similar results are given in reference 13. Without exception, various fuels investigated singly in an engine at two engine speeds and with hot-spot ignition showed sharp increases in the ignition temperatures of lean mixtures.

Zones of ignition and nonignition.—Two types of ignition may exist for the same fuel for certain oxygen-fuel ratios: (1) A low-temperature ignition practically independent of fuel concentration or oxygen-fuel ratio; and (2) a high-

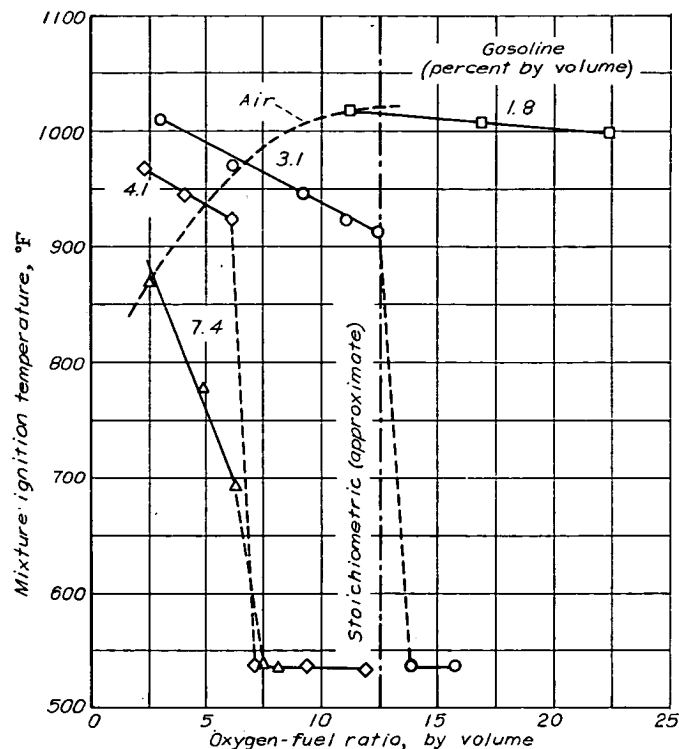


FIGURE 4.—Variation of mixture ignition temperature of gasoline with oxygen-fuel ratio for constant proportions of fuel. Method of ignition, dynamic heated tube; diluent component, nitrogen; supporting atmosphere flow, 230 cubic centimeters per minute at atmospheric pressure and temperature. (Data from reference 16.)

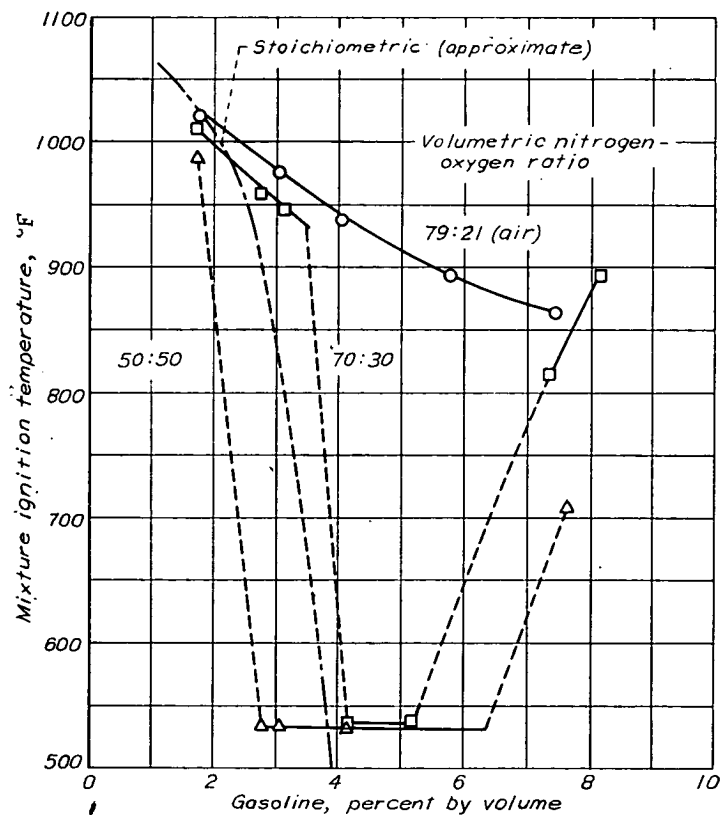
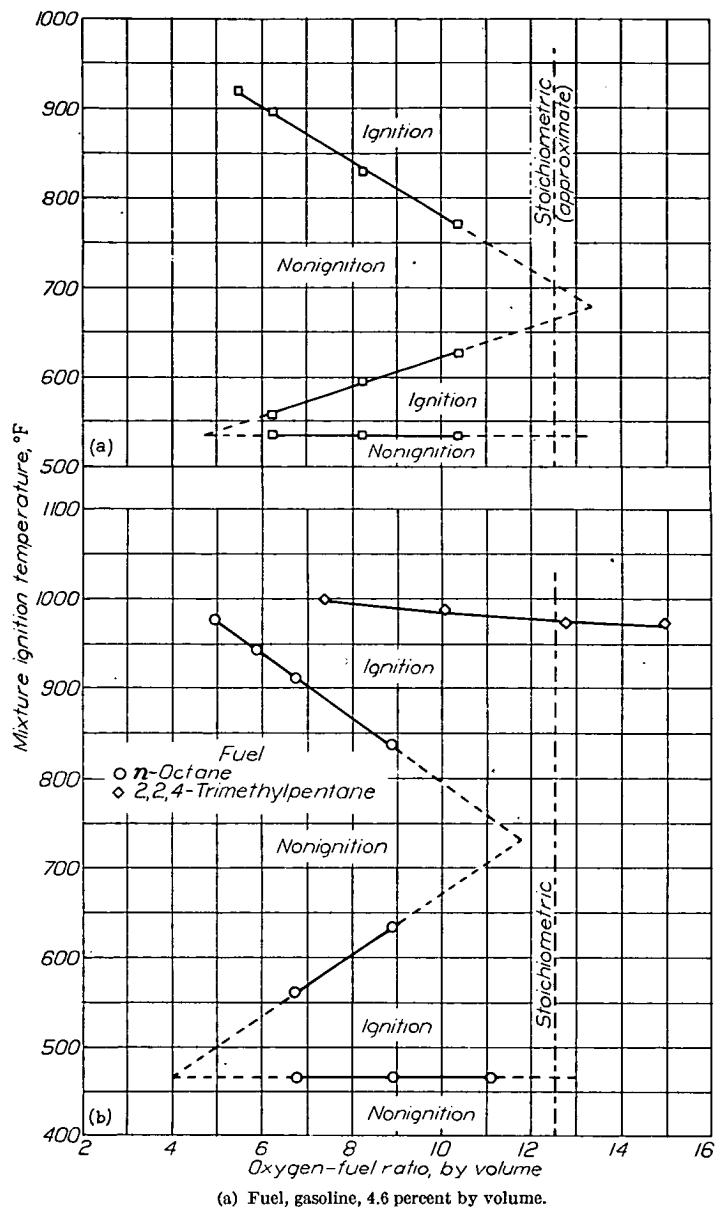


FIGURE 5.—Variation of mixture ignition temperature of gasoline with various proportions of fuel for constant nitrogen-oxygen ratios. Method of ignition, dynamic heated tube; supporting atmosphere flow, 230 cubic centimeters per minute at atmospheric pressure and temperature. (Data from reference 16.)

temperature ignition varying markedly with these variables. The two types of ignition for gasoline-oxygen-nitrogen mixtures are indicated in figures 4 and 5. The ignition temperature of gasoline in air corresponds to the high-temperature



(a) Fuel, gasoline, 4.6 percent by volume.
(b) Fuels, 2,2,4-trimethylpentane, 3.8 percent by volume and *n*-octane, 4.3 percent by volume.
FIGURE 6.—Ignition characteristics of mixtures of fuel, oxygen, and nitrogen. Method of ignition, dynamic heated tube; supporting atmosphere flow, 230 cubic centimeters per minute at atmospheric pressure and temperature. (Data from reference 16.)

type of ignition. Dynamic crucible tests reported in reference 17 indicate that readily ignited hydrocarbons, such as cetane, heptane, decane, and decahydronaphthalene exhibit zones of nonignition above the minimum ignition temperatures. Similar results, presented in reference 18, indicate that mixtures of straight-chain paraffins, containing three or more carbon atoms and air exhibit zones of nonignition. Oxidation-resistant hydrocarbons, such as benzene, toluene, and 2,2,4-trimethylpentane exhibit no nonignition zone. Ignition and nonignition zones as determined by the dynamic heated-tube method are shown in figure 6 for mixtures of gasoline, normal octane, and isooctane (2,2,4-trimethylpentane) with oxygen and nitrogen. For a fixed proportion of inflammable, the minimum mixture ignition temperature is independent of the oxygen-fuel ratio. The isooctane-oxygen mixture does not exhibit zones of ignition and nonignition, but the ignition temperature of the mixture decreases slightly with increasing oxygen-fuel ratios.

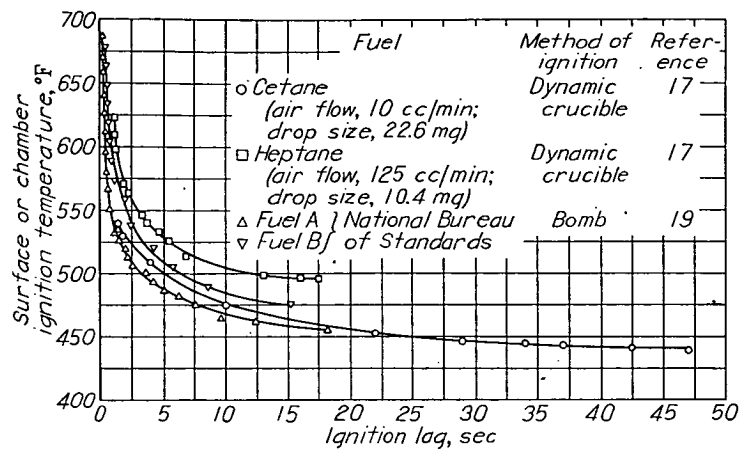


FIGURE 7.—Variation of ignition temperatures of various fuels in air with ignition lag. Cetane and heptane were ignited at atmospheric pressure.

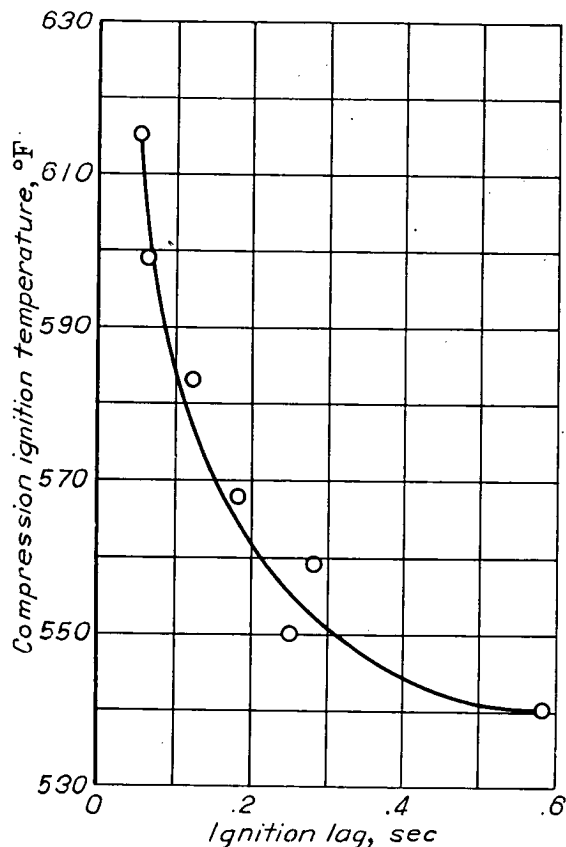


FIGURE 8.—Variation of compression ignition temperature of 1.42 percent by volume mixture of *n*-heptane and air with ignition lag. Method of ignition, adiabatic compression; final pressure, 176 pounds per square inch absolute. (Data from reference 4.)

The phenomenon of zones of nonignition may be due to the occurrence of low-temperature systems capable of cool-flame initiation. Research into the slow combustion of the higher hydrocarbons has shown that slow combustion reactions with accompanying incipient luminescence can be initiated at 300° to 400° F (reference 18). The luminescence increases with increasing temperatures until a cool flame appears. The cool flames move slowly about the reaction chamber, moving more slowly and becoming more diffuse with increasing temperature until they finally disappear leaving behind products of the incomplete combustion strongly aldehydic or peroxidic in character. At still higher

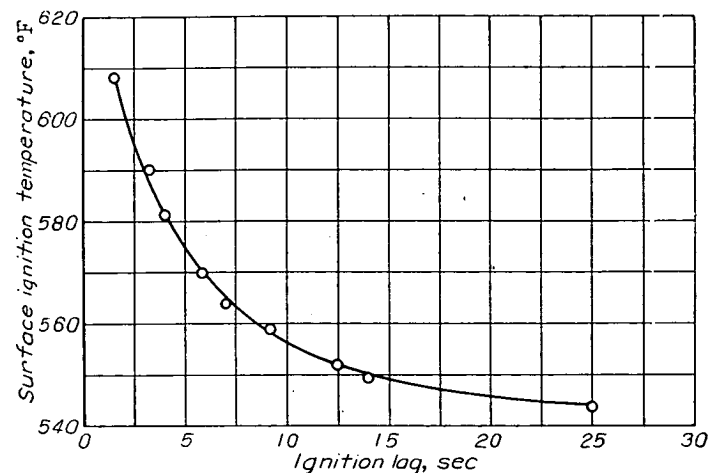


FIGURE 9.—Variation of surface ignition temperature of gasoline in flowing oxygen with ignition lag at atmospheric pressure. Method of ignition, dynamic crucible; ignition surface, platinum. (Data from reference 20.)

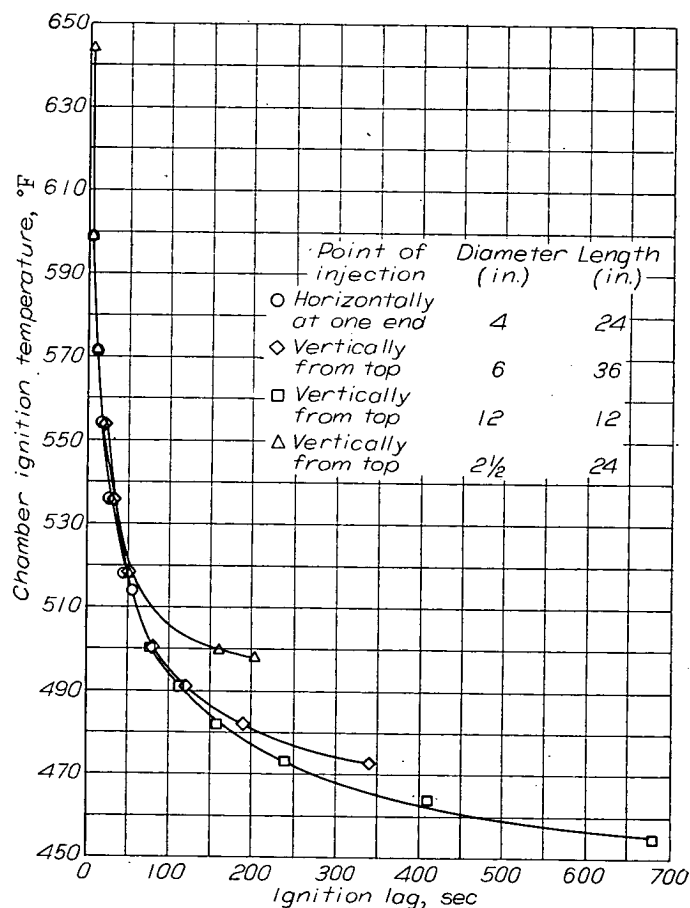


FIGURE 10.—Effect of ignition lag on chamber ignition temperature of aviation gasoline injected into quiescent air, in four different heated steel configurations, at atmospheric pressure. (Data from reference 12.)

temperatures, ignition and complete combustion of the mixture is possible.

EFFECT OF VARYING IGNITION LAG

Ignition lag of quiescent mixtures.—The ignition temperature of inflammable mixtures also depends on the ignition lag or induction period between introduction of the mixture to the ignition source and the first indication of inflammation. The effect of ignition lag on the surface or chamber ignition temperature of various mixtures is expressed in figures 7 to 10. The ignition temperatures of the

inflammable mixtures decrease almost hyperbolically with increasing ignition lags. The minimum ignition temperatures of the mixtures occur when the ignition temperature no longer decreases with increasing time lag. The effect of ignition lag on the ignition temperature of a mixture of *n*-heptane and air as determined by the adiabatic compression method is shown in figure 8. Figure 9 indicates the effect of ignition lag on the surface ignition temperature of gasoline in flowing oxygen as determined using the dynamic crucible method. The ignition temperature of gasoline in flowing oxygen after a 1.5-second lag is approximately 608° F, whereas the minimum ignition temperature, corresponding to a lag of 25 seconds, is approximately 544° F. The influence of ignition lag on the chamber ignition temperature of aviation gasoline injected into four different heated steel configurations is shown in figure 10. The minimum ignition temperature of aviation gasoline as determined from these data is approximately 455° F.

Data, from reference 21, which illustrate the effect of ignition lag on the ignition temperature of aviation gasoline ignited by the heated chamber method when the fuel (4.2 percent by volume in air) was injected from the top into a vertical steel cylinder 12 inches in diameter and 12 inches in height, are given in the following table:

Ignition lag (sec)	Chamber ignition temperature (° F)
7	600
12	570
672	450

Ignition lag of mixtures ignited by heated spheres and rods.—Ignition of inflammable mixtures has also been effected by means of heated spheres shot into the mixture. The effect of the diameter of the heated sphere on the surface ignition temperatures of three fuel-air mixtures ignited by heated quartz and platinum spheres shot into the mixtures at an average velocity of 13.1 feet per second is shown in figure 11 (a). Ignition temperatures of the mixtures decreased with increasing sphere diameter because of the greater ignition area. The criterion for ignition used in reference 22 is the requirement that the initial rate of heat production by the reaction should be greater than the heat loss by conduction. Using this criterion the investigator concludes that the following is theoretically true for each initial sphere velocity:

$$\frac{2(T_s - T_o)}{d} = Ke^{-\frac{A}{RT_s}} \quad (1)$$

where

A apparent energy of activation of mixture

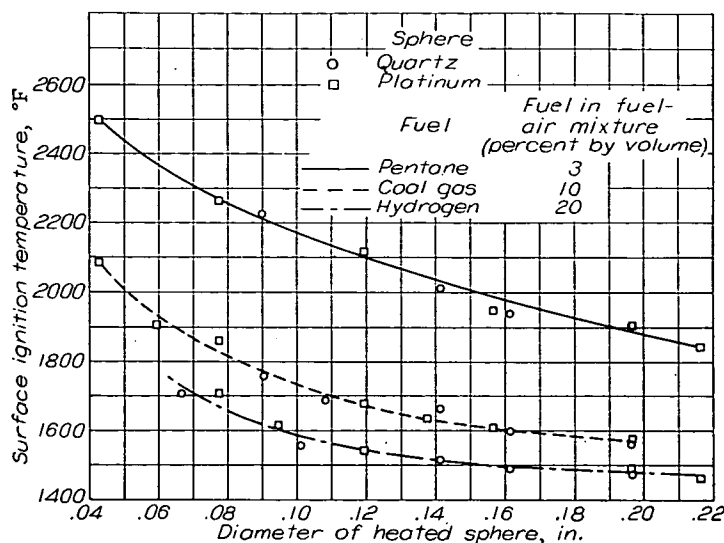
d diameter of sphere

K constant characterizing gas mixture and sphere material

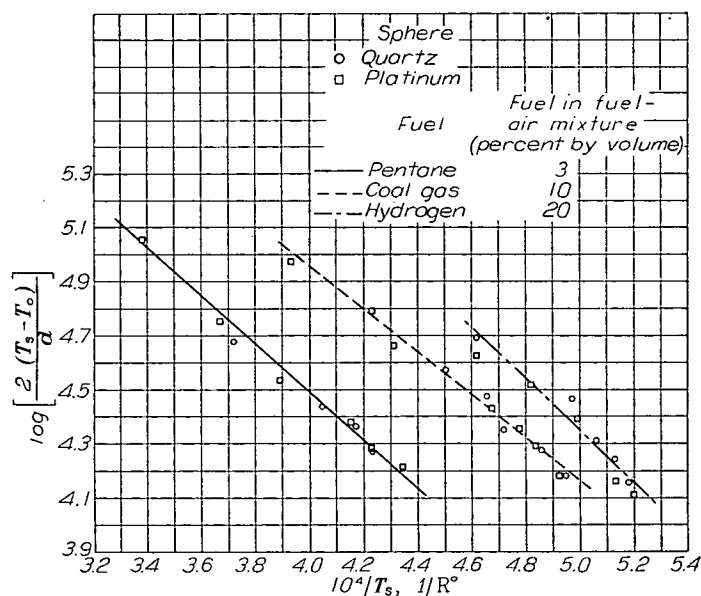
R gas constant

T_o mixture temperature

T_s heated sphere temperature



(a) Experimental data.



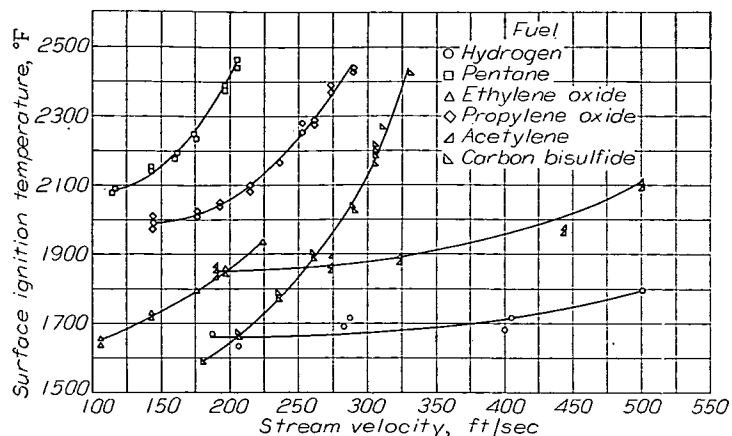
(b) Reduced data.

FIGURE 11.—Effect of sphere diameter on surface ignition temperature of three fuel-air mixtures, at atmospheric pressure, ignited by heated quartz and platinum spheres shot into mixture at average velocity of 13.1 feet per second. (Data from reference 22.)

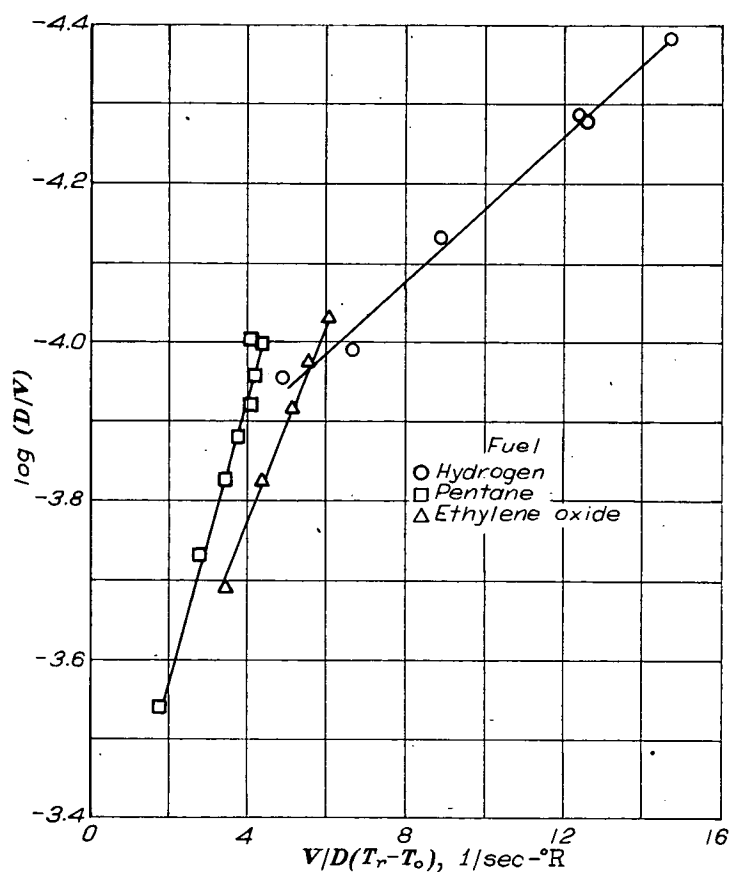
The data of figure 11 (a) have been replotted in figure 11 (b).

The same relative effect of velocity on the ignition temperatures of various inflammable mixtures is indicated in figure 12. The ignition source is at rest in this case and the mixture is in motion. The ignition temperature of the flowing mixture is the temperature of a rod situated in the center of a duct filled with flowing mixture and heated sufficiently to cause ignition of the mixture. The measured velocity is the velocity at the center of the duct. High mixture velocities correspond to short ignition lags; thus, the ignition temperature of an inflammable mixture increases with increasing velocity of the mixture.

The effect of variation of rod diameter on the ignition temperature of the mixture is expressed in figure 12 (b).



(a) Experimental data.



(b) Reduced data.

FIGURE 12.—Variation of surface ignition temperature of various fuels in air with gas-stream velocity. Ignition by $\frac{1}{4}$ -inch heated stainless-steel rods. Mixtures, stoichiometric at atmospheric pressure and temperature of 155° F. (Data from reference 23.)

For flowing mixtures ignited by heated metal rods, the following equation (reference 23) may be applied:

$$\frac{D}{V} e^{-\frac{E}{RB} \left[\frac{V}{D(T_r - T_o)} \right]} = C \quad (2)$$

where

B, C constants

D heated-rod diameter

E energy of activation of mixture

R gas constant

T_o flowing mixture temperature

T_r heated-rod temperature

V velocity of mixture flowing past heated rod

In cases of ignition of inflammable mixtures by both heated spheres and heated rods, the ignition effectiveness depends on the temperature of the ignition source, the size of the ignition source, and the duration of contact of the mixture with the ignition source. Factors related to these three criteria of ignition effectiveness are given in equation (2), which pertains to the ignition of flowing gas mixtures by heated metal rods. A factor that pertains to the duration of contact of the mixture with the ignition source is lacking in equation (1), which involves the ignition of quiescent gas mixtures by heated spheres shot into a mixture. This omission appears to be due to the author's criterion of ignition (reference 22) that recognized heat losses only by conduction.

Effect of turbulence.—Literature pertaining to the effect of turbulence on the ignition temperature of inflammable mixtures is not extensive. The available data contain no quantitative measurements of the amount of turbulence. Figure 13, which also includes the data of figure 2, indicates the effect of gentle turbulence excited by a small fan at the top of the ignition chamber on the ignition temperatures of natural-gas-air mixtures. Mild turbulence of a mixture decreases the mixture ignition temperatures. Subsequent investigation (reference 12), however, has contradicted this result. When the turbulence is sufficiently great, the

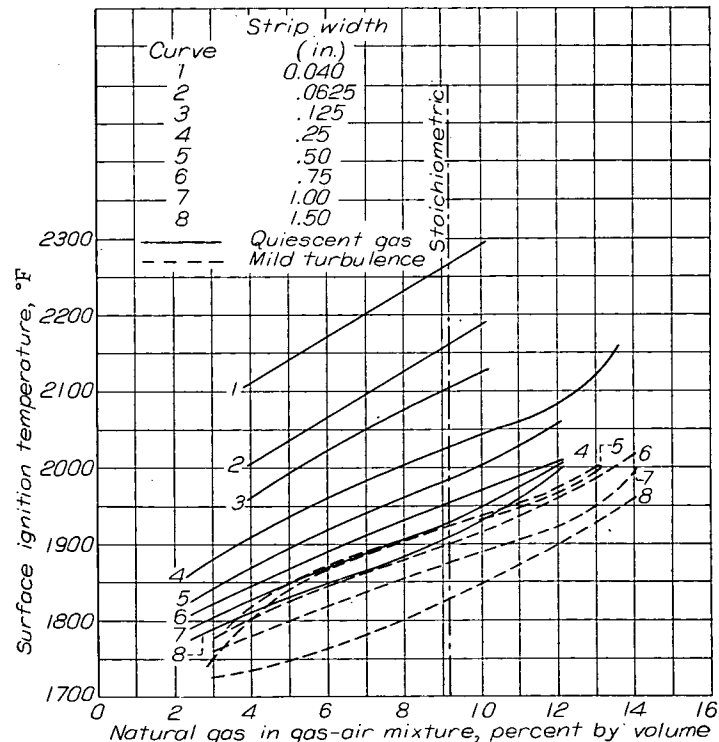


FIGURE 13.—Effect of mild turbulence on surface ignition temperature of various natural-gas-air mixtures. Mixtures ignited by electrically heated nickel strips cut from same sheet of No. 18 B and S gage commercial nickel. Length of strip, $\frac{3}{4}$ inches. (Data from reference 11.)

ignition temperatures are increased. According to figure 14, gentle stirring of an aviation-gasoline-air mixture increases the mixture ignition temperature approximately 30° F. Similar data have been found concerning the effect of mild turbulence on the ignition temperatures of stoichiometric mixtures of methane and oxygen, and hydrogen and oxygen.

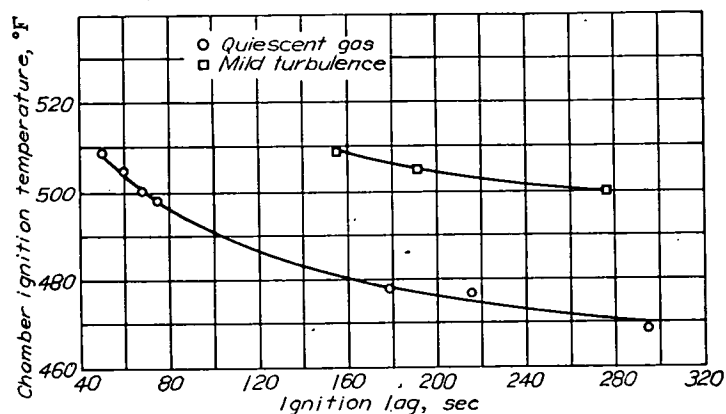


FIGURE 14.—Influence of mild turbulence on chamber ignition temperature of aviation gasoline injected into top of steel cylinder, 6 inches in diameter and 36 inches high, at atmospheric pressure. (Data from reference 12.)

EFFECT OF SURFACE CONDITION AND COMPOSITION

Open plates in air.—Various data are available concerning the ignition of fuels dropped on heated metal plates in open air. Generally, the data indicate that the ignition temperature of an inflammable liquid is much higher when dropped on a heated metal plate in air than when dropped into a heated metal tube that confines the inflammable mixture to a high-temperature region. A comparison between the ignition temperatures of various liquid aircraft inflammables in these two environments is included in table I (a). The ignition temperatures of various inflammable liquids dropped on an electrically heated sheet-iron or nickel plate in air are presented in table I (b). According to reference 24, the same results were obtained by vaporizing the liquids and passing the vapor over the heated plates. Data from reference 25 indicate that the ignition temperatures of gasoline, gasoline-benzene mixtures, Diesel fuel, and motor oil placed on heated iron plates varied from 1325° to 1400° F, but were lowered to 1000° to 1025° F when copper plates were used. Droplets of lubricating oil at 750° F falling into a heated steel pipe were ignited at approximately 840° F, but Diesel fuel at a temperature of 425° F under the same conditions did not ignite below 1200° F. The low ignition temperatures of lubricating oils are assumed to be due to the greater instability of the complex molecules. Lowering of the ignition temperatures of the inflammable liquids by using copper plates may be due to the greater thermal conductivity of the copper and differences in the catalytic activity of the two surfaces.

Various investigators (reference 26) have indicated that heated surfaces in air, below temperatures of 950° F, do not present serious fire hazards when sprayed with small amounts of oil. Reference 27 indicates that oil explosions occur only

when oil vapor is confined to a high-temperature region. Drops of oil placed on a plate heated to 1400° F would not ignite, but oil vapor ignited at 750° F. Similar results are presented in reference 26, which indicate that lubricating oils will not spontaneously ignite when dropped on either bare open metal plates or thin asbestos mats heated to 950° F. Light fractions, such as gasoline, immediately formed a ball and gradually distilled away. Heavier oils also vaporized (probably accompanied by simultaneous cracking); the vapors were diluted by air so quickly that no ignition could occur.

The results of actual tests conducted with a Napier Lion reciprocating engine are given in reference 12. Aviation gasoline vapors confined within the exhaust pipe and exposed to exhaust-pipe temperatures of 570° F ignited after a lag of 13 seconds. Outside the exhaust pipe, aviation gasoline did not ignite below temperatures of 850° F. Lubricating oil ignited at exhaust-pipe temperatures of 625° F. This apparent controversy over the ignition temperatures of oil or gasoline ignited by heated surfaces is due to the fact that liquid oils or gasolines falling on heated metal plates in open air are idealized experimental conditions that allow rapid diffusion of the inflammable vapors. The higher ignition temperature for the aviation gasoline is probably due to the rapid local cooling of the exhaust pipe in contact with the rapidly evaporating liquid. Heated metal pipes, exhaust manifolds, or enclosed heated areas of an aircraft engine may confine the inflammable vapors to high-temperature regions for ignition-lag periods corresponding to low-temperature ignition.

Varying surface compositions.—The ignition temperatures of inflammable mixtures ignited by heated surfaces are affected by the composition and the condition of the ignition surfaces. The effect of the composition of the ignition surfaces on the surface ignition temperatures of natural-gas-air mixtures is shown in figure 15. The various steels, copper, and Monel metal do not differ in their ignition effectiveness as much as catalytically active platinum differs from nickel. The abnormal effect of molybdenum is probably associated with the rapid oxidation properties of the metal. In all the experiments with molybdenum strips, dense clouds of oxides were formed. Tungsten, however, which oxidizes readily at high temperatures, did not oxidize vigorously until raised to temperatures higher than those required for surface ignition of the natural-gas-air mixtures. The following data, from reference 19, indicate the effect of the ignition-surface composition on the surface ignition temperatures of gasoline as determined in air by the dynamic crucible method:

Disk material (in silica crucible)	Surface ignition temperature (° F)
Platinum.....	878
Iron.....	833
Copper.....	788

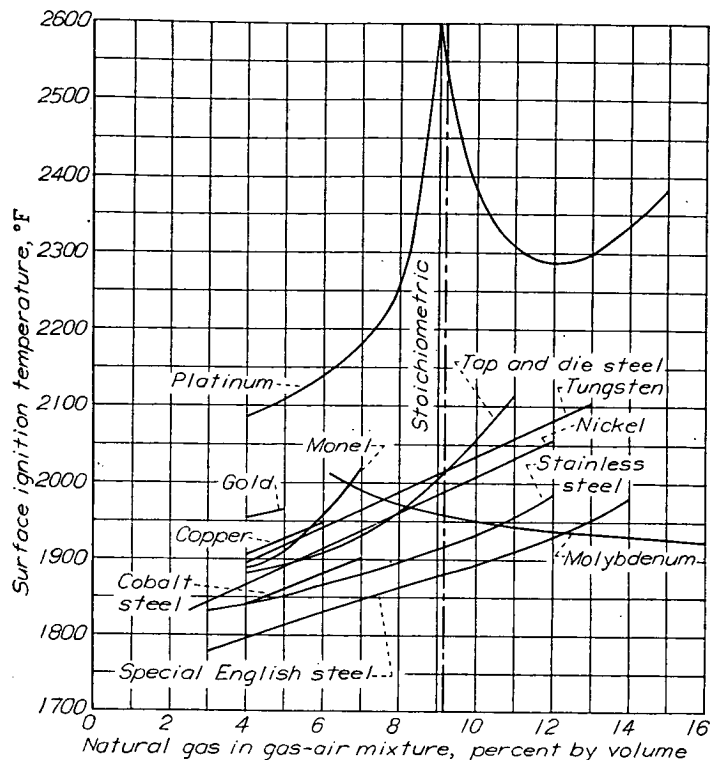


FIGURE 15.—Variation of surface ignition temperatures of natural-gas-air mixtures with percent of natural gas in mixture. Mixtures ignited by electrically heated metal strips. Size of metal strips, 4.25 by 0.50 by 0.04 inches. (Data from reference 11.)

Similar data are contained in table II concerning the effect of ignition-surface composition on the ignition temperatures of liquid fuels dropped into heated containers. In each case, catalytically inactive pyrex surfaces ignited inflammable mixtures at temperatures lower than those of metal surfaces. Discrepancies in ignition temperatures for the same substances in table II are probably due to different experimental techniques. Of the metal ignition surfaces with about the same degree of catalytic activity, those surfaces having the highest thermal conductivities had the lowest ignition temperatures. The effect of ignition-surface composition on the surface ignition temperatures of varying mixtures of a standard reference gasoline and benzene in air is indicated in figure 16. The ignition temperatures of the mixtures, excepting approximately 100-percent benzene, are 35° to 155° F greater for platinum surfaces than for quartz surfaces.

In similar experiments, the surfaces of greater catalytic activity or interstitial character must be hotter to ignite inflammable mixtures than surfaces of catalytic inactivity. Some references indicate that two types of combustion may be present in ignition by heated surfaces (1) a flameless, surface combustion, and (2) a gaseous combustion indicated by a flash. Combustion of a drop of liquid falling upon a surface that is not interstitial or porous in character or that is catalytically inactive occurs mostly in the gaseous phase with a minimum of surface combustion occurring. If the surface is highly catalytic or interstitial, however, it may be impossible for a flash to occur because most of the combustion may take place on the surface. Apparently, with all factors constant except the composition of the igniting surface, the surface ignition temperature of an inflammable mixture increases with increasing catalytic activity or interstitial character of the surface.

Rapid diffusion from the flameless reaction zones near the heated surfaces may explain the fact that higher surface ignition temperatures are required on catalytically active surfaces. Such rapid diffusion may diminish the concentration of fresh gas in the reaction zone below the limits for flame propagation at the surface temperature.

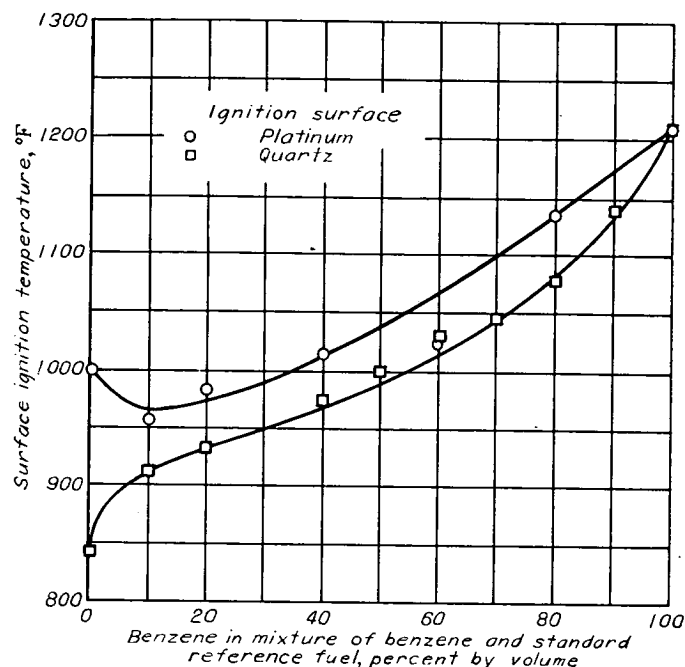


FIGURE 16.—Variation of surface ignition temperature of benzene-gasoline mixture in air at atmospheric pressure with various fuel compositions. Method of ignition, static crucible; ignition lag, 1 second. (Data from reference 28.)

The mixtures most difficult to ignite with the catalytically active platinum strips are the mixtures having approximately stoichiometric fuel-air ratios. With platinum ignition surfaces, the combustion products are almost completely oxidized, and the greatest thermal energy of the reaction occurs with mixtures of approximately stoichiometric fuel-air ratios. The thermal energy of the reaction accounts for the extremely high ignition temperatures of near stoichiometric mixtures of natural gas and air ignited by heated platinum surfaces. No effect of surface composition on the ignition temperatures of the three mixtures ignited by shot-in spheres of quartz or platinum is apparent in figure 11. Compared with platinum, quartz is catalytically inactive. The mixture composition in each case, however, is less or greater than the stoichiometric fuel-air ratio for each particular mixture; the mixtures are probably too far from the stoichiometric fuel-air ratios to be greatly affected by the composition of the ignition surface.

Varying surface conditions.—All the common metals having high melting points ignite inflammable mixtures with about equal facility except those on the surface of which scale or ash is formed. This scale or ash forming tendency is especially peculiar to iron. A possible explanation is that scale or ash upon a surface forms an insulating coating, which requires the surface to be heated to a much higher temperature for ignition than would otherwise be necessary. A gas film or layer on the inside of a container may act in the same manner.

According to reference 29, a stoichiometric mixture of methane and air introduced into a silica cylinder etched inside with hydrofluoric acid and maintained in a vacuum for 15 minutes ignited at 1085° F. The same mixture, placed in the cylinder following the evacuation of water vapor or carbon dioxide vapor, ignited at 1265° F. The surface ignition temperature of the mixture was increased by the adsorbed film of water vapor or carbon dioxide vapor.

EFFECT OF VARYING IGNITION SURFACE AREA

Heated plates.—Because the thermal energy required for ignition of an inflammable mixture is essentially constant, increasing the area of heat transfer results in a decrease of the surface temperature required for ignition of an inflammable mixture. This effect on a quiescent 7-percent mixture of natural gas in air ignited by electrically heated nickel surfaces is indicated in figure 17. From figure 17, it is apparent that a minimum surface ignition temperature exists for the mixture and this minimum ignition temperature cannot be decreased by indefinitely increasing the area of the ignition surface.

Heated wires.—Very fine wires electrically heated to incandescence may cause inflammation of explosive mixtures; but ignition of a methane-air mixture by an electrically heated platinum or tungsten wire is possible only in a narrow range of heating currents. Below this current range, flameless surface combustion occurs; above this range, the wires fuse without igniting the mixture. According to reference 30, the ignition of inflammable mixtures by heated platinum wires is not due to ionization, but to heat transfer alone, because ions do not appear until flame appears and no ions are formed during the flameless surface combustion.

Because of heat generated on its surface by flameless combustion, an electrically heated platinum wire surrounded by an inflammable mixture has a higher temperature than when surrounded by air. The mixture near the wire becomes heated and convection currents are set up. Ignition of the mixture occurs only if a portion of the mixture remains in contact with the heated wire for a period of time greater than the ignition lag. The convective effect is greatest with the mixtures having the highest thermal energies; that is, mixtures having compositions near stoichiometric fuel-

air ratios (reference 31). Thus stoichiometric mixtures are in contact with the heated wires for the shortest periods of time and consequently have higher ignition temperatures than mixtures of other compositions. The ignition temperatures of the mixtures decrease with increasing diameter of the heated ignition wires, because of the greater ignition area.

In the investigation of flameless surface combustion reported in reference 32, 0.02 to 0.09 cubic centimeter of hexane, cyclohexane, cyclohexene, and benzene were separately introduced into an air atmosphere in a 400-cubic-centimeter cylinder heated to 221° F that contained a small electrically heated platinum wire. Heating of the wire to 572° F resulted in sudden increases of the wire temperature for a short period of time prior to inflammation of the mixtures. According to reference 31, small, heated tungsten wires exposed to inflammable mixtures are quickly oxidized and burst into flames immediately at high wire temperatures. The ignition of the mixture is caused by flames rather than by a heated surface.

Data obtained from reference 33 indicate that the limiting minimum diameters of heated wires capable of igniting inflammable methane-air mixtures are approximately 0.0079 and 0.0354 inch for platinum and iron wires, respectively. For wires of lesser diameters than these, it is assumed that a layer of oxygen molecules forms on the heated wire, and thermal energy sufficient to ignite the mixture cannot be conducted through the layer. Even in an unburned mixture of methane and air, a spiral piano wire heated to redness has been found to oxidize.

Heated particles.—Ignition of inflammable mixtures by small heated spheres shot into the mixture has already been discussed. Friction sparks and fusion sparks both fall into the category of small heated particles.

According to reference 34, although natural-gas-air mixtures can be ignited by friction sparks, it is improbable that natural gas can be ignited in the field by friction sparks. Reference 35 indicates that explosive mixtures of gasoline vapors and air at atmospheric pressure and temperatures of 70° to 120° F would not ignite when exposed to sparks produced by the impact breaking of piano wire, contact of two pieces of hardened steel, steel in contact with a rotating emery wheel, or sparks from red-hot steel. Such sparks ordinarily lack the thermal energy required to ignite inflammable mixtures. Ordinary white friction sparks produced by grinding steel in air are actually small metal particles, which oxidize or burn in air after being initially heated by being torn off in the grinding process. These sparks will not ignite petroleum vapors unless the metal is held to the wheel for a long time to preheat the metal and thereby increase the thermal energy of the spark. Alloy steel, in general, produces red friction sparks of low temperature, whereas some special alloys such as are used in cigarette lighters produce bright white sparks capable of igniting gasoline. Fusion sparks are small fused-incandescent particles derived from metals contacting wires through which current is passing. Because the ignition temperature of a mixture ignited by small heated particles increases rapidly with decreasing particle diameter, fusion sparks ordinarily are very feeble ignition sources.

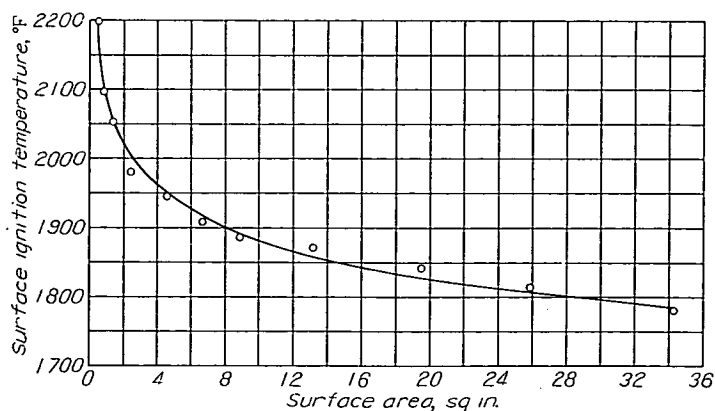


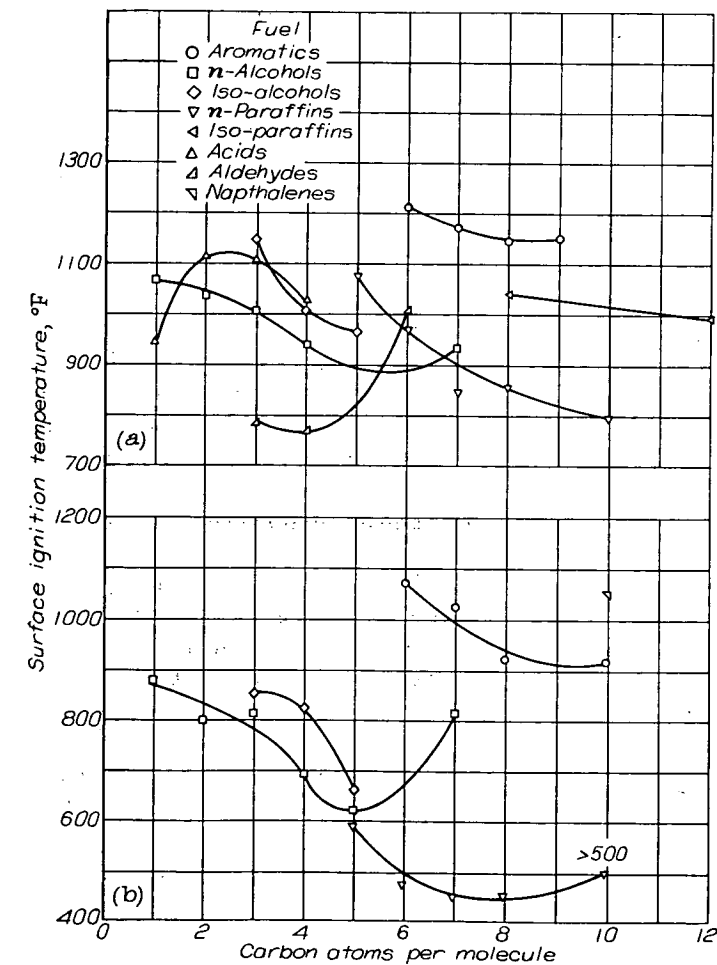
FIGURE 17.—Effect of surface area on surface ignition temperature of quiescent 7-percent mixture of natural gas and air. Ignition surface, electrically heated nickel. (Data from reference 11.)

EFFECT OF VARYING FUEL COMPOSITION

The ignition temperature of an inflammable mixture varies with the type of fuel and the amount of antiknock additives in the fuel. This report is primarily concerned with the effect of gasoline composition on ignition temperatures, but the effect of hydrocarbon families in addition to those that appear in gasoline are also included.

The variation of surface ignition temperature with the number of carbon atoms present per molecule for different hydrocarbon families is shown in figure 18. Under the same experimental conditions, the ignition temperatures of *n*-paraffin hydrocarbons decrease with a lengthening of the carbon chain, with the shortest chain hydrocarbons having the highest ignition temperatures. Iso-paraffins have higher ignition temperatures than the *n*-paraffins (fig. 18 (a)). The ignition temperatures of aromatic hydrocarbons are also higher than those of *n*-paraffins.

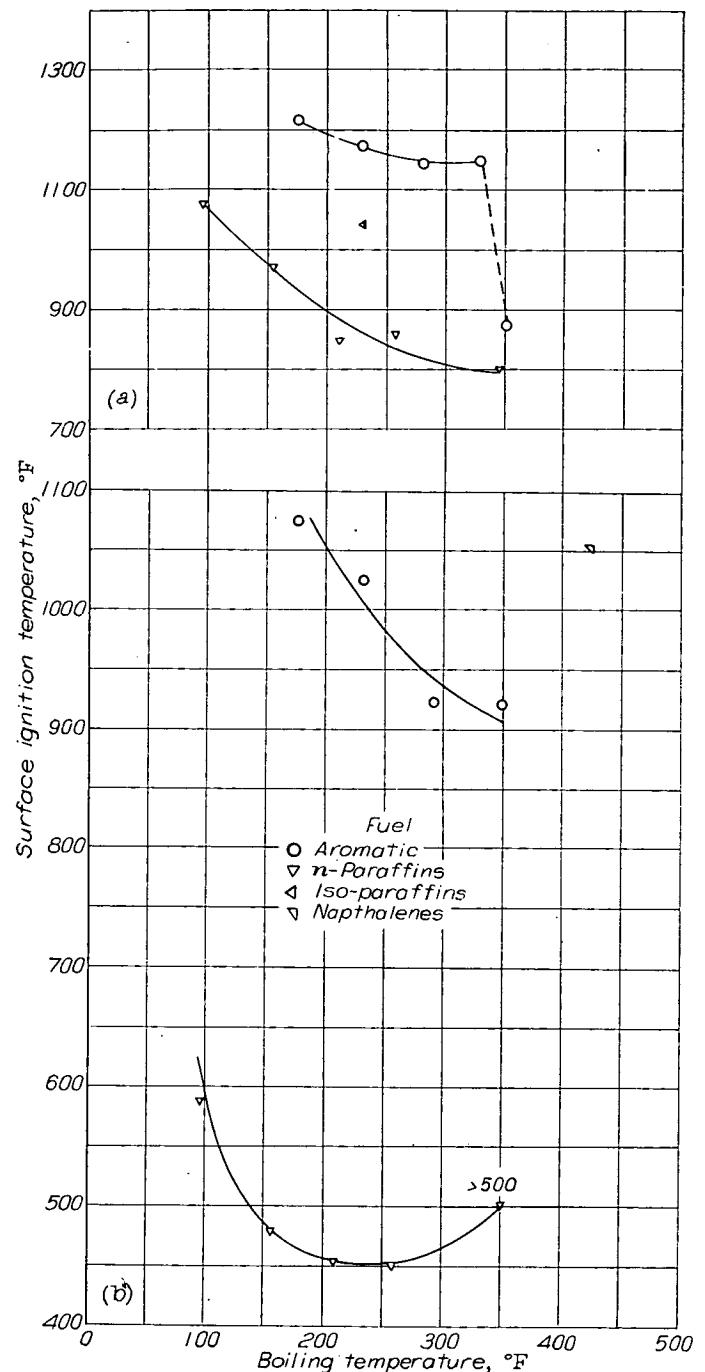
The surface ignition temperatures of various hydrocarbons are expressed as functions of the boiling temperatures in figure 19. Although the absolute value of the ignition temperature for the same hydrocarbon differs with the reference,



(a) Ignition temperatures determined by static crucible method. Ignition surface, platinum; ignition lag, 1 second; data from reference 36.

(b) Data from reference 37.

FIGURE 18.—Variation of surface ignition temperatures of several fuels in air at atmospheric pressure with number of carbon atoms per molecule.



(a) Method of ignition, static crucible; ignition surface, platinum; ignition lag, 1 second; data from reference 36.

(b) Data from reference 37.

FIGURE 19.—Effect of boiling temperature on surface ignition temperatures of several fuels in air at atmospheric pressure.

trends are similar for the same hydrocarbon families. The effect of fuel composition on the surface ignition temperatures of mixtures of isooctane and heptane ignited in air on a platinum surface by the static crucible method is shown in figure 20. In reference 38, it is stated that no exact relation exists between the octane number and the ignition temperature of a fuel, but figure 20 and table III indicate that the ignition temperatures of unleaded fuels generally increase with increasing octane number.

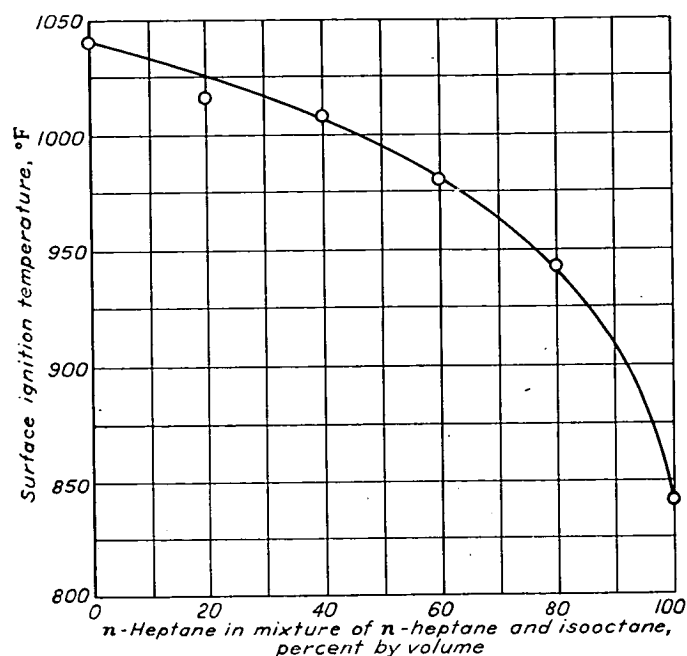


FIGURE 20.—Effect of varying fuel composition on surface ignition temperatures of *n*-heptane—iso-octane (2,2,4-trimethylpentane) mixture as determined in air at atmospheric pressure. Method of ignition, static crucible; ignition surface, platinum; ignition lag, 1 second. (Data from reference 28.)

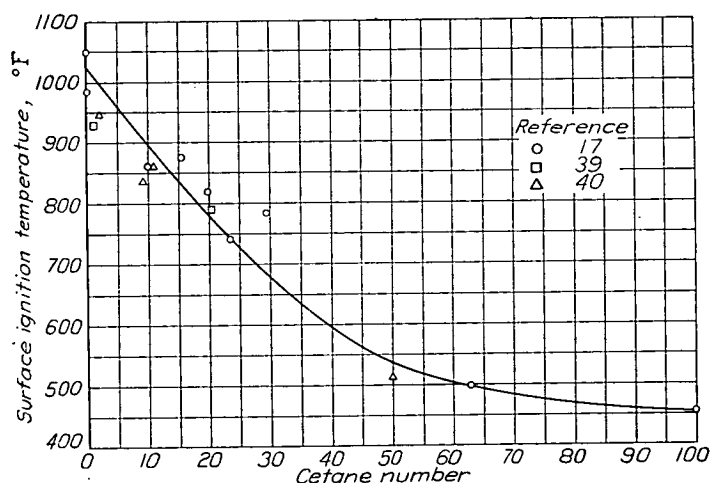


FIGURE 21.—Effect of cetane number on surface ignition temperature of various fuels in air at atmospheric pressure. Cetane numbers are converted A.S.T.M.-CFR numbers.

The ignition temperatures of unleaded fuels in air are also generally increased by additions of tetraethyl lead. Table IV shows the effect of additions of tetraethyl lead on the ignition temperatures of several fuels. A relation exists between the cetane numbers and the surface ignition temperatures of various fuels, as indicated by figure 21. The cetane numbers of the fuels were converted from octane numbers, given in reference 41, by the method given in reference 42. The ignition temperatures of the various fuels increase with decreasing fuel quality or cetane number.

EFFECT OF VARYING PRESSURE

Increased pressure decreases the ignition temperatures of inflammable mixtures. The effect of pressure on the ignition temperatures of various hydrocarbons is shown in figures 22 to 24. The chamber ignition temperature of gasoline in air

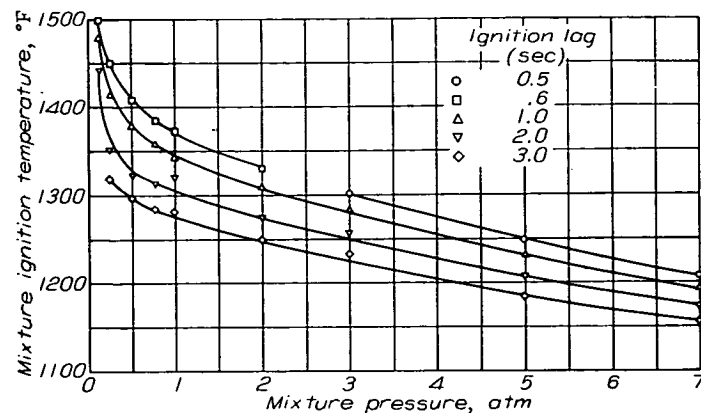


FIGURE 22.—Effect of mixture pressure on mixture ignition temperature of methane in air. Method of ignition, dynamic heated tube. (Data from reference 3.)

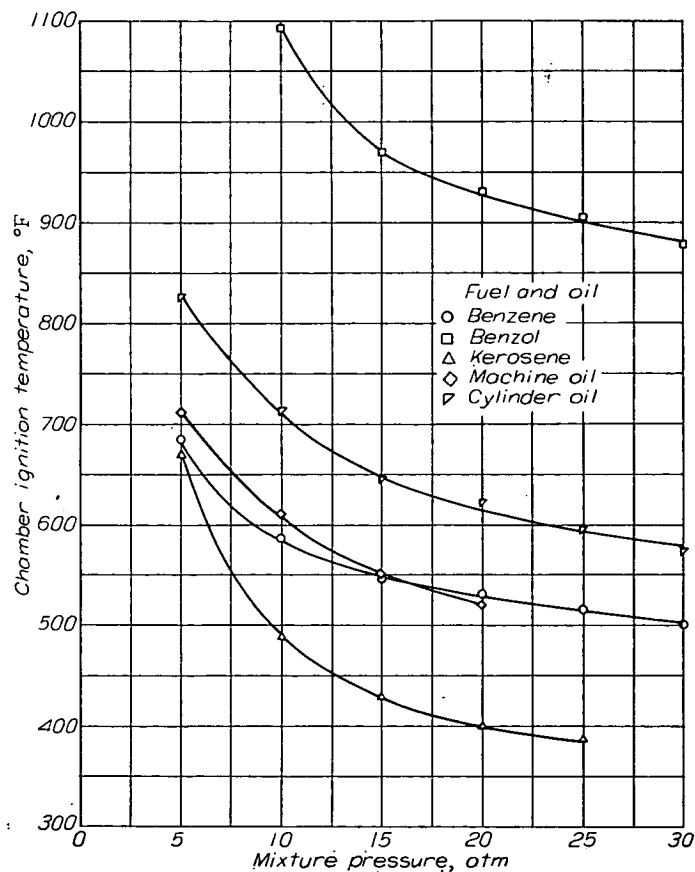


FIGURE 23.—Effect of mixture pressure on chamber ignition temperatures of several fuels and oils in air. Method of ignition, bomb. (Data from reference 43.)

as determined by the bomb method without spraying decreases from approximately 705° F at a pressure of 2.2 atmospheres to approximately 450° F at 10 atmospheres (fig. 24).

EFFECT OF DILUENTS

Figures 4 and 5 indicate the effect of nitrogen as a diluent on the mixture ignition temperatures of inflammable mixtures ignited by the dynamic heated-tube method. The ignition temperatures of the mixtures increase with decreasing amounts of oxygen. This effect is in general agreement with the statement that the ignition temperatures of fuels are generally less in oxygen than in air. In the low ignition-

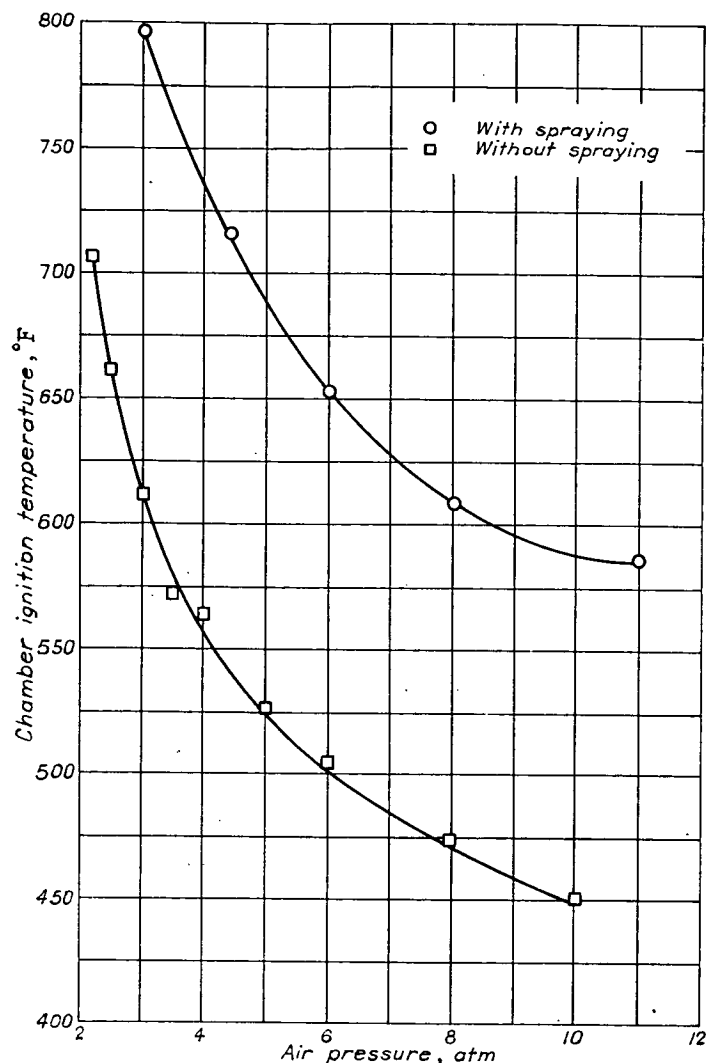


FIGURE 24.—Variation of chamber ignition temperature of gasoline in air with air pressure. Method of ignition, bomb. (Data from reference 43.)

temperature range of figures 4 and 5, however, the effect of inert gases is negligible. Regardless of the percentage of inert gas in the mixture, the ignition temperature in the low ignition-temperature range is approximately constant.

The ignition temperatures and limits of inflammability of gasoline and lubricating oils are dependent on the conditions and methods of experiment; these properties of gasolines and lubricating oils as obtained by different investigators are included in tables V and VI.

II—IGNITION BY ELECTRIC SPARKS AND ARCS

A study of ignition of inflammable mixtures by electric sparks and arcs involves an understanding of the nomenclature of spark-ignition systems. Definitions of terms encountered in the contemporary literature, which pertain to electric spark and arc ignition, are presented in the following paragraphs.

TERMINOLOGY

CAPACITANCE SPARKS

Capacitance sparks are produced by the discharge of charged condensers. According to reference 44, capacitance

sparks are usually bright, may be of short duration, and exhibit a spectrum that corresponds to the spectrum of the gas mixture in which the spark occurs. The current in a capacitance spark may be very large; however, the total energy transferred in the spark may be small because of the short duration of the spark. The total energy of a capacitance spark, expressed as the energy required in a circuit to produce the spark, is

$$E = \frac{1}{2} C (V_2^2 - V_1^2) \quad (3)$$

where

C capacitance of discharged condenser, farads

E energy dissipated in spark, joules

V_1 extinction potential remaining after spark has been dissipated because of weakening of field and depletion of ions, volts

V_2 circuit potential immediately prior to spark, volts

Generally, the extinction potential V_1 is so small that it can be disregarded. If V_2 is very small, however, V_1 may be important. Considerable energy losses may occur in the electric circuits through skin effects in the condenser plates and conductors especially if solid dielectrics are used. For this reason, it has been recommended that air condensers be used in electric circuits to determine the minimum spark-ignition energies of incendiary capacitance sparks.

INDUCTANCE SPARKS

Also known as low-tension sparks, inductance sparks are often obtained by breaking a wire in an inductive electric circuit or from magnetos or ignition coils. These sparks are generally not as bright in appearance as capacitance sparks and exhibit a spectrum corresponding to the spectrum of the vapor of the metal electrodes. The duration of inductance sparks can be relatively long compared with the duration of capacitance sparks; the total electric energy transferred in such sparks may therefore be large. Like the capacitance spark, the energy dissipated in an inductance spark is difficult to measure, but can be expressed as,

$$E = \frac{1}{2} L i^2 \quad (4)$$

where

E energy dissipated in spark, joules

i current in circuit prior to spark, amperes

L inductance of electric circuit, henrys

According to reference 44, the total energy dissipated in the spark may be less or greater than $\frac{1}{2} L i^2$. If the spark is extinguished before the available electromagnetic energy is dissipated, the total energy dissipated in the spark will be less than $\frac{1}{2} L i^2$. The total energy dissipated in the spark may be greater than $\frac{1}{2} L i^2$ if the potential across the break is enough to maintain the spark.

FUSION SPARKS

Fusion sparks are small incandescent particles and are treated in the section IGNITION BY HEATED SURFACES.

MINIMUM SPARK-IGNITION ENERGY

The minimum spark-ignition energy of an inflammable mixture is the total electric energy stored in an electric circuit at the initiation of the weakest spark just capable of ignition of the mixture. Essentially, this energy is the energy transmitted to the gas between the electrodes as heat and ionization.

QUENCHING DISTANCE

The quenching distance of any electrode configuration is the minimum electrode spacing below which the reaction initiated in a hydrocarbon-air mixture by an incendiary spark is quenched.

The amount of energy in any of the different type sparks

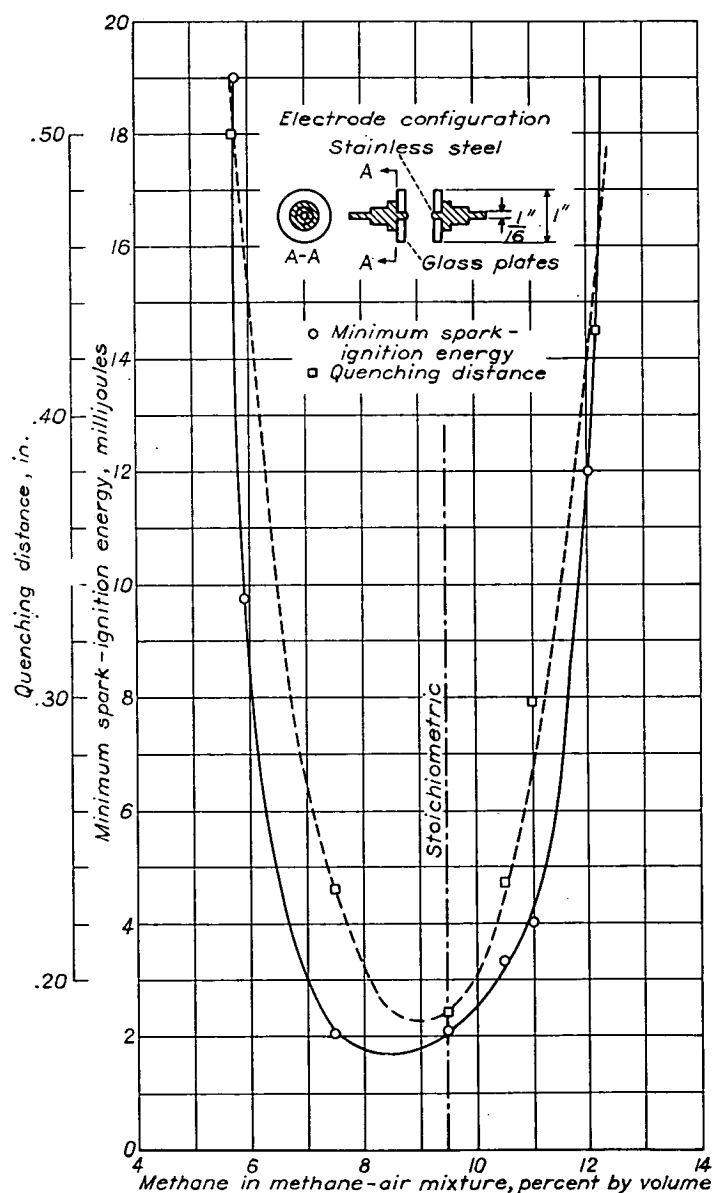


FIGURE 25.—Effect of mixture composition on minimum spark-ignition energy and quenching distance of methane-air mixtures at temperature of 77° F and pressure of $\frac{1}{3}$ atmosphere. (Data from reference 45.)

is difficult to measure exactly. Most of the results are therefore reported as the energy required in an electric circuit capable of producing an incendiary spark. This report is concerned with the minimum spark-ignition energy required to ignite a gasoline-air mixture and the minimum electrode spacing below which the reaction initiated in a gasoline-air mixture by an incendiary spark is quenched. These data are not extensive in the literature; however, knowledge pertaining to the spark ignition of the lighter hydrocarbons is quite extensive, and many of these results are included herein.

The spark-ignition process has been defined as an ionization process, a thermal process, or a combination of the two processes. Regardless of the definition of the type of process, the minimum spark-ignition energy of an inflammable mixture depends on mixture composition, pressure, temperature, velocity, spark duration, and electrode size, material, spacing, and configuration. Ignition lag, when applied to spark ignition, is approximately zero, because of the high temperature of the spark.

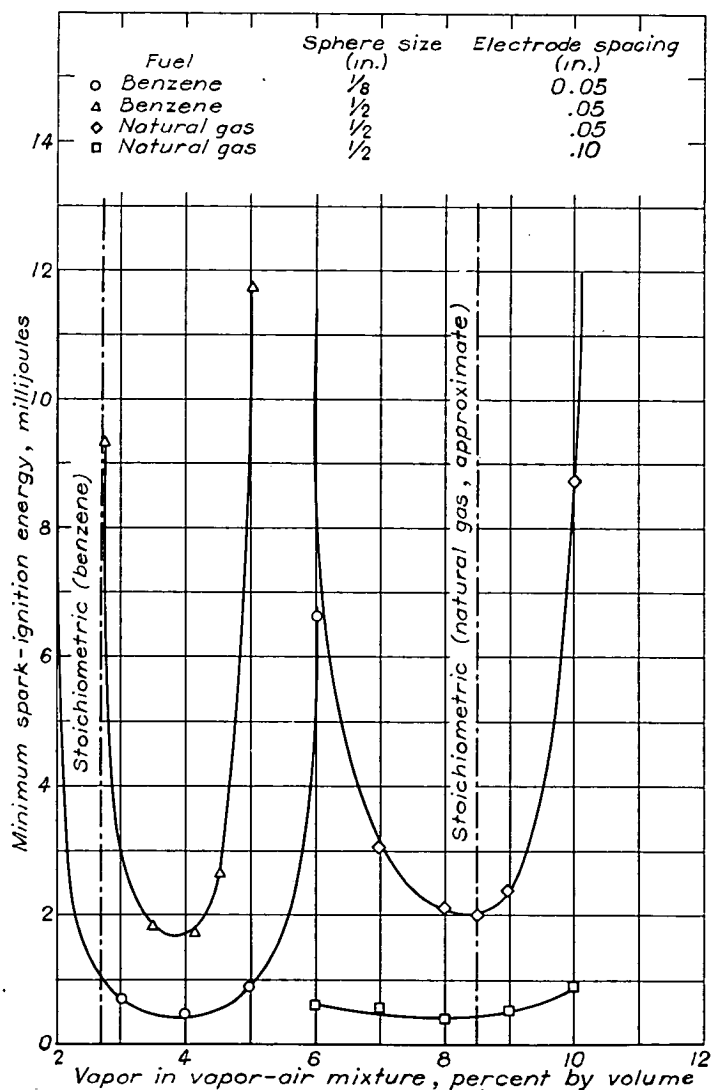


FIGURE 26.—Effect of mixture composition on minimum spark-ignition energy of benzene-air and natural-gas-air mixtures at room temperature and pressure of 1 atmosphere. Spherical stainless-steel electrode configurations. (Data from reference 46.)

EXPERIMENTAL RESULTS IN PUBLISHED LITERATURE

The ignition of fires in aircraft by electric sparks and arcs depends generally on the energy available in the spark. Whether the spark-ignition energy is capable of ignition depends on various experimental conditions. The manner in which results are presented here is similar to the manner in which heated-surface ignition results were presented; that is, the results are accompanied by data specifying the experimental conditions.

CAPACITANCE SPARKS

Effect of mixture composition.—The effect of mixture composition on the minimum spark-ignition energy of several different quiescent mixtures ignited by sparks from different electrode configurations is shown in figures 25 and 26. The curve of minimum spark-ignition energy, plotted against mixture composition for a methane-air mixture at a temperature of 77° F and a pressure of $\frac{1}{2}$ atmosphere (fig. 25), passes through a minimum of approximately 1.7 millijoules at approximately 8.4 percent by volume of methane in air, when $\frac{1}{16}$ -inch stainless-steel electrodes embedded in 1-inch glass disks are used. Similar values for other mixtures are given in the following table for an approximate temperature of 77° F and pressure of 1 atmosphere. Stainless-steel electrodes were used except for methane, which was investigated by using stainless-steel electrodes embedded in 1-inch glass disks.

Inflammable	Mixture vapor in air, (percent by volume)	Electrode diameter (in.)	Electrode configuration	Electrode spacing (in.)	Least energy (millijoules)	References
Benzene.....	3.85	$\frac{1}{16}$	Spherical.....	0.05	0.4	46
Benzene.....	3.85	$\frac{1}{16}$	Spherical.....	.05	1.7	46
Natural gas.....	8.00	$\frac{1}{16}$	Spherical.....	.10	0.4	46
Natural gas.....	8.40	$\frac{1}{16}$	Spherical.....	.05	2.0	46
Methane.....	10.00	$\frac{1}{16}$	Hemispherical.....	---	0.7	47

For each of these inflammable mixtures, well defined upper and lower limits of inflammability exist.

The data given in the following table (reference 48) indicate that the minimum spark-ignition energies of different quiescent hydrocarbon-air mixtures at 77° F and pressure of 1 atmosphere ignited by capacitance sparks from the same electrode configuration do not differ greatly:

Inflammable	Percent of fuel in fuel-air mixture by volume	Least energy (millijoules)
Methane.....	8.45	0.28
Ethane.....	6.61	.25
Propane.....	5.07	.26
n-Butane.....	4.53	.26
n-Hexane.....	3.64	.24
n-Heptane.....	3.36	.25
Cyclopropane.....	6.34	.18
Cyclohexane.....	3.94	.24
Benzene.....	4.67	.21
Diethyl ether.....	5.30	.19

The effect of mixture composition on the minimum spark-ignition energies of flowing propane-air mixtures at a pressure of 3 inches of mercury absolute is indicated in figure 27. The general shapes of the curves, for the three mixture velocities plotted, are much the same as the shapes of curves for various quiescent mixtures.

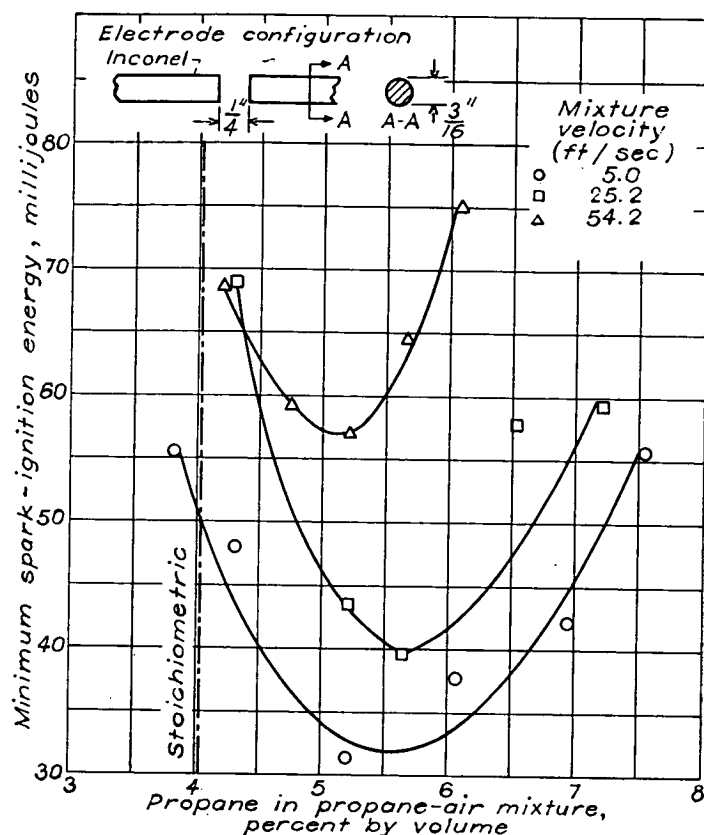
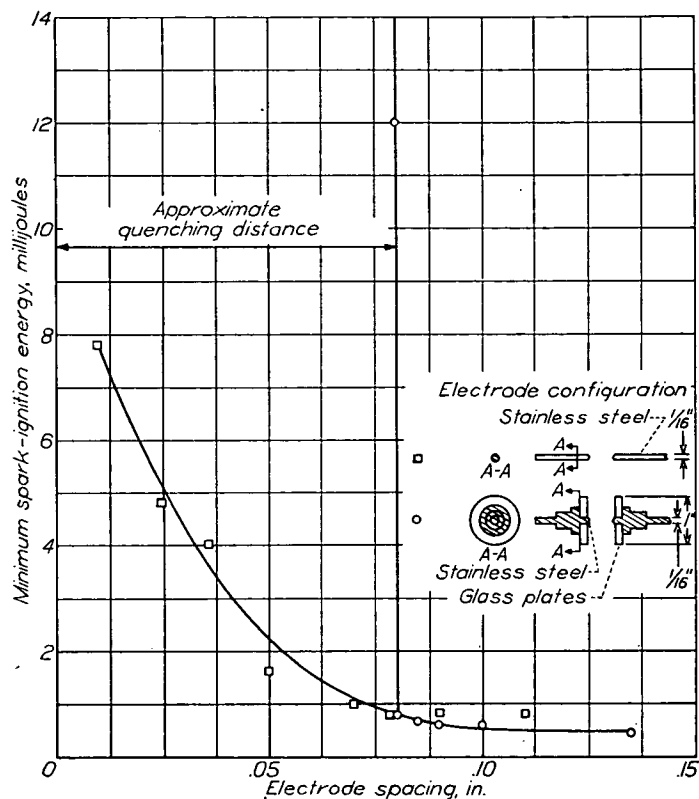


FIGURE 27.—Effect of mixture composition and velocity on minimum spark-ignition energy of flowing propane-air mixtures at temperature of 80° F and pressure of 3 inches mercury absolute. Electrode spacing, 0.25 inch; capacitance-spark duration, 600 to 900 microseconds. (Data from reference 49.)

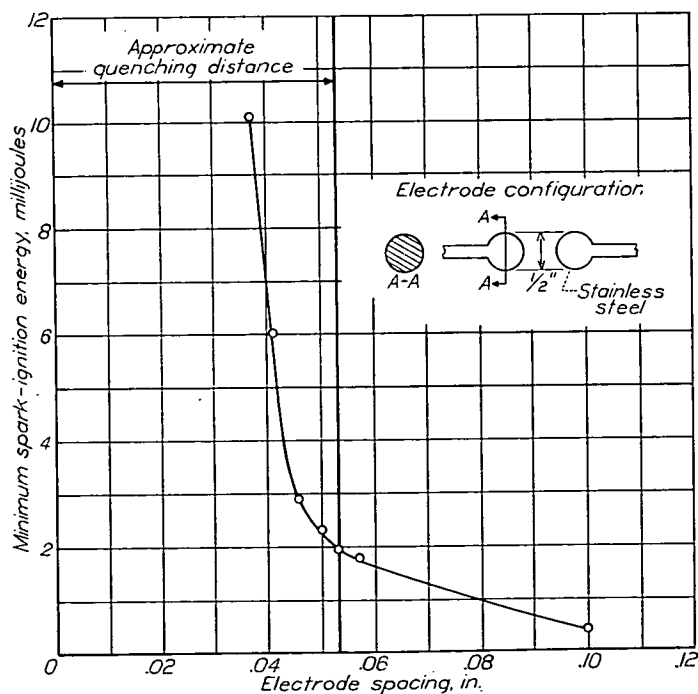
Effect of electrode spacing.—The effect of varying electrode spacings of different electrode configurations on the minimum spark-ignition energies of various inflammable mixtures is shown in figure 28. A critical electrode spacing exists for each electrode configuration. The minimum spark-ignition energy required to ignite a mixture is approximately constant for electrode spacings greater than the critical spacing. Greatly increased electrode spacings or spacings less than the critical spacing increase the required minimum spark-ignition energy of a mixture. According to reference 45, the critical electrode spacing represents the spacing below which the chemical reaction initiated by the initial inflammation is quenched by the cooling effect of the spark electrodes and is analogous to the diameter of the largest tube through which flame will not propagate. The minimum spark-ignition energies of a stoichiometric mixture of natural gas and air are approximately 0.45 millijoules for electrode spacings greater than the critical spacing or quenching distance of approximately 0.08 inch for two electrode configurations. When the electrode spacing is decreased to less than the quenching distance, the minimum spark-ignition energy increases gradually if rounded electrodes are used and abruptly if rounded electrodes embedded in glass are used. The abrupt change in the case of the disk electrode configuration may be due to the large cooling area of the glass disks. The quenching distance of one electrode configuration varies with composition of a methane-air mixture at a temperature of 77° F and a pressure of $\frac{1}{2}$ atmosphere, as indicated in figure 25. A quenching distance of approximately 0.089 inch exists for a glass-disk electrode configuration igniting a 9 percent (by volume) mix-

ture of methane and air at a temperature of 77° F and atmospheric pressure (reference 50).

Under the same atmospheric conditions, 1/2-inch spherical stainless-steel electrodes and 3/16-inch hemispherical brass electrodes have exhibited quenching distances of approximately 0.053 inch in the spark ignition of approximately 8.5 percent by volume mixtures of natural gas and air and

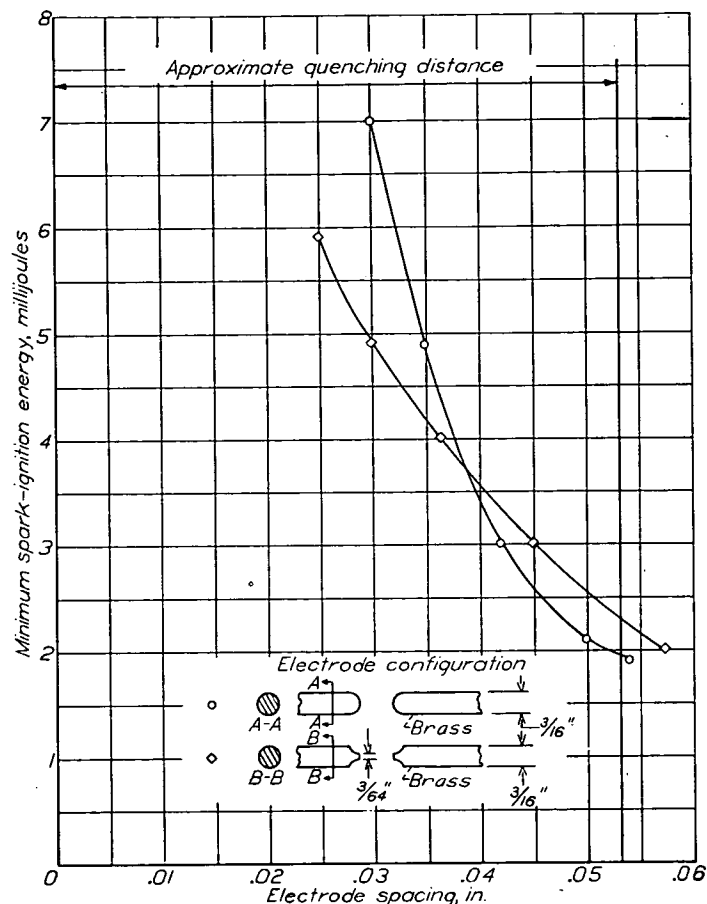


(a) Fuel, natural gas; data from reference 45.



(b) Fuel, Pittsburgh natural gas; data from reference 46.

FIGURE 28.—Effect of varying electrode spacing on minimum spark-ignition energy of approximately 8.5 percent by volume mixture of fuel and air at atmospheric pressure and temperature.



(c) Fuel, methane; data from reference 44.

FIGURE 28—Continued.

methane and air, respectively (fig. 28 (b) and 28 (c)). Quenching distances increase with increasing electrode size and thermal conductivity of the electrode material, owing to the increased cooling effect. Increasing the electrode size and electrode spacing for the same inflammable mixture increases the minimum spark-ignition energy required to ignite the mixture, because of the larger cooling area. For the same reason, the inflammability range is smaller when determined with large electrodes than with small ones (fig. 26).

Effect of electrode material and configuration.—With the exception of the relative effect of varying the thermal conductivity of the electrodes, varying the electrode material has no significant effect on the ignition of inflammable mixtures by capacitance sparks. Ignition of gaseous mixtures by sparks from platinum, nickel, zinc, aluminum, lead, brass, and steel electrodes was unaffected by changes in the electrode material (references 44 and 51). Ignition of the mixture was affected by the changes in electrode configuration.

Effect of spark potential.—Generally, the minimum spark-ignition energy of an inflammable mixture is practically independent of the spark potential of capacitance sparks. The same spark-ignition energy, regardless of the potential and capacitance changes in the circuit, ignites mixtures of the same composition. Some references indicate that the minimum spark-ignition energy required to ignite an inflammable mixture decreases with increasing spark potential. These data are insignificant in accordance with the results reported

in reference 50. When the sparking potential of capacitance sparks was varied from 1.6 to 5.8 kilovolts the minimum spark-ignition energies required to ignite methane-air mixtures was relatively unaffected by potential variations. Most of the references agree that it is the electrode configurations and electrode spacings that determine the minimum spark-ignition energies of capacitance sparks capable of igniting inflammable mixtures.

Effect of mixture velocity and spark duration.—According to reference 49, the minimum spark-ignition energy required to ignite flowing propane-air mixtures at low pressures increases approximately linearly with mixture velocity. This phenomenon is graphically shown in figure 29 for a 5.2 percent by volume mixture of propane in air at pressures of 2, 3, and 4 inches of mercury absolute. The relatively high ignition energies are due to the very low mixture pressures.

Minimum spark-ignition energies of flowing propane-air mixtures increase with spark duration as shown in figure 30. The relation is not linear, being approximately of the form

$$E = Kt^n \quad (5)$$

where

E minimum spark-ignition energy

K constant

n constant < 1

t spark duration

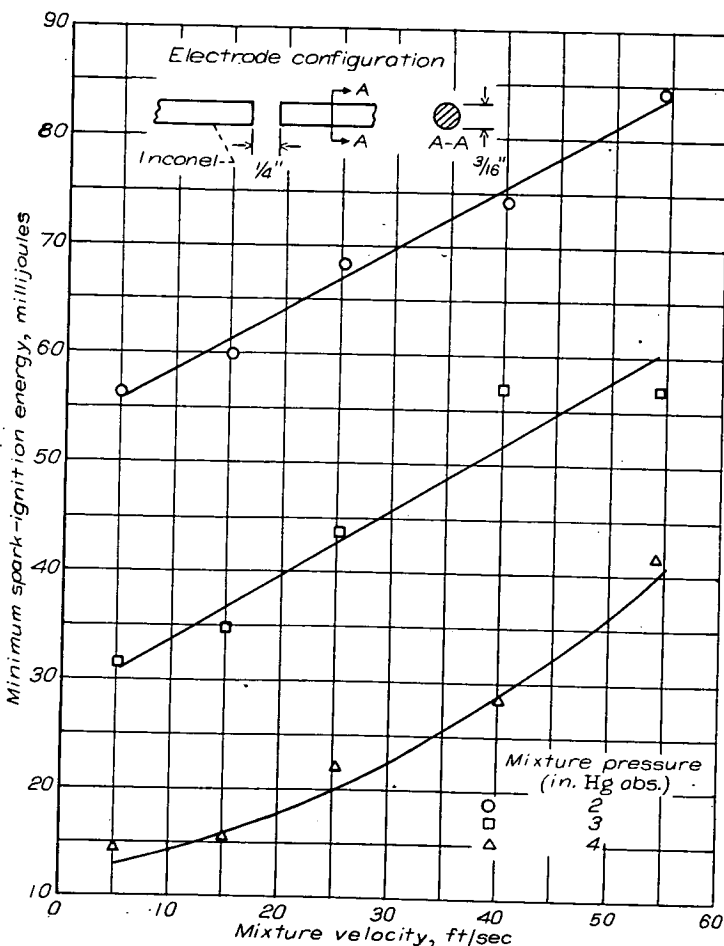


FIGURE 29.—Effect of mixture velocity and pressure on minimum spark-ignition energy of flowing 5.2 percent by volume mixture of propane in air at temperature of 80° F. Electrode spacing, 0.25 inch; capacitance-spark duration, 600 to 800 microseconds. (Data from reference 49.)

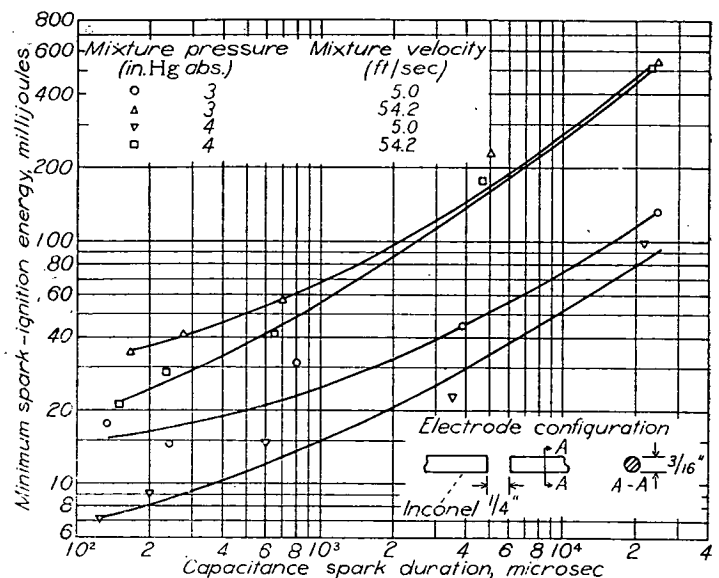


FIGURE 30.—Effect of capacitance-spark duration on minimum spark-ignition energy of flowing 5.2 percent by volume mixture of propane in air at temperature of 80° F. Electrode spacing, 0.25 inch. (Data from reference 49.)

Electrostatic sparks.—Electrostatic sparks are a special type of capacitance spark. Electrostatic charges capable of spark production can be accumulated by friction, impact, pressure, cleavage, induction, successive contact and separation of unlike surfaces, one of which is generally an insulator, and transference of inflammable fluids and gases from one container to another. Air temperatures appear to have no significant effects on the charges accumulated, but increasing humidity tends to prevent accumulation of high potential electrostatic charges. The potential of electrostatic charges accumulated by liquids flowing in tubes decreases with decreasing liquid pressures.

INDUCTANCE SPARKS

Effect of mixture composition.—The effect of mixture composition on the minimum spark-ignition energies of inductance sparks is indicated in figure 31 for two hydrocarbon-air mixtures. Because the electric circuit inductance is difficult to measure, the results are expressed as the minimum ignition current with a constant electrode spacing. The effect of mixture composition is approximately the same for inductance sparks as for capacitance sparks. Well-defined limits of inflammability exist with both types of igniting spark. Approximately 0.59 ampere primary current in the inductive circuit produces a spark capable of igniting an 8.3 percent by volume mixture of methane in air. An ignition current of approximately 0.475 ampere in the same electric circuit will produce an inductive spark capable of igniting a 4.2 percent by volume mixture of *n*-butane in air. These data were obtained at atmospheric pressure and temperature. According to available data, the ranges of inflammability determined for capacitance sparks are slightly greater than those determined for inductance spark ignition.

Effect of electrode spacing.—Because of the manner in which inductance sparks often occur, few data are available that indicate the effect of electrode spacing on the ignition of inflammable mixtures. The variation of spacing of inductance-

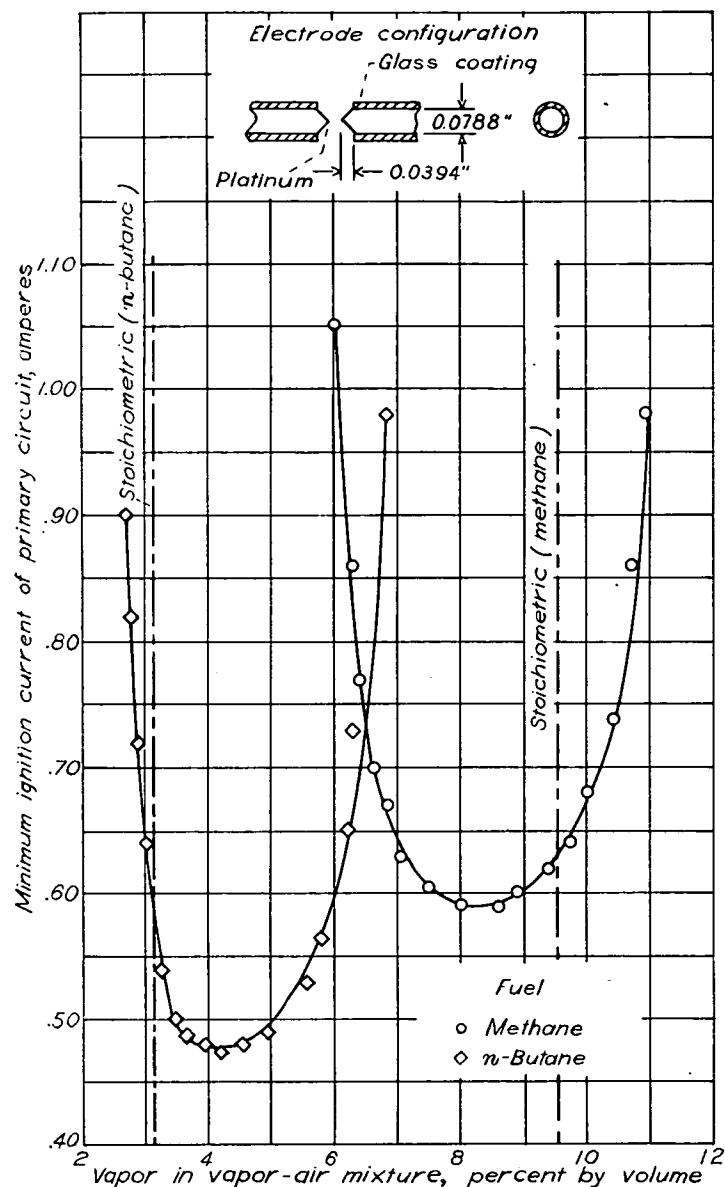


FIGURE 31.—Effect of mixture composition on minimum ignition currents of primary circuits producing inductance sparks capable of igniting methane-air and *n*-butane-air mixtures at atmospheric pressure and temperature. Electrode spacing, 0.0394 inch. (Data from reference 52.)

spark electrode configurations with mixture composition is indicated in figure 32 as being similar to the variation with capacitance-spark electrode configurations. A minimum electrode spacing of approximately 0.043 inch is indicated for an 8.5 percent by volume mixture of methane in air ignited by inductance sparks from the particular electrode configuration.

Effect of electrode material.—Unlike the effect of electrode material on capacitance-spark ignition, the minimum spark-ignition energies of inflammable mixtures ignited by inductance sparks are affected by material changes in the electrode configuration. Ethane-air and carbon-monoxide-air mixtures were ignited at atmospheric temperature and pressure using electrodes of platinum, nickel, copper, aluminum, and iron (reference 54). Decreasing density of the electrode material generally increased the ignitibility of the inflammable mixtures. The least amount of energy dissipated in an inductive spark capable of igniting an inflammable mixture occurs with spark electrodes of the lightest metal.

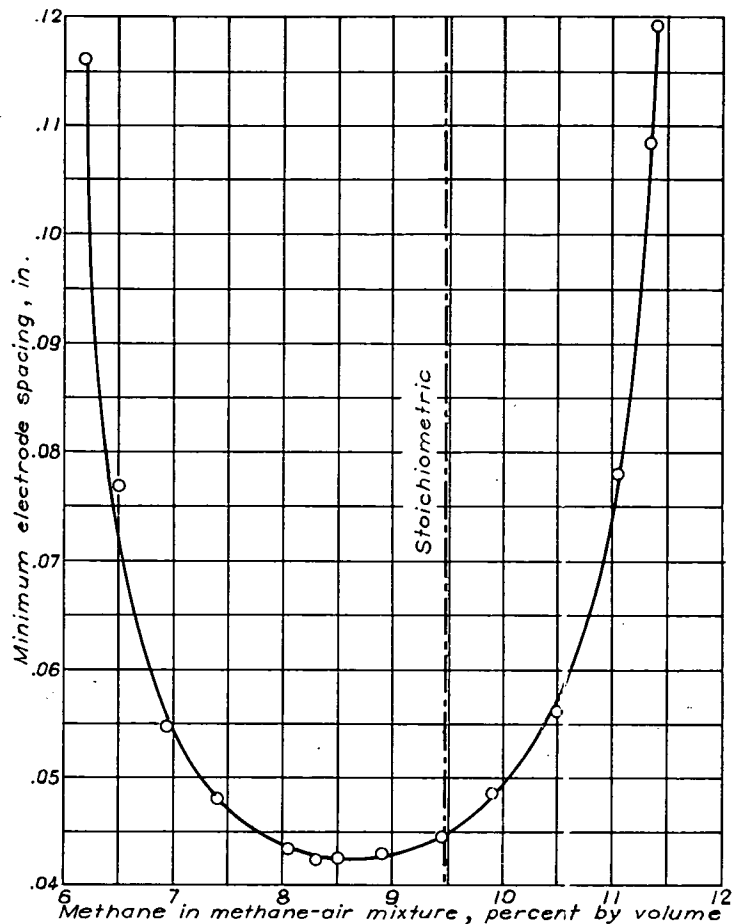


FIGURE 32.—Effect of mixture composition on minimum spacing of electrode configuration producing inductance sparks capable of igniting methane-air mixtures at atmospheric pressure and temperature. Electrode configuration, 0.394-inch-diameter platinum disk and sharply pointed platinum cone; primary circuit current, 1.0 ampere. (Data from reference 53.)

Effect of circuit inductance and potential.—The energy required for ignition of an inflammable mixture by inductance sparks depends on the inductance, the current, and the potential in the electric circuit prior to the spark. The amount of energy required to ignite the mixture is substantially constant, but larger amounts of energy may be indicated if a large portion of the measured energy is dissipated in the inductance coil as core losses. The following data from reference 55 indicate the effect inductance and current in an electric circuit have on the measured minimum spark-ignition energy required to ignite a mixture of coal gas and air at atmospheric pressure and temperature.

Number of layers in inductance-coil winding	Core of coil	Circuit inductance (henrys)	Circuit current (amperes)	Spark-ignition energy (millijoules)
14	Air	0.01	0.35	0.6
14	Straight iron bar	.07	.15	.8
14	Rectangular iron frame	.56	.09	2.3

Inductance sparks from electric circuits of low potential require more measured spark-ignition energy to ignite the same inflammable mixture than inductance sparks from similar electric circuits of higher potential, as indicated by the following data from reference 56 for inductance-spark

ignition of an 8.5 percent by volume mixture of methane in air at atmospheric temperature and pressure.

Circuit potential (volts d. c.)	Circuit inductance (henrys)	Circuit current (amperes)	Spark-ignition energy (millijoules)
4	0.02144	0.77	6.35
220	.02144	.35	1.30
4	.012	1.025	6.30
220	.012	.4	.96

The amount of energy dissipated in an inductance spark consists of two parts: the energy stored in the electric circuit prior to the spark and the continued supply of energy from the circuit potential. The amount of energy stored in the electric circuit prior to the inductance spark is the only spark-ignition energy measured. The spark-ignition energy supplied by the electric circuit potential to an inductance spark increases with increasing circuit potential. Thus, inductance sparks from electric circuits of low potential require greater measured amounts of spark-ignition energy $\frac{1}{2}Li^2$ than sparks from circuits of high potential to ignite similar inflammable mixtures.

Effect of type of current in circuit.—The type of current in the electric circuit, alternating or direct, has no significant

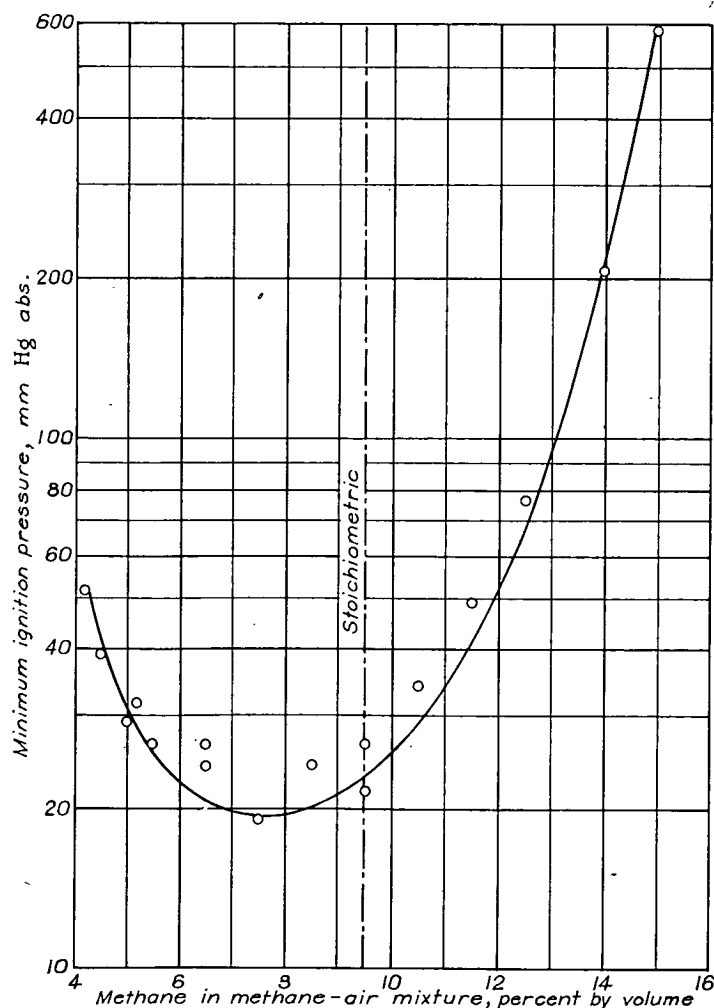


FIGURE 33.—Effect of mixture composition on minimum ignition pressures of methane-air mixtures at temperatures of 78° to 87° F. Method of ignition, bomb; electrode configuration, spark-plug; electrode spacing, 0.110 inch; capacitance-spark ignition energy, 8640 millijoules; volume of bomb, 750-cubic-centimeters. (Data from reference 58.)

effect on the ignition ability of inductance sparks. No difference in the ignition ability of alternating or direct-current inductance sparks has been found by the investigators of reference 57, who ignited fire-damp—air mixtures at atmospheric pressure and temperature. The following data from reference 56 indicate the negligible effect of current type on the incendiarity of inductance sparks igniting an 8.5-percent (by volume) methane-air mixture at atmospheric conditions:

Type of current	Current frequency (cps)	Ignition current (amperes)
Direct.....	---	0.415
Alternating.....	68	.410
Alternating.....	200	.420

EFFECT OF GAS PRESSURES AND TEMPERATURES

The effects of temperature and pressure are the same for both capacitance and inductance sparks; the following discussion therefore applies to all sparks.

Varying pressure.—The ignition of inflammable mixtures is significantly affected by varying mixture pressures and temperatures. The inflammability range of different mix-

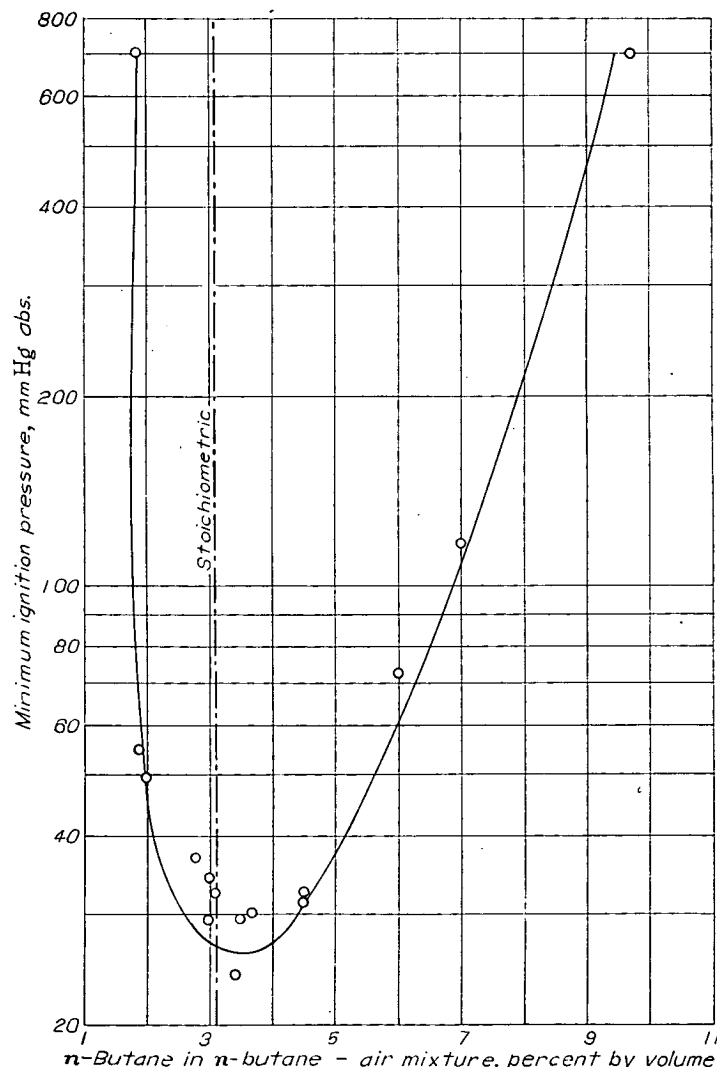


FIGURE 34.—Effect of mixture composition on minimum ignition pressures of *n*-butane-air mixtures at temperatures of 73° to 84° F. Method of ignition, bomb; electrode configuration, spark-plug; electrode spacing, 0.110 inch; capacitance-spark ignition energy, 8640 millijoules; volume of bomb, 750-cubic-centimeters. (Data from reference 58.)

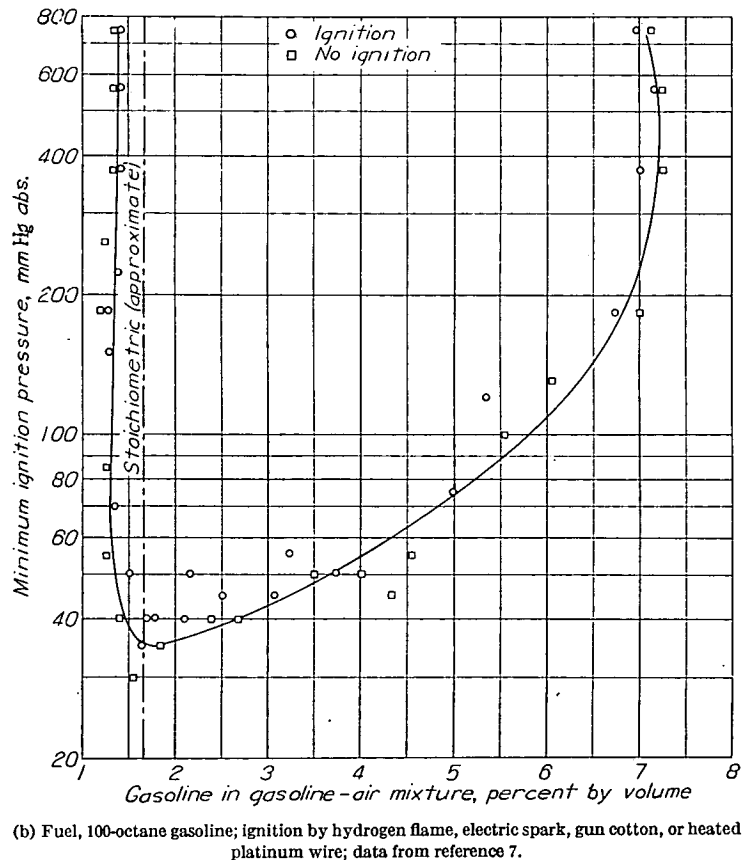
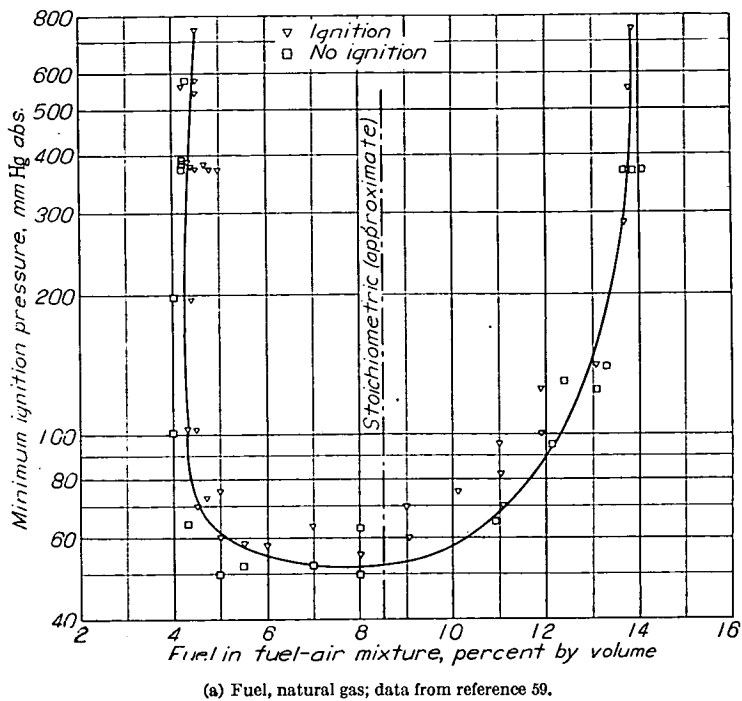


FIGURE 35.—Effect of mixture composition on minimum ignition pressures of two fuel-air mixtures.

tures is reduced by decreasing mixture temperatures and pressures. This effect of mixture pressure on the limits of inflammability of several inflammable mixtures is shown in figures 33 to 35. The minimum ignition pressure or the minimum mixture pressure of methane below which ignition

is no longer possible, exists at mixture compositions less than stoichiometric; reference 58 indicates that the same relation is true for hydrogen. The minimum ignition pressures of heavier hydrocarbons, such as propane and *n*-butane, occur at mixture compositions greater than stoichiometric. These phenomena may be due to the diffusivity of air being less than the diffusivities of methane or hydrogen, but greater than those of propane or *n*-butane. The minimum ignition pressures do not vary greatly for hydrocarbons in the range of molecular weights given in the following table taken from reference 58 which gives the minimum ignition pressures of hydrocarbon-air mixtures as determined in a 750-cubic-centimeter metal bomb (mixture temperature, 75° to 85° F; electrode spacing, 0.110 inch; capacitance-spark ignition energy, 8640 millijoules).

Hydrocarbon	Minimum ignition pressure (mm Hg abs.)
Hydrogen.....	11.5
Methane.....	19.0
Butadiene-1,3.....	21.0
Butenes-2.....	29.0
<i>n</i> -Butane and iso-butane.....	29.0
Benzene.....	28.0
2,4-Dimethyl-1,3-pentadiene.....	33.0
<i>n</i> -Nonane.....	33.0

Various investigators have found higher minimum ignition pressures for different hydrocarbons, as indicated in figure 35 (a), for natural gas. Differences in results are due to different experimental techniques. Little data expressing the effect of varying mixture pressures and temperatures on the incendiary properties of gasoline-air mixtures are available.

According to reference 60, the minimum ignition pressure of various gasoline-air mixtures ignited in vessels of 0.425, 1.09, and 125.0 cubic-foot capacity by sparks or hot platinum wires is 33 millimeters of mercury absolute at ordinary ambient temperatures. Close agreement with these results is evident in figure 35 (b). The minimum ignition pressure of mixtures of 100-octane gasoline and air ignited by four different methods at atmospheric temperatures is 35 milli-

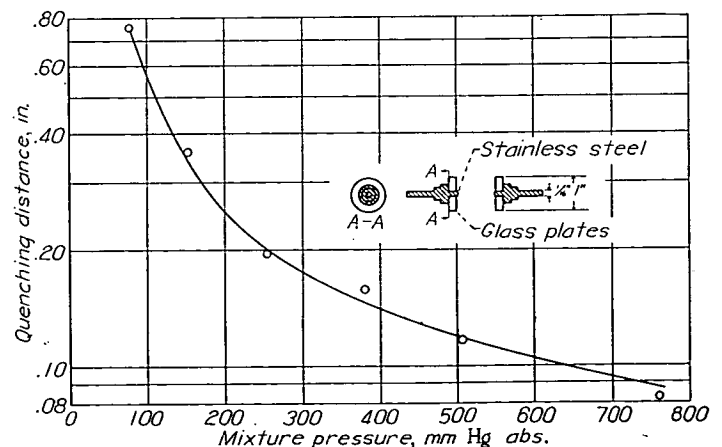


FIGURE 36.—Effect of mixture pressure of 8.5 to 9.5 percent by volume mixture of methane and air at temperature of 77° F on quenching distance of one electrode configuration. (Data from reference 50.)

meters of mercury absolute for a 1.75 percent by volume mixture of gasoline in air. The lower and upper limits of inflammability of 100-octane gasoline and air mixtures are relatively unaffected by mixture pressure changes above 60 and 300 millimeters of mercury absolute, respectively.

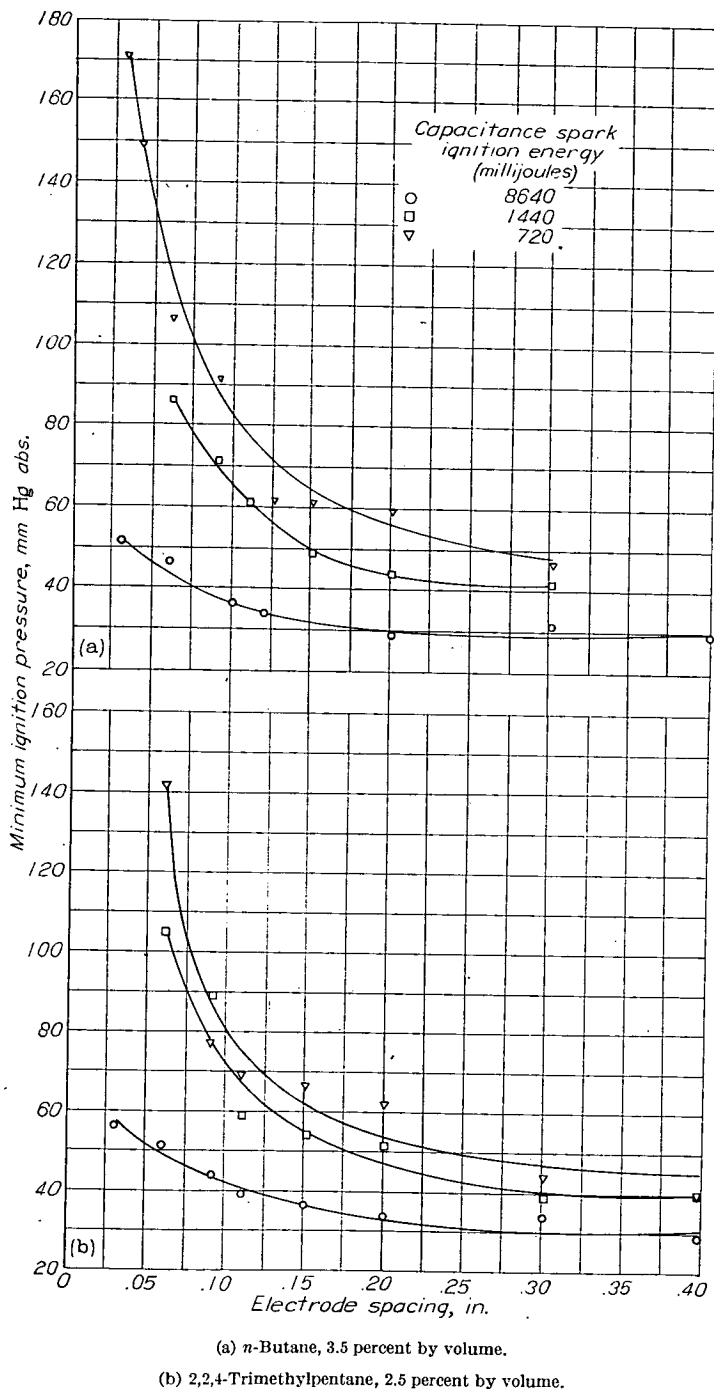


FIGURE 37.—Effect of electrode spacing on minimum ignition pressures of two fuel-air mixtures at temperatures of 70° to 75° F. Ignition by spark-plug-electrode configuration in 2-inch inside-diameter by 3½-inch glass reactor. (Data from reference 58.)

The effect of pressure on the quenching distances or electrode spacings of various electrode configurations producing incendiary sparks is shown in figures 36 and 37. For constant spark-ignition energies, the quenching distances, or electrode spacings, of incendiary-spark-producing electrode configurations increase with decreasing inflammable mixture pressures. This phenomenon is explained by assuming that a constant spark-ignition energy can initially ignite a constant mass of inflammable mixture. Thus, with decreasing mixture pressure, ignition of a larger volume of inflammable mixture occurs, and the electrode spacing must be increased to accommodate the increased mixture volume.

Variation of minimum spark-ignition energy with mixture pressure is expressed in figures 38 to 40. For constant electrode spacing, the minimum spark-ignition energy of a mixture increases with decreasing mixture pressure. The minimum spark-ignition energies of inflammable mixtures increase rapidly near the minimum ignition pressures of the mixtures (fig. 40).

Varying temperature.—Varying the temperature of inflammable mixtures from -50° to 300° F has a small effect on the minimum ignition pressures of the mixtures when ignition is initiated by high-energy capacitance sparks. As indicated in figure 41, the minimum ignition pressures of *n*-butane-air and isooctane-air mixtures decrease linearly with increasing mixture temperatures when ignition is effected by capacitance sparks having spark-ignition energies from 720 to 8640 millijoules. The minimum ignition pressures of *n*-butane-air and isooctane-air mixtures are approximately the same regardless of the spark-ignition energies at mixture temperatures of 300° F.

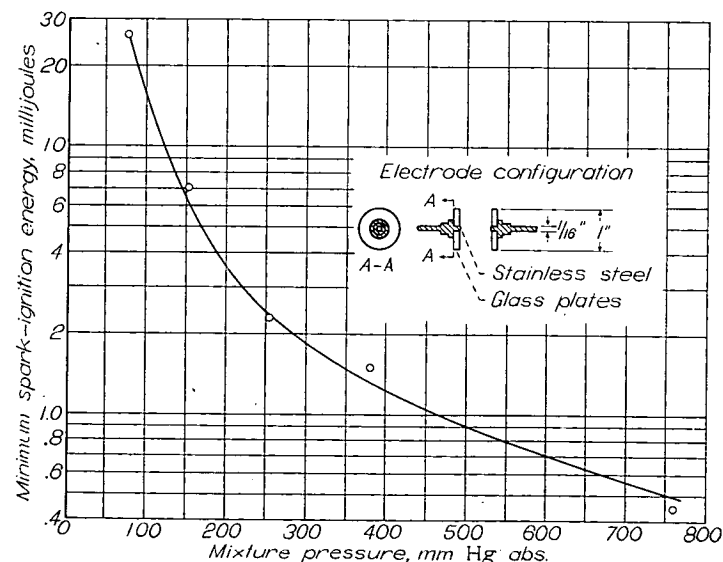


FIGURE 38.—Effect of pressure of 8.5 to 9.5 percent by volume mixture of methane and air at temperature of 77° F on minimum spark-ignition energy for one electrode configuration. (Data from reference 50.)

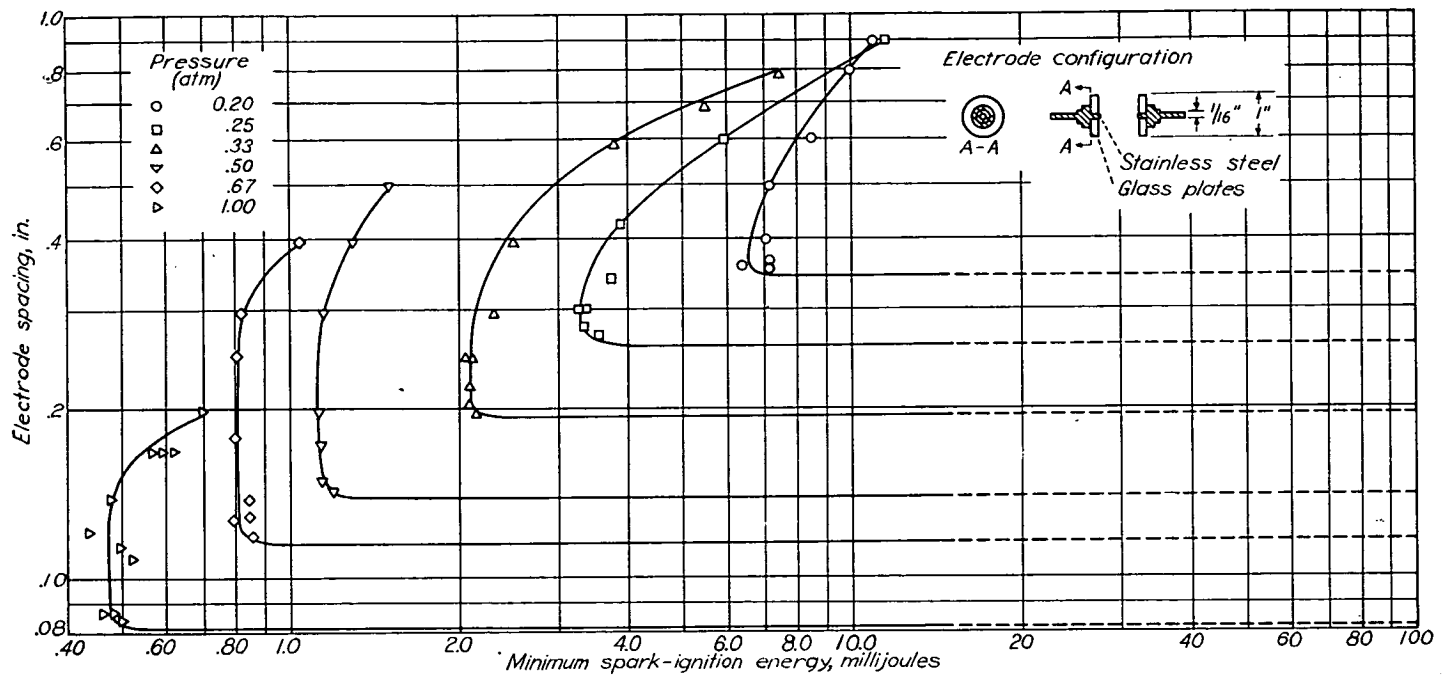


FIGURE 39.—Minimum spark-ignition energy of glass-flanged electrode configurations as functions of electrode spacing and pressure of 9.5 percent by volume mixture of methane and air at temperature of 77° F. (Data from reference 50.)

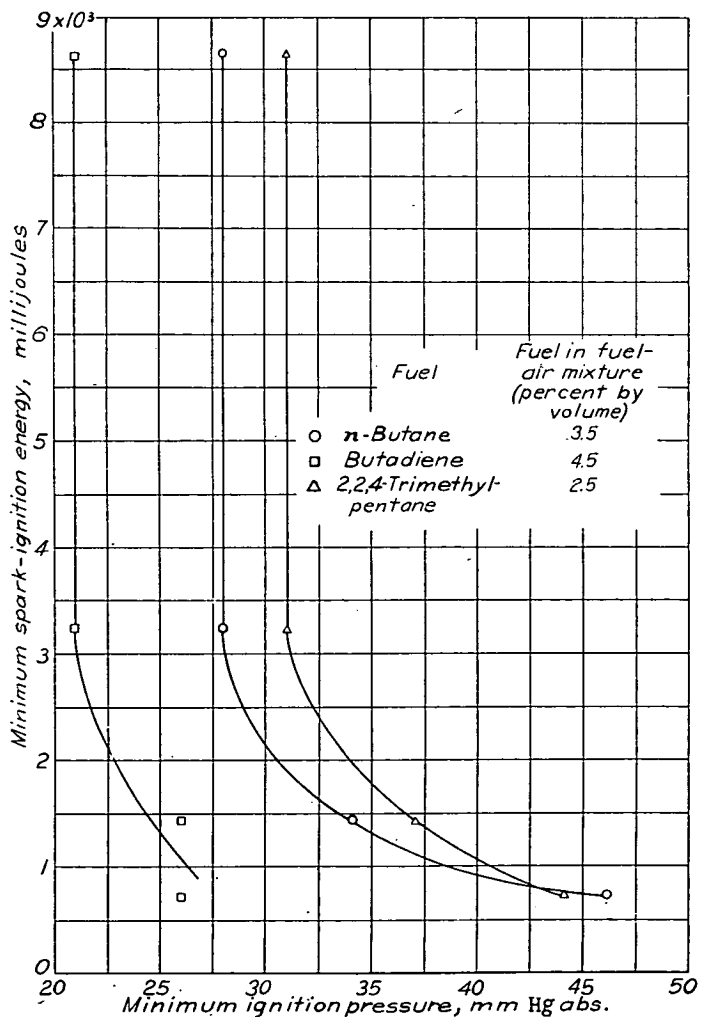


FIGURE 40.—Effect of mixture pressures on minimum spark-ignition energy of one electrode configuration. Electrode spacing, 0.110 inch; capacitance-spark voltage, 600 volts. All mixtures slightly richer than stoichiometric. (Data from reference 58.)

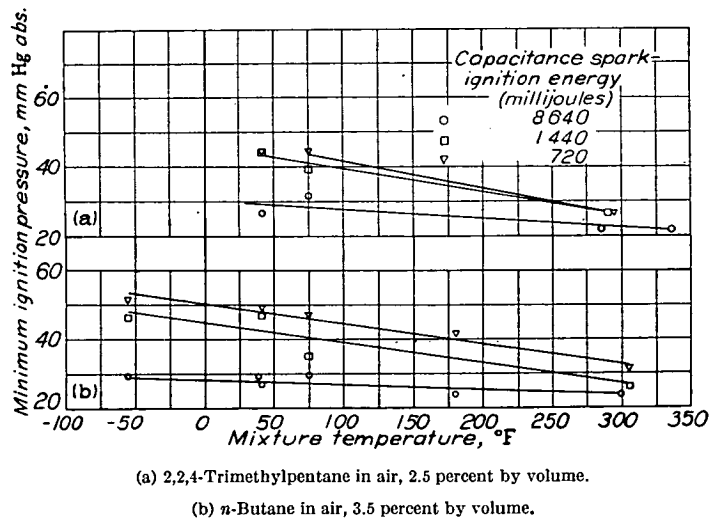


FIGURE 41.—Effect of mixture temperature on minimum spark-ignition pressures of 2,2,4-trimethylpentane and *n*-butane-air mixtures. Method of ignition, bomb; electrode configuration, spark-plug; electrode spacing, 0.110 inch; volume of bomb, 750-cubic-centimeters. (Data from reference 58.)

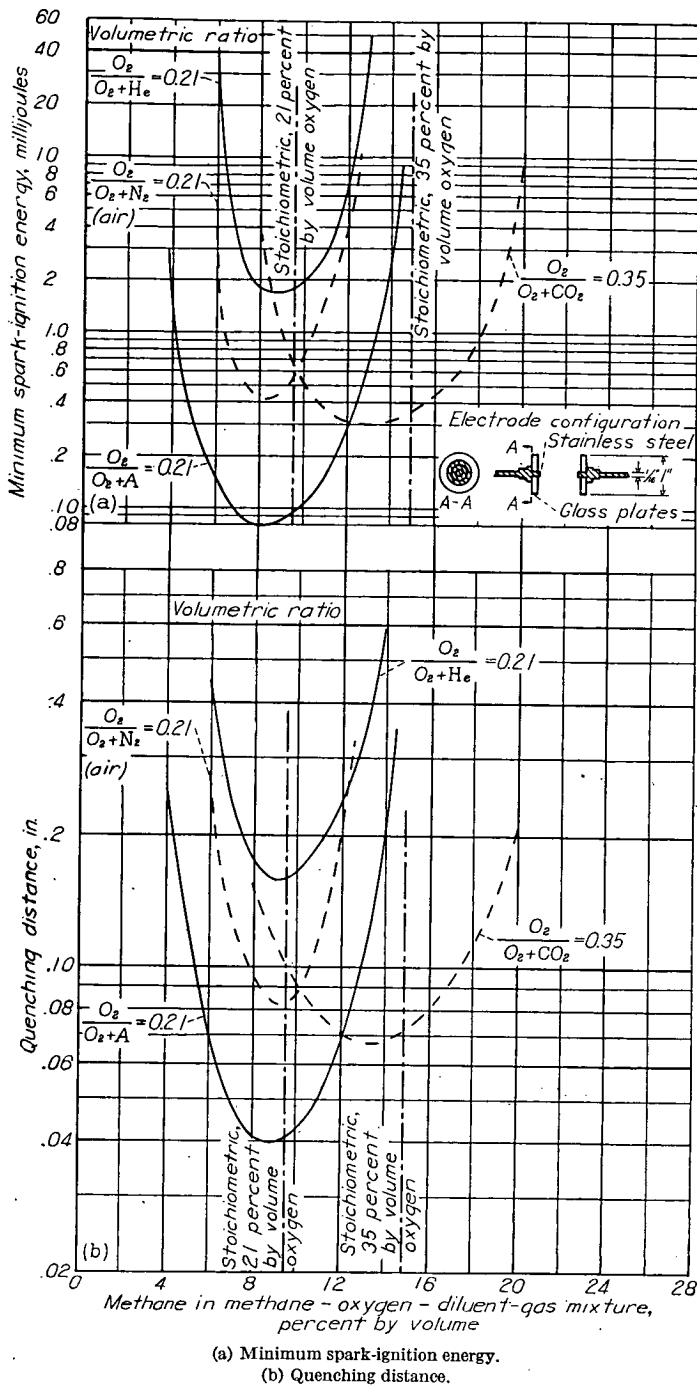


FIGURE 42.—Effect of diluent-gas addition on minimum spark-ignition energy and quenching distance of methane-oxygen-diluent-gas mixtures at temperature of 77° F and pressure of 1 atmosphere. (Data from reference 50.)

EFFECT OF DILUENTS

The presence of diluent gases in an inflammable mixture exerts a significant influence on its electric spark ignition. Like heated surface ignition, the spark-ignition energy required to ignite inflammable mixtures containing large percentages of diluents depends on the absorptive and conductive powers of the diluents and the proportion of inflammable gas to oxygen. Figure 42 indicates the effect of diluents on the minimum spark-ignition energies and quenching distances of methane-oxygen-diluent-gas mixtures. In figures 42 (a) and 42 (b), the relative effect of the diluent gases is the same. With the exception of the methane-oxygen-argon mixtures, the minimum ignition energies and quenching distances increase with increasing diffusivity of the oxygen-

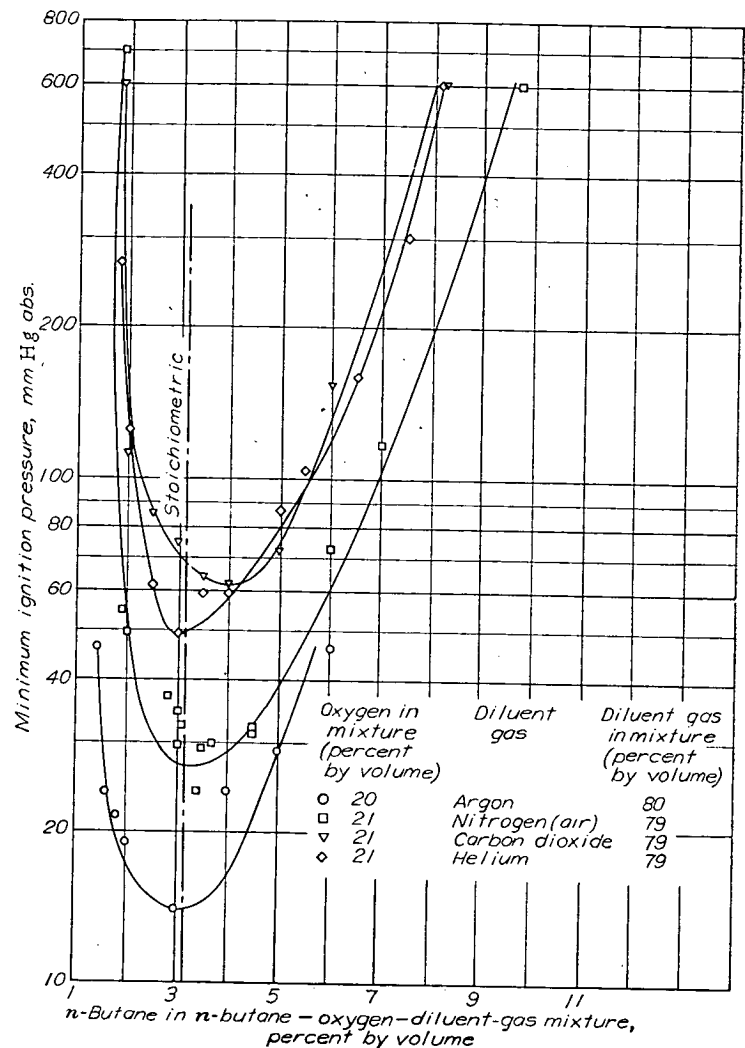


FIGURE 43.—Effect of diluent-gas addition on minimum ignition pressures of n-butane-oxygen-diluent-gas mixtures. Method of ignition, bomb; electrode spacing, 0.110 inch; capacitance-spark ignition energy, 8640 millijoules; volume of metal bomb, 750 cubic centimeters; mixture temperature, 74° to 82° F. (Data from reference 58.)

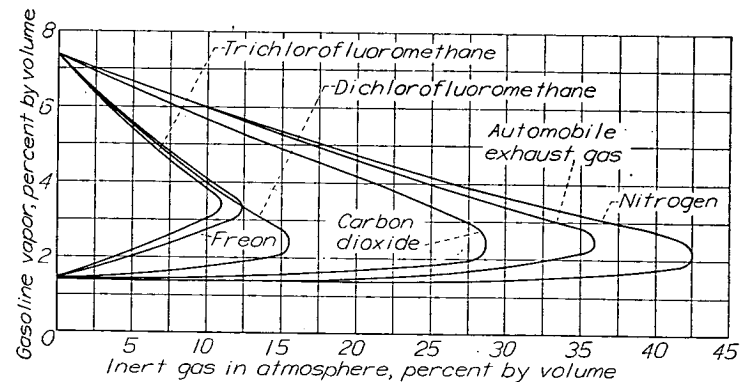


FIGURE 44.—Limits of inflammability of gasoline vapor in various air-diluent-gas atmospheres. Automobile-exhaust-gas composition: nitrogen, 85 percent by volume; carbon dioxide, 15 percent by volume. Gasolines tested: 73, 92, and 100 octane. Inflammable area covers entire area inside largest curve. (Data from reference 61.)

diluent-gas mixtures. This mixture diffusivity is expressed by

$$\alpha = \frac{k}{\rho c_p} \quad (6)$$

where

c_p mixture specific heat

k mixture thermal conductivity

α mixture thermal diffusivity

ρ mixture density

The minimum ignition pressures of mixtures of *n*-butane and oxygen, with argon, nitrogen, carbon dioxide, and helium as diluents are shown in figure 43. For all the inflammable mixtures containing diluents, the mixtures most and least ignitable are those containing argon and carbon dioxide, respectively. The effect of several diluents on the limits of inflammability of individual mixtures of 73-, 92-, and 100-octane gasoline and air is shown to be identical in figure 44. The lower limit of inflammability of the mixture is relatively unaffected, whereas the upper limit decreases approximately linearly with increasing amounts of diluents such as carbon dioxide, automobile exhaust gas, and nitrogen. Mixtures of 100-octane gasoline and air are rendered noninflammable by the addition of approximately 28.7, 36.0, and 42.5 percent by volume of carbon dioxide, automobile exhaust gas, or nitrogen, respectively. Halogenated hydrocarbons are more effective than any of these diluents in rendering gasoline-air mixtures noninflammable.

III—IGNITION BY FLAMES OR HOT GASES

Data pertaining to the ignition of inflammable mixtures by flames or hot gases are not extensive. Much of the available data pertain to the ignition of gas-air mixtures containing methane as the chief inflammable constituent. Whether the inflammable mixtures will or will not be ignited by flames or hot gases depends on mixture composition, duration of contact of the mixture with the ignition source, temperature of the ignition source, and size of the ignition source, in much the same manner as ignition by heated surfaces.

EFFECT OF VARYING MIXTURE COMPOSITION

The effect of mixture composition on the limiting diameters of openings for downward propagation of flames in methane-air mixtures is shown in figure 45. Limiting diameters occur with approximately stoichiometric compositions. The ignition of an inflammable mixture by a flame can be considered practically instantaneous. Methane-air mixtures exposed to flames 0.394, 0.492, and 0.591 inch in length exhibited ignition lags of 0.0068, 0.0042, and 0.0035 second, respectively; therefore, even though the ignition lag is practically zero for flame ignition, the lag decreases with increasing size of the ignition source. Minimum ignition lags have been found to occur with approximately stoichiometric mixtures.

LIMITING SIZE OF OPENINGS FOR FLAME PROPAGATION

The minimum quenching distances or the limiting diameters of tubes or holes in thin plates through which a flame will not propagate are expressed in figures 45 to 47. Minimum limiting tube and thin plate opening diameters of approximately 0.071 and 0.136 to 0.150 inch occurred for approximately stoichiometric mixtures of coal gas and methane with air, respectively. These limiting diameters for the downward propagation of flame in inflammable mixtures were determined for copper tubes having length-diameter ratios l/d of 10, and for circular openings in thin copper foil and mica plates having thicknesses of 0.0033 inch and 0.0024 to 0.0087 inch, respectively.

Some references (for example, references 62 and 67) indicate that the quenching of the flame is due solely to the

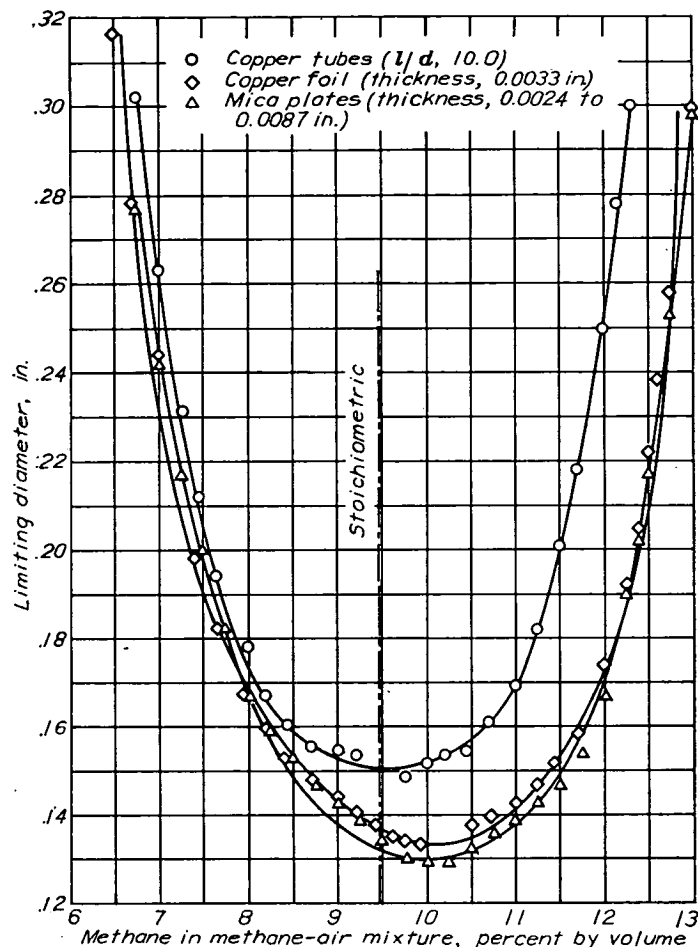


FIGURE 45.—Effect of mixture composition on limiting diameters of openings in copper and mica plates and copper tubes for downward flame propagation in methane-air mixtures at atmospheric pressure. (Data from reference 62.)

cooling effect of the unburned gases rather than a cooling effect due to varying thermal conductivity of the confining material. Data from reference 68 indicate that in addition to this effect the confining material and opening configuration may affect the quenching distance by varying the size of the dead space between the surface of the flame and the opening of the enclosure. The reduced inflammability range and the increased limiting diameter of opening for downward flame propagation determined with copper tubes instead of openings in copper foil may be due to an increase in the size of the dead space caused by the greater over-all cooling effect of the copper tubes. Investigations conducted with the glass tubes and the mica plates (references 62 and 63) gave approximately the same results as experiments in which copper tubes and plates were used.

No data were found in the survey of the literature to indicate the limiting sizes of openings for propagation of flames of gasoline-air mixtures. Reference 65, however, indicates that the quenching of laminar oxyhydrogen flames by solid surfaces is partly dependent on the factor k/SC , where

C heat capacity per unit volume of unburned mixture at quenching temperature

k thermal conductivity of unburned mixture at quenching temperature

S burning velocity of mixture

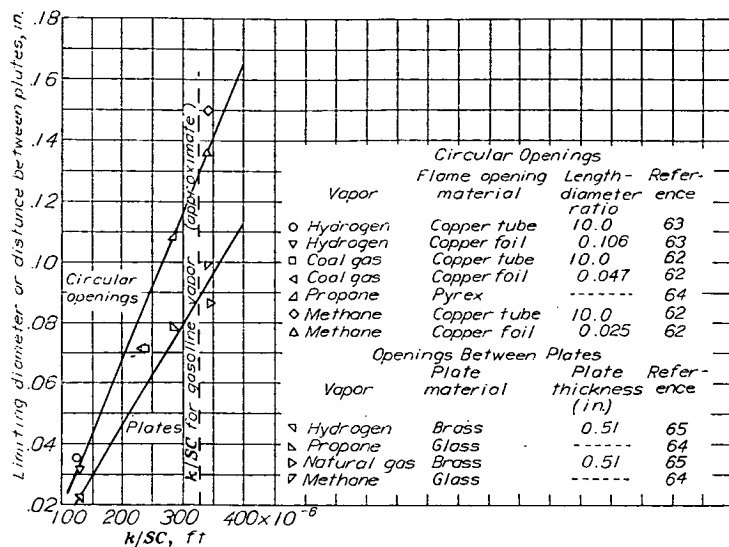


FIGURE 46.—Limiting diameters of circular openings and limiting distances between plates for downward propagation of flame in various vapor-air mixtures at atmospheric pressure.

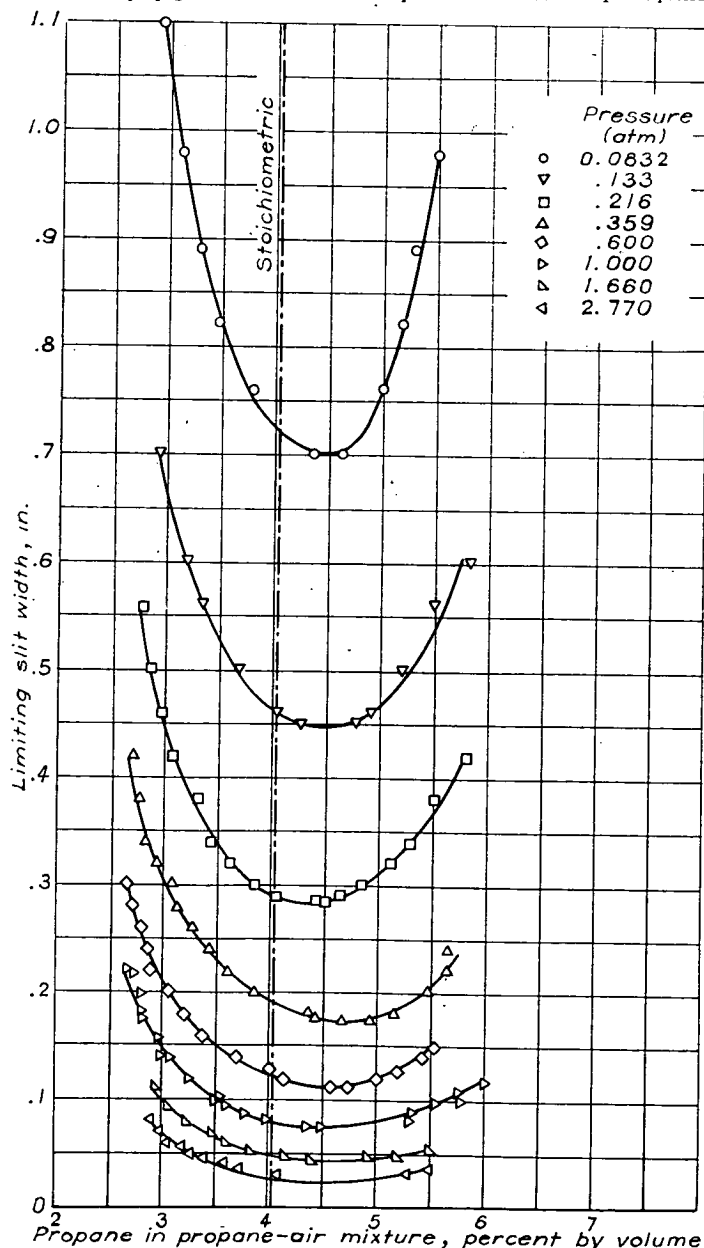


FIGURE 47.—Quenching of propane-air flames at various mixture pressures and temperature of 75° F; downward flame propagation. (Data from reference 66.)

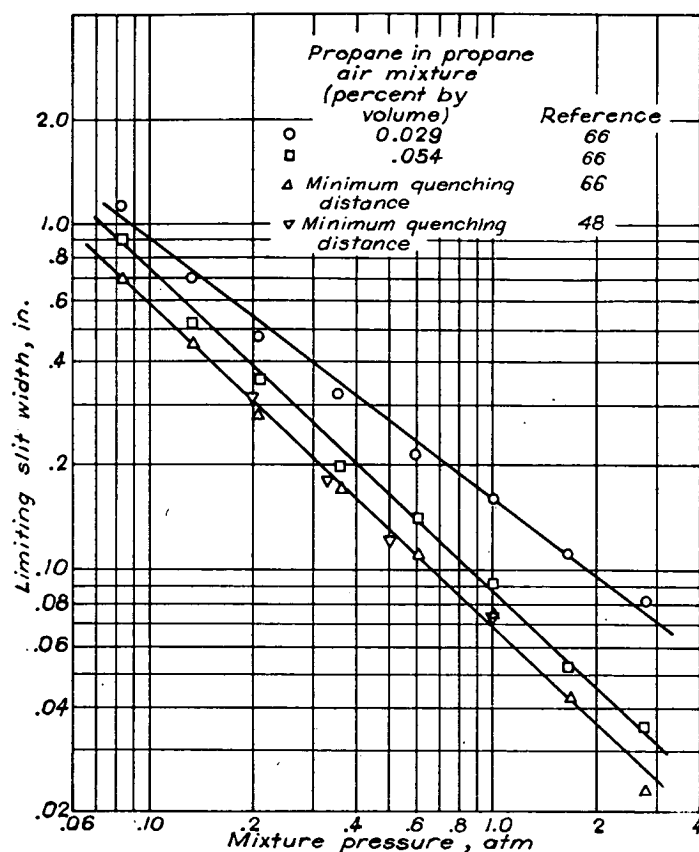


FIGURE 48.—Quenching of propane-air flames at various mixture pressures and temperature of 75° F. Downward-flame-propagation data, reference 66; spark-ignition data, reference 48.

Figure 46 was plotted, using values of k/SC calculated as indicated by reference 65. Experimental data, which pertain to the limiting size of openings for flame propagation in gasoline-air mixtures, are unavailable in the literature. Accordingly, k/SC was calculated for a stoichiometric n -octane-air mixture, which was assumed to be approximately the same as a stoichiometric gasoline-air mixture. This calculated value of k/SC indicates (fig. 46) that the limiting diameter of circular openings and the limiting distance between parallel plates for downward flame propagation in gasoline-air mixtures are approximately 0.13 and 0.09 inch, respectively.

EFFECT OF GAS PRESSURES AND TEMPERATURES

The effect of mixture pressure on the quenching of propane-air flames is indicated in figure 47. The configuration used consisted of a rectangular opening in $\frac{3}{16}$ -inch copper plates. The length-width ratio of the slit was always greater than 3.6. For all mixture pressures from 0.0832 to 2.77 atmospheres, the minimum limiting slit width for downward propagation of flame occurs with mixtures slightly richer than stoichiometric. As shown by figure 48, for any single mixture composition, the dependency of the limiting slit width on the mixture pressure is expressed approximately

by an equation of the form

$$w = Kp^n \quad (7)$$

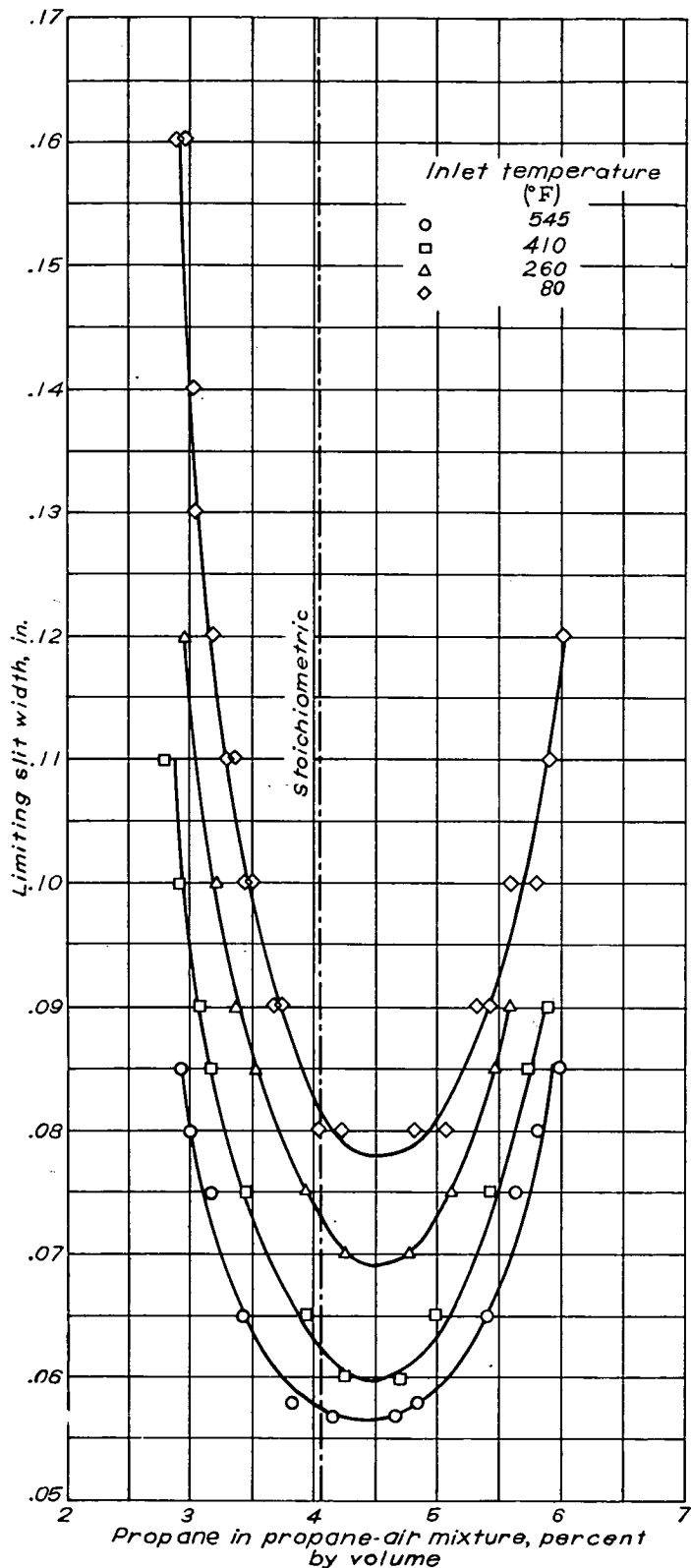


FIGURE 49.—Quenching of propane-air flames at various mixture temperatures (gas and plates at same temperature). Pressure, 14.3 pounds per square inch absolute; slit length-width ratio, > 3.6; slit jaws, $\frac{3}{16}$ -inch copper plates; downward flame propagation. (Data from reference 66.)

where

K constant

n negative exponent

p mixture pressure

w slit width

The results of investigations conducted with propane-air flames at atmospheric pressure to determine the effect of mixture temperature on the limiting slit width of the quenching configuration just described are presented in figure 49. The mixture and the plates forming the rectangular opening were at the same temperature. For all temperatures from 80° to 545° F, the minimum limiting slit width occurs with mixtures slightly richer than stoichiometric. The limiting slit width decreases with increasing mixture temperature. Similar data for brass-plate protective devices in natural-

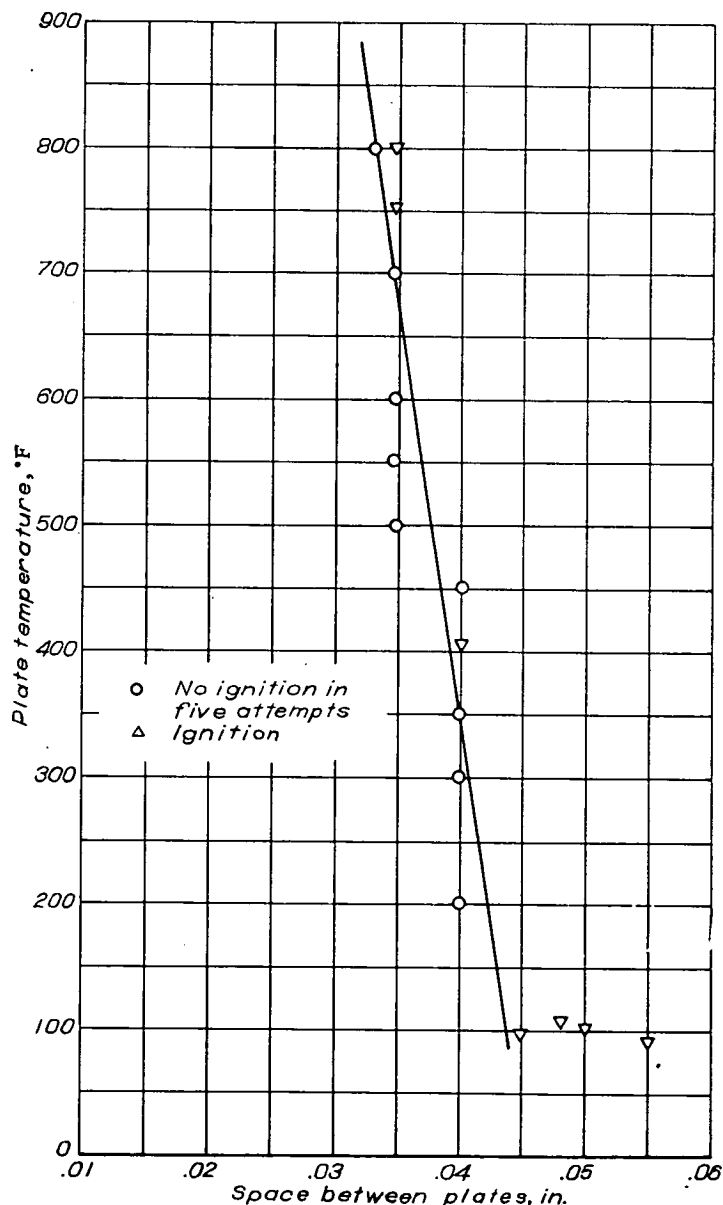


FIGURE 50.—Effect of plate temperature on performance of brass-plate protective device (plates, 0.09 in. by 4 in.) in inflammable atmospheres containing 8.6 to 9.5 percent by volume natural gas in air. (Data from reference 69.)

gas-air atmospheres are given in figure 50. The limiting distance between plates for a single mixture decreases linearly with increasing plate temperature.

EFFECT OF DILUENTS

The effect of diluents on the limiting diameter of openings for propagation of flames in methane-oxygen-nitrogen (air) and methane-oxygen-argon mixtures is shown in figure 51. The minimum limiting diameters of copper tubes (length-diameter ratio=10.0) are 0.150 and 0.087 inch, respectively, for flame propagation in the methane-air and methane-oxygen-argon mixtures. The cooling effect of the unburned gases is indicated by the smaller limiting diameters of openings for flame propagation in mixtures having the smaller heat capacities and thermal conductivities.

IV—RELATION OF PUBLISHED DATA TO AIRCRAFT-FIRE PROBLEMS

Fires in aircraft usually result from the ignition of inflammable vapors by heated surfaces, hot (electric) sparks and arcs, and flames or hot gases. The source of ignition can often be definitely determined for fires during flight or ground operation of aircraft, but the ignition source in crashes is often unknown, because the inflammables may sometimes be exposed to all the ignition sources simultaneously.

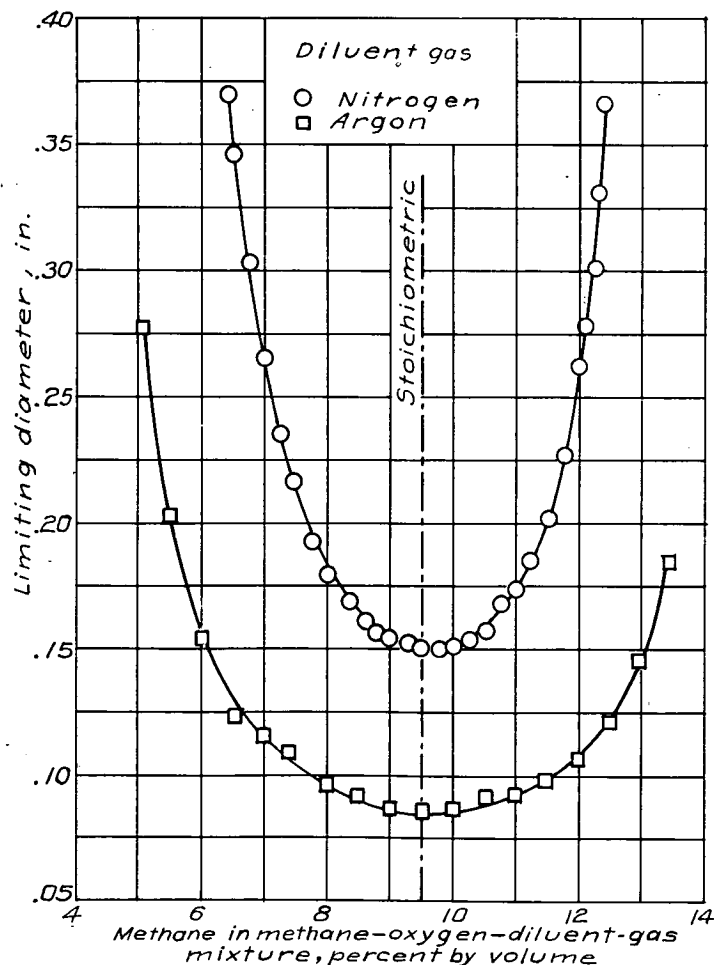


FIGURE 51.—Effect of diluent-gas additions on limiting diameters of openings for flame propagation in methane-oxygen-diluent-gas mixtures. Downward flame propagation through copper tubes; length-diameter ratio, 10.0; ratio of oxygen to diluent, 21:79. (Data from reference 63).

IGNITION BY HEATED SURFACES

Heated surfaces are common ignition hazards in aircraft environments. According to reference 70, hot exhaust ducts, combustion heaters, carburetor-air heaters, overheated cabin superchargers, moving parts overheated by friction, and short circuited or malfunctioning electric equipment have been responsible for the heated surface ignition of fires in airline aircraft in the decade prior to 1947. Of these surface-ignition hazards, the exhaust system is a continuous hazard during operation of the aircraft. The other ignition sources exist intermittently, generally because of malfunctioning or mechanical failure of the particular component.

A survey of air-transport crash records (reference 71) indicates that the inflammable most frequently initially involved in flight or ground fires and thus considered most hazardous of the liquids carried in aircraft is gasoline. Lubricating oil is not quite as hazardous as gasoline because of its much higher flash point and much lower volatility. Half the fires mentioned in the survey of flight fires involved either gasoline or lubricating oil as the initial inflammable.

In order to ignite any inflammable by a heated surface, the right proportions of the inflammable must contact the heated element. The temperature at which ignition occurs depends on the length of time the inflammable is in contact with the heated air or surface. According to the aforementioned survey, this contact is brought about during ground operation mainly by failure of the fuel plumbing system. Typical of fuel-plumbing-system failures are primer line failures, primer leaks, leaky hose connections, fuel pump leakage, and failure of the carburetor vent line. Serious accessory section fires in radial engines are often due to this type of failure, which allows inflammable vapors or fluids to escape and contact or drop on the hot exhaust stacks.

Flight fires are due primarily to engine failures, which usually are the result of structural faults of various components. Engine failures, especially in radial engines, often result in a rupture of the engine induction and exhaust systems. This type of failure can easily be brought about by engine cylinders displaced by broken connecting rods. When

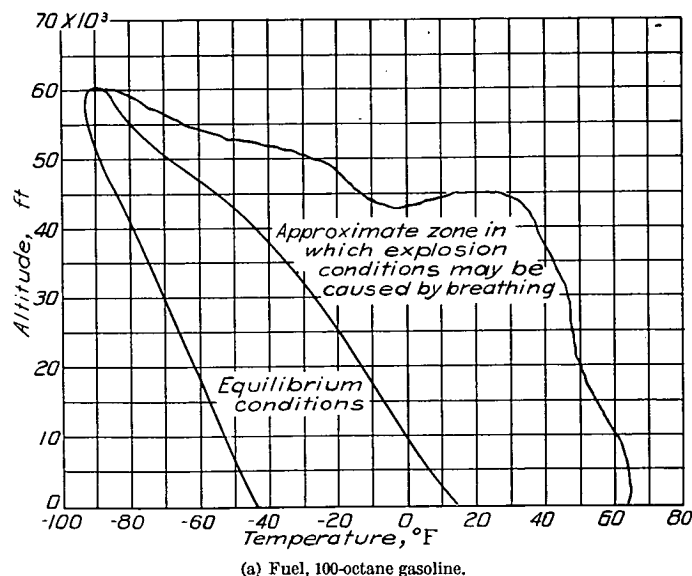


FIGURE 52.—Zones of inflammability of fuel in aircraft fuel tanks. (Data from reference 72.)

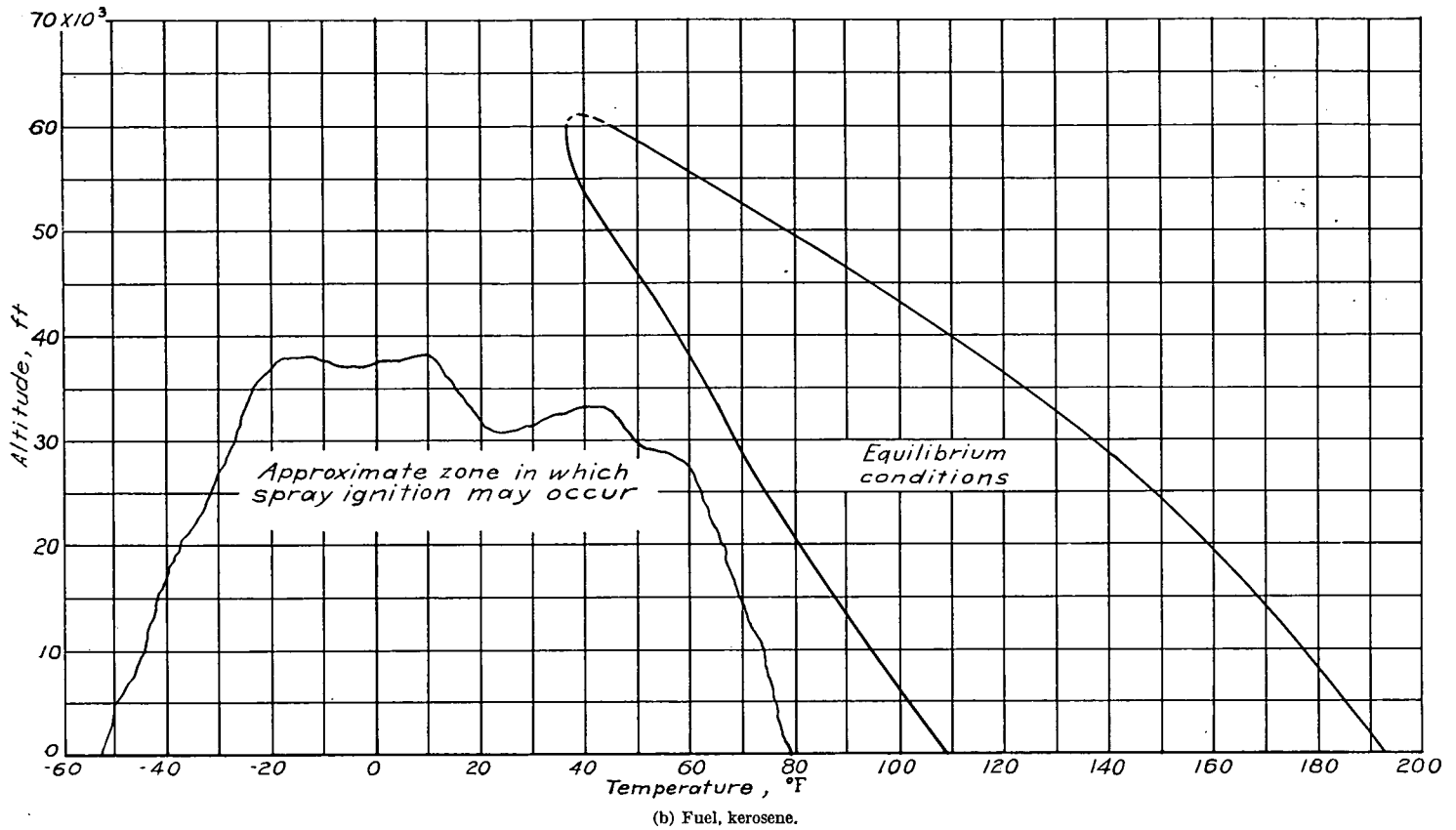


FIGURE 52.—Zones of inflammability of fuel in aircraft fuel tanks. (Data from reference 72.)

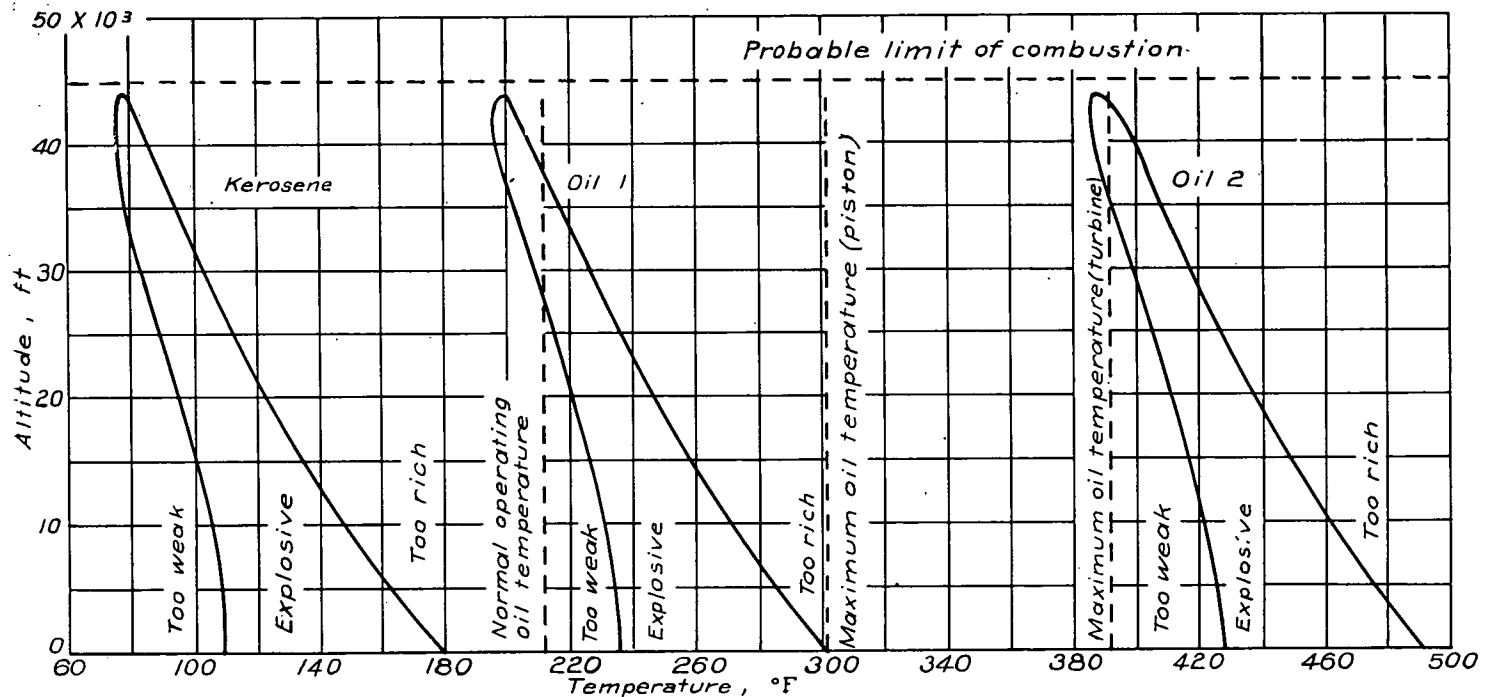


FIGURE 53.—Variation of inflammability of kerosene and two engine oils of similar viscosity but different composition with altitude and temperature. (Data from reference 72.)

such a displacement occurs, inflammable fuel-air mixtures and hot lubricating oil are exposed to hot surfaces, which are primarily components of the exhaust system. Usually, it is difficult to state whether inflammation results from ignition of the gasoline or the lubricating oil. Crash fires are generally due to structural failure of the engine, but are much more severe than flight fires in most cases.

In the range of its ignition temperatures (450° to 1325° F), gasoline can exist only in the vapor state. Any mixture

capable of flame propagation must lie within the limits of inflammability. According to figure 52, aircraft fuel-tank minimum temperatures of -44° and 108° F are explosively dangerous at sea level for 100-octane aviation gasoline and kerosene, respectively. The area marked "equilibrium conditions" indicates the limiting conditions at which inflammable mixtures exist in the fuel tank in equilibrium with the fuel. Breathing, which widens the limiting conditions for gasoline, is the influx of air into a fuel tank during a descent

due to increasing atmospheric pressure. For kerosene, which has a flash point generally greater than ambient temperatures, the ignition zone is widened by conditions under which an ignitable spray may be formed. Research carried out since the publication of figure 52 (b) has shown that the zone in which spray ignition of kerosene may occur is known to be larger than reference 72 indicates. A fuel mist or spray may be formed by the rupture of a fuel tank by a missile or the rupture of a high pressure fuel line such as are present in turbojet engines. Similar data are plotted in figure 53 for kerosene and for two lubricating oils of different composition but of the same viscosity.

In any occurrence of these fire hazards in aircraft, the ignition temperature and ignitibility of the fuel or lubricating oil will be determined by conditions described in the detailed discussion presented herein. The values of ignition temperature and ignitibility defined by these conditions serve as approximate indications of the relative temperatures at which heated surfaces become practical hazards in aircraft environments. Reference 12 indicates that aviation gasoline will not ignite in air when exposed to the outer surfaces of exhaust pipes at temperatures below 850° F. Injected into air inside hot exhaust pipes, aviation gasoline ignited at duct temperatures as low as 536° F. Lubricating oil on the outer surfaces of exhaust ducts ignited at a minimum duct temperature of 625° F. Thus, for complete immunity to fires ignited by the components of the exhaust system, the surface temperatures should not be greater than approximately 500° F. Although existing knowledge does not permit a complete description of the components of the exhaust system that serve as ignition sources, it is presumed in this discussion that the exhaust duct, the exhaust valves, and the cylinder interiors are the active components.

Volatile fuels necessary for engine operation under a variety of climatic conditions may be responsible for the relative ignition hazard of present-day grades of gasoline used in aircraft operation. As the volatility of a fuel increases, the range in which ignition sources are hazardous increases. Fuels of low volatility would result in more difficult starting of aircraft engines, but would seemingly reduce the ignition hazard of fuel in aircraft environments.

Reduction of the fire hazard of the exhaust system can be accomplished by:

- (1) Keeping the inflammables from the hot surfaces
- (2) Releasing some substance into and around the exhaust system in order to prevent ignition if a crash is imminent
- (3) Cooling the exhaust system below temperatures capable of ignition of gasoline or oil
- (4) Increasing the minimum ignition temperature of the fuel or oil by changing the fuel or oil composition or by change of material used in the construction of the exhaust ducts
- (5) Changing the volatility of the fuel.

An extinguishing system capable of spraying or flooding the exhaust ducts would aid in preventing combustion by simultaneously blanketing and lowering the temperature of the heated surfaces. According to reference 73, "it is most important that the extinguishant should be actually injected into enclosed hot spaces such as exhaust ducts, long exhaust

pipes or collector rings, turbine compartments and tailpipe ducts, and combination heater chambers."

The ignition temperature of gasoline is increased by the addition of small amounts of tetraethyl lead, benzene, and similar substances. Possibly other substances can be found that will increase the ignition temperature without reducing the performance of the fuel in the engine.

Changes in the surface composition of the exhaust-duct material may effectively increase the ignition temperature of the fuel. Dependent upon the mixture composition, highly catalytic surfaces such as platinum must be heated to temperatures 300° to 700° F greater than stainless steel to ignite inflammable mixtures. Use of platinum is obviated by its cost and therefore the aforementioned comments are only of academic interest.

Fuel-volatility changes apparently are desirable relative to the fire-hazard problem. According to tables III and IV, the surface-ignition temperatures of low-volatility safety fuels do not vary greatly from those of regular grades of aviation gasoline. Reduced fuel volatility would not increase the ignition temperature of the fuel, but would reduce the volume in which an ignition source is hazardous.

IGNITION BY ELECTRIC SPARKS AND ARCS

Sparks resulting from short circuits in wiring or from failures of starters, magnetos, or generators; sparks from the continued functioning of nonflameproofed electric apparatus; sparks from the continued rotation of damaged parts; discharges from the brushes of rotating equipment; sparks from ground friction; electrostatic sparks; and electric arcs created by the separation of metallic contacting portions of the electric circuit such as voltage regulators may all be regarded as potential spark-ignition hazards in aircraft.

Flight fires in transport aircraft are initiated with equal frequency by either electric sparks or the exhaust ducts (reference 71). The damage incurred by the electric spark-ignited fires, however, was generally confined to the electric insulation. In addition, reference 74 shows that self-clearing arcs are statistically more prevalent during the breaking of a load-carrying circuit. The clearing time was from 0.05 to 2.5 seconds, with a median value of 0.6 second. Compared with the exhaust system, electric sparks are relatively short, minor ignition hazards during flight operation.

High-energy concentrations at short-circuited points result in self-clearing faults, but are generally accompanied by a scattering of molten metal globules over a considerable area. Low-energy concentrations at short-circuited points may result in a permanent welded contact between the conductor and the airplane structure. Self-clearing faults are probably more dangerous than welded faults because they are accompanied by extremely high temperatures and considerable arcing. Welded faults, however, may disrupt the entire electric system and thus interfere with control of the airplane.

Continued rotation of generators is generally not significant in propeller-type aircraft in the event of crash. In jet-type aircraft, free running turbines may, however, continue rotation of such parts.

According to the survey (reference 71) of fires in transport aircraft, two of the 61 crash fires reported were initiated by

sliding friction. Inasmuch as data indicate that friction sparks ordinarily lack the thermal energy capable of igniting inflammable mixtures, the source of ignition of these two crash fires may have been surfaces heated to high temperatures by sliding friction rather than by friction sparks.

Electrostatic sparks generated in aircraft environments may be ignition hazards. Electrostatic charges having potentials of 50 to 2000 volts have been accumulated in air of 30- to 51-percent humidity at atmospheric pressure and temperature by benzene flowing through a 0.12-inch tube into an insulated receiver. The following table from reference 75 indicates the potentials of electrostatic charges accumulated in air of 63-percent humidity at atmospheric conditions by benzene flowing through tubes of different materials.

Tube material	Potential of accumulated charge (volts)
Iron	4000
Brass	3600
Aluminum	2900
Copper	2030

In air of 78-percent humidity, no charge was accumulated when the benzene flowed under its own pressure, but even a slight increase in the liquid pressure resulted in an accumulation of a high-potential electrostatic charge. At liquid pressures of 2 to 6 atmospheres, electrostatic charges having potentials of 4000 volts and greater were accumulated. Thus, unless the equipment is suitably grounded, practically any transfer of fuel in any environment is hazardous.

According to reference 76, the potential of the electrostatic charge accumulated during fuel transference is approximately the same for any grade of gasoline or kerosene, increases with the length of time of the transferring operation, increases with increasing fuel velocity, and is relatively unaffected by atmospheric conditions except that it decreases with increasing humidity of the air.

The capacitance of the human body has been determined (reference 77) as 0.0001 to 0.0004 microfarad. Values of 0.00028 to 0.00032 microfarad were consistently obtained for the capacitance of a person leaning against a wall. An individual charged to a potential of 10,000 volts would have an electrostatic spark energy of 15 millijoules. Because many inflammable mixtures have minimum spark-ignition energies less than 15 millijoules, and this amount of electrostatic spark energy is easily acquired by ungrounded personnel, equipment, and wiring, grounding precautions should be taken.

Most of the sparks occurring in aircraft environments ordinarily contain many times the energy required to ignite inflammable mixtures. Sparks of this type must be prevented from occurring in an inflammable atmosphere. According to reference 73, crash switches that cut off or isolate all electric equipment are a possibility. Batteries can no longer be regarded as the only power source in crashed airplanes. Although the generators in airplanes powered by reciprocating engines would be stopped during a crash by propeller stoppage, free-running turbines of jet-engine air-

craft may necessitate de-energizing fields or short circuiting the generator across its terminals if it would aid in stopping the turbine. Isolation of the electric system can stop electric fuel pumps and close solenoid valves supplying fuel to combustion heaters.

Short-circuit faults in aircraft can probably be protected by rapid acting selective fuses or circuit breakers that would isolate the fault from the rest of the circuit. In general, fuses are less desirable than circuit breakers because of their time-operating characteristics and the fact that they must be replaced after one operation. As indicated by reference 74, the isolating action must be very quick because the circuit voltage drops rapidly in the faulted circuit. This drop in voltage could cripple the entire electric system of the aircraft by allowing relays and contactors to open because of low voltage on the closing coils.

Improved design of the load-carrying cables might result in reduction of the spark-ignition hazard. This improvement might be accomplished by using a conduit containing the electric leads embedded in a solid insulating material. Another possibility in the design of the electric circuits is the inclusion of mechanically weak spots surrounded by inert materials such as powders. During crashes, cable separation would occur at the weak points with the energy released by any sparks being dissipated in inert atmospheres.

IGNITION BY FLAMES OR HOT GASES

Exposure of inflammable vapors to flames or hot gases constitutes the third important fire hazard in aircraft environments. Large, disastrous fires may result from the exposure of inflammable mixtures to exhaust flames and gases or flames initiated by electric spark or heated surface-ignition sources. Possible reduction of these ignition hazards seems more difficult than the reduction of spark or heated surface hazards. Use of the cooled exhaust gases to inert engine nacelles and fuel-tank compartments would prevent contact between inflammable mixtures and hot exhaust gases or flames during flight or ground operation. Simultaneously, the exhaust gases would be inerting hazardous portions of the plane. During a crash, however, such inerting may be ineffective. A possible solution to the problem of flames or hot gases during crashes may be quick acting, crash-actuated mechanisms capable of flooding the engine and exhaust system with fire extinguishing agents.

CONCLUSIONS

An analysis of the available literature included in the survey resulted in the following conclusions:

The inflammability ranges of most hydrocarbon-air mixtures decreased with decreasing mixture pressures and temperatures. The minimum spark-ignition energies of inflammable mixtures and the quenching distances of electrode configurations both increased with decreasing mixture pressures. As a result of these effects, sea-level pressures and temperatures of inflammable mixtures were the most critical design considerations, relative to the reduction of fire hazards in aircraft environments.

Generally, the ignition temperatures of hydrocarbons in air increased with different variables as shown in the following table:

Decreasing variable	Increasing variable
Ignition lag Igniting surface area Fuel quality or cetane number Boiling point or number of carbon atoms in n-paraffins	Mixture turbulence Mixture velocity Surface catalytic activity Tetraethyl lead additions

Ignition temperatures of hydrocarbons were also increased if scale or ash existed on the igniting surfaces or if the inflammables were dropped on heated surfaces in the open air. The ignition temperatures of hydrocarbons containing the same number of carbon atoms were lowest for normal paraffins followed by paraffinic isomers and aromatics.

Gasoline exhibited zones of ignition and nonignition for certain oxygen-fuel ratios much in the same manner as normal octane. The ignition temperature of gasoline in air depends on the degree of confinement of the mixture by the experimental apparatus. Dependent on the experimental conditions and the degree of confinement, the ignition temperatures of gasoline in air may range from 450° to 1325° F. The lower and upper limits of inflammability of 100-octane gasoline in air at atmospheric conditions were approximately 1.40 percent and 7.40 percent by volume, respectively, and the minimum ignition pressure was approximately 35 millimeters of mercury absolute. The minimum diameter of a hole or tube through which a gasoline-air mixture flame would not propagate downwards was approximately 0.13 inch as determined from interpolation of data of known mixtures.

Minimum spark-ignition energies of inflammable mixtures were greatly dependent upon the electrode configurations; these energies increased with increasing size of electrodes and mixture velocity and decreased with increasing electrode spacings. Minimum spark-ignition energies as low as 0.28 millijoule were capable of ignition of methane-air mixtures. Electrode material had no visible effect on mixture ignition by capacitance sparks, but the minimum ignition energies of inductance sparks decreased with decreasing electrode-material density. The measured minimum ignition energies of inductance sparks decreased with increasing spark potentials, and were unaffected by alternating or direct currents. Although high humidity content of the air tends to prevent accumulations of electrostatic charges capable of producing sparks having sufficient energy to ignite vapor-air mixtures, practically any transfer of fuel in any environment was hazardous unless the equipment was suitably grounded.

An increase in the amount of diluent gases in a mixture increased the ignition temperature. In hydrocarbon-oxygen-diluent-gas mixtures containing either argon, carbon dioxide, helium or nitrogen as the diluent component, lowest values of quenching distances, minimum ignition energies, and minimum ignition pressures were exhibited with inflammable mixtures containing argon as the diluent gas. Highest values of quenching distances and minimum ignition energies were exhibited with inflammable mixtures containing helium as the diluent gas; highest values of minimum ignition pressures

were exhibited with inflammable mixtures containing carbon dioxide as the diluent. In order to render any mixture of 100-octane gasoline and air noninflammable, approximately 42.5, 36.0, and 28.7 percent by volume of nitrogen, automobile exhaust gas, or carbon dioxide, respectively, were necessary.

Application of the survey data to the aircraft-fire problem indicated the possibility of reducing aircraft-fire hazards by means of the following remedial measures:

- (1) Prevention of contact of fuels or fuel vapors with the hot exhaust surfaces
- (2) Release of an extinguishing agent inside of and around the exhaust system if a crash is imminent
- (3) Reduction of the temperature of the exhaust duct or gases below the surface-ignition temperature of gasoline and lubricating oil in the event of a crash
- (4) Increase of the surface-ignition temperatures of gasoline and lubricating oil
- (5) Use of fuels of reduced volatility
- (6) Elimination of the electrical generating system as an ignition hazard in the event of a crash
- (7) Inerting of engine nacelles and wing compartments.

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, February 27, 1950.

REFERENCES

1. Moore, Harold: Spontaneous Ignition Temperatures of Liquid Fuels for Internal Combustion Engines. *Jour. Soc. Chem. Ind.*, vol. XXXVI, no. 3, Feb. 15, 1917, pp. 109-112.
2. Anon.: The Spontaneous Ignition-Temperatures of Liquid Fuels. *Engineering*, vol. CIX, Jan. 30, 1920, pp. 151-152.
3. Dixon, H. B., and Higgins, W. F.: On the Ignition-Point of Gases at Different Pressures. *Memoirs and Proc. Manchester Literary & Phil. Soc. (Manchester Memoirs)*, vol. LXX, 1925-26, pp. 29-36.
4. Tizard, H. T., and Pye, D. R.: Experiments on the Ignition of Gases by Sudden Compression. *Phil. Mag. and Jour. Sci.*, vol. XLIV, no. 259, July 1922, pp. 79-121.
5. Mason, Walter, and Wheeler, Richard Vernon: The Ignition of Gases. Part IV. Ignition by a Heated Surface. Mixtures of the Paraffins with Air. *Jour. Chem. Soc. Trans. (London)*, vol. CXXV, pt. II, 1924, pp. 1869-1875.
6. Newitt, D. M., and Townend, D. T. A.: Combustion Phenomena of Hydrocarbons. *The Science of Petroleum*, vol. IV, Oxford Univ. Press (London), 1938, pp. 2860-2883.
7. Jones, G. W., and Spolan, I.: Inflammability of Gasoline Vapor-Air Mixtures at Low Pressures. *R. I. 3966, Bur. Mines*, Oct. 1946.
8. White, Albert Greville: Limits for the Propagation of Flame in Inflammable Gas-Air Mixtures. Part III. The Effect of Temperature on the Limits. *Jour. Chem. Soc. Trans. (London)*, vol. CXXVII, pt. I, 1925, pp. 672-684.
9. Briand, M., Dumanois, P., et Laffitte, P.: Sur l'influence de la température sur les limites d'inflammabilité de quelques vapeurs combustibles. *Comptes Rendus*, T. 197, Juillet 24, 1933, P. 322-323.
10. Briand, Marius: Influence de la température sur les limites d'inflammabilité de mélanges de vapeurs combustibles avec l'air. *Ann. l'Office Nat. Combustibles Liquides*, Ann. X, No. 6, Nov.-Déc. 1935, P. 1129-1164.
11. Guest, P. G.: Ignition of Natural Gas-Air Mixtures by Heated Surfaces. *Tech. Paper 475, Bur. Mines*, 1930.

12. Glendinning, W. G.: Possible Cause of Aircraft Fires on Crash. R. & M. No. 1375, British A. R. C., Jan. 1930.
13. Spencer, R. C.: Preignition Characteristics of Several Fuels under Simulated Engine Conditions. NACA Rep. 710, 1941.
14. Patterson, Stewart: The Ignition of Inflammable Gases by Hot Moving Particles. Phil. Mag. and Jour. Sci., vol. 28, no. 186, ser. 7, July 1939, pp. 1-23.
15. Serruys, Max: Experimental Study of Ignition by Hot Spot in Internal Combustion Engines. NACA TM 873, 1938.
16. Dykstra, F. J., and Edgar, Graham: Spontaneous Ignition Temperature of Liquid Hydrocarbons at Atmospheric Pressure. Ind. and Eng. Chem., vol. 26, no. 5, May 1934, pp. 509-510.
17. Sortman, Charles W., Beatty, Harold A., and Heron, S. D.: Spontaneous Ignition of Hydrocarbons. Ind. and Eng. Chem., vol. 33, no. 3, March 1941, pp. 357-360.
18. Townend, D. T. A.: The Present Era in Combustion. Chem. and Ind. (London), no. 44, Nov. 10, 1945, pp. 346-351.
19. Bridgeman, Oscar, and Marvin, Charles F., Jr.: Auto-Ignition Temperatures of Liquid Fuels. Ind. and Eng. Chem., vol. 20, no. 11, Nov. 1928, pp. 1219-1223.
20. Moore, Harold: Spontaneous Ignition-Temperatures of Liquid Fuels. The Auto. Eng., vol. X, no. 138, May 1920, pp. 199-204.
21. Ridley, H. G.: A Study of the Auto-Ignition of Fuel and Oil with Respect to XB-29 Exhaust Tunnel Wall Temperatures. Rep. No. D-4789, Boeing Aircraft Co., May 31, 1943.
22. Silver, Robert S.: The Ignition of Gaseous Mixtures by Hot Particles. Phil. Mag. and Jour. Sci., vol. 23, no. 156, ser. 7, April 1937, suppl., pp. 633-657.
23. Mullen, James W., II, Fenn, John B., and Irby, Moreland R.: The Ignition of High Velocity Streams of Combustible Gases by Heated Cylindrical Rods. Third Symposium on Combustion and Flame and Explosion Phenomena, The Williams & Wilkins Co. (Baltimore), 1949, pp. 317-329.
24. Jeufroy, A.: Études sur l'inflammabilité des carbures d'hydrogène. Chim. & Ind., Numéro Spéciale, Fév. 1929, P. 271(271C).
25. Sperling, Karl: Entzündungsmöglichkeiten von Mineralölen. Oel und Kohle, Jahrg. 37, Heft 18, Mai 8, 1941, S. 329-333.
26. Wiezevich, P. J., Whiteley, J. M., and Turner, L. B.: Spontaneous Ignition of Petroleum Fractions. Ind. and Eng. Chem., vol. 27, no. 2, Feb. 1935, pp. 152-155.
27. Dallas, A. W., and Hansberry, H. L.: Determination of Means to Safeguard Aircraft from Powerplant Fires in Flight. Part I. Tech. Development Rep. No. 33, CAA, Sept. 1943.
28. Masson, Henry James, and Hamilton, William F.: A Study of Auto-Ignition Temperatures. III-(a) Mixtures of Pure Substances, (b) Gasolines. Ind. and Eng. Chem., vol. 21, no. 6, June 1929, pp. 544-549.
29. Audibert, Étienne: Sur l'inflammation des mélanges grisouteux par élévation de la température. Comptes Rendus, T. 218, Jan. 10, 1944, P. 77-79.
30. Morgan, J. D.: An Experiment on the Combustion of an Inflammable Gas Mixture by a Hot Wire. Phil. Mag. and Jour. Sci., vol. 15, no. 98, ser. 7, Feb. 1933, suppl., pp. 440-442.
31. Shepherd, W. C. F., and Wheeler, R. V.: The Ignition of Gases by Hot Wires. Paper No. 36, Safety in Mines Research Board (London), 1927.
32. Aubert, M., et Pignot, A.: Sur l'inflammation par point chaud des mélanges carbures-air. Ann. l'Office Nat. Combustibles Liquides, Ann. 6, No. 5, Sept.-Oct. 1931, P. 819-828.
33. Couriot, et Meunier, Jean: Action d'un conducteur électrique incandescent sur les gaz qui l'entourent. Comptes Rendus, T. 145, Déc. 9, 1907, P. 1161-1163.
34. Hoerl, Arthur E., Jr.: Ignition of Gases and Vapors by Sparks. Gas, vol. 23, no. 12, Dec. 1947, pp. 60-63.
35. Hankins, G. A.: Ignition of Explosive Mixtures of Petrol Vapour and Air. Appendix I of R. & M. No. 796, British A. R. C., Sept. 1920.
36. Masson, Henry James, and Hamilton, William F.: A Study of Auto-Ignition Temperatures. II-Pure Compounds. Ind. and Eng. Chem., vol. 20, no. 8, Aug. 1928, pp. 813-816.
37. The Associated Factory Mutual Fire Insurance Cos.: Properties of Flammable Liquids, Gases, and Solids. Ind. and Eng. Chem., vol. 32, no. 6, June 1940, pp. 880-884.
38. Bayan, S.: La résistance des carburants à la détonation et leur température d'autoinflammation. Chim. et Ind., Vol. 38, No. 1, Juillet 1937, P. 52 (24D).
39. Edgar, Graham: Ignition Temperatures of Aircraft Combustible Liquids. SAE Jour. (Trans.), vol. 45, no. 1, July 1939, p. 294.
40. Thompson, Norman J.: Auto-Ignition Temperatures of Flammable Liquids. Ind. and Eng. Chem., vol. 21, no. 2, Feb. 1929, pp. 134-139.
41. Nygaard, E. M., Crandall, G. S., and Berger, H. G.: Means of Improving Ignition Quality of Diesel Fuels. Jour. Inst. Petroleum (London), vol. 27, no. 214, Oct. 1941, pp. 348-368.
42. Egloff, Gustav, and Van Arsdell, P. M.: Octane Rating Relationships of Aliphatic, Alicyclic, Mononuclear Aromatic Hydrocarbons, Alcohols, Ethers, and Ketones. Jour. Inst. Petroleum (London), vol. 27, no. 210, April 1941, pp. 121-138; discussion, pp. 139-142.
43. Tausz, J., and Schulte, F.: Determination of Ignition Points of Liquid Fuels under Pressure. NACA TM 299, 1925.
44. Morgan, J. D.: Principles of Ignition. Isaac Pitman & Sons, Ltd. (London), 1942.
45. Lewis, Bernard, and von Elbe, Guenther: Ignition and Flame Stabilization in Gases. A. S. M. E. Trans., vol. 70, no. 4, May 1948, pp. 307-314; discussion, pp. 314-316.
46. Guest, P. G.: Apparatus for Determining Minimum Energies for Electric Spark Ignition of Flammable Gases and Vapors. R. I. 3753, Bur. Mines, May 1944.
47. Lewis, Bernard, and von Elbe, Guenther: Ignition of Explosive Gas Mixtures by Electric Sparks. II. Theory of the Propagation of Flame from an Instantaneous Point Source of Ignition. Jour. Chem. Phys., vol. 15, no. 11, Nov. 1947, pp. 803-808.
48. Blanc, M. V., Guest, P. G., von Elbe, Guenther, and Lewis, Bernard: Ignition of Explosive Gas Mixtures by Electric Sparks. Pt. III. Minimum Ignition Energies and Quenching Distances of Mixtures of Hydrocarbons and Ether with Oxygen and Inert Gases. Third Symposium on Combustion and Flame and Explosion Phenomena, The Williams & Wilkins Co. (Baltimore), 1949, pp. 363-367.
49. Swett, Clyde C., Jr.: Spark Ignition of Flowing Gases. I-Energies to Ignite Propane-Air Mixtures in Pressure Range of 2 to 4 Inches Mercury Absolute. NACA RM E9E17, 1949.
50. Blanc, M. V., Guest, P. G., von Elbe, Guenther, and Lewis, Bernard: Ignition of Explosive Gas Mixtures by Electric Sparks. I. Minimum Ignition Energies and Quenching Distances of Mixtures of Methane, Oxygen, and Inert Gases. Jour. Chem. Phys., vol. 15, no. 11, Nov. 1947, pp. 798-802.
51. Boyle, A. R., and Llewellyn, F. J.: The Electrostatic Ignitability of Various Solvent Vapour-Air Mixtures. Jour. Soc. Chem. Ind., vol. 66, no. 3, March 1947, pp. 99-102.
52. Wheeler, Richard Vernon: The Ignition of Gases. Part III. Ignition by the Impulsive Electrical Discharge. Mixtures of the Paraffins with Air. Jour. Chem. Soc. Trans. (London), vol. CXXV, pt. II, 1924, pp. 1858-1867.
53. Wheeler, Richard Vernon: The Ignition of Gases. Part I. Ignition by the Impulsive Electrical Discharge. Mixtures of Methane and Air. Jour. Chem. Soc. Trans. (London), vol. CXVII, pt. II, 1920, pp. 903-917.
54. Thornton, W. M.: The Reaction Between Gas and Pole in the Electrical Ignition of Gaseous Mixtures. Proc. Roy. Soc. (London), vol. XCII, no. A634, ser. A, Oct. 1, 1915, pp. 9-22.
55. Morgan, John David: The Ignition of Explosive Gases by Electric Sparks. Jour. Chem. Soc. Trans. (London), vol. CXV, 1919, pp. 94-104.
56. Morgan, J. D.: The Thermal Theory of Gas Ignition by Electric Sparks. Phil. Mag. and Jour. Sci., vol. 49, no. 290, ser. 6, Feb. 1925, pp. 323-336.
57. Wheeler, R. V.: The Electric Ignition of Firedamp: Alternating and Continuous Currents Compared. Paper No. 20, Safety in Mines Research Board (London), 1926.

58. Lakin, W. P., Thwaites, H. L., Skarstrom, C. W., and Baum, A. W.: Fourth Annual Rep. on Fundamental Studies of Combustion. Rep. No. RL-5M-48 (69), Esso Labs. (Standard Oil Development Co.), Nov. 30, 1948.
59. Jones, G. W., and Kennedy, R. E.: Inflammability of Natural Gas: Effect of Pressure Upon the Limits. R. I. 3798, Bur. Mines, Feb. 1945.
60. Nuckolls, A. H., Matson, A. F., and Dufour, R. E.: Propagation of Flame in Gasoline Vapor-Air Mixtures at Pressures Below Atmospheric. Bull. of Res. No. 7, Underwriters' Labs., Inc., March 1939.
61. Jones, G. W., and Gilliland, W. R.: Extinction of Gasoline Flames by Inert Gases. R. I. 3871, Bur. Mines, April 1946.
62. Holm, John M.: On the Initiation of Gaseous Explosions by Small Flames. Phil. Mag. and Jour. Sci., vol. 14, no. 89, ser. 7, July 1932, pp. 18-56.
63. Holm, John M.: On the Ignition of Explosive Gaseous Mixtures by Small Flames. Phil. Mag. and Jour. Sci., vol. 15, no. 98, ser. 7, Feb. 1933, suppl., pp. 329-359.
64. Harris, Margaret E., Grumer, Joseph, von Elbe, Guenther, and Lewis, Bernard: Burning Velocities, Quenching, and Stability Data on Nonturbulent Flames of Methane and Propane with Oxygen and Nitrogen. Third Symposium on Combustion and Flame and Explosion Phenomena, The Williams & Wilkins Co. (Baltimore), 1949, pp. 80-89.
65. Friedman, Raymond: The Quenching of Laminar Oxyhydrogen Flames by Solid Surfaces. Third Symposium on Combustion and Flame and Explosion Phenomena, The Williams & Wilkins Co. (Baltimore), 1949, pp. 110-120.
66. Friedman, Raymond, and Johnston, W. C.: The Wall-Quenching of Laminar Propane Flames as a Function of Pressure, Temperature, and Air-Fuel Ratio. Res. Rep. R-94451-4-B, Westinghouse Res. Labs. (E. Pittsburgh, Pa.), Nov. 9, 1949.
67. Jost, Wilhelm: Explosion and Combustion Processes in Gases. McGraw-Hill Book Co., Inc., 1946.
68. Forsyth, J. S., and Garside, J. E.: The Mechanism of Flash-Back of Aerated Flames. Third Symposium on Combustion and Flame and Explosion Phenomena, The Williams & Wilkins Co. (Baltimore), 1949, pp. 99-102.
69. Grupp, George W.: Diesel Ignition Hazard Tests for Mine Locomotives. Diesel Power and Diesel Trans., vol. 27, no. 3, March 1949, pp. 46-49.
70. Baird, John W.: How Powerplant Installation Troubles Hampered Airline Operations-1933-47. SAE Jour., vol. 57, no. 1, Jan. 1949, pp. 62-67.
71. Pesman, Gerard J.: Analysis of Multiengine Transport Airplane Fire Records. NACA RM E9J19, 1950.
72. Glendinning, W. G., and Drinkwater, J. W.: The Prevention of Fire in Aircraft. R. A. S. Jour., vol. 51, no. 439, July 1947, pp. 616-641; discussion, pp. 641-650.
73. Bennett, Neill G.: How to Cut Crash-Fire Dangers. Aviation Week, vol. 50, no. 25, June 20, 1949, pp. 27-30.
74. Hutton, J. G., and Bogiages, P. C.: Electric Power Supply Protection on Aircraft. Gen. Elec. Rev., vol. 51, no. 4, April 1948, pp. 27-32.
75. Holde, D.: The Danger of Electrically Excited Ignition of Flammable Liquids. Chem. Abs., vol. 19, no. 2, Jan. 20, 1925, p. 400. (From Farben-Zeitung, vol. 29, 1924, pp. 1891-1893.)
76. Cooper-Key, A., et al.: Forty-ninth Annual Report H. M. Inspectors of Explosives. Chem. Abs., vol. 19, no. 13, July 10, 1925, p. 2132. (From Pamphlet 51, H. M. Stationery Office (London), 1925.)
77. Brown, F. W., Kusler, D. J., and Gibson, F. D.: Sensitivity of Explosives to Initiation by Electrostatic Discharges. R. I. No. 3852, Bur. Mines, Jan. 1946.
78. Heron, S. D., and Beatty, Harold A.: Aviation Fuels—Present and Future Developments. Proceedings of Ninth Mid-Year Meeting of American Petroleum Institute, sec. III. Vol. 20M (III), pub. by A. P. I. (New York), 1939.
79. Egerton, A., and Gates, S. F.: The Effect of Metallic Vapours on the Ignition of Substances. Jour. Inst. Petroleum Tech. (London), vol. 13, 1927, pp. 244-255.
80. Anon.: Report of Subcommittee XXVIII on Autogenous Ignition of Petroleum Products. A. S. T. M. Proc., vol. 30, pt. 1, 1930, pp. 788-792.
81. Huff, Wilbert J.: Annual Report of the Explosives Division, Fiscal Year 1941. R. I. 3602, Bur. Mines, Jan. 1942.
82. Alt, Otto: Combustion of Liquid Fuels in Diesel Engine. NACA TM 281, 1924.
83. Mondain-Monval, P., et Quanquin, B.: Température d'inflammation spontanée des mélanges gazeux d'air et hydrocarbures saturés. Influence de la pression et du chauffage préalable. Comptes Rendus, T. 189, Nov. 25, 1929, P. 917-919.
84. Anon.: National Fire Codes. Vol. I. Nat. Fire Protection Assoc. (Boston, Mass.), 1948.
85. Coffey, S., and Birchall, T.: Spontaneous Ignition Temperatures of Hydrocarbon-Air Mixtures. Jour. Soc. Chem. Ind., vol. 53, no. 11, March 16, 1934, pp. 245-247.
86. Sullivan, M. V., Wolfe, J. K., and Zisman, W. A.: Flammability of the Higher Boiling Liquids and Their Mists. Ind. and Eng. Chem., vol. 39, no. 12, Dec. 1947, pp. 1607-1614.
87. Jones, G. W.: Inflammation Limits and Their Practical Application in Hazardous Industrial Operations. Chem. Rev., vol. 22, no. 1, Feb. 1938, pp. 1-26.

BIBLIOGRAPHY

- Anon.: Airplane Crash Fire Fighting Manual. NFPA (Boston), 1945.
- Anon.: Joint Discussion on the Ignition of Gases. The Colliery Guardian and Jour. Coal and Iron Trades, vol. CXXX, no. 3375, Sept. 4, 1925, pp. 555-556.
- Anon.: Physical Constants of Paraffin Hydrocarbons. Petroleum Refiner, vol. 25, no. 5, May 1946, p. 130.
- Anon.: The Correlation of Cetane Number with Other Physical Properties of Diesel Fuels. Jour. Inst. Petroleum (London), vol. 30, no. 247, July 1944, pp. 193-197.
- Allen, Augustine O., and Rice, O. K.: The Explosion of Azomethane. Jour. Am. Chem. Soc., vol. 57, no. 2, Feb. 6, 1935, pp. 310-317.
- Anfenger, M. B., and Johnson, O. W.: Friction Sparks. Vol. 22 (1) (1941) of Proc. 22d Ann. Meeting American Petroleum Inst., sec. 1, Nov. 1941, pp. 54-56.
- Boussu, Roger-G.: Limite d'inflammabilité des vapeurs du système alcoolessence et d'un système triple à base d'alcool et d'essence. Comptes Rendus, T. 175, Juillet 3, 1922, P. 30-32.
- Burrell, G. A., and Robertson, I. W.: Effects of Temperature and Pressure on the Explosibility of Methane-Air Mixtures. Tech. Paper 121, Bur. Mines, 1916.
- Callendar, H. L.: Dopes and Detonation (1926). Engineering, vol. CXXIII, Feb. 4, 1927, pp. 147-148.
- Chappuis, J., et Pignot, A.: Limite d'inflammabilité des mélanges gazeux. Chim. & Ind., Numéro Spécial, Fév. 1929, P. 228 (228C)-230 (230C).
- Chappuis, James, et Pignot, A.: Sur la compression de gaz de ville. Comptes Rendus, T. 185, Déc. 19, 1927, P. 1486-1488.
- Clark, G. L., and Thee, W. C.: Present Status of the Facts and Theories of Detonation. Ind. and Eng. Chem., vol. 17, no. 12, Dec. 1925, pp. 1219-1226.
- Cleveland Laboratory Aircraft Fire Research Panel: Preliminary Survey of the Aircraft Fire Problem. NACA RM E8B18, 1948.
- Coward, Hubert Frank, and Brinsley, Frank: The Dilution-limits of Inflammability of Gaseous Mixtures. Part I. The Determination of Dilution-limits. Part II. The Lower Limits for Hydrogen, Methane and Carbon Monoxide in Air. Jour. Chem. Soc. Trans. (London), vol. CV, pt. II, 1914, pp. 1859-1885.
- Coward, H. F., and Guest, P. G.: Ignition of Natural Gas—Air Mixtures by Heated Metal Bars. Jour. Am. Chem. Soc., vol. 49, no. 10, Oct. 1927, pp. 2479-2486.
- Gardner, William H.: Carelessness with Gasoline. Fire Engineering, vol. 93, no. 9, Sept. 1940, pp. 446-448.
- Hansberry, Harvey L.: Aircraft Power-Plant Fire Protection. Aero. Eng. Rev., vol. 3, no. 10, Oct. 1944, pp. 9-16, 23-27.

- Hélier, M. H.: Recherches sur les combinaisons gazeuses. *Ann. Chim. et Phys.*, T. X, 7th Sér., 1897, P. 521-556.
- Jennings, Burgess H., and Obert, Edward F.: *Internal Combustion Engines*. International Textbook Co. (Scranton, Pa.), 1944.
- Jones, G. W.: Explosion and Fire Hazards of Combustible Anesthetics. R. I. 3443, Bur. Mines, April 1939.
- Jones, G. W., and Kennedy, R. E.: Limits of Inflammability of Natural Gases Containing High Percentages of Carbon Dioxide and Nitrogen. R. I. 3216, Bur. Mines, June 1933.
- Jones, G. W., and Scott, G. S.: Limits of Inflammability and Ignition Temperatures of Naphthalene. R. I. 3881, Bur. Mines, May 1946.
- Kintz, G. M.: Some Information on the Causes and Prevention of Fires and Explosions in the Petroleum Industry. I. C. 7150, Bur. Mines, April 1941.
- Köhler, L.: Einordnung von Kraftstoffen nach der Cetanzahl. *Oel und Kohle*, Jahrg. 37, Heft 44, Nov. 22, 1941, S. 903-905.
- Lewis, Bernard, and von Elbe, Guenther: Physics of Flame and Explosions of Gases. *Jour. Appl. Phys.*, vol. 10, no. 6, June 1939, pp. 344-359.
- Mack, Edward, Boord, C. E., and Barham, H. N.: Calculation of Flash Points for Pure Organic Substances. *Ind. and Eng. Chem.*, vol. 15, no. 9, Sept. 1923, pp. 963-965.
- Mackeown, S. S., and Wouk, Victor: Electrical Charges Produced by Flowing Gasoline. *Ind. and Eng. Chem.*, vol. 34, no. 6, June 1942, pp. 659-664.
- Mallard, et Le Chatelier: Recherches expérimentales et théoriques sur la combustion des mélanges gazeux explosifs. *Ann. des Mines*, T. IV, Sér. 8, 1883, P. 274-568.
- Mason, Walter, and Wheeler, Richard Vernon: The Ignition of Gases. Part II. Ignition by a Heated Surface. Mixtures of Methane and Air. *Jour. Chem. Soc. Trans. (London)*, vol. CXXI, pt. II, 1922, pp. 2079-2091.
- Mason, Walter, and Wheeler, Richard Vernon: The "Uniform Movement" During the Propagation of Flame. *Jour. Chem. Soc. Trans. (London)*, vol. CXI, pt. II, 1917, pp. 1044-1057.
- Masson, Henry James, and Hamilton, William F.: A Study of Auto-Ignition Temperatures. *Ind. and Eng. Chem.*, vol. 19, no. 12, Dec. 1927, pp. 1335-1338.
- Morgan, John David, and Wheeler, Richard Vernon: Phenomena of the Ignition of Gaseous Mixtures by Induction Coil Sparks. *Jour. Chem. Soc. Trans. (London)*, vol. CXIX, pt. I, 1921, pp. 239-251.
- Müller, Bruno: Die Reibungselektrizität in Rohrleitungen als Ursache von Benzinexplosionen. *Chemiker-Zeitung*, Jahrg. 47, Nr. 14, Feb. 1, 1923, S. 97-98.
- Naylor, C. A., and Wheeler, R. V.: The Ignition of Gases. VIII. Ignition by a Heated Surface. (a) Mixtures of Ethane, Propane, or Butane with Air: (b) Mixtures of Ethylene, Propylene, or Butylene with Air. *Jour. Chem. Soc. (London)*, pt. II, 1933, pp. 1240-1247.
- Naylor, C. A., and Wheeler, R. V.: The Ignition of Gases. Part IX. Ignition by a Heated Surface. Mixtures of Methane and Air at Reduced Pressures. *Jour. Chem. Soc. (London)*, pt. II, 1935, pp. 1426-1430.
- Naylor, Clement Albert, and Wheeler, Richard Vernon: The Ignition of Gases. Part VI. Ignition by a Heated Surface. Mixtures of Methane with Oxygen and Nitrogen, Argon, or Helium. *Jour. Chem. Soc. (London)*, pt. II, 1931, pp. 2456-2467.
- Oehmichen, M.: La pression de vapeur et les limites d'inflammabilité de l'essence à indice d'octane de 87, aux températures de +40 à -50°C. *Chim. et Ind.*, Vol. 47, No. 1, Jan. 1942, P. 45 (13D).
- Parker, Albert: Influence of Increase of Initial Temperature on the Explosiveness of Gaseous Mixtures. *Jour. Chem. Soc. Trans. (London)*, vol. CIII, pt. I, 1913, pp. 934-940.
- Paterson, Clifford C., and Campbell, Norman: Some Characteristics of the Spark Discharge and Its Effect in Igniting Explosive Mixtures. *Proc. Phys. Soc. London*, vol. XXXI, Dec. 1918-Aug. 1919, pp. 168-227; discussion, pp. 227-228.
- Payman, William, and Wheeler, Richard Vernon: The Propagation of Flame through Tubes of Small Diameter. Part II. *Jour. Chem. Soc. Trans. (London)*, vol. CXV, 1919, pp. 36-45.
- Porter, Horace C.: Ignition and Ignitibility. *Ind. and Eng. Chem.*, vol. 32, no. 8, Aug. 1940, pp. 1034-1036.
- Reutenauer, Georges: Concentrations limites d'inflammation et inflammabilité des hydrocarbures. *Comptes Rendus*, T. 212, Juin 30, 1941, P. 1148-1150.
- Riffkin, J.: The Ignition Quality of Diesel Fuels. *Engineering*, vol. CXLVII, no. 3808, Jan. 6, 1939, pp. 1-4.
- Scott, G. S., Jones, G. W., and Scott, F. E.: Determination of Ignition Temperatures of Combustible Liquids and Gases. *Anal. Chem.*, vol. 20, no. 3, March 1948, pp. 238-241.
- Stavenhagen, A., und Schuchard, E.: Über des Verhalten von explosiblen Gasgemischen bei niederen Drucken. *Zeitschr. ange. Chem.*, Bd. I, Jahrg. 33, Aufsatzteil, Nov. 16, 1920, S. 286-287.
- Tanaka, Yoshio, Nagai, Yuzaburo, and Akiyama, Kinsei: Lower Limit of Inflammability of Ethyl Alcohol, Ethyl Ether, Methyl Cyclohexane and Their Mixtures. *Rep. Aero. Res. Inst., Tôkyô Imperial Univ.*, vol. II, no. 21, March 1927, pp. 235-246. (In English.)
- Terres, E., und Plenz, F.: Über den Einfluss des Druckes auf die Verbrennung explosiver Gas-Luftmischungen. *Jour. f. Gasbeleuchtung und Wasserversorgung*, Jahrg. LVII, Nr. 47, Nov. 21, 1914, S. 990-995; fortsetzung, Nr. 48, Nov. 28, 1914, S. 1001-1007; fortsetzung, Nr. 49, Dez. 5, 1914, S. 1016-1019.
- Thornton, W. M.: The Ignition of Gases by Condenser Discharge Sparks. *Proc. Roy. Soc. (London)*, vol. XCI, no. A623, ser. A, Nov. 2, 1914, pp. 17-22.
- Thornton, W. M.: The Ignition of Gases by Hot Wires. *Phil. Mag. and Jour. Sci.*, vol. XXXVIII, no. 227, Nov. 1919, pp. 613-633.
- Thornton, W. M.: The Mechanism of Gas Ignition. *Rep. 91st Meeting, British Assoc. Advancement of Sci. (Liverpool)*, Sept. 12-19, 1923, pp. 469-470.
- Townend, Donald T. A., and Mandlekar, M. R.: The Influence of Pressure on the Spontaneous Ignition of Inflammable Gas-Air Mixtures. I-Butane-Air Mixtures. *Proc. Roy. Soc. (London)*, vol. CXXI, no. A844, ser. A, Aug. 1933, pp. 484-493.
- Townend, Donald T. A., and Mandlekar, M. R.: The Influence of Pressure on the Spontaneous Ignition of Inflammable Gas-Air Mixtures. II-Pentane-Air Mixtures. *Proc. Roy. Soc. (London)*, vol. CXLIII, no. A848, ser. A, Dec. 4, 1934, pp. 168-176.
- Tréer, M. F.: Ignition Temperature of Pure Hydrocarbons and Motor Fuels. *Fuel in Sci. and Practice*, vol. XIX, no. 6, July 1940, pp. 149-152.
- van der Hoeven, H. W.: Flammability of Propane-Air Mixtures. *Ind. and Eng. Chem.*, vol. 29, no. 4, April 1937, pp. 445-446.
- Viallard, R.: Ignition of Explosive Gas Mixtures by Electric Sparks. Minimum Ignition Energies. *Jour. Chem. Phys.*, vol. 16, no. 5, May 1948, p. 555.
- von Elbe, Guenther, and Lewis, Bernard: Theory of Ignition, Quenching and Stabilization of Flames of Nonturbulent Gas Mixtures. Third Symposium on Combustion and Flame and Explosion Phenomena, The Williams & Wilkins Co. (Baltimore), 1949, pp. 68-79.
- Walls, N. S., and Wheeler, R. V.: The Ignition of Firedamp by Momentary Flames. Part I. Rintoul, W., and White, A. G.: The Ignition of Firedamp by Momentary Flames. Part II. Paper No. 24, Safety in Mines Res. Board (London), 1926.
- Yannaquis: The Influence of Temperature on the Limits of Inflammability of Alcohols. *Chem. Abs.*, vol. 23, no. 21, Nov. 10, 1929, p. 5319. (From *Ann. l'Office Nat. Combustibles Liquides*, vol. 4, 1929, pp. 303-316.)

TABLE I—SURFACE IGNITION TEMPERATURES OF SEVERAL INFLAMMABLES IN AIR

(a) Ignition surfaces, heated 6-inch-diameter steel tube and open steel plate (reference 72)

Fuel	Ignition temperature (°F)		Flash point (°F)
	Tube (a)	Plate	
Gasoline (unleaded).....	473	>1238	-44
100-Octane aviation gasoline (leaded).....	734	>1238	-44
Kerosene.....	417	1202	108
Hydraulic fluid.....	455	752	302
Lubricating oil.....	572	806	482

(a) Heated-surface ignition environments.

(b) Ignition surfaces, heated nickel or iron plates (reference 24)

Plate, 0.5 or 1.0 mm by 45 mm by 140 mm; temperatures measured by iron-constantan thermocouple soldered to plate.]

Fuel	Surface ignition temperature (°F)
Benzene.....	1400
Toluene.....	1409
Ethyl alcohol.....	1274
Cyclohexane.....	1125
n-Pentane.....	1085
n-Heptane.....	1058
n-Decane.....	1085
Rumanian aviation gasoline.....	1085
Shale-oil gasoline.....	1040
Green mineral oil.....	977
Dimethylcyclohexene.....	833

TABLE II—EFFECT OF IGNITION-SURFACE COMPOSITION ON SURFACE IGNITION TEMPERATURES OF SEVERAL FUELS IGNITED IN AIR BY STATIC CRUCIBLE METHOD

(a) Data from reference 40

[All configurations are 106-cc cylinders except pyrex surface, which is 125-cc spherica configuration.]

Fuel	Surface ignition temperature (°F)			
	Pyrex	Copper	Low-carbon steel	Chromium
n-Hexane.....	479	510	516	513
n-Heptane.....	452	481	484	483
Isobutyl alcohol.....	825	861	902	862
Gasoline (Socony).....	497	534	550	559

(b) Data from reference 36

[Ignition volume, 1180 cc; ignition lag, 1 sec.]

Fuel	Surface ignition temperature (°F)	
	Platinum	Pyrex
n-Pentane (technical grade).....	1074	1009
n-Hexane.....	968	959
n-Octane.....	856	815
Isooctane.....	1042	999
Isododecane.....	993	930
Benzene.....	1213	1144
Toluene.....	1171	1119
Mesitylene.....	1150	1110
p-Xylene.....	1144	1103
Methyl alcohol.....	1065	997
Ethyl alcohol (abs.).....	1035	1011
n-Propyl alcohol.....	1004	979
Benzyl alcohol.....	936	829
Isopropyl alcohol.....	1148	990
Isobutyl alcohol.....	1008	972
Isoamyl alcohol.....	964	840
Ethylene glycol.....	972	855

TABLE III—EFFECT OF OCTANE NUMBER ON SURFACE IGNITION TEMPERATURE OF UNLEADED LIQUID FUELS IGNITED IN AIR BY DYNAMIC CRUCIBLE METHOD OF IGNITION (REFERENCE 78)

Fuel	Octane number	Surface ignition temperature (°F) (*)	Flash point (°F)	A.S.T.M. distillation (°F)			Reid vapor pressure (lb at 100° F)	
				Initial boiling point	Percentage evaporated			
					10	50		90
Certified 2,2,4-trimethylpentane.....	100	970		211	211	211	1.5	
Paraffinic safety fuel.....	99	930	110	330	340	342	about 0.1	
Straight-run gasoline plus isooctane plus isopentane (commercial blend).....	85	910		158	212	257	7.0	
Diisobutylene (mixture of 2 isomers).....	b 84 to 100+	880		220	max.	max.	max.	
Aromatic safety fuel.....	75	790	110	327	338	354	about 0.1	
Straight-run aviation gasoline.....	74	830			158	212	7.0	
					max.	max.	max.	
Certified n-heptane.....	0	490 to 530 c 790		209	209	209	1.6	

(a) Air-flow rate, 50 cc/min at atmospheric pressure and temperature; fuel addition, 0.01 ml from pipette at 2-minute intervals; ignition surface, stainless steel.

(b) Depends on knock test method.

(c) Nonignition zone from 540 to 760° F.

TABLE IV—EFFECT OF ADDITION OF TETRAETHYL LEAD ON SURFACE IGNITION TEMPERATURE OF SEVERAL FUELS AS DETERMINED BY DYNAMIC CRUCIBLE METHOD

(a) Data from references 79 and 17

Fuel	Ignition temperature without TEL (°F)	Ignition temperature increase with TEL (°F)	Reference
Pentane.....	959	a 135	79
Isohexane.....	977	a 83	79
Heptane.....	806	a 149	79
Gasoline (Shell).....	860	a 148	79
Benzene.....	1274	a 32	79
Cyclohexane.....	995	a 49	79
Methylcyclohexane.....	878	a 166	79
Cetane.....	450	b 482	17
n-Heptane.....	498	b 341	17
Isooctane.....	985	b 92	17
Isododecane.....	932	b 63	17

a 0.25 percent by volume TEL added (approximately 9.5 cc/gal). (Air-flow rate, 330 cc/min; fuel drop size, 0.012-0.013 cc; ignition surface, iron.
b 3 cc/gal TEL added. (Air-flow rate, 54 cc/min; fuel drop size, 8 mg; ignition surface, stainless steel.

(b) Surface ignition temperatures obtained from reference 78

Fuel	Ignition temperature without TEL (°F)	Ignition temperature increase with TEL (°F)	Flash point (°F)	A. S. T. M. distillation (°F)				Reid vapor pressure (lb at 100° F)
				Initial boiling point	Percentage evaporated			
					10	50	90	
Certified n-heptane.....	490-530			209	209	209	209	1.6
	^b 790	40						
Certified 2,2,4-trimethylpentane.....	970	90		211	211	211	211	1.5
Paraffinic safety fuel.....	930	60	110	330	340	342	344	about 0.1
Aromatic safety fuel.....	790	90	110	327	338	354	381	about 0.1
Straight-run aviation gasoline.....	830	40			158	212	257	7.0
					max.	max.	max.	max.
Straight-run gasoline plus isooctane plus isopentane (commercial blend).....	910	50			158	212	257	7.0
					max.	max.	max.	max.

a Air-flow rate, 50 cc/min at atmospheric pressure and temperature; fuel addition, 0.01 ml from pipette at 2-minute intervals; ignition surface, stainless steel; 3 cc/gal TEL added.

b Nonignition zone from 540 to 760° F.

TABLE V—IGNITION TEMPERATURES OF VARIOUS FUELS AND OILS

[All results from experiments conducted at atmospheric pressure and temperature unless otherwise noted.]

Fuel	Fuel specification	Reference	Ignition method	Ignition atmosphere	Igniting surface	Ignition lag (sec)	Ignition temperature (°F)
Gasoline.....	Socony.....	40	(a)	Air.....	Pyrex.....		b 508
Aviation gasoline.....	62.0° A. P. I.; initial boiling point, 111° F.....	80	(a)	Air.....	Pyrex.....	50	532
Gasoline.....	Socony.....	40	(a)	Air.....	Copper.....		c 534
Gasoline.....	Socony.....	40	(a)	Air.....	Low carbon steel.....		c 550
Aviation gasoline.....	63.6° A. P. I.; boiling range, 111°-324° F.....	80	(a)	Air.....	Pyrex.....	6	554
Gasoline.....	Socony.....	40	(a)	Air.....	Chromium.....		c 559
Aviation gasoline.....	66.8° A. P. I.; unleaded; 73 octane.....	80	(a)	Air.....	Pyrex.....	13	563
Gasoline.....	66.8° A. P. I.; unleaded; 73 octane.....	61, 81	(a)	Air.....	Pyrex.....		570
Aviation gasoline.....	63.8° A. P. I.; boiling range, 109°-327° F.....	80	(a)	Air.....	Pyrex.....	6-7	590
Aviation gasoline.....	63.8° A. P. I.; boiling range, 106°-325° F.....	80	(a)	Air.....	Pyrex.....	8	594
Aviation gasoline.....	63.9° A. P. I.; boiling range, 135°-325° F.....	80	(a)	Air.....	Pyrex.....		624
Aviation gasoline.....	63.8° A. P. I.; boiling range, 109°-327° F.....	f 80	(a)	Air.....	Pyrex.....	4-5	640
Gasoline.....	68.9° A. P. I.; unleaded; 92 octane.....	61, 81	(a)	Air.....	Pyrex.....		734
Gasoline.....	66.1° A. P. I.; unleaded; 100 octane.....	61, 81	(a)	Air.....	Pyrex.....		804
Gasoline.....	Mean boiling point, 167° F.....	82	(d)	Oxygen.....	Steel.....		545
Gasoline.....	Unleaded; 75 octane; aromatic.....	39	(d)	Air.....			790
Safety fuel.....	Unleaded; 99 octane; paraffinic.....	39	(d)	Air.....			930
Aviation gasoline.....	Leaded; 100 octane.....	39	(d)	Air.....			960
Aviation gasoline.....	Straight run.....	78	(d)	Air.....	Steel.....	<60	830
Gasoline.....		83		Air.....		3	626
Gasoline.....	Unleaded; flash point, -44° F.....	16	(e)	Air.....	Pyrex.....		865
Gasoline.....		72	(f)	Air.....	Steel.....		473
Aviation gasoline.....	Leaded; 100 octane; flash point, -44° F.....	12	(f)	Air.....	Steel.....	7	599
Aviation gasoline.....		72	(f)	Air.....	Steel.....		734
Gasoline.....	From shale oil.....	25	(e)	Air.....	Copper.....		1004
Gasoline.....	Rumanian.....	24	(e)	Air.....	Iron or nickel.....		1040
Aviation gasoline.....	Unleaded; flash point, -44° F.....	72	(e)	Air.....	Iron or nickel.....		1085
Gasoline.....	Leaded; 100 octane; flash point, -44° F.....	72	(e)	Air.....	Steel.....		>1238
Aviation gasoline.....		25	(e)	Air.....	Steel.....		>1238
Gasoline.....	Boiling range, 100°-400° F; flash point, -50° to -45° F; specific gravity, 0.75.....	37, 84		Air.....	Iron.....		1328
Gasoline.....		38		Oxygen.....			495
Gasoline.....		85		Air.....			536
Gasoline.....	Shell.....	79	(d)	Air.....	Iron.....		608
Gasoline.....	Water white.....	39	(d)	Air.....			860
Kerosene.....	Flash point, 108° F.....	72	(f)	Air.....	Steel.....		520
Kerosene.....	Flash point, 108° F.....	72	(e)	Air.....	Steel.....		417
Kerosene.....	Flash point, 100°-165° F.....	37, 84		Air.....			1202
Kerosene.....	SAE 60, mid-continent solvent extracted.....	39	(d)	Air.....			490
Aircraft lubricating oil.....	Flash point, >500° F.....	78	(d)	Air.....	Steel.....	<60	750
Lubricating oil.....	Flash point, 482° F.....	72	(f)	Air.....	Steel.....		572
Lubricating oil.....	Flash point, 482° F.....	72	(e)	Air.....	Steel.....		806
Naval lubricating oil.....	N. S. 2135.....	86	(b)	Air.....	Pyrex.....		685
Turbine lubricating oil.....	Open-cup flash point, 400° F.....	84					700

a Static crucible method.

b Average of five trials.

c Lowest temperature measured.

d Dynamic crucible method.

e Dynamic heated-tube method.

f Liquid dropped into heated iron or steel tube (static).

g Liquid dropped on heated metal plate.

h ASTM D286-30.

TABLE VI—LIMITS OF INFLAMMABILITY OF GASOLINES AND KEROSENE IN AIR

[All results are from investigations conducted at atmospheric pressure and temperature unless otherwise noted.]

Fuel	Specification	Reference	Ignition source	Ignition environment	Lower limit (a)	Upper limit (a)
Gasoline.....		87	Spark or flame.....	Vertical tube.....	1.40	6.90
Gasoline.....	Specific gravity, 0.75; closed-cup flash point, -50° to -45° F.....	37, 84			1.3	6.0
Gasoline.....	Unleaded; 100 octane; 66.1° A. P. I.; RVP, 3.25 lb/sq in. at 77° F.....	61	Alcohol flame.....	Vertical tube.....	1.45	7.40
Gasoline.....	Unleaded; 92 octane; 68.9° A. P. I.; RVP, 3.52 lb/sq in. at 77° F.....	61	Alcohol flame.....	Vertical tube.....	1.50	7.60
Gasoline.....	Unleaded; 73 octane; 66.8° A. P. I.; RVP, 3.87 lb/sq in. at 77° F.....	61	Alcohol flame.....	Vertical tube.....	1.50	7.60
Kerosene.....	Closed-cup flash point, 100°-165° F.....	37, 84	Alcohol flame.....	Vertical tube.....	1.50	7.60
					1.16	6.0

a Inflammability limits are volumetric percentages of fuel in fuel-air mixture.

REPORT 1020

MEASUREMENTS OF AVERAGE HEAT-TRANSFER AND FRICTION COEFFICIENTS FOR SUBSONIC FLOW OF AIR IN SMOOTH TUBES AT HIGH SURFACE AND FLUID TEMPERATURES¹

By LEROY V. HUMBLE, WARREN H. LOWDERMILK, and LELAND G. DESMON

SUMMARY

An investigation of forced-convection heat transfer and associated pressure drops was conducted with air flowing through smooth tubes for an over-all range of surface temperature from 535° to 3050° R, inlet-air temperature from 535° to 1500° R, Reynolds number up to 500,000, exit Mach number up to 1, heat flux up to 150,000 Btu per hour per square foot, length-diameter ratio from 30 to 120, and three entrance configurations. Most of the data are for heat addition to the air; a few results are included for cooling of the air. The over-all range of surface-to-air temperature ratio was from 0.46 to 3.5.

Correlation of the measured average heat-transfer and friction coefficients with heat addition by conventional methods wherein the physical properties of the air were evaluated at the average air temperature resulted in considerable decreases in both the Nusselt number and the friction coefficient at constant Reynolds number in the turbulent region, as the ratio of surface temperature to air temperature was increased. The effect of surface-to-air temperature ratio was eliminated by evaluating the physical properties of air, including density, in the Reynolds number, the Nusselt number, and the friction coefficient at a temperature higher than the average air temperature.

Correlation of the heat-transfer coefficients was also affected by increases in the average air temperature, produced by increasing the inlet-air temperature, which resulted in a decrease in the Nusselt number. This effect of temperature level was reduced by using the smallest variation of thermal conductivity with temperature available in the literature and was eliminated when the thermal conductivity was arbitrarily assumed to vary as the square root of temperature.

INTRODUCTION

A large amount of data is available in the literature on forced-convection heat transfer from surfaces to fluids. Most of these data, however, have been obtained at relatively low surface temperatures and heat-flux densities, and do not extend into the range of high temperature and flux that is of interest in many current engineering applications. Inasmuch as convective heat transfer is a boundary-layer phenomenon, the severe temperature and velocity gradients

in the fluid film adjacent to the surface (and attendant variation in fluid properties), which accompany heat transfer at high flux densities, make extrapolation of existing data for low flux densities uncertain. This possibility has been previously recognized; for example, in references 1 and 2 it is indicated that heat transfer and friction may depend on both surface and fluid temperatures.

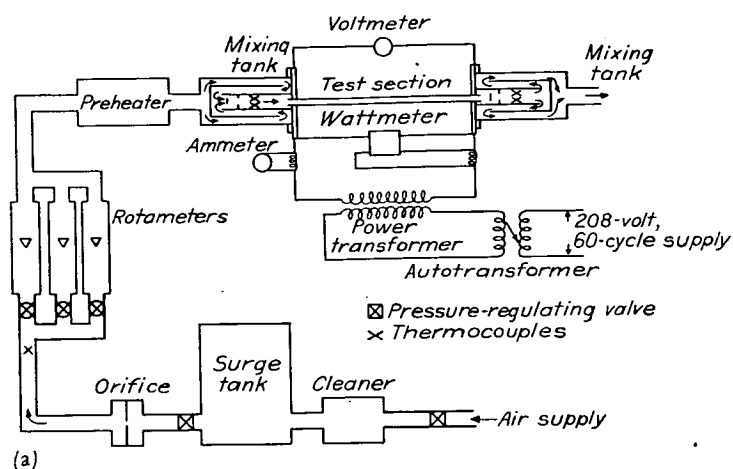
An experimental investigation was undertaken at the NACA Lewis laboratory during 1948-50 to obtain heat-transfer and related pressure-drop information for a wide range of surface and fluid temperatures and heat flux. As part of the general program, an investigation was made with air flowing through smooth tubes. The effects of such variables as surface temperature, inlet-air temperature, and tube-entrance configuration on heat transfer and pressure drop have been investigated and are reported in references 3 to 6. The results of references 3 to 6 are summarized herein and, in addition, previously unpublished data showing the effect of tube length-diameter ratio and inlet-air temperature on heat-transfer and friction coefficients at high surface temperature and heat flux are presented. Most of the data are for heat addition to the air; however, a few previously unpublished results are included for heat extraction from the air.

APPARATUS

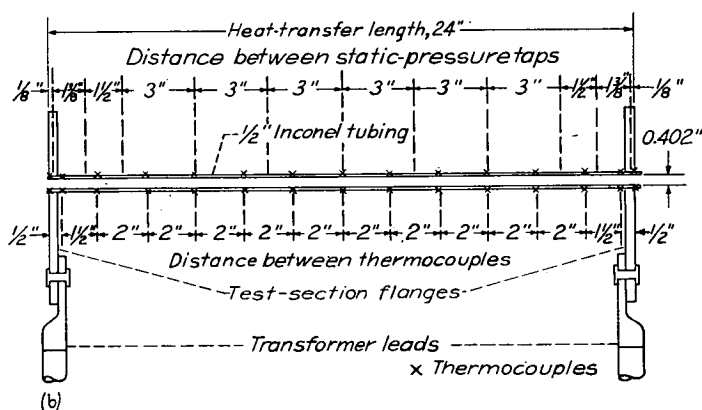
ARRANGEMENT

A schematic diagram of the equipment used in the investigation is shown in figure 1 (a). Compressed air was supplied through a pressure-regulating valve, cleaner, and surge tank to a second pressure-regulating valve where the flow rate was controlled. From this second valve, the air flowed through metering devices and a preheater into a mixing tank, which consisted of three concentric passages so arranged that the air made three passes through the tank before entering the test section. Baffles were provided in the central passage to insure thorough mixing of the air before it entered the test section. From the test section the air flowed through a second mixing tank and was discharged to the atmosphere. The test section, mixing tanks, and adjoining piping were thermally insulated.

¹ Supersedes NACA RM E7L31, "Heat Transfer from High-Temperature Surfaces to Fluids. I—Preliminary Investigation with Air in Inconel Tube with Rounded Entrance, Inside Diameter of 0.4 Inch, and Length of 24 Inches" by Leroy V. Humble, Warren H. Lowdermilk, and Milton Grele, 1948; NACA RM ESL03, "Heat Transfer from High-Temperature Surfaces to Fluids. II—Correlation of Heat-Transfer and Friction Data for Air Flowing in Inconel Tube with Rounded Entrance" by Warren H. Lowdermilk and Milton D. Grele, 1949; NACA RM E50E23, "Influence of Tube-Entrance Configuration on Average Heat-Transfer Coefficients and Friction Factors for Air Flowing in an Inconel Tube" by Warren H. Lowdermilk and Milton D. Grele, 1950; NACA RM E50H23, "Correlation of Forced-Convection Heat-Transfer Data for Air Flowing in Smooth Platinum Tube with Long-Approach Entrance at High Surface and Inlet-Air Temperatures" by Leland G. Desmon and Eldon W. Sams, 1950.



(a) Schematic diagram of setup.



(b) Typical test section showing thermocouple and pressure-tap locations.

FIGURE 1.—Arrangement of apparatus for heating air.

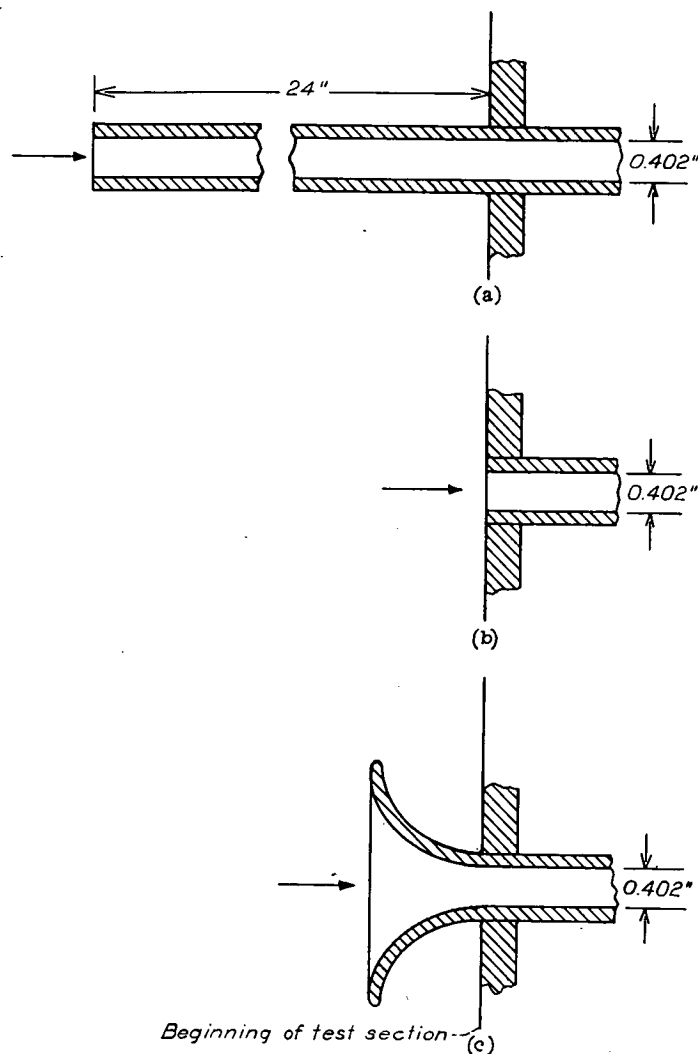
The temperature of the air entering and leaving the test section was measured by thermocouples located downstream of the mixing baffles in the entrance and exit mixing tanks, respectively.

For the runs with heat addition to the air, electric power was supplied to the test section from a 208-volt, 60-cycle supply line through an autotransformer and a step-down power transformer. The low-voltage leads from the power transformer were connected to flanges on the test section by flexible copper cables. A voltmeter, an ammeter, and a wattmeter were provided to measure the electric-power input to the test section.

For the runs with heat extraction from the air, the electric-power supply was disconnected and the test section was surrounded with a water jacket. The warmed air from the preheater was cooled in the water-jacketed test section.

TEST SECTIONS

Most of the investigation was conducted using test sections fabricated from commercial Inconel tubing. The range of tube-wall temperature was extended beyond that attainable with Inconel by using one test section made of platinum (reference 6), which was installed in a separate setup similar to that shown in figure 1 (a). One of the test sections used



(a) Long approach.
(b) Right-angle edge.
(c) Bellmouth.

FIGURE 2.—Entrances to test sections.

for heating the air is illustrated schematically in figure 1 (b). Flanges were attached to both ends of the tube to provide electric contact with the power supply. The dimensions of the various tubes used as test sections in the investigation were as follows:

Tube material	Inside diameter (in.)	Outside diameter (in.)	Heat-transfer length (in.)	Length-diameter ratio
Inconel.....	0.402	0.500	12	30
Do.....	.402	.500	24	60
Do.....	.402	.500	48	120
Platinum.....	.525	.685	24	46

Outside-wall temperatures were measured at a number of stations along the tube by means of thermocouples and self-balancing indicating-type potentiometers. Chromel-alumel thermocouples were used with the Inconel test sections, and platinum-platinum-rhodium thermocouples, with the platinum test section. In general, at each station along the tube, two thermocouples were located diametrically opposite and

the average of the two temperatures was taken as the outside-wall temperature at the station. Static-pressure taps were located at intervals along each of the Inconel test sections.

Investigations were made with heat addition to the air using three types of entrance on the 24-inch Inconel tube (reference 5). The entrances, shown in figure 2, were (a) long approach, (b) right-angle edge, and (c) bellmouth.

The cooling data were obtained with an Inconel test section having a length-diameter ratio of 60 and a bellmouth entrance.

PROCEDURE

The following general procedure was used in obtaining the experimental data: For the heating runs, the inlet-air temperature was set at the desired value, the inlet-air pressure was adjusted to give the minimum desired flow rate (minimum Reynolds number), and the electric-power input was adjusted to give the desired tube-wall temperature. After equilibrium conditions had been attained, electric-power input, flow rate, temperatures, and pressures were recorded. The air flow and the power input were then increased in increments (to increase the Reynolds number while maintaining constant wall temperature) and data were recorded after each incremental increase. The foregoing procedure was repeated for a range of tube-wall temperature with each of the various test sections and entrance configurations.

For the cooling runs, the inlet-air pressure was adjusted to give the minimum desired air-flow rate at the desired inlet-air temperature. The water-flow rate in the water jacket was set at a constant value. After equilibrium conditions were obtained, the data were recorded as in the heating runs. The inlet-air pressure was then incrementally varied to cover a range of Reynolds number while maintaining constant inlet-air temperature. The foregoing procedure was repeated for a range of inlet-air temperature.

The over-all range of conditions for which data were obtained is summarized in the following table:

Reference	Tube material	Length-diameter ratio	Entrance type	Maximum bulk Reynolds number	Average tube-wall temperature (°R)	Inlet-air temperature (°R)	Maximum exit Mach number
3	Inconel	60	Bellmouth	250,000	680-1700	535	1.0
4	do.	60	do.	500,000	535-2050	535	1.0
5	do.	60	Long approach, right angle	375,000	535-1960	535	1.0
6	Platinum	46	Long approach, right angle	320,000	980-3050	535-1160	0.6
(a)	Inconel	30	Bellmouth	390,000	535-1685	535	1.0
(a)	do.	120	do.	282,000	535-1850	535-1400	1.0
(a), (b)	do.	60	do.	250,000	535-585	535-1500	1.0

* Previously unpublished.
 b Cooling data.

The over-all range of heat-flux density encountered in the investigation was from 500 to 150,000 Btu per hour per square foot of heat-transfer area.

SYMBOLS

The following symbols are used in the report:

c_p	specific heat of air at constant pressure (Btu/(lb) (°F))
D	inside diameter of test section (ft)
f	average friction coefficient
f_f, f_s	modified average friction coefficient
G	mass velocity (mass flow per unit cross-sectional area) (lb/(hr) (sq ft))
g	acceleration due to gravity (4.17×10^8 ft/hr ²)
h	average heat-transfer coefficient (Btu/(hr) (sq ft) (°F))
k	thermal conductivity of air (Btu/(hr) (sq ft) (°F/ft))
k_t	thermal conductivity of test-section material (Btu/(hr) (sq ft) (°F/ft))
L	heat-transfer length of test section (ft)
P	absolute total or stagnation pressure (lb/(sq ft))
p	absolute static pressure (lb/(sq ft))
Δp	over-all static-pressure drop across test section (lb/(sq ft))
Δp_f	friction static-pressure drop across test section (lb/(sq ft))
Q	rate of heat transfer to air (Btu/hr)
R	gas constant for air (53.35 ft-lb/(lb) (°F))
r	radius of test section (ft)
S	heat-transfer area of test section (sq ft)
T	total or stagnation temperature (°R)
T_b	average bulk temperature, defined by $(T_1 + T_2)/2$, (°R)
T_f	average film temperature, defined by $(T_s + T_b)/2$, (°R)
$T_{0.75}$	average film temperature, defined by $T_b + 0.75(T_s - T_b)$, (°R)
T_o	average outside-wall temperature of test section (°R)
T_s	average inside-wall (surface) temperature of test section (°R)
t	static temperature (°R)
V	velocity (ft/hr)
V_{av}	average velocity, defined by G/ρ_{av} , (ft/hr)
W	air flow (lb/hr)
x	distance from entrance of test section (ft)
γ	ratio of specific heats of air
μ	absolute viscosity of air (lb/(hr) (ft))
ρ	density of air (lb/(cu ft))
ρ_{av}	average density of air defined by $(p_1 + p_2)/R(t_1 + t_2)$, (lb/(cu ft))
$c_p \mu / k$	Prandtl number
GD/μ	Reynolds number
$\rho_f V_b D / \mu_f$	modified Reynolds number
$\rho_{0.75} V_b D / \mu_{0.75}$	
$\rho_s V_b D / \mu_s$	
hD/k	Nusselt number

Subscripts:

1	test-section entrance
2	test-section exit
<i>i</i>	inner surface of test section
<i>o</i>	outer surface of test section
<i>b</i>	bulk (when applied to properties, indicates evaluation at average bulk temperature T_b)
<i>f</i>	film (when applied to properties, indicates evaluation at average film temperature T_f)
0.75	film (when applied to properties, indicates evaluation at $T_{0.75}$)
<i>s</i>	surface (when applied to properties, indicates evaluation at average inside-wall temperature of test section T_s)

METHOD OF CALCULATION

Physical properties of air.—The physical properties of air used in the present investigation are shown in figure 3 as functions of temperature. The solid curves are from references 7 and 8, and are in agreement with the values of reference 9. The dashed curve for thermal conductivity is from reference 10. The dotted curve represents values of thermal conductivity assumed herein to vary as the square root of temperature and its use will be discussed in a subsequent section. The thermal conductivities from references 7 to 10 were

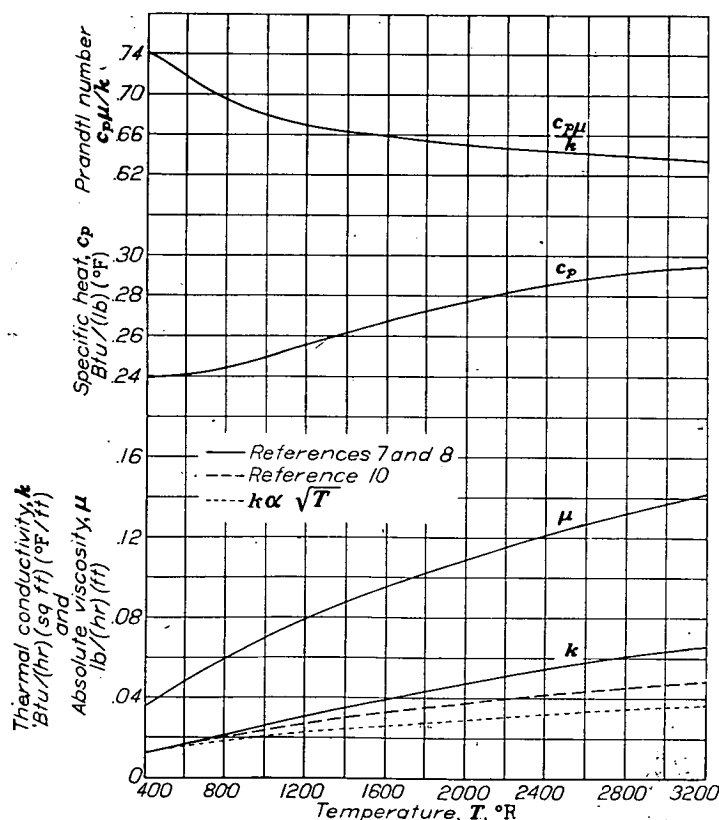


FIGURE 3.—Variation of thermal conductivity, absolute viscosity, specific heat, and Prandtl number of air with temperature.

extrapolated considerably beyond the range of experimental data in the reference reports and have been extrapolated still further in figure 3 to cover the temperature range of the present investigation. The upper temperature limits of the experimental data upon which values of thermal conductivity from references 7, 8, and 10 are based are 1050°, 1360°, and 1215° R, respectively.

Except where otherwise stated, the physical property data of references 7 and 8 are used in the present report.

Determination of surface temperature.—The average outside-wall temperature of the test section T_o was obtained by measuring the area under a curve of the temperature distribution (as obtained from thermocouple readings) along the heat-transfer length and dividing the area by this length.

For the heating runs, the average inside-wall (surface) temperature T_s was then calculated by the following equation (derived in reference 11):

$$T_s = T_o - \frac{Q}{2\pi L k_i (r_o^2 - r_i^2)} \left(r_o^2 \ln \frac{r_o}{r_i} - \frac{r_o^2 - r_i^2}{2} \right) \quad (1)$$

In equation (1), the assumptions are made that heat is generated uniformly across the tube-wall thickness and that the heat flow from every point in the tube material is radially inward. For the present investigation, the radial temperature drop through the tube wall, calculated from equation (1), was very small compared with the difference between the average inside-wall and the air bulk temperature.

For the cooling runs, the thermocouples were imbedded in the test-section wall and were assumed to measure the inside-wall temperature, inasmuch as the calculated temperature drops through the wall were small.

Heat-transfer coefficients.—The average heat-transfer coefficient h was computed from the experimental data by the relation

$$h = \frac{W c_{p,a} (T_2 - T_1)}{S (T_s - T_b)} \quad (2)$$

The use of the total air temperature T_b instead of adiabatic wall temperature was considered justified because the difference between these temperatures was small compared with the difference between the average surface and air (or adiabatic wall) temperatures.

Correlation of the average heat-transfer coefficients with the pertinent variables is discussed in the section **RESULTS AND DISCUSSION**.

Friction coefficients.—Friction data were obtained with and without heat transfer. Average friction coefficients were calculated from the experimental pressure-drop data as follows: The friction pressure drop was obtained by subtracting the calculated momentum pressure drop from the measured static-pressure drop across the test section. Thus

$$\Delta p_{fr} = \Delta p - \frac{G}{g} (V_2 - V_1) = \Delta p - \frac{G^2 R}{g} \left(\frac{t_2}{p_2} - \frac{t_1}{p_1} \right) \quad (3)$$

where t_1 and t_2 , the absolute static temperatures at entrance and exit of the test section, respectively, were, in general, calculated from the measured values of air flow, static pressure, and total temperature by the following equation, which is obtained by combining the perfect gas law, the equation of continuity, and the energy equation:

$$t = -\frac{\gamma g}{(\gamma-1)R} \left(\frac{p}{G}\right)^2 + \sqrt{\left[\frac{\gamma g}{(\gamma-1)R} \left(\frac{p}{G}\right)^2\right]^2 + 2T \frac{\gamma g}{(\gamma-1)R} \left(\frac{p}{G}\right)^2} \quad (4)$$

For the bellmouth entrance, it was found that the entrance static temperature t_1 could be represented with sufficient accuracy by the relation

$$t_1 = T_1 \left(\frac{p_1}{P_1}\right)^{\frac{\gamma-1}{\gamma}} \quad (5)$$

In equation (5) the total pressure at the test-section entrance P_1 was assumed to be equal to the static pressure in the entrance mixing tank, where the velocity was negligible.

Average friction coefficients were calculated by dividing the friction pressure drop by four times the length-diameter ratio of the tube times a dynamic pressure. The density in the dynamic pressure was computed in three ways, resulting in three definitions of friction coefficient.

The first of these friction coefficients was based on a density evaluated at the average static pressure and temperature of the air in the tube; thus

$$f = \frac{\Delta p_{fr}}{4 \frac{L}{D} \frac{\rho_{av} V_{av}^2}{2g}} = \frac{g \rho_{av} \Delta p_{fr}}{2 \frac{L}{D} G^2} \quad (6)$$

where

$$\rho_{av} = \frac{1}{R} \left(\frac{p_1 + p_2}{t_1 + t_2}\right) \quad (7)$$

In the other two definitions of friction coefficient, the density was evaluated at the film temperature T_f and at the average surface temperature T_s , resulting in equations (8) and (9), respectively:

$$f_f = \frac{\Delta p_{fr}}{4 \frac{L}{D} \frac{\rho_f V_{av}^2}{2g}} = \left(\frac{\rho_{av}}{\rho_f}\right) f = \left(\frac{2T_f}{t_1 + t_2}\right) f \quad (8)$$

$$f_s = \frac{\Delta p_{fr}}{4 \frac{L}{D} \frac{\rho_s V_{av}^2}{2g}} = \left(\frac{\rho_{av}}{\rho_s}\right) f = \left(\frac{2T_s}{t_1 + t_2}\right) f \quad (9)$$

Correlation of the friction data is discussed in the following section.

RESULTS AND DISCUSSION

AXIAL WALL-TEMPERATURE DISTRIBUTIONS

Representative axial wall-temperature distributions obtained with heat addition to the air in an Inconel tube having a length-diameter ratio of 60 and a bellmouth entrance

are shown in figure 4. The outside-wall temperature is plotted against the ratio of the distance from the tube entrance to the heat-transfer length x/L for five different rates of heat transfer to the air. The rate of heat transfer to the air, the air flow, the average inside tube-wall temperature, and the temperature rise of the air are tabulated.

The wall temperature increases almost linearly over the major portion of the tube length, and the slope of this part of the curve increases with an increase in rate of heat input. The large axial-temperature gradients at the entrance and exit of the test section are the result of conduction losses through the electric connector flanges and cables.

The curves in figure 4 are typical; similar trends were obtained with other entrances and for other length-diameter ratios.

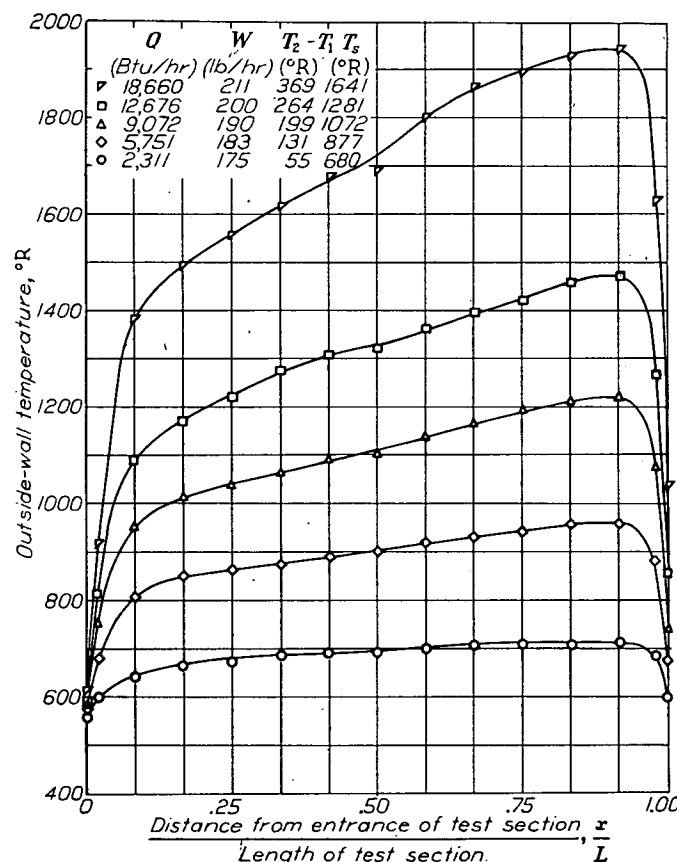


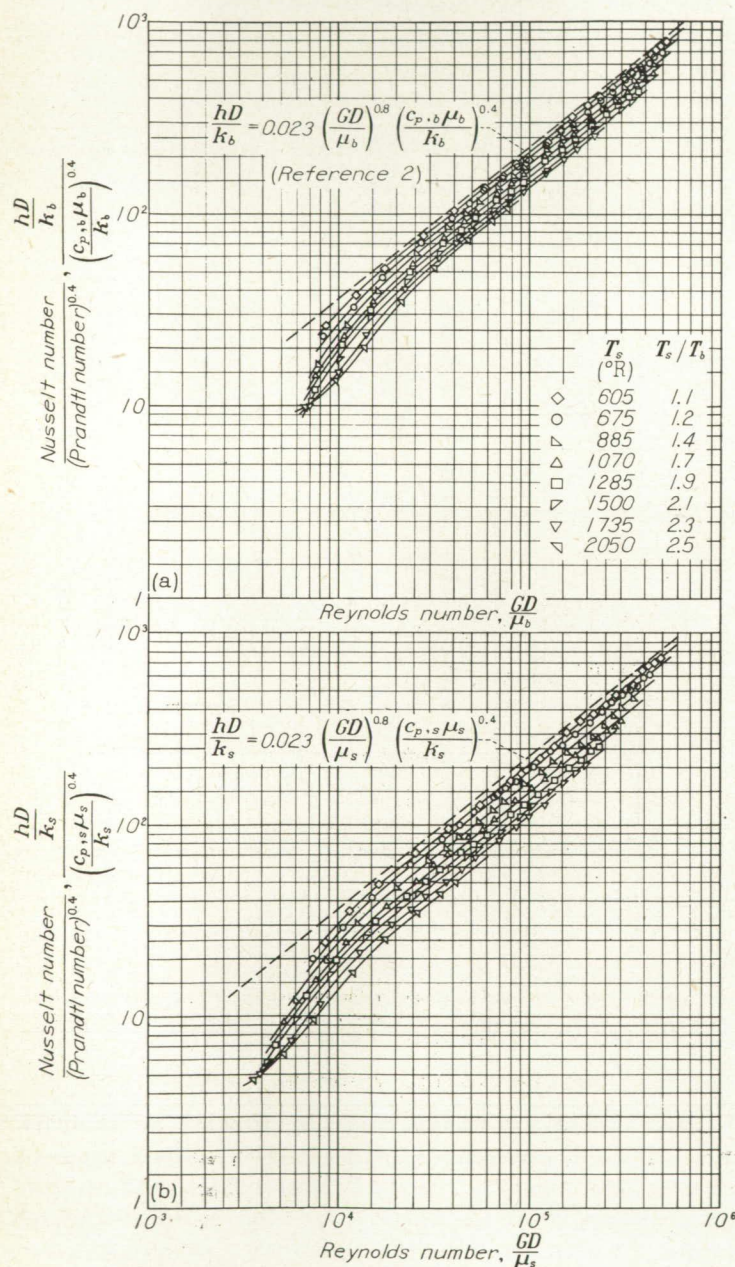
FIGURE 4.—Representative outside-wall temperature distribution for various amounts of heat input to air. Reynolds number, approximately 140,000; Inconel tube; length-diameter ratio, 60; bellmouth entrance; inlet-air temperature, 535° R.

CORRELATION OF HEAT-TRANSFER COEFFICIENTS

Conventional correlation based on average air temperature.—Forced-convection, turbulent-flow, heat-transfer coefficients are generally correlated with other pertinent variables by means of the familiar relation in which Nusselt number divided by Prandtl number to a power (generally 0.4) $\frac{hD}{k} / \left(\frac{c_p \mu}{k}\right)^{0.4}$ is plotted against Reynolds number GD/μ , and in which the physical properties of the fluid (specific heat, viscosity, and thermal conductivity) are usually eval-

uated at the average fluid temperature. Results obtained by heating air flowing through an Inconel tube having a length-diameter ratio of 60 and a bellmouth entrance (reference 4) are plotted in this manner in figure 5 (a). The data shown are for an inlet-air total temperature of 535° R and cover a range of Reynolds number from about 7000 to 500,000. Included for comparison is the average line obtained by McAdams (reference 2) from correlation of the results of various investigators. The equation corresponding to this reference line is

$$\frac{hD}{k_b} = 0.023 \left(\frac{GD}{\mu_b} \right)^{0.8} \left(\frac{c_{p,b} \mu_b}{k_b} \right)^{0.4} \quad (10)$$



(a) Physical properties of air evaluated at average bulk temperature.

(b) Physical properties of air evaluated at average surface temperature.

FIGURE 5.—Conventional methods of correlating heat-transfer coefficients. Inconel tube; length-diameter ratio, 60; bellmouth entrance; inlet-air temperature, 535° R.

A family of parallel lines representing the various surface-to-air temperature ratios and having slopes of about 0.8 at Reynolds numbers above approximately 25,000 is obtained.

The data for low surface temperatures (T_s , 605° R) are in reasonable agreement with the reference line; however, as the surface temperature is raised the data fall progressively below the reference line. For example, at a Reynolds number of 70,000, an increase in T_s/T_b from 1.1 to 2.5 (corresponding to an increase in surface temperature from 605° to 2050° R) results in a decrease in the ordinate parameter of about 38 percent. Inasmuch as the corresponding change in Prandtl number to the 0.4 power is small, about the same percentage decrease occurs in Nusselt number.

The results shown in figure 5 (a) are typical of those obtained with other entrance configurations and length-diameter ratios.

Conventional correlation based on surface temperature.—

The heat-transfer data of figure 5 (a) are replotted in figure 5(b), wherein the physical properties of the air are evaluated at the average inside-wall temperature T_s . The line representing equation (10), but with the physical properties evaluated at T_s , is included for comparison. The data again show a separation with surface temperature level, which in this case is greater than when the properties are evaluated at the bulk temperature. The use of any temperature between these extremes would, of course, cause an intermediate amount of separation of the data.

In references 1 and 12, heat-transfer coefficients were predicted for fully developed flow of fluids having a Prandtl number of 1.0 in smooth tubes. The analyses took into consideration the radial variation of fluid properties and indicated an effect of surface-to-fluid temperature ratio analogous to that shown in figure 5.

Effect of surface-to-fluid temperature ratio.—The magnitude of the effect on the heat-transfer correlations of surface-to-fluid temperature ratios greater than 1.0 is illustrated more clearly in figure 6. Lines are shown for which the air properties are evaluated at bulk temperature T_b , film temperature T_f , and surface temperature T_s . The lines corresponding to evaluation of properties at T_b and T_s are cross plots of the data at a Reynolds number of 100,000 from figures 5 (a) and 5 (b), respectively. The line corresponding to evaluation of air properties at T_f was obtained from a plot similar to those of figure 5.

A series of straight lines is obtained, which intersect at T_s/T_b of 1.0, at which point the ordinate parameter has the value predicted by equation (10). The lines have increasingly greater negative slopes as the temperature at which the fluid properties are evaluated is increased. For example, an increase in T_s/T_b from 1.0 to 2.5 results in a decrease in the ordinate parameter of about 40 percent when the properties are evaluated at T_b , of 48 percent when evaluated at T_f , and of 54 percent when evaluated at T_s .

Appreciable error can be made (fig. 6) if conventional heat-transfer correlation equations are used to compute heat-transfer coefficients for surface-to-fluid temperature ratios appreciably greater than 1.0.

Modified correlation based on surface temperature.—

The slope of the line in figure 6 corresponding to evaluation of the fluid properties at T_s is -0.8 . The ordinate in this case is $(hD/k_s)/(c_{p,s}\mu_s/k_s)^{0.4}$ and the corresponding Reynolds number is defined by GD/μ_s . Inasmuch as at $T_s/T_b=1.0$ equation (10) applies, it might be expected that the effect of T_s/T_b could be eliminated by incorporating the term $(T_b/T_s)^{0.8}$ in equation (10) with the properties evaluated at T_s . Thus

$$\frac{hD}{k_s} = 0.023 \left(\frac{GD}{\mu_s} \right)^{0.8} \left(\frac{T_b}{T_s} \right)^{0.8} \left(\frac{c_{p,s}\mu_s}{k_s} \right)^{0.4} \quad (11)$$

If the difference between total and static temperatures is neglected,

$$\frac{T_b}{T_s} = \frac{\rho_s}{\rho_b} \quad (12)$$

and the following relation is obtained by letting $G = \rho_b V_b$:

$$\left(\frac{GD}{\mu_s} \right) \left(\frac{T_b}{T_s} \right) = \frac{\rho_s V_b D}{\mu_s} \quad (13)$$

Equation (13) defines a modified form of Reynolds number in which the conventional mass velocity G is replaced by the product of air density evaluated at the average surface temperature ρ_s and air velocity evaluated at the average bulk temperature V_b . Substituting equation (13) in equation (11) gives

$$\frac{hD}{k_s} = 0.023 \left(\frac{\rho_s V_b D}{\mu_s} \right)^{0.8} \left(\frac{c_{p,s}\mu_s}{k_s} \right)^{0.4} \quad (14)$$

The data of figure 5 are replotted in figure 7. Nusselt

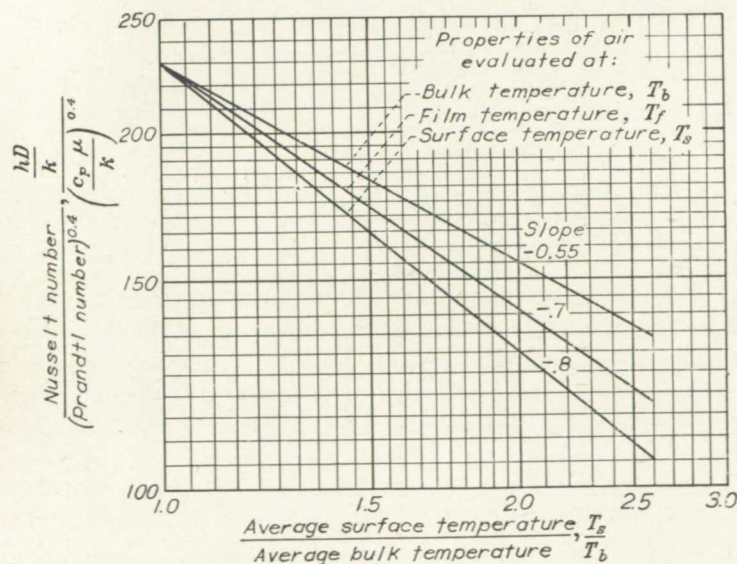


FIGURE 6.—Variation of Nusselt number divided by Prandtl number to 0.4 power with ratio of surface-to-bulk temperature using conventional methods of correlating heat-transfer coefficients. Reynolds number, 100,000; Inconel tube; length-diameter ratio, 60; bellmouth entrance; inlet-air temperature, 535° R.

number divided by Prandtl number to the 0.4 power is plotted against the modified Reynolds number $\rho_s V_b D / \mu_s$.

The use of the modified Reynolds number eliminates the trends with respect to surface-to-bulk temperature ratio, so that for the range of surface temperature and Reynolds number shown the data can be represented with good accuracy by a single line. For modified Reynolds numbers above about 10,000, the data can be represented very well by equation (14).

Effect of entrance configuration.—Heating data obtained with an Inconel tube having a length-diameter ratio of 60 and both long-approach and right-angle-edge entrances are shown in figure 8. The coordinates are the same as in figure 7, and the line representing the data of figure 7 (bellmouth entrance) is included in the figure.

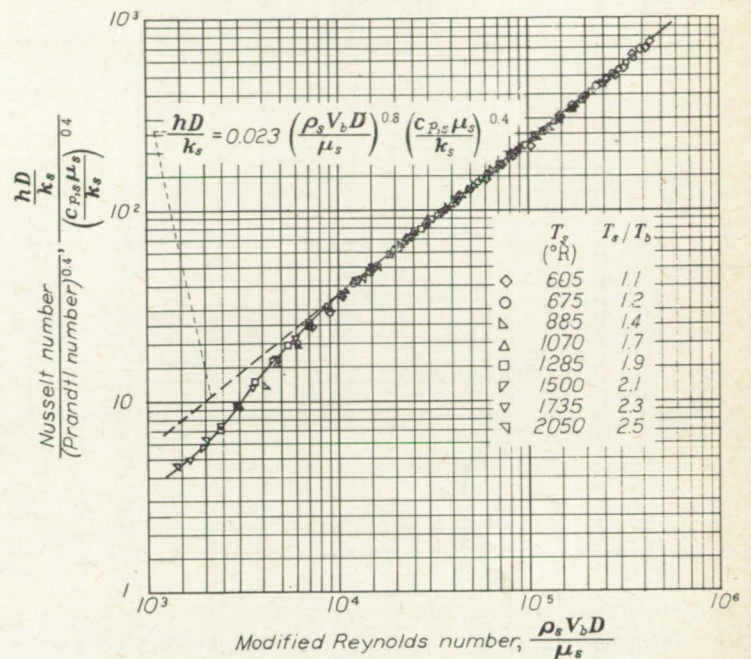


FIGURE 7.—Modified method of correlating heat-transfer coefficients. Inconel tube; length-diameter ratio, 60; bellmouth entrance; inlet-air temperature, 535° R; physical properties of air evaluated at surface temperature.

The long-approach and bellmouth entrances tend to provide fully developed and uniform velocity profiles, respectively, at the inlet to the test section. The right-angle-edge entrance is typical of most tube-type heat exchangers.

As shown in figure 8, the data obtained with the long-approach and right-angle-edge entrances can be represented with good accuracy by the line which best fitted the data for the bellmouth entrance shown in figure 7. It may be concluded that the average heat-transfer coefficient is negligibly affected by the entrance configurations investigated, at least for length-diameter ratios of the order of 60 or greater.

Effect of length-diameter ratio.—Heating data obtained with Inconel tubes having length-diameter ratios of 30 and 120 and with a platinum tube having a length-diameter ratio of 46 (reference 6) are shown in figure 9 (a). The coordinates are the same as in figures 7 and 8 and the line (shown solid) representing the data of figures 7 and 8, which

are for a length-diameter ratio of 60, is included. The Inconel tubes had bellmouth entrances, and the platinum tube had a long-approach entrance; however, as indicated in figure 8, the effect of entrance configuration is negligible.

A small trend with respect to length-diameter ratio is evident in the turbulent region. The data for length-diameter ratios less than 60 tend to fall parallel to, but above, the reference line, whereas the data for a length-diameter ratio of 120 fall below the line. This effect of length-

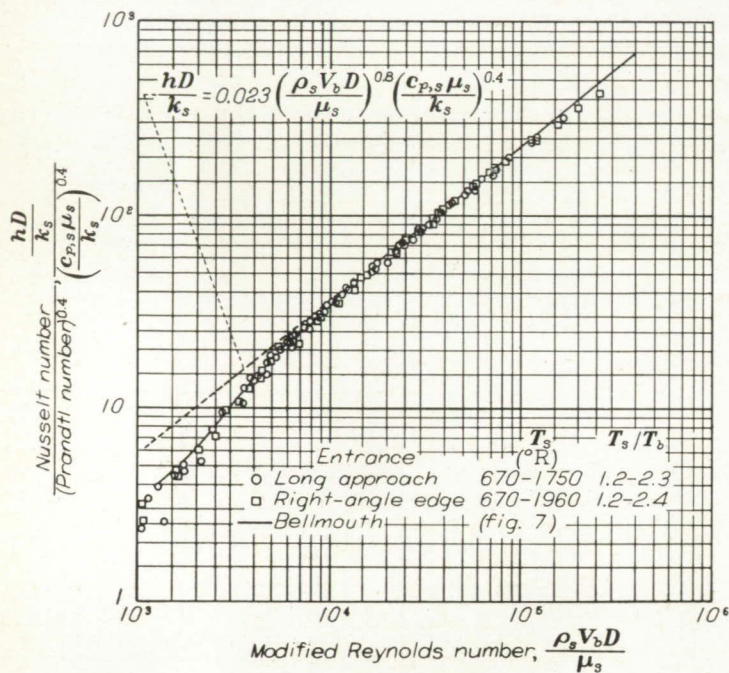


FIGURE 8.—Effect of entrance configuration using modified method of correlating heat-transfer coefficients. Inconel tube; length-diameter ratio, 60; inlet-air temperature, 535° R; physical properties of air evaluated at surface temperature.

diameter ratio has been observed by other investigators (see, for example, references 1 and 13) and was accounted for by including a power function of the length-diameter ratio in the general correlation equation. The exponent 0.054 is quoted in reference 1 and the value 0.1 is quoted in reference 13. The exponent 0.1 applies best for the present data and accordingly the data of figures 7 and 9 (a), corrected for the effect of length-diameter ratio, are replotted in figure 9 (b).

The correction for length-diameter ratio appreciably improves the correlation for modified Reynolds numbers above approximately 10,000, and in this region the data can be represented by the following equation:

$$\frac{hD}{k_s} = 0.034 \left(\frac{\rho_s V_b D}{\mu_s} \right)^{0.8} \left(\frac{c_{p,s} \mu_s}{k_s} \right)^{0.4} \left(\frac{L}{D} \right)^{-0.1} \quad (15)$$

The constant 0.034 is 0.023 (equation (14)) multiplied by $60^{0.1}$, where 60 is the length-diameter ratio of the tube used in obtaining the data upon which equation (14) is based.

The data for modified Reynolds numbers below about 10,000 show a trend with respect to length-diameter ratio and cannot be represented by a single line.

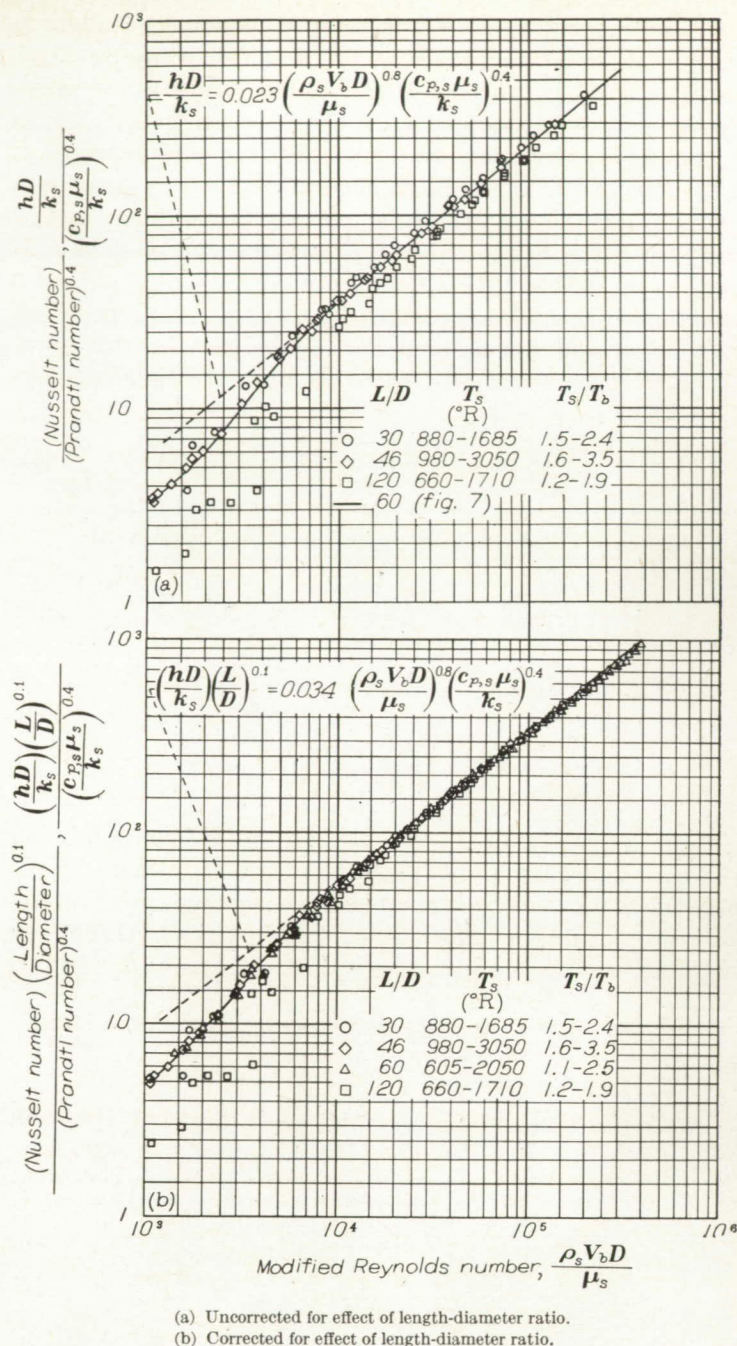


FIGURE 9.—Effect of ratio of length to diameter using modified method of correlating heat-transfer coefficients. Inconel and platinum tubes; inlet-air temperature, 535° R; physical properties of air evaluated at surface temperature.

The data shown in figure 9 for the platinum tube (L/D , 46) were obtained for surface temperatures up to 3050° R and surface-to-air temperature ratios up to 3.5; it is of interest that, for this extended range of temperature, the data are correlated with negligible scatter by use of the modified Reynolds number.

Effect of air temperature.—The data presented thus far were obtained with an inlet-air temperature of about 535° R. A wide range of surface temperature and corresponding surface-to-air temperature ratio was investigated. In order to determine whether the correlation applied for a wider

range of air temperature, tests were made in which the inlet-air temperature, and hence average air temperature, was varied. The results are shown in figure 10.

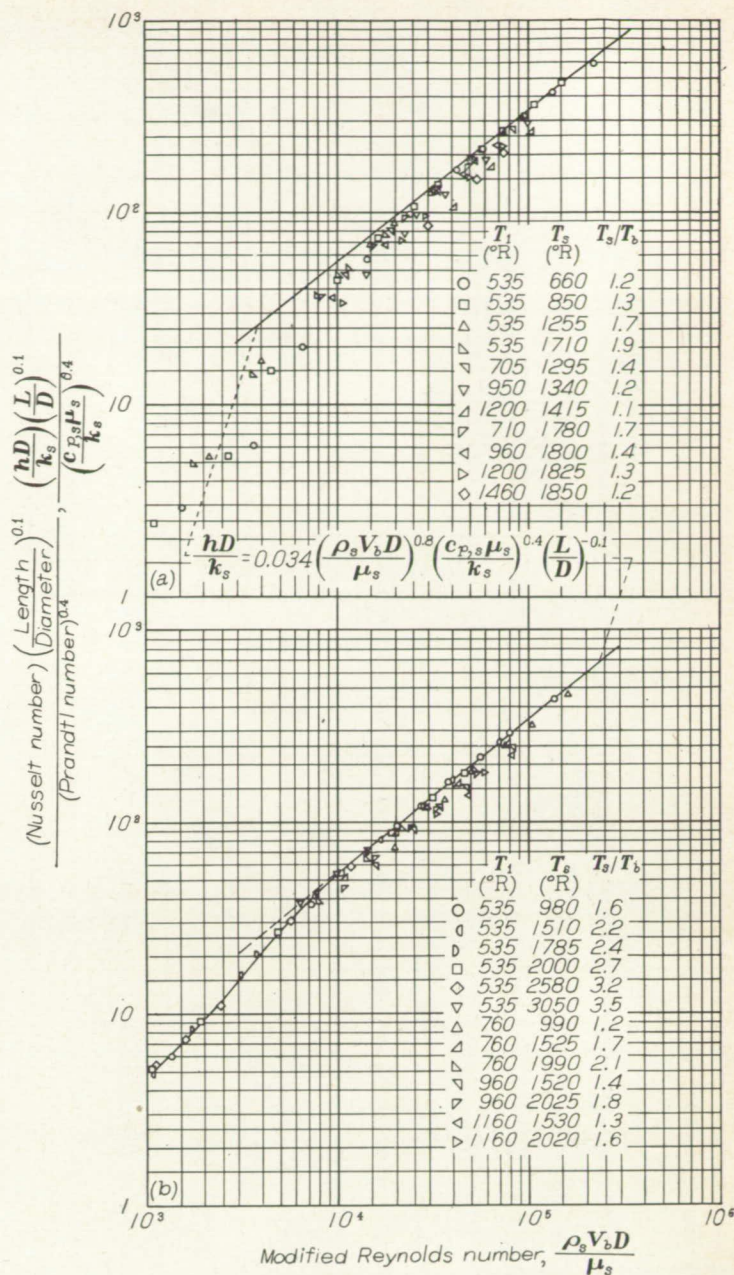
Inspection of figure 10 indicates that the ordinate parameter decreases as the inlet-air temperature and the corresponding average air temperature increase. For example, in figure 10 (a), at a Reynolds number of approximately 100,000 and a surface temperature of approximately 1800° R, an increase in inlet-air temperature from 535° to 1460° R (with a corresponding decrease in T_s/T_b from 1.9 to 1.2) results in a reduction in the ordinate parameter of about 24 percent. The ordinate parameter would decrease further if the inlet temperature were increased until T_s/T_b was approximately 1.0. At constant values of T_s/T_b , the ordinate parameter varies approximately as temperature to the -0.2 power.

As much as one-fourth of the observed separation of the data with temperature level may be caused by experimental error in measuring air temperature at the high temperature levels. Inasmuch as there is a considerable variation in the reported values of thermal conductivity at high temperature, as is evident in figure 3, the effect may be partly or entirely the result of using improperly extrapolated values of thermal conductivity.

The values of conductivity from references 7 and 8, which have been used thus far in correlating the present data, vary approximately as the 0.85 power of the absolute temperature. The values from reference 10 vary approximately as the 0.73 power of the temperature. Inasmuch as the ordinate parameter of figure 10 varies inversely as conductivity to the 0.6 power, the conductivity would have to vary as temperature to the 0.85 power minus 0.2 divided by 0.6, or to the 0.5 power, in order to completely eliminate the effect of temperature level in figures 10 (a) and (b).

Inspection of figures 5 and 6 indicates that the use of smaller values of thermal conductivity at high temperatures in correlation of the data would also result in a decrease of the effect of surface-to-air temperature ratio, so that correlation of the data according to equation (15), in which the fluid physical properties and density are evaluated at the surface temperature, would result in an overcorrection of the data with increasing values of surface-to-air temperature ratio. A relation of the form of equation (15) can, however, be used for correlation of the data by evaluating the fluid physical properties and density at a reference temperature lower than the surface temperature. For example, if conductivity is assumed to vary as temperature to the 0.73 power, the effect of surface-to-air temperature ratio can be decreased by using a reference temperature $T_{0.75}$ midway between T_f and T_s . If conductivity is assumed to vary as the square root of temperature, the effect can be eliminated by using the film temperature T_f as the reference temperature.

In order to illustrate the effect of using smaller values of thermal conductivity at high temperatures, the data of



(a) Physical properties of air evaluated at surface temperature; thermal conductivity proportional to $T^{0.85}$; Inconel tube; length-diameter ratio, 120.

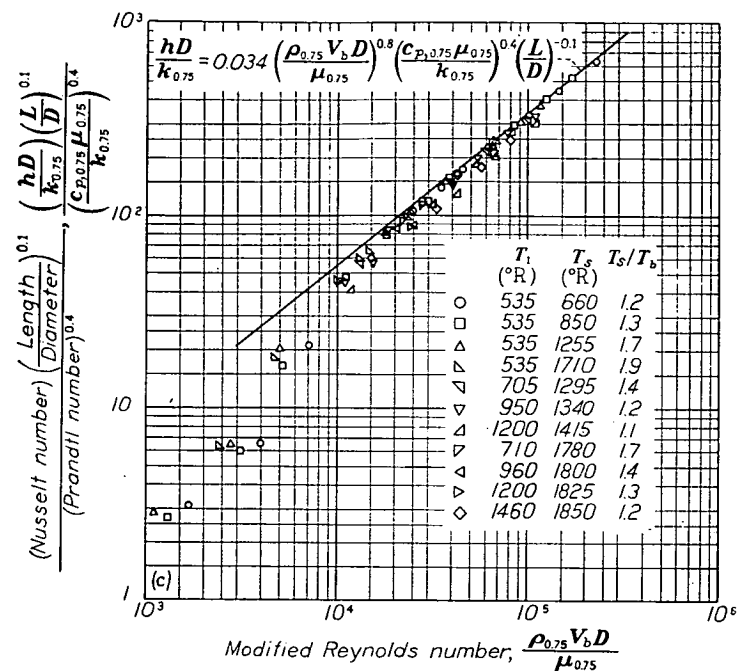
(b) Physical properties of air evaluated at surface temperature; thermal conductivity proportional to $T^{0.85}$; platinum tube; length-diameter ratio, 46.

FIGURE 10.—Effect of inlet-air temperature using modified method of correlating heat-transfer coefficients.

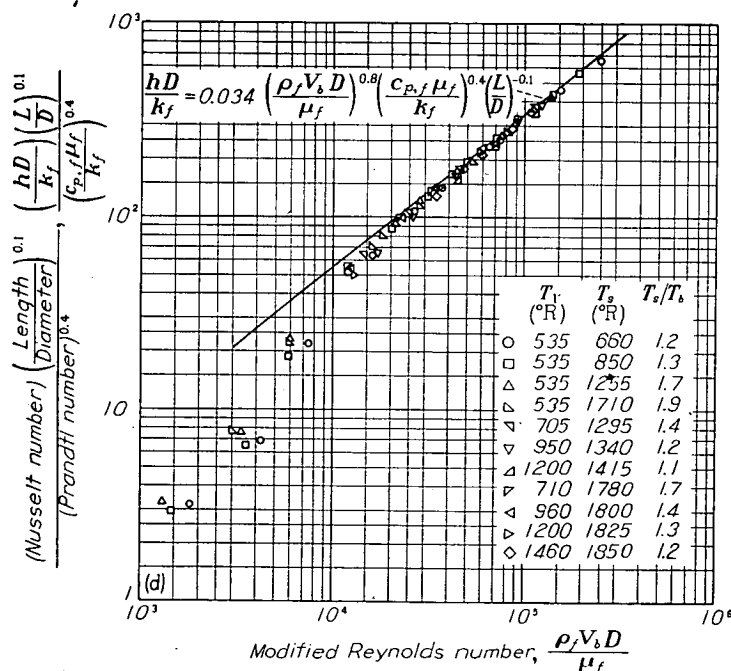
figure 10 (a) are replotted in figure 10 (c) using the conductivity data of reference 10, and with the physical properties and density evaluated at the film temperature $T_{0.75}$. The line representing equation (15), which for the lower reference temperature becomes

$$\frac{hD}{k_{0.75}} = 0.034 \left(\frac{\rho_{0.75} V_b D}{\mu_{0.75}} \right)^{0.8} \left(\frac{c_{p,0.75} \mu_{0.75}}{k_{0.75}} \right)^{0.4} \left(\frac{L}{D} \right)^{-0.1} \quad (15a)$$

is included for comparison.



(c) Physical properties of air evaluated at film temperature $T_{b,75}$; thermal conductivity proportional to $T^{0.75}$; Inconel tube.



(d) Physical properties of air evaluated at film temperature T_f ; thermal conductivity proportional to $\sqrt{T}$; Inconel tube.

FIGURE 10.—Concluded. Effect of inlet-air temperature using modified method of correlating heat-transfer coefficients.

For an inlet-air temperature of 1460° R, a surface temperature of 1850° R, and a Reynolds number of 85,000, the ordinate parameter is about 15 percent below the reference line. Comparison of figures 10 (a) and 10 (c) indicates that the use of the lower values of conductivity from reference 10 reduces the effect of temperature level by approximately one-half, or for constant values of T_s/T_b , the ordinate parameter varies as temperature to the -0.1 power.

The data of figure 10 (b), although not shown, would fall within the total separation of the data in figure 10 (c).

Figure 10 (d) is included to illustrate the effect of using still lower values of thermal conductivity at high temperatures. The data are the same as in figures 10 (a) and 10 (c), but in figure 10 (d) the conductivity is assumed to vary as the square root of temperature and the physical properties and density are evaluated at the film temperature T_f . The line representing equation (15), which for the reference film temperature T_f becomes

$$\frac{hD}{k_f} = 0.034 \left(\frac{\rho_f V_b D}{\mu_f} \right)^{0.8} \left(\frac{c_{p,f} \mu_f}{k_f} \right)^{0.4} \left(\frac{L}{D} \right)^{-0.1} \quad (15b)$$

is included for comparison.

In figure 10 (d) it is indicated that both the effects of air temperature and surface-to-air temperature ratio are eliminated by use of the assumed conductivity-temperature relation and by evaluation of the physical properties and density at the film temperature. The data can be represented with good accuracy by equation (15b).

No justification of the arbitrarily assumed conductivity-temperature relation shown in figure 3 can be given other than that it results in the best correlation of the data reported herein.

Effect of heat extraction.—In reference 12, it is predicted that with heat extraction from the air the Nusselt number would increase as the heat-flux density increases, or as the surface-to-air temperature ratio decreases. It was further found that the effect could be eliminated by evaluating the physical properties and density at a reference temperature midway between T_b and T_s .

Tests were conducted to determine whether or not the effect of T_s/T_b on correlation of heat-transfer coefficients was the same for heat extraction as for heat addition. These results were obtained with an Inconel tube having a length-diameter ratio of 60 and a bellmouth entrance for a range of T_s/T_b from 0.46 to 0.82. The data are correlated in the conventional manner in figure 11 using thermal conductivities from references 7 and 8 with the physical properties evaluated at the average temperature T_b . Included is the line representing equation (10) modified for variation in L/D , which becomes

$$\frac{hD}{k_b} = 0.034 \left(\frac{GD}{\mu_b} \right)^{0.8} \left(\frac{c_{p,b} \mu_b}{k_b} \right)^{0.4} \left(\frac{L}{D} \right)^{-0.1} \quad (16)$$

The effect of a decrease in T_s/T_b from 0.82 to 0.46 on the correlation of the data is negligible, and the data can be adequately represented by equation (16).

The data in figure 11 do not necessarily show that no effect of T_s/T_b exists for cooling inasmuch as the use of a smaller variation of conductivity with temperature would result in an increase in Nusselt number for a decrease in T_s/T_b and hence introduce an effect of T_s/T_b into the data.

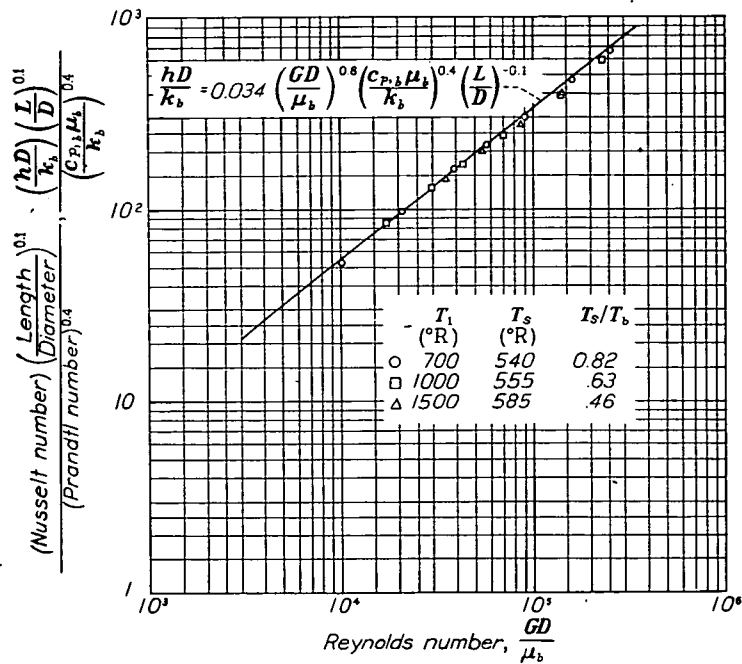
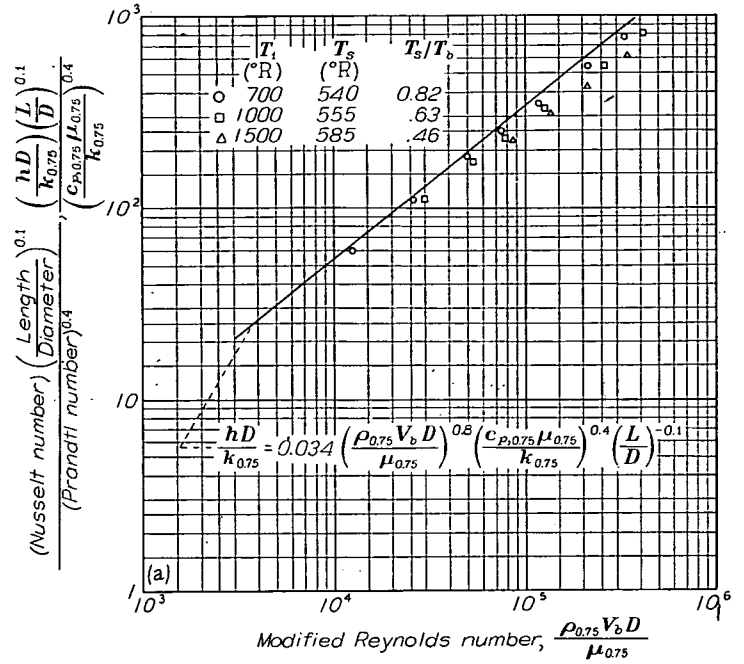


FIGURE 11.—Conventional method of correlating heat-transfer coefficients with heat extraction from air. Inconel tube; length-diameter ratio, 60; bellmouth entrance.

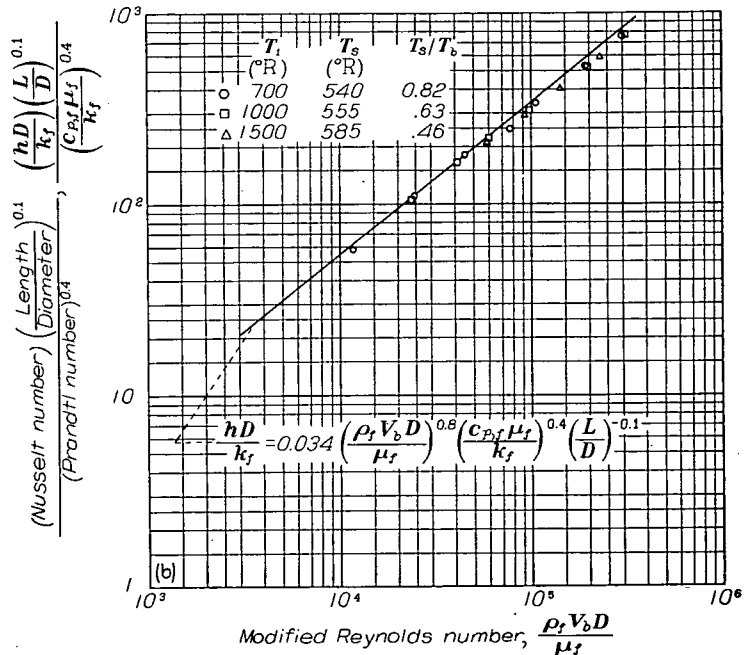
The effects of using smaller variations of conductivity are shown in figures 12 (a) and 12 (b) with the data based on the conductivity data of reference 10 and on the assumption that conductivity varies as the square root of temperature, respectively. In figure 12 (a), the data are correlated according to equation (15a) for heat addition. A considerable overcorrection for the effects of T_s/T_b is obtained, which indicates that the effect of T_s/T_b for cooling is smaller than that obtained for heating. In figure 12 (b), however, where the data are correlated according to equation (15b) for heat addition, the separation of the data is very small, indicating that for the assumption that conductivity varies as the square root of temperature the effect of T_s/T_b is the same for both the heating and cooling data, and that both the cooling and the heating data can be represented by equation (15b).

Summary plots of heat-transfer data.—Heat-transfer data have been shown for a range of Reynolds number, surface temperature, inlet-air temperature, entrance configuration, and length-diameter ratio, and for heating and cooling of the air. The results are summarized in figure 13, which shows on single plots data from previous figures, which include all the foregoing variables.

The data are correlated in figure 13 (a) by the modified method using the thermal conductivity data of references 7 and 8 with the physical properties and density evaluated at T_s . The line representing equation (15) is included for comparison. As previously shown, all the heating data obtained with an inlet-air temperature of 535° R regardless of length-diameter ratio, surface temperature, or entrance configuration can be represented with good accuracy by the



(a) Physical properties of air evaluated at film temperature $T_{0.75}$; thermal conductivity proportional to $T_{0.75}$.

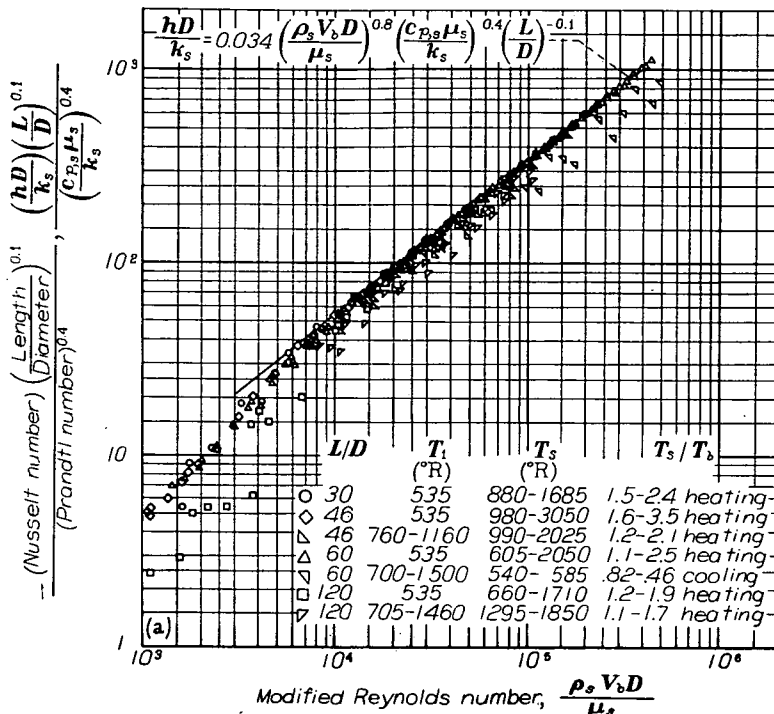


(b) Physical properties of air evaluated at film temperature T_f ; thermal conductivity proportional to $\sqrt{T}$.

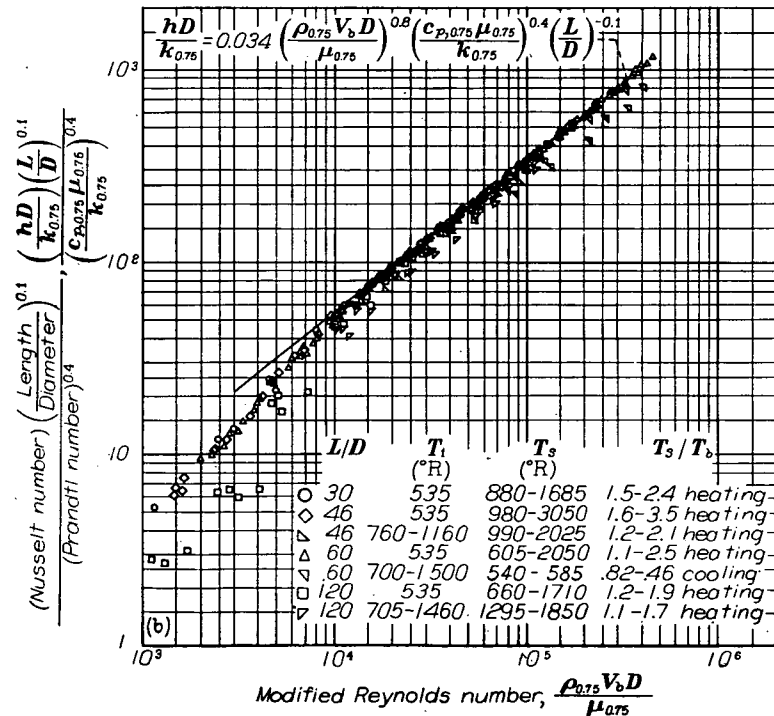
FIGURE 12.—Modified method of correlating heat-transfer coefficients with heat extraction from air.

reference line. The high inlet-air temperature data and the cooling data fall below the reference line, so that equation (15) does not apply with desirable accuracy for all the conditions investigated.

The data of figure 13 (a) are replotted in figure 13 (b) using the thermal conductivity data from reference 10, and with the physical properties and density evaluated at $T_{0.75}$.



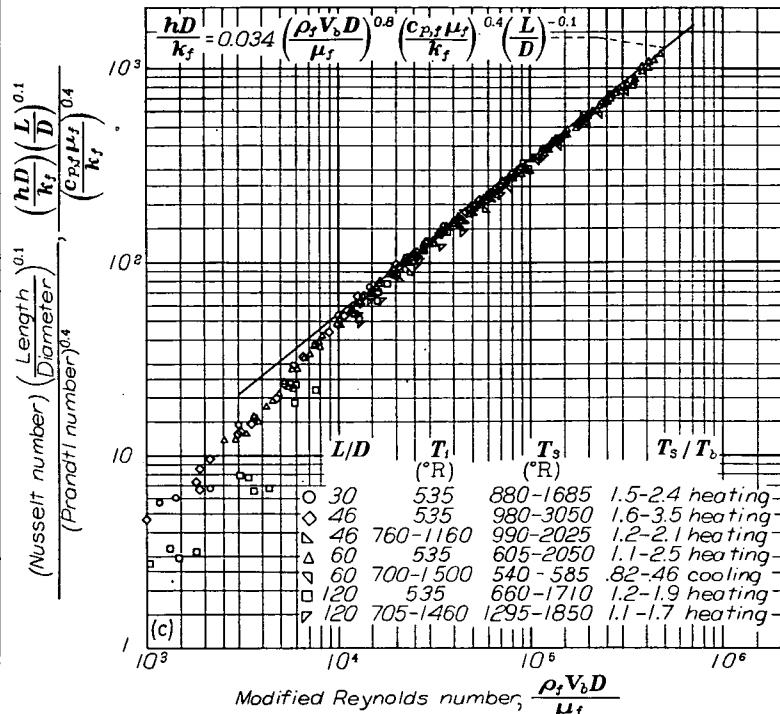
(a) Physical properties of air evaluated at surface temperature T_s ; thermal conductivity proportional to $T_s^{0.85}$.



(b) Physical properties of air evaluated at film temperature $T_{0.75}$; thermal conductivity proportional to $T_{0.75}^{0.73}$.

FIGURE 13.—Summary plot of experimental data using modified method of correlating heat-transfer coefficients.

The line representing equation (15a) is included. Trends with temperature level and with T_s/T_b are reduced compared with figure 13 (a). The high inlet-air temperature data and the cooling data again deviate farthest from the reference line.



(c) Physical properties of air evaluated at film temperature T_f ; thermal conductivity proportional to $\sqrt{T}$.

FIGURE 13.—Concluded. Summary plot of experimental data using modified method of correlating heat-transfer coefficients.

Figure 13 (c) shows the data of figures 13 (a) and 13 (b) replotted using the assumption that thermal conductivity varies as the square root of temperature, and with the physical properties and density evaluated at T_f . The line representing equation (15b) is included. The conductivity-temperature relation assumed in figure 13 (c) results in the best correlation of all the present data. Trends with respect to temperature level and with T_s/T_b for both heat addition and heat extraction are almost completely eliminated and the data can be represented by equation (15b).

CORRELATION OF FRICTION COEFFICIENTS

Friction coefficients without heat transfer.—Average half-friction coefficients $f/2$ calculated by equation (6) for flow without heat transfer are plotted against Reynolds number GD/μ_b in figure 14 (a). Included for comparison is the line representing the Kármán-Nikuradse relation between friction coefficient and Reynolds number for incompressible turbulent flow, which is

$$\frac{1}{\sqrt{8 \frac{f}{2}}} = 2 \log \left(\frac{GD}{\mu_b} \sqrt{8 \frac{f}{2}} \right) - 0.8 \quad (17)$$

and the line representing the equation for laminar flow in circular tubes, which is

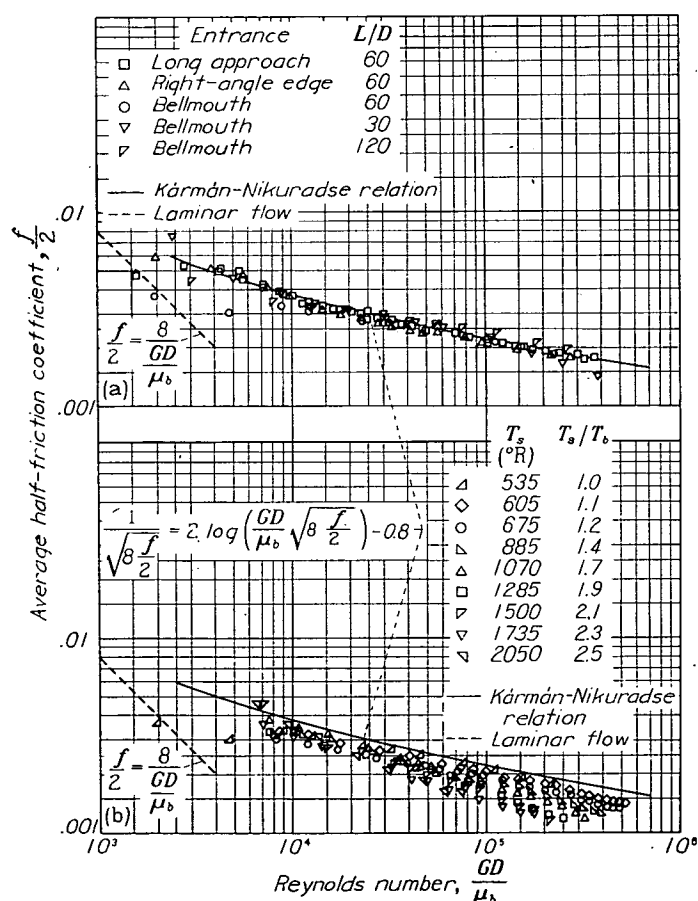
$$\frac{f}{2} = \frac{8}{GD \mu_b} \quad (18)$$

Reasonably good agreement with the Kármán-Nikuradse line is obtained for all five configurations in the range of Reynolds number corresponding to turbulent flow. It is of interest that the Kármán-Nikuradse relation, which was derived for incompressible flow, applies with good accuracy to compressible flow, inasmuch as the present data cover a range of tube-exit Mach number up to 1.0.

Figure 14 (a) illustrates the agreement of the data obtained in the present investigation without heat transfer with the results of other investigators, which are in general agreement with equations (17) and (18).

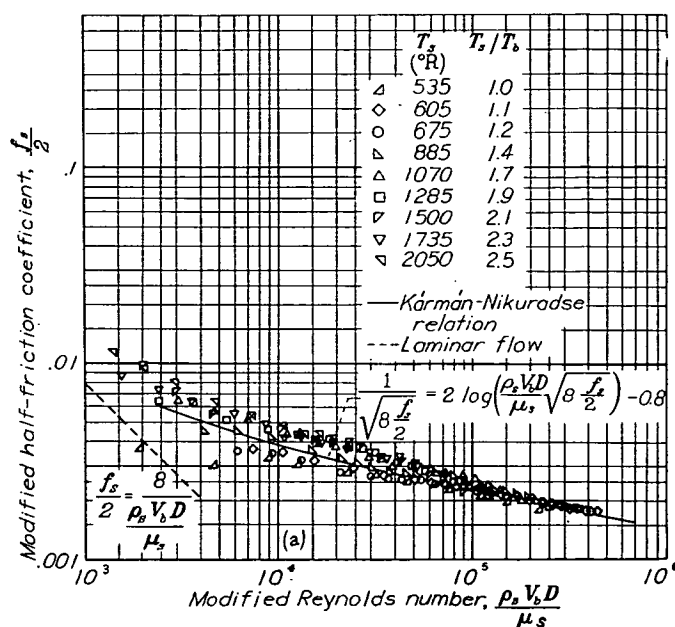
Friction coefficients with heat addition.—Representative friction data, obtained with heat addition to the air, are shown in figure 14 (b). Data for no heat transfer (T_s/T_b , 1.0), and the Kármán-Nikuradse and laminar flow lines are included for comparison. These data are for an Inconel tube having a length-diameter ratio of 60 and a bellmouth entrance, and were obtained simultaneously with the heat-transfer data of figure 5.

The data for low surface temperatures (T_s , 605° and 675° R) agree fairly well with those for no heat addition. As the surface temperature is increased, however, $f/2$ decreases at high Reynolds numbers and stays the same or increases at



(a) No heat transfer; Inconel tubes; inlet-air temperature, 535° R.
(b) Heat addition; Inconel tube; length-diameter ratio, 60; bellmouth entrance; inlet-air temperature, 535° R.

FIGURE 14.—Variation of average half-friction coefficient with Reynolds number.



(a) Viscosity and density evaluated at surface temperature T_s .

FIGURE 15.—Variation of modified half-friction coefficient with modified Reynolds number with heat addition. Inconel tube; length-diameter ratio, 60; bellmouth entrance; inlet-air temperature, 535° R.

low Reynolds numbers, so that the data cannot be represented by a single line. These results are typical of those obtained with other entrances and length-diameter ratios.

Modified friction coefficient.—Various methods of defining the Reynolds number and the dynamic pressure on which to base the average friction coefficient were tried in an attempt to eliminate the trends shown in figure 14 (b). One such method was to base the dynamic pressure on a density evaluated at the surface temperature and a velocity evaluated at the average air temperature and to plot the resulting friction coefficient against the modified Reynolds number $\rho_s V_b D / \mu_s$. The friction coefficient defined in the foregoing way f_s is given by equation (9).

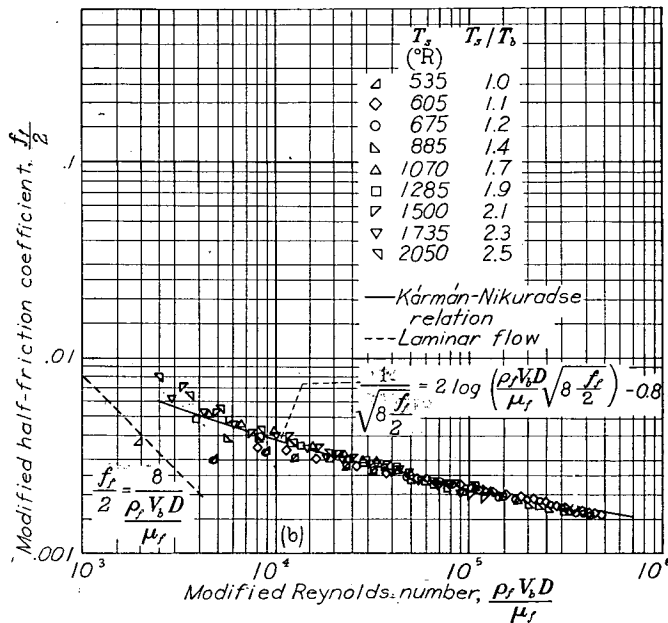
The results of this method of correlating the friction data are illustrated in figure 15 (a), where $f_s/2$ is plotted against $\rho_s V_b D / \mu_s$. The data are the same as in figure 14 (b), and curves representing the Kármán-Nikuradse and the laminar flow equations are again included.

In general, the friction coefficient is overcorrected and increases with an increase in T_s/T_b at constant modified Reynolds number. Figure 15 (a) is typical of the results obtained with other entrances and length-diameter ratios.

The friction factor was also defined on the basis of a density evaluated at the film temperature T_f (equation (8)). The resulting friction coefficient was then plotted against the modified Reynolds number with the density and viscosity evaluated at the film temperature T_f . The data of figure 15 (a) are replotted in this manner in figure 15 (b). The trends with T_s/T_b are eliminated for Reynolds numbers above 20,000 and the data can be represented with reasonable accuracy by the Kármán-Nikuradse relation using modified parameters.

Effect of entrance configuration.—Friction data obtained simultaneously with the heat-transfer data of figure 8 using an Inconel tube (length-diameter ratio, 60) with long-approach and right-angle-edge entrances are shown in figure 16 (a). Data for no heat transfer are included. At Reynolds numbers above 20,000, the data of figure 16 (a) are in reasonable agreement with the Kármán line, which was seen in figure 15 (b) to fit the bellmouth-entrance data with good accuracy. The data for the right-angle-edge entrance fall slightly below the reference line. No particular significance can be attached to this small deviation, however, and it is felt that the average friction coefficients for all three entrances can be approximated with sufficient accuracy at modified Reynolds numbers greater than about 20,000 by the relation

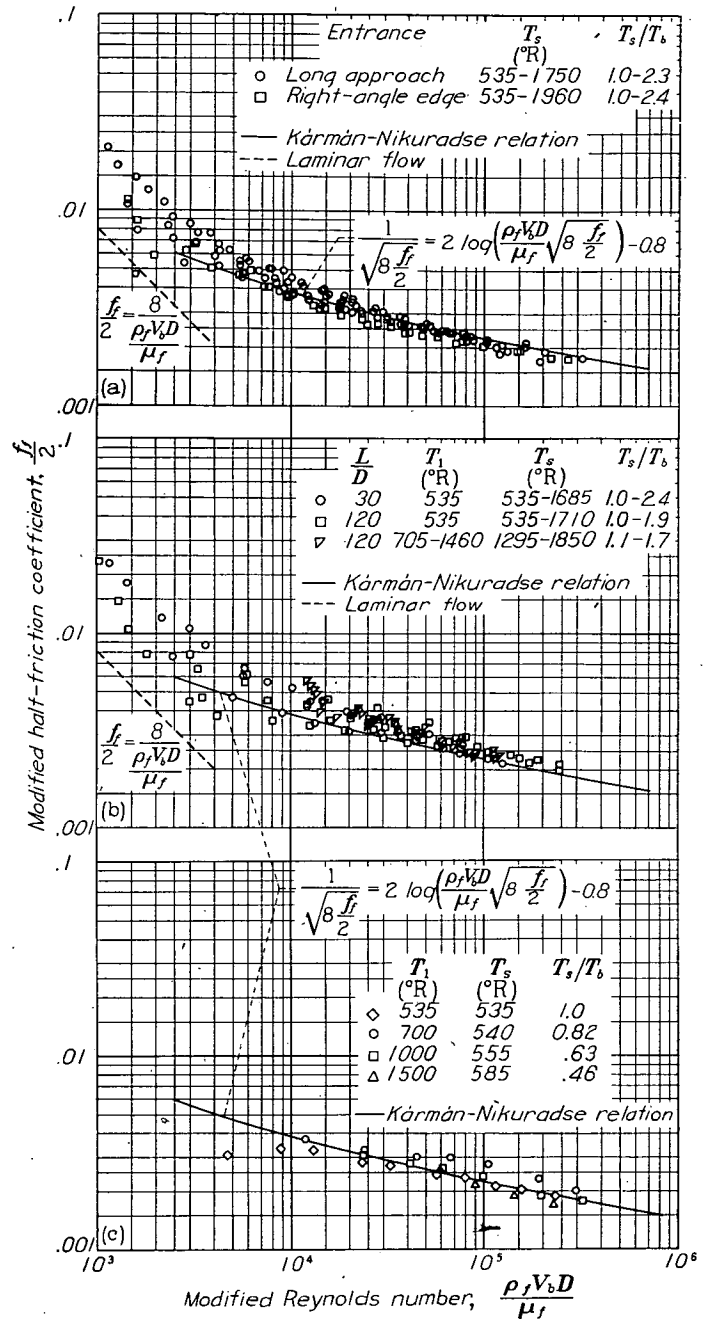
$$\frac{1}{\sqrt{8 \frac{f_f}{2}}} = 2 \log \left(\frac{\rho_f V_b D}{\mu_f} \sqrt{8 \frac{f_f}{2}} \right) - 0.8 \quad (19)$$



(b) Viscosity and density evaluated at film temperature T_f .

FIGURE 15.—Concluded. Variation of modified half-friction coefficient with modified Reynolds number with heat addition. Inconel tube; length-diameter ratio, 60; bellmouth entrance; inlet-air temperature, 535° R.

Effect of length-diameter ratio and inlet-air temperature.—Friction data obtained with heat addition to the air for length-diameter ratios of 30 and 120 and for inlet-air temperatures from 535° to 1460° R are shown in figure 16 (b). The data of figure 16 (b), which are for a bellmouth entrance, fall above the Kármán line, whereas the bellmouth data in figure 15 (b), which correspond to a length-diameter ratio of 60, fall on the line. No particular significance can be attached to these differences, however, and it is felt that the data for the range of length-diameter ratio and inlet-air temperature considered can be represented with sufficient accuracy by equation (19).



(a) Effect of entrance configuration with heat addition. Inconel tube; length-diameter ratio, 60; inlet-air temperature, 535° R.

(b) Effect of length-diameter ratio and inlet-air temperature with heat addition. Inconel tubes.

(c) Effect of heat extraction. Inconel tube; length-diameter ratio, 60.

FIGURE 16.—Variation of modified half-friction coefficient with modified Reynolds number.

Effect of heat extraction.—Friction coefficients obtained simultaneously with the data of figure 11 for heat extraction from the air are shown in figure 16 (c). Data for no heat transfer are included. The coordinates and the reference line are the same as in figures 15 (b) to 16 (b). The data can be represented with reasonable accuracy by the reference line for turbulent flow.

SUMMARY OF RESULTS

The results of this investigation of heat transfer and associated pressure drops conducted with air flowing through smooth tubes for a range of surface-to-air temperature ratio from 0.46 to 3.5, surface temperature from 535° to 3050° R, inlet-air temperature from 535° to 1500° R, Reynolds number up to 500,000, exit Mach number up to 1.0, length-diameter ratio from 30 to 120, and for three entrance configurations may be summarized as follows:

1. Conventional methods of correlating average heat-transfer coefficients, wherein specific heat, viscosity, and thermal conductivity are evaluated at the average air temperature, resulted in a considerable and progressive separation of the data with increased surface-to-air temperature ratio for heat addition to the air. Evaluation of the physical properties at the film or surface temperature, instead of the bulk temperature, aggravated the separation.

2. Satisfactory correlation of heating data obtained with inlet-air temperatures near 535° R was obtained for the range of surface temperature considered when the Reynolds number was modified by substituting the product of air density evaluated at the surface temperature and velocity evaluated at the average total air temperature for the conventional mass flow per unit cross-sectional area, and the properties of the air were evaluated at the average surface temperature.

3. Correlation of the heating data obtained at inlet-air temperatures above 535° R by the modified method resulted in a decrease in Nusselt number for an increase in average air temperature, which was eliminated by the assumption that the thermal conductivity of air varies as the square root of temperature and by the evaluation of the physical properties of air including density at a temperature midway between air and surface temperatures instead of at the surface temperature.

4. With the assumption that thermal conductivity varies as the square root of temperature, data obtained for cooling of the air indicated that the effect of surface-to-air temperature ratio was the same as for heat addition, so that both heating and cooling data can be represented by the same equation.

5. The use of different entrances to the test section (that is, a long-approach, a right-angle-edge, or a bellmouth entrance) had negligible effect on average heat-transfer coefficients, at least for length-diameter ratios of 60 or greater.

6. Variation in tube length-diameter ratio had a relatively small effect on average heat-transfer coefficients, which was satisfactorily accounted for by including a parameter for length-diameter ratio in the correlation equation.

7. Friction coefficients for no heat transfer, calculated from a dynamic pressure based on an average air density, were in good agreement with those obtained by other investigators. The corresponding friction coefficients obtained

with heat addition to the air showed a considerable effect of surface-to-air temperature ratio for the Reynolds numbers in the turbulent region.

8. All the friction data could be represented with reasonable accuracy by the Kármán-Nikuradse relation for incompressible turbulent flow by using a friction coefficient calculated from a dynamic pressure based on a density evaluated at the film temperature and the modified Reynolds number in which the density and viscosity were evaluated at a temperature midway between the air and surface temperatures.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, December 31, 1950

REFERENCES

1. Nusselt, Wilhelm: Der Einfluss der Gastemperatur auf den Wärmeübergang im Rohr. *Techn. Mech. u. Thermodynamik*, Bd. 1, Nr. 8, Aug. 1930, S. 277-290.
2. McAdams, William H.: *Heat Transmission*. McGraw-Hill Book Co., Inc., 2d ed., 1942.
3. Humble, Leroy V., Lowdermilk, Warren H., and Grele, Milton: *Heat Transfer from High-Temperature Surfaces to Fluids. I—Preliminary Investigation with Air in Inconel Tube with Rounded Entrance, Inside Diameter of 0.4 Inch, and Length of 24 Inches*. NACA RM E7L31, 1948.
4. Lowdermilk, Warren H., and Grele, Milton D.: *Heat Transfer from High-Temperature Surfaces to Fluids. II—Correlation of Heat-Transfer and Friction Data for Air Flowing in Inconel Tube with Rounded Entrance*. NACA RM E8L03, 1949.
5. Lowdermilk, Warren H., and Grele, Milton D.: *Influence of Tube-Entrance Configuration on Average Heat-Transfer Coefficients and Friction Factors for Air Flowing in an Inconel Tube*. NACA RM E50E23, 1950.
6. Desmon, Leland G., and Sams, Eldon W.: *Correlation of Forced-Convection Heat-Transfer Data for Air Flowing in Smooth Platinum Tube with Long-Approach Entrance at High Surface and Inlet-Air Temperatures*. NACA RM E50H23, 1950.
7. Tribus, Myron, and Boelter, L. M. K.: *An Investigation of Aircraft Heaters. II—Properties of Gases*. NACA ARR, Oct. 1942.
8. Boelter, L. M. K., and Sharp, W. H.: *An Investigation of Aircraft Heaters. XXXII—Measurement of Thermal Conductivity of Air and of Exhaust Gases Between 50° and 900° F*. NACA TN 1912, 1949.
9. Keenan, Joseph H., and Kaye, Joseph: *Thermodynamic Properties of Air*. John Wiley & Sons, Inc., 1945.
10. Gröeber, H., und Erk, S.: *Die Grundgesetze der Wärmeübertragung*. Julius Springer (Berlin), 1933.
11. Bernardo, Everett, and Eian, Carroll S.: *Heat-Transfer Tests of Aqueous Ethylene Glycol Solutions in an Electrically Heated Tube*. NACA ARR E5F07, 1945.
12. Deissler, Robert G.: *Analytical Investigation of Turbulent Flow in Smooth Tubes with Heat Transfer with Variable Fluid Properties for Prandtl Number of 1*. NACA TN 2242, 1950.
13. Cholette, Albert: *Heat Transfer—Local and Average Coefficients for Air Flowing Inside Tubes*. *Chem. Eng. Prog.*, vol. 44, no. 1, Jan. 1948, pp. 81-88.

REPORT 1021

ANALYSIS OF PLANE-PLASTIC-STRESS PROBLEMS WITH AXIAL SYMMETRY IN STRAIN-HARDENING RANGE¹

By M. H. LEE WU

SUMMARY

A simple method is developed for solving plane-plastic-stress problems with axial symmetry in the strain-hardening range which is based on the deformation theory of plasticity employing the finite-strain concept. The equations defining the problems are first reduced to two simultaneous nonlinear differential equations involving two dependent variables: (a) the octahedral shear strain, and (b) a parameter indicating the ratio of principal stresses. By multiplying the load and dividing the radius by an arbitrary constant, it is possible to solve these problems without iteration for any value of the modified load. The constant is determined by the boundary condition.

This method is applied to a circular membrane under pressure, a rotating disk without and with a central hole, and an infinite plate with a circular hole. Two materials, Inconel X and 16-25-6, the octahedral shear stress-strain relations of which do not follow the power law, are used. Distributions of octahedral shear strain, as well as of principal stresses and strains, are obtained. These results are compared with the results of the same problems in the elastic range. The variation of load with maximum octahedral shear strain of the member is also investigated.

The following conclusions can be drawn:

- 1. Inasmuch as the ratios of the principal stresses remain essentially constant during loading for the materials considered, the deformation theory is applicable to this group of problems.*
- 2. In plastic deformation, the distributions of the principal strains and of the octahedral shear strain are less uniform than in the elastic range, although the distributions of the principal stresses were more uniform. The stress-concentration factor around the hole is reduced with plastic deformation, but a high strain-concentration factor occurs.*
- 3. For the rotating disk and the infinite plate the deformation that can be sustained by the member before failure depends mainly on the maximum octahedral shear strain of the material.*
- 4. The added load that the member could sustain between the onset of yielding and failure depended mainly on the octahedral shear stress-strain relations of the material.*

INTRODUCTION

In the design of turbine rotors, it is desirable to know the detailed stress and strain distributions in the strain-hardening

range and the increase in load that can be sustained between the onset of yielding and failure. It is also desirable to know the effects of a notch or a hole in a turbine rotor or other machine members that are stressed in the strain-hardening range. If a member is thin, it can be analyzed on the basis of plane stress. For problems of this type for ideally plastic material, Nadai obtained solutions for a thin plate with a hole and a flat ring radially stressed (reference 1), and Nadai and Donnell obtained a solution for a rotating disk (reference 2). For materials having strain-hardening characteristics, a solution of plane-stress problems has been obtained by Gleyzal for a circular membrane under pressure (reference 3). The concept of infinitesimal strain was used and the solution was obtained by an iterative procedure with a good first approximate solution. The plastic laws were always satisfied by using a chart given in reference 3. In reference 4, a trial-and-error method is given for a rotating disk with very small plastic strain, in which the elastic stresses and strains are used as the first approximate values. An experimental investigation of high-speed rotating disks is given in reference 5; distributions of plastic strains (logarithmic strains) for different types of disk are measured. Reference 6 gives an experimental investigation of the burst characteristics of rotating disks; stress at the center of the disk is calculated by assuming that the material behaves elastically at the burst speed; the average tangential stress along the radius at burst speed is also calculated.

A simple method of solving plane-plastic-stress problems with axial symmetry in the strain-hardening range for finite strains was developed at the NACA Lewis laboratory during 1949-50. This method is based on the deformation theory of Hencky and Nadai (references 7 to 9), which is derived for the condition of constant directions and ratios of the principal stresses during loading. The equations of equilibrium, strain, and plastic law are reduced to two simultaneous nonlinear differential equations involving three variables, one independent and two dependent, that can be integrated numerically to any desired accuracy. These variables are the proportionate radial distance, the octahedral shear strain, and a parameter α that indicates the ratio of principal stresses. The magnitude of variation in calculated values of the parameter α with change in load directly indicates whether the deformation theory is applicable to the problem.

¹ Supersedes NACA TN 2217, "Analysis of Plane-Stress Problems With Axial Symmetry in Strain-Hardening Range" by M. H. Lee Wu, 1950.

The method developed is applied to: (1) a circular membrane under pressure, in order to compare results obtained by this method with those obtained by Gleyzal (reference 3); (2) rotating disks without and with a circular central hole, in order to investigate plastic deformation in such disks and the effects of the hole; and (3) an infinite plate with a circular hole or a flat ring radially stressed, in order to investigate the effects of the hole in the strain-hardening range.

In the investigation of (2) and (3), two materials, Inconel X and 16-25-6, with different strain-hardening characteristics were used in order to determine the effect of the octahedral shear stress-strain curve on plastic deformation. The octahedral shear stress of these two materials is not a power function of the octahedral shear strain, so that more general information can be obtained. Distributions of stresses and strains of the same problems in the elastic range are also calculated for purposes of comparison.

Acknowledgment is made to Professor D. C. Drucker for his discussion of this work and for his suggestion to examine whether the logarithmic strain could be applied correctly to the present problems and his suggestion to plot the stress-strain curves of Inconel X and 16-25-6 on a logarithmic scale in order to show that these materials do not obey the power law.

SYMBOLS

The following symbols are used in this report:

A, B, C, D, E, F	coefficients of nonlinear differential equations; functions of α, γ , and $\frac{r}{k}$
a	initial radius of hole
b	initial outside radius of membrane, rotating disk, or flat ring
c	initial outside radius of plate, very large compared with radius a
G, H, J, L	trigonometric functions of α
h	instantaneous thickness of membrane, rotating disk, or plate
h_{init}	initial thickness of membrane, disk, or plate
K_1, K_2	arbitrary loading constants
k	constant having a dimension of length
p	pressure on membrane
r	radial coordinate of undeformed membrane, disk, or plate
s	arc length
u	radial displacement
w	axial displacement
z	axial coordinate
α	parameter indicating ratio of principal stresses
γ	octahedral shear strain
ϵ	logarithmic strain (natural strain), logarithm of instantaneous length divided by initial length of element
θ	angular coordinate

ρ	mass per unit volume
σ	true normal stress, normal force per unit instantaneous area
τ	octahedral shear stress
ω	angular velocity
Subscripts:	
b	at radius b
c	at radius c
o	at center for member without hole; at radius a for member with concentric circular hole
1, 2, 3	principal directions in general
r, θ, z	principal directions: radial, tangential, and axial directions

STRESS-STRAIN RELATIONS IN PLASTIC DEFORMATION

The deformation theory of plasticity for ideally plastic materials was developed by Hencky from the theory of Saint Venant-Levy-Mises for the cases in which the directions and the ratios of principal stresses remain constant during loading (reference 7). Nadai extended the theory to include materials having strain-hardening characteristics (references 8 and 9). The conditions for the deformation theory have been emphasized by Nadai (reference 9, p. 209), Ilyushin (references 10 and 11), Prager (reference 12), and Drucker (reference 13). Experiments conducted by Davis (reference 14), Osgood (reference 15), and others on thin tubes subjected to combined loads with the directions and the ratios of the principal stresses constant throughout the tube and remaining constant during loading show that good results can be expected from the deformation theory.

In more recent experiments on thin tubes by Fraenkel (reference 16) and Davis and Parker (reference 17), it has been shown that even with considerable variation of the ratios of principal stresses during loading the strains obtained from the experiments were in good agreement with the strains predicted by use of the deformation theory. Further experimental investigation is needed to determine the extent to which the variation of ratios of principal stresses is permissible with the deformation theory. However, when the variation is small (approximately 10 percent over the strain-hardening range), the deformation theory can be expected to give good results.

In the present problems with axial symmetry, the directions of the axes of the principal stresses remain fixed during loading and it is probable that the ratios of principal strains and of principal stresses also remain approximately constant. The deformation theory previously discussed is therefore used. The stress-strain relations are as follows:

$$\epsilon_1 + \epsilon_2 + \epsilon_3 = 0 \quad (1)$$

$$\frac{\sigma_1 - \sigma_2}{\epsilon_1 - \epsilon_2} = \frac{\sigma_2 - \sigma_3}{\epsilon_2 - \epsilon_3} = \frac{\sigma_3 - \sigma_1}{\epsilon_3 - \epsilon_1} \quad (2)$$

$$\tau = \tau(\gamma) \quad (3)$$

where

$$\tau = \frac{1}{3} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2} \quad (4a)$$

$$\gamma = \frac{2}{3} [(\epsilon_1 - \epsilon_2)^2 + (\epsilon_2 - \epsilon_3)^2 + (\epsilon_3 - \epsilon_1)^2]^{1/2} \quad (4b)$$

From equations (1), (2), and (4a) or (4b), the following relations are obtained:

$$\epsilon_1 = \frac{1}{3} \frac{\gamma}{\tau} \left[\sigma_1 - \frac{1}{2} (\sigma_2 + \sigma_3) \right]$$

$$\epsilon_2 = \frac{1}{3} \frac{\gamma}{\tau} \left[\sigma_2 - \frac{1}{2} (\sigma_3 + \sigma_1) \right]$$

$$\epsilon_3 = \frac{1}{3} \frac{\gamma}{\tau} \left[\sigma_3 - \frac{1}{2} (\sigma_1 + \sigma_2) \right]$$

For plane-stress problems $\sigma_3 = 0$. It is convenient to use cylindrical coordinates for the problems considered; the principal directions 1, 2, and 3 in the preceding equations become radial, circumferential, and axial directions, respectively. The equations thus become

$$\epsilon_r + \epsilon_\theta + \epsilon_z = 0 \quad (1a)$$

$$\tau = \frac{\sqrt{2}}{3} (\sigma_r^2 - \sigma_r \sigma_\theta + \sigma_\theta^2)^{1/2} \quad (5a)$$

$$\gamma = 2 \sqrt{\frac{2}{3}} (\epsilon_r^2 + \epsilon_r \epsilon_\theta + \epsilon_\theta^2)^{1/2} \quad (5b)$$

and

$$\epsilon_r = \frac{1}{3} \frac{\gamma}{\tau} \left(\sigma_r - \frac{1}{2} \sigma_\theta \right) \quad (6a)$$

$$\epsilon_\theta = \frac{1}{3} \frac{\gamma}{\tau} \left(\sigma_\theta - \frac{1}{2} \sigma_r \right) \quad (6b)$$

$$\epsilon_z = \frac{1}{3} \frac{\gamma}{\tau} \left[-\frac{1}{2} (\sigma_r + \sigma_\theta) \right] = -(\epsilon_r + \epsilon_\theta) \quad (6c)$$

When σ_r and σ_θ are expressed in terms of ϵ_r and ϵ_θ , there is obtained

$$\left. \begin{aligned} \sigma_r &= 2 \frac{\tau}{\gamma} (2\epsilon_r + \epsilon_\theta) \\ \sigma_\theta &= 2 \frac{\tau}{\gamma} (2\epsilon_\theta + \epsilon_r) \end{aligned} \right\} \quad (7)$$

Because large deformations in the strain-hardening range will be considered, the concept that the change of dimension of an element is infinitesimal compared with the original dimension of the element is not accurate enough. Hence, the finite-strain concept, which considers the instantaneous dimension of the element, is used. (The equations of infinitesimal strains considered as special cases of finite strains are given in appendix A.) The stress is then equal

to the force divided by the instantaneous area and the strains are defined by the following equation:

$$\delta(\epsilon_j) = \frac{\delta(l_j)}{l_j}$$

where l_j is the instantaneous length of a small element having the original length of $(l_j)_0$ and j is any principal direction. During plastic deformation, the plastic strains at a particular state depend on the path by which that state is reached. For the paths along which the ratios of principal stresses remain constant during loading, however, the octahedral shear stress-strain relation, the value of the octahedral shear strain, and the values of the principal strains are defined by the initial and final states (references 14, 15, and reference 9, p. 209); $\delta(\epsilon_j)$ is then an exact differential and

$$\epsilon_j = \log_e \frac{l_j}{(l_j)_0} \text{ or } e^{\epsilon_j} = \frac{l_j}{(l_j)_0} \quad (8)$$

It should be noted that the condition under which equation (8) was obtained is also one of the conditions under which the deformation theory is derived; as long as the deformation theory is applicable, equation (8) can also be used.

EQUATIONS OF EQUILIBRIUM AND STRAINS INVOLVING DISPLACEMENTS CIRCULAR MEMBRANE UNDER PRESSURE

Equations of equilibrium and equations of strain are derived for a circular membrane under pressure. The membrane considered is so thin that bending stress can be neglected (reference 18, p. 576). Figure 1 shows the membrane clamped at the rim and subjected to a pressure p and

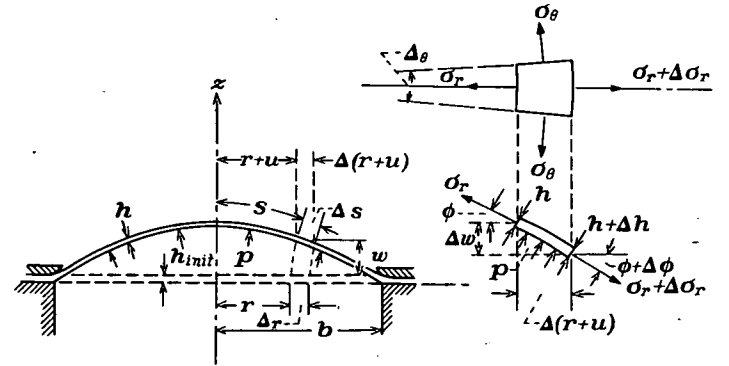


FIGURE 1.—Thin circular membrane (under pressure) and its element in deformed state.

a small element defined by $\Delta\theta$ and Δs taken at radius $r+u$ in the deformed state. In the undeformed state, the same element would be at radius r and defined by $\Delta\theta$ and Δr . The dotted lines represent an undeformed membrane. The instantaneous thickness of the element and the stresses acting on the element are also shown in the figure. Two principal stresses are σ_r and σ_θ , and ϕ is the angle between σ_r and the original radial direction.

Equations of equilibrium.—When all the forces acting on the element in the direction of σ_r are summed up, the following equation of equilibrium is obtained:

$$\sigma_r(r+u)h\Delta\theta - (\sigma_r + \Delta\sigma_r)[r+u+\Delta(r+u)]\Delta\theta(h+\Delta h) \cos \Delta\varphi + 2\sigma_\theta \Delta s \left(h + \frac{1}{2}\Delta h\right) \sin \frac{\Delta\theta}{2} \cos \varphi - p\Delta s(r+u)\Delta\theta \sin \frac{\Delta\varphi}{2} = 0$$

When $\Delta(r+u)$ approaches zero as a limit, the differential equation of equilibrium may be obtained:

$$(r+u) \frac{d(\sigma_r h)}{d(r+u)} = h(\sigma_\theta - \sigma_r) \quad (9)$$

A cap of the membrane bounded by radius $r+u$ and the forces acting on it are shown in figure 2. Summing up the forces in the z -direction yields

$$p\pi(r+u)^2 = \sigma_r \frac{dw}{ds} 2\pi h(r+u)$$

or

$$\left[\frac{dw}{d(r+u)}\right]^2 = \frac{1}{\left[\frac{2h\sigma_r}{p(r+u)}\right]^2 - 1} \quad (10)$$

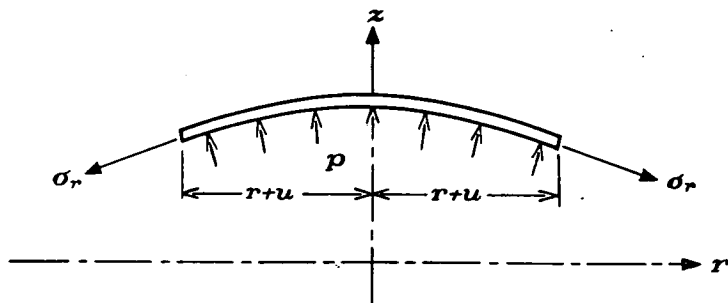


FIGURE 2.—Cap of membrane with radius $r+u$ in deformed state.

Equations of strains.—Inasmuch as the element at radius r , defined by $\Delta\theta$ and Δr in the undeformed state, is moved by the application of pressure p (fig. 1) to radius $r+u$ and defined by $\Delta\theta$ and Δs , by use of equation (8) the strains are

$$\epsilon_r = \log_e \frac{ds}{dr}$$

$$\epsilon_\theta = \log_e \frac{r+u}{r}$$

$$\epsilon_z = \log_e \frac{h}{h_{init}}$$

Then

$$\epsilon_r = \frac{d(r+u)}{dr} \left\{ 1 + \left[\frac{dw}{d(r+u)} \right]^2 \right\}^{1/2} \quad (11a)$$

$$\epsilon_\theta = \frac{r+u}{r} \quad (11b)$$

$$\epsilon_z = \frac{h}{h_{init}} \quad (11c)$$

ROTATING DISK

Equation of equilibrium.—A disk of radius b and thickness h , rotating about its axis with angular speed ω , and an element

taken at radius $r+u$, defined by $\Delta\theta$ and $\Delta(r+u)$, are shown in figure 3 with all the external forces acting on the element.

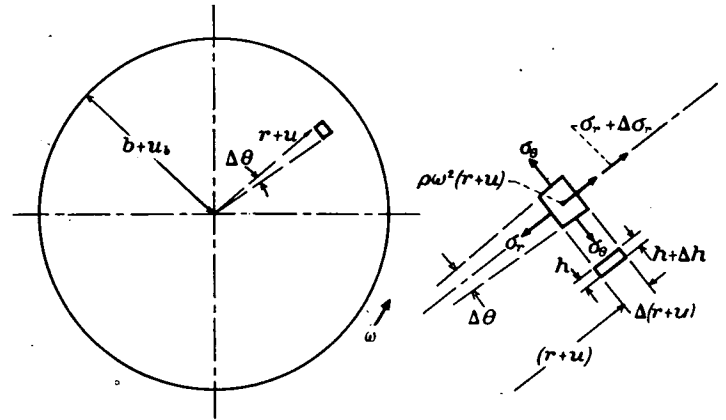


FIGURE 3.—Rotating disk and its element.

Summing up all forces acting on the element in the radial direction yields

$$\begin{aligned} &\sigma_r(r+u)h\Delta\theta - (\sigma_r + \Delta\sigma_r)[r+u+\Delta(r+u)]\Delta\theta(h+\Delta h) + \\ &2\sigma_\theta[\Delta(r+u)] \left(h + \frac{1}{2}\Delta h\right) \sin \frac{\Delta\theta}{2} - \\ &\omega^2 \left[r+u + \frac{1}{2}\Delta(r+u)\right] \frac{\rho\pi[(r+\Delta r)^2 - r^2]\Delta\theta}{2\pi} h_{init} = 0 \end{aligned}$$

When $\Delta(r+u)$ approaches zero as a limit, the following equation of equilibrium is obtained:

$$(r+u) \frac{d(\sigma_r h)}{d(r+u)} = (\sigma_\theta - \sigma_r)h - \rho\omega^2 r^2 h_{init} \frac{r+u}{r} \frac{dr}{d(r+u)} \quad (12)$$

Equations of strains.—The strains are

$$\epsilon_r = \frac{d(r+u)}{dr} \quad (13a)$$

$$\epsilon_\theta = \frac{r+u}{r} \quad (13b)$$

$$\epsilon_z = \frac{h}{h_{init}} \quad (13c)$$

INFINITE PLATE WITH CIRCULAR HOLE OR FLAT RING RADIALLY STRESSED

An infinite plate uniformly stressed in its plane in all directions and having a circular hole is shown in figure 4. The whole system is equivalent to a very large circular plate of radius c with a small concentric circular hole radially subjected to the same uniform stress σ on the outer boundary. The solution obtained for such a plate within any radius b can also be considered as a solution of a flat ring with outer radius b and inner radius a , that is, uniformly loaded at the outer boundary with the radial stress σ_b obtained in the plate solution.

The equations for this case can be obtained in a manner similar to the two previous cases, or by simply setting dw/dr and w equal to zero in equations for the membrane, or by setting ω equal to zero in the equation for the rotating disk.

EQUATIONS OF EQUILIBRIUM AND COMPATIBILITY IN TERMS OF PRINCIPAL STRESSES AND STRAINS

CIRCULAR MEMBRANE UNDER PRESSURE

A set of ten independent equations (equations (1a), (3), (5b), (6a), (6b), (9), (10), (11a), (11b), and (11c)) involving the ten unknowns σ_r , σ_θ , ϵ_r , ϵ_θ , ϵ_z , γ , τ , h , u , and w define the problem of the circular membrane under pressure. If equation (11b) is differentiated with respect to r and combined with equation (11a),

$$r \frac{d\epsilon_\theta}{dr} = \frac{e^{(\epsilon_r - \epsilon_\theta)}}{\left\{ 1 + \left[\frac{dw}{d(r+u)} \right]^2 \right\}^{1/2}} - 1 \quad (14)$$

Substituting equation (10) in equation (14) to eliminate w yields the following equation of compatibility:

$$r \frac{d\epsilon_\theta}{dr} = e^{(\epsilon_r - \epsilon_\theta)} \left\{ 1 - \left[\frac{p(r+u)}{2h\sigma_r} \right]^2 \right\}^{1/2} - 1 \quad (15)$$

Equations (9) and (15) can be simplified by using equations (11) to eliminate u and h , which results in

$$r \frac{d\sigma_r}{dr} + \sigma_r r \frac{d\epsilon_z}{dr} = (\sigma_\theta - \sigma_r) e^{(\epsilon_r - \epsilon_\theta)} \left\{ 1 - \left[\frac{rpe^{(\epsilon_\theta - \epsilon_z)}}{2h_{init}\sigma_r} \right]^2 \right\}^{1/2} \quad (16)$$

and

$$r \frac{d\epsilon_\theta}{dr} = e^{(\epsilon_r - \epsilon_\theta)} \left\{ 1 - \left[\frac{rpe^{(\epsilon_\theta - \epsilon_z)}}{2h_{init}\sigma_r} \right]^2 \right\}^{1/2} - 1 \quad (17)$$

The ten equations defining this problem are now reduced to seven independent equations, (1a), (6a), (6b), (5b), (3), (16), and (17), with the seven unknowns σ_r , σ_θ , ϵ_r , ϵ_θ , ϵ_z , τ , and γ .

The solution of the problem is simplified by introducing an arbitrary constant k into equations (16) and (17):

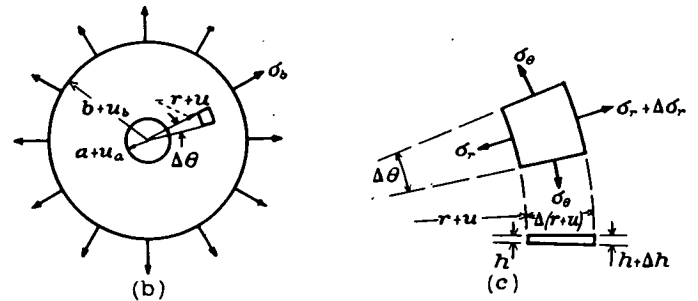
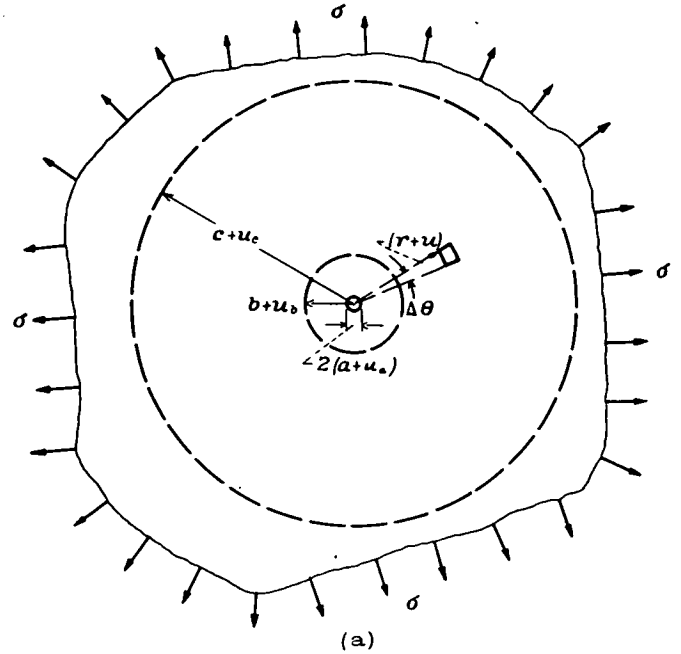
$$\left. \begin{aligned} r \frac{d\sigma_r}{d\left(\frac{r}{k}\right)} + \sigma_r r \frac{d\epsilon_z}{d\left(\frac{r}{k}\right)} &= (\sigma_\theta - \sigma_r) e^{(\epsilon_r - \epsilon_\theta)} \left\{ 1 - \left[\frac{pk}{h_{init}k} \frac{r}{2\sigma_r} e^{(\epsilon_\theta - \epsilon_z)} \right]^2 \right\}^{1/2} \\ r \frac{d\epsilon_\theta}{d\left(\frac{r}{k}\right)} &= e^{(\epsilon_r - \epsilon_\theta)} \left\{ 1 - \left[\frac{pk}{h_{init}k} \frac{r}{2\sigma_r} e^{(\epsilon_\theta - \epsilon_z)} \right]^2 \right\}^{1/2} - 1 \end{aligned} \right\} \quad (18)$$

where k is any arbitrary unknown constant with the dimension of length. By use of the two parameters r/k and pk/h_{init} , it is possible to solve the problem in a simple, direct way without the use of the iteration. This will be further discussed in the section "Methods of Numerical Integration."

ROTATING DISK

For the rotating disk there are nine independent relations (equations (1a), (3), (5b), (6a), (6b), (12), (13a), (13b), and (13c)) with the nine unknowns σ_r , σ_θ , ϵ_r , ϵ_θ , ϵ_z , γ , τ , h , and u . If equation (13b) is differentiated with respect to r and combined with equation (13a), the following compatibility equation is obtained:

$$r \frac{d\epsilon_\theta}{dr} = e^{(\epsilon_r - \epsilon_\theta)} - 1 \quad (19)$$



(a) Infinite plate with circular hole uniformly stressed in its plane in all directions.
(b) Flat ring radially stressed.
(c) Element.

FIGURE 4.—Infinite plate with circular hole, flat ring radially stressed, and its element in deformed state.

As in the case of the membrane, u and h can be eliminated from the equilibrium equation (12) by using equations (13), which yields

$$r \frac{d\sigma_r}{dr} + \sigma_r r \frac{d\epsilon_z}{dr} = (\sigma_\theta - \sigma_r) e^{(\epsilon_r - \epsilon_\theta)} - \rho \omega^2 r^2 e^{(-\epsilon_z)} \quad (20)$$

The nine equations defining this problem are now reduced to seven independent equations, (1a), (6a), (6b), (5b), (3), (19), and (20), with the seven unknowns σ_r , σ_θ , ϵ_r , ϵ_θ , ϵ_z , τ , and γ .

The solution of the problem is made simpler by introducing an arbitrary constant k into equations (19) and (20):

$$\left. \begin{aligned} r \frac{d\sigma_r}{d\left(\frac{r}{k}\right)} + \sigma_r r \frac{d\epsilon_z}{d\left(\frac{r}{k}\right)} &= (\sigma_\theta - \sigma_r) e^{(\epsilon_r - \epsilon_\theta)} - \rho (\omega k)^2 \left(\frac{r}{k}\right)^2 e^{(-\epsilon_z)} \\ r \frac{d\epsilon_\theta}{d\left(\frac{r}{k}\right)} &= e^{(\epsilon_r - \epsilon_\theta)} - 1 \end{aligned} \right\} \quad (21)$$

By use of the parameters r/k and ωk instead of r and ω , a simple direct solution is possible for any arbitrary value of ωk with k to be determined by the boundary condition.

INFINITE PLATE WITH CIRCULAR HOLE OR
FLAT RING RADIALLY STRESSED

The equations of equilibrium and compatibility for the infinite plate with a circular hole or the flat ring radially stressed are:

$$\left. \begin{aligned} \frac{r}{a} \frac{d\sigma_r}{d\left(\frac{r}{a}\right)} + \sigma_r \frac{r}{a} \frac{d\epsilon_z}{d\left(\frac{r}{a}\right)} &= (\sigma_\theta - \sigma_r) e^{(\epsilon_r - \epsilon_\theta)} \\ \frac{r}{a} \frac{d\epsilon_\theta}{d\left(\frac{r}{a}\right)} &= e^{(\epsilon_r - \epsilon_\theta)} - 1 \end{aligned} \right\} \quad (22)$$

The problem is defined by equations (22) together with equations (1a), (6a), (6b), (5b), and (3) (seven equations with seven unknowns).

EQUATIONS OF EQUILIBRIUM AND COMPATIBILITY IN
TERMS OF OCTAHEDRAL SHEAR STRAIN AND PARAM-
ETER INDICATING RATIO OF PRINCIPAL STRESSES

In the preceding section, displacements are eliminated from the equations, which result in seven equations involving the seven unknown quantities σ_r , σ_θ , ϵ_r , ϵ_θ , ϵ_z , τ , and γ . The quantity ϵ_z can be expressed in terms of ϵ_r and ϵ_θ (from equation (1a)). Two of the four unknowns σ_r , σ_θ , ϵ_r , and ϵ_θ can be eliminated by using equations (6a) and (6b) or (7). The quantity τ is a known function of γ that is experimentally determined. The problem is then reduced to one involving three unknowns. Obtaining the solution of the resulting equations is not, however, a simple matter; the iterative process is usually needed.

It is proposed that this can be avoided by using the following transformation:

$$\sigma_\theta + \sigma_r = 3\sqrt{2}\tau \sin \alpha$$

$$\sigma_\theta + \sigma_r = \sqrt{6}\tau \cos \alpha$$

or

$$\left. \begin{aligned} \sigma_r &= \sqrt{\frac{3}{2}}\tau(\sqrt{3}\sin \alpha - \cos \alpha) \\ \sigma_\theta &= \sqrt{\frac{3}{2}}\tau(\sqrt{3}\sin \alpha + \cos \alpha) \end{aligned} \right\} \quad (23)$$

Then σ_r and σ_θ satisfy equation (5a), because the yielding surfaces are ellipses according to the deformation theory. The octahedral shear stress τ , a function of γ , in the preceding equations varies with r/k and also with loading. Such a transformation has been used for ideally plastic material ($\tau = \text{constant}$) by Nadai in the section "Yielding in Thin Plate With Circular Hole or Flat Rings Radially Stressed"

(reference 1, p. 189) and for a rotating disk (reference 2). From equations (6a), (6b), and (23), the principal strains also can be expressed in terms of γ and α :

$$\left. \begin{aligned} \epsilon_r &= \frac{\gamma}{2\sqrt{2}}(\sin \alpha - \sqrt{3}\cos \alpha) \\ \epsilon_\theta &= \frac{\gamma}{2\sqrt{2}}(\sin \alpha + \sqrt{3}\cos \alpha) \end{aligned} \right\} \quad (24)$$

The equations of equilibrium and compatibility for the three problems considered herein are then obtained in terms of γ and α in the following form:

$$\left. \begin{aligned} A \frac{r}{k} \frac{d\alpha}{d\left(\frac{r}{k}\right)} + B \frac{r}{k} \frac{d\gamma}{d\left(\frac{r}{k}\right)} &= C \\ D \frac{r}{k} \frac{d\alpha}{d\left(\frac{r}{k}\right)} + E \frac{r}{k} \frac{d\gamma}{d\left(\frac{r}{k}\right)} &= F \end{aligned} \right\} \quad (25)$$

where the coefficients A , B , C , D , E , and F are functions of α , γ , and r/k . For the circular membrane under pressure, from equation (18),

$$\left. \begin{aligned} A &= (\sqrt{3}\cos \alpha + \sin \alpha) - (\sqrt{3}\sin \alpha - \cos \alpha) \frac{\gamma \cos \alpha}{\sqrt{2}} \\ B &= (\sqrt{3}\sin \alpha - \cos \alpha) \left(\frac{\gamma}{\tau} \frac{d\tau}{d\gamma} - \frac{\gamma \sin \alpha}{\sqrt{2}} \right) \frac{1}{\gamma} \\ C &= 2(\cos \alpha) e^{(-\sqrt{\frac{3}{2}}\gamma \cos \alpha)} \\ D &= (\sqrt{3}\sin \alpha - \cos \alpha) \gamma \\ E &= -(\sqrt{3}\cos \alpha + \sin \alpha) \\ F &= 2\sqrt{2} \\ &\left\{ 1 - e^{(-\sqrt{\frac{3}{2}}\gamma \cos \alpha)} \left[1 - \frac{e^{\sqrt{\frac{3}{2}}(\sqrt{3}\sin \alpha + \cos \alpha)\gamma}}{6\tau^2(\sqrt{3}\sin \alpha - \cos \alpha)^2} \left(\frac{r}{k}\right)^2 \left(\frac{pk}{h_{init}}\right)^2 \right]^{1/2} \right\} \end{aligned} \right\} \quad (26)$$

For the rotating disk, from equation (21),

$$\left. \begin{aligned} A &= (\sqrt{3}\cos \alpha + \sin \alpha) - (\sqrt{3}\sin \alpha - \cos \alpha) \frac{\gamma \cos \alpha}{\sqrt{2}} \\ B &= (\sqrt{3}\sin \alpha - \cos \alpha) \left(\frac{\gamma}{\tau} \frac{d\tau}{d\gamma} - \frac{\gamma \sin \alpha}{\sqrt{2}} \right) \frac{1}{\gamma} \\ C &= 2(\cos \alpha) e^{(-\sqrt{\frac{3}{2}}\gamma \cos \alpha)} - \sqrt{\frac{2}{3}} \rho(\omega k)^2 \frac{1}{\tau} \left(\frac{r}{k}\right)^2 e^{\frac{\gamma}{\sqrt{2}}\sin \alpha} \\ D &= (\sqrt{3}\sin \alpha - \cos \alpha) \gamma \\ E &= -(\sqrt{3}\cos \alpha + \sin \alpha) \\ F &= 2\sqrt{2} \left[1 - e^{(-\sqrt{\frac{3}{2}}\gamma \cos \alpha)} \right] \end{aligned} \right\} \quad (27)$$

For the infinite plate with a circular hole, from equation (22),

$$\left. \begin{aligned} A &= (\sqrt{3} \cos \alpha + \sin \alpha) - (\sqrt{3} \sin \alpha - \cos \alpha) \frac{\gamma \cos \alpha}{\sqrt{2}} \\ B &= (\sqrt{3} \sin \alpha - \cos \alpha) \left(\frac{\gamma}{\tau} \frac{d\tau}{d\gamma} - \frac{\gamma \sin \alpha}{\sqrt{2}} \right) \frac{1}{\gamma} \\ C &= 2 (\cos \alpha) e^{(-\sqrt{\frac{3}{2}} \gamma \cos \alpha)} \\ D &= (\sqrt{3} \sin \alpha - \cos \alpha) \gamma \\ E &= -(\sqrt{3} \cos \alpha + \sin \alpha) \\ F &= 2\sqrt{2} \left[1 - e^{(-\sqrt{\frac{3}{2}} \gamma \cos \alpha)} \right] \end{aligned} \right\} \quad (28)$$

With these transformations, the solution of the problems is reduced to simply a numerical integration of the two simultaneous differential equations (equations (25)) involving the two unknowns γ and α . Furthermore, the parameter γ , being the octahedral shear strain, directly indicates the stage of plastic deformation at any point under any load. (In plastic problems, according to the deformation theory, the individual stress and strain distributions cannot give as clear a picture of the stage of plastic deformation as can the octahedral shear strain.) Also, the parameter α indicates the ratio of the principal stresses or strains. At any point, if α remains constant during loading, the ratio of principal stresses at that point remains fixed. The value of α obtained at each point in the calculation during loading directly indicates whether or not the deformation theory is applicable to the problem.

The value of α is known at the boundaries or the center. This value can be determined from equations (23) and (24). For the circular membrane under pressure, when $r/b=0$,

$$\sigma_r = \sigma_\theta$$

$$\alpha = \frac{\pi}{2} = 1.5708$$

when $r/b=1$,

$$\epsilon_\theta = 0$$

$$\alpha = \frac{2}{3} \pi = 2.0944$$

For the rotating disk without a hole, when $r/b=0$,

$$\sigma_r = \sigma_\theta$$

$$\alpha = \frac{\pi}{2} = 1.5708$$

when $r/b=1$,

$$\sigma_r = 0$$

$$\alpha = \frac{\pi}{6} = 0.5236$$

For the rotating disk with a hole, when $r/a=1$ and $r/b=1$,

$$\sigma_r = 0$$

$$\alpha = \frac{\pi}{6} = 0.5236$$

For the infinite plate with a circular hole, when $r/a=1$,

$$\sigma_r = 0$$

$$\alpha = \frac{\pi}{6} = 0.5236$$

when r/a approaches c/a or a value that is large compared with 1,

$$\sigma_r = \sigma_\theta$$

$$\alpha = \frac{\pi}{2} = 1.5708$$

METHODS OF NUMERICAL INTEGRATION

Two methods are developed to solve the differential equations (25). In the first method, the differential equations are numerically integrated along r/k , which is considered the independent variable. (In the second method, α is considered the independent variable.) Because many terms in the equations are trigonometric functions of α , the use of α as the independent variable considerably reduces the work of computation.

Numerical integration with r/k as independent variable.—Equation (25) can be written in the following forms:

$$\left. \begin{aligned} \frac{r}{k} \frac{d\alpha}{d\left(\frac{r}{k}\right)} &= \frac{CE - FB}{AE - DB} \\ \frac{r}{k} \frac{d\gamma}{d\left(\frac{r}{k}\right)} &= \frac{FA - CD}{EA - BD} \end{aligned} \right\} \quad (29)$$

At any point, if α and γ are known, $\frac{d\alpha}{d(r/k)}$ and $\frac{d\gamma}{d(r/k)}$ can be

calculated by equations (29). At the boundaries or the center, α is known, but γ is determined by the load. Only one value (unknown) of γ corresponding to a particular load exists on each boundary; therefore it is difficult to start the numerical integrations on the boundary with the correct value of γ corresponding to a given load. Also, in plastic problems covering the strain-hardening range, the method of superposition is invalid. Usually, a method of iteration is used to solve the problem (for example, references 3 and 4). In the method presented herein, an arbitrary but unknown constant k has been introduced in equations (18), (21), and (22). For the cases considered, the terms in the equations that involve load are always multiplied by r , so that $\left(\frac{pr}{h_{init}}\right)^2$ can be written as $\left(\frac{pk}{h_{init}}\right)^2 \left(\frac{r}{k}\right)^2$ in equations (18) and (26) and $(\omega r)^2$ as $(\omega k)^2 \left(\frac{r}{k}\right)^2$ in equations (21) and (27).

The numerical integration then can be started at the inner boundary (or at the center if there is no circular hole at the center) by using the known values of α_o , a desired value of γ_o , and an arbitrary value of $\left(\frac{pk}{h_{init}}\right)^2$ for the membrane or of $(\omega k)^2$ for the rotating disk. The numerical integrations can then be carried out, obtaining values of α and γ at different values of r/k , until α progressively reaches the value that satisfies the outer boundary condition. Because the value of r is known at the boundaries, the value of k can be determined for the selected value of γ_o . The number of points and the formulas used in the calculation depend on the accuracy required (references 19 and 20). If the formula for evaluating definite integrals is applied after using the forward integration formula (references 19 and 20), great accuracy can easily be obtained.

The procedure used herein to obtain solutions is the same for each problem. Calculations are started from the inner boundary (or from the center if there is no circular hole at the center) with the known value of α_o , the desired value of γ_o , and the arbitrary loading term. The parameter α_o is equal to $\pi/2$ at $r/b=0$ for the membrane and for the solid rotating disk and is equal to $\pi/6$ at $r/a=1$ for the infinite plate with a circular hole and for the rotating disk with a hole. The arbitrary loading terms are $\left(\frac{pk}{h_{init}}\right)^2$ and $(\omega k)^2$ for the membrane and the rotating disk, respectively. Then $\left[\frac{d\alpha}{d(r/k)}\right]_o$ and $\left[\frac{d\gamma}{d(r/k)}\right]_o$, corresponding to α_o and γ_o at the inner boundary or the center, are obtained from equations (29). The following formulas for forward integration are used to determine the first approximate values of α and γ at the next point (α_1^* and γ_1^*):

$$\begin{aligned}\alpha_1^* &= \alpha_o + \left[\left(\frac{r}{k} \right)_1 - \left(\frac{r}{k} \right)_o \right] \left[\frac{d\alpha}{d\left(\frac{r}{k} \right)} \right]_o \\ \gamma_1^* &= \gamma_o + \left[\left(\frac{r}{k} \right)_1 - \left(\frac{r}{k} \right)_o \right] \left[\frac{d\gamma}{d\left(\frac{r}{k} \right)} \right]_o\end{aligned}\quad (30a)$$

By substitution of α_1^* and γ_1^* into equation (29), approximate values of $\left[\frac{d\alpha}{d\left(\frac{r}{k}\right)}\right]_1$ and $\left[\frac{d\gamma}{d\left(\frac{r}{k}\right)}\right]_1$ are obtained and

the second approximate values of α_1 and γ_1 (α_1^{**} and γ_1^{**}) can be computed from the following formulas:

$$\begin{aligned}\alpha_1^{**} &= \alpha_o + \frac{1}{2} \left[\left(\frac{r}{k} \right)_1 - \left(\frac{r}{k} \right)_o \right] \left\{ \left[\frac{d\alpha}{d\left(\frac{r}{k} \right)} \right]_o + \left[\frac{d\alpha}{d\left(\frac{r}{k} \right)} \right]_1^* \right\} \\ \gamma_1^{**} &= \gamma_o + \frac{1}{2} \left[\left(\frac{r}{k} \right)_1 - \left(\frac{r}{k} \right)_o \right] \left\{ \left[\frac{d\gamma}{d\left(\frac{r}{k} \right)} \right]_o + \left[\frac{d\gamma}{d\left(\frac{r}{k} \right)} \right]_1^* \right\}\end{aligned}\quad (30b)$$

The values of α_1^{**} and γ_1^{**} are substituted into equation (29)

again in order to calculate the values of $\left[\frac{d\alpha}{d\left(\frac{r}{k}\right)}\right]_1$ and $\left[\frac{d\gamma}{d\left(\frac{r}{k}\right)}\right]_1$. By use of the following formulas for evaluating

definite integrals, the volumes of α_1 and γ_1 are calculated:

$$\begin{aligned}\alpha_1 &= \alpha_o + \frac{1}{2} \left[\left(\frac{r}{k} \right)_1 - \left(\frac{r}{k} \right)_o \right] \left\{ \left[\frac{d\alpha}{d\left(\frac{r}{k} \right)} \right]_o + \left[\frac{d\alpha}{d\left(\frac{r}{k} \right)} \right]_1 \right\} \\ \gamma_1 &= \gamma_o + \frac{1}{2} \left[\left(\frac{r}{k} \right)_1 - \left(\frac{r}{k} \right)_o \right] \left\{ \left[\frac{d\gamma}{d\left(\frac{r}{k} \right)} \right]_o + \left[\frac{d\gamma}{d\left(\frac{r}{k} \right)} \right]_1 \right\}\end{aligned}\quad (30c)$$

This procedure is applied to the next point, and so forth, until the value of α reaches the required value of α_b at the outside boundary ($\alpha_b = 2/3 \pi$ at $r/b=1$ for the membrane, $\alpha_b = \pi/6$ at $r/b=1$ for the rotating disk, and $\alpha_b = \pi/2$ at $r/a=c/a$ for the thin plate with a circular hole). Inasmuch as

$$\left(\frac{r}{k} \right)_{a-\alpha_b} = \frac{b}{k}$$

the loading terms are determined as follows:

For the membrane,

$$\left(\frac{pb}{h_{init}} \right)^2 = \left(\frac{pk}{h_{init}} \right)^2 \left(\frac{b}{k} \right)^2 \quad (31a)$$

For the rotating disk,

$$(\omega b)^2 = (\omega k)^2 \left(\frac{b}{k} \right)^2 \quad (31b)$$

For the infinite plate with a circular hole,

$$t_c = \sigma h_c \left(\frac{r+u}{r} \right)_c = \sigma h_{init} e^{-\epsilon_r} = \sigma h_{init} e^{-\frac{\gamma_c}{2\sqrt{2}}} \quad (31c)$$

or for the flat ring radially stressed at the outside diameter b ,

$$t_b = (\sigma_r)_b h_b \left(\frac{r+u}{r} \right)_b = (\sigma_r)_b h_{init} e^{-\frac{\gamma_b}{2\sqrt{2}} (\sin \alpha_b - \sqrt{3} \cos \alpha_b)} \quad (31d)$$

where t_c and t_b are the tensions per unit original circumferential length at $r=c$ and $r=b$, respectively.

Numerical integration with α as independent variable.—Equations (29) can be written in the following forms:

$$\begin{aligned}\frac{d\gamma}{d\alpha} &= \frac{FA - CD}{CE - FB} \\ \frac{d\left(\frac{r}{k} \right)}{d\alpha} &= \frac{AE - DB}{CE - BF} \frac{r}{k}\end{aligned}\quad (32)$$

By use of equations (26) to (28) and expansion of $e^{f(\alpha, \gamma)}$ into a series, the following equations are obtained:

For the circular membrane, from equations (26),

$$\left. \begin{aligned} CE-BF &= 2GJ - \left[2\sqrt{3}HJg\left(\alpha, \gamma, \frac{\gamma}{\tau} \frac{d\tau}{d\gamma}\right)f_1(\alpha, \gamma) + 2\sqrt{3} \right] j\left(\alpha, \gamma, \frac{r}{k}, \frac{pk}{h_{init}}\right) \\ AF-CD &= 2\sqrt{2}L - 2HJ\gamma - 2\sqrt{2}Lf_2(\alpha, \gamma)j\left(\alpha, \gamma, \frac{r}{k}, \frac{pk}{h_{init}}\right) \\ AE-BD &= -L^2 - J^2g\left(\alpha, \gamma, \frac{\gamma}{\tau} \frac{d\tau}{d\gamma}\right) \end{aligned} \right\} \quad (33)$$

For the rotating disk, from equations (27),

$$\left. \begin{aligned} CE-BF &= -2HL - 2\sqrt{3}HJg\left(\alpha, \gamma, \frac{\gamma}{\tau} \frac{d\tau}{d\gamma}\right)f_1(\alpha, \gamma) + L\frac{K_2}{\tau}\left(\frac{r}{k}\right)^2f_3(\alpha, \gamma) \\ AF-CD &= \left\{ 8H^2 - 2\sqrt{3}HL[1-f_1(\alpha, \gamma)] + J\frac{K_2}{\tau}\left(\frac{r}{k}\right)^2f_3(\alpha, \gamma) \right\} \gamma \\ AE-BD &= -L^2 - J^2g\left(\alpha, \gamma, \frac{\gamma}{\tau} \frac{d\tau}{d\gamma}\right) \end{aligned} \right\} \quad (34)$$

For the infinite plate with a circular hole, from equations (28),

$$\left. \begin{aligned} CE-BF &= -2HL - 2\sqrt{3}HJg\left(\alpha, \gamma, \frac{\gamma}{\tau} \frac{d\tau}{d\gamma}\right)f_1(\alpha, \gamma) \\ AF-CD &= \{ 8H^2 - 2\sqrt{3}HL[1-f_1(\alpha, \gamma)] \} \gamma \\ AE-BD &= -L^2 - J^2g\left(\alpha, \gamma, \frac{\gamma}{\tau} \frac{d\tau}{d\gamma}\right) \end{aligned} \right\} \quad (35)$$

where

$$G = \sin \alpha$$

$$H = \cos \alpha$$

$$J = \sqrt{3} \sin \alpha - \cos \alpha$$

$$L = \sqrt{3} \cos \alpha + \sin \alpha$$

$$K_1 = \left(\frac{pk}{h_{init}}\right)^2$$

$$K_2 = \sqrt{\frac{2}{3}} \rho(\omega k)^2$$

and

$$f_1(\alpha, \gamma) = \frac{1}{\sqrt{\frac{3}{2}}(\cos \alpha)\gamma} \left[1 - e^{-\sqrt{\frac{3}{2}}(\cos \alpha)\gamma} \right]$$

$$= 1 - \frac{1}{2} \sqrt{\frac{3}{2}}(\cos \alpha)\gamma + \frac{1}{4}(\cos^2 \alpha)\gamma^2 - \dots$$

$$f_2(\alpha, \gamma) = e^{-\sqrt{\frac{3}{2}}(\cos \alpha)\gamma}$$

$$= 1 - \sqrt{\frac{3}{2}}(\cos \alpha)\gamma + \frac{1}{2} \times \frac{3}{2}(\cos^2 \alpha)\gamma^2 -$$

$$\frac{1}{4} \sqrt{\frac{3}{2}}(\cos^3 \alpha)\gamma^3 + \dots$$

$$= 1 - \sqrt{\frac{3}{2}}(\cos \alpha)\gamma f_1(\alpha, \gamma)$$

$$f_3(\alpha, \gamma) = e^{\frac{\gamma}{\sqrt{2}} \sin \alpha}$$

$$= 1 + \left(\frac{1}{\sqrt{2}} \sin \alpha\right)\gamma + \frac{1}{2} \left(\frac{1}{\sqrt{2}} \sin \alpha\right)^2 \gamma^2 +$$

$$\frac{1}{6} \left(\frac{1}{\sqrt{2}} \sin \alpha\right)^3 \gamma^3 + \dots$$

$$g\left(\alpha, \gamma, \frac{\gamma}{\tau} \frac{d\tau}{d\gamma}\right) = \frac{\gamma}{\tau} \frac{d\tau}{d\gamma} - \sqrt{\frac{3}{2}} \frac{\gamma}{\sqrt{3} \sin \alpha - \cos \alpha} = \frac{\gamma}{\tau} \frac{d\tau}{d\gamma} - \sqrt{\frac{3}{2}} \frac{\gamma}{J}$$

$$j\left(\alpha, \gamma, \frac{r}{k}, \frac{pk}{h_{init}}\right) = \left[1 - \frac{e^{\sqrt{\frac{3}{2}}\gamma(\sqrt{3} \sin \alpha + \cos \alpha)}}{6\tau^2(\sqrt{3} \sin \alpha - \cos \alpha)^2} \left(\frac{r}{k}\right)^2 \left(\frac{pk}{h_{init}}\right)^2 \right]^{1/2}$$

$$= \left[1 - \frac{K_1}{6J^2} \left(\frac{1}{\tau} \frac{r}{k}\right)^2 \frac{f_3(\alpha, \gamma)}{f_2(\alpha, \gamma)} \right]^{1/2}$$

The symbols G , H , J , and L are trigonometric functions of α only; K_1 and K_2 are arbitrary loading constants. The symbols f_1 , f_2 , f_3 , and g are functions of α and γ ; j is a function of α , γ , and r/k .

For the solution of an infinite plate with a circular hole α is the independent variable. The procedure of numerical integration is similar to that used in the method in which r/k is the independent variable. The first four terms of the series of $e^{f(\alpha, \gamma)}$ are used; the accuracy of the result is the same as that of the previous method, but computation is reduced by one half.

Both methods presented herein are used to obtain the solutions for the given values of γ_o . The purpose of the present paper is to obtain solutions for the entire strain-hardening range and the methods developed are very convenient for this purpose. If, however, a solution for only a particular value of loading is required, it can be obtained by interpolating between values obtained from two or three solutions corresponding to loading near the specified value.

NUMERICAL EXAMPLES

Membrane.—In order to compare the results for the circular membrane obtained by the method developed herein with those obtained by Gleyzal (reference 3), one numerical solution for infinitesimal strain is calculated by using the $\tau(\gamma)$ curve of the tensile test in figure 1 of reference 3. Inasmuch as reference 3 states that: "For simplicity, strain will be taken to mean conventional strain $(ds - ds_o)/ds_o$ where ds and ds_o are final and initial arc length, respectively.", equations (25a) and (36) given in appendix A for infinitesimal strain are used. The calculation is started at $r/k=0.005$. Values of $\alpha_o=1.5708$, $\gamma_o=0.0299$, and $pk/h_{int}=55,920$ are used.

Rotating disk.—Numerical solutions for the rotating disk

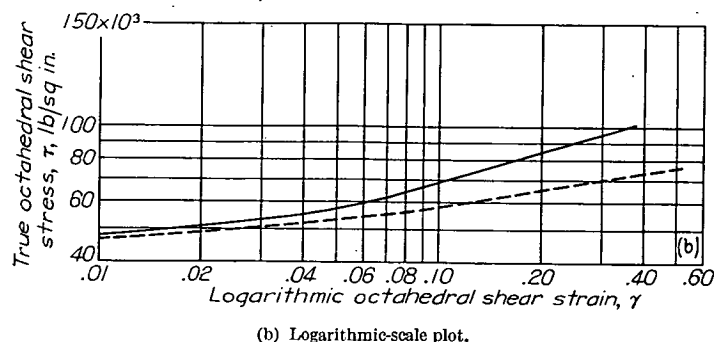
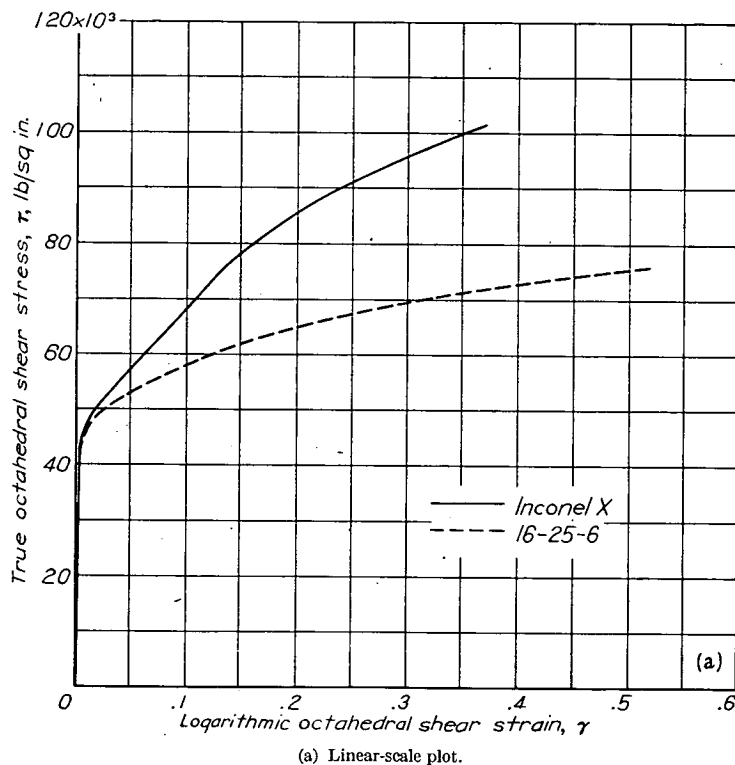


FIGURE 5.—Octahedral shear stress-strain curves.

for finite strain (equations (25) and (27)) are calculated. The $\tau(\gamma)$ curves of two materials, Inconel X and 16-25-6, are plotted in figure 5 (a). These data were supplied by W. F. Brown, Jr., H. Schwartzbart, and M. H. Jones. The same $\tau(\gamma)$ curves are plotted on logarithmic coordinates in figure 5 (b). These materials, Inconel X and 16-25-6, for which τ is not a power function of γ , were chosen so that more general information can be obtained. The given octahedral shear stress-strain curves (fig. 5) of these two materials have not been corrected for the triaxiality and nonuniform stress distribution introduced by necking and consequently do not represent the exact stress-strain relation after necking of these two materials. The solutions obtained from the $\tau(\gamma)$ curves of the tensile test after necking can, however, represent the solutions corresponding to materials having the exact $\tau(\gamma)$ curves shown in figure 5 and for simplicity such materials are herein referred to as "Inconel X" and "16-25-6".

The calculation for the solid rotating disk is started at $r/k=0.005$, as in the case of the membrane.

Three solutions are also obtained for the rotating disk with a central hole, using Inconel X. Calculations are started at $r/a=1$.

All numerical examples for the rotating disk are given in the following table:

Solid rotating disk		
Material	γ_o	$K_2 = \sqrt{\frac{2}{3}} \rho (\omega k)^2$
Inconel X	0.04	1×10^3
	.1152	1×10^3
	.30	1×10^3
16-25-6	0.04	1×10^3
	.1152	1×10^3
	.30	2.5×10^3

Rotating disk with central hole		
Material	γ_o	$\sqrt{\frac{2}{3}} \rho (\omega a)^2$
Inconel X	0.30	1×10^4
	.30	2×10^3
	.30	4×10^2

Infinite plate with circular hole.—The calculations for the infinite plate with a circular hole are carried out for the case in which $\sigma_r=0$ at $r/a=1$. The value of α_o at $r/a=1$ is then equal to 0.5236. (When σ_r is different from 0 at $r/a=1$, the corresponding value of α_o should be used.) The same materials as in the previous problem are considered. The numerical examples are:

Material	γ_o
Inconel X	0.04
	.1152
	.1871
	.30
16-25-6	0.04
	.1871
	.30

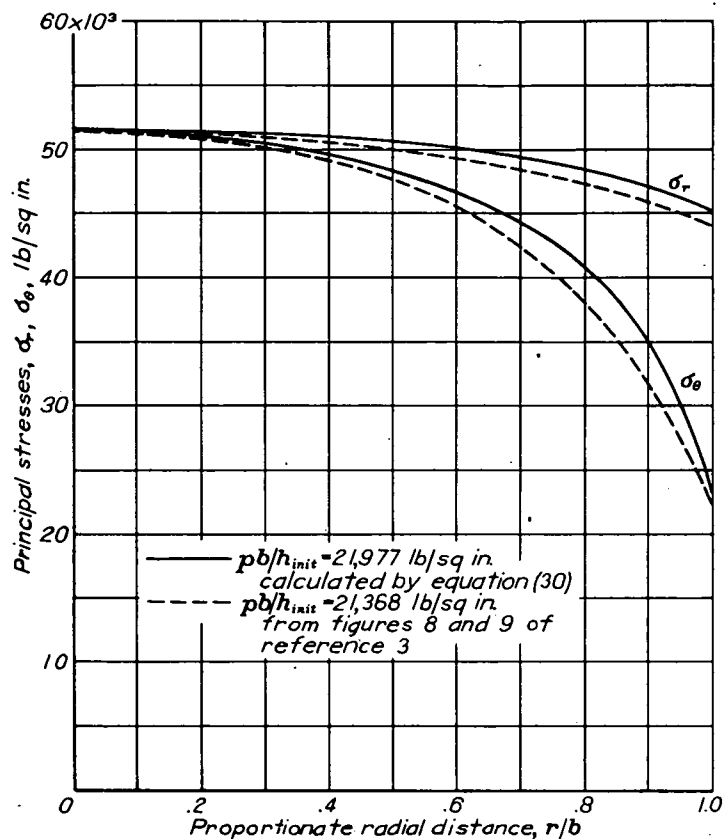


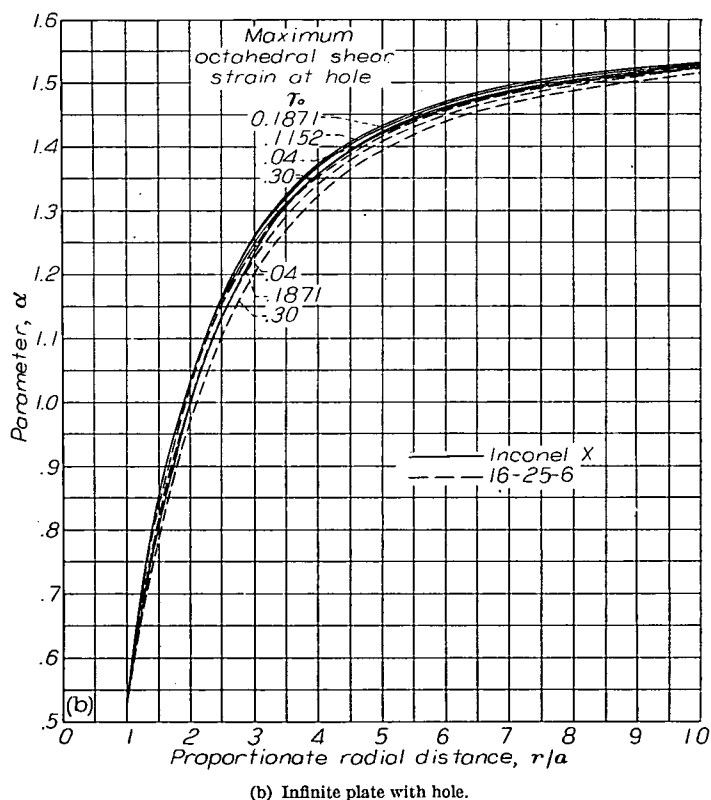
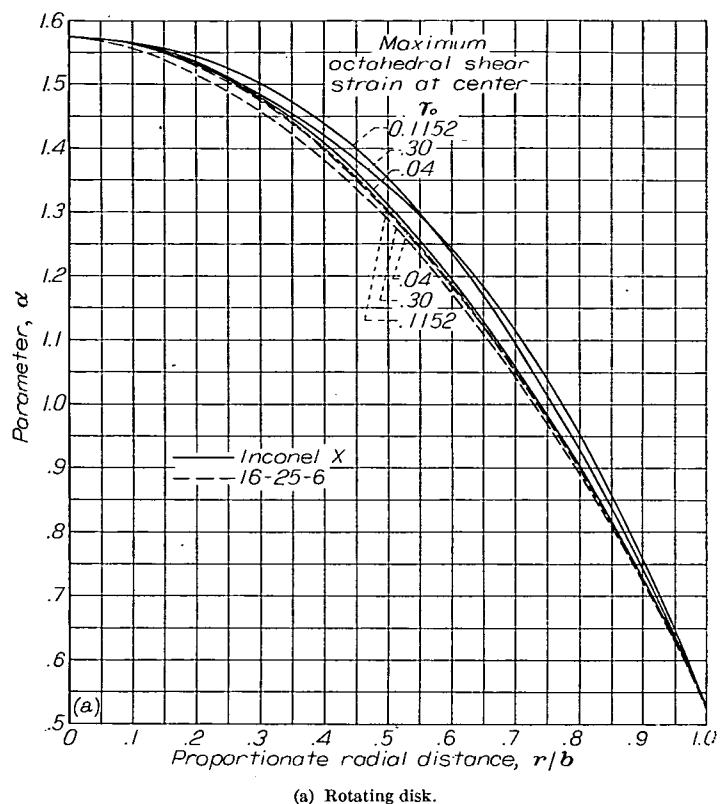
FIGURE 6.—Variation of principal stresses with proportionate radius distance.

RESULTS AND DISCUSSIONS

The radial and circumferential stresses σ_r and σ_θ , respectively, obtained for the circular membrane are plotted against r/b in figure 6. Two curves, taken from reference 3, corresponding to calculations for about the same pressure used in the present calculation, are included in the figure for comparison. In the present calculation, the $\tau(\gamma)$ curve given in figure 1 of reference 3 and the same infinitesimal-strain definition based on the original dimension are used. The initial thickness h_{init} is also used for consistency in the calculation rather than the instantaneous thickness h , which is used in reference 3.

The variations of α with the radius for the rotating disk and with the radius for the infinite plate with a circular hole are plotted in figures 7 (a) and 7 (b), respectively, for different loads and materials. The variations of α with γ_o (or loading) at various radii for the rotating disk and the infinite plate with a circular hole are plotted in figures 8 (a) and 8 (b), respectively. Similar curves for the ratio of the principal stresses σ_r/σ_θ are shown in figures 9 (a), 9 (b), and 10. Comparison of figure 7 and figures 9 (a) and 9 (b) shows that the variations of α with radius are very similar to the variations of σ_r/σ_θ with radius, although the relation between α and σ_r/σ_θ is not linear.

Numerical examples for a membrane with a large strain are not calculated herein, because the result of reference 3 is sufficient to give an approximate variation of the ratios of principal stresses along the radius during loading, although the infinitesimal-strain concept is used. The variations of the ratio of principal stresses with radius for different loads, based on the values of σ_r and σ_θ given in figures 8 and 9 of reference 3, are calculated and plotted on figure 9 (c).

FIGURE 7.—Variations of parameter α with proportionate radius distance for Inconel X and 16-25-6.

The values of σ_r are plotted against σ_θ at various radii under different loads for the rotating disk and the infinite plate with a circular hole in figures 11 (a) and 11 (b). The heavy solid and dashed curves represent the values of σ_r and σ_θ at different radii for any given load and are called loading curves. The loading curve moves away from the

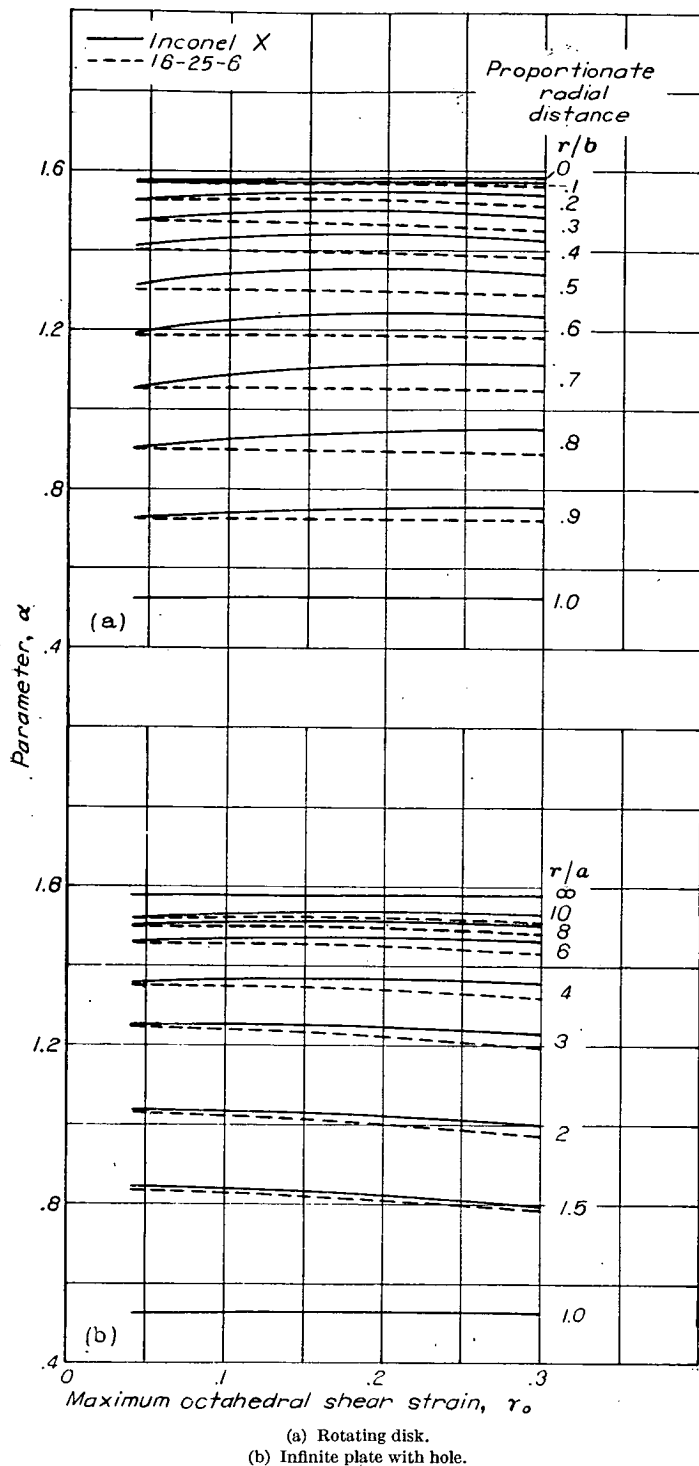


FIGURE 8.—Variation of parameter α with maximum octahedral shear strain at different radii.

origin with increasing load. The light solid and dotted lines connecting the different loading curves at a given radius and extending to the origin represent the values of σ_r and σ_θ at different loads for any given radius and are called loading paths. Also shown in the figures are the yielding surfaces, which are ellipses under the deformation theory.

A clear picture of the variation of the ratios of principal stresses in this group of problems with different loads for Inconel X, 16-25-6, and the material used in reference 3 is given in figures 7 to 11. It is evident that the ratios of principal stresses remain essentially constant during loading.

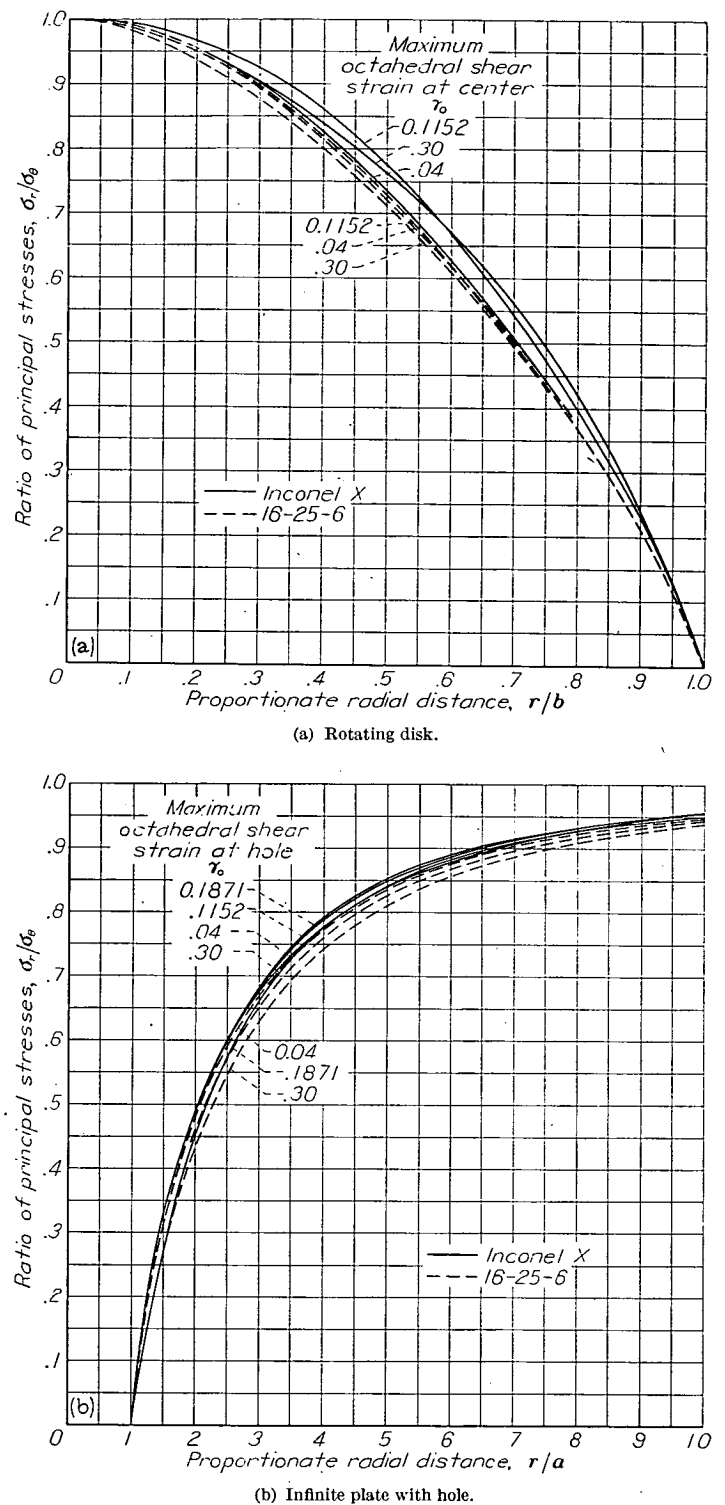
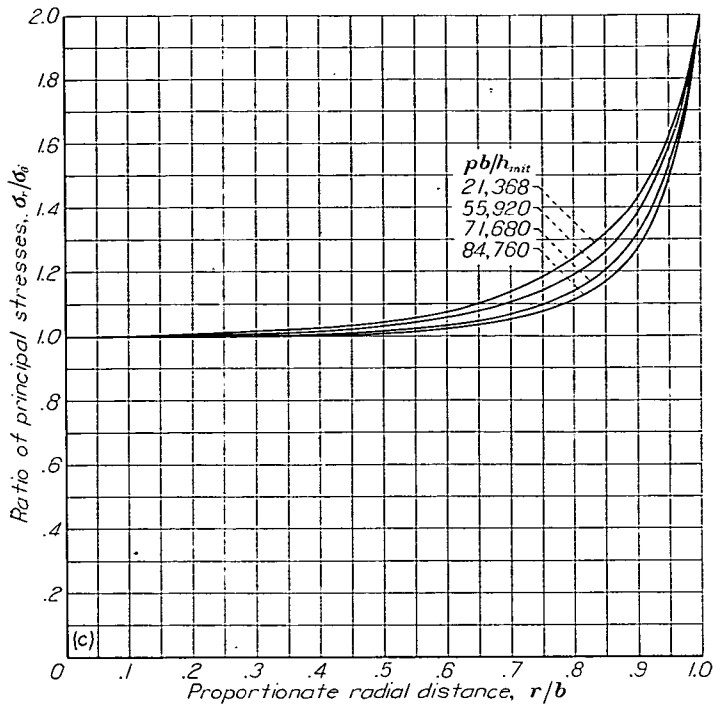


FIGURE 9.—Variations of ratio of principal stresses with proportionate radial distance.

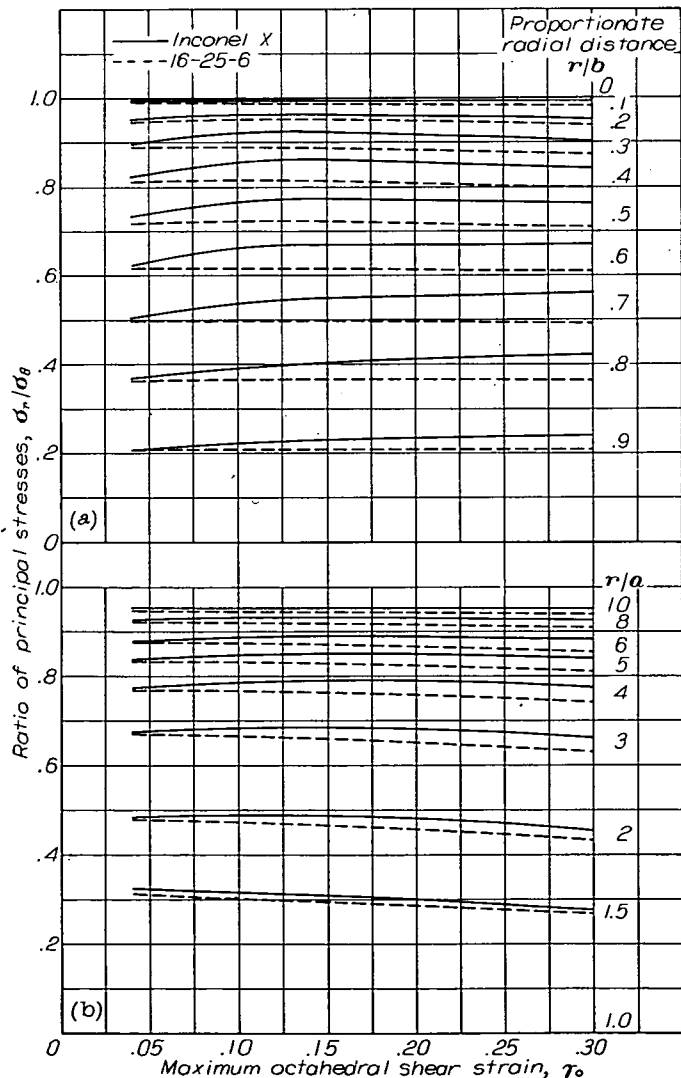
The deformation theory is therefore applicable to this group of problems, at least for the materials considered.

The variations of γ and γ/γ_o with radius are plotted in figures 12 and 13, respectively, for the rotating disk and the infinite plate with a circular hole. It is interesting to note that the curves in figure 13 for different loads for the same material are quite close together. For different materials, the curves of figures 7 and 9 are also close, but the curves of figure 13 are not as close together.



(c) Circular membrane under pressure.

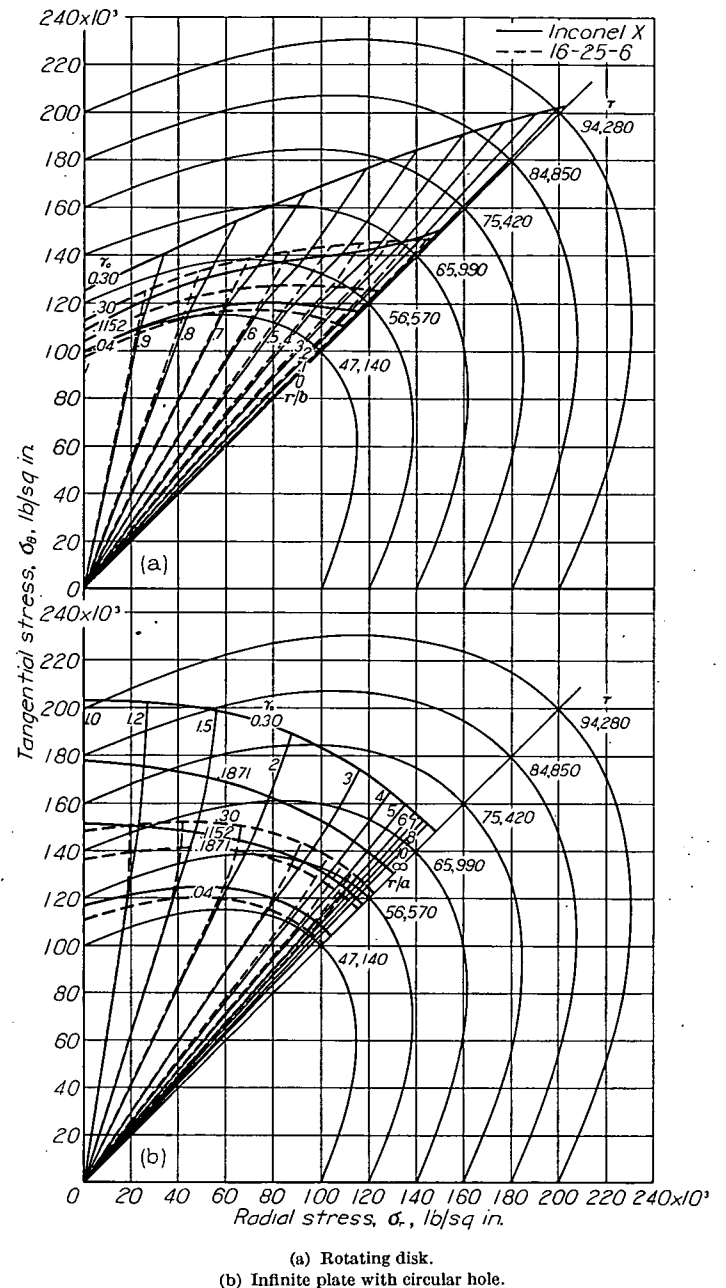
FIGURE 9.—Concluded. Variations of ratio of principal stresses with proportionate radial distance.



(a) Rotating disk.

(b) Infinite plate with hole.

FIGURE 10.—Variation of ratio of principal stresses with maximum octahedral shear strain at different radii.



(a) Rotating disk.

(b) Infinite plate with circular hole.

FIGURE 11.—Loading curves, loading paths, and yielding surfaces.

The distributions of principal stresses and principal strains along the radius for the rotating disk and for the infinite plate with a circular hole are plotted in figures 14 and 15, respectively. For comparison, the variations of $\sigma_\theta/(\sigma_\theta)_b$, $\epsilon_\theta/(\epsilon_\theta)_b$, and γ/γ_b with radius for both the elastic and the plastic range are plotted in figures 16 and 17. (The equations for the elastic case are given in appendix B.) If only the stress distributions for the elastic and plastic cases are compared (figs. 16 (a) and 17 (a)), it is seen that the stresses are more uniform in the plastic state; but if the distributions of the principal strains and the octahedral shear strain for the elastic and the plastic cases are compared (figs. 16 (b), 16 (c), 17 (b), and 17 (c)), it is evident that a less-uniform strain distribution is obtained in the plastic state. It is of special interest in the case of the finite plate with a hole to note that with plastic deformation the stress-(tangential stress) concentration factor around the hole is reduced; instead there is a high concentration in principal strain and in octahedral shear

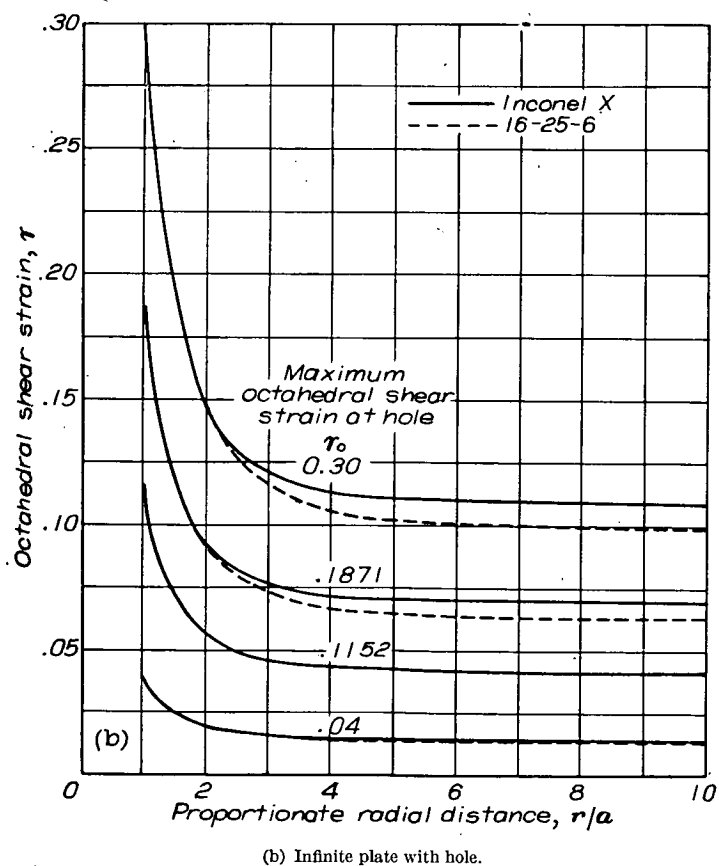
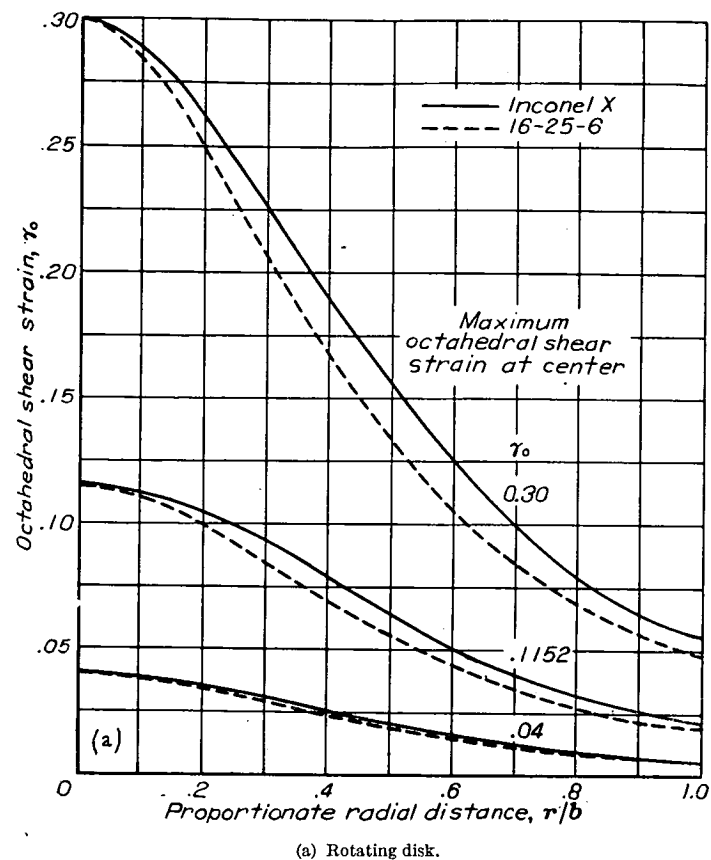


FIGURE 12.—Variation of octahedral shear strain with proportionate radial distance.

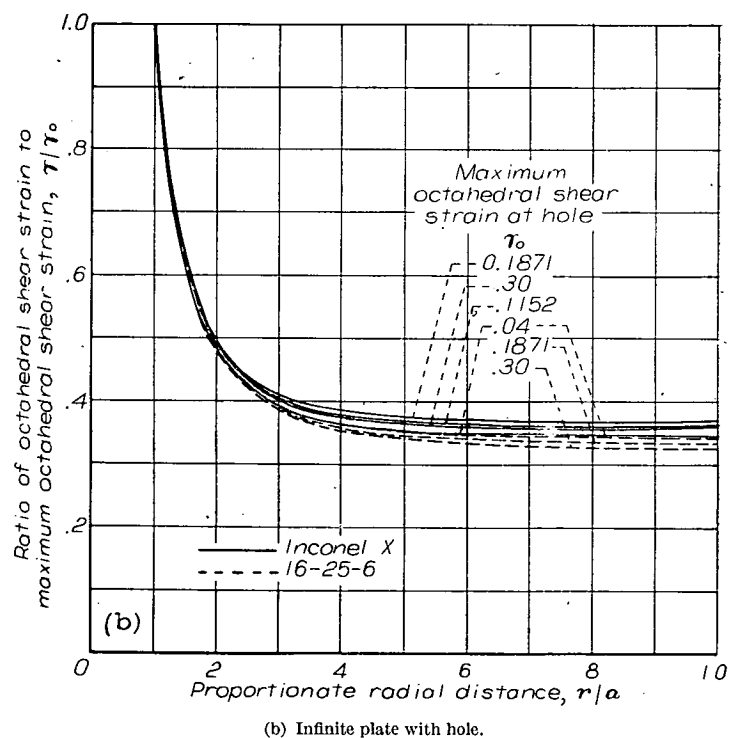
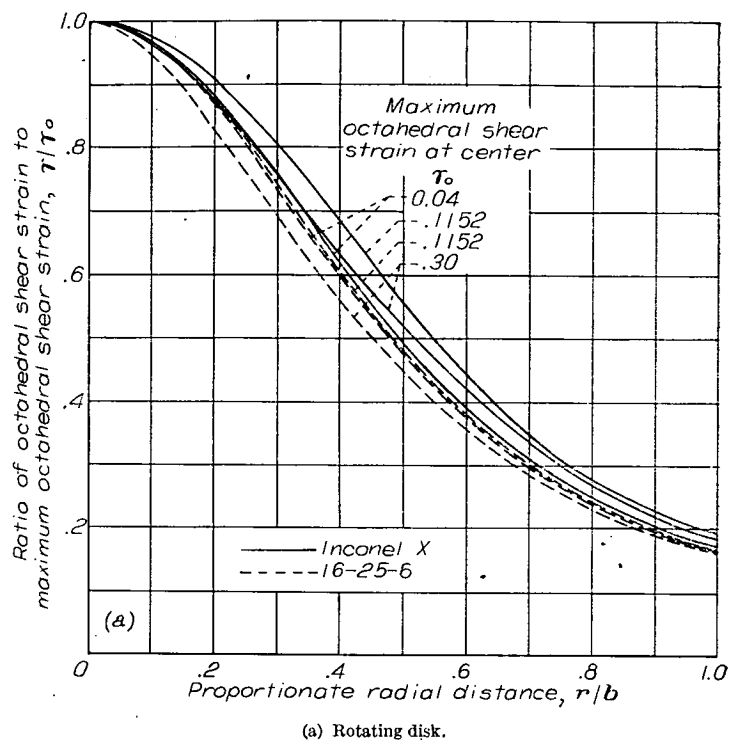
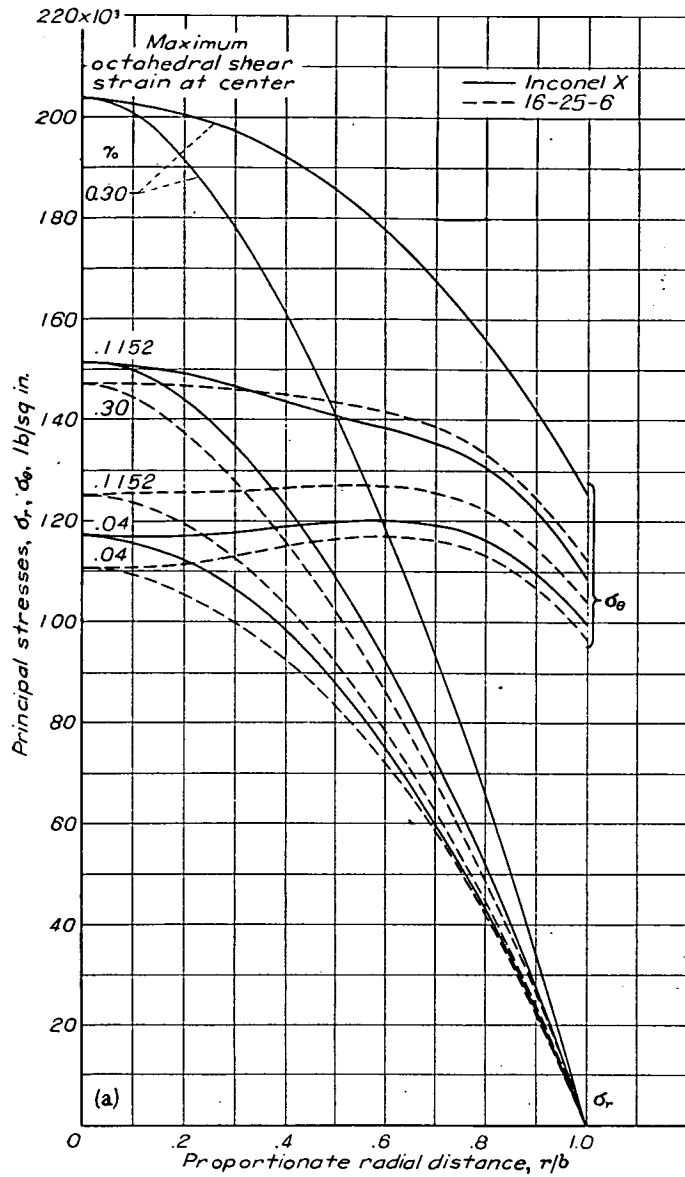
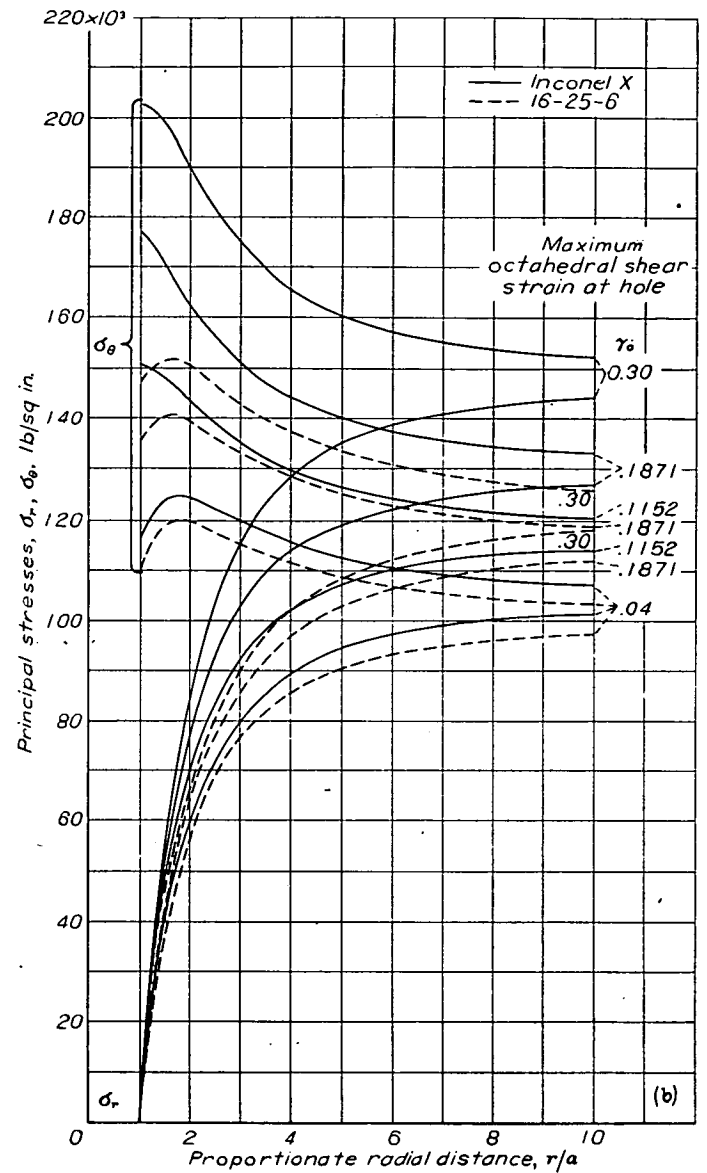


FIGURE 13.—Variation of ratio of octahedral shear strain to maximum octahedral shear strain with proportionate radial distance.



(a) Rotating disk.



(b) Infinite plate with hole.

FIGURE 14.—Variation of principal stresses with proportionate radial distance.

strain. A similar conclusion regarding the concentration factor around a circular hole in a tension panel is given in references 21 and 22.

The quantities $\sigma_\theta/(\sigma_\theta)_0$, $\sigma_r/(\sigma_r)_0$, $\epsilon_r/(\epsilon_r)_0$, and $\epsilon_\theta/(\epsilon_\theta)_0$ along the radius for the rotating disk and $\sigma_\theta/(\sigma_\theta)_0$ and $\epsilon_\theta/(\epsilon_\theta)_0$ for the infinite plate with a circular hole are plotted in figures 18 and 19, respectively. The curves representing $\sigma_r/(\sigma_r)_0$, $\epsilon_r/(\epsilon_r)_0$, and $\epsilon_\theta/(\epsilon_\theta)_0$ for Inconel X and 16-25-6 and different values of γ_0 are close together; but the curves of $\sigma_\theta/(\sigma_\theta)_0$ are quite far apart for the two materials, as well as for different values of γ_0 .

The relation between the rotating-speed function $\rho(\omega b)^2$

and γ_0 for the rotating disk and the relation between the tension per unit original circumferential length t_b/h_{init} and γ_0 for the infinite plate with a hole are plotted in figures 20 (a) and 20 (b), respectively. It is shown in these figures that $\rho(\omega b)^2$ and t_b/h_{init} increase considerably for Inconel X and increase only slightly for 16-25-6 as the value of γ_0 increases from 0.04 to 0.30.

Figures 7, 13, and 16 to 19 show that for the plate with a hole, the variations of α , γ/γ_0 , $\epsilon_r/(\epsilon_r)_0$, and $\epsilon_\theta/(\epsilon_\theta)_0$ with radius are essentially independent of the value of γ_0 for the plate and the $\tau(\gamma)$ curve of the material, at least for the materials considered. These results show that the deformation that

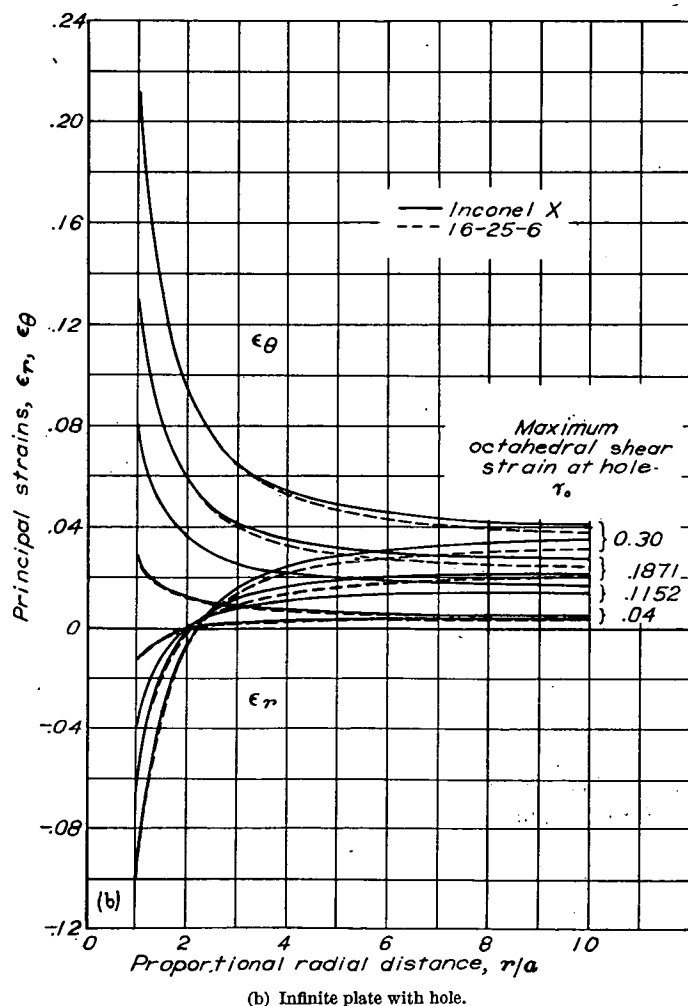
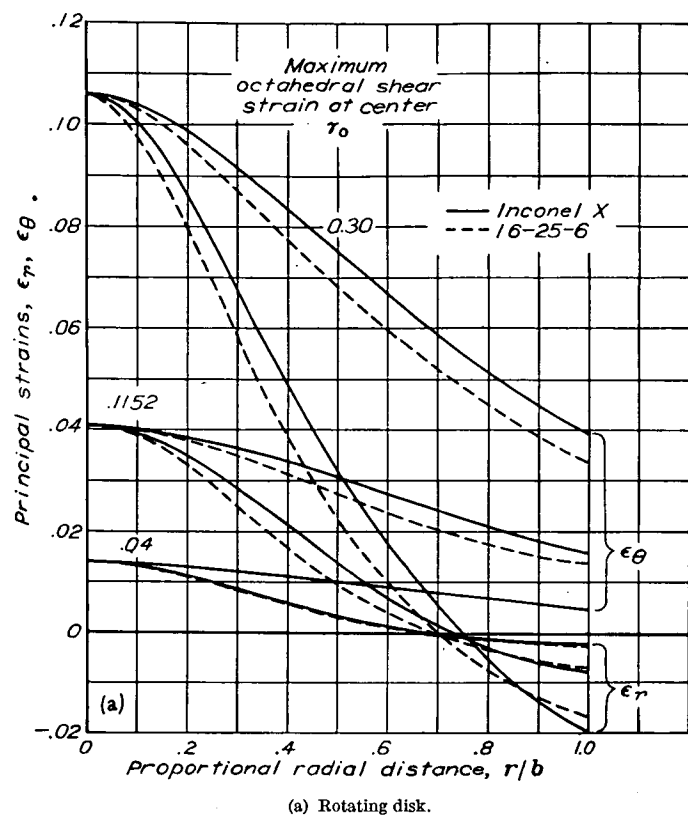
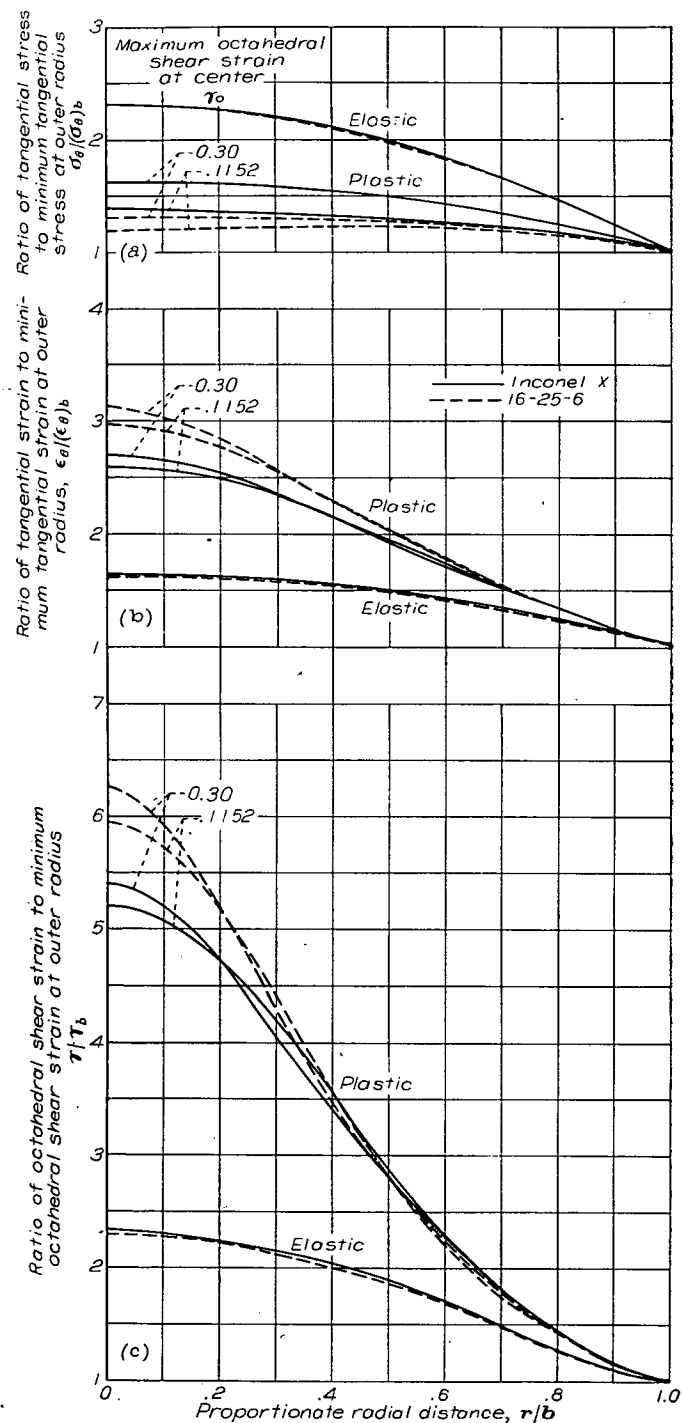
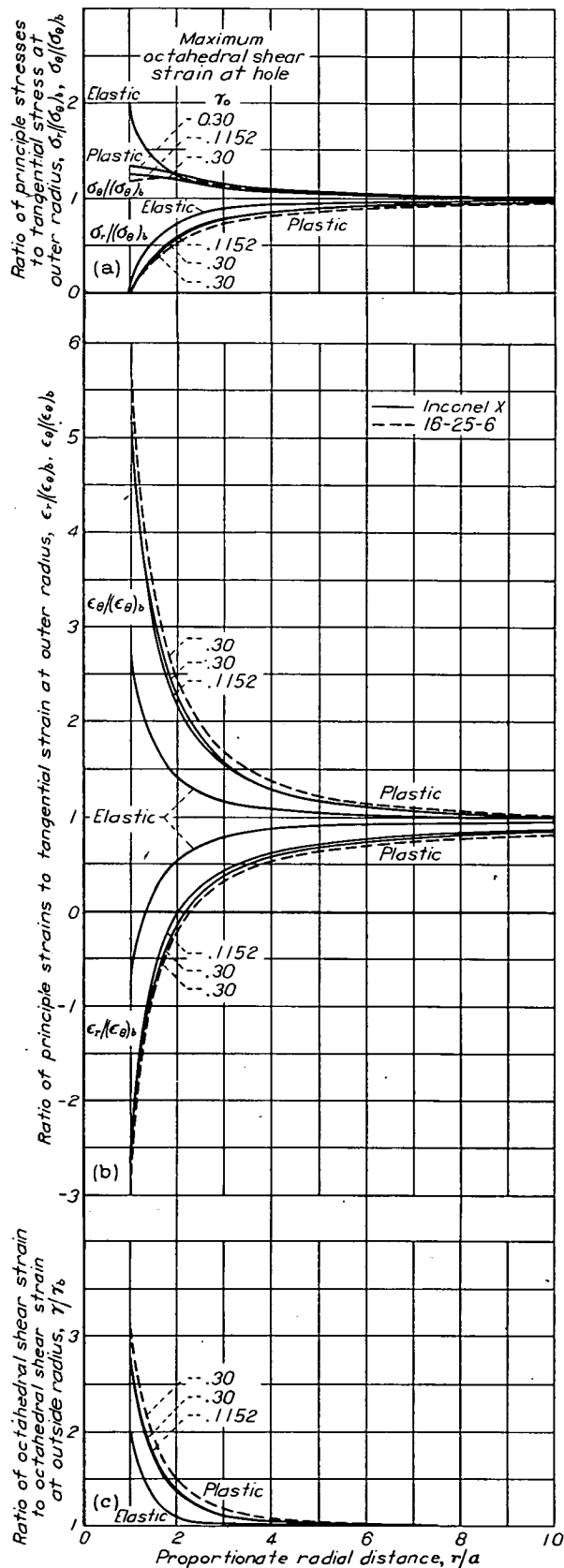


FIGURE 15.—Variation of principal strains with proportionate radial distance:



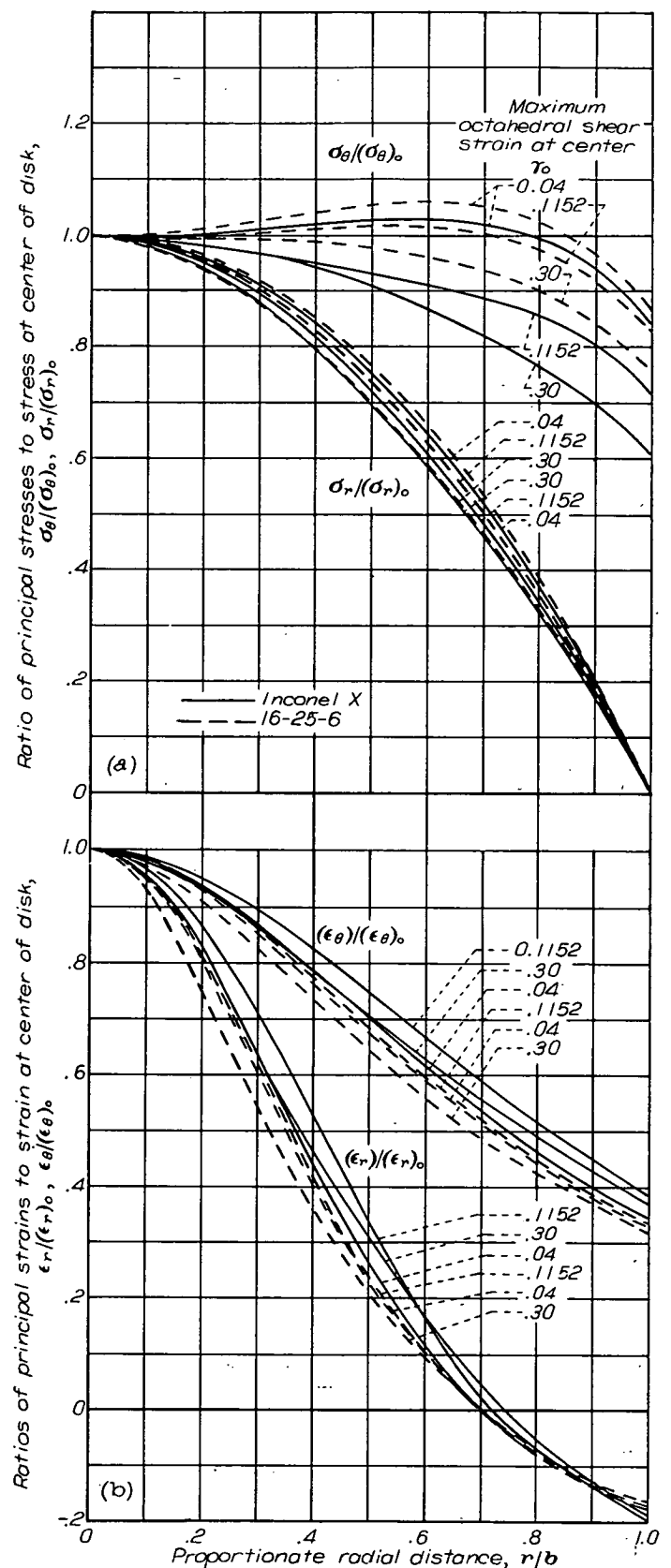
- (a) Variation of ratio of tangential stress to minimum tangential stress at outer radius with proportionate radial distance.
 (b) Variation of ratio of tangential strain to minimum tangential strain at outer radius with proportionate radial distance.
 (c) Variation of ratio of octahedral shear strain to minimum octahedral shear strain at outer radius with proportionate radial distance.

FIGURE 16.—Comparison of results obtained in elastic and plastic range for rotating disk.



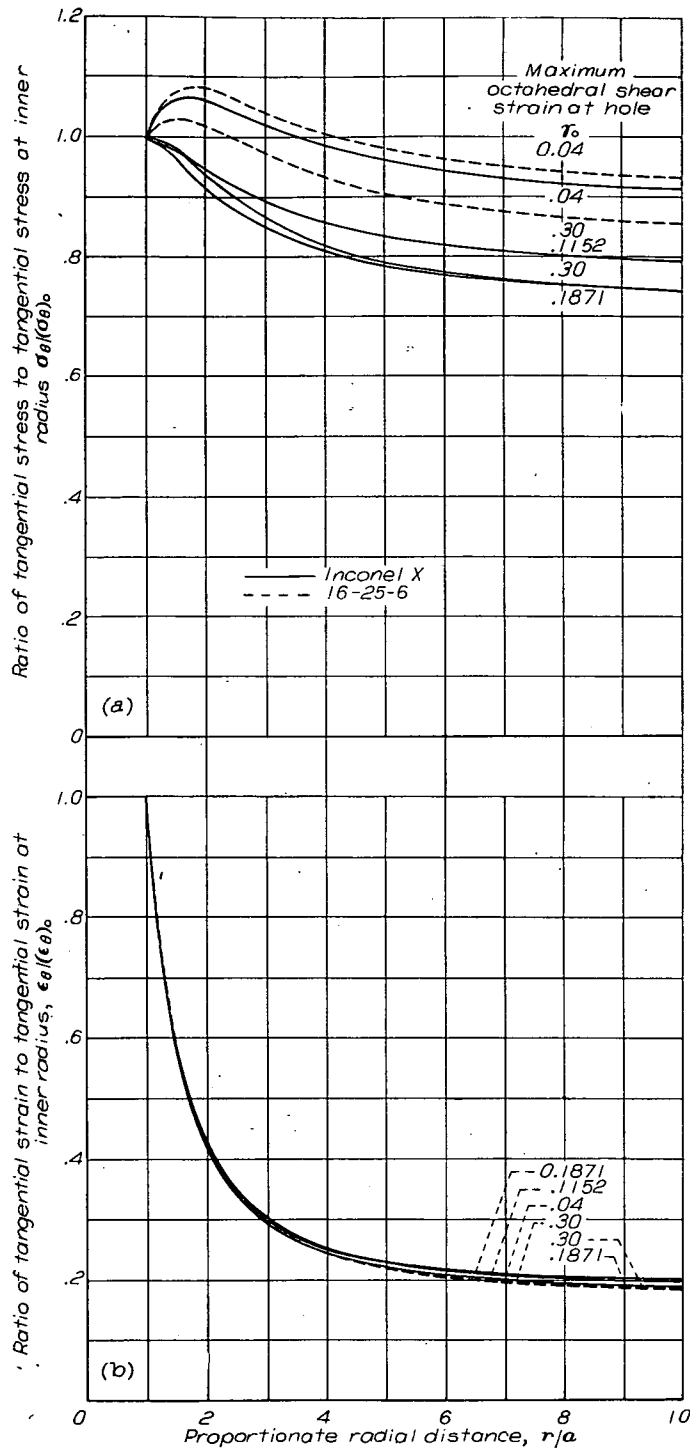
(a) Variation of ratio of principal stresses to tangential stress at outer radius with proportionate radial distance.
 (b) Variation of ratio of principal strains to tangential strain at outer radius with proportionate radial distance.
 (c) Variation of ratio of octahedral shear strain to octahedral shear strain at outer radius with proportionate radial distance.

FIGURE 17.—Comparison of results obtained in elastic and plastic range for infinite plate with circular hole.



(a) Variation of ratio of principal stresses to stress at center of disk with proportionate radial distance.
 (b) Variation of ratios of principal strains to strain at center of disk with proportionate radial distance.

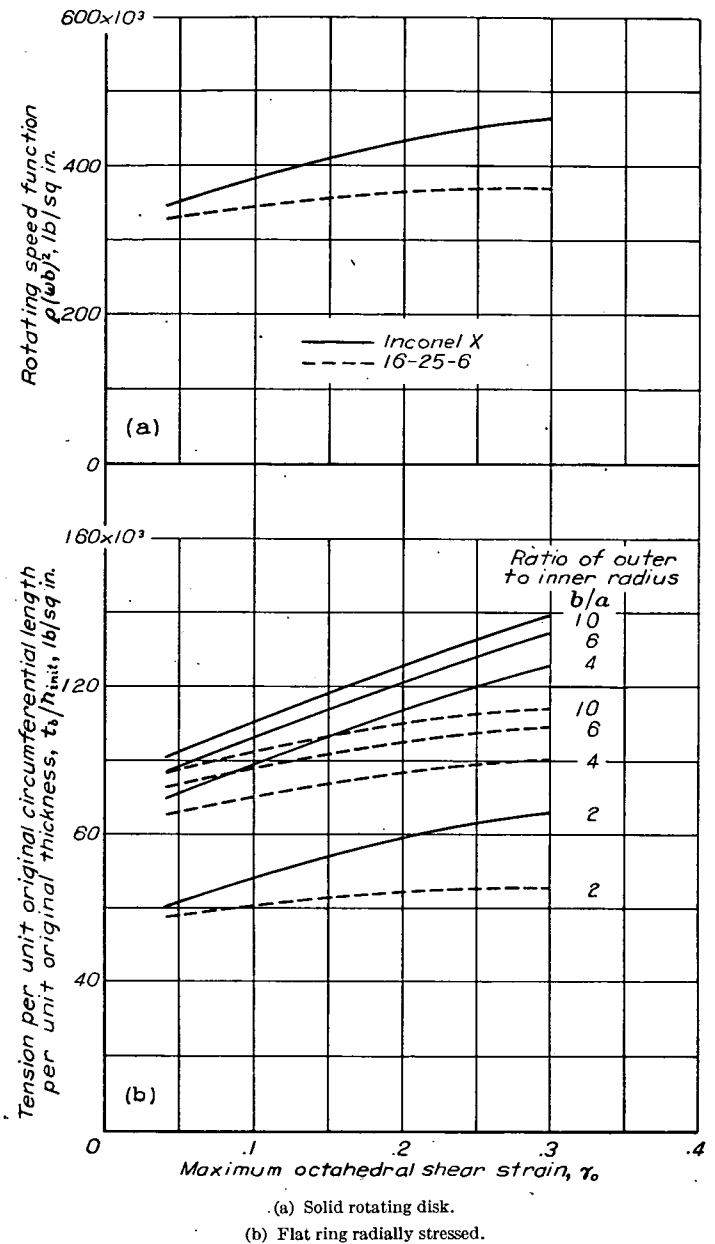
FIGURE 18.—Comparison of stress and strain distributions for different materials and for different maximum octahedral shear strains of rotating disk.



(a) Variation of ratio of tangential stress to tangential stress at inner radius with proportionate radial distance.
(b) Variation of ratio of tangential strain to tangential strain at inner radius with proportionate radial distance.

FIGURE 19.—Comparison of stress and strain distributions for different materials and for different maximum octahedral shear strains of infinite plate with circular hole.

can be accepted by the plate before failure depends mainly on the maximum octahedral shear strain (or ductility) of the material, which would not be true if the strain distributions were a function of the $\tau(\gamma)$ curve. For the rotating disk,

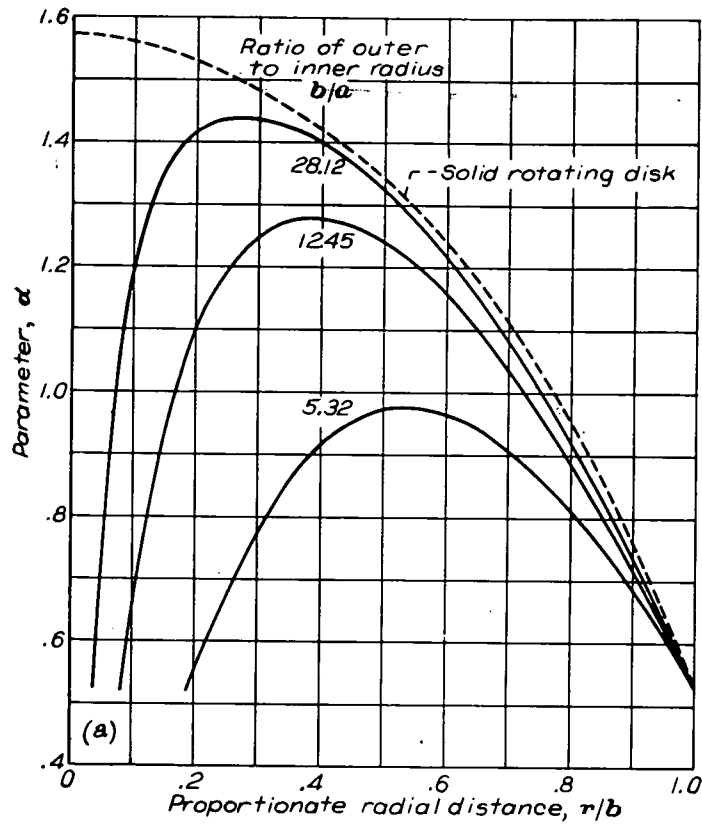
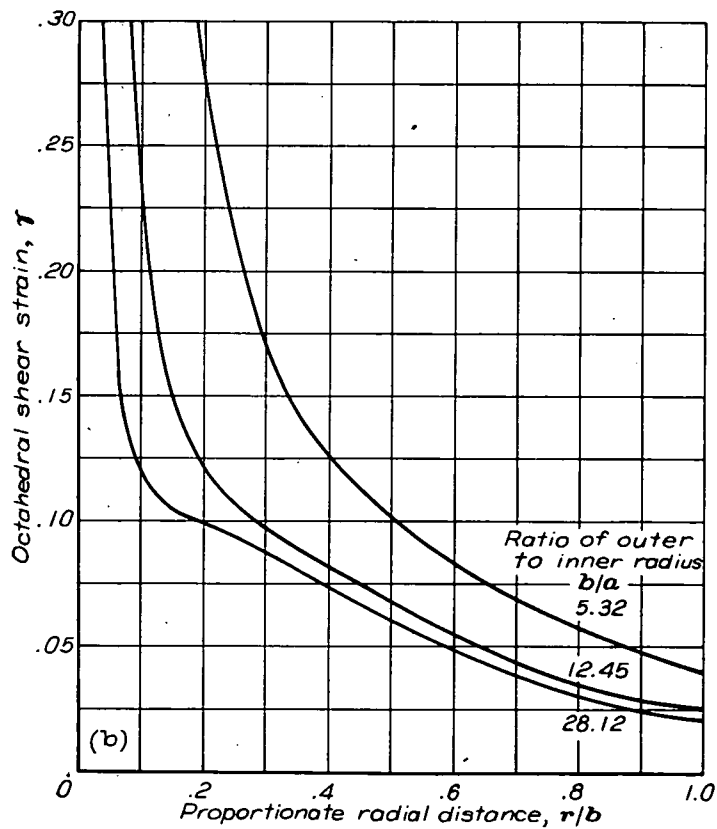


(a) Solid rotating disk.
(b) Flat ring radially stressed.

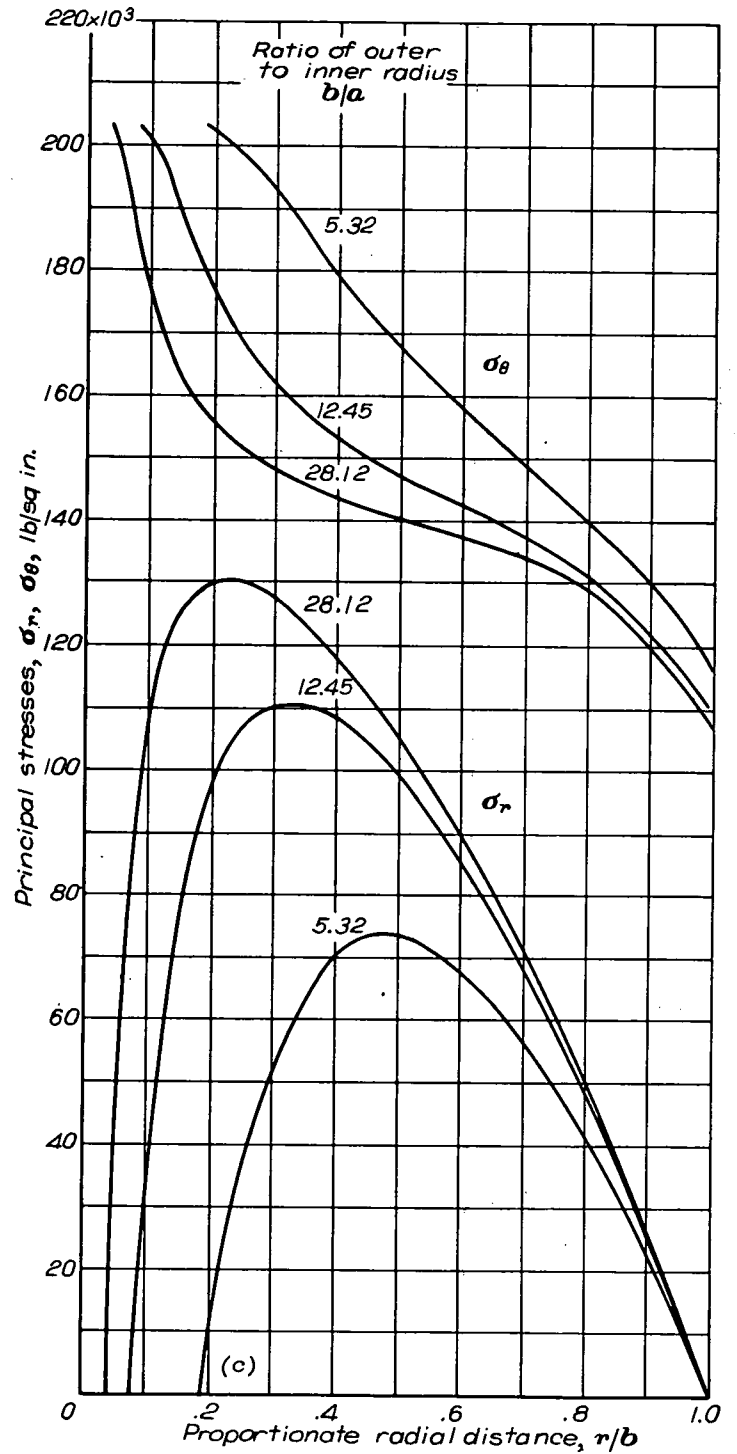
FIGURE 20.—Relation between rotating-speed function and maximum octahedral shear strain for disk and relation between tension per unit original circumferential length for plate.

however, a slight effect of γ_0 and the $\tau(\gamma)$ curve is apparent on the strains; this effect seems to be caused by the body-force term of the disk.

The stress distribution that will determine the load which a member can sustain is now considered. Figures 16 to 19 show that the variation of $\sigma_\theta/(\sigma_\theta)_0$ with radius depends on the $\tau(\gamma)$ curve of the material and on the value of γ_0 for the member. Figure 20 indicates that the load also depends on the $\tau(\gamma)$ curve. It therefore follows that the added load that the member can sustain between the onset of yielding and failure depends on the $\tau(\gamma)$ curve of the material. The octahedral shear (or effective) stress and strain curve of the material should be used as a criterion in selecting a material

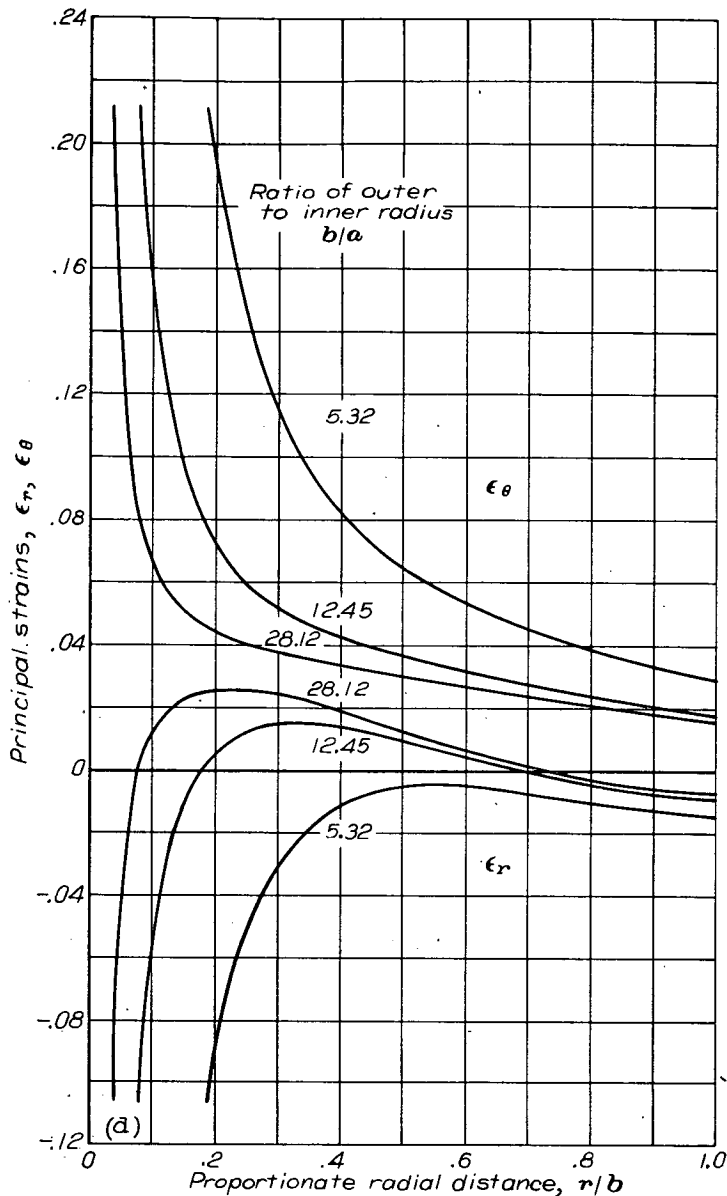

 (a) Variation of parameter α with proportionate radial distance.


(b) Variation of octahedral shear strain with proportionate radial distance.



(c) Variation of principal stresses with proportionate radial distance.

 FIGURE 21.—Rotating disk with hole. Inconel X; γ_0 , 0.3000.



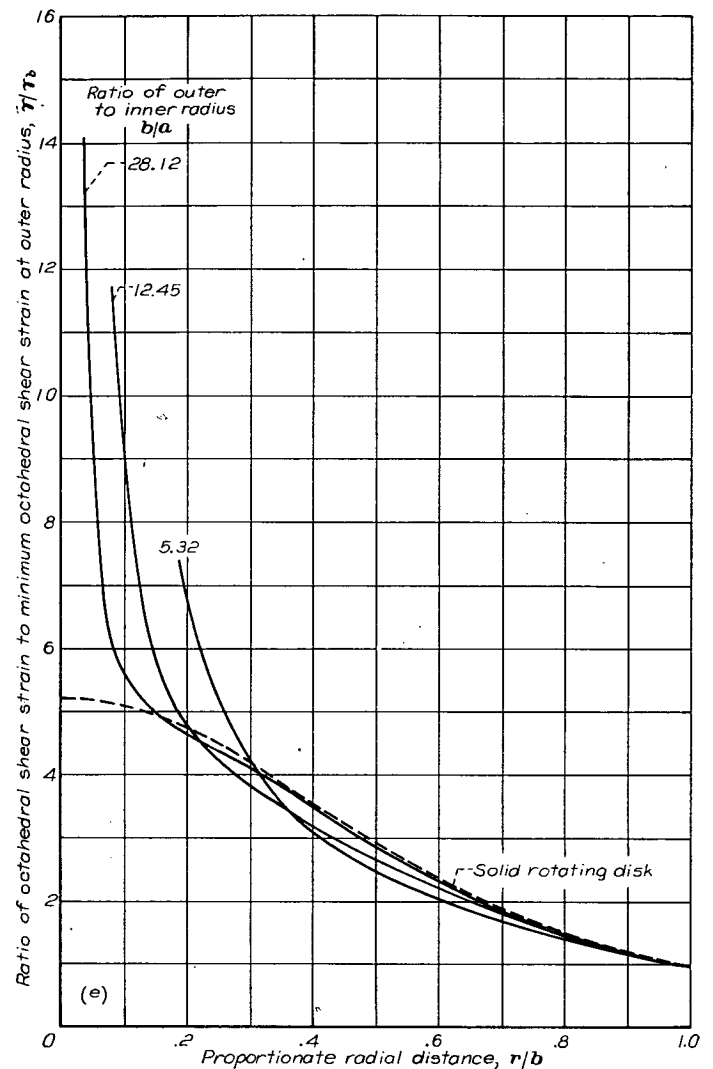
(d) Variation of principal strains with proportionate radial distance.

FIGURE 21.—Continued. Rotating disk with hole. Inconel X; γ_0 , 0.3000.

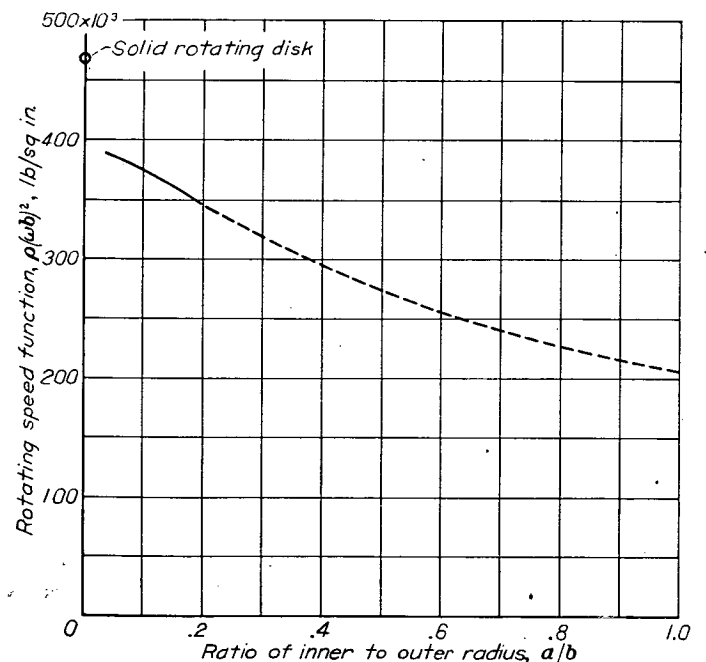
for a particular member under a particular loading condition, because consideration of the maximum octahedral shear strain only (or ductility only) of the material is insufficient.

The variations of α , γ , σ_r , σ_θ , ϵ_r , ϵ_θ , and γ/γ_b with radius for three rotating disks with a hole are shown in figure 21. The values of the ratios of outer to inner radius b/a of these three disks equal 5.32, 12.45, and 28.12. These disks were made of Inconel X and had a maximum octahedral shear strain γ_0 of 0.3 at the inner radius of the disk. The tangential stress σ_θ , the tangential strain ϵ_θ , and the octahedral shear strain γ are much less uniform for the disk with a hole than for a solid rotating disk. The ratio of maximum to minimum octahedral shear strain γ_0/γ_b is equal to 7.41 for a disk with $b/a=5.32$, 11.75 for a disk with $b/a=12.45$, and 14.1 for a disk with $b/a=28.12$; for a solid disk of the same material, the ratio γ_0/γ_b is about 5.3.

The load, rotating-speed function $\rho(\omega b)^2$, for disks of Inconel X reaching a maximum octahedral shear strain γ_0 of 0.3 at the inner radius of the disk and having different ratios of inner to outer radius a/b is represented by the solid curve



(e) Variation of ratio of octahedral shear strain to minimum octahedral shear strain at outer radius with proportionate radial distance.

FIGURE 21.—Concluded. Rotating disk with hole. Inconel X; γ_0 , 0.3000.FIGURE 22.—Variation of load (function of speed) with ratio of inner to outer radius of rotating disk with hole. Inconel X; $\gamma_0=0.3000$.

in figure 22. The dashed curve in figure 22 is obtained by extending this solid curve toward $a/b=1$, where the value of $\rho(\omega b)^2$ can be determined by considering a rotating ring with $a/b \rightarrow 1$. The figure indicates approximately how the load $\rho(\omega b)^2$ varies with disks having different ratios of inner to outer radius and reaching the same maximum octahedral shear strain at the inner radius of the disk. The value of $\rho(\omega b)^2$ for a solid rotating disk made of Inconel X with $\gamma_0=0.3$ at the center of the disk is indicated in the same figure.

CONCLUSIONS

The results obtained for a membrane, a rotating disk without and with a hole, and an infinite plate with a hole strained in the strain-hardening range in which the elastic strains are negligible compared with the plastic strains for Inconel X and 16-25-6 in the absence of time and temperature effects and unloading show that:

(1) The method developed not only accurately solves the plane-plastic-stress problems with axial symmetry in a simple manner but also shows clearly the octahedral shear strain distribution and the ratio of principal stresses during loading.

(2) The ratio of the principal stresses in the cases investigated remained essentially constant during loading and, consequently, the deformation theory is applicable to this group of problems for the materials considered.

(3) The distributions of principal strains and octahedral shear strains in the plastic state are less uniform than those in the elastic state, although the distributions of tangential stresses appear more uniform in the plastic state. The stress concentration factor around a hole is reduced in the plastic state, but instead there is a high concentration of principal strain and octahedral shear strain.

(4) The ratios of the strains along the radius to their maximum value are essentially independent of the value of the maximum octahedral shear strain of the member and the octahedral shear stress-strain curve of the material. Hence, the deformation that can be sustained by the member before failure depends mainly on the maximum octahedral shear strain (or ductility) of the material.

(5) The stress distributions depend on the octahedral shear stress-strain curve of the material. Hence, the added load that the member can sustain between the onset of yielding and failure depends mainly upon the octahedral shear (or effective) stress-strain curve in the strain-hardening range of the material.

LEWIS FLIGHT PROPULSION LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

CLEVELAND, OHIO, February 28, 1950

APPENDIX A

EQUATIONS OF EQUILIBRIUM AND COMPATIBILITY FOR INFINITESIMAL STRAIN IN TERMS OF α AND γ

The final forms of the equilibrium and compatibility equations for small strains are given in this section. The concept of infinitesimal strain is defined as follows: The changes of dimensions are small compared with the original dimensions but are large enough so that the elastic strain can be neglected. The equations presented can be obtained either by direct derivation as was done previously or by reducing the equations for finite strains through expanding the $e^{f(\alpha, \gamma)}$ terms in series and neglecting the small terms. For infinitesimal strain, the coefficients (functions of α and γ) A , B , C , D , E , and F of equations (25) are each denoted by a superscript prime but the coefficients (functions of α and γ) are simpler than those for large strain.

$$A' \frac{r}{k} \frac{d\alpha}{d\left(\frac{r}{k}\right)} + B' \frac{r}{k} \frac{d\gamma}{d\left(\frac{r}{k}\right)} = C' \quad (25a)$$

$$D' \frac{r}{k} \frac{d\alpha}{d\left(\frac{r}{k}\right)} + E' \frac{r}{k} \frac{d\gamma}{d\left(\frac{r}{k}\right)} = F'$$

For the circular membrane under pressure,

$$\left. \begin{aligned} A' &= \sqrt{3} \cos \alpha + \sin \alpha \\ B' &= (\sqrt{3} \sin \alpha - \cos \alpha) \frac{1}{\tau} \frac{d\tau}{d\gamma} \\ C' &= 2 \cos \alpha \\ D' &= (\sqrt{3} \sin \alpha - \cos \alpha) \gamma \\ E' &= -(\sqrt{3} \cos \alpha + \sin \alpha) \\ F' &= 2\sqrt{3} \gamma \cos \alpha + \frac{\sqrt{2}}{6} \left[\frac{\frac{pk}{h_{ini}} \frac{r}{k}}{\tau(\sqrt{3} \sin \alpha - \cos \alpha)} \right]^2 \end{aligned} \right\} \quad (36)$$

For the rotating disk,

$$\left. \begin{aligned} A' &= \sqrt{3} \cos \alpha + \sin \alpha \\ B' &= (\sqrt{3} \sin \alpha - \cos \alpha) \frac{1}{\tau} \frac{d\tau}{d\gamma} \\ C' &= 2 \cos \alpha - \sqrt{\frac{2}{3}} \rho(\omega k)^2 \left(\frac{r}{k}\right)^2 \frac{1}{\tau} \\ D' &= (\sqrt{3} \sin \alpha - \cos \alpha) \gamma \\ E' &= -(\sqrt{3} \cos \alpha + \sin \alpha) \\ F' &= 2\sqrt{3} (\cos \alpha) \gamma \end{aligned} \right\} \quad (37)$$

For the infinite plate with a circular hole,

$$\left. \begin{aligned} A' &= \sqrt{3} \cos \alpha + \sin \alpha \\ B' &= (\sqrt{3} \sin \alpha - \cos \alpha) \frac{1}{\tau} \frac{d\tau}{d\gamma} \\ C' &= 2 \cos \alpha \\ D' &= (\sqrt{3} \sin \alpha - \cos \alpha) \\ E' &= -(\sqrt{3} \cos \alpha + \sin \alpha) \\ F' &= 2\sqrt{3} (\cos \alpha) \gamma \end{aligned} \right\} \quad (38)$$

For small strains, the coefficients A' , B' , C' , D' , E' , and F' are used in equation (29) instead of A , B , C , D , E , and F ; respectively.

APPENDIX B

EQUATIONS FOR ROTATING DISK AND INFINITE PLATE
WITH CIRCULAR HOLE IN ELASTIC RANGE

ROTATING DISK

For a solid rotating disk with the radial stress at the periphery ($r=b$) equal to zero, the principal stresses can be expressed by the following equations (reference 23, p. 68):

$$\left. \begin{aligned} \sigma_r &= \frac{3+\nu}{8} \rho \omega^2 (b^2 - r^2) \\ \sigma_\theta &= \frac{3+\nu}{8} \rho \omega^2 b^2 - \frac{1+3\nu}{8} \rho \omega^2 r^2 \end{aligned} \right\} \quad (39)$$

where ν is Poisson's ratio. At $r=b$,

$$(\sigma_\theta)_b = \frac{1}{4} \rho \omega^2 b^2 (1-\nu)$$

Dividing equation (39) by $(\sigma_\theta)_b$ yields

$$\left. \begin{aligned} \frac{\sigma_r}{(\sigma_\theta)_b} &= \frac{3+\nu}{2(1-\nu)} \left[1 - \left(\frac{r}{b} \right)^2 \right] \\ \frac{\sigma_\theta}{(\sigma_\theta)_b} &= \frac{3+\nu}{2(1-\nu)} \left[1 - \frac{1+3\nu}{3+\nu} \left(\frac{r}{b} \right)^2 \right] \end{aligned} \right\} \quad (39a)$$

The stress-strain relations of plane-stress problems in the elastic range are:

$$\left. \begin{aligned} \epsilon_r &= \frac{1}{E} (\sigma_r - \nu \sigma_\theta) \\ \epsilon_\theta &= \frac{1}{E} (\sigma_\theta - \nu \sigma_r) \end{aligned} \right\} \quad (40)$$

where E is the modulus of elasticity in tension and compression.

Substituting equations (39) into equations (40) yields:

$$\left. \begin{aligned} \epsilon_r &= \frac{1}{8E} (1-\nu)(3+\nu)(\rho \omega^2 b^2) \left[1 - \frac{3(1+\nu)}{3+\nu} \left(\frac{r}{b} \right)^2 \right] \\ \epsilon_\theta &= \frac{1}{8E} (1-\nu)(3+\nu)(\rho \omega^2 b^2) \left[1 - \frac{1+\nu}{3+\nu} \left(\frac{r}{b} \right)^2 \right] \end{aligned} \right\} \quad (40a)$$

or

$$\left. \begin{aligned} \frac{\epsilon_r}{(\epsilon_\theta)_b} &= \frac{3+\nu}{2} \left[1 - \frac{3(1+\nu)}{3+\nu} \left(\frac{r}{b} \right)^2 \right] \\ \frac{\epsilon_\theta}{(\epsilon_\theta)_b} &= \frac{3+\nu}{2} \left[1 - \frac{1+\nu}{3+\nu} \left(\frac{r}{b} \right)^2 \right] \end{aligned} \right\} \quad (40b)$$

The equations for the octahedral shear stress and strain given by equations (4a), (4b), and (5a) can be applied to both the elastic and the plastic ranges, but equation (5b) can be applied only in the plastic range. The octahedral shear strain in the elastic range can be calculated by equation (4b) or by using the following equation:

$$\gamma = \frac{2(1+\nu)}{E} \tau = \frac{2(1+\nu)}{E} \frac{\sqrt{2}}{3} (\sigma_r^2 - \sigma_r \sigma_\theta + \sigma_\theta^2)^{1/2} \quad (41)$$

Substitute equations (39) in equation (41) to obtain:

$$\gamma = \frac{\sqrt{2}}{12E} \frac{(1+\nu)}{(3+\nu)} (\rho \omega^2 b^2) \left[(3+\nu)^2 - 4(1+\nu)(3+\nu) \left(\frac{r}{b} \right)^2 + (7+2\nu+7\nu^2) \left(\frac{r}{b} \right)^4 \right]^{1/2} \quad (41a)$$

or

$$\frac{\gamma}{\gamma_b} = \frac{1}{2(1-\nu)} \left[(3+\nu)^2 - 4(1+\nu)(3+\nu) \left(\frac{r}{b} \right)^2 + (7+2\nu+7\nu^2) \left(\frac{r}{b} \right)^4 \right]^{1/2} \quad (41b)$$

The value of Poisson's ratio ν for the two materials are:

$$\nu = 0.29 \text{ for Inconel X (reference 24)}$$

$$\nu = 0.286 \text{ for 16-25-6 (reference 25)}$$

INFINITE PLATE WITH CIRCULAR HOLE

For a uniformly loaded infinite plate with a circular hole, the principal stresses are (reference 23, p. 56):

$$\left. \begin{aligned} \sigma_r &= \frac{A}{r^2} + 2C \\ \sigma_\theta &= -\frac{A}{r^2} + 2C \end{aligned} \right\} \quad (42)$$

where A and C are arbitrary constants. For the plate considered herein, the boundary conditions are:

$$\begin{aligned} \sigma_r &= 0 & \text{at } r=a \\ \sigma_r &= (\sigma_r)_b & \text{at } r=b \end{aligned}$$

These boundary conditions are used to determine the arbitrary constants A and C , which yield

$$\left. \begin{aligned} \sigma_r &= \frac{(\sigma_r)_b}{1 - \left(\frac{a}{b} \right)^2} \frac{\left(\frac{r}{a} \right)^2 - 1}{\left(\frac{r}{a} \right)^2} \\ \sigma_\theta &= \frac{(\sigma_r)_b}{1 - \left(\frac{a}{b} \right)^2} \frac{\left(\frac{r}{a} \right)^2 + 1}{\left(\frac{r}{a} \right)^2} \end{aligned} \right\} \quad (42a)$$

or

$$\left. \begin{aligned} \frac{\sigma_r}{(\sigma_\theta)_b} &= \frac{1}{1 + \left(\frac{a}{b} \right)^2} \frac{\left(\frac{r}{a} \right)^2 - 1}{\left(\frac{r}{a} \right)^2} \\ \frac{\sigma_\theta}{(\sigma_\theta)_b} &= \frac{1}{1 + \left(\frac{a}{b} \right)^2} \frac{\left(\frac{r}{a} \right)^2 + 1}{\left(\frac{r}{a} \right)^2} \end{aligned} \right\} \quad (42b)$$

Substituting equations (42a) into equations (40) yields

$$\left. \begin{aligned} \epsilon_r &= \frac{1}{E} \frac{(\sigma_r)_b}{1 - \left(\frac{a}{b}\right)^2} \frac{(1-\nu)\left(\frac{r}{a}\right)^2 - (1+\nu)}{\left(\frac{r}{a}\right)^2} \\ \epsilon_\theta &= \frac{1}{E} \frac{(\sigma_r)_b}{1 - \left(\frac{a}{b}\right)^2} \frac{(1-\nu)\left(\frac{r}{a}\right)^2 + (1+\nu)}{\left(\frac{r}{a}\right)^2} \end{aligned} \right\} \quad (43)$$

or

$$\left. \begin{aligned} \frac{\epsilon_r}{(\epsilon_\theta)_b} &= \frac{(1-\nu)\left(\frac{r}{a}\right)^2 - (1+\nu)}{\left[(1-\nu) + (1+\nu)\left(\frac{a}{b}\right)^2\right]\left(\frac{r}{a}\right)^2} \\ \frac{\epsilon_\theta}{(\epsilon_\theta)_b} &= \frac{(1-\nu)\left(\frac{r}{a}\right)^2 + (1+\nu)}{\left[(1-\nu) + (1+\nu)\left(\frac{a}{b}\right)^2\right]\left(\frac{r}{a}\right)^2} \end{aligned} \right\} \quad (43a)$$

Substituting equations (42a) into equation (41) yields

$$\gamma = \frac{2\sqrt{2}(1+\nu)}{3E} \frac{(\sigma_r)_b}{1 - \left(\frac{a}{b}\right)^2} \left[\frac{\left(\frac{r}{a}\right)^4 + 3}{\left(\frac{r}{a}\right)^4} \right]^{1/2} \quad (44)$$

or

$$\frac{\gamma}{\gamma_b} = \left\{ \frac{\left(\frac{r}{a}\right)^4 + 3}{\left[1 + 3\left(\frac{a}{b}\right)^4\right]\left(\frac{r}{a}\right)^4} \right\}^{1/2}$$

REFERENCES

1. Nadai, A.: *Plasticity*. McGraw-Hill Book Co., Inc., 1931.
2. Nadai, A., and Donnell, L. H.: Stress Distribution in Rotating Disks of Ductile Material after the Yield Point Has Been Reached. *A. S. M. E. Trans.*, vol. 51, pt. I, APM-51-16, 1929, pp. 173-180; discussion, pp. 180-181.
3. Gleyzal, A.: Plastic Deformation of a Circular Diaphragm under Pressure. *Jour. Appl. Mech.*, vol. 15, no. 3, Sept. 1948, pp. 288-296.
4. Millenson, M. B., and Manson, S. S.: Determination of Stresses in Gas-Turbine Disks Subjected to Plastic Flow and Creep. NACA Rep. 906, 1948. (Formerly NACA TN 1636.)
5. MacGregor, C. W., and Tierney, W. D.: Developments in High-Speed Rotating Disk Research at M. I. T. *Welding Jour. Suppl.*, vol. 27, no. 6, June 1948, pp. 303S-309S.
6. Holms, Arthur G., and Jenkins, Joseph E.: Effect of Strength and Ductility on Burst Characteristics of Rotating Disks. NACA TN 1667, 1948.
7. Hencky, Heinrich: Zur Theorie plastischer Deformationen und der hierdurch in Material hervorgerufenen Nachspannungen. *Z. f. a. M. M.*, Bd. 4, Heft 4, Aug. 1924, S. 323-334.
8. Nadai, A.: Theories of Strength. *A. S. M. E. Trans.*, vol. 55, APM-55-15, 1933, pp. 111-129.
9. Nadai, A.: Plastic Behavior of Metals in the Strain-Hardening Range. Part I. *Jour. Appl. Phys.*, vol. 8, no. 3, March 1937, pp. 205-213.
10. Ilyushin, A. A.: Relation between the Theory of Saint Venant-Levy-Mises and the Theory of Small Elastic-Plastic Deformations. RMB-1, trans. by Appl. Math. Group, Brown Univ., for David W. Taylor Model Basin (Washington, D. C.), 1945.
11. Ilyushin, A. A.: The Theory for Small Elastic-Plastic Deformations. RMB-17, trans. by Grad. Div. Appl. Math., Brown Univ., for David W. Taylor Model Basin (Washington, D. C.), 1947. (Contract NObs-34166.)
12. Prager, W.: Strain Hardening Combined Stresses. *Jour. Appl. Phys.*, vol. 16, no. 12, Dec. 1945, pp. 837-840.
13. Drucker, D. C.: A Reconsideration of Deformation Theories of Plasticity. *A. S. M. E. Trans.*, vol. 71, no. 5, July 1939, pp. 587-592.
14. Davis, E. A.: Increase of Stress with Permanent Strain and Stress-Strain Relations in the Plastic State for Copper under Combined Stresses. *Jour. Appl. Mech.*, vol. 10, no. 4, Dec. 1943, pp. A187-A196.
15. Osgood, W. R.: Combined-Stress Tests on 24S-T Aluminum Alloy Tubes. *Jour. Appl. Mech.*, vol. 14, no. 2, June 1947, pp. A147-A153.
16. Fraenkel, S. J.: Experimental Studies of Biaxially Stressed Mild Steel in the Plastic Range. *Jour. Appl. Mech.*, vol. 15, no. 3, Sept. 1948, pp. 193-200.
17. Davis, H. E., and Parker, E. R.: Behavior of Steel under Biaxial Stress as Determined by Tests on Tubes. *Jour. Appl. Mech.*, vol. 15, no. 3, Sept. 1948, pp. 201-215.
18. Lankford, W. T., Low, J. R., and Gensamer, M.: The Plastic Flow of Aluminum Alloy Sheet under Combined Loads. *Trans. A. I. M. E., Inst. Metals Div.*, vol. 171, 1947, pp. 574-604.
19. Bickley, W. G.: Formulae for Numerical Integration. *The Math. Gazette*, vol. XXIII, no. 256, Oct. 1939, pp. 352-359.
20. Milne, William Edmund: *Numerical Calculus*. Princeton Univ. Press, 1949.
21. Griffith, George E.: Experimental Investigation of the Effects of Plastic Flow in a Tension Panel with a Circular Hole. NACA TN 1705, 1948.
22. Stowell, Elbridge Z.: Stress and Strain Concentration at a Circular Hole in an Infinite Plate. NACA TN 2073, 1950.
23. Timoshenko, S.: *Theory of Elasticity*. McGraw-Hill Book Co., Inc., 1934.
24. Anon.: *Nickel and Nickel Alloys*. The International Nickel Co., Inc. (New York), 1947.
25. Fleischmann, Martin: 16-25-6 Alloy for Gas Turbines. *Iron Age*, vol. 157, no. 3, Jan. 17, 1946, pp. 44-53; cont., vol. 157, no. 4, Jan. 24, 1946, pp. 50-60.

REPORT 1022

TEMPERATURE DISTRIBUTION IN INTERNALLY HEATED WALLS OF HEAT EXCHANGERS COMPOSED OF NONCIRCULAR FLOW PASSAGES¹

By E. R. G. ECKERT and GEORGE M. LOW

SUMMARY

In the walls of heat exchangers composed of noncircular passages, the temperature varies in the circumferential direction because of local variations of the heat-transfer coefficients. A prediction of the magnitude of this variation is necessary in order to determine the region of highest temperature and in order to determine the admissible operating temperatures.

A method for the determination of these temperature distributions and of the heat-transfer characteristics of a special type of heat exchanger is developed. The heat exchanger is composed of polygonal flow passages and the passage walls are uniformly heated by internal heat sources. The coolant flow within the passages is assumed to be turbulent. The circumferential variation of the local heat-transfer coefficients is estimated from flow measurements made by Nikuradse, postulating similarity between velocity and temperature fields. Calculations of temperature distributions based on these heat-transfer coefficients are carried out and results for heat exchangers with triangular and rectangular passages are presented.

INTRODUCTION

The conventional recuperative type of heat exchanger consists of passages for two liquids or gases separated by a heating surface. Heat from an outside source is carried with a fluid flowing through one of the passages and is transferred in the heat exchanger to a second fluid flowing through the other passage. Very often such a heat exchanger is composed of a large number of tubes, with the two liquids flowing inside and over the outside of the tubes, respectively.

The regenerative type of heat exchanger has passages for one fluid only. During the heating period, heat from an outside source is carried to the heat exchanger by a hot fluid and is stored within the solid walls of the passages. This heat is then given off to a cold fluid, which passes through the heat exchanger during the cooling period.

In this report, a heat exchanger is considered that differs from the regenerative type only by the fact that the heat is generated by heat sources within the passage walls and is transferred to a coolant flowing continuously through the passages. The passage walls of this heat exchanger are assumed to be flat plates assembled to form a honeycomb; thus the flow passages formed in this manner have a polygonal cross section. A cross-sectional view of a typical heat

exchanger of this type is shown in figure 1. The exchanger is composed of a number of plates *a*, which form the coolant passages *b*. The flow of the coolant is normal to the cross section shown. High temperatures may be anticipated near the corners *c* of the passages, inasmuch as the rate of heat transfer there is expected to be poor. A theoretical investigation of the temperature distribution in such a heat exchanger was made at the NACA Lewis laboratory during 1950 and is presented herein.

The basis for this investigation is a knowledge of local heat-transfer coefficients in passages of noncircular cross section. Some information is available on the average heat-transfer coefficients in such tubes (references 1 and 2). The result of these investigations is essentially that the expressions derived for the heat-transfer coefficients in circular tubes apply for other cross sections as well, provided the diameter is replaced by the hydraulic diameter. (The hydraulic diameter is defined as four times the cross-sectional flow area divided by the circumference of the passage.)

A knowledge of local heat-transfer coefficients in noncircular passages is important not only in the present problem, but also for several other engineering problems, such as the determination of local temperatures in the walls of air-cooled turbine blades. No information on local heat-transfer coefficients was found in the available literature, however. These values are therefore estimated from flow measurements made by Nikuradse (reference 3) on the basis of the similarity between temperature and velocity fields.

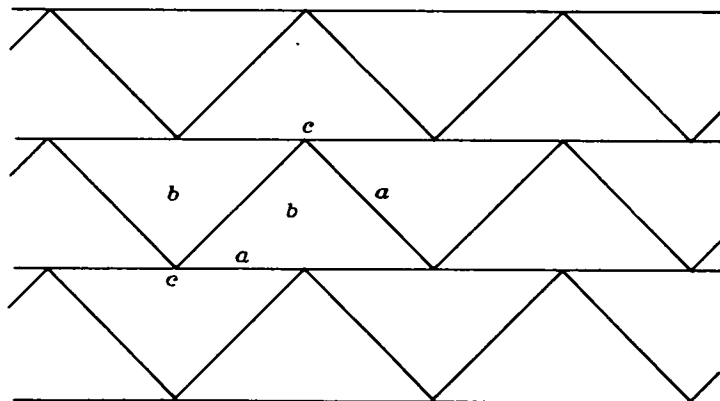


FIGURE 1.—Cross section through typical heat exchanger composed of noncircular flow passages.

¹ Supersedes NACA TN 2257, "Temperature Distribution in Internally Heated Walls of Heat Exchangers Composed of Noncircular Flow Passages" by E. R. G. Eckert and George M. Low, 1951.

SYMBOLS

The following symbols are used in this report:

A	cross-sectional flow area (sq ft)
A^*	ratio of wall area to flow area (dimensionless)
C	internal circumference of passage (ft)
C^*	C/D (dimensionless)
c_p	specific heat at constant pressure, Btu/(lb) (°F)
D	hydraulic diameter, $4A/C$, (ft)
g	acceleration due to gravity (ft)/(sec ²)
h	local heat-transfer coefficient, Btu/(sec) (sq ft) (°F)
$\bar{h}$	average heat-transfer coefficient, $\frac{1}{C} \int_0^C h \, dx$, Btu/(sec) (sq ft) (°F)
h^*	$\bar{h}/h$ (dimensionless)
k	thermal conductivity of wall material, Btu/(sec) (ft) (°F)
k_g	thermal conductivity of coolant, Btu/(sec) (ft) (°F)
k^*	k/k_g (dimensionless)
N	residual value (dimensionless)
Nu	Nusselt number, $\bar{h}D/k_g$, (dimensionless)
n	coordinate normal to passage wall (ft)
n^*	n/D (dimensionless)
Pr	Prandtl number, ν/α , (dimensionless)
p	pressure (lb)/(sq ft)
q	local rate of heat transfer, $h\theta$, Btu/(sec) (sq ft)
q^*	$h^* \theta^*$ (dimensionless)
R	radial coordinate (ft)
Re	Reynolds number, uD/ν , (dimensionless)
r	rate of internal heat generation, Btu/(sec) (cu ft)
s	wall thickness (ft)
s^*	s/D (dimensionless)
T	local total temperature of coolant, °F
T_B	bulk total temperature of coolant, °F
T^*	Tk/rD^2 (dimensionless)
t	local wall temperature, °F
t^*	tk/rD^2 (dimensionless)
u, v	velocity components in x - and y -direction, respectively (ft/sec)
x, y, z	Cartesian coordinates
x^*	x/D (dimensionless)
y^*	y/D (dimensionless)
z^*	z/D (dimensionless)
α	thermal diffusivity, $k_g/\rho g c_p$, (sq ft)/(sec)
β	angle subtended by two adjacent sides of polygonal passage (deg)
Δ	increment of length (ft)
Δ^*	Δ/D (dimensionless)
δ	increment or difference
ϵ_H	turbulent diffusivity of heat (sq ft)/(sec)
ϵ_M	turbulent diffusivity of momentum (sq ft)/(sec)
θ	temperature difference, $t - T_B$, °F
θ^*	$\theta k/rD^2$ (dimensionless)
ν	kinematic viscosity (sq ft)/(sec)
ρ	mass density (lb) (sec ²)/(ft ⁴)
τ_w	local wall shear stress (lb)/(sq ft)

Subscripts:

c	conditions for equivalent circular tube
m	conditions at center of flow passage
n	conditions normal to passage wall
s	conditions at wall surface

Superscript:

$*$	dimensionless quantity
-----	------------------------

ASSUMPTIONS

If this analysis were to be made without any simplifying assumptions, the simultaneous solution of the equations of motion of the coolant and of the heat flow in the coolant and in the passage walls would be required. The following assumptions are made in order to make these equations amenable to solution without seriously curtailing the results of the analysis:

(1) The Prandtl number of the coolant used in the heat exchanger is in the neighborhood of 1. This condition is well fulfilled by gases and by water above a temperature of 200° F, excluding the neighborhood of the critical point.

(2) The passages of the heat exchanger are long enough so that in the cross section investigated the flow is fully developed, which means that the velocity profile does not change its shape in the direction of the tube axis.

(3) The rate of heat generation in the walls of the heat exchanger is uniform. As a consequence of this condition, the temperature within the coolant and the walls increases linearly in a downstream direction provided the flow is thermally developed. For a fluid with a Prandtl number of 1, the points of thermal and velocity development in a tube practically coincide provided that the heating of the tube starts at the entrance section.

(4) The temperature gradient along the tube axis is assumed small as compared with the gradients in any cross section of the passage.

(5) The thermal conductivity of the solid material is large as compared with the thermal conductivity of the coolant. This condition is always fulfilled for metal walls regardless of the type of coolant used provided that assumption (1) applies and for nonmetallic walls if the coolant is a gas. For the case of nonmetallic walls and liquid coolants, the applicability of the calculations presented in this report must be checked in each individual case. Furthermore, the temperature differences within the passage wall at any one cross section are postulated to be small as compared with the temperature differences between the wall and the core of the coolant. As a consequence of the assumption listed in this paragraph, the heat transport within the coolant, normal to the tube axis and parallel to the walls, is small as compared with the heat conduction within the walls and may therefore be neglected. The heat transport within the coolant, normal to the tube axis and normal to the walls, is of course taken into account.

(6) The turbulent diffusivity of momentum ϵ_M and that of heat ϵ_H are equal.

FLOW IN TUBES WITH NONCIRCULAR CROSS SECTION

A thorough investigation of the flow through tubes with noncircular cross sections was made by Nikuradse (reference 3). This investigation is used in the present report as the basis for estimating local heat-transfer coefficients.

The flow of water through tubes of several different shapes, as shown in figure 2, was investigated in reference 3. Three of the tubes had triangular cross sections; an equilateral triangle, an isosceles right triangle, and a right triangle with the sides enclosing the 90° angle having a ratio of 1 to 2.32. One tube had a trapezoidal cross section and two tubes were circular with one and two grooves, respectively. In a second report, Nikuradse investigated tubes with a rectangular cross section; however, only a summary of this report is available (reference 4). The hydraulic diameter of the passages varied from 0.3 to 0.6 inch and the length-to-hydraulic-diameter ratio varied from 100 to 200. The Reynolds number based on the hydraulic diameter and the mean velocity ranged from 77,000 to 120,000. Because it is known that the shape of a turbulent velocity profile changes only slightly with Reynolds number, the results of the calculations should be applicable for a fairly large range of Reynolds number in the turbulent region. The velocity profiles were measured

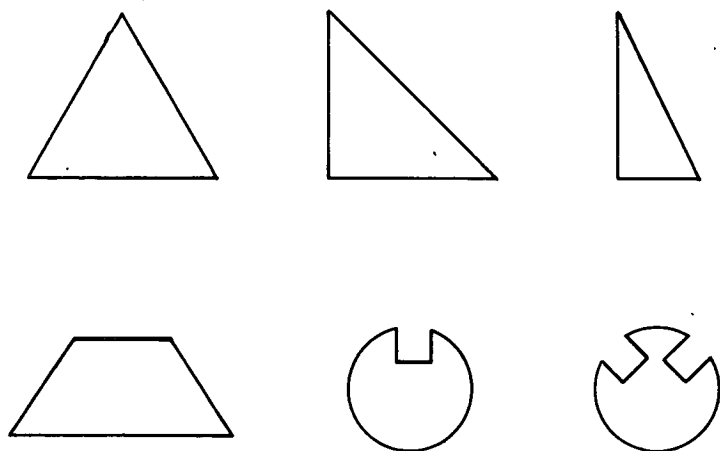


FIGURE 2.—Cross sections of passages investigated in reference 3.

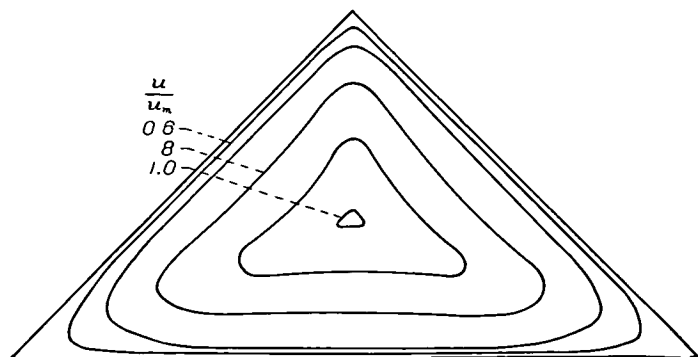


FIGURE 3.—Velocity contours in triangular passage as measured in reference 3. Reynolds number, 81,000.

with a small total-head tube in a cross section near the downstream end of the tube.

Results of reference 3 for the isosceles right triangle are shown in figures 3 to 5. Similar results for all other cross

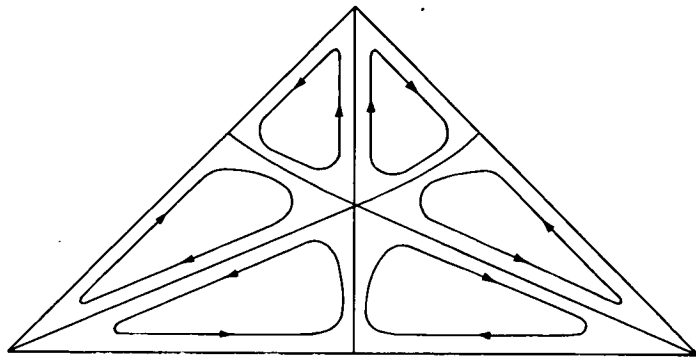


FIGURE 4.—Secondary flow in triangular passage deduced from measurements of reference 3.

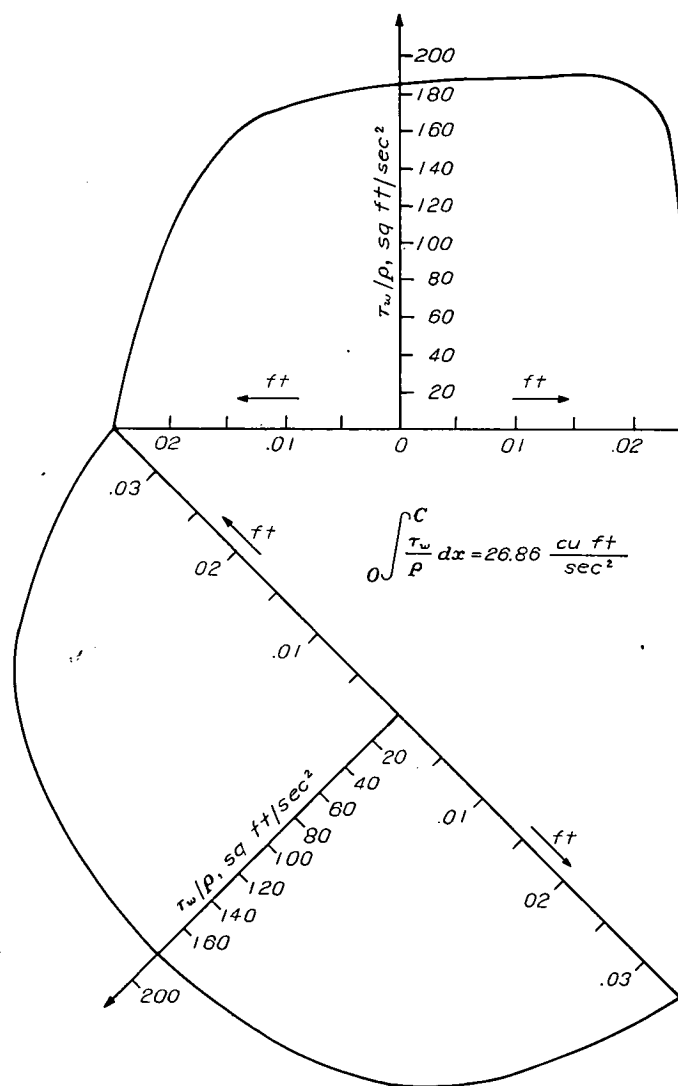


FIGURE 5.—Shear stress distribution on circumference of triangular passage as determined in reference 3.

sections can be found in the original report. In figure 3 the lines of constant velocity are presented as contour lines. These contours are indented near the center of each side of the passage. The conclusion of reference 3 is that these indentations indicate a secondary flow normal to the main flow direction. A qualitative sketch of this secondary flow is presented in figure 4. Prandtl (reference 5) has ascribed the secondary flow to a turbulent mixing motion within the fluid, which is more intense in the direction parallel to the wall than normal to it. This secondary flow tends to equalize the velocities and temperatures within any cross section of a noncircular tube and is therefore favorable for the similarity consideration that will be used to deduce the temperature field from the measured velocity field. On the other hand, because no quantitative knowledge of the secondary flow exists, an exact theoretical calculation of the temperature field in noncircular passages is impossible. The local wall shear stresses were computed by Nikuradse, using the measured velocity profiles together with the assumption that the Blasius pipe resistance law of the turbulent velocity profile for circular tubes applies also for noncircular passages on normals to the walls (fig. 5). A check by Nikuradse of the calculated average shear stresses against the measured pressure drop showed good agreement.

SIMILARITY BETWEEN VELOCITY AND TEMPERATURE PROFILES FOR $Pr=1$

It is well known that the velocity and temperature profiles in the boundary layer of a fluid with a Prandtl number of 1 are similar in shape in the absence of a pressure gradient. This similarity is immediately apparent from a comparison of the momentum and energy equations of the boundary layer for two-dimensional steady flow along a flat plate (reference 6):

$$\rho \left(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = \frac{\partial}{\partial y} \left[\rho (\nu + \epsilon_M) \frac{\partial u}{\partial y} \right] \quad (1)$$

$$\rho \left(u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = \frac{\partial}{\partial y} \left[\rho (\nu + \epsilon_H) \frac{\partial T}{\partial y} \right] \quad (2)$$

where T represents the total temperature, the Prandtl number is equal to 1, and the specific heat is constant. The effect of internal friction on the temperature profile is taken into account by basing equation (2) on the total temperature.

Because it is assumed that the turbulent diffusivity of momentum ϵ_M and of heat ϵ_H are equal, equations (1) and (2) are similar. The solutions of the equations, namely, the velocity and temperature profiles, are therefore also similar, provided the boundary conditions are similar. The boundary conditions for the velocity field on a flat plate are that the velocity is 0 along the wall and has a constant value outside the boundary layer. Similar boundary conditions for the temperature field are that the temperature is constant both along the wall and in the main stream.

It is therefore evident that similarity between velocity and temperature profiles in the boundary layer over a flat

plate exists in an exact mathematical sense. The same is not true, however, for fully developed pipe flow. The balance of the forces and of the heat energy on a stationary annular volume element with a radius R , thickness dR , and length (in the direction of the tube axis) dz , leads to the following equations for fully developed flow in a circular tube:

$$\frac{1}{R} \frac{\partial}{\partial R} \left[R \rho (\nu + \epsilon_M) \frac{\partial u}{\partial R} \right] = \frac{dp}{dz} \quad (3)$$

$$\frac{1}{R} \frac{\partial}{\partial R} \left[R \rho (\alpha + \epsilon_H) \frac{\partial T}{\partial R} \right] = \rho u \frac{\partial T}{\partial z} \quad (4)$$

where the heat-conduction term in the z -direction is neglected in equation (4). It is apparent that equations (3) and (4) are not similar, even if the viscosity and diffusivity terms in the two equations are equal. The following type of analysis can be made, however:

The turbulent diffusivity of momentum ϵ_M can be calculated from equation (3), provided that the velocity profile $u=f(R)$ and the pressure drop dp/dz are experimentally known. If it is again assumed that ϵ_M and ϵ_H are equal, the temperature profile can then be calculated from equation (4). Such an analysis, based on constant property values and with internal friction neglected, was carried out by Latzko (reference 7). The results of this analysis were used to calculate the relation between the temperatures and the velocities for hydrodynamically and thermally developed flow of a fluid with $Pr=1$. Figure 6 represents a plot of the local temperature-difference ratio $(T-t_s)/(T_m-t_s)$ against the local velocity ratio u/u_m . It can be seen that this relation is very nearly linear. This linearity means that the velocity and temperature profiles are similar as far as practicality is con-

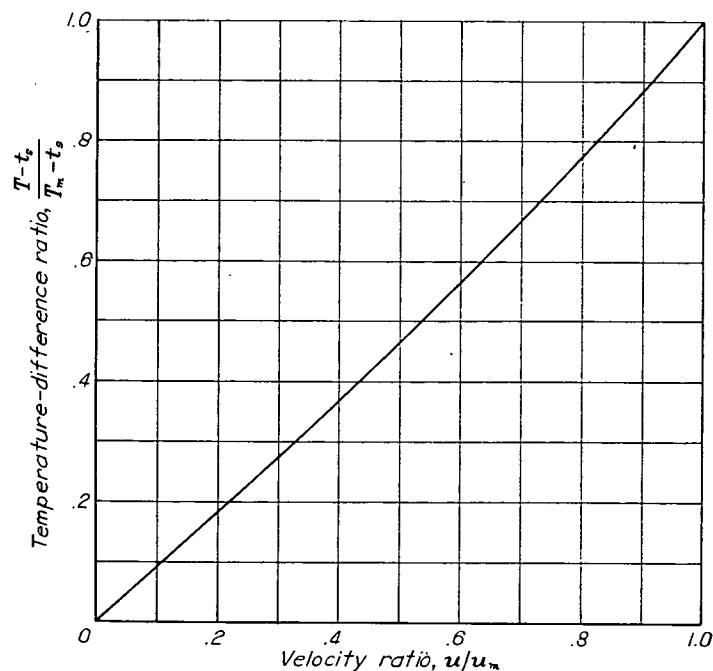


FIGURE 6.—Relation between local temperatures and local velocities in fully developed turbulent region of circular tube. Based on reference 7.

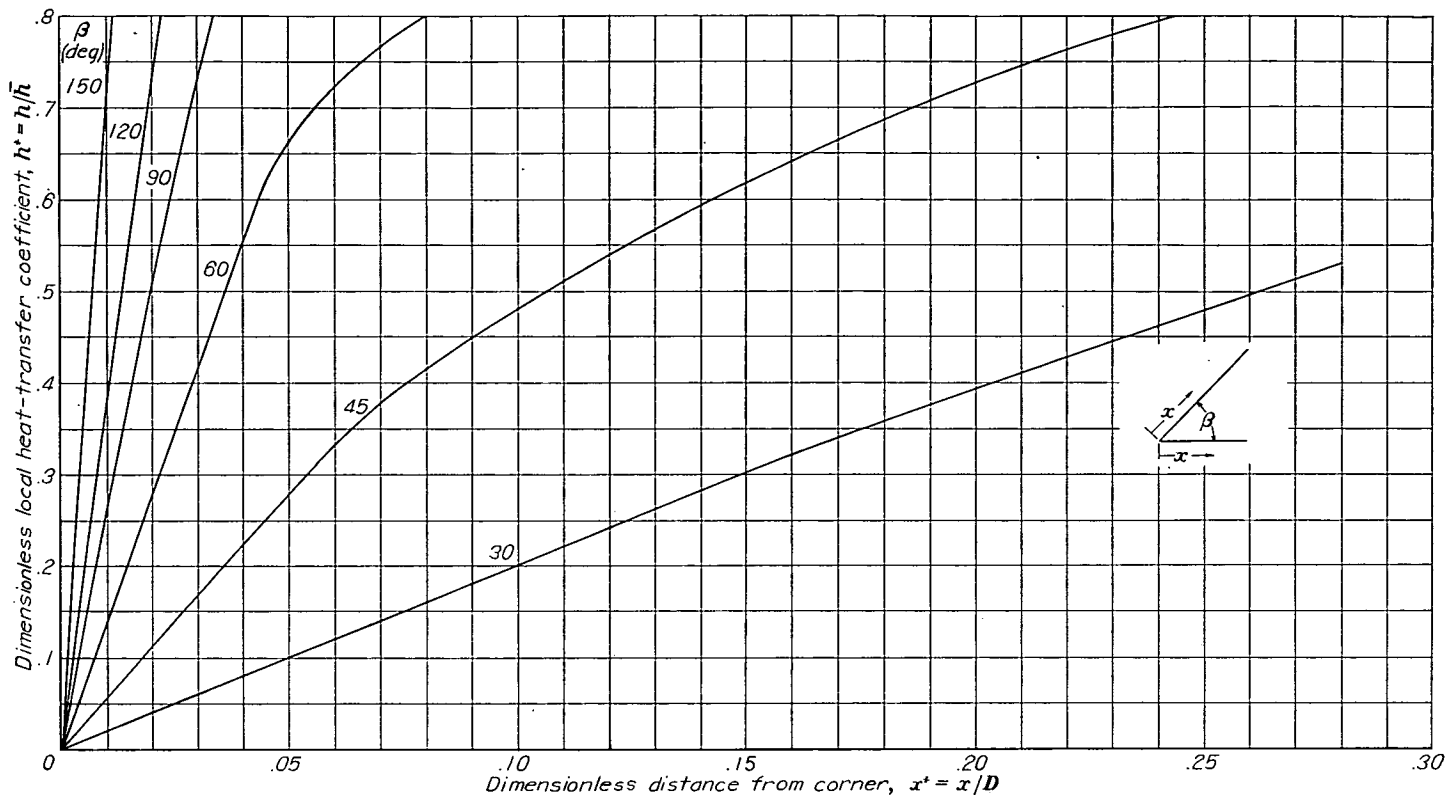


FIGURE 7. Local heat-transfer coefficients in corners of noncircular passages.

cerned, even though they are not similar in a strict mathematical sense. Most of the recent theoretical investigations of turbulent heat transfer are therefore based on the assumption of similarity between these profiles for a fluid with a Prandtl number of 1.

It is known that the fully developed turbulent velocity profile in a circular tube is such that the essential part of the velocity change from 0 at the wall to a maximum value at the center occurs in a narrow strip around the periphery of the tube. The investigations of references 3 and 4 show that this condition is also true for tubes of polygonal cross section, as long as the included angles between two adjacent passage walls are not very small. Nikuradse further shows that the law for the velocity variation on a normal to the wall established for turbulent flow in circular tubes holds also for the noncircular passages in this region of essential velocity variation. It can therefore be expected that the similarity between temperature and velocity profiles will also be fulfilled reasonably well on normals to the walls for noncircular tubes and fluids with a Prandtl number of 1, although a balance of forces and energies similar to equations (3) and (4) shows that the similarity cannot exist exactly in regions very close to the corners.

An immediate result of the similarity between velocity and temperature profiles is the fact that the local wall shear stress determined by the velocity gradient at the wall is proportional to the local heat-transfer coefficient determined by the temperature gradient at the wall. (The same result can be

obtained from Reynolds' analogy.) The proportionality of these values, together with the knowledge that the average heat-transfer coefficients for circular and noncircular tubes with the same hydraulic diameter are equal, can be used to obtain local heat-transfer coefficients in noncircular tubes from the wall-shear-stress data of references 3 and 4. A generalization of this result is possible because, for a given Reynolds number, the shear stress distribution in the vicinity of a corner of the passage depends primarily on the conditions near that corner and not on the shape of the passage.

The local heat-transfer coefficients, as presented in figure 7, were obtained by correlating the results of references 3 and 4. In this figure the ratio h^* of the local heat-transfer coefficient to the mean value is plotted against the dimensionless distance from the corner x^* . The included angle of two adjacent passage walls is the parameter for the curves. Some of these corner angles were represented on more than one of the cross sections investigated in reference 3, and for these angles the curves of h^* against x^* agree reasonably well.

The application of figure 7 can best be explained with the aid of an example. Suppose it is desired to find the distribution of h^* over the short leg of an isosceles right triangle, as shown in figure 8. The two ends of the curve can immediately be transposed from figure 7. The remainder of the curve is then extrapolated so that the area under the curve divided by length $\overline{AB}$ is equal to 1. This procedure gives a reasonably good approximation as long as the maximum value of h^* is not considerably greater than 1.

DIMENSIONLESS VARIABLES

With the distribution of the local heat-transfer coefficient around the circumference of the passages established, the only problem remaining to be solved is the calculation of the heat-conduction process within the passage walls. Before this problem is taken up, an investigation is made to determine the dimensionless moduli on which the temperature distribution in a heat exchanger of the type under consideration depends. This investigation may be useful as a basis for experimental investigations. None of the simplifying assumptions, as summarized previously, is necessary for the development in this section of the report.

The heat conduction within the solid walls of a heat exchanger with internal heat generation is described by Poisson's equation

$$k\left(\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} + \frac{\partial^2 t}{\partial z^2}\right) + r = 0 \quad (5)$$

The number of parameters for this equation can be reduced by the use of the following dimensionless values:

$$t^* = \frac{tk}{rD^2} \quad x^* = \frac{x}{D} \quad y^* = \frac{y}{D} \quad z^* = \frac{z}{D} \quad (6)$$

With these values, equation (5) can be rewritten

$$\frac{\partial^2 t^*}{\partial x^{*2}} + \frac{\partial^2 t^*}{\partial y^{*2}} + \frac{\partial^2 t^*}{\partial z^{*2}} + 1 = 0 \quad (7)$$

The temperature field that results under the influence of the internal heat generation in the solid walls depends on how the heat is transferred from the wall surfaces to the coolant. The heat must be conducted within the solid material to the surface and from there into the coolant. Along the surface, therefore, the following boundary condition exists:

$$k\left(\frac{\partial t}{\partial n}\right)_s = k_s\left(\frac{\partial T}{\partial n}\right)_s \quad (8)$$

where n represents the direction normal to the surface. In terms of dimensionless variables, this equation becomes

$$k^*\left(\frac{\partial t^*}{\partial n^*}\right)_s = \left(\frac{\partial T^*}{\partial n^*}\right)_s \quad (9)$$

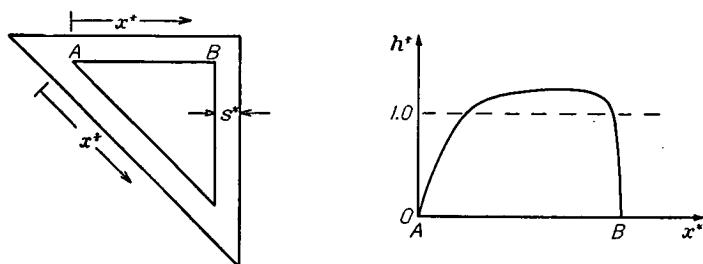


FIGURE 8.—Determination of local heat-transfer coefficients in triangular heat-exchanger passage.

The heat transfer within the coolant by conduction and convection is governed by an energy equation similar to equation (4) and the flow of the coolant is determined by a momentum equation (similar to equation (3)) and the corresponding continuity equation. Textbooks on heat transfer (for example, reference 8) show that the dimensionless temperature described by the energy equation of the coolant depends, for low-velocity flow and constant property values, on the Reynolds number Re and the Prandtl number Pr . At high velocities and variable property values, there is an additional influence of the Mach number and other dimensionless expressions characterizing the temperature dependency on the property values. By neglecting the last-mentioned influences and summarizing all the factors that influence the heat flow in the solid material and in the coolant, a functional relation of the following type can be deduced:

$$\theta^* = f(Re, Pr, k^*, x^*, y^*, z^*) \quad (10)$$

where the temperature difference θ^* is introduced because only temperature differentials appear in the equations. The function f as expressed by this equation depends on the geometric configuration of the heat-exchanger passages.

When it can be assumed that the heat-transfer process from the walls to the coolant is not influenced by the temperature distribution within the wall, the number of factors influencing the problem can be considerably reduced. This assumption is made throughout the calculations in this report and seems reasonable as long as previously mentioned assumption (5) holds. In this case equation (8) can be replaced by

$$k\left(\frac{\partial t}{\partial n}\right)_s = -h(t_s - T_B) \quad (11)$$

where h is a known function of the flow parameters. By introducing the ratio h^* of the local heat-transfer coefficient to the value averaged over the circumference of the passage and the Nusselt number $Nu = \bar{h}D/k_g$ based on this average value, and by changing to the temperature difference θ , equation (11) can be transformed to

$$\left(\frac{\partial \theta^*}{\partial n^*}\right)_s = -Nu \frac{h^*}{k^*} \theta_s^* \quad (12)$$

The dimensionless temperature within the solid walls of the heat exchanger is determined by equations (7) and (12) and can be presented as a function of the following kind:

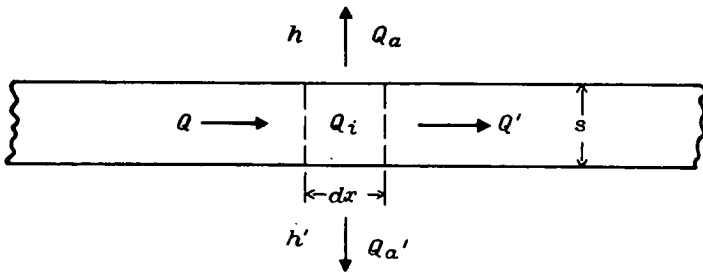
$$\theta^* = f\left(Nu \frac{h^*}{k^*}, x^*, y^*, z^*\right) \quad (13)$$

This dimensionless temperature difference depends only on the dimensionless local coordinates and on the parameter $Nu \frac{h^*}{k^*}$. Temperature distributions and heat-transfer characteristics for any given geometric configuration are therefore presented as a one-parameter family of curves.

CALCULATION OF TEMPERATURE DISTRIBUTIONS WITHIN PASSAGE WALLS

The problem of calculating temperature distributions within the passage walls can be classified in two general categories, depending on the relative thickness of these walls. If the dimensionless wall thickness s^* (see fig. 8) is small as compared with the length of a wall $\overline{AB}$, the temperature differences in the direction normal to the wall surface are small as compared with the temperature difference in the direction x^* parallel to the surface. Only the temperature differences in this parallel direction need therefore be considered. Hereinafter this special case of the problem is referred to as the "one-dimensional problem." On the other hand, if s^* becomes large, the heat flow in both the x^* and s^* directions must be considered. This second and more general problem is referred to as the "two-dimensional problem."

One-dimensional problem.—Consider a thin plate of thickness s that is a section of the heat-exchanger passage walls. Heat is generated uniformly throughout the plate at a rate r . The local plate (or wall) temperature t is assumed to be varying in the x -direction only. The bulk temperature of the coolant is T_B and the thermal conductivities of the wall and coolant are k and k_g , respectively. The local coefficients of heat transfer to the coolant above and below the plate are denoted by h and h' , as shown in the following sketch:



The heat balance for an element of volume with the dimensions dx and s and of unit depth is

$$Q + Q_i = Q' + Q_a + Q_a' \quad (14)$$

where

$$Q = -ks \frac{dt}{dx}$$

$$Q' = -ks \left(\frac{dt}{dx} + \frac{d^2t}{dx^2} dx \right)$$

$$Q_i = rs dx$$

$$Q_a + Q_a' = (h + h')(t - T_B) dx$$

With the preceding values of Q , equation (14) becomes

$$\frac{d^2t}{dx^2} - \frac{h + h'}{ks} (t - T_B) + \frac{r}{k} = 0 \quad (15)$$

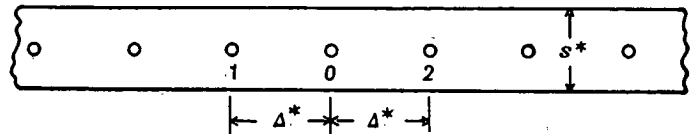
Equation (15) can be expressed in terms of dimensionless variables as

$$\frac{d^2\theta^*}{dx^{*2}} - \frac{h^* + h'^*}{s^*k^*} Nu \theta^* + 1 = 0 \quad (16)$$

This equation expresses the dimensionless wall temperature θ^* as a function of the dimensionless coordinate x^* and a single parameter $(h^* + h'^*)Nu/s^*k^*$. For any given geometric configuration (which determines h^* and h'^* as functions of x^*), the dimensionless wall temperature is a function only of the average Nusselt number Nu , the dimensionless wall thickness s^* , and the ratio of the thermal conductivities k^* .

In general, equation (16) must be solved by numerical means inasmuch as h^* and h'^* are experimentally determined functions of x^* . Several numerical methods of solution can be applied and two of these are considered here. The relaxation method (see, for example, reference 2, pp. 365-379, or reference 9) has the advantage that it is easy to apply and that computational errors are immediately apparent. A second method of solution, based on the Runge-Kutta method (reference 10) is presented in the appendix. This method is preferable when results of high accuracy are desired. It is less convenient, however, than the relaxation method because computational errors are much more difficult to detect.

For the purpose of solving equation (16) by the relaxation method, the equation is first expressed in finite-difference form. Consider a grid, or net of points, placed into the wall, any two adjacent points being separated by a small but finite distance Δ^* , as indicated in the following sketch:



The second derivative of θ^* at an arbitrary point 0 can be expressed in terms of the temperatures θ^* at this point and the two adjacent points as follows:

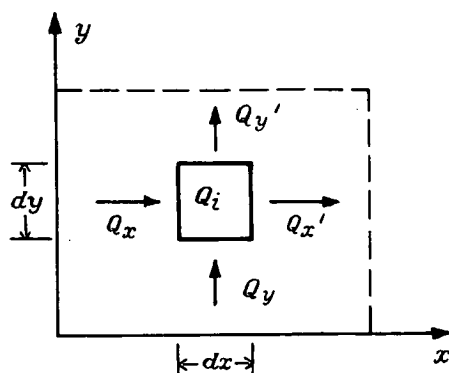
$$\begin{aligned} \left(\frac{d\theta^*}{dx^*} \right)_0 &\approx \frac{\theta_1^* - \theta_0^*}{\Delta^*} & \left(\frac{d\theta^*}{dx^*} \right)_2 &\approx \frac{\theta_0^* - \theta_2^*}{\Delta^*} \\ \frac{d^2\theta^*}{dx^{*2}} &\approx \frac{1}{\Delta^{*2}} \left[\left(\frac{d\theta^*}{dx^*} \right)_0 - \left(\frac{d\theta^*}{dx^*} \right)_2 \right] \\ &\approx \frac{1}{\Delta^{*2}} (\theta_1^* + \theta_2^* - 2\theta_0^*) \end{aligned} \quad (17)$$

With this value for the second derivative, equation (16) becomes

$$\theta_1^* + \theta_2^* - \theta_0^* \left[2 + \frac{Nu}{s^*k^*} (h^* + h'^*)_0 \Delta^{*2} \right] + \Delta^{*2} = 0 \quad (18)$$

The relaxation method, used for solving this equation, is subsequently discussed.

Two-dimensional problem.—Consider a slab of homogeneous solid material with unit thickness, as shown in the following diagram:



Heat is again generated within the material at a uniform rate r . The heat balance for an element of volume with the dimensions dx and dy and of unit depth is

$$Q_x + Q_y + Q_i = Q_x' + Q_y' \quad (19)$$

A constant flow of heat through the slab normal to the plane of the preceding sketch does not influence this heat balance. Local deviations from this constant flow of heat are neglected.

The individual terms in equation (19) are

$$\begin{aligned} Q_x &= -k dy \frac{\partial t}{\partial x} \\ Q_x' &= -k dy \left(\frac{\partial t}{\partial x} + \frac{\partial^2 t}{\partial x^2} dx \right) \\ Q_y &= -k dx \frac{\partial t}{\partial y} \\ Q_y' &= -k dx \left(\frac{\partial t}{\partial y} + \frac{\partial^2 t}{\partial y^2} dy \right) \\ Q_i &= r dx dy \end{aligned}$$

Equation (19) can therefore be written

$$\frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 t}{\partial y^2} = -\frac{r}{k} \quad (20)$$

Or, in terms of the dimensionless variables,

$$\frac{\partial^2 \theta^*}{\partial x^{*2}} + \frac{\partial^2 \theta^*}{\partial y^{*2}} = -1 \quad (21)$$

If the slab is bounded by a coolant whose bulk temperature is T_B and if the local surface heat-transfer coefficient at a point s on the surface is h , the boundary condition at that point is

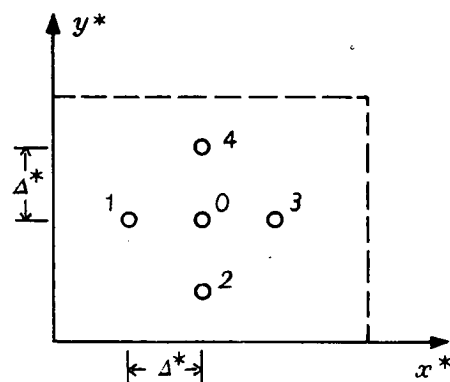
$$k \left(\frac{\partial t}{\partial n} \right)_s = -h(t - T_B)_s \quad (22)$$

where n is the direction normal to the surface. Equation (22) can again be written in terms of the dimensionless variables

$$\left(\frac{\partial \theta^*}{\partial n^*} \right)_s = -\frac{Nu h^*}{k^*} \theta_s^* \quad (23)$$

Equation (21) together with boundary condition (23) fully describes the two-dimensional problem. All physical variables are again grouped into a single parameter $h^* Nu/k^*$, which appears in the boundary condition. In order to apply the relaxation method of solution to these expressions, they are first converted to difference equations.

For this purpose a rectangular grid is placed into the slab, as indicated in the following diagram:



Adjacent net points are separated by a distance Δ^* . The derivatives of θ^* at an arbitrary point 0 can be expressed in terms of the temperature function at surrounding points. Thus the first derivatives are

$$\begin{aligned} \left(\frac{\partial \theta^*}{\partial x^*} \right)_0 &\approx \frac{\theta_1^* - \theta_0^*}{\Delta^*} & \left(\frac{\partial \theta^*}{\partial x^*} \right)_3 &\approx \frac{\theta_0^* - \theta_3^*}{\Delta^*} \\ \left(\frac{\partial \theta^*}{\partial y^*} \right)_0 &\approx \frac{\theta_2^* - \theta_0^*}{\Delta^*} & \left(\frac{\partial \theta^*}{\partial y^*} \right)_4 &\approx \frac{\theta_0^* - \theta_4^*}{\Delta^*} \end{aligned}$$

and the second derivatives become

$$\begin{aligned} \frac{\partial^2 \theta^*}{\partial x^{*2}} &\approx \frac{\theta_1^* + \theta_3^* - 2\theta_0^*}{\Delta^{*2}} \\ \frac{\partial^2 \theta^*}{\partial y^{*2}} &\approx \frac{\theta_2^* + \theta_4^* - 2\theta_0^*}{\Delta^{*2}} \end{aligned}$$

With these values, equation (21) becomes

$$\theta_1^* + \theta_2^* + \theta_3^* + \theta_4^* - 4\theta_0^* + \Delta^{*2} = 0 \quad (24)$$

The boundary condition can be evaluated by assuming that points 1, 0, and 3 lie on the surface of the wall. Point 4 then lies in the stream and its temperature must be expressed by the normal derivative at point 0.

$$\left(\frac{\partial \theta^*}{\partial y^*} \right)_0 \approx \frac{\theta_4^* - \theta_0^*}{\Delta^*}$$

or, with the use of the value of $\partial\theta^*/\partial y^*$ as given by equation (23),

$$\theta_4^* = \theta_0^* \left(1 - \frac{Nu h^*}{k^*} \Delta^* \right) \quad (25)$$

Along the boundary, therefore, the following equation applies:

$$\theta_1^* + \theta_2^* + \theta_3^* - \theta_0^* \left(3 + \frac{Nu h^*}{k^*} \Delta^* \right) + \Delta^{*2} = 0 \quad (26)$$

Equation (24) together with boundary condition (26) can again be solved by means of the relaxation method.

Solution by relaxation method.—A heat exchanger composed of a large number of rectangular passages will be discussed. The walls of the heat exchanger are made of a homogeneous material in which heat is generated at a uniform rate r . Figure 9 represents a cut through the heat exchanger so that the flow of coolant is in a direction normal to the cut. In the discussion that follows, it is assumed that the geometry of the configuration is given in the dimensionless system of coordinates.

A complete discussion of the relaxation method is not presented herein, inasmuch as it is generally available elsewhere. (See, for example, reference 9.) The essential features of the method can be outlined as follows:

Suppose it is desired to solve a given finite-difference equation over a certain area of integration. The equation is of the following type:

$$f(\theta^*) = 0 \quad (29)$$

The solution of equation (29) must also satisfy prescribed conditions at the boundary of the area of integration.

First, it is necessary to select a number of net points covering the entire area of integration. The distance Δ^* between net points is arbitrary, with the accuracy of the final solution increasing as the distance between points is decreased. Next, values of θ^* are assumed at each net point. If by chance these assumed values of θ^* are the correct values, then they satisfy the appropriate finite-difference equation at all net points.

In general, however, the assumed values of the function do not satisfy the difference equation and the left side of equation (29) is equal to some residual value N instead of zero. At any given net point θ^* must then be adjusted in order to make N vanish at that point. This adjustment of θ^* also changes the residuals at adjacent net points. However, if this process of adjustment is started at the point at which the absolute value of N is greatest and is then repeated for points at which the value of the residual is successively less, the correct values of θ^* for the entire net eventually are obtained.

In applying this method to the heat exchanger under consideration, it is first assumed that the wall thickness s^* is small, so that the one-dimensional solution applies.

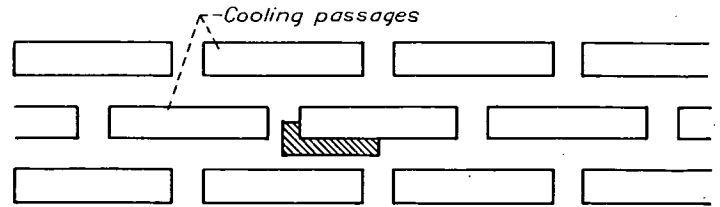


FIGURE 9.—Thick-walled heat exchanger with rectangular passages. (Shaded region is considered in calculations.)

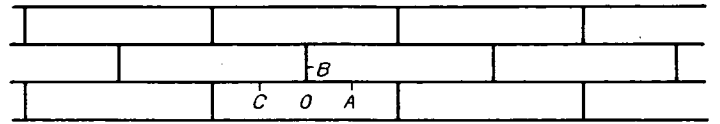


FIGURE 10.—Thin-walled heat exchanger with rectangular passages. (Calculation is confined to region ABC.)

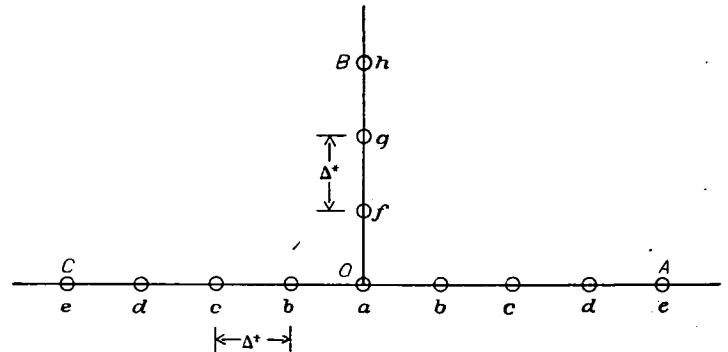


FIGURE 11.—Relaxation net for rectangular thin-walled heat exchanger.

The actual configuration to be discussed is shown in figure 10. Because points A, B, and C are points of symmetry, only the region bounded by these points need be considered. Furthermore, the distribution of temperature is also symmetric about point O with respect to AO and OC. An enlarged view of the section under consideration is represented in figure 11.

The section has been subdivided into a number of net points $a, b, c, \dots, h$, each separated by a distance Δ^* from adjacent points. The first step in the analysis is the assumption of temperatures at all net points, which can be done by setting up a balance between the total heat produced within the walls and the total outflow of heat from the walls. In terms of the dimensionless variables, this heat balance becomes

$$\frac{1}{C^*} \frac{Nu}{h^* s^*} \int_0^{C^*} \theta^* (h^* + h'^*) dx^* = 1 \quad (30)$$

If, for the initial assumption, $h^* = h'^* = 1$ and θ^* is constant, the following expression is obtained:

$$\theta^* = \frac{s^* k^*}{2Nu} \quad (31)$$

The value of θ^* given by equation (31) is assumed to exist at all net points. The appropriate difference equation at points b, c, d, f , and g is (from equation (18))

$$\theta_1^* + \theta_2^* - \theta_0^* \left[2 + \frac{Nu}{s^* k^*} (h^* + h'^*)_0 \Delta^{*2} \right] + \Delta^{*2} = N_0 \quad (32)$$

where the subscript 0 refers to the point at which the equation is to be applied and subscripts 1 and 2 refer to adjacent points. At point a , which is influenced by three points, the following equation is required:

$$2\theta_b^* + \theta_f^* - \theta_a^* \left[3 + \frac{Nu}{s^* k^*} (h^* + h'^*)_a \Delta^{*2} \right] + \Delta^{*2} = N_a \quad (33)$$

At point e , which is a point of symmetry, the following expression applies:

$$2\theta_a^* - \theta_e^* \left[2 + \frac{Nu}{s^* k^*} (h^* + h'^*)_e \Delta^{*2} \right] + \Delta^{*2} = N_e \quad (34)$$

A similar expression applies at point h . With these equations, the residuals N are calculated at each net point. The point at which N has the largest absolute value is then selected and θ^* at that point is adjusted so that the residual vanishes. The effect of this adjustment on the residuals at adjacent points is calculated with the aid of the appropriate finite-difference equation.

The process is repeated for the point at which the next largest value of the residual appears. Eventually, if the process is repeated often enough, the residuals at all net points approach zero. The final adjusted values of θ^* then satisfy the appropriate equations at all net points.

These values of θ^* should also satisfy the heat balance as given by equation (30). This equation can therefore be used to check the validity of the final temperature distribution. If the check is unsatisfactory, it is necessary to continue the solution of the finite-difference equations by using a smaller net spacing.

The same heat-exchanger configuration is now investigated without making the assumption that the wall thickness is small. The two-dimensional equations are therefore applied. An enlarged view of the shaded portion of figure 9 is shown in figure 12. The lines $\overline{CD}$, $\overline{FG}$, and $\overline{GA}$ are lines of symmetry, and there is no flow of heat across these lines. The temperature distribution along line $\overline{DEF}$ will be symmetric about point E , but there will be a flow of heat across this line. A large number of net points are selected to cover the entire section of the configuration shown in figure 12. (Only

a few of these points are presented in the fig.) An initial value for the temperature along lines $\overline{AB}$ and $\overline{BC}$ is obtained by setting up a balance between the total heat produced within the section and the flow of heat to the gas. In terms of the dimensionless variables, this heat balance becomes

$$\frac{1}{C^*} \int_0^{C^*} h^* \theta^* dx^* = \frac{A^*}{4} \frac{k^*}{Nu} \quad (35)$$

If, for the initial assumption, $h^*=1$ and θ^* is constant, then the following expression for θ^* is obtained:

$$\theta^* = \frac{A^*}{4} \frac{k^*}{Nu} \quad (36)$$

It can now be assumed that the temperature as given by equation (36) exists along the surface of the wall and that a somewhat higher temperature exists at internal points. The assumed temperatures are then adjusted by means of the relaxation method.

The appropriate finite-difference equation for internal points, such as point e , is written as follows:

$$\theta_b^* + \theta_f^* + \theta_n^* + \theta_a^* - 4\theta_e^* + \Delta^{*2} = N_e \quad (37)$$

For points along the boundary, such as point b , the equation is

$$\theta_a^* + \theta_e^* + \theta_c^* - \theta_b^* \left(3 + \frac{Nu}{k^*} h_b^* \Delta^* \right) + \Delta^{*2} = N_b \quad (38)$$

and for points on a line of symmetry, such as point f ,

$$\theta_c^* + 2\theta_e^* + \theta_i^* - 4\theta_f^* + \Delta^{*2} = N_f \quad (39)$$

The appropriate equation for points along $\overline{DEF}$ can be determined by taking advantage of the antisymmetry about this line. At point k , for instance, the equation becomes

$$\theta_n^* + \theta_i^* + \theta_m^* + \theta_j^* - 4\theta_k^* + \Delta^{*2} = N_k \quad (40)$$

Expressions of the type (37) to (40) apply at all net points. The method of determining the actual temperature distribution is the same as the method outlined in the previous section.

The final temperatures along the surface of the wall can be checked with the aid of equation (35). If the check is unsatisfactory, a finer net spacing is required.

RESULTS

One-dimensional solution.—The one-dimensional solution was applied to heat exchangers composed of rectangular and triangular passages, respectively. In each case the passages were staggered in order to minimize the expected hot spots.

The rectangular configuration is represented by figure 10. The height-to-width ratio of each passage in this configuration is 1 to 5. Figure 1 represents the triangular configuration. Each passage in this configuration is an isosceles right triangle.

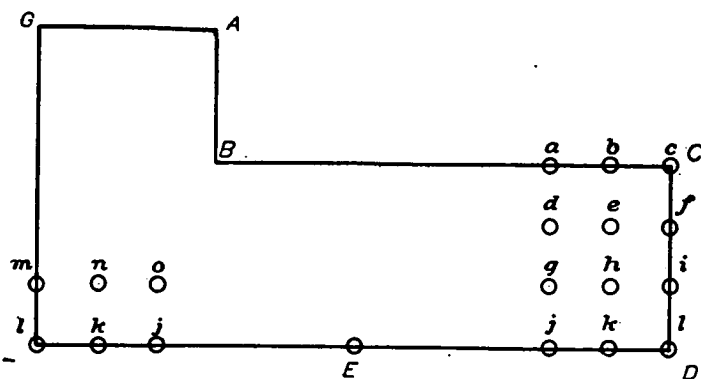


FIGURE 12.—Relaxation net for rectangular thick-walled heat exchanger.

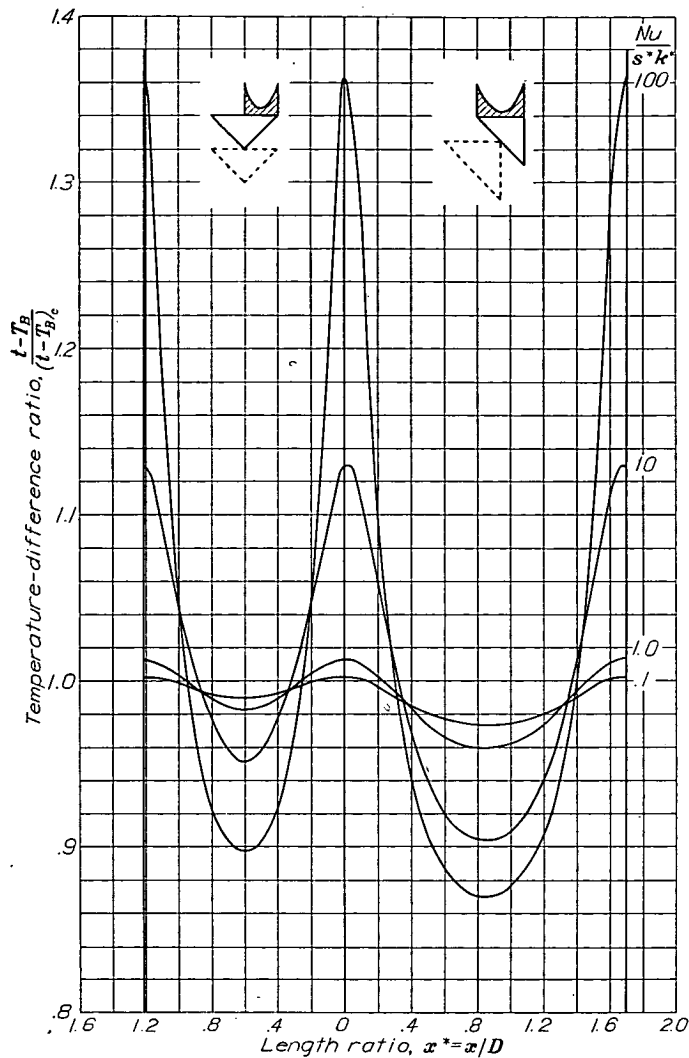


FIGURE 13.—Temperature distribution for passage of thin-walled triangular heat exchanger (fig. 1).

It has previously been stated that, for each configuration, the temperature distribution in the passage walls is a function only of the parameter Nu/s^*k^* . Accordingly, temperature distributions were calculated for several values of this parameter, ranging from 0.1 to 100. This range of values is believed to include all values actually encountered.

Temperature distributions were calculated according to the method outlined in the previous section of this report and were then checked by equation (30). Local rates of heat transfer $q^* \equiv h^*\theta^*$ were calculated for this purpose. According to equation (30), the mean value of q^* when multiplied by $2Nu/k^*s^*$ should equal 1. Because the finite-difference method of solution is essentially an approximate method, the results were not expected to be exact. All results presented in this section of the report, however, were held to an error of less than 6 percent. The final temperatures were multiplied by a constant scale factor in order to satisfy equation (30) exactly.

Temperature distributions and local rates of heat transfer are presented for the triangular heat exchanger in figures 13 and 14 and for the rectangular heat exchanger in figures 15

and 16. In the temperature plots (figs. 13 and 15), the temperature difference $(t - T_B)$ is referred to the temperature difference $(t - T_B)_c$ for a thin-walled circular tube with the same hydraulic diameter and an internally heated wall with the same physical properties. The temperature difference for the circular tube is given in the nondimensional form by equation (31). Similarly, the ordinate in figures 14 and 16 refers the rate of heat transfer q to the rate of heat transfer from the wall of a thin circular tube q_c . It is believed that

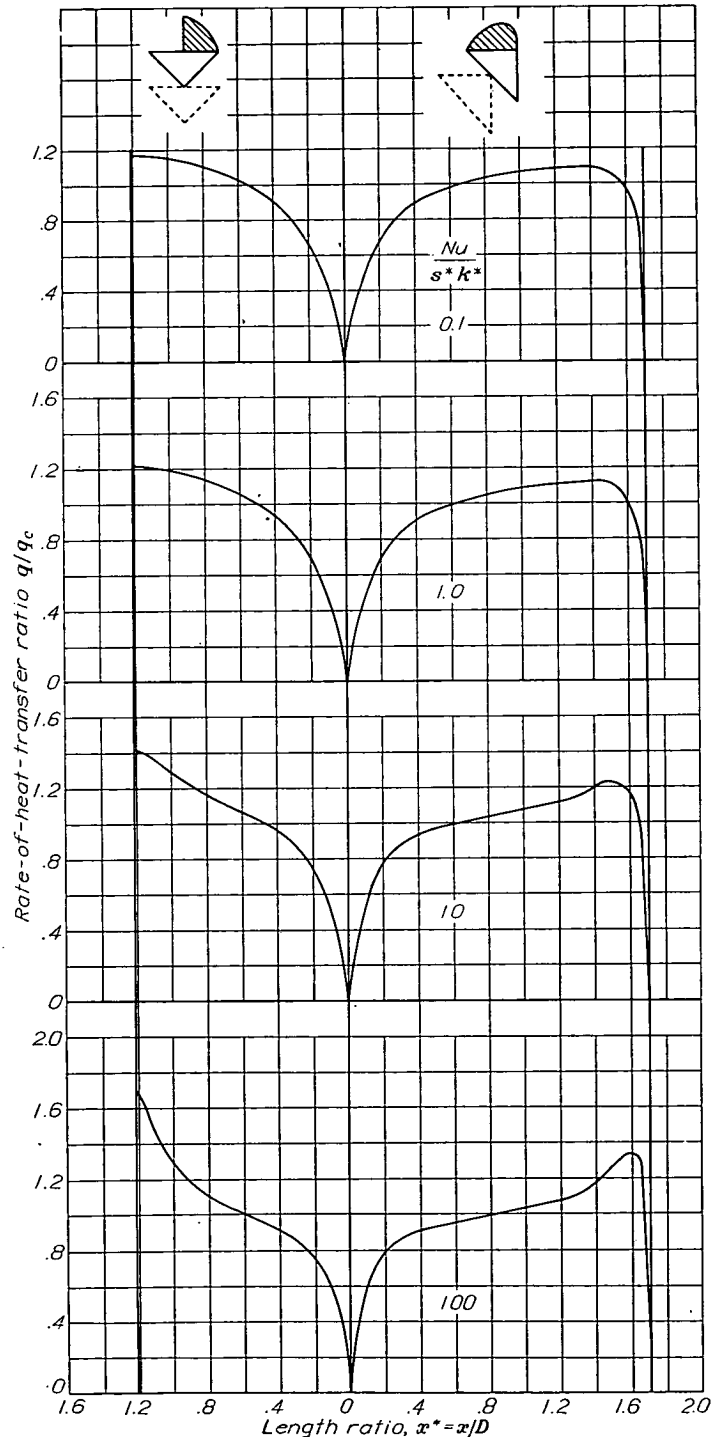


FIGURE 14.—Local rate of heat transfer from passage of thin-walled heat exchanger (fig. 1).

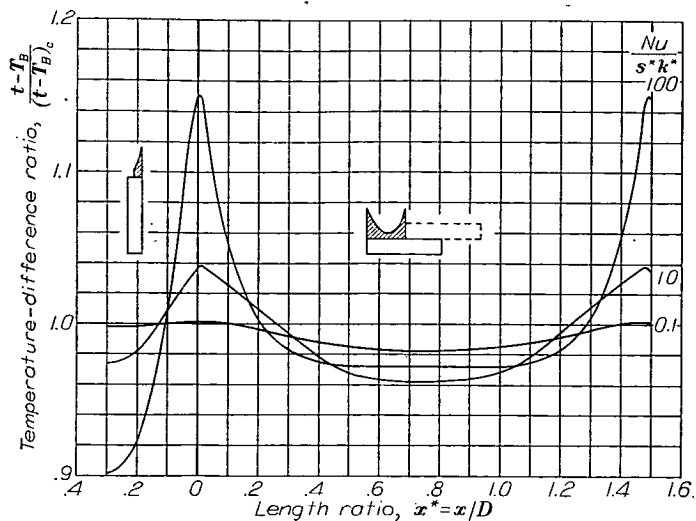


FIGURE 15.—Temperature distribution for passage of thin-walled rectangular heat exchanger (fig. 10).

the interpretation of the temperature and heat-transfer curves is simplified by this final change of ordinates. Temperature distributions for the triangular configuration are shown in figure 13. These curves are presented for values of $Nu/s*k^*$ of 0.1, 1.0, 10, and 100. As was expected, the temperatures peak near the corners of the passage. The largest temperature differences are encountered for large values of the parameter $Nu/s*k^*$. A large value of this parameter indicates a low conductivity of the wall material or a small wall thickness. Both of these conditions are conducive to low rates of heat exchange within the wall. Local rates of heat transfer corresponding to the temperature distributions just discussed are presented in figure 14.

Temperature and heat-transfer curves for the rectangular configuration are shown in figures 15 and 16. The results for this configuration are similar to those for the triangular configuration.

In order to determine when the one-dimensional solution can be used, it is necessary to obtain some information on the temperature difference that exists across the passage walls on normals to the surfaces and to compare this temperature difference with the temperature differences along the surfaces. An estimate of the temperature difference δt_n across the wall may be obtained by calculating this value for a flat plate. The result of this calculation is

$$\delta t_n = \frac{r}{8k} s^2 \quad (41)$$

The equation can be changed to the dimensionless values to yield

$$\delta t_n^* = \frac{s^{*2}}{8} \quad (42)$$

This temperature difference can again be referred to the temperature difference $(t - T_B)_c$ as follows:

$$\frac{\delta t_n}{(t - T_B)_c} = \frac{s^* Nu}{4k^*} \quad (43)$$

The one-dimensional solution applies as long as this value is small as compared with the temperature differences presented in figures 13 and 15.

Two-dimensional solution.—The two-dimensional solution for the temperature field within the heat exchanger depends on two parameters; namely, the dimensionless wall thickness s^* and the value $Nu h^*/k^*$. In addition, the time required to obtain the solution for a special case is much longer than for the one-dimensional solution. Only one example was therefore calculated; namely, the temperature distribution within the walls of a heat exchanger composed of rectangular passages, as shown in figure 9. The ratio of the two side lengths of the rectangle is 1 to 5. The ratio of the hydraulic diameter to the short side of this passage is 1.67 and the dimensionless wall thickness is 0.6. Figure 17 presents the results of the calculation using the relaxation method with a network of 88 points. Lines of constant temperature (isotherms) are shown within the portion of the heat-exchanger walls that is shaded in figure 9. As may be seen, the heat flow within this wall is mainly in the direction normal to the wall surfaces. The maximum temperature differences on any normal to the surface are not very different from the value in a flat plate as calculated in equation (42). The temperature differences along the surface of the wall are

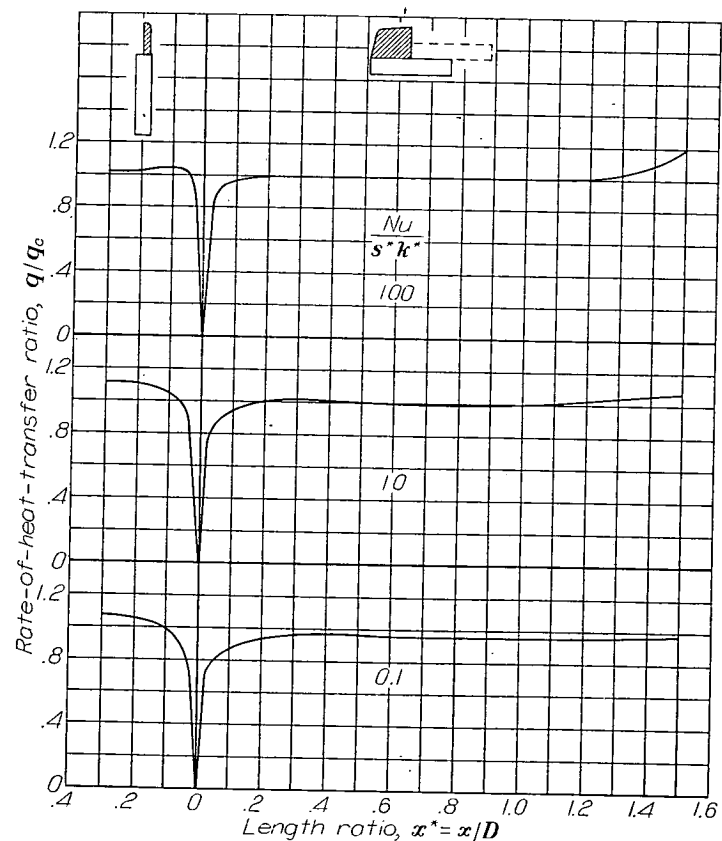


FIGURE 16.—Local rate of heat transfer from passage of thin-walled rectangular heat exchanger (fig. 10).

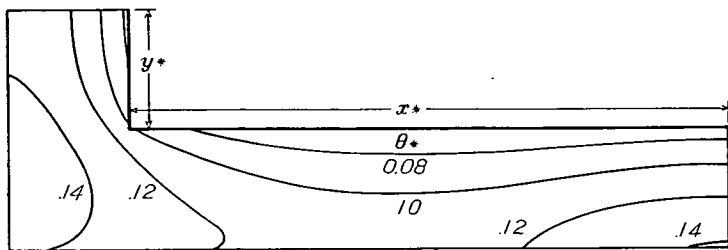


FIGURE 17.—Isotherms in internally heated walls of thick-walled rectangular heat exchanger. (Shaded portion of fig. 9 shown.) Nu/s^*k^* , 10.0; ratio of flow area to total area, 0.416; s^* , 0.6; y^*/x^* , 1/5.

appreciably smaller than the temperature differences across the wall. The values of the one-dimensional solution cannot therefore be expected to apply to this case. Actually, the temperature difference found in figure 17 along the surface of the wall is considerably greater than for the corresponding one-dimensional case. In addition, heat is generated within the corner area of the rectangular-wall configuration and has to be conducted away along comparatively long paths. An additional temperature increase can therefore be found within this corner area.

CONCLUSIONS

A method to calculate the temperature distribution in a heat exchanger composed of noncircular flow passages with internally heated walls has been presented.

Local heat-transfer coefficients along the circumference of the heat-exchanger passages were obtained from flow measurements made by Nikuradse, assuming similarity between the velocity and temperature fields. The heat-transfer coefficients, as determined in this manner, decrease sharply near the corners of the passages and vanish at the corners. This decrease becomes more pronounced as the corner angle

is increased. Near the corners these coefficients are essentially influenced only by the magnitude of the included angle of the corner.

It was shown that the dimensionless temperature distribution within the passage walls depends on a single parameter, provided the dimensionless wall thickness is small. Numerical evaluations for triangular- and rectangular-passage configurations for a wide range of the aforementioned parameter were carried out. The results of these evaluations are presented so that temperature differences arising in the walls of heat exchangers with the investigated passage shapes for any condition within the range of practical interest can be read off one of several curves. These curves show a temperature increase near the corners of the passages. This increase becomes more pronounced for high Nusselt numbers based on the average heat-transfer coefficient, for low wall-thickness ratios, and for low ratios of the conductivity of the wall to the conductivity of the coolant.

The dimensionless temperature distribution for thick-walled heat exchangers depends on the wall-thickness ratio in addition to the property parameter. Because the time required to calculate temperature distributions in the whole field of interest determined by the two parameters is prohibitive, only a specific example was evaluated numerically. The method of calculation is presented in great detail, however, so that evaluations for other interesting cases can be carried out.

LEWIS FLIGHT PROPULSION LABORATORY
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS
CLEVELAND, OHIO, October 4, 1950

APPENDIX

SOLUTION BY RUNGE-KUTTA METHOD

Equation (16) is of the following form:

$$\frac{d^2 \theta^*}{dx^{*2}} - f(x^*) \theta^* + 1 = 0 \quad (A1)$$

The boundary conditions are

$$\left. \begin{aligned} \left(\frac{d\theta^*}{dx^*} \right)_{x^*=a} &= 0 \\ \left(\frac{d\theta^*}{dx^*} \right)_{x^*=b} &= 0 \end{aligned} \right\} \quad (A2)$$

where a and b are points of symmetry.

In general, a solution of equation (A1) could be obtained by assuming an initial value of θ^* at point a in addition to the first of the boundary conditions (A2) and by working the solution toward point b using a numerical method of integration. If the boundary condition at point b is not satisfied,

the solution has to be repeated with a new initial value. This trial-and-error process is tedious and can be avoided by splitting equation (A1) in such a manner as to keep an undetermined constant c in the solution. This constant is finally determined by satisfying the second boundary condition. Let

$$\theta^* = \theta_1^* + c\theta_2^* \quad (A3)$$

$$\frac{d^2 \theta_1^*}{dx^{*2}} - f(x^*) \theta_1^* + 1 = 0$$

$$\frac{d^2 \theta_2^*}{dx^{*2}} - f(x^*) \theta_2^* = 0$$

where

$$\begin{aligned} \theta_1^*(a) &= 0 & \theta_2^*(a) &= 1 \\ \left(\frac{d\theta_1^*}{dx^*} \right)_a &= 0 & \left(\frac{d\theta_2^*}{dx^*} \right)_a &= 0 \end{aligned}$$

These expressions still satisfy equation (A1) and the first of the boundary conditions (A2). After the solutions for θ_1^* and θ_2^* have been numerically obtained, the constant c is determined so that the boundary condition at point b is fulfilled; thus

$$\left(\frac{d\theta_1^*}{dx^*}\right)_b + c\left(\frac{d\theta_2^*}{dx^*}\right)_b = 0$$

or

$$c = -\frac{\left(\frac{d\theta_1^*}{dx^*}\right)_b}{\left(\frac{d\theta_2^*}{dx^*}\right)_b}$$

The final temperature distribution is given by equation (A3).

The temperature distribution in a thin-walled heat exchanger with triangular passages was calculated in this manner (for $Nu/s^*k^*=10$) and was compared with the corresponding curve in figure 13. The Runge-Kutta method (reference 10) was used for the numerical integration. This particular comparison showed differences of 2.2 percent of the temperature ratio near the corners of the passages and smaller deviations elsewhere. The temperature curves are uncertain to the same order of magnitude, however, because

of the freedom in the extrapolation of the heat-transfer-coefficient curves. The simpler relaxation method is therefore regarded satisfactory for the present purpose.

REFERENCES

1. McAdams, William H.: Heat Transmission. McGraw-Hill Book Co., Inc., 2d ed., 1942.
2. Jakob, Max: Heat Transfer. John Wiley & Sons, Inc., 1949.
3. Nikuradse, J.: Untersuchungen über turbulente Strömungen in nicht kreisförmigen Rohren. Ingenieur-Archiv, Bd. 1, 1930, S. 306-332.
4. Nikuradse, J.: Geschwindigkeitsverteilung in turbulenten Strömungen. VDI Zeitschr., Bd. 70, Nr. 37, Sept. 11, 1926, S. 1229-1230.
5. Prandtl, L.: Turbulent Flow. NACA TM 435, 1927.
6. Kalikhman, L. E.: Heat Transmission in the Boundary Layer. NACA TM 1229, 1949.
7. Latzko, H.: Heat Transfer in a Turbulent Liquid or Gas Stream. NACA TM 1068, 1944.
8. Eckert, E. R. G.: Introduction to the Transfer of Heat and Mass. McGraw-Hill Book Co., Inc., 1950, pp. 128-140.
9. Emmons, H. W.: The Numerical Solution of Heat-Conduction Problems. Trans. A.S.M.E., vol. 65, no. 6, Aug. 1943, pp. 607-612.
10. Scarborough, James B.: Numerical Mathematical Analysis. Johns Hopkins Press (Baltimore), 1930.

REPORT 1023

DIFFUSION OF CHROMIUM IN ALPHA COBALT-CHROMIUM SOLID SOLUTIONS¹

By JOHN W. WEETON

SUMMARY

Diffusion of chromium in α cobalt-chromium solid solutions was investigated in the range 0 to 40 atomic percent at temperatures of 1360°, 1300°, 1150°, and 1000° C. The diffusion coefficients were found to be relatively constant within the composition range covered by each specimen.

The activation heat of diffusion was determined to be 63,600 calories per mole. This value agrees closely with the value of 63,400 calories per mole calculated by means of the Dushman-Langmuir equation. The exponential equation relating diffusion coefficient D to temperature T is as follows:

$$D = 0.443e^{-\frac{63,600}{RT}}$$

where R is the gas constant.

When compared with the diffusion data previously obtained by other investigators for most alloy systems, the diffusion rates of chromium in α cobalt-chromium were found to be low; considerably higher temperatures were required to produce diffusion coefficients of the same order of magnitude as were previously found for most substitutional alloys. This behavior is congruous with the fact that cobalt-chromium based alloys such as Haynes Stellite 21 have good high-temperature characteristics.

Chromium diffusivity from α cobalt-chromium alloys into pure cobalt is greater than chromium diffusivity from high-chromium α -alloys to low-chromium α -alloys for all concentration gradients studied.

INTRODUCTION

Several important high-temperature alloys (Haynes Stellite 21, X-40, 61, and 422-19) consist primarily of cobalt and chromium; Haynes Stellite 21 has been one of the most extensively used turbojet-blade alloys in the United States. Because of the importance of cobalt-chromium based alloys and because diffusion controls many reactions within solid metals, an investigation was conducted at the NACA Lewis laboratory to determine the diffusion coefficients of chromium into α cobalt-chromium solid solutions, to determine the dependence of these coefficients on concentration, and to evaluate the basic constants Q and A in the exponential equation relating temperature and diffusion (reference 1)

$$D = Ae^{-\frac{Q}{RT}}$$

where

A constant, (cm²/sec)

D diffusion coefficient, (cm²/sec)

Q activation heat of diffusion, (cal/mole)

R gas constant, (cal/(mole) (°K))

T temperature, (°K)

The method employed consisted in pressure-welding cobalt and cobalt-chromium bars unlike in composition, annealing

these joined bars at constant temperatures to cause an inter-diffusion of atoms, and determining the distribution of chromium through the diffusion zones thus formed by machining and chemically analyzing several successive turnings through this zone.

Diffusion coefficients were determined for 11 specimens at the nominal temperatures and composition ranges shown in the following table:

CHROMIUM CONTENT OF WELDED SPECIMENS
(ATOMIC PERCENT)

1360° C			1300° C			1150° C			1000° C		
Specimen	Composition		Specimen	Composition		Specimen	Composition		Specimen	Composition	
	Low-Cr half	High-Cr half		Low-Cr half	High-Cr half		Low-Cr half	High-Cr half		Low-Cr half	High-Cr half
1	0	22.20	3	0	28.00	6	13.15	38.65	8	0	24.92
2	9.60	28.06	4	9.60	41.15	7	0	28.06	9	9.80	39.97
11	0	28.00	5	9.98	39.20						
			10	0	28.15						

With the exception of specimens 1 and 2, which were originally intended for annealing at 900° C, the specimens were made to cover most of the range of the α field in the cobalt-chromium equilibrium diagram (fig. 1).

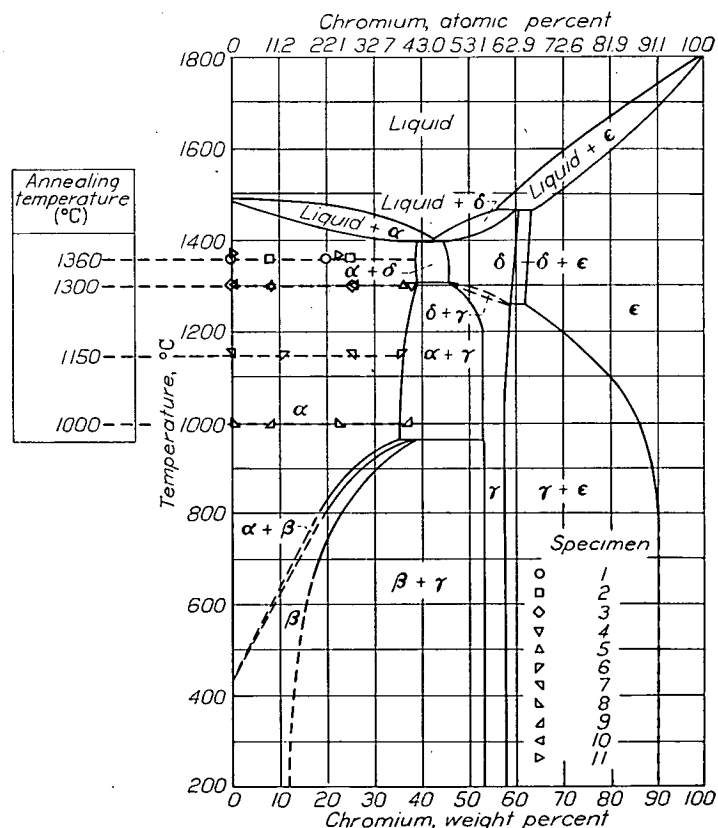


FIGURE 1.—Cobalt-chromium equilibrium diagram from reference 2. Also drawn are composition ranges at annealing temperatures used in this investigation.

¹ Supersedes NACA TN 2218, "Diffusion of Chromium in α Cobalt-Chromium Solid Solutions" by John W. Weeton, 1950.

Diffusion coefficients were calculated by the Grube method (reference 3), although the Matano method (reference 3) was also used in two cases.

APPARATUS AND PROCEDURE

Materials.—Cobalt rondels and electrolytic chromium of high purity were used as raw materials. Spectrographic and wet chemical analyses were made of the rondels. Colorimetric analyses, made for elements that were shown to be present by the spectrograph but were not detectable by wet chemical methods, are less than 0.01 percent. The results of the analysis of cobalt rondels and of an analysis of electrolytic chromium made by the supplier, the Bureau of Mines, are given in the following table:

IMPURITIES IN COBALT AND CHROMIUM

Cobalt rondels (percent)		Electrolytic chromium (percent)	
Ni	0.38	Fe	0.31
Fe	.10	O	.17
Cu	.08		
Mn	.01		
Sn	.01		
Ag	.01		
Pb	.01		
Ti	.01		
V	.01		
C	*, 14, .20		

* Determined by commercial laboratory.

It will be shown that the carbon content was considerably reduced during melting.

Melting and casting.—Cobalt and cobalt-chromium melts of nominally 0, 10, 25, and 35 percent-by-weight chromium (0, 11, 27, and 38 atomic percent) were made in zirconium silicate crucibles, using a 100-kilowatt, 9600-cycle-per-second induction unit. The rondels were first melted with no protective atmosphere and were held at temperatures of 1593° to 1649° C (2900° to 3000° F) for 5 to 10 minutes to burn out the carbon. As a result of this procedure, the carbon was reduced to less than 0.06 percent in most cases, but the oxide content of the metal was probably increased.

Where chromium additions were required, an atmosphere of argon was kept over the surface of the melt from the time the addition was made until the pouring time, a total period of 10 to 20 minutes. Chromium was not charged with the cobalt because chromium prevents oxidation and removal of almost all the carbon.

Castings were made in copper molds and, except for the heads, were shaped like truncated cones 4 inches long, 1¼ inches in diameter at the top, and ⅞ inch in diameter at the bottom. Chemical analyses made from both ends of the castings containing chromium indicated that the churning action of the induction current produced an excellent degree of chemical homogeneity; differences in chromium analyses made from the large and the small ends of castings ranged from 0.02 to 0.15 percent.

Heat treatment prior to forging.—As a precautionary measure, the samples containing chromium were soaked at 1176° to 1231° C (2150° to 2250° F) for ½ hour and then oil-quenched. This treatment was intended to reduce the

possibility of cracking the samples if insufficient soaking time were allowed prior to forging.

Forging.—The tapered castings, which were forged at a commercial laboratory, were upset one or two times and were fullered out to an approximately uniform diameter of 1 inch. The reduction in length occurring when the castings were upset varied from approximately 20 to 50 percent. Forging at temperatures from 982° to 1093° C (1800° to 1900° F) reduced the grain size from grains as large as ¼ inch to smaller, more uniformly shaped grains that varied from A. S. T. M. grain size 5 to small macroscopic grains approximately ⅛ inch.

Homogenizing heat-treatment.—The samples containing chromium were annealed at 1204° C (2200° F) for 3 hours in a helium atmosphere for further homogenization. Results presented herein indicate that this treatment does little more than stress-relieve the specimens.

Machining of bars to be welded.—Cylinders ⅞ inch in diameter and 1¼ to 2¼ inches in length were machined from the forged bars. Both ends of these cylinders were surface ground to produce flatness and one end of each bar was then lapped by a gage-block manufacturer until it could be "wrung" to another bar.

Welding of dissimilar bars or cylinders.—Lapped specimens of unlike compositions were placed in the holders shown in figure 2. The upper holder was bolted to a 120,000-pound tensile machine and the lower one was placed on the table of the machine. A load of 5000 pounds was applied after the cylinders were centered in an induction coil and the holders were alined. The specimens were heated with a portable, 35-kilowatt, 360,000-cycle-per-second, spark-gap induction unit.

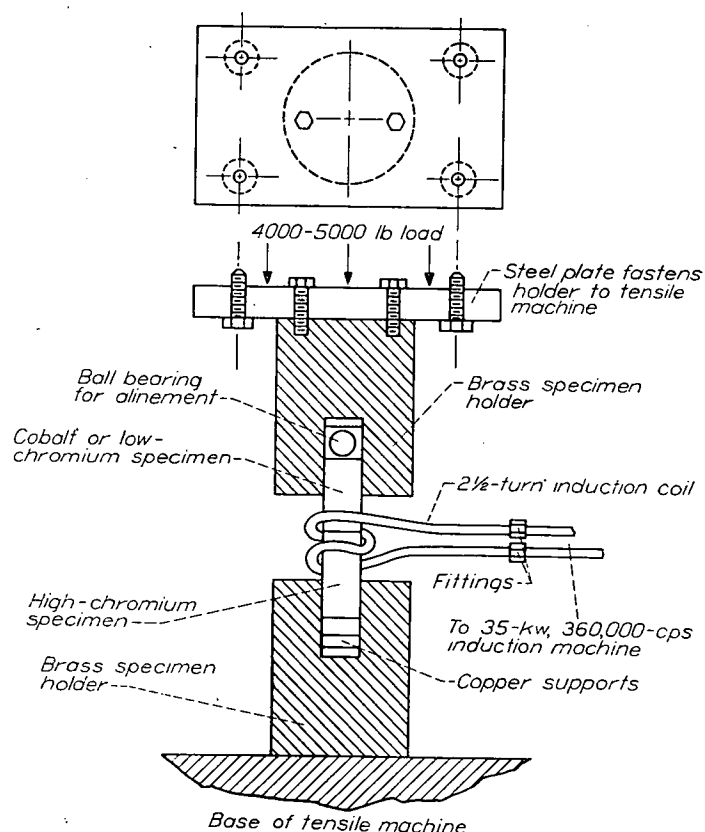


FIGURE 2.—Pressure-welding apparatus.

During the heat-up time, which varied from $\frac{1}{2}$ to $1\frac{1}{2}$ minutes, the tendency for the load to increase as thermal expansion of the cylinders took place was reduced by the tensile-machine operator, who kept the pressure reasonably constant (within ± 200 pounds). Because a skin effect could be produced by raising the specimen temperature too rapidly, the power to the induction machine was turned on and off to allow an even heat distribution across the weld surface. When the temperature, which was read with an optical pyrometer, reached a point between 982° and 1038° C (1800° to 1900° F), the load was reduced to 4000 pounds to prevent excessive bulging. This load was maintained during the welding period of $1\frac{1}{2}$ to $2\frac{1}{2}$ minutes at 1121° to 1204° C (2050° to 2200° F). Bulging was kept at a minimum to prevent curving of the interface. Uneven heating or misalignment of specimens caused bent or distorted welds. Satisfactory welds were strong enough to be hammered and bent.

First welding attempts were unsuccessful because the surfaces were not lapped to a gage-block finish and, as a result, considerable oxides were formed in the welds. An attempt was also made to surround the apparatus shown in figure 2 with a vacuum-tight shell into which argon or helium could be bled during welding. However, in spite of insulating the container and the coil from the work, extensive arcing in these atmospheres prevented welding. This method was then abandoned in favor of the one used.

Metallographic examination after welding.—Flat planes were ground lengthwise on the welded specimens for metallographic examination (fig. 3). In the etched and unetched

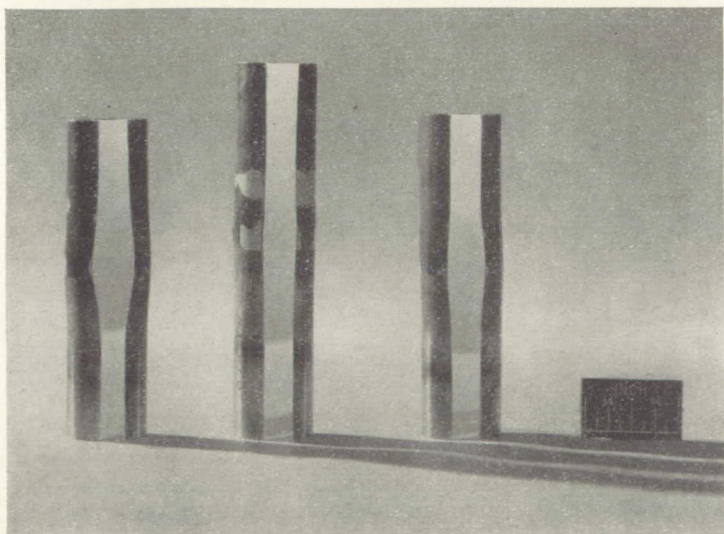


FIGURE 3.—Pressure-welded specimens with flat planes polished for metallographic examination.

conditions, the weld interfaces were examined for oxides, weld curvature, and other imperfections. Examples of satisfactory and unsatisfactory welds are shown in figure 4. Diffusion zones formed during welding were measured with a micrometer eyepiece and a research metallograph. The thickness of these zones, ranging from less than 0.0001 to 0.00046 inch (<0.00254 to 0.0117 mm), were considered negligible (fig. 4 (b)).

Machining prior to anneal.—Specimens $1\frac{1}{4}$ inches long and $\frac{1}{16}$ to $\frac{1}{8}$ inch in diameter, which included the weld interface, were turned from the welded specimens. Care was exercised during machining to make certain that the axes of these cylinders were perpendicular to the weld interfaces.

Calibration of instruments.—The thermocouple wires used during the diffusion anneal, subsequently to be described, were taken from a coil calibrated by the National Bureau of Standards. The potentiometer used to read the temperature was calibrated at the Lewis laboratory.

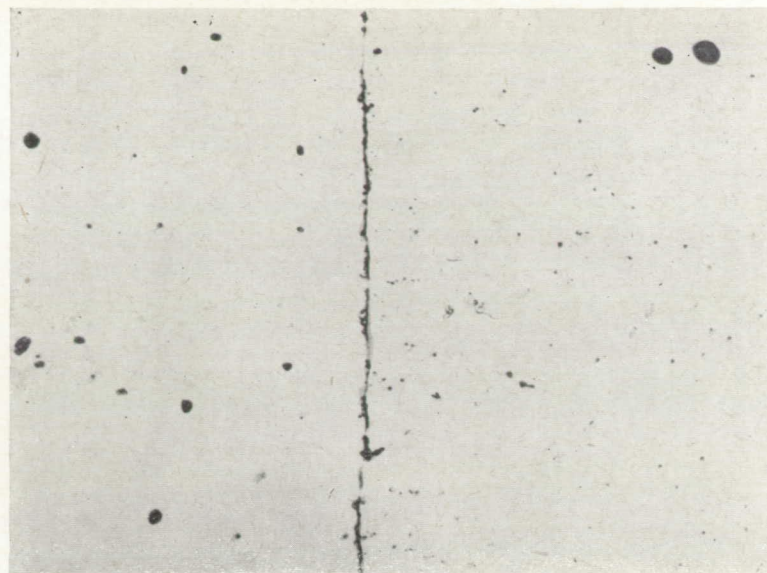
Diffusion anneal.—

1. **Furnaces:** Most of the specimens were annealed in a furnace using silicon carbide heating elements controlled by a self-balancing, self-standardizing, indicating pyrometer. The furnace, an eight-element unit, and the controller were wired so that four elements cycled on and off while the other four received power at all times. A zirconium silicate tube was placed in the furnace as shown in figure 5. The specimens to be annealed were placed in a porous-brick holder along with a platinum-platinum-rhodium thermocouple. A period of 1 to 3 hours was required to stabilize the furnace temperature after loading. Fluctuations of more than $\pm 4^\circ$ C ($\pm 7.2^\circ$ F) were rare and were of short duration. Time-temperature plots were made for all of the specimens except those annealed at 1000° C for 87.7 days. A typical time-temperature plot for specimen 4 is shown in figure 6. The plots were integrated with a planimeter to find a weighted average temperature, which was then corrected for the thermocouple wire calibration and converted to $^\circ$ C (table I).

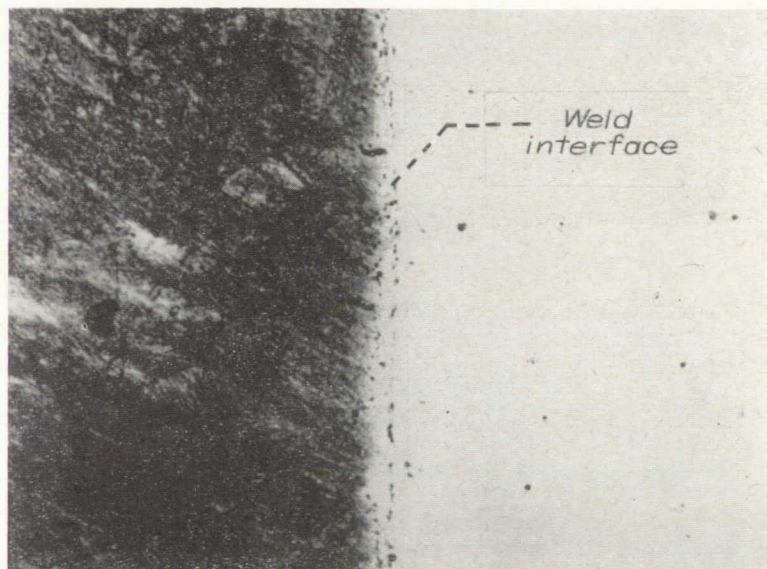
Specimens 8 and 9, which were annealed at 1000° C, were first annealed for a month in the silicon-carbide-element furnace and then transferred to a platinum-wound-resistance furnace. This furnace and a commercial light-beam-galvanometer-type photoelectric-cell controller were located in a room kept at constant temperature. Except for a temporary breakdown, the furnace was controlled within $\pm 4^\circ$ C ($\pm 7.2^\circ$ F).

2. **Atmospheres:** During loading of the furnace, a large quantity of argon was blown through the tube to prevent oxidation of the specimen. The gas flow was decreased after the stopper was sealed in the loading end of the zirconium silicate tube. The argon flow was completely shut off, the rubber tube to the bubble jar was clamped shut, and the zirconium silicate tube and the argon line, including the drying towers, were evacuated by a pump capable of evacuating the system to a pressure of 1 millimeter of mercury when no leaks were present. This evacuation seated the rubber stopper and made it possible to locate any leaks in the thermocouple and the stopper seals. Argon was then bled into the evacuated zirconium silicate tube, which was then re-evacuated. This procedure was repeated five times to flush the tube. The vacuum pump was then closed and argon was allowed to pass through the system and out of the bubble jar.

Specimens 3 and 4 were placed in the furnace without the aid of the vacuum pump but upon removal after annealing were appreciably scaled (table I). After the system was evacuated, the drying towers (commercial metallic towers) were found to be leaking. Laboratory-type glass drying



(a)



(b)

- (a) Unsatisfactory weld interface. Etchant, none; magnification, $\times 750$. Examination of unetched polished flat plane shows almost unbroken oxide layer in interface.
- (b) Satisfactory weld interface. Etchant, 10-percent nitric acid in ethyl alcohol; magnification, $\times 750$. Examination of heavily etched surface reveals approximately half of diffusion zone (distance between dark area and weld interface, approximately 0.00012 in. (0.0028 mm) wide) caused by welding. Etchant has attacked only cobalt half of weld.

FIGURE 4.—Appearance of satisfactory and unsatisfactory welds. (Magnification increased 10.4 percent in printing.)

towers were then installed and scaling was somewhat reduced. As an added precaution in annealing specimens 1, 2, 5, and 8 to 11, a mixture of hydrogen and helium, rather than argon, was passed through the zirconium silicate tube for a few hours after the system was sealed in order to scavenge oxygen from the bricks inside the tube. Argon was used after the hydrogen-helium purge and scaling was decreased to a negligible amount, especially when it is considered that loading and unloading the furnace could account for almost all the scale indicated for these specimens (table I).

Examination and reduction of diameter after anneal.—

After the anneal, flat surfaces for metallographic examination were again ground on the specimens so that an estimate of the reduction in diameter needed to eliminate surface effects could be made. In most cases, $\frac{1}{32}$ to $\frac{3}{32}$ inch of metal was removed. Flat surfaces were again ground and examined (fig. 7), and where surface scaling was appreciable a thin turning was chemically analyzed for comparison with the original analyses. Oxide accumulations in the interface zone were also observed (fig. 8). The specimens were heavily etched and approximate measurements of total diffusion

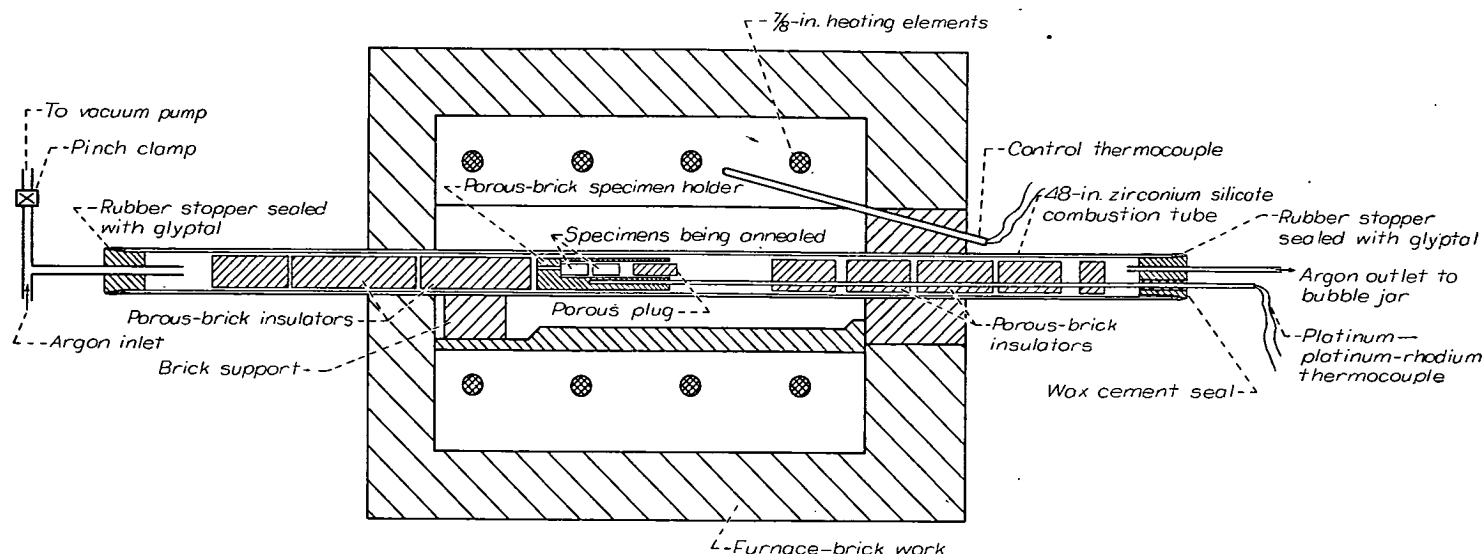


FIGURE 5.—Sectional view of annealing furnace showing specimen holder and thermocouple placement.

zones were made with a micrometer eyepiece. This measurement was made to determine the number of machining cuts required to cover the diffusion zone.

Machining successive layers through diffusion zone.—Before machining successive layers through the diffusion zone, excess metal was removed from the low-chromium half of the specimen to establish a reference surface from which cuts for analyses were made. From 0.1 to 0.125 inch (2.54 to 3.175 mm) of metal was left between the weld interface (determined metallographically) and the reference surface. Tungsten carbide cutting tools were used for all machining operations.

A series of turnings was made across the entire cross section starting at the reference surface perpendicular to the axis of the cylinder. After each turning, the distance machined was measured and the chips were gathered for chemical analysis. The cuts farthest from the interface were 0.010 inch thick (0.254 mm); cuts were reduced to a thickness from 0.003 to 0.005 inch (0.0762 to 0.127 mm) in the diffusion zone. In all the specimens except specimen 3, the entire zone was covered by the smallest cuts made.

Three methods of machining successive cuts through the diffusion zone were used. In the first method, used for specimens 2 to 7, the cylinders with machined reference surfaces were chucked in a large rigid lathe. The head of this lathe contained a large cast-iron face plate that was machined for use as a surface plate. All cuts less than 0.007 inch thick were measured from the surface plate with a surface dial gage that could be read to 0.0001 inch and estimated to 0.00001 inch. Thus, removing the work from the chuck after each cut was unnecessary. A calibration of the dial showed that cuts of 0.003 inch needed no corrections and that cuts of 0.005 inch were in error by as little as 0.00006 inch, or approximately 1 percent. If human errors are taken into consideration, the maximum machining errors were believed to be no more than 2 percent. These

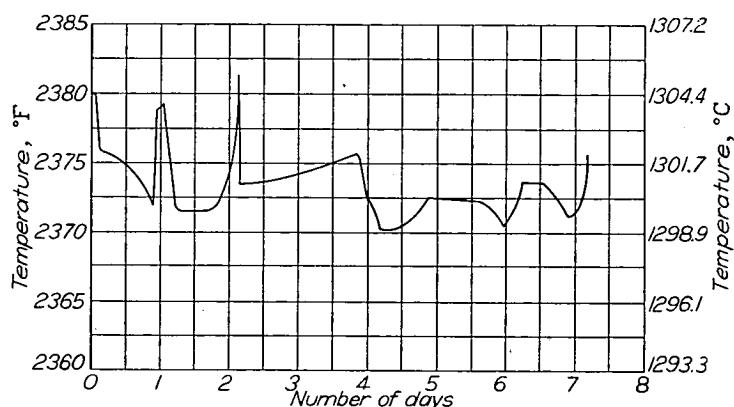
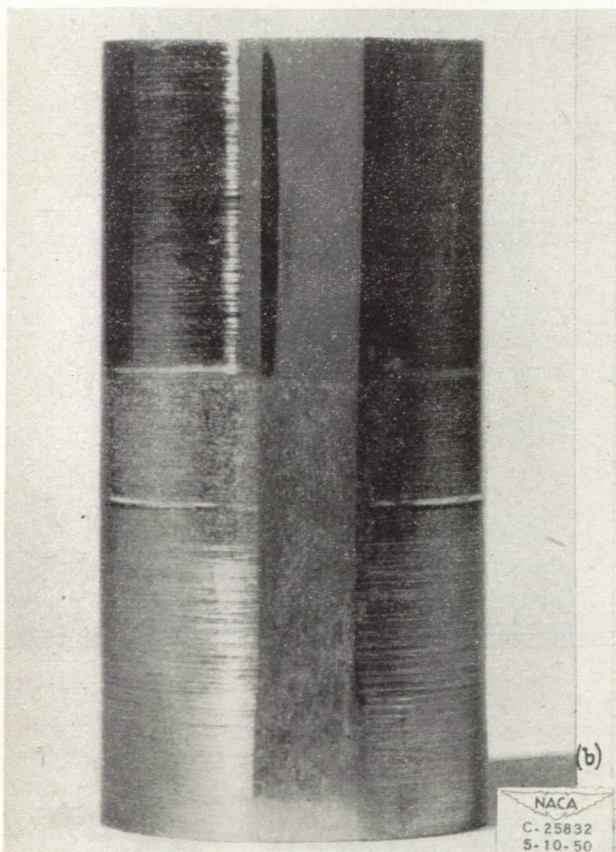
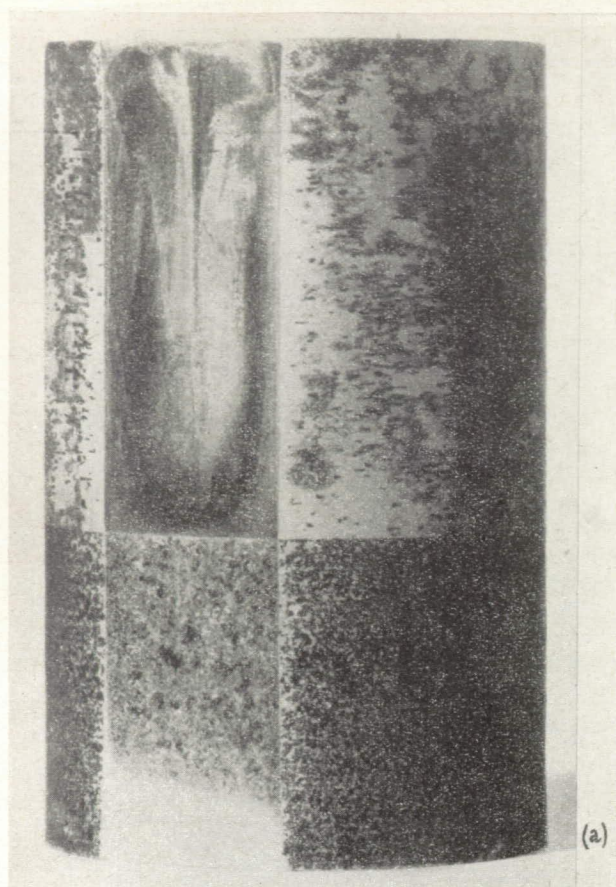


FIGURE 6.—Typical time-temperature plot for diffusion-anneal period in annealing furnace (specimen 4). Temperatures have not been corrected with platinum-platinum-rhodium wire calibration.

dial corrections were considered negligible and have been omitted in the calculations.

The second method, similar to the preceding one, was used for specimens 1, 8, and 9. The chips were machine off with a high-precision vertical jig borer (SIP Hydrostatic-7), which was kept in an air-conditioned room, and a calibrated surface gage was used to measure the cuts. Machining errors were measured with a micrometer and were determined by comparing the total amount of metal removed after making all the necessary cuts with the sum of all the small cuts measured with the surface dial gage. In specimen 8, with a total of 48 cuts, the error was 0.25 percent, whereas in specimen 9, the error was 2 percent in the same number of cuts.

The third method, used for specimens 10 and 11, was the most accurate as well as the most complicated. The vertical jig borer was again used, but the thickness of each cut was measured by an electronic height gage that measured the differences in height between stacks of gage blocks and the specimen surface to 0.00001 inch. As a further assurance that no discrepancies would occur, a height gage (surface



(a) Scale present after an anneal of several days. Scaling occurred chiefly during loading and unloading of annealing furnace.
 (b) Specimen machined to smaller diameter than specimen in figure 7 (a). Another flat has been ground for metallographic examination and turnings have been taken from lower portion for chemical analysis.

FIGURE 7.—Specimen after diffusion anneal.

plate checker) was also used in conjunction with the electronic gage. With this gage, it was possible to measure the over-all height of the specimen to 0.00001 inch after each layer was machined off. Differences in over-all height were compared with the thickness of each cut as measured with the electronic height gage and the stacks of gage blocks. In the two specimens so machined, the error was 0 percent in 36 cuts for specimen 11 and 0.025 percent in 32 cuts for specimen 10.

Chemical analyses.—At least two and usually three analyses by a persulfate-oxidation method were made at the Lewis laboratory for every sample machined from the diffusion zone.

Methods of calculating diffusion coefficients.—The diffusion coefficients for all specimens were calculated by the Grube method (reference 3). In addition, calculations using the Matano method (reference 3) were made for specimens 3 and 5. The Matano method was not used throughout because it was extremely time consuming and the results closely paralleled those calculated by the Grube method.

The metric system was used in plotting concentration-penetration curves and diffusion coefficients were calculated in this system to conform with previous work. The data points were plotted at distances halfway between successive cuts, which were measured from a reference plane. Leveling-off points of the curves were drawn from mathematical averages of several data points, some of which are not shown in the figures.

Diffusion coefficients were calculated from the concentration-penetration curves using the following equation (Grube method):

$$\frac{C-C_0}{C_1-C_0} = \frac{1}{2} \left(1 \pm \frac{2}{\sqrt{\pi}} \int_0^{\omega} e^{-\omega^2} d\omega \right)$$

where

C	chromium concentration, atomic percent
C_0	chromium concentration of low-chromium bar, atomic percent
C_1	chromium concentration of high-chromium bar, atomic percent
ω	$= \frac{x}{2\sqrt{Dt}}$
D	diffusion coefficient, (cm ² /sec)
x	distance from Grube interface, (cm)
t	time, (sec)

Positive values were used for calculations made from the high-chromium portions of the concentration-penetration curves to the right of the Grube interface, whereas negative values were used for calculations made from the low-chromium portions of the curves to the left of the Grube interface.

The Grube interface is the distance in the diffusion zone at which the concentration is halfway between the lower and upper concentrations C_0 and C_1 , respectively (figs. 9 (d) and 9 (f)).

Let

$$H\left(\frac{x}{2\sqrt{Dt}}\right) = \pm \frac{2}{\sqrt{\pi}} \int_0^{\omega} e^{-\omega^2} d\omega$$

$$\omega = \frac{x}{2\sqrt{Dt}}$$

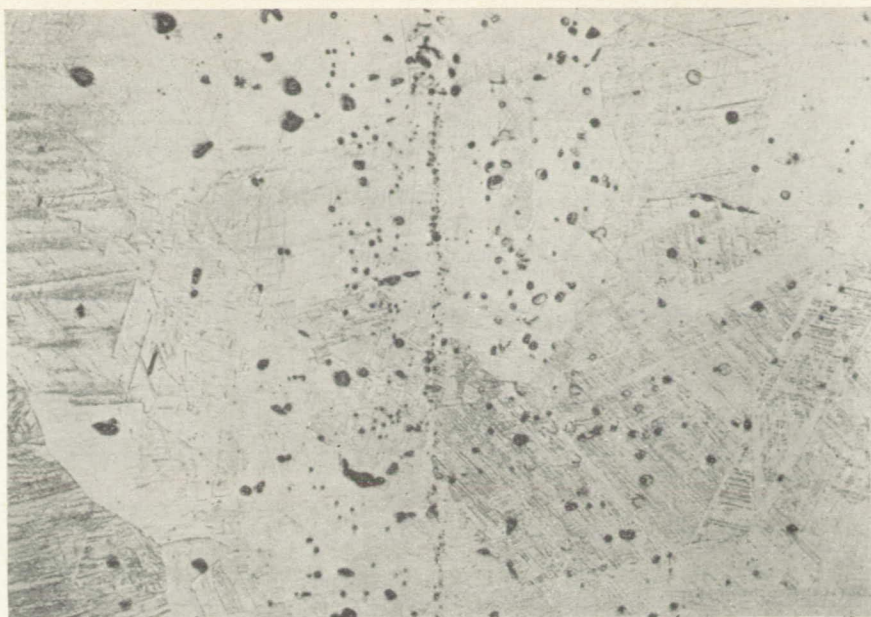


FIGURE 8.—Central portion of diffusion zone of specimen 10. Annealed 7.17 days at 1297° C; etchant, 5-percent nitric acid in ethyl alcohol followed by aqua regia in glycerine; magnification, $\times 100$. Photograph shows accumulation of oxides near center of diffusion zone. (Magnification increased 6.2 percent in printing.)

Then

$$\frac{C-C_0}{C_1-C_0} = \frac{1}{2} \left[1 \pm H(\omega) \right]$$

$$H(\omega) = \pm 2 \left(\frac{C-C_0}{C_1-C_0} - 0.5 \right)$$

The value of ω may be obtained from probability tables and inasmuch as

$$\omega = \frac{x}{2\sqrt{Dt}}$$

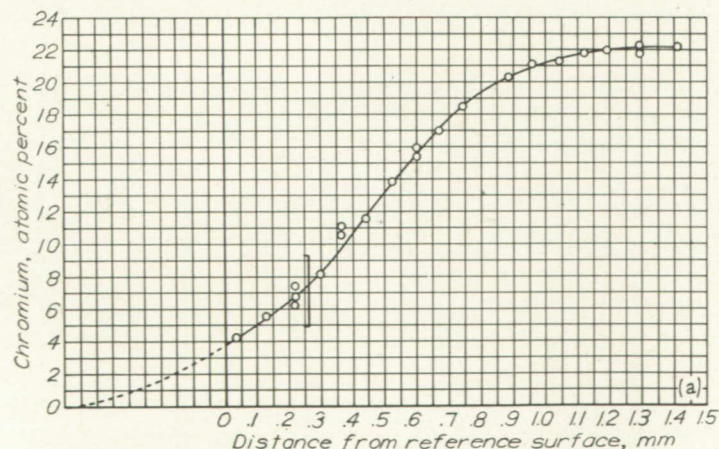
then

$$D = \frac{x^2}{\omega^2} \frac{1}{4t}$$

The final equation indicates that D is very sensitive to x , the machined distance.

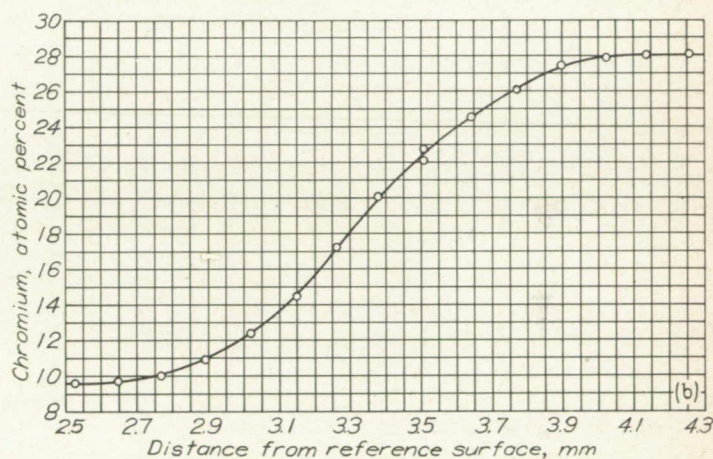
RESULTS

Concentration-penetration curves of all specimens are plotted in figure 9. Several of these curves, such as those for specimens 3, 4, 7, 8, and 10 (figs. 9 (d), 9 (e), 9 (i), 9 (j), and 9 (g), respectively) are very symmetrical. When data from these curves are plotted on a probability graph in the form

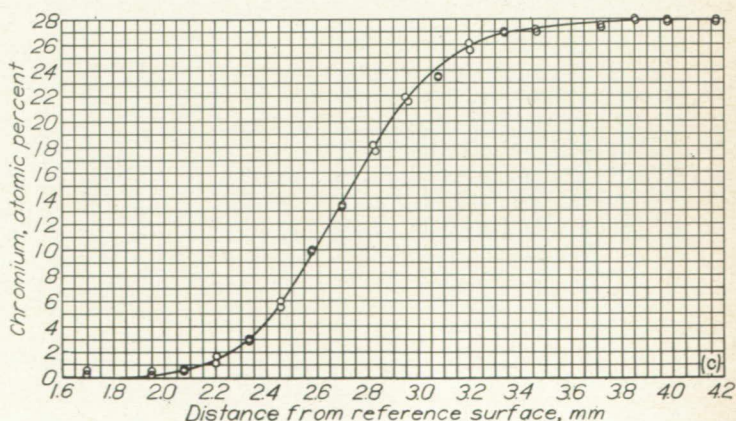


(a) Specimen 1; annealed at 1360° C for 3.384×10^5 seconds (3.92 days). Data to left of bracket are inaccurate because of quench crack that formed after diffusion took place.

FIGURE 9.—Concentration-penetration curves.



(b) Specimen 2; annealed at 1360° C for 3.384×10^5 seconds (3.92 days).



(c) Specimen 11; annealed at 1360° C for 3.321×10^5 seconds (3.84 days).

FIGURE 9.—Continued. Concentration-penetration curves.

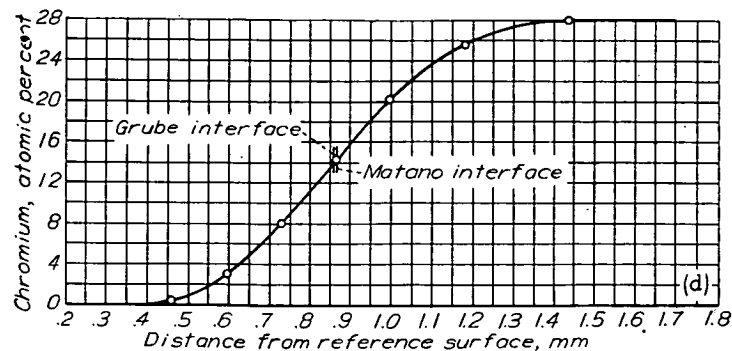
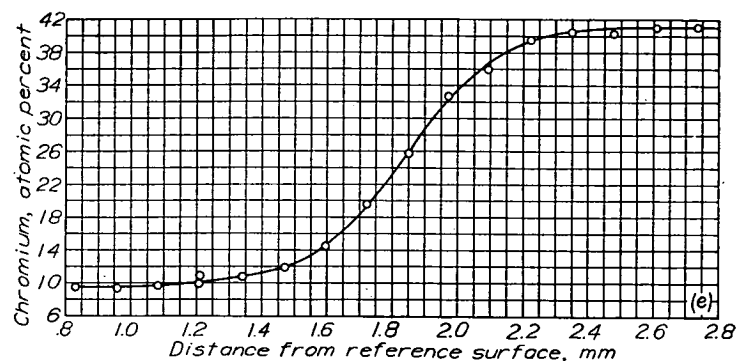
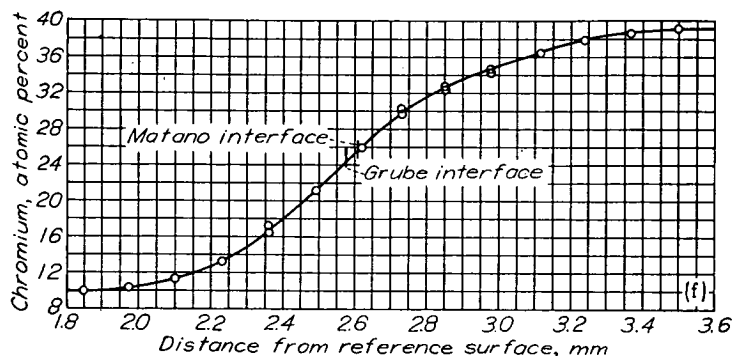
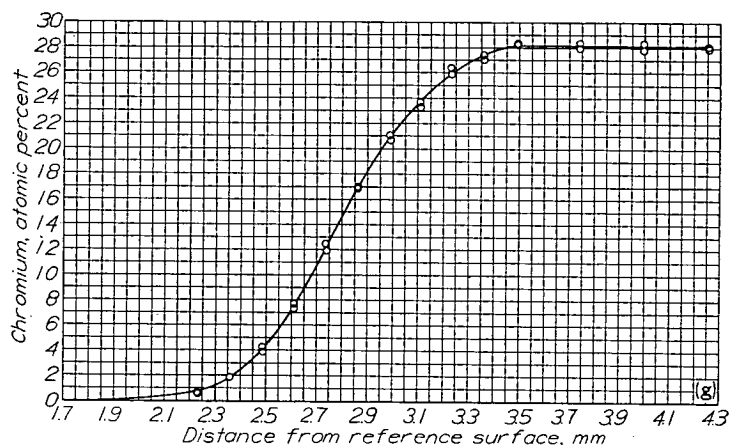
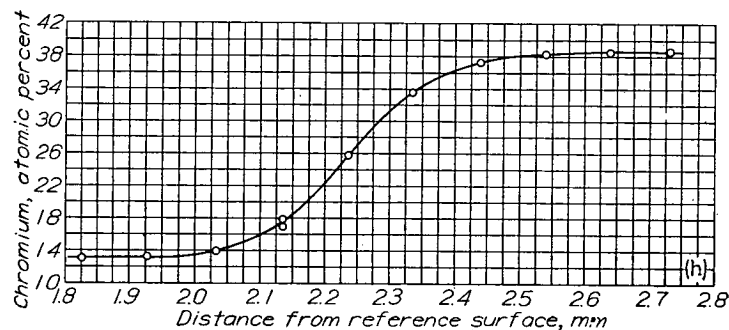
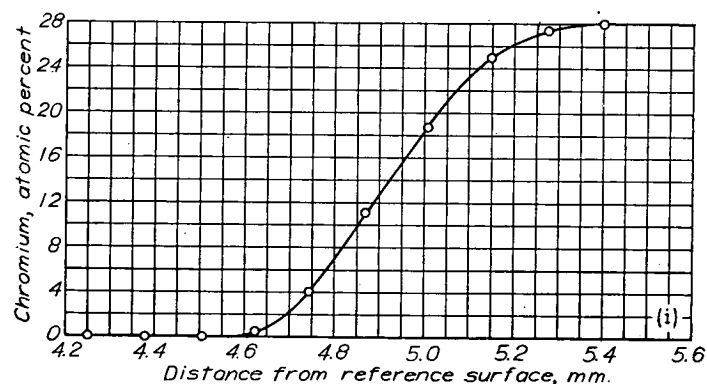
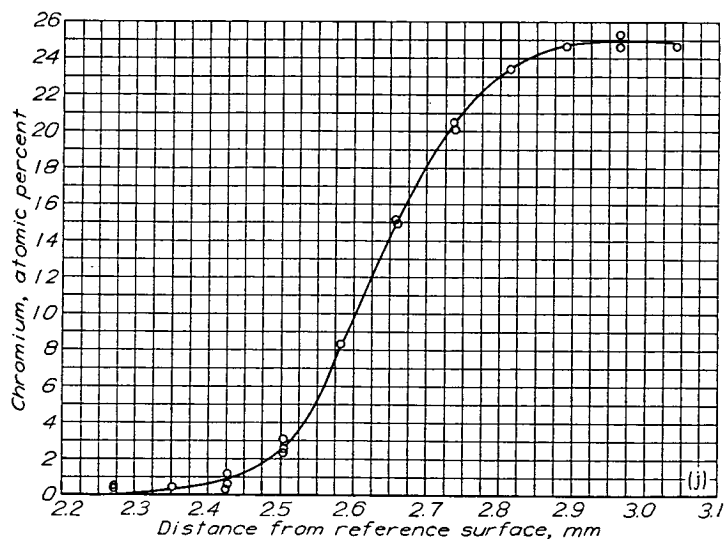
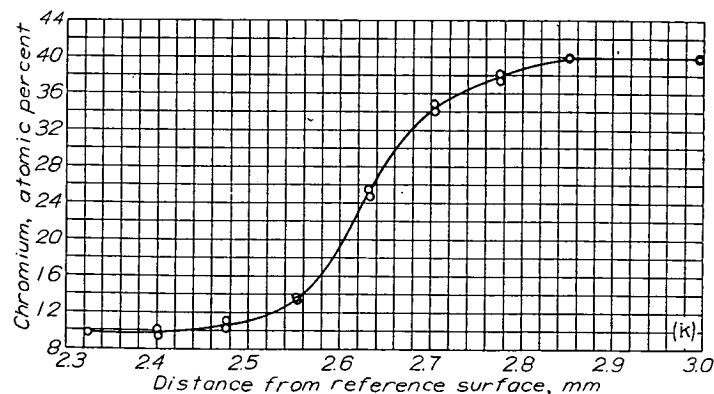
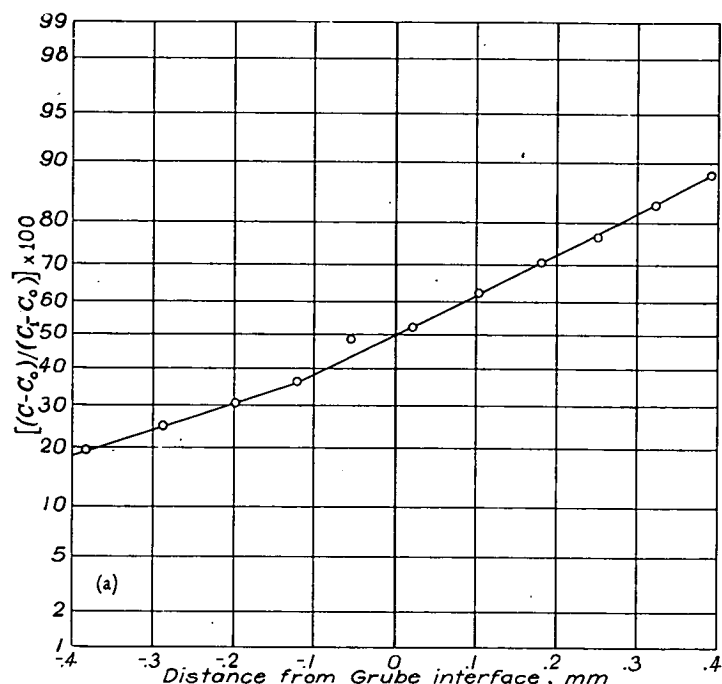
(d) Specimen 3; annealed at 1303° C for 3.522×10^4 seconds (4.08 days).(e) Specimen 4; annealed at 1298° C for 6.216×10^4 seconds (7.19 days).(f) Specimen 5; annealed at 1299° C for 9.549×10^4 seconds (11.05 days).(g) Specimen 10; annealed at 1297° C for 6.198×10^4 seconds (7.17 days).(h) Specimen 6; annealed at 1150° C for 1.0446×10^6 seconds (12.09 days).(i) Specimen 7; annealed at 1151° C for 2.152×10^6 seconds (24.91 days).(j) Specimen 8; annealed at 1000° C for 7.664×10^6 seconds (87.70 days).(k) Specimen 9; annealed at 1000° C for 7.664×10^6 seconds (87.70 days).

FIGURE 9.—Concluded. Concentration-penetration curves.

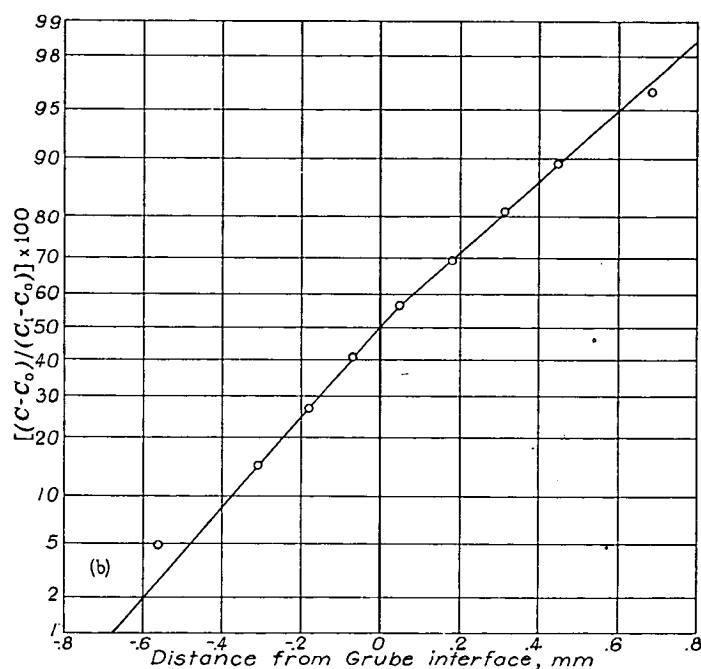
$(C-C_0)/(C_1-C_0)$ against the distance from the Grube interface, approximately straight lines are obtained (fig. 10). In these cases, the diffusion coefficient D is almost invariant with the concentration C . The curves pass through the point $(x=0, \frac{C-C_0}{C_1-C_0} \times 100 = 50)$.

Other curves, such as that for specimen 5 (fig. 9 (f)), bend slightly from a smooth curve in the upper right, or high-chromium portions, of the diffusion curves. These bends also show up in the probability plots (fig. 10 (f)). Deviations from

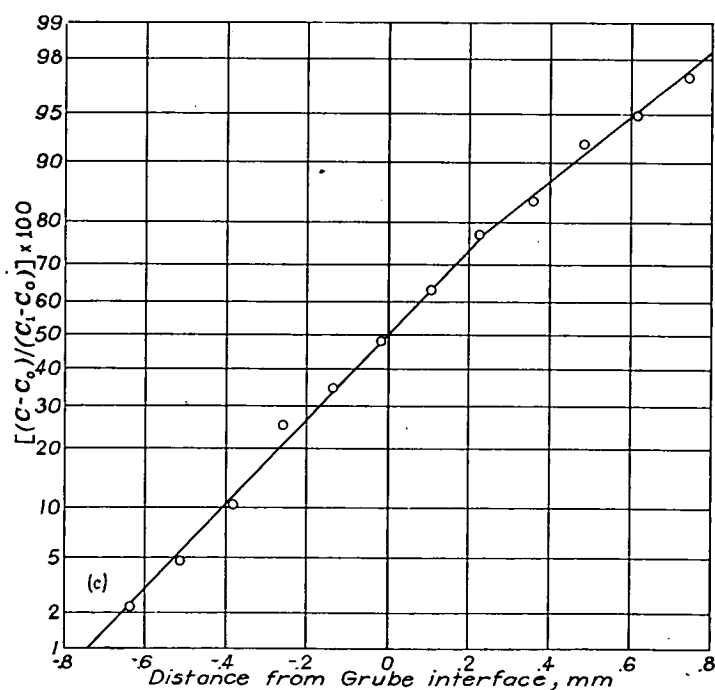
smooth concentration-penetration curves and from straight lines in the probability plots could be a function of concentration. The deviations, however, are believed to be caused by oxide segregations that may have been present in the high-chromium half of the welded specimens before the diffusion anneal or that may have formed during the annealing by migrations of oxides that were originally randomly scattered throughout the matrix of the metal. The metallographic examinations and the smooth lower portions of most of the curves indicate that migration is the most probable case.



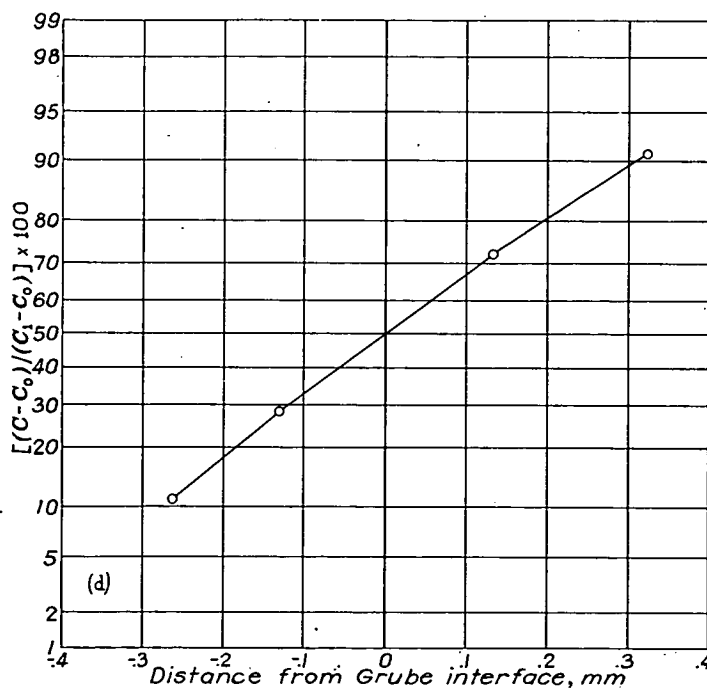
(a) Specimen 1; annealed at 1360° C for 3.92 days.



(b) Specimen 2; annealed at 1360° C for 3.92 days.

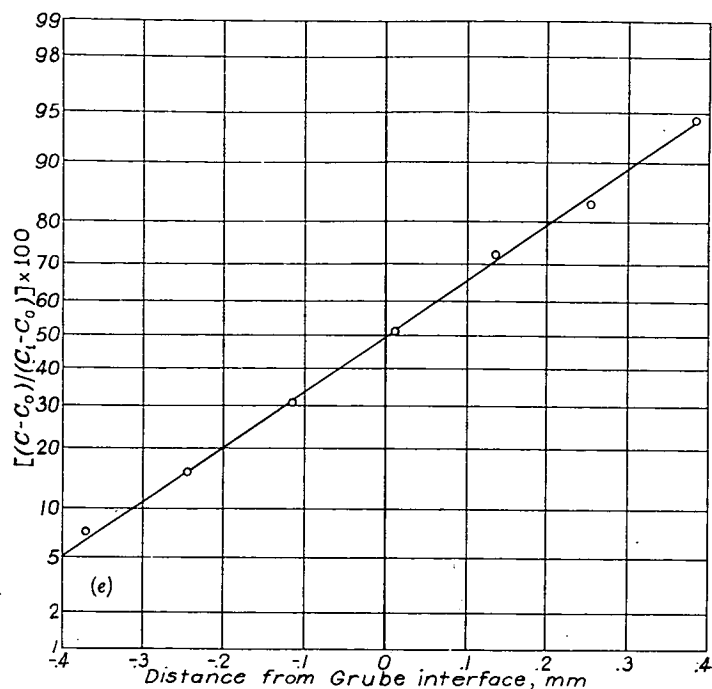


(c) Specimen 11; annealed at 1369° C for 3.84 days.

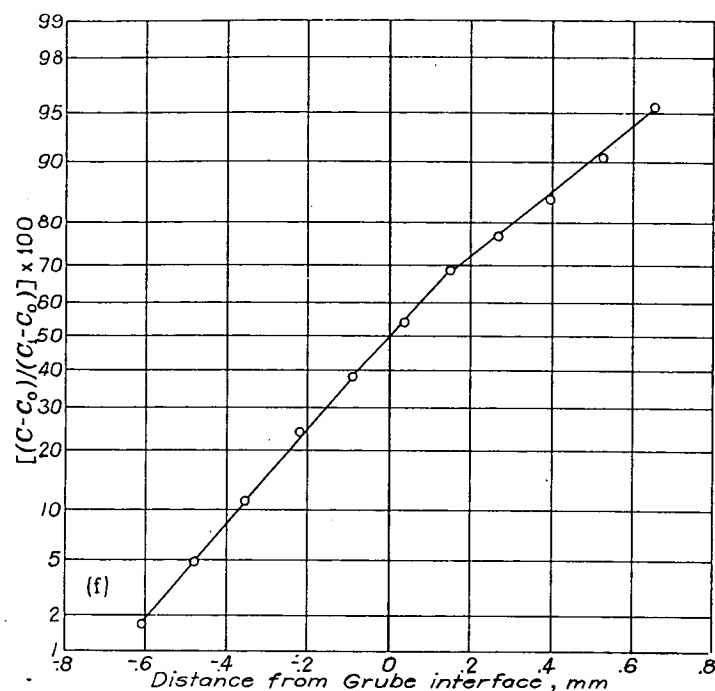


(d) Specimen 3; annealed at 1303° C for 4.08 days.

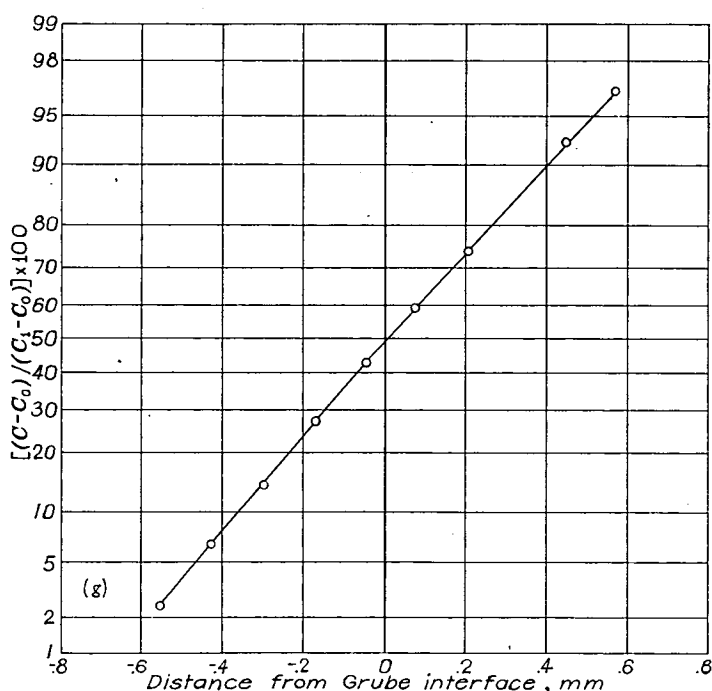
FIGURE 10.—Plots of $(C-C_0)/(C_1-C_0)$ against distance from Grube interface.



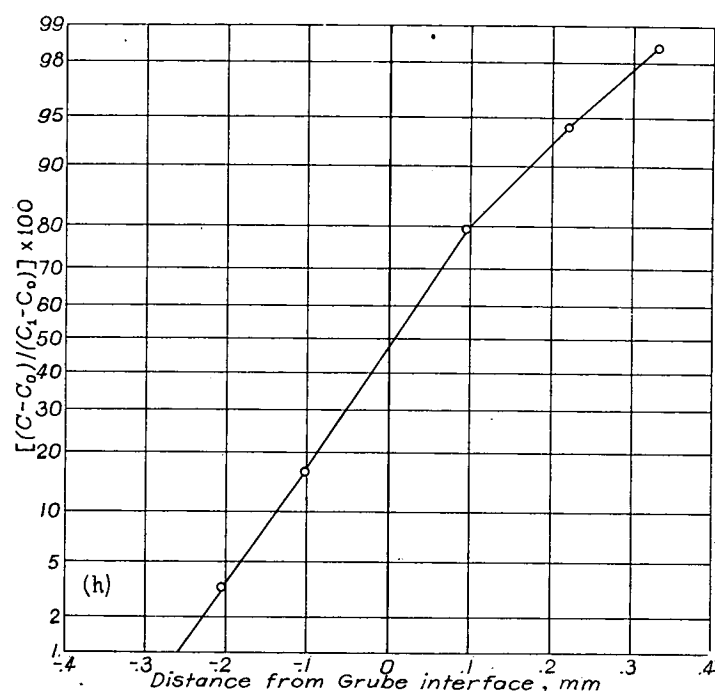
(e) Specimen 4; annealed at 1298° C for 7.19 days.



(f) Specimen 5; annealed at 1299° C for 11.05 days.



(g) Specimen 10; annealed at 1297° C for 7.17 days.



(h) Specimen 6; annealed at 1150° C for 12.09 days.

FIGURE 10.—Continued. Plots of $(C-C_0)/(C_1-C_0)$ against distance from Grube interface.

Diffusion coefficients calculated by the Grube method from the concentration-penetration curves are plotted against concentration in figure 11. Values of diffusion coefficients obtained from the extreme ends of the diffusion-penetration curves and from portions near the Grube interfaces were omitted from the plot because they are inherently inaccurate. The degree of reproducibility of data appears good, as shown in the cases of specimens 3 and 10, and 4 and 5, which were annealed at $1300^\circ \pm 3^\circ$ C.

Specimens 1 and 2, prior to annealing at 1360° C, were annealed at 900° C for several weeks. Meanwhile, diffusion

coefficients were obtained at 1300° and 1150° C and rough calculations from these preliminary data showed that an annealing time of about 8 years at 900° C would be required to produce a diffusion zone comparable in thickness to the smallest zone obtained at the higher temperatures. Inasmuch as the annealing time at 900° C was insignificantly small, the specimens were reannealed at 1360° C to obtain additional data. Because the specimens were originally intended for heat treatment at 900° C, the composition range covered by specimens 1 and 2 is small compared with what would be possible at 1360° C.

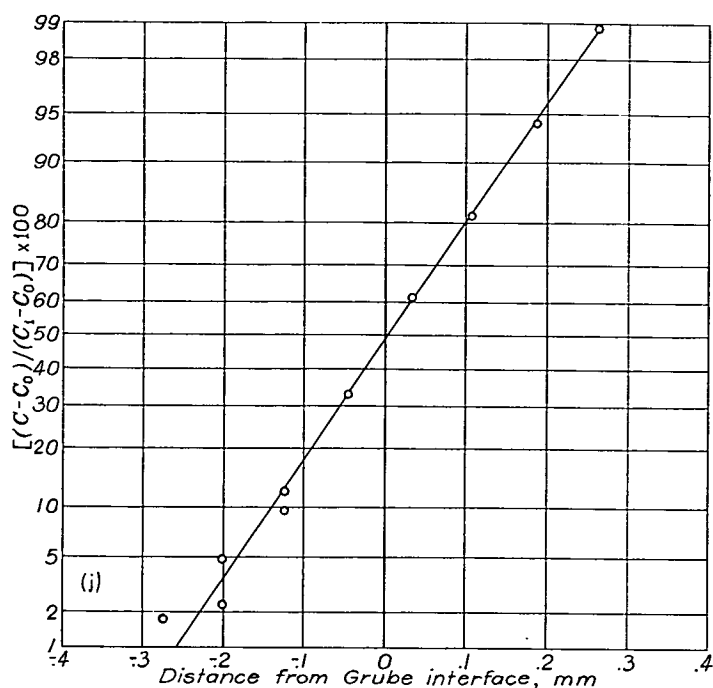
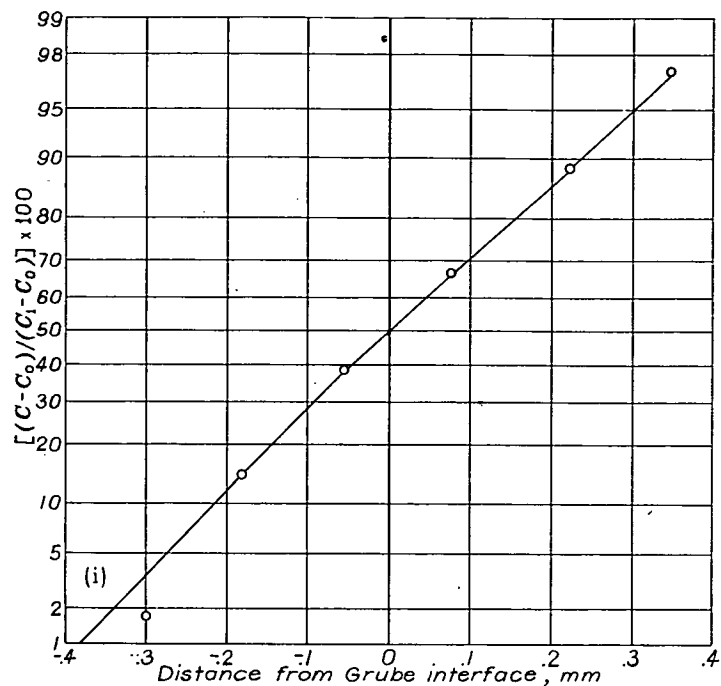
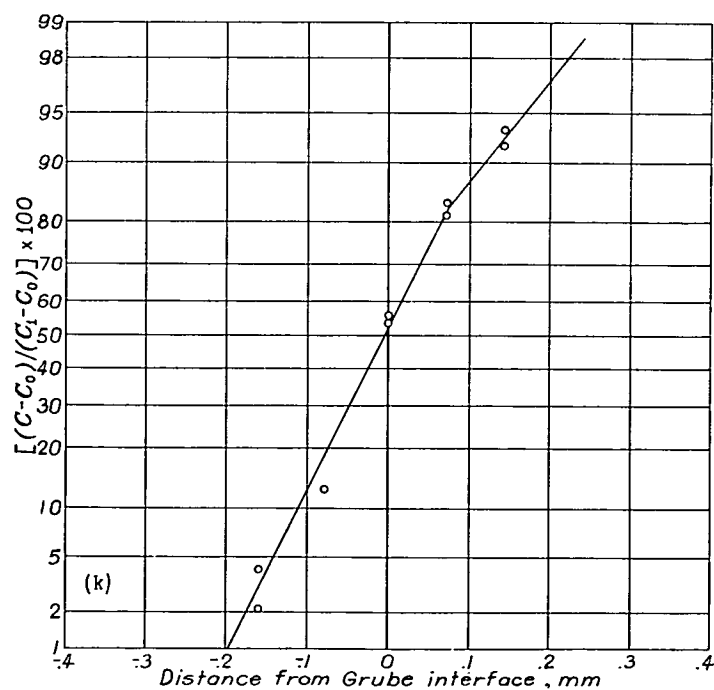
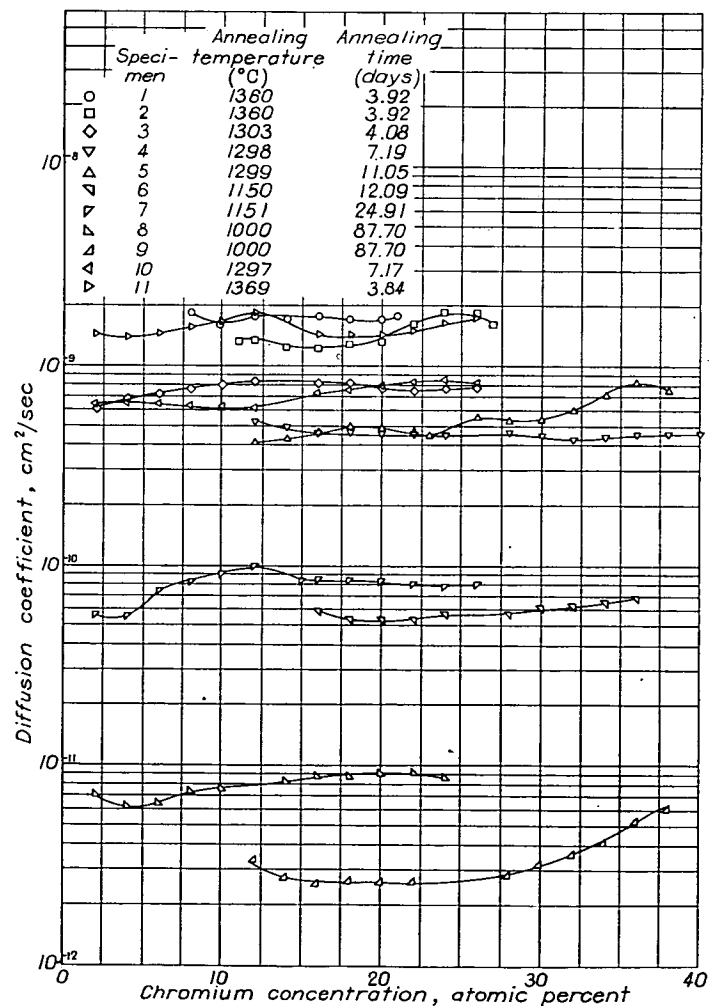

 FIGURE 10.—Continued. Plots of $(C-C_0)/(C_1-C_0)$ against distance from Grube interface.

 FIGURE 10.—Concluded. Plots of $(C-C_0)/(C_1-C_0)$ against distance from Grube interface.


FIGURE 11.—Diffusion coefficients against concentration. (Data to left of 8 atomic percent not plotted for specimen 1 because of quench crack that formed after diffusion took place.)

The upper half of the curve of specimen 1 (fig. 9 (a)) is satisfactory and calculations were made from it by the Grube method. The lower portion, however, was inaccurate as a result of a quench crack formed by cooling the specimen in an air blast as it was removed from the annealing furnace.

The composition of specimen 9 extends into the $\alpha + \gamma$ field (fig. 1), but because it is close to the α -phase boundary, only a small percentage of γ would be present. Since the γ -phase was not observed during the metallographic examinations, the calculated diffusion coefficients would therefore be reasonably accurate.

In spite of several metallographic inspections, a strip of oxide $\frac{1}{2}$ inch deep extending halfway about the periphery of specimen 11 was noticed as it was being machined. Approximately 5.4 percent of the area of the interface was blocked. Inasmuch as this area is a small portion of the total area, the data for specimen 11 were not discarded.

A plot of diffusion coefficients against concentration (fig. 12) shows the small differences between values determined by the Grube and the Matano methods.

The relation between diffusion coefficients and reciprocal temperatures are shown in figure 13. The upper and lower curves are plotted from data obtained for 16-atomic-percent chromium because these values are close to averages of the flat portions of the curves of figure 11. The upper curve represents diffusion coefficients obtained from specimens of cobalt welded to cobalt-chromium alloys, whereas the lower curve represents alloys approximately 9- to 14-atomic-percent chromium welded to alloys of approximately 28- to 41-atomic-percent chromium.

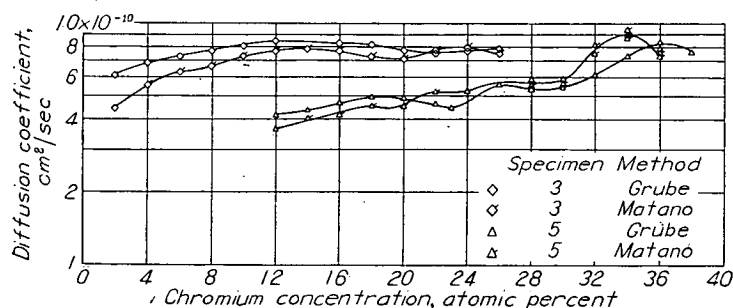


FIGURE 12.—Comparison of diffusion coefficients calculated by the Grube and Matano methods.

By use of the dashed curve, which was drawn halfway between the upper and lower solid curves in figure 13, and of the exponential equation (reference 1)

$$D = Ae^{-\frac{Q}{RT}}$$

the activation heat of diffusion Q and the constant A were determined to be

$$Q = 63,600 \text{ (cal/mole)}$$

$$A = 0.443 \text{ (cm}^2\text{/sec)}$$

The value of Q was determined from the slope of the dashed curve because

$$2.303 \frac{(\log D_1 - \log D_2)}{\frac{1}{T_1} - \frac{1}{T_2}} = -\frac{Q}{R}$$

and A was determined from the preceding exponential equation, using the value of Q obtained and values of D and $1/T$ chosen from the dashed curve.

The diffusion behavior of chromium in α cobalt-chromium alloys may be compared with that of other metal systems in figure 14, where values from the dashed curve of figure 13 are replotted for the comparison. The curves for the other alloy systems were taken from the graphs of references 3 to 9.

DISCUSSION OF RESULTS

Diffusion coefficients are relatively constant within the concentration ranges of each specimen. Appreciable differences between diffusion coefficients, however, were obtained from specimens of pure cobalt welded to α cobalt-chromium alloys and specimens of low-chromium α -alloys welded to high-chromium α -alloys at each nominal annealing temperature; the diffusion coefficients for specimens of pure cobalt welded to α cobalt-chromium alloys were larger (figs. 11 and 13).

The differences in diffusion coefficients are not the results of such experimental variables as:

- (1) Differences in sources of materials because single supplies of cobalt and chromium were used in this study
- (2) Differences in annealing temperatures because specimens 1 and 2 and specimens 8 and 9 were annealed together at identical temperatures and the remaining specimens were annealed at temperatures very close to the desired nominal temperatures
- (3) Differences due to the closeness of the high-chromium specimens to the α - γ region because specimens 1 and 2 were well within the α region

A somewhat similar observation may be made from figure 14, where curves 4 and 5 represent the diffusion of pure copper into pure gold, and copper from a gold-copper alloy into gold, respectively.

The experimental data give additional confirmation to the Dushman-Langmuir equation

$$D = \frac{Q}{Nh} \delta^2 e^{-\frac{Q}{RT}}$$

where

D diffusion coefficient, (cm²/sec)

Q activation heat of diffusion, (cal/mole)

N Avogadro's number

h Planck's constant, (cal-sec)

δ distance of closest approach of atoms, $\frac{a_0}{\sqrt{2}}$, (cm)

a_0 lattice parameter

R gas constant, (cal/(mole)(°K))

T temperature, (°K)

The following value of Q has been calculated by arbitrarily selecting a point on the dashed curve of figure 13 to obtain a D value for a given temperature:

$$Q = 63,400 \text{ (cal/mole)}$$

Values used in the calculation are as follows:

$$D = 6 \times 10^{-11} \text{ (cm}^2\text{/sec)}$$

$$\frac{1}{^\circ K} = 7.1 \times 10^{-4}$$

$$a_0 = 3.54 \times 10^{-8} \text{ (cm) (reference 10)}$$

The value of Q compares favorably with the value of $Q = 63,600$ (cal/mole) obtained from the slope of the dashed curve.

The diffusion rates of chromium in α cobalt-chromium were very low compared with those of most alloy systems for which diffusion data had been previously obtained. For

example, it can be determined from figure 14 that a high temperature (1031°C) is required to produce $\log D = -11$ for chromium in cobalt, whereas annealing temperatures of 362° , 628° , and 784°C are required for Mg in Al, Zn in Cu, and Pt in Au, respectively. The sluggishness of diffusion is congruous with the fact that cobalt-chromium base alloys have good high-temperature characteristics.

The activation heat of diffusion for chromium in α cobalt-chromium alloys is greater than that for most systems previously investigated, with tungsten alloys being the chief exceptions (reference 11). Higher activation heats of diffusion show up in the form of steeper slopes on the curve of $\log D$ against $1/^\circ K$ (fig. 14).

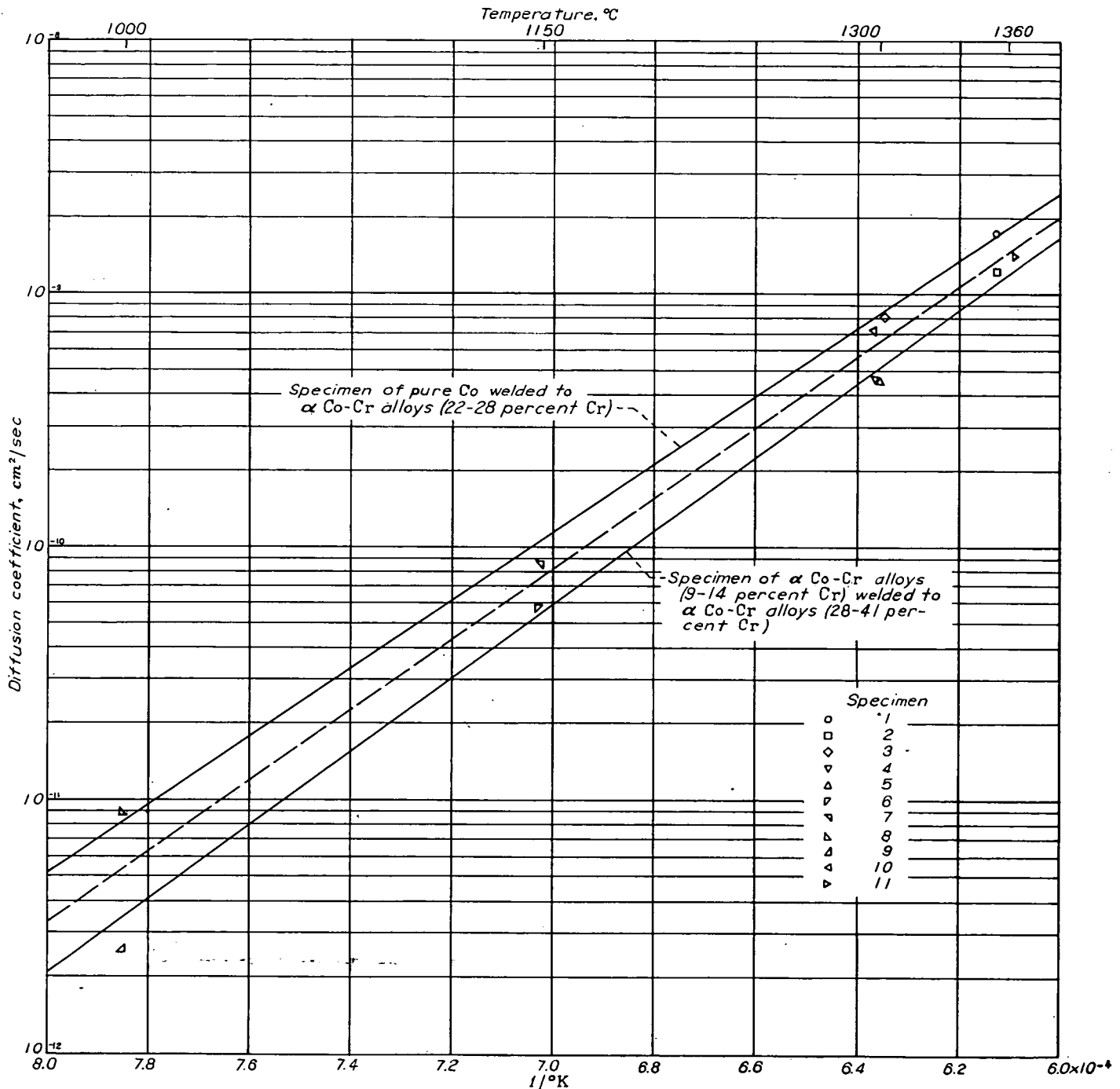


FIGURE 13.—Diffusion coefficients against $1/^\circ K$. Diffusion coefficients were plotted from data for 16-atomic-percent chromium.

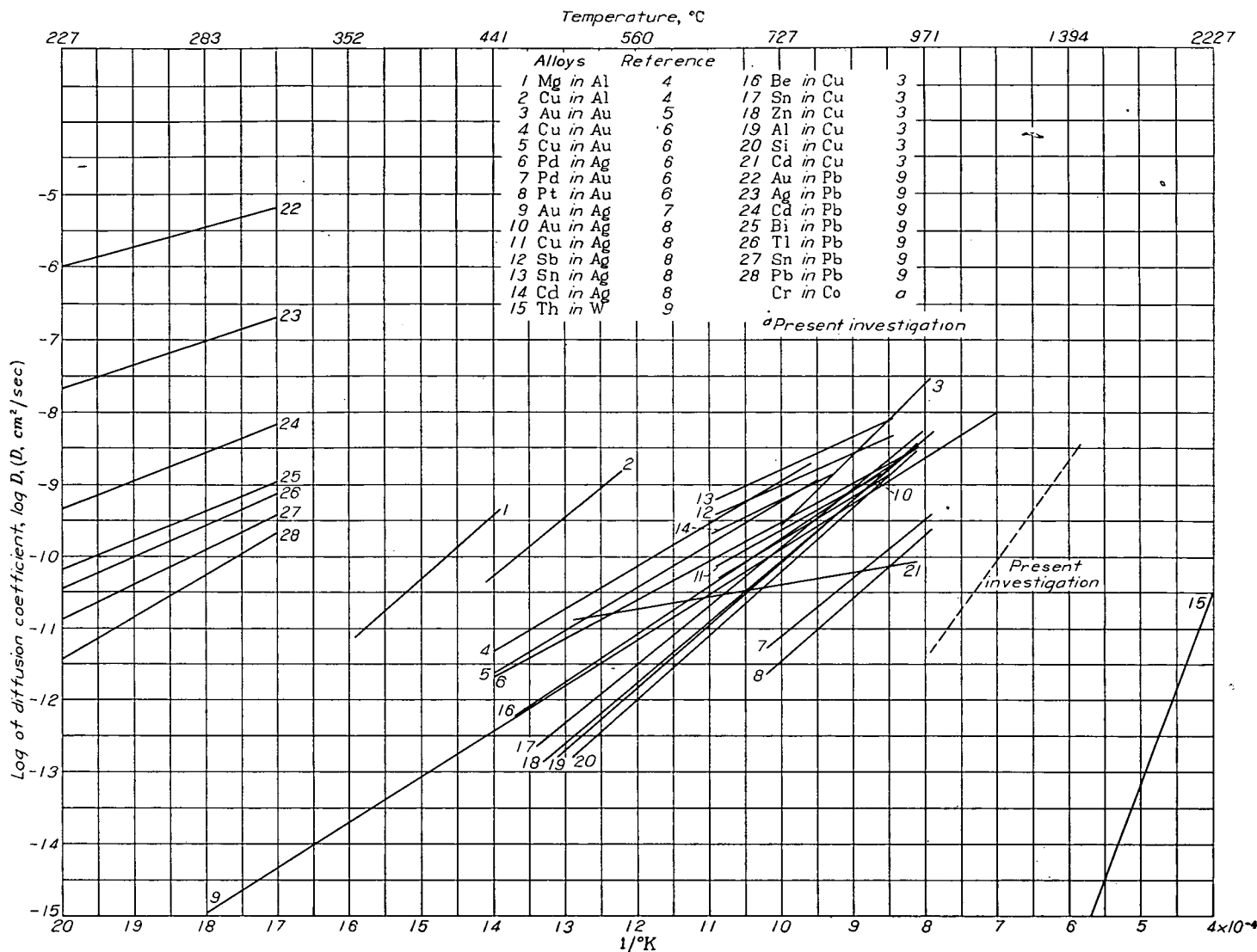


FIGURE 14.—Comparison of diffusion behavior of chromium in α cobalt-chromium alloys with other substitutional solid-solution alloys.

SUMMARY OF RESULTS

Diffusion of chromium in α cobalt-chromium solid solutions was investigated at temperatures of 1360°, 1300°, 1150°, and 1000° C and the following results obtained:

1. The exponential equation relating diffusion coefficient D and temperature T is as follows:

$$D = 0.443e^{-\frac{63,600}{RT}}$$

where R is the gas constant. The activation heat of diffusion is 63,600 calories per mole.

2. When compared with the diffusion data previously obtained by other investigators for most alloy systems, the diffusion rates of chromium in α cobalt-chromium were found to be low; considerably higher temperatures were required to produce diffusion coefficients of the same order

of magnitude as were previously found for most substitutional alloys.

3. Diffusion coefficients are relatively constant within the concentration ranges covered by each specimen.

4. Chromium diffusivity from α cobalt-chromium alloys into pure cobalt is greater than chromium diffusivity from high-chromium α -alloys to low-chromium α -alloys for all concentration gradients studied.

5. The results of this investigation are further confirmation of the Dushman-Langmuir equation. The value for the activation heat of diffusion calculated from this equation agrees closely with the experimentally determined value (63,400 cal/mole as calculated against 63,600 cal/mole obtained from experimentation).

LEWIS FLIGHT PROPULSION LABORATORY

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

CLEVELAND, OHIO, June 23, 1950

REFERENCES

1. Wells, Cyril, and Mehl, Robert F.: Rate of Diffusion of Carbon in Austenite, in Plain Carbon, in Nickel and in Manganese Steels. Trans. A. I. M. E., Iron and Steel Div., vol. 140, 1940, pp. 279-306.
2. Elsea, A. R., Westerman, A. B., and Manning, G. K.: The Cobalt-Chromium Binary System. Metals Tech., vol. 15, no. 4, June 1948, pp. 1-24.
3. Rhines, Frederick N., and Mehl, Robert F.: Rates of Diffusion in the Alpha Solid Solutions of Copper. Trans. A. I. M. E., Inst. Metals Div., vol. 128, 1938, pp. 185-221; discussion, pp. 221-222.
4. Brick, R. M., and Phillips, Arthur: Diffusion of Copper and Magnesium into Aluminum. Trans. A. I. M. E., Inst. Metals Div., vol. 124, 1937, pp. 331-347; discussion, pp. 347-350.
5. McKay, H. A. C.: The Self-Diffusion Coefficient of Gold. Trans. Faraday Soc (London), vol. XXXIV, 1938, pp. 845-849.
6. Jost, W.: Die Diffusionsgeschwindigkeit einiger Metalle in Gold und Silber. Zeitschr. Phys. Chem., Bd. 21, Heft 1 und 2, Abt. B, April 1933, S. 158-160.
7. Jost, W.: Über den Platzwechselmechanismus in festen Körpern. Die Diffusion von Gold und Silber. Zeitschr. Phys. Chem., Bd. 9, Heft 1, Abt. B, Juli 1930, S. 73-82.
8. Jost, W.: Diffusion und Chemische Reaktion in festen Stoffen. Theodor Steinkopff (Dresden und Leipzig), 1937, S. 115. (Reproduced by Edwards Bros. (Ann Arbor, Mich.), 1943.)
9. Mehl, Robert F.: Diffusion in Solid Metals. Trans. A. I. M. E., Inst. Metals Div., vol. 122, 1936, pp. 11-56.
10. Anon.: Metals Handbook, 1948 Edition. Am. Soc. Metals (Cleveland), 1948, p. 1191.
11. Barrer, Richard M.: Diffusion in and through Solids. Cambridge Univ. Press (London), 1941, p. 275.

TABLE I—ANNEALING DATA

Specimen	Weighted average annealing temperature (°C)	Annealing time		Approximate thickness of scale formed during annealing (in.)
		(days)	(sec)	
1	1360.4	3.917	3.384×10^4	0.002
2	1360.4	3.917	3.384×10^3	.002
3	1302.9	4.08	3.522×10^3	$\frac{1}{16}$
4	1297.8	7.194	6.216×10^3	$\frac{1}{32}$
5	1299.2	11.05	9.549×10^3	<.001
6	1149.7	12.09	1.0446×10^4	.010
7	1150.5	24.907	2.152×10^4	$\frac{1}{64}$
8	1000.3	87.704	7.664×10^4	.001-.002
9	1000.3	87.704	7.664×10^4	.001-.002
10	1296.9	7.174	6.198×10^3	.002
11	1368.9	3.844	3.321×10^3	.003

REPORT 1024

CALCULATION OF THE LATERAL CONTROL OF SWEEP AND UNSWEEP FLEXIBLE WINGS OF ARBITRARY STIFFNESS¹

By FRANKLIN W. DIEDERICH

SUMMARY

A method similar to that of NACA Rep. 1000 is presented for calculating the effectiveness and the reversal speed of lateral-control devices on swept and unswept wings of arbitrary stiffness. Provision is made for using either stiffness curves and root-rotation constants or structural influence coefficients in the analysis. Computing forms and an illustrative example are included to facilitate calculations by means of the method.

The effectiveness of conventional aileron configurations and the margin against aileron reversal are shown to be relatively low for swept wings at all speeds and for all wing plan forms at supersonic speeds.

INTRODUCTION

Adequate lateral control constitutes one of the more significant design requirements for airplanes. The ability of the airplane to enter a roll is determined by the control power and is measured by the maximum available rolling moment resulting from lateral-control deflection. A measure of the degree of lateral maneuverability is the helix angle at the wing tips corresponding to the highest rate of roll; the lateral maneuverability depends both on the control power and the damping in roll.

The control power and the damping in roll are affected by structural flexibility. Control deflection ordinarily gives rise to aerodynamic loads which tend to deform the wing structure in such a way as to reduce the loads on it and thus to reduce the control power. If the dynamic pressure of the air stream is sufficiently high, the amount of lift which results from the structural deformation may be sufficient to nullify the effect of the control deflection. The speed and dynamic pressure corresponding to this condition are known as the aileron reversal speed or reversal dynamic pressure, since at a slightly higher dynamic pressure a control deflection in a given direction would result in a rolling moment in a direction opposite to that of the moment on a similar rigid wing.

The present report is concerned with an analysis of these problems for swept and unswept wings of arbitrary stiffness. The method is based on the analysis of loading of flexible wings presented in reference 1. Since suitable aerodynamic influence coefficients are not yet available for antisymmetric

lift distributions, aerodynamic-induction effects on the lift distribution are taken into account only as an over-all correction and as a slight reduction of the load at the tip, as in reference 1. The method is formulated in such a manner, however, that aerodynamic influence coefficients may easily be included as soon as they become available.

The numerical analysis required in any given practical case constitutes an extension of the calculations outlined in reference 1. The computing forms for the additional calculations required for an analysis of lateral-control effectiveness or reversal are included in the present report. Their use is described in the section entitled "Application of the Method," which may be read without reference to the derivation of the method. The material presented in reference 1 which is pertinent to the present analysis is included herein. As an example illustrating the method, the lateral-control effectiveness and reversal of the wing considered in reference 1 are analyzed in this report. The reversal speeds of several wings derived from this wing by shifting the elastic axis and rotating the wing are calculated to demonstrate some general effects of sweep on the aileron reversal speed.

SYMBOLS

A	aspect ratio (b^2/S)
$[A], [A']$	aeroelastic matrices defined by equations (7) and (10)
$[\bar{A}], [\bar{A}']$	auxiliary aeroelastic matrices defined by equations (8) and (11)
$[A_R]$	aileron-reversal matrix defined by equation (18)
a	section aerodynamic center, measured from leading edge, fraction of chord
b	wing span, inches
b'	wing span less fuselage width, inches
c	chord measured parallel to the air stream, inches
$\bar{c}$	average wing chord, inches (S/b)
c_l	section lift coefficient (l/qc)
$c_{l\alpha}$	section lift-curve slope, per radian
$c_{m_{c/4}}$	section pitching-moment coefficient referred to quarter-chord point
$C_{L\alpha}$	wing lift-curve slope, per radian

¹ Supersedes NACA RM LSH24a, "Calculation of the Effects of Structural Flexibility on Lateral Control of Wings of Arbitrary Plan Form and Stiffness" by Franklin W. Diederich, 1948.

$C_{L_{\alpha_e}}$	effective lift-curve slope for twist distributions, per radian	S	total wing area including part of wing covered by fuselage, square inches
C_l	rolling-moment coefficient $\left(\frac{\text{Rolling moment}}{qSb}\right)$	s_m	trace of matrix $[A_R]^m$
$[C_3']$	matrix converting torques due to distributed loads to torques due to concentrated torques	t	running torque due to air load about axes perpendicular to the plane of symmetry, inch-pounds per inch
$[C_4']$	matrix converting bending moments due to distributed loads to bending moments due to concentrated loads	w	fuselage width, inches
cp_s	section center of pressure due to aileron deflection measured from leading edge of chord, fraction of chord	w_e	distance between the effective root and the innermost complete section of the torsion box perpendicular to the elastic axis, inches
EI	bending stiffness in planes perpendicular to the elastic axis, pound-inches ²	y	lateral ordinate measured from plane of symmetry, inches
e	location of elastic axis measured from leading edge, fraction of chord	α_s	angle of attack due to structural deformation, radians
e_1	dimensionless distance along chord from reference axis to section aerodynamic center ($e-a$)	α_δ	angle of attack equivalent to unit aileron deflection $\left(\frac{dc_l/d\delta}{dc_l/d\alpha}\right)$
e_2	dimensionless distance along chord from reference axis to section center of pressure due to aileron deflection (cp_s-e)	δ	aileron deflection measured in planes parallel to the direction of flight, radians
g	factor proportional to the rolling-moment coefficient due to aileron deflection defined by equation (16)	ϵ	moment-arm ratio (e_2/e_1)
GJ	torsional stiffness in planes perpendicular to the elastic axis, pound-inches ²	η	lateral distance from wing root, inches
$[I], [I']$	integrating matrices for single integration from tip to root	Λ	angle of sweepback (measured to the reference axis unless specified otherwise), degrees
$[II], [II']$	integrating matrices for double integration from tip to root	$[\Phi_P]$	influence-coefficient matrix for wing twist in planes parallel to the air stream due to concentrated unit loads applied at the reference axis, radians per pound
$[I'_1], [II'_1]$	first rows of matrices $[I']$ and $[II']$, respectively	$[\Phi_T]$	influence-coefficient matrix for wing twist in planes parallel to the air stream due to concentrated unit torques applied in planes parallel to the air stream, radians per inch-pound
$[II'_0]$	integrating matrix defined by equation (14)		
k	dimensionless parameter $\left(\frac{(GJ)_r}{(EI)_r} \frac{b'/2}{e_1 c_r \cos^2 \Lambda} \tan \Lambda\right)$	Subscripts:	
κ	wing lift-curve-slope ratio ($C_{L_{\alpha_e}}/C_{L_{\alpha}}$)	$c/2$	midchord
l	running air load per unit length perpendicular to the plane of symmetry, pounds per inch	D	divergence
M_0	free-stream Mach number	fw	flexible wing
$\frac{pb}{2V}$	wing-tip helix angle	i	inboard end of aileron
$Q_{\alpha_M}, Q_{\alpha_T}$	root-twist constants	o	outboard end of aileron
q	dynamic pressure, pounds per square inch	p	damping in roll
q^*	dimensionless dynamic pressure $\left(\frac{C_{L_{\alpha}} q (b'/2)^2 e_1 c_r^2 \cos \Lambda}{(GJ)_r}\right)$	R	reversal
$q^\dagger$	reduced dynamic pressure $\left(C_{L_{\alpha}} q \frac{b'}{2} c_r\right)$	R_0	reversal of unswept wing
$\bar{q}$	dimensionless dynamic pressure $\left(\frac{C_{L_{\alpha}} q (b'/2)^3 c_r \tan \Lambda}{(EI)_r \cos \Lambda}\right)$	r	at root or effective root
		rw	rigid wing
		sub	subsonic
		spr	supersonic
		w	wing exclusive of fuselage
		δ	due to (unit) aileron deflection
		Matrix notation:	
		$\{ \}$	column matrix
		$[\]$	row matrix
		$[\]$	square matrix
		$[\]$	diagonal matrix

[]'' double transpose of a matrix: first about the principal diagonal, then about the other diagonal

[1] unit matrix

$$[1_1'] = \begin{bmatrix} 0 & 0 & 0 & 0 & . & . \\ 1 & 0 & 0 & 0 & . & . \\ 1 & 0 & 0 & 0 & . & . \\ 1 & 0 & 0 & 0 & . & . \\ 1 & 0 & 0 & 0 & . & . \\ . & . & . & . & . & . \end{bmatrix}$$

DERIVATION OF THE METHOD

ASSUMPTIONS

The assumptions made in reference 1 are that all deformations and angles of attack are small and that the wing deformations either are known in the form of structural influence coefficients or can be calculated from simple beam theory in conjunction with rotations of a flexible root. In addition the assumption is made in the present report that the angle between the aileron and the wing is constant along the span of the aileron and that, in the absence of suitable aerodynamic influence coefficients, aeroelastic effects due to aileron deflection can be calculated on the basis of a modified strip theory.

AIR LOADS

The lift on a wing section of unit width parallel to the direction of flight may be expressed in terms of the loading coefficient cc_l/c_r as

$$\{l\} = qc_r \left\{ \frac{cc_l}{c_r} \right\} \quad (1)$$

In the case of a wing at zero angle of attack with ailerons deflected the lift may be considered to consist of two parts: one due to structural deformation and one due to aileron deflection. The loading coefficients for these two parts are best treated separately.

The part of the lift due to the antisymmetric structural twist can be written in terms of the local values of twist as

$$\left\{ \frac{cc_l}{c_r} \right\} = C_{L_\alpha} [Q_a] \{\alpha_s\} \quad (2)$$

in terms of suitable antisymmetric aerodynamic influence coefficients $[Q_a]$. Since no such coefficients are available at present, modified strip theory may be used, as in reference 1, with some saving in labor but at some sacrifice in accuracy. With this modified strip theory

$$\left\{ \frac{cc_l}{c_r} \right\} = C_{L_\alpha} \left[\frac{c}{c_r} \right] \{\alpha_s\} \quad (3)$$

where for subsonic speeds the approximate value of effective lift-curve slope C_{L_α} may be obtained from the equation

$$C_{L_\alpha} = c_{l_\alpha} \frac{A \cos \Lambda}{A + 4 \cos \Lambda}$$

and for supersonic speeds,

$$C_{L_\alpha} = \frac{4 \cos \Lambda_{c/2}}{\sqrt{M_0^2 \cos^2 \Lambda_{c/2} - 1}}$$

c_{l_α} being the lift-curve slope of the section perpendicular to the quarter-chord line at a Mach number equal to $M_0 \cos \Lambda$.

The lift due to aileron deflection should be calculated by a fairly accurate method (see references 2 and 3). This lift distribution may be expressed as

$$\begin{aligned} \left\{ \frac{cc_l}{c_r} \right\}_\delta &= \alpha_\delta \delta C_{L_\alpha} \left\{ \frac{cc_l}{c_r C_{L_\alpha}} \right\}_\delta \\ &= \alpha_\delta \delta C_{L_\alpha} \left[\frac{c}{c_r} \right] \left\{ \frac{c_l}{C_{L_\alpha}} \right\}_\delta \end{aligned}$$

where the coefficients $\left(\frac{c_l}{C_{L_\alpha}} \right)_\delta$ are the values of c_l calculated for a unit effective aileron deflection $\alpha_\delta \delta$ and divided by C_{L_α} .

The combined lift due to twist and aileron deflection is then

$$\{l\} = qc_r C_{L_\alpha} \left[\frac{c}{c_r} \right] \left\{ \alpha_s + \alpha_\delta \delta \left\{ \frac{c_l}{C_{L_\alpha}} \right\}_\delta \right\} \quad (4)$$

The torque per unit width of span is the product of the lift per unit width and the local moment arm. For the lift due to twist the moment arm is $e_1 c$; for the lift due to aileron deflection the moment arm is $-e_2 c$. (See fig. 1.) Consequently, the torque may be written as

$$\{t\} = qc_r^2 C_{L_\alpha} e_1 \left[\frac{e_1}{e_1} \left(\frac{c}{c_r} \right)^2 \right] \left\{ \alpha_s - \alpha_\delta \delta [\epsilon] \left\{ \frac{c_l}{C_{L_\alpha}} \right\}_\delta \right\} \quad (5)$$

where ϵ is the ratio of the moment arms e_2/e_1 .

THE AEROELASTIC EQUATION

Method employing stiffness curves.—The lifts and torques per unit width given in the preceding section can be integrated to obtain bending moments and accumulated torques about axes perpendicular and parallel to the plane of symmetry and, hence, about axes perpendicular and parallel to the elastic axis. Integration of these torques and moments yields the twists and local dihedrals, which can be combined to yield the desired angle of attack due to

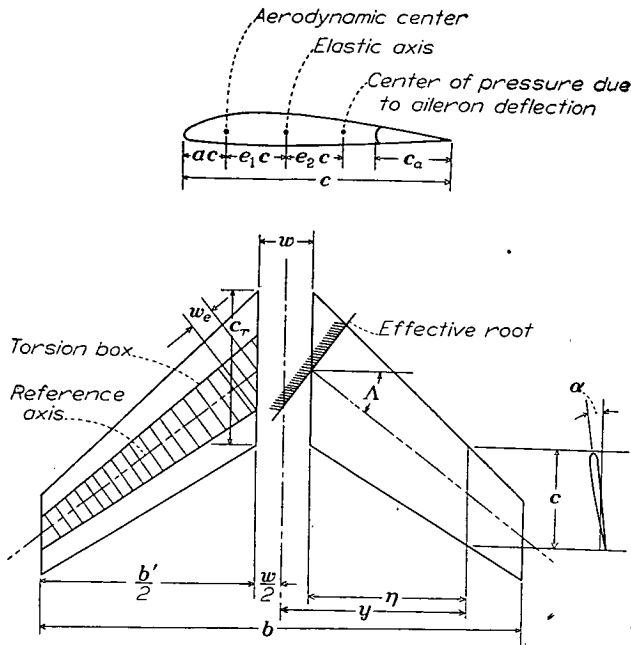


FIGURE 1.—Definition of geometrical parameters used in the analysis.

structural deformation α_s . (See reference 1.) The resulting equation is

$$\{\alpha_s\} = \kappa q^* \left\{ [A] \{\alpha_s\} - \alpha_s \delta [\bar{A}] \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\} \right\} \quad (6)$$

where κ , the ratio of the effective to the actual wing lift-curve slope, is

$$\kappa = \frac{A + 2 \cos \Lambda}{A + 4 \cos \Lambda}$$

for subsonic flow and has a value of 1 for supersonic flow, q^* is the dimensionless dynamic pressure, $[A]$ is the aeroelastic matrix defined in reference 1 (for subsonic flow) as

$$[A] = \left[[I]'' \left[\frac{(GJ)_r}{GJ} \right] + \frac{w_e}{b'/2} (Q_{\alpha_T} - Q_{\alpha_M} \tan \Lambda) [1_1'] \right. \\ \left. + \frac{(GJ)_r}{(EI)_r} \tan^2 \Lambda [I]'' \left[\frac{(EI)_r}{EI} \right] [I'] \left[\frac{e_1}{e_{1r}} \left(\frac{c}{c_r} \right)^2 \right] \right. \\ \left. - \left[k [I]'' \left[\frac{(EI)_r}{EI} \right] - \frac{w_e}{b'/2} \frac{b'/2}{e_{1r} c_r \cos^2 \Lambda} Q_{\alpha_M} [1_1'] \right] [II'] \left[\frac{c}{c_r} \right] \right] \quad (7)$$

and $[\bar{A}]$ is an auxiliary aeroelastic matrix defined (for subsonic flow) by

$$[\bar{A}] = \left[[I]'' \left[\frac{(GJ)_r}{GJ} \right] + \frac{w_e}{b'/2} (Q_{\alpha_T} - Q_{\alpha_M} \tan \Lambda) [1_1'] \right. \\ \left. + \frac{(GJ)_r}{(EI)_r} \tan^2 \Lambda [I]'' \left[\frac{(EI)_r}{EI} \right] [I'] \left[\frac{e_1}{e_{1r}} \left(\frac{c}{c_r} \right)^2 \right] [\epsilon] \right. \\ \left. + \left[k [I]'' \left[\frac{(EI)_r}{EI} \right] - \frac{w_e}{b'/2} \frac{b'/2}{e_{1r} c_r \cos^2 \Lambda} Q_{\alpha_M} [1_1'] \right] [II'] \left[\frac{c}{c_r} \right] \right] \quad (8)$$

For supersonic flow, matrices $[I]$ and $[II]$ are used instead of matrices $[I']$ and $[II']$, provided that the decrease in lift

at the tip is known; otherwise, matrices $[I']$ and $[II']$ may be used in supersonic flow also as a fair approximation.

Method employing structural influence coefficients.—If the angle-of-attack changes due to unit concentrated normal loads and torques $[\Phi_P]$ and $[\Phi_T]$ have been determined in static tests of the actual structure or calculated by a method such as that of reference 4, they may be used in an aeroelastic analysis in the manner indicated in reference 1. The pertinent aeroelastic equation for the lateral-control problem is

$$\{\alpha_s\} = \kappa q^* \left\{ [A'] \{\alpha_s\} - \alpha_s \delta [\bar{A}'] \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\} \right\} \quad (9)$$

where $[A']$ is the aeroelastic matrix defined in reference 1 as

$$[A'] = \left[e_{1r} c_r [\Phi_T] [C_3'] \left[\frac{e_1}{e_{1r}} \left(\frac{c}{c_r} \right)^2 \right] + [\Phi_P] [C_4'] \left[\frac{c}{c_r} \right] \right] \quad (10)$$

$[\bar{A}']$ is an auxiliary aeroelastic matrix defined by

$$[\bar{A}'] = e_{1r} c_r [\Phi_T] [C_3'] \left[\frac{e_1}{e_{1r}} \left(\frac{c}{c_r} \right)^2 \right] [\epsilon] - [\Phi_P] [C_4'] \left[\frac{c}{c_r} \right] \quad (11)$$

q^* is the reduced dynamic pressure defined by

$$q^* = C_{L_{\alpha}} q \frac{b'}{2} c_r$$

and the load-conversion matrices $[C_3']$ and $[C_4']$ are defined and given in reference 1.

SOLUTION OF THE AEROELASTIC EQUATION

Calculation of the control effectiveness.—The aerodynamic loading corresponding to a given aileron deflection at a given dynamic pressure can be obtained by writing equation (6) or equation (9) in the form

$$[[1] - \kappa q^* [A]] \{\alpha_s\} = -\kappa q^* \alpha_s \delta [\bar{A}] \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\} \quad (12)$$

Once the right-hand side of equation (12) is evaluated it may be regarded as a set of knowns which in conjunction with the coefficients of the matrix $[[1] - \kappa q^* [A]]$ permits a solution for the unknowns $\{\alpha_s\}$. The loading corresponding to $\{\alpha_s\}$ may then be obtained from equation (4) and hence the net rolling moment due to aileron deflection and the resulting wing deformations:

$$\text{Rolling moment} = 2 \left(\frac{b'}{2} \right)^2 [II'_0] \{\alpha_s\} \quad (13)$$

where

$$[II'_0] = [II'_1] + \frac{w}{b'} [I'_1] \quad (14)$$

As indicated in reference 1, the solution of the set of simultaneous equations represented by equation (12) may be carried out by any conventional method of solving simultaneous equations or by an iteration procedure; the choice of method depending primarily on the preference of the computer.

Calculation of the aileron reversal speed.—The aileron reversal speed can be obtained by calculating the rolling moment due to aileron deflection, as indicated in the

preceding paragraph, and plotting it against the speed or dynamic pressure. The value or values of the speed or dynamic pressure for which the rolling moment is zero constitute the aileron reversal speeds or dynamic pressures. However, a more direct procedure, similar to that used to find the divergence speed in reference 1, can be derived as follows.

The term $\alpha_s \delta \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s$ in equation (12) can be expressed in terms of $\{\alpha_s\}$ by means of equations (4) and (13) since at the aileron-reversal condition the rolling moment is equal to zero. Hence,

$$[II']_0 \left[\frac{c}{c_r} \right] \left\{ \{\alpha_s\} + \alpha_s \delta \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s \right\} = 0$$

or

$$\alpha_s \delta = - \frac{[II']_0 \left[\frac{c}{c_r} \right] \{\alpha_s\}}{[II']_0 \left[\frac{c}{c_r} \right] \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s}$$

Multiplying both sides of this equation by $\left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s$ yields

$$\alpha_s \delta \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s = - \frac{1}{g} \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s [II']_0 \left[\frac{c}{c_r} \right] \{\alpha_s\} \quad (15)$$

where

$$g = [II']_0 \left[\frac{c}{c_r} \right] \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s \quad (16)$$

Substitution of equation (15) in equation (12) yields

$$\{\alpha_s\} = \kappa q^* [A_R] \{\alpha_s\} \quad (17)$$

where the aileron-reversal matrix $[A_R]$ is defined by

$$[A_R] = [A] + \frac{1}{g} [A] \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s [II']_0 \left[\frac{c}{c_r} \right] \quad (18)$$

The value of the dimensionless dynamic pressure κq^* at reversal can be found by iterating equation (17) or by expanding the determinant of the matrix $[1] - \kappa q^* [A_R]$ and setting the resulting polynomial in κq^* equal to 0, as described in reference 1.

APPLICATION OF THE METHOD

SELECTION OF THE PARAMETERS

The geometric and structural parameters used in the calculation of the lateral-control effectiveness and related aerodynamic properties of a wing are the same as those used in the calculations of the aerodynamic loading described in reference 1. If the root-rotation constants are different for symmetric and antisymmetric loadings, those for antisymmetric loadings should be used for calculating the $[A]$ matrix used in this report. Similarly, the section lift-curve slope, wing lift-curve slopes, and local aerodynamic-center values are chosen for the Mach number of interest, as described in reference 1, except for the calculation of the aileron reversal speed, as discussed in the section entitled "Calculation of the Aileron Reversal Speed."

The values of α_s and cp_s are best obtained from experimental section data at the appropriate Mach number $M_0 \cos \Lambda$. These values, in terms of commonly available quantities, are

$$\alpha_s = \frac{\frac{dc_l}{d\delta}}{\frac{dc_l}{d\alpha}}$$

and

$$cp_s = 0.25 - \frac{\frac{dc_{m_{c/4}}}{d\delta}}{\frac{dc_l}{d\delta}}$$

Theoretical thin-airfoil values of these parameters are given in figure 2 for subsonic and supersonic speeds. Insufficient information exists at present to permit an accurate correction of these data for finite-span effects in all cases. In a qualitative sense, the section value of α_s is known to be a useful approximation to the actual value required in the calculations of this report, except for wings of very low aspect ratio, for which the value of α_s tends to be somewhat larger than the section value.

The values of cp_s on a wing tend to be further rearward than the values obtained from section data at subsonic speeds. The finite-span values may be estimated by calculating lift distributions for a given aileron deflection both by unmodified strip theory and by a rational finite-span method, such as that of reference 2 or of reference 3; if the local lift given by strip theory is assumed to act at the section value of cp_s , and if the difference in the lifts given by the two theories is assumed to act at the local aerodynamic center, the chordwise location of the resultant of these two forces may be considered to be the three-dimensional value of cp_s . On the basis of this approximation $e_2 = -e_1$ at all points on the wing not covered by the aileron.

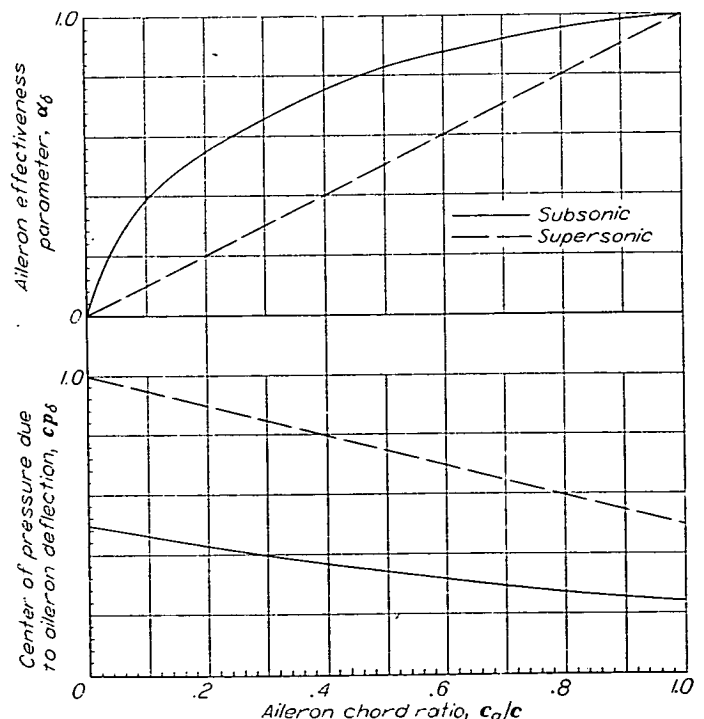


FIGURE 2.—Theoretical thin-airfoil values of the aileron force parameters.

The values of $\left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_i$ required in this report may also be obtained for subsonic speeds by the methods of references 2 or 3 by calculating spanwise lift distributions in the form $\frac{cc_l}{2b}$ for $\alpha\delta=1$ and multiplying the results by $\frac{2b}{cC_{L_{\alpha_e}}}$. A simpler but less accurate way of estimating these values consists in using the modified strip theory, which is also used in this report for the calculation of the lift due to structural deformations, until suitable aerodynamic influence coefficients are available. This approximation implies that the elements of $\left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_i$ are 0 for stations not covered by the aileron and 1 for stations covered by the aileron. However, in order to take into account the location of the inboard and outboard extremities of the aileron with the relatively few stations used in the analysis, equivalent values of $\alpha\delta$ have to be used. These values, referred to as equivalent δ values, are given in figure 3. They are intended to give a rounded-off distribution of $\left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_i$ which has approximately the same area and the same moment about the root as the unmodified strip-theory distribution. The equivalent δ values of figure 3 pertain to actual values of $\alpha\delta$ equal to 1; they apply to ailerons which extend from $\frac{\eta_i}{b'/2}$ to the tip but can be combined to apply to any aileron configuration. Several examples are listed in the following table for the six-point method, the values of $\alpha\delta$ being 1 and the equivalent values being read from figure 3(a) as 0.716 for $\frac{\eta_i}{b'/2}=0.55$ and as 0.293 for $\frac{\eta_i}{b'/2}=0.95$:

Aileron ends	Case 1	Case 2	Case 3	Case 4	Case 5
$\frac{\eta_i}{b'/2}$	0.55	0.95	0.55	0	0
$\frac{\eta_o}{b'/2}$	1.00	1.00	.95	1.00	.55
$\frac{\eta}{b'/2}$	$\left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}$ according to modified strip theory				
	Case 1	Case 2	Case 3	Case 4	Case 5
0	0	0	0	1	1
.20	0	0	0	1	1
.40	0	0	0	1	1
.60	.716	0	.716	1	.284
.80	1	0	1	1	0
.90	1	.293	.707	1	0

The values of case 3 are obtained from those of cases 1 and 2 and the values for case 5, from the ones of cases 1 and 4.

CALCULATION OF THE MATRICES

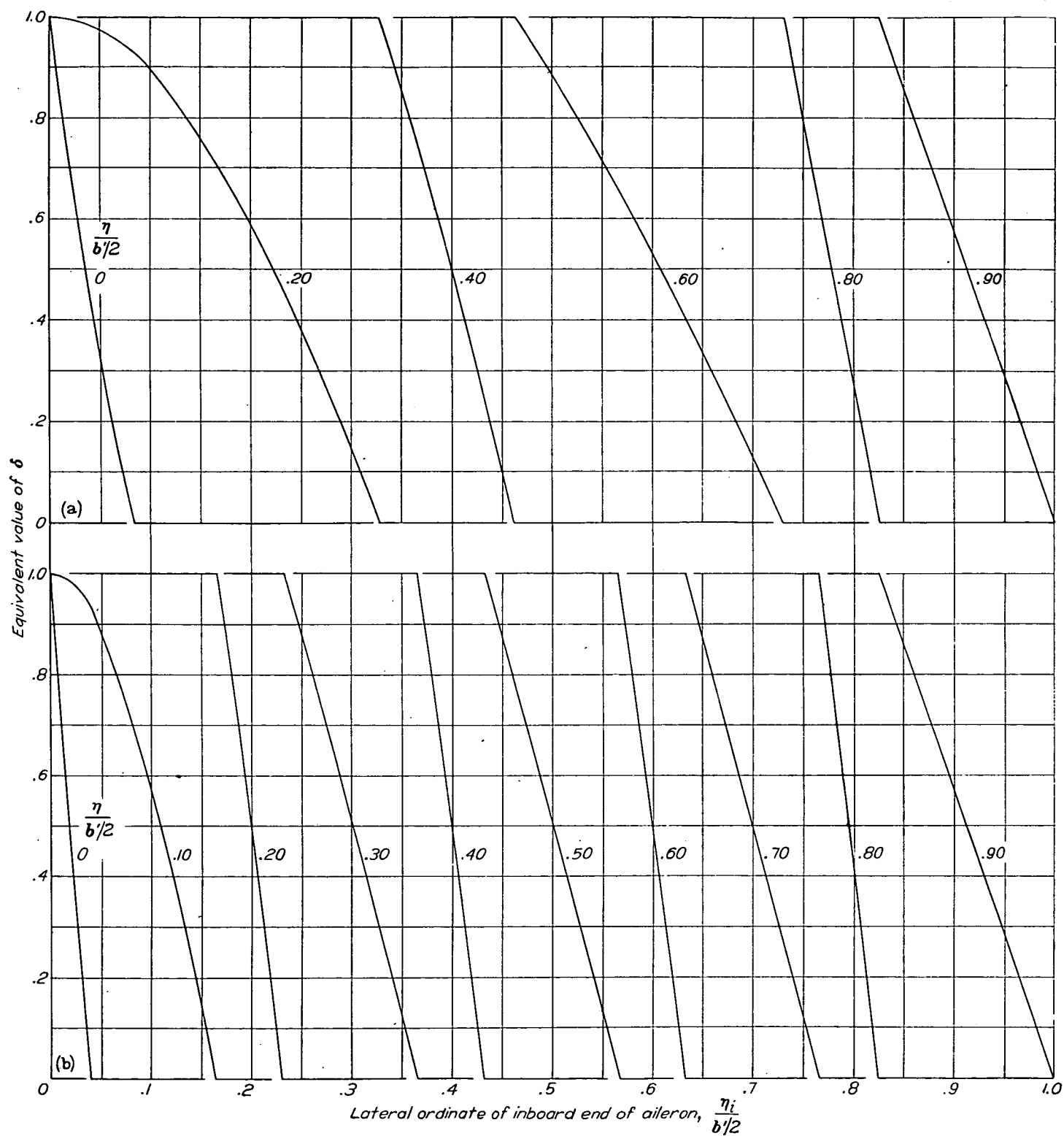
A brief introduction to matrix methods is given in the appendix of reference 1; a fuller treatment is given in reference 5. The numerical constants required in the method of this report are given in reference 1 and in the present report for 6-point and 10-point solutions. The elements of the matrices $[I]$ and $[I']$ are given in table I and those of the matrices $[II]$ and $[II']$, in table II. The matrix $[I]''$ is essentially the double transpose of the matrix $[I]$ and its elements are given in table III. If the structural influence-coefficient method is used, the required matrices $[C_3']$ and $[C_4']$ may be obtained from reference 1.

The matrix $[A]$ is calculated as described in reference 1 and, if desired, by means of the computing forms given in reference 1. The steps in the computation may be summarized as follows:

Step	Operation
①	$[I]'' \left[\frac{(GJ)_r}{GJ} \right]$
②	$[I]'' \left[\frac{(EI)_r}{EI} \right]$
③	$\frac{(GJ)_r}{(EI)_r} \tan^2 \Lambda$ [②]
④	$\frac{w_e}{b'/2} (Q_{\alpha_T} - Q_{\alpha_M} \tan \Lambda)$ [①']
⑤	[①]+[③]+[④]
⑥	$[I'] \left[\frac{e_1}{e_{1r}} \left(\frac{c}{c_r} \right)^2 \right]_{sub}$
⑦	[⑤][⑥]
⑧	k [②]
⑨	$\frac{w_e}{b'/2} \frac{b'/2}{e_{1r} c_r \cos^2 \Lambda} Q_{\alpha_M}$ [①']
⑩	[⑧]-[⑨]
⑪	$[II'] \left[\frac{c}{c_r} \right]$
⑫	[⑩][⑪]
⑬	$[A] = [⑦] - [⑫]$

where the notation [⑦], for instance, refers to the matrix calculated in step 7. For supersonic speeds four additional steps are required:

⑥a	$[I] \left[\frac{e_1}{e_{1r}} \left(\frac{c}{c_r} \right)^2 \right]_{spr}$
⑦a	[⑤] [⑥a]
⑫a	$\frac{(e_{1r})_{sub}}{(e_{1r})_{spr}}$ [⑫]
⑬a	$[A] = [⑦a] - [⑫a]$



(a) Six-point method.
(b) Ten-point method.
FIGURE 3.—Equivalent values of δ .

To proceed with the calculation of lateral control and of the aileron reversal speed, the auxiliary aeroelastic and the aileron-reversal matrices are then calculated as shown in table IV; the numbering of the steps indicated in the upper left corner of each block is a continuation of the numbering of the steps required to calculate the matrix $[A]$. The auxiliary aeroelastic matrix is obtained in step 15 and the aileron-reversal matrix, in step 20. The value of g required in step 16 is obtained by *post*multiplying the row $[II'_0] \left[\frac{c}{c_r} \right]$ obtained in the step immediately above step 16 by the column $\left\{ \frac{c_i}{C_{L_{\alpha_e}} \delta} \right\}$. On the other hand, the square matrix of step 18 is obtained by *pre*multiplying the row matrix $[\textcircled{17}]$, which is the same row matrix $[II'_0] \left[\frac{c}{c_r} \right]$ multiplied by $\frac{1}{g}$, by the column $\left\{ \frac{c_i}{C_{L_{\alpha_e}} \delta} \right\}$.

A separate set of calculations from step 14 to step 20, inclusive, has to be performed for subsonic and supersonic speeds; the steps for supersonic speeds can be labeled steps 14a to 20a to follow the pattern set in reference 1. For a 10-point solution, forms similar to those of table IV can easily be drawn up. For the influence-coefficient method a set of computing instructions and computing forms can be based on equations (10) and (11) in the same way as the instructions and forms discussed in this section are based on equations (7), (8), and (18).

Special cases arise when any or all of the values of e_1 or of e_2 are zero. If only e_{1r} is zero, the value of e_1 at some other point can be used as a reference throughout the analysis and the parameter q^* can be redefined accordingly. The first column of the matrix $[\textcircled{19}]$ is calculated in this case by multiplying the first column of the matrix $[\textcircled{5}] [I']$ by the ratio of the value e_2 to the reference value of e_1 . Similarly, if some other value of e_1 is zero, say the n th along the span, e_{1r} is used as a reference but the n th column of $[\textcircled{19}]$ is calculated by multiplying the n th column of the matrix $[\textcircled{5}] [I']$ by $\frac{e_{2n}}{e_{1r}} \left(\frac{c_n}{c_r} \right)^2$, where e_{2n} and c_n are the values of e_2 and c at the n th station.

If e_1 is zero along the entire span, some of the computing instructions given in this report, as well as the ones given in reference 1, must be modified somewhat. In table VI (a) of reference 1 the matrix $\left[\frac{e_2}{e_{2r}} \left(\frac{c}{c_r} \right)^2 \right]$ can be entered in the space provided for the $\left[\frac{e_1}{e_{1r}} \left(\frac{c}{c_r} \right)^2 \right]$ matrix. Some of the instructions of table VI(b) of reference 1 and table IV of this report are then modified as follows:

Step 6	$[I'] \left[\frac{e_2}{e_{2r}} \left(\frac{c}{c_r} \right)^2 \right]$
Step 9	$\frac{(EI)_r}{(GJ)_r} \frac{w_e}{b'/2} \frac{Q_{\alpha_M}}{\tan \Lambda} [1,']$
Step 10	$[\textcircled{9}] - [\textcircled{2}]$

Step 11	As is
Step 12	Omit
Step 13	$[A]_{e_1=0} = [\textcircled{10}] [\textcircled{11}]$
Step 14	$\frac{e_{2r} c_r \cos^2 \Lambda}{b'/2} \frac{(EI)_r}{(GJ)_r} \frac{1}{\tan \Lambda} [\textcircled{7}]$
Step 15	$[\bar{A}] = [\textcircled{14}] - [\textcircled{13}]$

All other instructions are unaffected.

If e_2 is zero along the span, table IV of this report may be modified as follows:

Step 15	$[\bar{A}] = [\textcircled{12}]$
---------	----------------------------------

Step 14 may be omitted in this case; all other steps in table IV are unaffected.

Similar modifications must also be made if the influence-coefficient method is used.

CALCULATION OF THE AILERON REVERSAL SPEED

The matrix $[A_R]$ is iterated in table V(a) to calculate the critical value of the parameter κq^* and hence the critical speed. The calculation has to be performed once for subsonic speeds and, if the airplane is to fly at transonic and supersonic speeds, once for supersonic speeds. From these critical values, from the definition of the parameters κ and q^* , and from the lift-curve slope the dynamic pressure required for aileron reversal q_R may be calculated and plotted as a function of Mach number. If the actual dynamic pressure for the altitudes of interest is also plotted on the same chart, the lowest intersection of the reversal with a true-dynamic-pressure line will give the reversal Mach number and dynamic pressure at the altitude of the true-dynamic-pressure line.

The matrices $[A_R]$ calculated for the special cases mentioned in the preceding section do not all yield the critical value of the parameter κq^* . When the value of e_1 is zero at the root, the critical value of the parameter κq^* based on the reference value of e_1 will be obtained. If e_1 is zero at some other point along the span or if e_2 is zero along the entire span, critical values of the parameter κq^* will be obtained. In the case where e_1 is zero along the entire span, iteration of the matrix $[A_R]$ calculated by following the instructions of the preceding section will yield the value of the parameter $\kappa \bar{q}$ at reversal.

In some of these special cases, and possibly in other cases as well, the iteration procedure may not converge. In those cases the lowest value of the parameters $(\kappa q^*)_R$ or $(\kappa \bar{q})_R$ is imaginary, so that there is no physical reversal speed corresponding to this value, and the wing under consideration is likely to be safe against reversal (in the speed range under consideration). If the lowest value of the parameter κq^* has the sign opposite to that of the value of e_{1r} (or the other value of e_1 used as a reference) or if the critical value of $\kappa \bar{q}$ has the sign opposite to that of the sweep angle Λ , the reversal dynamic pressure is negative. In that case the wing also is likely to be safe against reversal, since a negative dynamic pressure cannot be obtained at any real speed.

However, if the wing is to operate at dynamic pressures which correspond to values of κq^* or $\kappa \bar{q}$ much larger than the absolute values of the critical values obtained by iteration, the next higher eigenvalues of the aileron-reversal matrix may have to be calculated by a method such as that given in reference 5, page 143. If the next higher eigenvalue is real and of appropriate sign, it defines the critical aileron reversal speed.

Instead of iterating the matrix $[A_R]$ to calculate $(\kappa q^*)_R$ the determinant of the matrix $[1] - \kappa q^*[A_R]$ may be expanded and equated to zero, as noted in reference 1. The result is an equation of the type

$$C_n(\kappa q^*)^n + C_{n-1}(\kappa q^*)^{n-1} + \dots + C_1(\kappa q^*) + 1 = 0$$

where n is the order of the matrix—that is, 6 or 10 in the case of a 6-point or 10-point solution, respectively. Solution of this equation yields n values of κq^* ; the lowest real value with the appropriate sign is the one that defines the critical reversal speed. Instead of actually expanding the determinant, however, the coefficients $C_1, C_2, \dots, C_n$ can be obtained in terms of the traces of the powers of the matrix $[A_R]$, the trace of a matrix being the sum of the elements on its principal diagonal, and the n th power of the matrix $[A_R]$ being the matrix obtained by multiplying $[A_R]$ by itself $n-1$ times. If s_m is the trace of $[A_R]^m$, then

$$C_1 = -s_1$$

$$C_2 = -\frac{1}{2}(C_1 s_1 + s_2)$$

$$C_3 = -\frac{1}{3}(C_2 s_1 + C_1 s_2 + s_3)$$

$$\dots$$

$$C_n = -\frac{1}{n}(C_{n-1}s_1 + C_{n-2}s_2 + \dots + C_1s_{n-1} + s_n)$$

Unless certain types of automatic computing machinery are available, the iteration procedure is generally preferable to the procedure based on the expansion of the determinant.

CALCULATION OF CONTROL POWER AND MANEUVERABILITY

The calculation of the twist distribution for a given aileron deflection may be carried out in table V (b), which is similar to table VI (b) of reference 1. The matrix $[1] - \kappa q^*[A]$ is entered at the left, and the column $\left\{ \frac{c_i}{C_{L_{\alpha_e}}} \right\}$ is entered at the right. This column is premultiplied by the $[\bar{A}]$ matrix obtained in step 15 or step 15a and is entered in the second column at the right, which in turn is multiplied by $-\kappa q^*$ to yield the third column. The simultaneous equations with the coefficients at the left and the knowns at the right (the third column) are then solved for the unknown α_s values. If preferred, an iterative solution of the type discussed in reference 1 may be used instead of Crout's method (reference 6) for which table V is set up. A computing form similar to that of table VII (c) of reference 1 may be used for this purpose.

If the same values of κq^* are selected as were used in the calculation of the aerodynamic loading by the method of reference 1, the $[1] - \kappa q^*[A]$ matrix is already available. If, in addition, Crout's method of solving simultaneous equations has been used to solve the simultaneous equations, part of the auxiliary matrix is also available so that calculation of the α_s values for the aileron loading requires very little time. However, the iterative solution does not have this advantage.

In some of the special cases discussed in the preceding sections, care must be taken to use the proper parameters in conjunction with the matrices calculated for these special cases. In the case where e_{1r} is zero, the values of κq^* must be based on the reference value of e_1 selected in calculating the matrix; in the case where e_1 is zero along the entire span, the parameter $\kappa \bar{q}$ must be used instead of κq^* in table V (b).

The resulting α_s values may be added algebraically to the values of $\left\{ \frac{c_i}{C_{L_{\alpha_e}}} \right\}_{\delta}$, multiplied by $\frac{c_r}{\bar{c}}$ and $\left[\frac{c}{c_r} \right]$ as indicated in steps 4, 5, and 6 of table V(b), and plotted over the span

to yield the net aerodynamic load distribution $\left\{ \frac{cc_i}{\bar{c}C_{L_{\alpha_e}}} \right\}_{\delta_{fw}}$ which pertains to a unit value of $\alpha_s \delta$ on the flexible wing. The rolling-moment coefficient due to this forcing loading (over both wings) may be obtained from a dimensionless form of equation (13)

$$C_{l_w} = \frac{1}{2} C_{L_{\alpha_e}} \left(\frac{b'}{b} \right)^2 |II'|_{01} \left\{ \frac{cc_i}{\bar{c}C_{L_{\alpha_e}}} \right\}_{\delta_{fw}} \quad (19)$$

This coefficient, which is a direct measure of the rolling power, is seen to be dependent only on q/q_D (except for the factor $C_{L_{\alpha_e}}$) since $\frac{q}{q_D} = \frac{\kappa q^*}{(\kappa q^*)_D}$ and $(\kappa q^*)_D$ is constant for a given speed range.

The rolling maneuverability depends not only on the rolling power but also on the damping in roll. The rate of roll per unit aileron deflection (measured in a plane parallel to the plane of symmetry) is given by

$$\left(\frac{pb}{2V} \right)_{\delta=1} = \alpha_s \frac{C_{l_b}}{C_{l_p}} \quad (20)$$

where C_{l_b} is the forcing coefficient calculated from equation (19) (if the contributions of the pressures on the fuselage are neglected) and C_{l_p} is the damping coefficient calculated from equation (19) with a column of values of $\left\{ \frac{cc_i}{\bar{c}C_{L_{\alpha_e}}} \right\}$ calculated

by the method of reference 1 for a case where $\alpha_s = \frac{y}{b/2}$. If modified strip theory is used, the desired column is

$$\left\{ \frac{cc_i}{\bar{c}C_{L_{\alpha_e}}} \right\} = \frac{c_r}{\bar{c}} \left[\frac{c}{c_r} \right] \{ \{ \alpha_s \} + \{ \alpha_e \} \}$$

where α_e is the structural deformation associated with the given values of α_s .

ILLUSTRATIVE EXAMPLE

The method described in the preceding sections has been used to analyze the lateral maneuverability of the wing considered in the illustrative example of reference 1. The required additional parameters of this wing are presented in table VI (a), which follows the form of table IV (a). Modified strip theory has been used for calculating $\left\{ \frac{c_l}{C_{L_{\alpha_e}} } \right\}_\delta$. The equivalent value of δ at the station $\frac{\eta}{b'/2} = 0.4$ is obtained from figure 3 for the given values of $\frac{\eta_i}{b'/2}$ and $\frac{\eta_o}{b'/2}$. The auxiliary aeroelastic matrix for the subsonic case has been calculated by following the form of table IV(b); the resulting matrix is shown in table VI (b).

The aileron-reversal matrix for the subsonic case is calculated by means of the form of table IV (c). Several of the steps, as well as the result, are shown in table VI (c) for the subsonic case. In these calculations the contribution of the matrix $[I'_1]$ to the matrix $[II'_0]$ has been neglected, so that the matrix $[II'_1]$ has been used instead of the matrix $[II'_0]$, a procedure which is not recommended in general. Iteration of the aileron-reversal matrix (by means of the form of table V(a) or otherwise) yields a value of $(\kappa q^*)_R = 2.364$. A similar calculation for supersonic speeds yields a value of $(\kappa q^*)_R = 0.1280$. From these two values and the definition of the parameters κ and q^* the dynamic pressure required for reversal has been calculated and is plotted against Mach number in figure 4. Also shown in figure 4 for comparison are the dynamic pressures required for divergence as well as the actual dynamic pressures at sea level and at an altitude of 25,000 feet. Where the dynamic pressure required for reversal is less than the actual dynamic pressure, the aileron control is reversed. For the example wing, reversal is likely to occur at a Mach number of approximately 1.3 at sea level.

The aerodynamic loading due to aileron deflection has been calculated by means of the form of table V(b). For the subsonic case and for $\kappa q^* = 0.552$ the $[[1] - \kappa q^*[A]]$ matrix is that shown in table X(b) of reference 1. The three columns to be entered at the right of table V(b), as well as the four columns obtained at the bottom of table V(b), are as follows:

①	②	③	⑥	⑤	④	Final matrix
$\left\{ \frac{c_l}{C_{L_{\alpha_e}} } \right\}$	$[A] \{ \textcircled{1} \}$	$-\kappa q^* \{ \textcircled{2} \}$	$\frac{c_r}{\bar{c}} \{ \textcircled{6} \}$	$\left[\frac{c}{c_r} \right] \{ \textcircled{4} \}$	$\{ \textcircled{1} \} + \{ \alpha_s \}$	$\{ \alpha_s \}$
0	0	0	0	0	0	0
0	.1890	-.1043	-.080	-.063	-.070	-.0699
.265	.5100	-.2815	.067	.053	.065	-.1999
1	.8217	-.4536	.599	.473	.661	-.3392
1	1.0273	-.5671	.435	.344	.552	-.4477
1	1.0656	-.5882	.379	.300	.523	-.4773
$[II'_1] \{ \textcircled{6} \} = 0.179$						

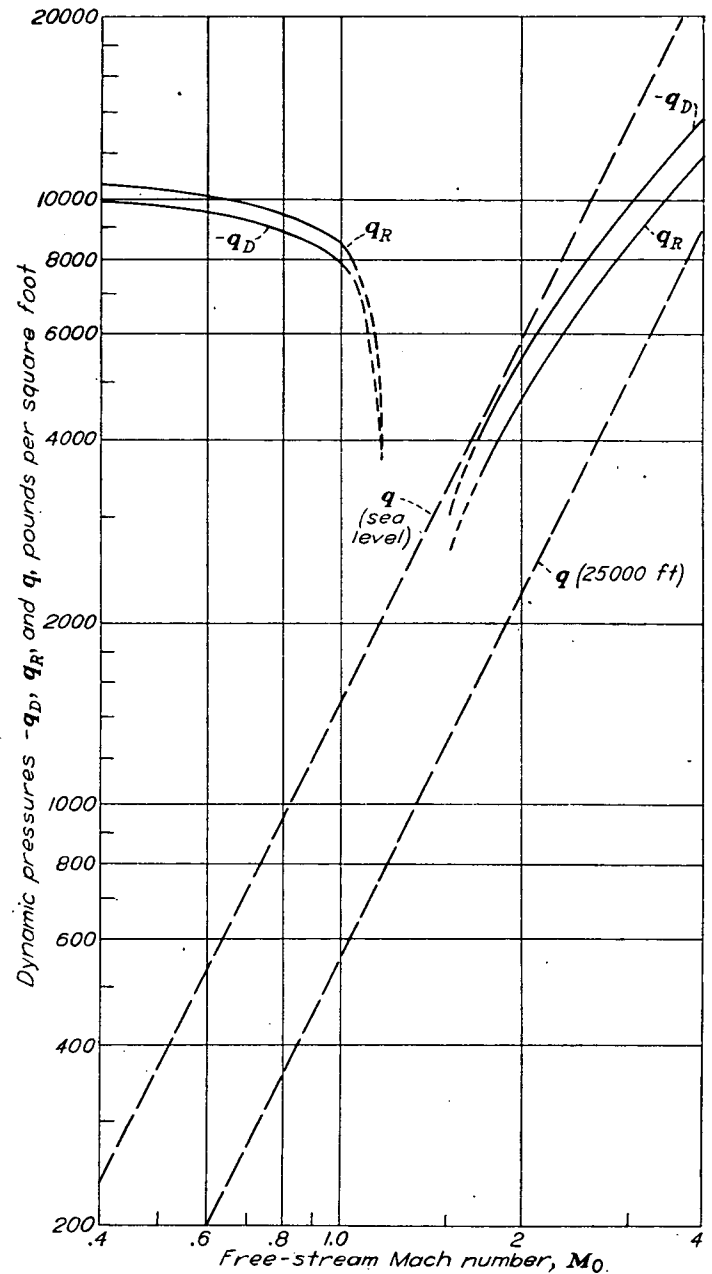
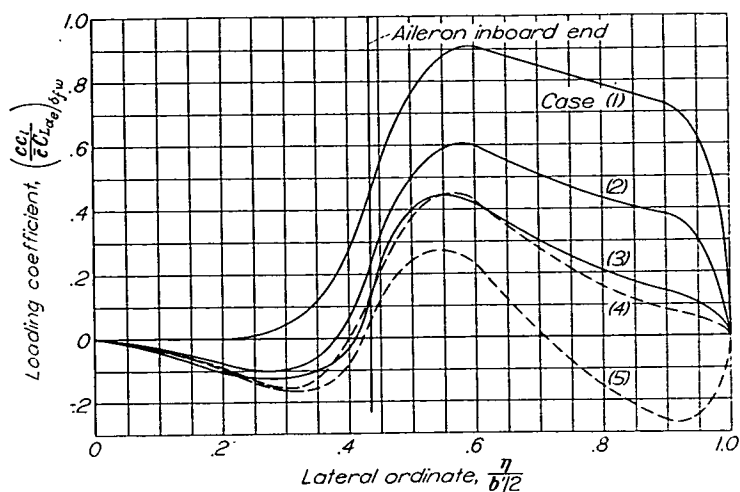


FIGURE 4.—Effect of Mach number on the divergence and aileron-reversal dynamic pressures of the example wing.

The lift distribution due to aileron deflection

$$\left\{ \frac{cc_l}{\bar{c}C_{L_{\alpha_e}} } \right\}_\delta = \frac{c_r}{\bar{c}} \left[\frac{c}{c_r} \right] \left\{ \frac{c_l}{C_{L_{\alpha_e}} } \right\}_\delta$$

obtained by using the modified strip-theory values of $\left\{ \frac{c_l}{C_{L_{\alpha_e}} } \right\}_\delta$ is plotted in figure 5. For the flexible wing the lift distributions due to the calculated twist distributions, such as the one shown in the next to the last column of the foregoing tabulation, must be added algebraically to the lift distribution due to aileron deflection. This addition is best performed by first plotting the lift distributions due to twist



Case	q/q_D	q/q_R	Condition
1	0	0	Subsonic
2	-0.25	0.234	Subsonic
3	-0.50	0.467	Subsonic
4	-0.50	0.577	Supersonic
5	-1.00	1.154	Supersonic

FIGURE 5.—Loading of example wing with aileron deflected.

separately and then adding them point for point to the lift distribution due to aileron deflection. The net distributions obtained in this manner for several cases are shown in figure 5. The distribution for case 5 (supersonic speeds, $\frac{q}{q_D} = -1.00$) indicates that the wing is operating at a speed above its reversal speed; from the given values of q^*_D and q^*_R the dynamic pressure for the case of $\frac{q}{q_D} = -1.00$ can be seen to exceed by 15.4 percent the dynamic pressure required for aileron reversal.

The rolling-moment coefficient is obtained from equation (19) or by adding the moments corresponding to the aileron-distribution curve and the twist curve algebraically. (As stated previously, the contribution of $[I'_1]$ to $[II'_0]$ has been neglected in these calculations.) The ratio of the flexible-wing rolling-moment coefficient obtained in this manner to the corresponding rigid-wing rolling-moment coefficient is plotted in figure 6 (a) against the ratio $-q/q_D$; for the rigid wing the value of C_{l_i} at subsonic speeds is

$$(C_{l_i})_{rw} = 0.070 C_{L_{\alpha_e}}$$

and at supersonic speeds is

$$(C_{l_i})_{rw} = 0.026 C_{L_{\alpha_e}}$$

The lateral maneuverability is calculated by means of equation (20) with the damping coefficients calculated in reference

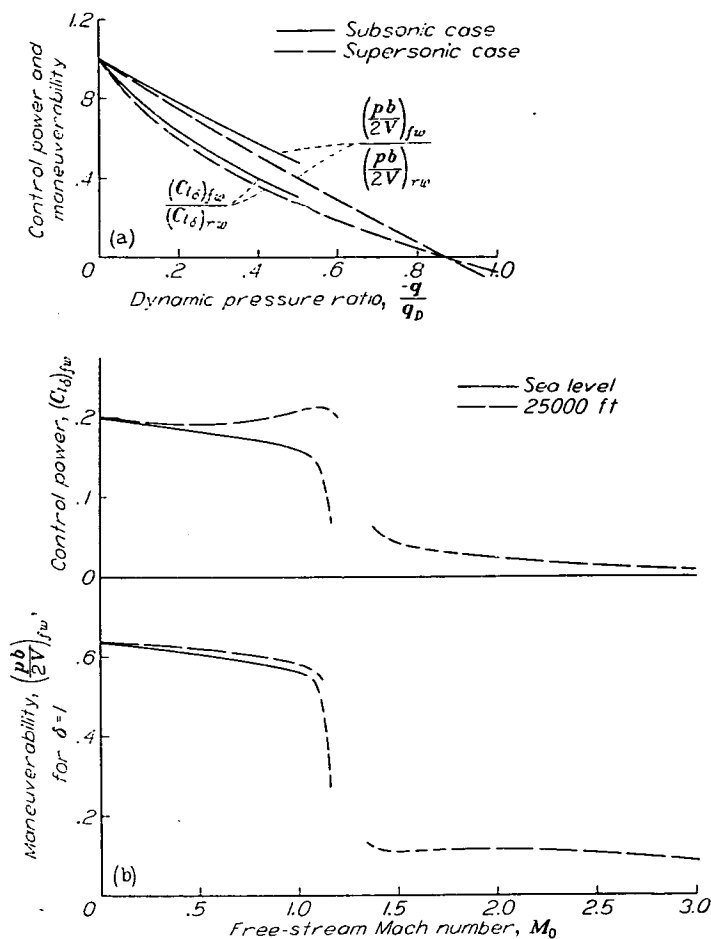


FIGURE 6.—Control power and effectiveness of example wing.

1 and is also plotted in figure 6 (a) as a fraction of the rigid-wing value, which at subsonic speeds is

$$\left(\frac{pb}{2V}\right)_{rw} = 0.634$$

and at supersonic speeds is

$$\left(\frac{pb}{2V}\right)_{rw} = 0.232$$

Both the maneuverability and the control power become zero at a value of $\frac{q}{q_D} = -0.87$, which is the ratio of the reversal to the divergence dynamic pressure at supersonic speeds, as is shown in figure 4.

Since the ratio q/q_D has been determined as a function of altitude and Mach number in figure 4, the parameters of figure 6 (a) can be plotted as functions of altitude and Mach number, as has been done in figure 6 (b). The maneuverability and, to a lesser extent, the control power are relatively low at supersonic speeds, particularly at low altitudes. Since at high speeds even a small value of $pb/2V$ implies a fairly large

value of the rate of roll p , this situation is not necessarily alarming. The wing in question should have adequate control at all speeds for altitudes greater than about 20,000 feet.

DISCUSSION

The method of this report is based in essence on a numerical integration by means of matrices of the differential equations of structural equilibrium. The actual stiffness distributions, root rotations, and the lift and pitching-moment distributions of the undeformed wing can be taken into account as accurately as they are known. The commonly made simplification of treating the wing as an aggregate of constant-chord segments with all flexibility concentrated at the ends and all forces at the midpoint of the segments is not resorted to in this report. No time-consuming graphical integrations nor trial and error procedures are used. The aileron reversal speed is calculated by means of an iteration, but each cycle of this iteration consists of a single matrix multiplication so that the entire procedure is straightforward in application, and usually the results converge rapidly. If preferred, the iteration procedure can be replaced by an expansion of the determinant of the matrix $[[1] - \kappa q^*[A_R]]$, as outlined in this report.

The purpose of this section is to discuss the assumptions and limitations of the method of this report and to indicate the effect of certain design variables on the aileron reversal speed by means of the results of a few calculations for the example wing and some others related to it.

ASSUMPTIONS AND LIMITATIONS OF THE METHOD

The discussion of the aerodynamic and structural assumptions in reference 1 is also pertinent to the analysis of this report. This discussion may be summarized as follows: All angles of attack, structural deformations, and control deflections must be sufficiently small to give rise only to linear aerodynamic and structural forces. When structural influence coefficients are used, no further assumptions are necessary concerning the structural deformations; when the stiffness curves are used, elementary beam theory as corrected by root rotations must be applicable. If elementary beam theory is inapplicable—that is, if shear deformations, shear lag, and bending-torsion interaction cannot be neglected—a more refined method than elementary beam theory, such as the method of reference 4, can be used to calculate structural influence coefficients which can be used in the method of the present report. When suitable aerodynamic influence coefficients are available, no further assumptions need be made concerning the aerodynamic forces; if no such coefficients are available, the assumption must be made that modified strip theory is sufficiently accurate to calculate the aeroelastic effects of interest.

In the present report additional aerodynamic assumptions must be made, primarily, because although accurate aerodynamic information can be used in the method of this report such information is not available in many instances. For instance, no suitable aerodynamic influence coefficients are available as yet for antisymmetric lift distributions so that modified strip theory has to be used for the lift due to

structural deformation. For the lift due to aileron deflection, which is used in the form of the coefficients $\left(\frac{c_l}{C_{L\alpha_e}}\right)_\delta$, the best

available information should be used; for unswept wings of moderate or high aspect ratio the method of reference 2 gives accurate results, and for swept wings or wings of low aspect ratio the method of reference 3, with certain modifications explained in reference 6, may be used to calculate the desired coefficients. However, information concerning the parameters α_δ and cp_δ for wings of finite span is very meager; the suggested means of estimating them give results which must be used with some caution. If experimental results are available for these parameters, they can, of course, be used in this method.

Modified strip theory should not in general be used to calculate the coefficients $\left(\frac{c_l}{C_{L\alpha_e}}\right)_\delta$, as was done in the illustra-

tive example. If it is desired to use this approximation, the equivalent δ values of figure 3 may be used to obtain a suitable fairing of the lift-distribution curve at the aileron ends. However, these equivalent values are premised on the use of the $[I']$ and $[II']$ matrices and should not be employed for any other purpose than that indicated herein.

Two additional structural assumptions are also made in this report. In the first place, the angle δ between the wing and the aileron is assumed to be constant along the span. This assumption appears to have been made in almost all of the published investigations into the problem of lateral-control reversal and appears to have yielded satisfactory results; the shorter the aileron and the greater the number of points at which the aileron is supported and at which its hinge moment is taken out, the more nearly true is the assumption. Also, the control linkage is assumed to be stiff so that the aileron angle for a given stick displacement is independent of the dynamic pressure. However, in order to account for the control-linkage deflection, it is necessary only to calculate the ratio of the true aileron angle at a given dynamic pressure to that at zero dynamic pressure for the same stick position. The calculated control moment and maneuverability must then be reduced by this factor to get values of these quantities for a given stick displacement. Since deformations of the control linkage only affect the aileron effectiveness, they have no bearing on the reversal speed. On the other hand, these deformations may lead to aileron divergence for wings with heavily overbalanced ailerons. This problem, as well as the problem of wing-aileron divergence, has not been considered in the present analysis.

The fuselage and tail do not contribute any appreciable amounts to either the control or the damping moment so that their effects may ordinarily be neglected for the purpose of lateral-control calculations. Similarly, the effect of wing camber does not enter into the problem, because the only important effect of camber is to give the flexible wing a symmetrical lift distribution if it is set at the angle of attack which would give zero lift for the rigid wing; this symmetrical lift distribution has no effect on the lateral-control problem.

As in reference 1, the effects of the inertia loading on the aerodynamic loading have not been considered explicitly in the analysis of this report. As pointed out in reference 1, however, the structural deformations due to the inertia loading may be calculated conveniently by means of the integrating matrices and then be considered as part of the geometrical angles of attack. This procedure may be applied in the case of a rolling wing to determine the change in rolling moment for a unit rolling acceleration at any given Mach number and dynamic pressure. This rolling moment must be taken into account in estimating the rolling accelerations due to a given forcing moment at any time before the steady-roll condition is reached.

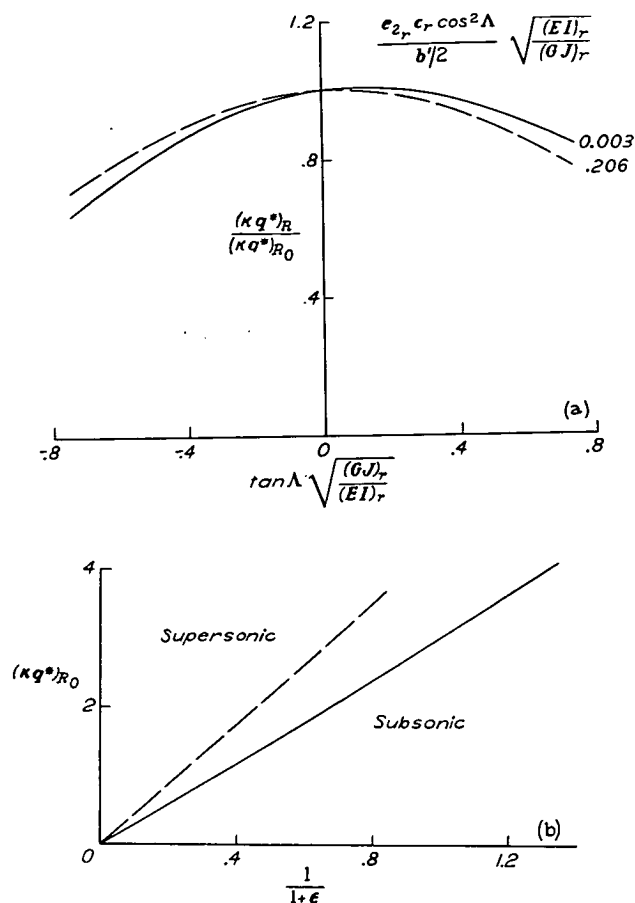
At transonic speeds there is considerable uncertainty in the aerodynamic parameters. The control power is directly proportional to the value of the parameter $c_{l_\delta} = \frac{dc_l}{d\delta}$, which may be quite low in the transonic region

due to the fact that the aileron is located in a region of separated flow. The method of this report is applicable to this case if the value of c_{l_δ} is known for the rigid wing and if the aerodynamic forces due to aileron deflection and due to twist can be superimposed linearly. If, for instance, the decrease in this parameter due to flow separation is 40 percent at a given Mach number and if the loss in control power due to wing flexibility amounts to 20 percent, then the total loss is 52 percent. However, the loss in maneuverability due to the decrease in c_{l_δ} may be much less than the loss in control power, since a decrease in c_{l_δ} is usually accompanied by a decrease in c_{l_α} and $C_{L_{\alpha_e}}$ and hence in the coefficient of damping in roll.

Should the value of the parameter c_{l_δ} decrease to zero or reverse, aileron reversal will occur. This type of reversal is altogether different from the type of reversal discussed in this report since it is due entirely to aerodynamic action, whereas the reversal of concern in this report is due to aeroelastic action. Both types of reversal are largely independent of each other; aerodynamic reversal is likely to occur at a given Mach number regardless of the stiffness of the wing, whereas aeroelastic reversal will occur ordinarily at a different speed which is unaffected by the aerodynamic effectiveness.

EFFECTS OF SOME DESIGN VARIABLES ON THE AILERON REVERSAL SPEED

Some general effects of sweep and of the moment arms e_1 and e_2 on the aeroelastic reversal speed may be of interest. The ratio of the reversal parameter $(\kappa q^*)_R$ of a given wing to that of the unswept wing obtained by rotating the given wing $(\kappa q^*)_{R_0}$ is shown in figure 7 (a) plotted against a function of the sweep angle for subsonic and supersonic speeds; the two curves were obtained by considering the wing of the illustrative example to be rotated in such a manner as to keep the parameters $\frac{e_1 c_r \cos^2 \Lambda}{b'/2}$ and $\frac{(EI)_r}{(GJ)_r}$ as well as the chord, stiffness, and moment-arm distributions e_1 and e_2 constant.



(a) Effect of sweep on reversal parameter $(\kappa q^*)_R$.
(b) Effect of moment-arm ratio ϵ on reversal parameter $(\kappa q^*)_R$ for unswept wings.
FIGURE 7.—Effects of sweep and moment-arm ratio on reversal speed.

Both sweepback and sweepforward apparently tend to decrease the reversal parameter and hence the reversal speed. At supersonic speeds or, more specifically, at small values of the parameter $\frac{e_2 c_r \cos^2 \Lambda}{b'/2} \sqrt{\frac{(EI)_r}{(GJ)_r}}$ the reversal speed for the sweptforward wing is somewhat lower than that of the sweptback wing; whereas at higher values of the parameter the variation of the reversal speed with the sweep parameter $\tan \Lambda \sqrt{\frac{(GJ)_r}{(EI)_r}}$ is more nearly symmetrical with respect to the zero-sweep case. There are some indications that this behavior is not typical of all wings but rather is due to the fairly large variation of the values of e_1 , e_2 , and ϵ over the span of the example wing. In general it appears that, for small values of the moment-arm parameter $\frac{e_2 c_r \cos^2 \Lambda}{b'/2} \sqrt{\frac{(EI)_r}{(GJ)_r}}$, the variation of the reversal speed with the sweep parameter $\tan \Lambda \sqrt{\frac{(GJ)_r}{(EI)_r}}$ should be nearly symmetrical and that, for large values of the moment-arm parameter, the reversal speed should tend to be lower for sweptback wings than for sweptforward wings.

The variation of the reversal speed of an unswept wing with the moment-arm ratio is shown in figure 7 (b) for wings which have the same distributions of the parameters e_1/e_1 , and e_2/e_2 , along the span but have different values of e_1 , and

e_2 . The parameter $(\kappa q^*)_{R_0}$ is plotted against the ratio $\frac{1}{1+\epsilon}$, where the value of ϵ is selected at the midaileron station. It is seen that the plot is linear for both the subsonic and supersonic case. The difference in these cases is due to the different variations of e_1 and e_2 along the span; if the variations were the same or if e_1 and e_2 were constant along the span, the two lines of figure 7 (b) would coincide. Since the reversal parameter $(\kappa q^*)_{R_0}$ is proportional to $\frac{1}{1+\epsilon}$ and since the reversal dynamic pressure is directly proportional to the reversal parameter and inversely proportional to the value e_1 , (by definition of the parameter κq^*), it follows that the reversal dynamic pressure is approximately proportional to the ratio $\frac{1}{e_1+e_2}$. From figure 1 it is seen that the sum of e_1 and e_2 represents the distance from the aerodynamic center to the center of pressure of the lift due to aileron deflection and is independent of the location of the elastic axis. This fact corroborates the commonly made observation that the reversal speed is independent of the location of the elastic axis in the case of unswept wings.

The control power and maneuverability cannot be related to the structural and geometric parameters in as relatively simple a manner as the reversal speed. The control power is a function of both the ratio q/q_R and the ratio q_R/q_D ; it normally decreases with q/q_R , the rate of decrease being slow at first and then more rapid for positive values of q_R/q_D (which would generally be obtained for unswept and sweptforward wings) and being rapid at first and then slower for negative values of q_R/q_D (which would generally be obtained for sweptback wings). The variation of the maneuverability should generally be similar to that of the control power since the damping coefficient decreases (or in the case of unswept and sweptforward wings increases) steadily with q/q_D and is independent of q_R/q_D .

From the calculations for the example wings it appears that the control power and maneuverability of sweptback wings tend to be relatively low, particularly at supersonic speeds. A combination of high sweep and large moment arm e_2 may lead to an undesirably low maneuverability. Of course, any increase in the purely aerodynamic effectiveness α_s of the aileron-airfoil combination results in a proportional increase in the lateral-control effectiveness. At supersonic speeds α_s is proportional to the aileron-chord ratio c_a/c , so that an increased aileron chord results in greater maneuverability; at subsonic speeds an increase in the aileron chord is less effective. Another obvious means of raising the reversal speed and of increasing the control power is to increase either the torsional stiffness or the bending stiffness of the structure. In some cases, however, the increase of the reversal parameter $(\kappa q^*)_R$ due to a change in the parameter $\tan A \sqrt{\frac{(GJ)_r}{(EI)_r}}$ (see fig. 7 (a)) produced by a decrease in the torsional stiffness $(GJ)_r$ may

be so rapid as to cause a net increase in the reversal speed.

If the sweep, the moment arm e_2 , the stiffness, and the aileron effectiveness cannot be changed sufficiently to increase the maneuverability, it may be necessary to resort to unconventional control devices. Leading-edge ailerons, for instance, have negative values of the moment arm e_2 , so that wings equipped with them tend to reverse at very high speeds, if at all. This type of configuration has the additional advantage of relatively high effectiveness at transonic speeds. The effectiveness of leading-edge ailerons at subsonic speeds is so low, however, that they would have to be used in conjunction with trailing-edge ailerons to assure satisfactory lateral control at low subsonic speeds; furthermore, they pose some other aerodynamic as well as structural and mechanical problems. Similarly, from an aeroelastic point of view spoilers appear attractive because they tend to have small or negative values of e_2 , but they also pose certain design problems. Consequently, both these devices require careful consideration before they are used to alleviate aeroelastic difficulties in any specific case.

CONCLUDING REMARKS

A method has been presented for calculating the effectiveness and the speed-of reversal of lateral control as well as of the aerodynamic loading and the rolling moment produced by aileron deflection on swept flexible wings of arbitrary stiffness.

It has been shown that the aileron reversal speed decreases with both sweptback and sweptforward wings and that the effectiveness of conventional aileron configurations on sweptback wings at supersonic speeds tends to be relatively low. The control effectiveness and the resulting maneuverability of the airplane may be increased by varying some of the design parameters such as the structural stiffness and, if necessary, resorting to unconventional control devices, such as leading-edge ailerons or spoilers.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., April 5, 1951.

REFERENCES

1. Diederich, Franklin W.: Calculation of the Aerodynamic Loading of Swept and Unswept Flexible Wings of Arbitrary Stiffness. NACA Rep. 1000, 1950.
2. Multhopp, H.: Die Berechnung der Auftriebsverteilung von Tragflügeln. Luftfahrtforschung, Bd. 15, Lfg. 4, April 6, 1938, pp. 153-169. (Available as R.T.P. Translation No. 2392, British M.A.P.)
3. Weissinger, J.: The Lift Distribution of Swept-Back Wings. NACA TM 1120, 1947.
4. Kuhn, Paul: Deformation Analysis of Wing Structures. NACA TN 1361, 1947.
5. Frazer, R. A., Duncan, W. J., and Collar, A. R.: Elementary Matrices and Some Applications to Dynamics and Differential Equations. The Macmillan Co., 1946.
6. DeYoung, John: Spanwise Loading for Wings and Control Surfaces of Low Aspect Ratio. NACA TN 2011, 1950.

(a) SIX-POINT SOLUTION

[I']

$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0.06667	0.26667	0.13333	0.26667	0.09333	0.15085
.2	-.01667	.13333	.15000	.26667	.09333	.15085
.4	0	0	.06667	.26667	.09333	.15085
.6	0	0	-.01667	.13333	.11000	.15085
.8	0	0	0	0	.02667	.15085
.9	0	0	0	0	-.01886	.09333

 $[I']$

$\frac{\eta}{b'/2}$	.0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0.03333	0.13333	0.06667	0.13333	0.06667	0.13333	0.06667	0.13333	0.06000	0.15085
.1	— .00833	.06667	.07500	.13333	.06667	.13333	.06667	.13333	.06000	.15085
.2	0	0	.03333	.13333	.06667	.13333	.06667	.13333	.06000	.15085
.3	0	0	— .00833	.06667	.07500	.13333	.06667	.13333	.06000	.15085
.4	0	0	0	0	.03333	.13333	.06667	.13333	.06000	.15085
.5	0	0	0	0	— .00833	.06667	.07500	.13333	.06000	.15085
.6	0	0	0	0	0	0	.03333	.13333	.06000	.15085
.7	0	0	0	0	0	0	— .00833	.06667	.06833	.15085
.8	0	0	0	0	0	0	0	0	.02667	.15085
.9	0	0	0	0	0	0	0	0	— .01886	.09333

TABLE II.—VALUES OF THE INTEGRATING MATRICES $[II]$ AND $[II']$

(a) SIX-POINT SOLUTION

 $[II]$

$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9	1.0
0	0	0.05333	0.05333	0.16000	0.08000	0.12000	0.03333
.2	-.00167	.01000	.02500	.10667	.06000	.09333	.02667
.4	0	0	0	.05333	.04000	.06667	.02000
.6	0	0	-.00167	.01000	.01833	.04000	.01333
.8	0	0	0	0	0	.01333	.00667
.9	0	0	0	0	-.00042	.00250	.00292
1.0	0	0	0	0	0	0	0

 $[II']$

$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0.05333	0.05333	0.16000	0.07314	0.13792
.2	-.00167	.01000	.02500	.10667	.05448	.10775
.4	0	0	0	.05333	.03581	.07758
.6	0	0	-.00167	.01000	.01548	.04741
.8	0	0	0	0	-.00152	.01724
.9	0	0	0	0	-.00108	.00419

(b) TEN-POINT SOLUTION

 $[II']$

$\frac{\eta}{b'/2}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0	0.013333	0.013333	0.040000	0.026667	0.066667	0.040000	0.093333	0.046476	0.137920
.1	-.000417	.002500	.006251	.026667	.020000	.053333	.033333	.080000	.040476	.122835
.2	0	0	0	.013333	.013333	.040000	.026667	.066667	.034477	.107750
.3	0	0	-.000417	.002500	.006251	.026667	.020000	.053333	.028476	.092665
.4	0	0	0	0	0	.013333	.013333	.040000	.022476	.077580
.5	0	0	0	0	-.000417	.002500	.006251	.026667	.016477	.062495
.6	0	0	0	0	0	0	0	.013333	.010476	.047410
.7	0	0	0	0	0	0	-.000417	.002500	.004060	.032325
.8	0	0	0	0	0	0	0	0	-.001523	.017240
.9	0	0	0	0	0	0	0	0	-.001077	.004190

TABLE III.—VALUES OF THE INTEGRATING MATRIX $[II']$

(a) SIX-POINT SOLUTION

$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2	.08333	.13333	-.01667	0	0	0
.4	.06667	.26667	.06667	0	0	0
.6	.06667	.26667	.15000	.13333	-.01667	0
.8	.06667	.26667	.13333	.26667	.06667	0
.9	.06667	.26667	.13333	.26667	.10833	.06667

(b) TEN-POINT SOLUTION

$\frac{\eta}{b'/2}$	0	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	0	0	0	0	0	0	0	0	0	0
.1	.04167	.06667	-.00833	0	0	0	0	0	0	0
.2	.03333	.13333	.03333	0	0	0	0	0	0	0
.3	.03333	.13333	.07500	.06667	-.00833	0	0	0	0	0
.4	.03333	.13333	.06667	.13333	.03333	0	0	0	0	0
.5	.03333	.13333	.06667	.13333	.07500	.06667	-.00833	0	0	0
.6	.03333	.13333	.06667	.13333	.06667	.13333	.03333	0	0	0
.7	.03333	.13333	.06667	.13333	.06667	.13333	.07500	.06667	-.00833	0
.8	.03333	.13333	.06667	.13333	.06667	.13333	.06667	.13333	.03333	0
.9	.03333	.13333	.06667	.13333	.06667	.13333	.06667	.13333	.07500	.06667

TABLE IV.—FORM FOR COMPUTATION OF AUXILIARY AEROELASTIC AND AILERON-REVERSAL MATRICES

(a) PARAMETERS PERTINENT TO LATERAL-CONTROL CALCULATIONS

$$\frac{\eta_i}{b'/2} = \underline{\hspace{2cm}}$$

$$\frac{\eta_o}{b'/2} = \underline{\hspace{2cm}}$$

$$\alpha_s = \underline{\hspace{2cm}}$$

$\frac{\eta}{b'/2}$	$\left(\frac{c_l}{C_{L_{\alpha_e}}}\right)_s$	cp_s	e_2	ϵ
0				
.2				
.4				
.6				
.8				
.9				

[ϵ]						
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0		0	0	0	0	0
.2	0		0	0	0	0
.4	0	0		0	0	0
.6	0	0	0		0	0
.8	0	0	0	0		0
.9	0	0	0	0	0	

(b) CALCULATION OF THE AUXILIARY AEROELASTIC MATRIX

[$\textcircled{7}$] [ϵ]						
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2						
.4						
.6						
.8						
.9						

[$\overline{A}$] = [$\textcircled{12}$] + [$\textcircled{14}$]						
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2						
.4						
.6						
.8						
.9						

(c) CALCULATION OF THE AILERON-REVERSAL MATRIX

$\frac{w}{b'} = \underline{\hspace{2cm}}$						
[II'_0] = [II'_1] + $\frac{w}{b'}$ [I'_1]						
[II'_0] $\left[\frac{c}{c_r}\right]$						
$g = \left[[II'_0] \left[\frac{c}{c_r}\right] \right] \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s$ $g = \underline{\hspace{2cm}}$						
$\frac{1}{g} [II'_0] \left[\frac{c}{c_r}\right]$						

[$\textcircled{17}$] $\left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s$						
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2						
.4						
.6						
.8						
.9						

[$\textcircled{15}$] [$\textcircled{18}$]						
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2						
.4						
.6						
.8						
.9						

[A_R] = [$\textcircled{13}$] + [$\textcircled{19}$]						
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2						
.4						
.6						
.8						
.9						

(a) REVERSAL

(b) CONTROL POWER

[A _B]						
$\frac{\eta}{b^2/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2			c			
.4						
.6						
.8						
.9						

[illegible]

$\{\alpha_s\}$						
$\frac{\eta}{b'/2}$	(1)	(2)	(3)	(4)	(5)	(6)
0	0	0	0	0	0	0
.2	.3000					
.4	.5000					
.6	.7000					
.8	.9000					
.9	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

AUXILIARY MATRICES						
0	1.0000	0	0	0	0	0
.2						
.4						
.6						
.8						
.9						

$[\mathcal{A}_B]\{\alpha_s\}$						
$\frac{\eta}{b'/2}$	(1)	(2)	(3)	(4)	(5)	(6)
0	0	0	0	0	0	0
.2						
.4						
.6						
.8						
.9						

$[II'0]$
$\{II'0\} \{ \textcircled{6} \} =$ _____

[illegible]
$$(\kappa Q^*)_R =$$

TABLE VI.—COMPUTATION OF AUXILIARY AEROELASTIC AND AILERON-REVERSAL MATRICES FOR THE EXAMPLE WING AT SUBSONIC SPEEDS

(a) PARAMETERS PERTINENT TO LATERAL-CONTROL CALCULATIONS

$$\frac{\eta_i}{b'/2} = 0.434 \quad \frac{\eta_o}{b'/2} = 1.000$$

$$\alpha_s = 0.547$$

$\frac{\eta}{b'/2}$	$\left(\frac{c_l}{C_{L_{\alpha_e}}}\right)_s$	cp_s	c_2	ϵ
0	0	0.250	-0.2020	-1.0000
.2	0	.250	-.1990	-1.0000
.4	.265	.458	.0111	.0563
.6	1.000	.458	.0136	.0701
.8	1.000	.458	.0160	.0833
.9	1.000	.458	.0173	.0906

(b) CALCULATION OF THE AUXILIARY AEROELASTIC MATRIX

(15)	$[A] = [(12)] + [(13)]$					
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2	.00414	-.00671	.02541	.08337	.03542	.06348
.4	.00774	-.02181	.04563	.21248	.09966	.18577
.6	.00774	-.02181	.04041	.29601	.17065	.34430
.8	.00774	-.02181	.03443	.31802	.20199	.49812
.9	.00774	-.02181	.03443	.31802	.19534	.54309

(c) CALCULATION OF THE AILERON REVERSAL MATRIX

$[II']_1 \left[\frac{c}{c_r} \right]$					
0	0.04826	0.04325	0.11456	0.04549	0.07917
(16)	$\varrho = [II']_1 \left[\frac{c}{c_r} \right] \left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s$ $\varrho = 0.2507$				
(17)	$\frac{1}{\varrho} [II']_1 \left[\frac{c}{c_r} \right]$				
0	0.1925	0.1725	0.4570	0.1815	0.3158

(18)	$\left\{ \frac{c_l}{C_{L_{\alpha_e}}} \right\}_s [(17)]$					
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2	0	0	0	0	0	0
.4	0	.0510	.0457	.1211	.0481	.0837
.6	0	.1925	.1725	.4570	.1815	.3158
.8	0	.1925	.1725	.4570	.1815	.3158
.9	0	.1925	.1725	.4570	.1815	.3158

(20)	$[A_R] = [(13)] + [(19)]$					
$\frac{\eta}{b'/2}$	0	.2	.4	.6	.8	.9
0	0	0	0	0	0	0
.2	-.00414	.04310	.02543	.02357	.00433	.00372
.4	-.00774	.12000	.09385	.10715	.01583	.00695
.6	-.00774	.18000	.15503	.26327	.04642	-.00732
.8	-.00774	.21958	.18440	.38811	.10027	.00400
.9	-.00774	.22695	.19101	.40563	.10911	.04635

REPORT 1025

EXPERIMENTAL AND THEORETICAL STUDIES OF AREA SUCTION FOR THE CONTROL OF THE LAMINAR BOUNDARY LAYER ON AN NACA 64A010 AIRFOIL¹

By ALBERT L. BRASLOW, DALE L. BURROWS, NEAL TETERVIN, and FIORAVANTE VISCONTI

SUMMARY

A low-turbulence wind-tunnel investigation was made of an NACA 64A010 airfoil having a porous surface to determine the reduction in section total-drag coefficient that might be obtained at large Reynolds numbers by the use of suction to produce continuous inflow through the surface of the airfoil (area suction). In addition to the experimental investigation, a related theoretical analysis was made to provide a basis of comparison for the test results.

Full-chord laminar flow was maintained by application of area suction up to a Reynolds number of approximately 20×10^6 . At this Reynolds number, combined wake and suction drags of the order of 38 percent of the drag for a smooth and fair NACA 64A010 airfoil without boundary-layer control were obtained. The minimum experimental values of suction-flow coefficient for full-chord laminar flow were of the same order of magnitude as the theoretical values and decreased with an increase in Reynolds number in the same manner as the theoretical values. It seems likely from the results that attainment of full-chord laminar flow by means of continuous suction through a porous surface will not be precluded by a further increase in Reynolds number provided that the airfoil surfaces are maintained sufficiently smooth and fair and provided that outflow of air through the surface is prevented.

Although area suction was able to overcome the destabilizing effects of an adverse pressure gradient such as that which occurs over the rear portion of an airfoil, area suction does not appear to stabilize the boundary layer completely for relatively large disturbances such as those which might be caused by protuberances that have a height comparable to the boundary-layer thickness.

INTRODUCTION

The stability theory for the incompressible laminar boundary layer is an analysis of the damping or amplification of vanishingly small two-dimensional aerodynamically possible disturbances in the boundary layer (reference 1). A possible definition in the physical sense of a small disturbance is one that does not produce transition from laminar to turbulent flow at its origin in contrast with a large

disturbance, which does cause immediate transition. Small disturbances may either amplify as they progress downstream and eventually grow large enough to cause turbulence or they may be damped and cause no change in the downstream flow; if small disturbances of all frequencies are damped rather than amplified, the laminar boundary layer is considered stable (reference 2).

Theoretical investigations have been made of the characteristics of flows past a flat plate through which there is a small normal velocity and, in addition, the stability theory has been used to calculate the stability of the laminar boundary layer for this type of flow. Examples of some of this theoretical work can be found in references 3 to 7 and in British work (not generally available). The results of these analyses indicate that a small normal velocity into the surface at all points along the surface has a large stabilizing effect on the laminar layer. Inasmuch as there appear to be no data that show this effect experimentally, an investigation of the effect of area suction on the boundary-layer stability is being made in the Langley low-turbulence pressure tunnel. Three suction arrangements were investigated on an NACA 64A010 airfoil that had porous sintered-bronze surfaces. Measurements were made up to a Reynolds number of approximately 20×10^6 and included wake drags, suction-flow quantities, suction losses, and a few boundary-layer velocity profiles.

In order to provide a basis of comparison for the measured suction flows, the stability of the laminar boundary layer was calculated for two important cases of chordwise suction distribution for the test airfoil. Theoretical results are also presented for a flat plate with uniform suction. The calculations were made by combining Schlichting's theory for the computation of the laminar boundary layer (reference 8) with Lin's theory for the determination of the stability of the laminar velocity profile (reference 1). Suction quantities necessary to keep the boundary layer neutrally stable at all points along the airfoil chord were calculated for Reynolds numbers of 6×10^6 , 15×10^6 , and 25×10^6 . The minimum suction quantities required to keep the boundary layer stable were obtained also for the case where the inflow velocity is constant over the entire surface.

¹ Supersedes NACA TN 1905, "Experimental and Theoretical Studies of Area Suction for the Control of the Laminar Boundary Layer on a Porous Bronze NACA 64A010 Airfoil" by Dale L. Burrows, Albert L. Braslow, and Neal Tetervin, 1949, and NACA TN 2112, "Further Experimental Studies of Area Suction for the Control of the Laminar Boundary Layer on a Porous Bronze NACA 64A010 Airfoil" by Albert L. Braslow and Fioravante Visconti, 1950.

SYMBOLS

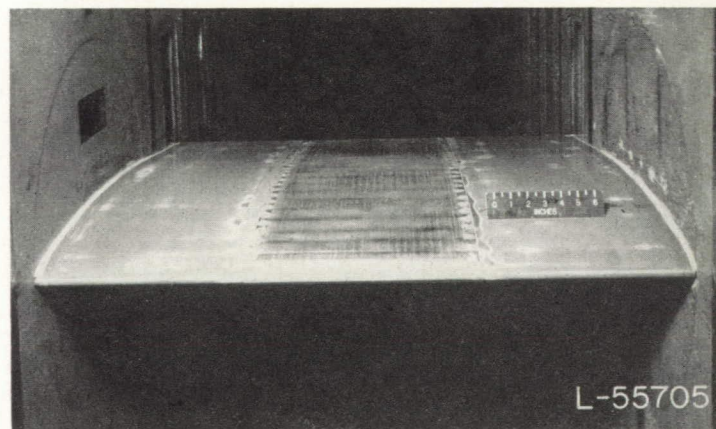
α_0	section angle of attack
c	airfoil chord
b	span of porous surface
x	distance along chord from leading edge of airfoil
s	distance along surface from leading edge of airfoil
y	distance normal to surface of airfoil
ρ_0	free-stream mass density
U_0	free-stream velocity
q_0	free-stream dynamic pressure $\left(\frac{1}{2} \rho_0 U_0^2\right)$
U	local velocity parallel to surface at outer edge of boundary layer
u	local velocity parallel to surface and inside boundary layer
Q	total quantity rate of flow through both airfoil surfaces
C_Q	suction-flow coefficient $\left(\frac{Q}{bcU_0}\right)$
H_0	free-stream total pressure
H_i	total pressure in model interior
p	local static pressure on airfoil surface
S	airfoil pressure coefficient $\left(\frac{H_0 - p}{q_0}\right)$
R	free-stream Reynolds number based on airfoil chord
C_P	suction-air pressure-loss coefficient $\left(\frac{H_0 - H_i}{q_0}\right)$
c_{d_w}	section wake-drag coefficient
c_{d_s}	section suction-drag coefficient $(C_Q C_P)$
c_{d_T}	section total-drag coefficient $(c_{d_s} + c_{d_w})$
δ^*	displacement thickness $\left(\int_0^\infty \left(1 - \frac{u}{U}\right) dy\right)$
θ	momentum thickness $\left(\int_0^\infty \frac{u}{U} \left(1 - \frac{u}{U}\right) dy\right)$
$R_{\delta^*} = \frac{U \delta^*}{\nu}$	
ν	kinematic viscosity
v_0	velocity through airfoil surface (for suction, $v_0 < 0$; for blowing, $v_0 > 0$)
Δp	static pressure drop across porous surface
c_{v_0}	porosity factor, length ² $\left(\left \frac{v_0}{\Delta p}\right \mu t\right)$
μ	absolute viscosity
t	thickness of porous material
$z = \left(\frac{\theta}{c}\right)^2 \frac{U_0 c}{\nu}$	(reference 8)
G	function of k and k_1 (reference 8)
$k = z \frac{d \frac{U}{U_0}}{d \frac{s}{c}}$	(reference 8)
$k_1 = f_1 \sqrt{z}$	(reference 8)
$f_1 = \frac{-v_0}{U_0} \sqrt{\frac{U_0 c}{\nu}}$	(reference 8)

K	profile shape parameter (reference 8)
$\eta = \frac{y}{\delta_1}$	(reference 8)
δ_1	measure of boundary-layer thickness (reference 8)
u_c	velocity of disturbance in boundary layer
$R_{\delta^* cr}$	value of R_{δ^*} at which disturbance is neither damped nor amplified

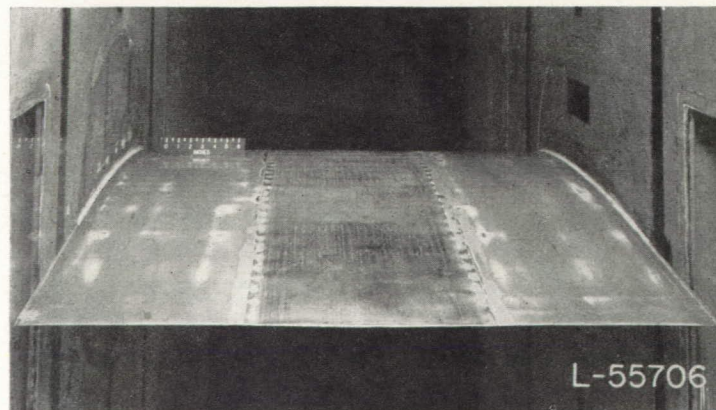
MODEL

Photographs of the 3-foot-chord by 3-foot-span model mounted in the Langley low-turbulence pressure tunnel are presented as figure 1. The model was formed to the NACA 64A010 profile, ordinates for which are presented in reference 9. The theoretical pressure distribution of this airfoil at zero angle of attack is presented in figure 2.

A sketch showing the details of the model construction is presented as figure 3. The model was constructed with two hollow cast-aluminum end sections which were machined to contour and connected to an under-contour hollow center casting that served as a base support for the bronze skin. The skin was directly supported on a chordwise arrangement of $\frac{1}{4}$ -inch spanwise rods which were attached to the under-contour casting. The chordwise locations of these rods are



(a) Leading edge.



(b) Trailing edge.

FIGURE 1.—NACA 64A010 airfoil model with porous sintered-bronze surface mounted for area-suction studies in Langley low-turbulence pressure tunnel.

shown as figure 4. This arrangement of support rods, in which the rods made essentially line contact with the skin, was intended to provide as much open area as possible on the inner side of the skin so that very little of the skin would be blanked off from the suction flow. The center casting was perforated with 1-inch holes over the center portion and 1-inch slits at the model leading and trailing edges to provide a passageway for the air from the skin into the inner chamber of the hollow casting. This model configuration is herein referred to as configuration 1.

The model was made such that the chordwise flow could be altered by installing orifices in the model base casting as shown in figure 4. Flow between compartments formed by the $\frac{1}{4}$ -inch rods could be prevented by sealing the rods to

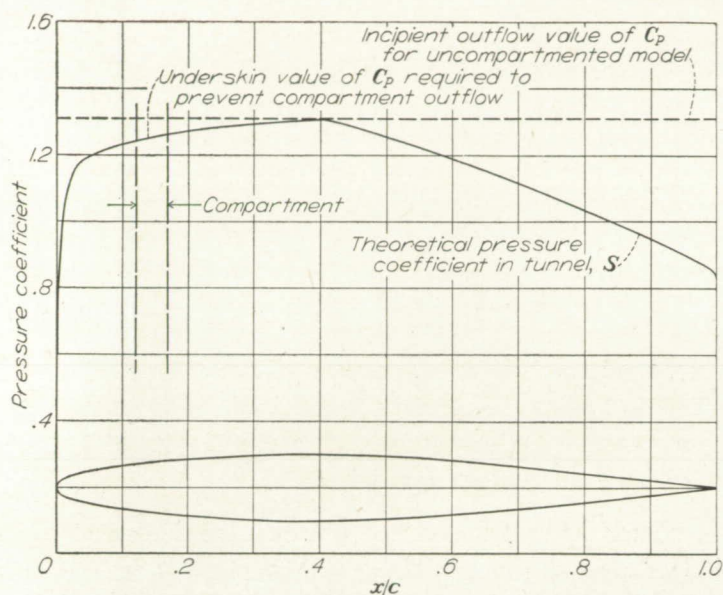


FIGURE 2.—External-pressure and suction-pressure distributions for porous bronze NACA 64A010 airfoil model. $\alpha_0 = 0^\circ$.

the skin with rubber cement. The model arrangement with orifices and compartment seal is referred to herein as configuration 2. A photograph which shows model configuration 2 with the skin removed is shown as figure 5.

For configurations 1 and 2, the upper and lower surfaces of the 13-inch center section of the span were constructed from a continuous sheet of porous sintered bronze with a single spanwise joint at the model trailing edge, which was fastened only at the spanwise edges to $\frac{1}{2}$ -inch inner end plates; consequently, a 12-inch span of the skin was left open to suction. In order to prevent outflow from the upper and lower leading-edge compartments, the leading-edge skin of configuration 2 was saturated with lacquer for a distance of 1 inch ($\frac{s}{c} = 0.028$) from the leading edge on both upper and lower surfaces.

A third suction arrangement was investigated (configuration 3) wherein the orifices in the model base casting were removed and a low-porosity skin was installed. Flow between compartments was not prevented by sealing the rods to the skin. Two bronze sheets were used to form the center part of the airfoil surfaces from the trailing edge to the 2-percent-chord station. A sheet of duralumin was formed around the leading-edge contour and butted to the bronze. The skin was fastened at the butt joint in addition to being fastened along the spanwise edges to the $\frac{1}{2}$ -inch inner end plates; the leading edge was glazed and faired with hard-drying putty up to the 5-percent-chord station.

The same sintered-bronze sheet was used in configurations 1 and 2 and was fabricated of spherical bronze powder that was specified to be small enough to pass through a 200-mesh screen but too large to pass through a 400-mesh screen. The thickness of the sheet was found to vary about ± 0.010 inch from a mean of 0.080 inch. The sintered-bronze sheets of configuration 3 were approximately 0.090-inch thick and

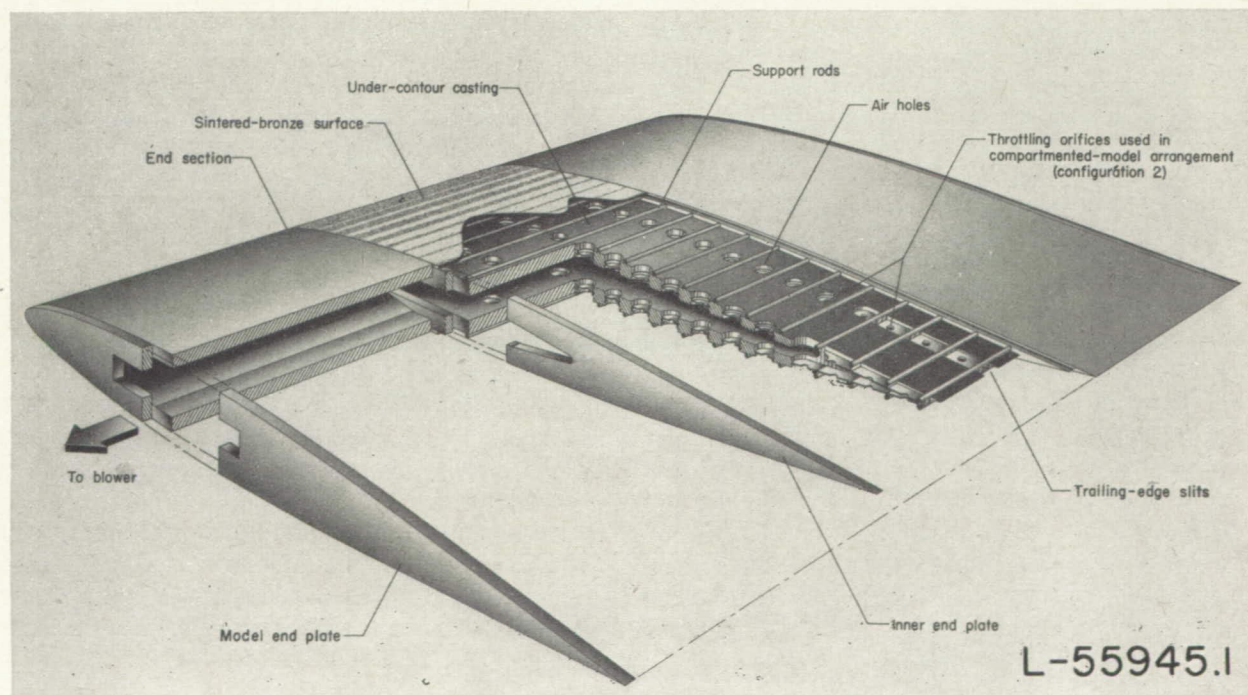


FIGURE 3.—Construction of porous bronze NACA 64A010 airfoil model.

L-55945.1

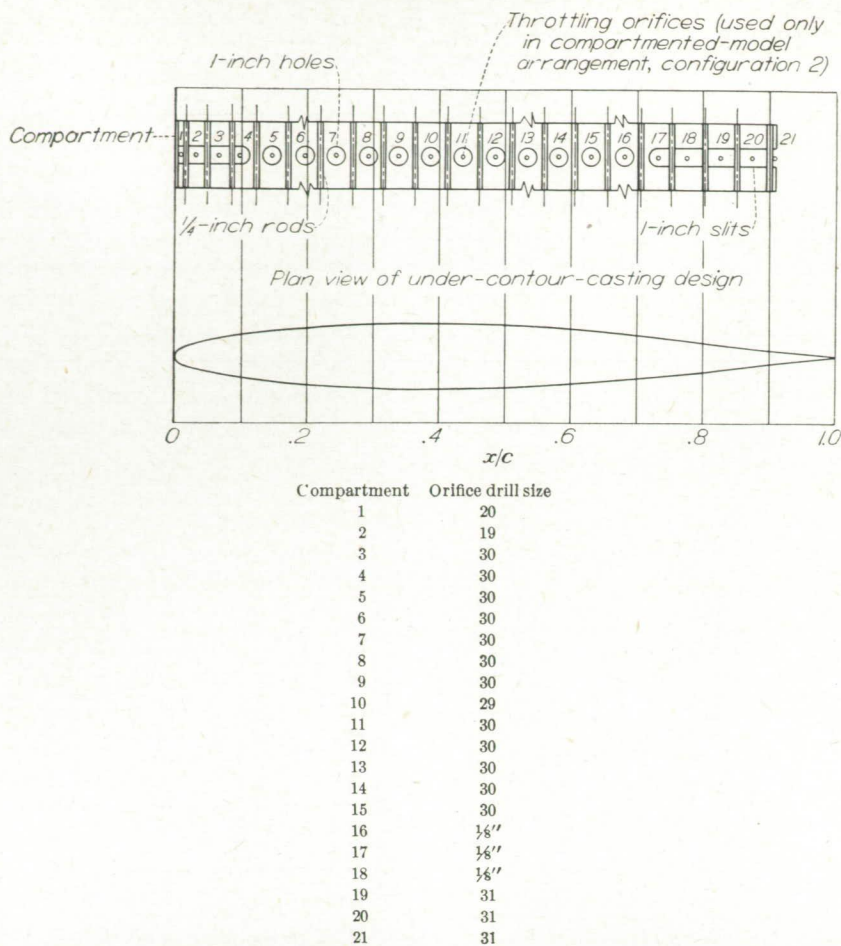


FIGURE 4.—Chordwise locations of support rods and throttling orifices of porous bronze NACA 64A010 airfoil model.



FIGURE 5.—NACA 64A010 airfoil model with porous sintered-bronze surface removed.

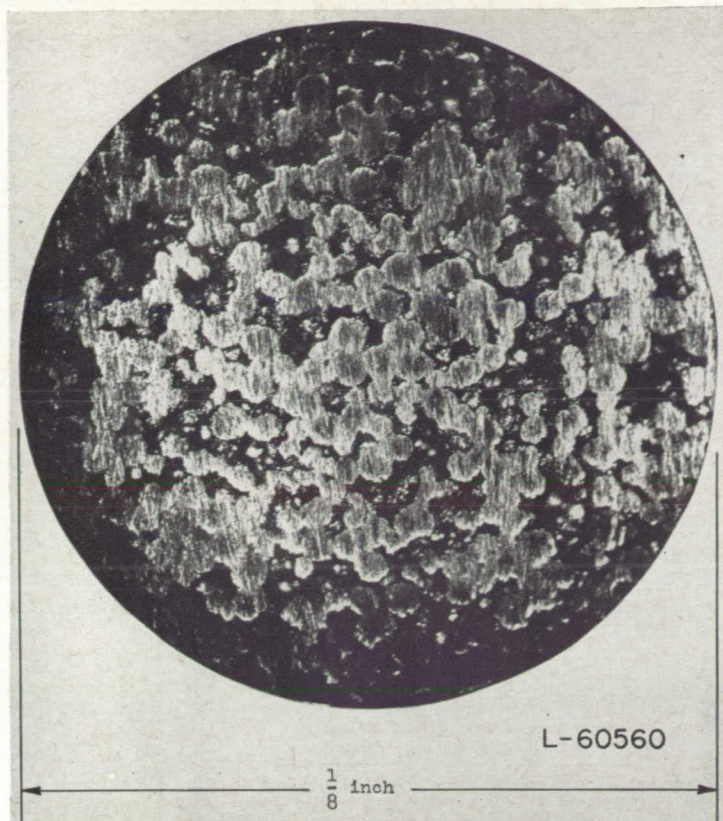
were fabricated of spherical bronze powder which was small enough to pass through a 400-mesh screen.

The porosity of the sintered-bronze material was such that the flow quantity varied directly with the pressure drop, as is characteristic of dense filters. With air at standard conditions the measured porosity of the skin of configurations 1 and 2 and the skin of configuration 3 was such that an applied suction of 0.06 and of 1.84 pounds per square inch, respectively, induced a velocity of 0.5 foot per second through the material; these values amount to porosity factors c_{r_0} equal

to 1.44×10^{-10} and 0.0525×10^{-10} square foot which are independent of the material thickness and the viscosity and density of the flow medium provided that the flow through the material is purely viscous. The outer surfaces of the sintered bronze were sanded to reduce local surface irregularities with a resultant decrease in porosity at local points because of a "smearing over" of metal particles. Frequent vacuum cleaning of the surfaces eliminated large changes in the porosity with time because of dust clog.

Photomicrographs of the sanded skin of configurations 1 and 2 are presented as figure 6 to give a visual indication of the porosity. Figure 6(a) is representative of about 80 percent of the sanded surface. Figures 6(b) and 6(c) indicate the amount by which the metal was smeared as a result of excessive sanding on poorly sintered areas. Less sanding was required for the skin used on configuration 3 inasmuch as the original surface texture was much better than that of configurations 1 and 2. Porosity measurements made after sanding, however, indicated that the average porosity for the whole airfoil was not affected appreciably by the sanding operations.

As a result of a low modulus of elasticity of the material and the variations in thickness, the airfoil contour of configurations 1 and 2 as tested was quite wavy. Absolute variations from the true profile were not measured; however,



(a) Representative of approximately 80 percent of total area.

FIGURE 6.—Photomicrographs of sanded sintered-bronze surface as tested.

a relative waviness survey was made at various spanwise stations with a three-point indicating mechanism (fig. 7). An estimate of the degree of waviness of the bronze surface may be obtained by comparing the profiles for the bronze surface with the profiles of the cast-aluminum end sections; the profiles of the end sections varied no more than ± 0.003 inch from the true airfoil profile. No waviness survey was made on model configuration 3; however, the surfaces were somewhat smoother and fairer than the surfaces of configurations 1 and 2.

APPARATUS AND TESTS

The model was tested in the Langley low-turbulence pressure tunnel and was mounted as shown in figure 1. A detailed description of this tunnel is given in reference 10. Flow measurements for the suction air were made by means of an orifice plate in the suction duct. The suction flow was taken through one of the model end plates and was regulated by varying the blower speed and the diameter of the orifice which was used to measure suction flow.

A static-pressure tube was used to measure the suction pressure in the inner compartment of the center casting. Since the velocity was low, the measured static pressure was very nearly equal to the total pressure. These data were used to obtain the total pressure lost by the suction air in passing from the free stream to the inner chamber of the model and were also used to give an indication of outflow which occurred when the pressure inside the model was greater than the lowest pressure on the airfoil surface.



(b) Representative of approximately 19 percent of total area.

FIGURE 6.—Continued.



(c) Representative of approximately 1 percent of total area.

FIGURE 6.—Concluded.

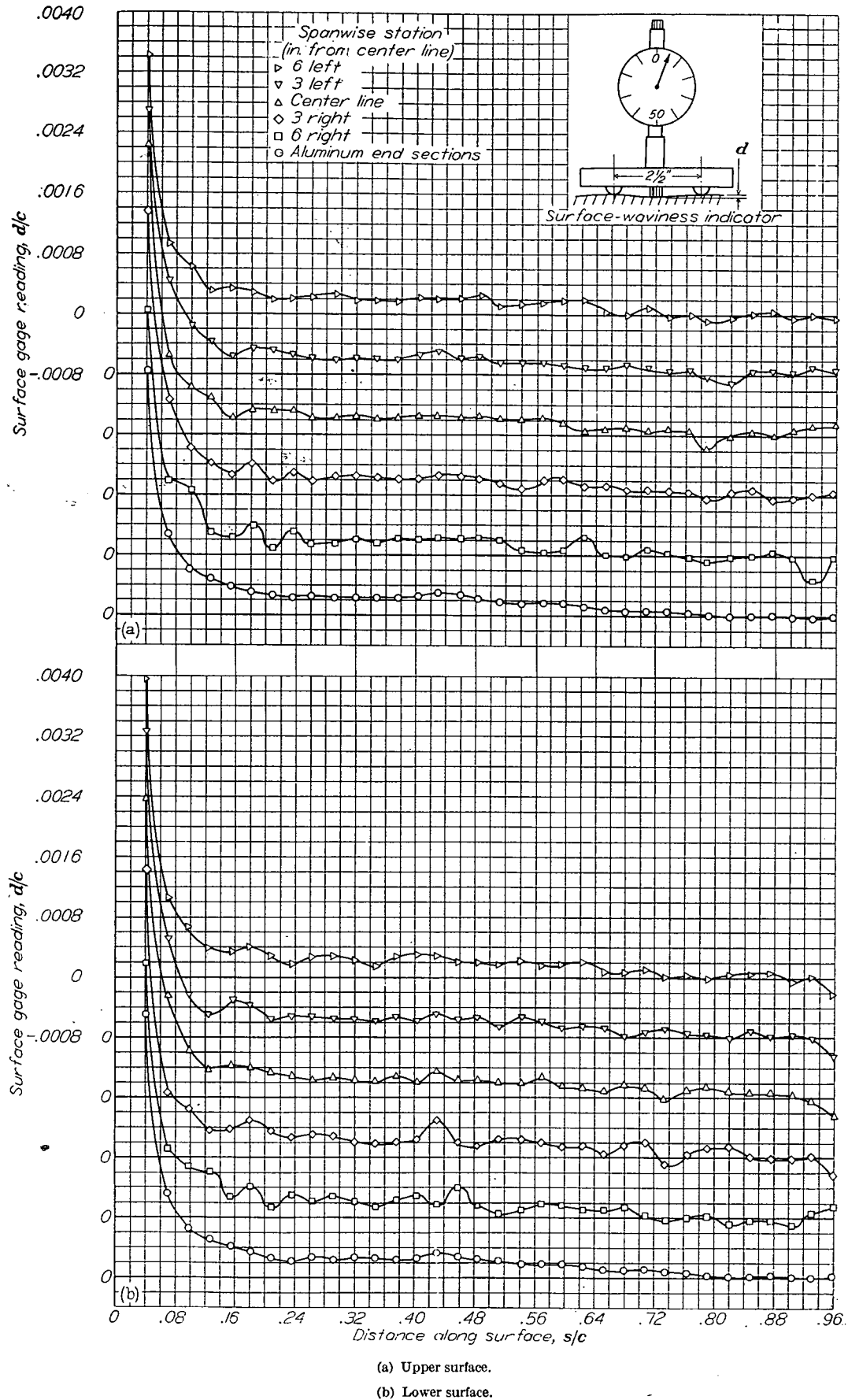


FIGURE 7.—Chordwise surface-waviness survey for various spanwise positions across porous bronze NACA 64A010 airfoil model. Configurations 1 and 2.

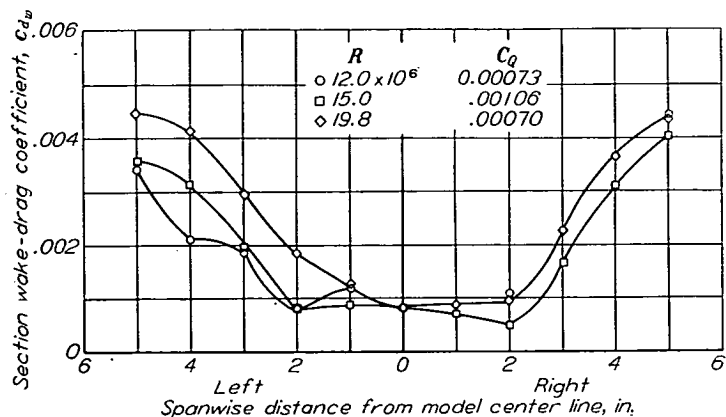


FIGURE 8.—Spanwise variation of section wake-drag coefficient on porous bronze NACA 64A010 airfoil model for three Reynolds numbers and suction-flow coefficients. $\alpha_0 = 0^\circ$; configuration 3.

For configurations 1 and 3, the skin was the only resistance to the suction; that is, air passages between the skin and the inner chamber were large enough to make the internal losses low in comparison with the pressure drop through the skin. The use of flow-control orifices in each compartment of configuration 2 had the disadvantage that any one set of orifices such as shown in figure 4 was able to produce a near-uniform chordwise inflow distribution only at free-stream Reynolds numbers below the design value, which for this case was 6.0×10^6 . No attempt was made to measure the flow in each compartment because of the exploratory nature of the tests. To obtain an indication of outflow through the skin, however, the suction pressure was measured in compartment 5 under the upper-surface skin (fig. 4). Because of the peculiarities of the throttling system, calculations indicated that compartment 5 would be the most critical to outflow.

Spanwise surveys of section wake-drag coefficient over the part of the model covered with the porous skin indicated appreciable variations in wake drag. Examples of these variations are shown in figure 8 for configuration 3. The large variations, occurring only over the outer portions of the bronze skin, were due to disturbances originating at or near the junctures between the bronze skin and the solid model end sections; over a spanwise region of approximately 4 inches at the center of the bronze skin, the measured wake-drag coefficients were rather constant. The wake drags and boundary-layer measurements presented herein were obtained at the center region of the model and are believed to correspond to the true two-dimensional conditions and to be essentially uninfluenced by the flow disturbances outside this region. All wake drags were measured with a survey rake located at about 70 percent of the chord behind the model trailing edge. Boundary-layer measurements were obtained with a conventional multitube "mouse" (reference 11) located on the center line of the upper surface of configuration 1 at 83 percent of the chord. Station 0.83c was the most rearward position at which the mouse could be mounted conveniently. The tests were made for Reynolds numbers up to 19.8×10^6 and for suction-flow coefficients up to 0.01 with the model set at zero angle of attack.

RESULTS AND DISCUSSION

THEORETICAL

In order to provide a standard of comparison for the experimental results, the characteristics of the laminar boundary layer were calculated for flow into the surface of the NACA 64A010 airfoil at an angle of attack of 0° . The minimum suction quantities necessary to keep the laminar boundary layer neutrally stable over the entire surface at Reynolds numbers of 6×10^6 , 15×10^6 , and 25×10^6 were computed. The stability of the boundary layer was also investigated for cases in which the velocity through the surface was everywhere the same.

The boundary-layer velocity profiles and thicknesses were calculated by the Schlichting method (reference 8), an approximate method. The velocity profiles of the Schlichting method are a single-parameter family of curves for which the parameter depends on the velocity of flow into the surface, the pressure gradient along the surface, the boundary-layer thickness, and the kinematic viscosity of the fluid. The single-parameter family of velocity profiles is used with the boundary-layer-momentum equation to obtain a first-order differential equation. In the calculations, the differential equation of reference 8 was integrated by Euler's step-by-step method. In the process of integrating the differential equation, the boundary-layer profiles and boundary-layer thicknesses are found at each point along the wing surface. The lengths of the steps in the integration process were about the same for all the calculations and were so small that halving them made no important difference for the no-suction case and a Reynolds number of 25×10^6 .

In order to begin the calculation, the value of $\frac{dz}{ds}$, the rate of change of a representative boundary-layer-thickness parameter, at $x=0$ (reference 8) was taken as zero. This value of $\frac{dz}{ds}$ was found from the equation

$$\left(\frac{dz}{ds/c} \right)_{\frac{s}{c}=0} = \left[\frac{z \left(\frac{d^2 U/U_0}{ds/c^2} \right) \frac{\partial G}{\partial k} + \frac{df_1}{ds/c} \sqrt{z} \frac{\partial G}{\partial k_1}}{\frac{dU/U_0}{ds/c} \left(1 - \frac{\partial G}{\partial k} \right) - \frac{f_1}{2} \frac{\partial G}{\partial k_1} \frac{1}{\sqrt{z}}} \right]_{\frac{s}{c}=0}$$

which was obtained by applying L'Hospital's Rule to the equation for $\frac{dz}{ds/c}$, equation (30) of reference 8. In order to continue the calculations as far as the trailing edge of the airfoil, it was necessary to modify slightly the Schlichting method by extrapolation of the curves in figures 5 and 6 of reference 8 beyond the value of k for which the Schlichting method breaks down. The recommendation of reference 8 that separation be assumed to exist when k equals -0.0682 was ignored in order to avoid the contradiction that the boundary-layer profile can become more convex and, at the same time, approach separation.

Lin's approximate formula (reference 1) was used to calculate the Reynolds number $R_{\delta^*_{cr}}$ at which any Schlichting velocity profile is neutrally stable. The stability theory as originally derived assumed that the boundary-layer thickness and velocity distribution did not vary with distance along the surface. Pretsch (reference 12), however, showed that the rates of variations in the thickness and velocity distribution which normally occur can have only a second-order effect on the stability. Lin's stability theory may be used, therefore, to calculate the stability of boundary layers in the presence of pressure gradients. When combined with the Schlichting method, Lin's formula becomes

$$R_{\delta^*_{cr}} = \frac{\left[1 - K\left(2 - \frac{6}{\pi}\right)\right] 25 \left(\frac{du/U}{d\eta}\right)_1}{u_c^4}$$

where u_c is equal to the value of u for which

$$-\pi \left(\frac{du/U}{d\eta}\right)_1 \left[3 - \frac{2 \left(\frac{du/U}{d\eta}\right)_1 \eta}{u/U}\right] \frac{u/U \left(\frac{d^2u/U}{d\eta^2}\right)}{\left(\frac{du/U}{d\eta}\right)^3} = 0.58$$

and where the subscript 1 denotes "at surface." The application of Lin's formula to the Schlichting profiles results in the curve of figure 9 which shows the variation of the critical Reynolds number $R_{\delta^*_{cr}}$ with change in the Schlichting velocity-profile shape parameter K . In figure 10 are shown velocity profiles for three values of K .

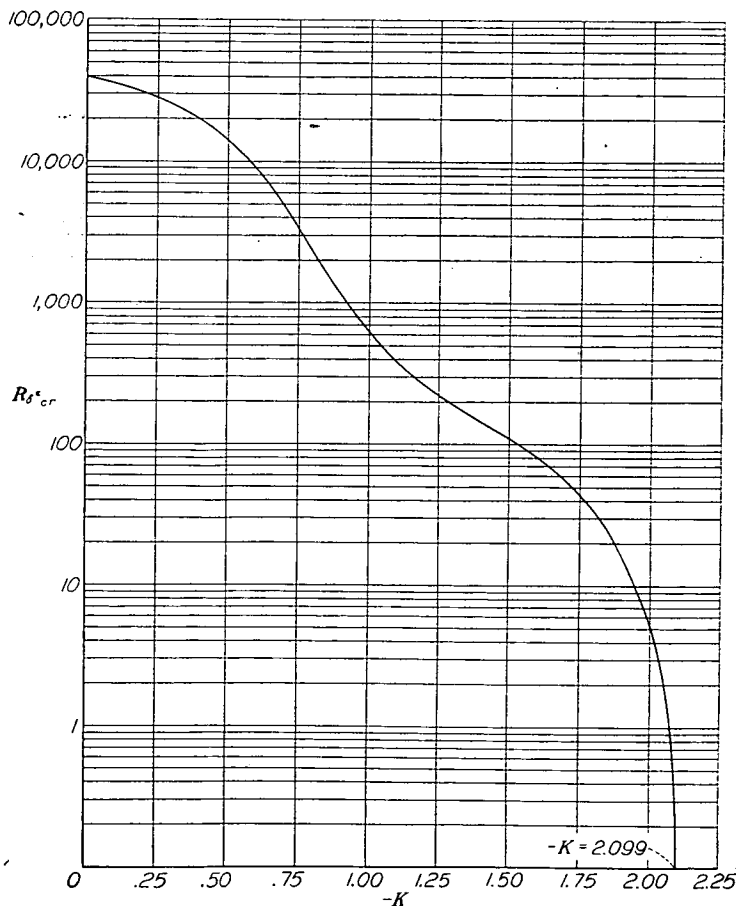


FIGURE 9.—Theoretical variation of critical boundary-layer Reynolds number with boundary-layer shape parameter.

A special procedure was used to calculate the distribution of the velocity of flow through the surface that was necessary to keep the boundary layer neutrally stable, $R_{\delta^*} = R_{\delta^*_{cr}}$. For these cases, suction was begun at the first station at which the boundary layer would become unstable without suction. The special method depends on the fact that the condition of neutral stability, $R_{\delta^*_{cr}} = R_{\delta^*}$, can be written as

$$R_{\delta^*_{cr}} = \frac{U}{U_0} \sqrt{z} \sqrt{R} \frac{\delta^*}{\theta}$$

or

$$\frac{R_{\delta^*_{cr}}}{\delta^*/\theta} = \phi(K) = \frac{U}{U_0} \sqrt{z} \sqrt{R}$$

where $R_{\delta^*_{cr}}$, $\frac{\delta^*}{\theta}$, and ϕ are functions only of K . The numerical value of the function $\phi(K)$ depends on $\frac{U}{U_0} \sqrt{z} \sqrt{R}$. With a known value of z at the station at which suction begins, K can be found from the value of $\frac{U}{U_0} \sqrt{z} \sqrt{R}$. All the quantities necessary to proceed to the next station by the step-by-step integration process can be calculated once K is determined. The calculation of these quantities provides the value of the local-suction-velocity ratio v_0/U_0 .

In figure 11 are presented the curves of R_{δ^*} and $R_{\delta^*_{cr}}$ as determined from the calculation for the suction flow required to keep the boundary layer neutrally stable at all points on the airfoil at a Reynolds number of 15×10^6 . The calculated

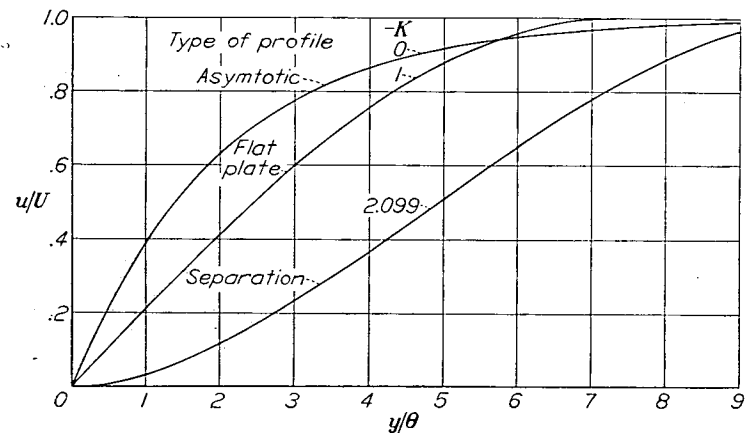


FIGURE 10.—Several boundary-layer profiles for values of shape parameter K as calculated by Schlichting.

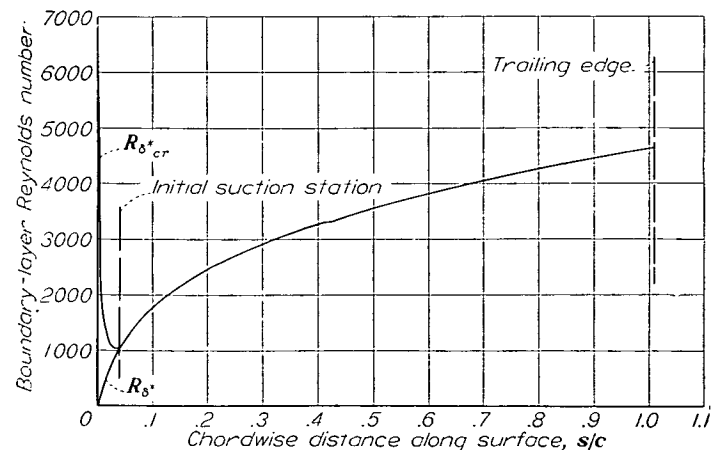


FIGURE 11.—Theoretical chordwise variation of boundary-layer Reynolds number on the NACA 64A010 airfoil for $R_{\delta^*} = R_{\delta^*_{cr}}$; $R = 15.0 \times 10^6$; $C_d = 0.000405$; $\alpha = 0^\circ$.

variation of v_0/U_0 over one surface is shown in figure 12. The suction flow required to keep the boundary layer neutrally stable decreases slowly as the region of falling pressure on the airfoil surface is traversed. When the region of rising pressure is entered, the required suction rises rapidly and continues to increase to the trailing edge. The summary of the results of the computations for the minimum suction quantities is presented in figure 13.

In figure 14 are presented curves of $R_{\delta^*_{cr}}$ and R_{δ^*} for cases where v_0/U_0 is the same over the entire surface. The figures illustrate that by a sufficient increase in the value of the suction parameter $\frac{v_0}{U_0} \sqrt{R}$ the position at which the boundary layer first becomes unstable on the NACA 64A010 airfoil

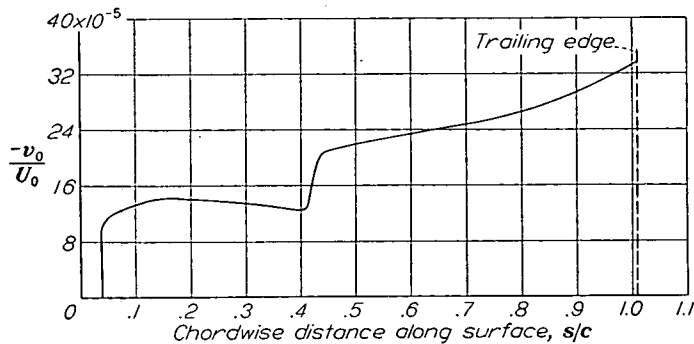


FIGURE 12.—Theoretical chordwise variation of minimum suction velocity ratio required to produce neutral stability at all points along the chord of the NACA 64A010 airfoil. $R=15.0 \times 10^6$; $\alpha_0=0^\circ$; $C_u=0.000405$.

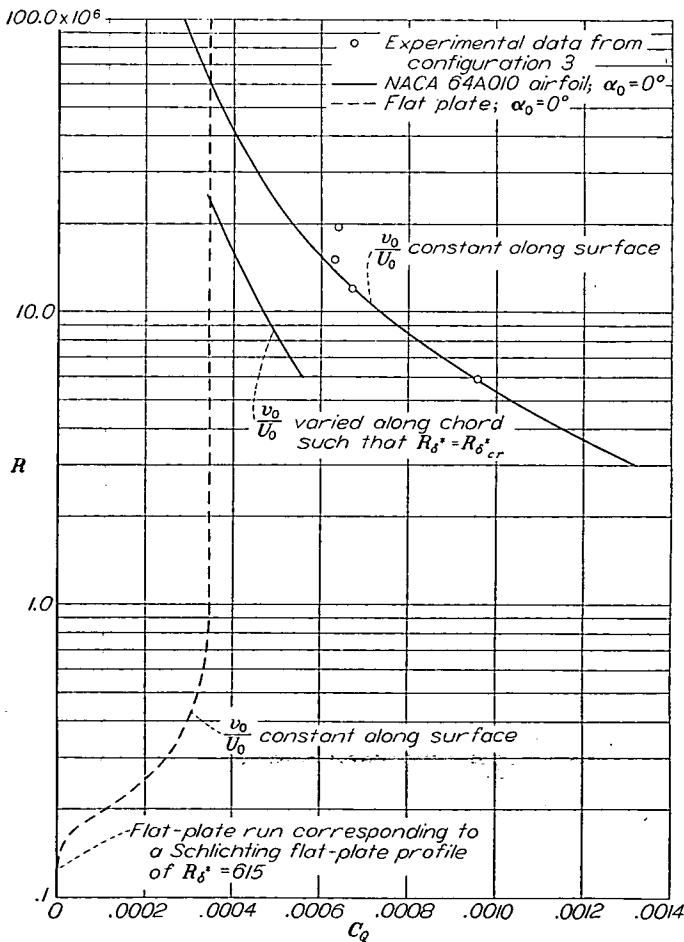


FIGURE 13.—Theoretical variation of free-stream Reynolds number with minimum suction-flow coefficient required to produce full-chord laminar stability.

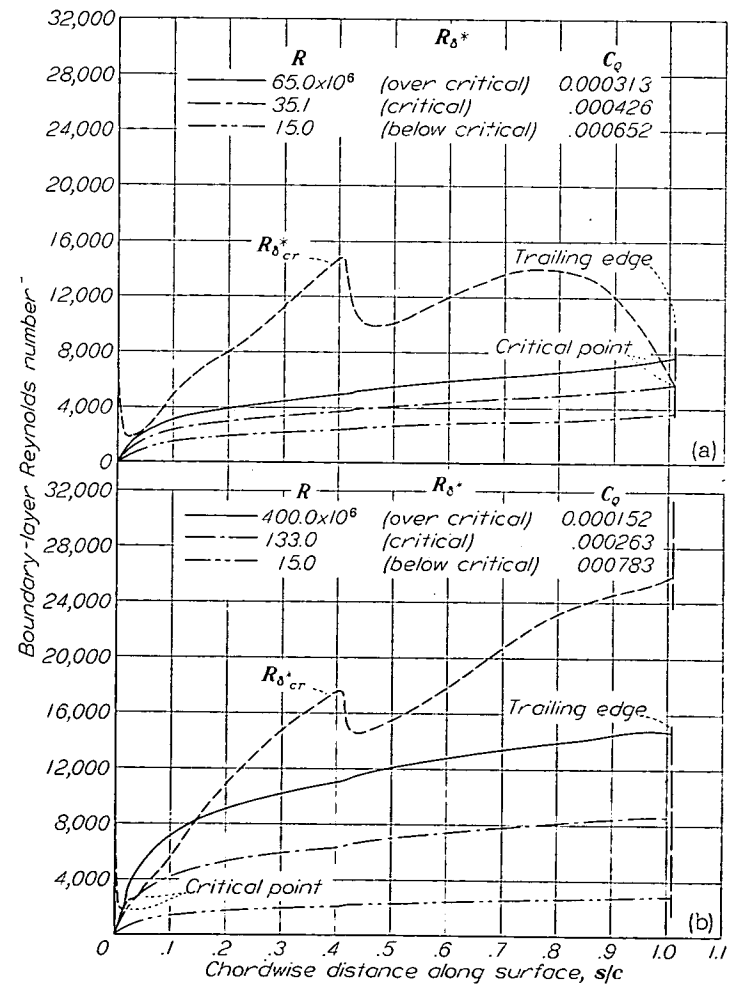
can be made to jump from the trailing edge to the region near the leading edge. It may be of interest to note from figure 14 (a) that the Reynolds number can be increased considerably and that the suction coefficient can be decreased appreciably without exposing much of the trailing surface to turbulent flow.

The curves of $R_{\delta^*_{cr}}$ and R_{δ^*} can be found at any Reynolds number, when they are known at one Reynolds number and one value of v_0/U_0 , by noting that if $\frac{v_0}{U_0} \sqrt{R}$ does not vary with Reynolds number then the $R_{\delta^*_{cr}}$ curve is independent of Reynolds number and

$$\frac{R_{\delta^*_2}}{R_{\delta^*_1}} = \sqrt{\frac{R_2}{R_1}} \quad (1)$$

Thus, for a given distribution of $\frac{v_0}{U_0} \sqrt{R}$, which results in a fixed distribution of $R_{\delta^*_{cr}}$, the R_{δ^*} curve for the same distribution of $\frac{v_0}{U_0} \sqrt{R}$ but some other R can be found by equation (1).

At some value of R , R_{δ^*} will touch and yet not cross the $R_{\delta^*_{cr}}$ curve at only one point. (See fig. 14.) This



(a) Critical point near trailing edge; $\frac{v_0}{U_0} \sqrt{R}=1.25$.

(b) Critical point near leading edge; $\frac{v_0}{U_0} \sqrt{R}=1.50$.

FIGURE 14.—Theoretical chordwise variation of boundary-layer Reynolds number and critical boundary-layer Reynolds number for the NACA 64A010 airfoil at three free-stream Reynolds numbers for constant chordwise inflow velocity. $\alpha_0=0^\circ$.

procedure results in one point on the curve of figure 13 which shows the variation of R with the minimum C_q required to produce full-chord laminar stability. Other points on the curve of figure 13 may be found by application of the foregoing procedure to other choices of $\frac{v_0}{U_0} \sqrt{R}$ which result in other $R_{s,cr}$ distributions.

Because the first theoretical studies of the effects of suction on boundary-layer stability were made for the flow over a flat plate, it was thought of interest to include the curve for the flat plate in figure 13 wherein it is illustrated that no suction is required to keep the flow stable on a flat plate when the Reynolds number is sufficiently low. This result is in contrast to that for the airfoil which, because of the adverse

pressure gradient over its rear portion, requires suction at all Reynolds numbers to maintain laminar flow to the trailing edge. Figure 13, however, indicates the rather surprising result that in order to keep full-chord laminar flow at large Reynolds numbers the NACA 64A010 airfoil requires smaller values of C_q than are required for a flat plate. This outcome seems reasonable because, near the leading edge where both the flat plate and the airfoil become critical at high Reynolds numbers, the airfoil profits from the existence of a favorable pressure gradient which increases the critical boundary-layer Reynolds number and decreases the actual boundary-layer Reynolds number over that of the flat plate for the same free-stream Reynolds number.

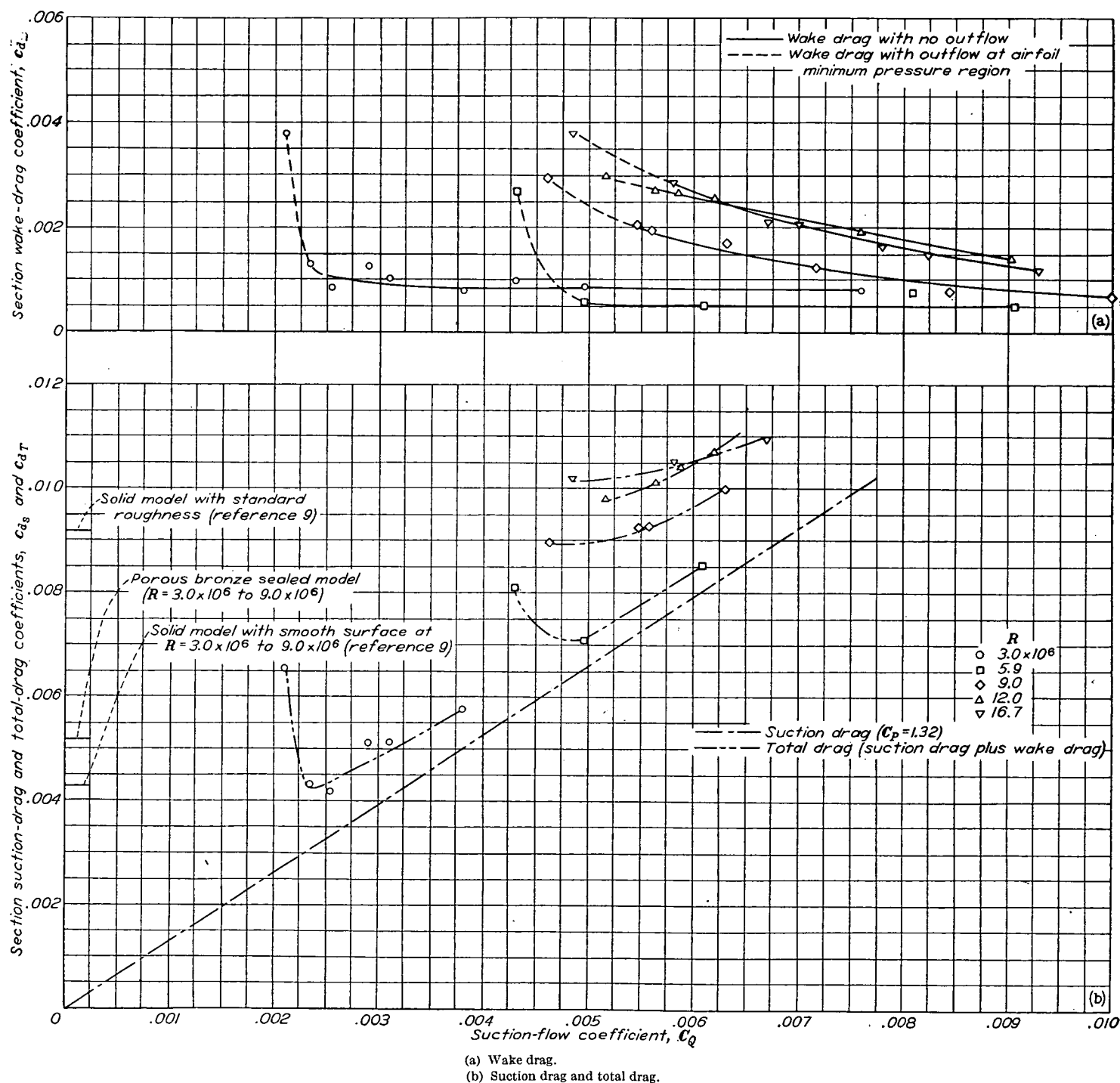


FIGURE 15.—Variation of section drag coefficients with suction-flow coefficient for porous bronze NACA 64A010 airfoil model. $\frac{C_{q0}}{c_l} = 7.2 \times 10^{-3}$; $\alpha = 0^\circ$; configuration 1.

EXPERIMENTAL CONFIGURATION 1

Wake drag.—The variation of section wake-drag coefficient with suction-flow coefficient is shown in figure 15(a) for Reynolds numbers up to 16.7×10^6 for configuration 1. The static pressure in the interior of this model arrangement was everywhere the same.

At a Reynolds number of 3.0×10^6 the section wake-drag coefficient for $C_q > 0.0028$ was constant and equal to about 0.0008. For a Reynolds number of 5.9×10^6 and for $C_q > 0.0050$ the wake-drag coefficient remained practically constant at 0.0005. The wake drag at Reynolds numbers of 9.0×10^6 , 12.0×10^6 , and 16.7×10^6 , however, decreased steadily with increase in C_q but never became less than the lowest drags obtained at a Reynolds number of 5.9×10^6 , even for a suction-flow coefficient as high as 0.010.

The rapid increase of c_{d_w} that occurred with decreasing C_q for values of C_q slightly less than 0.0024 at a Reynolds number of 3.0×10^6 and for values of C_q less than about 0.0048 at a Reynolds number of 5.9×10^6 was caused by a rapid forward shift of the point of transition from laminar to turbulent flow. The curves in figure 16 show that at 83 percent of the chord, at a Reynolds number of 6.0×10^6 , the boundary layer on the upper surface was laminar for a C_q of 0.0048 but was turbulent for only a slightly lower C_q of 0.0046. Other boundary-layer surveys, not presented herein, indicated that, whenever a rapid increase in drag coefficient accompanied a small decrease in flow coefficient, the boundary layer changed rapidly from laminar to turbulent over a large portion of the surface.

Outflow through the surface occurred when the static pressure inside the skin was greater than the minimum static pressure on the outside of the airfoil. The curves of c_{d_w} against C_q are shown in figure 15 (a) as dash lines when outflow occurred locally and as solid lines when inflow occurred over the whole airfoil. At Reynolds numbers of 3.0×10^6 and 5.9×10^6 the values of minimum C_q required to prevent outflow are seen to be very near the value of C_q at which the drag changes rapidly. It seems probable that outflow produced the rapid forward shift of the transition point

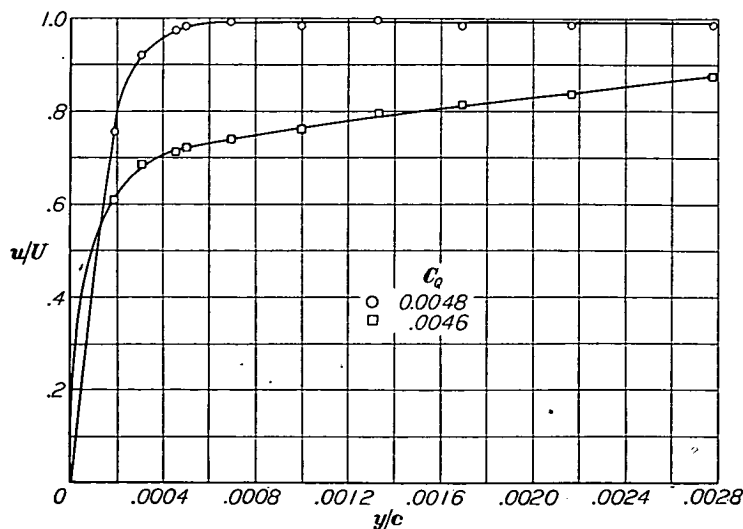


FIGURE 16.—Boundary-layer surveys for two values of suction-flow coefficient at station 0.83c on upper surface of porous bronze NACA 64A010 airfoil model. Configuration 1; $R = 6.0 \times 10^6$; $\alpha = 0^\circ$.

at least for Reynolds numbers up to 5.9×10^6 . From the plots of c_{d_w} against C_q the lowest C_q for which no outflow occurred at each Reynolds number is seen to increase with increasing Reynolds number. This increase of the minimum value of C_q with Reynolds number is a result of the linear variation of flow velocity through the sintered bronze with pressure drop across the surface; the minimum C_q for no outflow increases with the free-stream Reynolds number. A more detailed treatment of this subject is presented in appendix A.

Wake-drag measurements for Reynolds numbers of 9.0×10^6 , 12.0×10^6 , and 16.7×10^6 as shown in figure 15 (a) indicated that laminar flow was probably not maintained over the complete chord of the model. The gradual decrease of wake-drag coefficient with increasing flow coefficient may have been caused either by a gradual rearward movement of the transition point with increasing flow coefficient or by a mere reduction in the size of the turbulent boundary layer. The extreme thinness of the boundary layer at these high Reynolds numbers prevented an accurate determination of its shape even near the trailing edge. It should be noted that full-chord laminar flow was not maintained even though the suction pressures were sufficient to prevent outflow. Theoretical calculations for a Reynolds number of 16.7×10^6 for the configuration and inflow distribution tested indicated that the laminar boundary layer should have been very stable.

The basis for a possible explanation of the transition difficulties at the higher Reynolds number is indicated in reference 13. It is shown therein that the presence of a surface projection will cause premature transition of a laminar boundary layer when the Reynolds number based on the height of the projection and the velocity at the top of the projection exceeds a critical value that is dependent on the geometry of the projection.

Although the numerous protuberances on the bronze skin were sanded to very small dimensions, calculations indicated that the relatively large amount of suction near the leading edge of the model so thinned the boundary layer (especially at the higher Reynolds numbers) that even the particles forming the material probably projected completely through the boundary layer. Because of the high velocity at the top of the particles at the high Reynolds numbers, it seems entirely possible that the critical Reynolds number for the roughness was exceeded in spite of its small height, and thus large disturbances were introduced into the boundary-layer flow. The relatively high drag coefficients measured under such conditions of roughness show that the suction-type profile is not stable to sufficiently large disturbances. The relative stability of the suction-type and no-suction-type profiles to finite disturbances is an important problem for future research.

The difficulties associated with obtaining low drag at high values of the Reynolds number are seen from the foregoing discussion to result from nonuniformity of the chordwise inflow distribution (excessive suction near leading edge) and from the surface irregularities of the material. If the Reynolds number is increased by increasing the value of U_0/ν while the size of the model is held constant, the inflow distribution becomes increasingly nonuniform (see equation (A4), appendix A) and the ratio of the size of the roughness

to the boundary-layer thickness increases. Both of these effects are unfavorable for obtaining low drag at high Reynolds numbers. On the other hand, if c_{r0}/t is held constant (no change in the type of material used) and if the Reynolds number is increased by increasing the chord of the model, the inflow distribution will remain unchanged but the ratio of the size of the roughness to the boundary-layer thickness will vary inversely as the square root of the Reynolds number. For this reason, it seems likely that favorable results can be obtained more easily with a large model than with a small one.

Total drag.—The measured pressure-loss coefficient for the suction air was used to calculate the section suction-drag coefficient and the results are shown in figure 15 (b). The suction-drag coefficient based on the model chord was calculated as $C_P \times C_Q$, which is the drag equivalent of the power required to pump the suction air back to free-stream total pressure. (See appendix B.) In this method of calculating suction drag, the over-all pumping efficiency is considered to be equal to that of the main propulsive system.

As may be seen in figure 15 (b), the variation of c_d was assumed to be linear with C_Q for all Reynolds numbers because of the small variation in C_P which averaged about 1.32. Inasmuch as there is no induced drag on a two-dimensional model, the total drag is the sum of the suction and wake drags. As shown in figure 15 (b), the minimum total drag at a Reynolds number of 3.0×10^6 occurred at a C_Q of about 0.0024, which was slightly less than the minimum C_Q required to prevent outflow; the minimum total-drag coefficient of 0.0042 is an inappreciable improvement over 0.0043, the drag coefficient of a solid NACA 64A010 airfoil with a smooth and faired surface at a Reynolds number of 3.0×10^6 (shown in figure 15 (b) as taken from reference 9). The porous model, however, did not have a completely smooth and faired surface, and a test made with the surface sealed resulted in a no-suction-drag coefficient of approximately 0.0052. The value of 0.0052 is probably somewhat low inasmuch as the surface was sealed with external applications of water glass and wax; as a result, the surface texture was unavoidably improved over that for the unsealed condition although the waviness of the surface was probably not affected. The no-suction-drag coefficient, therefore, would probably be somewhat greater than 0.0052 but less than 0.0092, the value for the extreme condition of standard leading-edge roughness on the solid airfoil (reference 9). The minimum total-drag coefficient increased with increasing Reynolds number with the result that, at a Reynolds number only slightly greater than 3.0×10^6 , no decrease in total drag was obtained by suction. A large proportion of the total drag consisted of suction drag because of the excessive amounts of air required at the leading-edge and trailing-edge surfaces of the airfoil in order to prevent outflow at the minimum pressure point.

The total amount of air that needs to be withdrawn in order to prevent outflow can be greatly reduced by applying suction separately to small portions of the airfoil surface. This purpose was accomplished by dividing the underskin region into separate compartments (configuration 2) as described previously in the section entitled "Model."

CONFIGURATION 2

Wake drag.—Wake-drag tests made on the compartmented model (configuration 2) previous to the tests reported herein indicated that the nose should be sealed for about 1 inch back from the leading edge (fig. 4) in order to obtain low drags at reasonable suction coefficients. The character of the flow at the leading edge before the nose was sealed was not clearly established, but early transition probably occurred because of excessive local outflow near the nose where the external-pressure variation over the first compartment was very large. A better understanding of these outflow difficulties may be gained from the discussion on compartmentation in appendix A.

As shown in figure 17 (a), the section wake-drag coefficient for the model with sealed nose and compartments was less than 0.0010 (for Reynolds numbers as high as 7.8×10^6) for suction-flow coefficients that ranged from 0.0015 at a Reynolds number of 3.0×10^6 to about 0.0035 at a Reynolds number of 7.8×10^6 . At a Reynolds number of 9.1×10^6 the wake drag remained greater than 0.0030 even for flow coefficients as high as 0.0040. Failure to obtain lower wake drags was believed to result from the presence of outflow. Although no outflow and low wake drags might have been obtained at higher suction-flow coefficients, the total drag would not have been reduced because of the excessive suction drags that would have resulted from the large suction quantities.

A comparison of figures 15 (a) and 17 (a) indicates that for equal Reynolds numbers the minimum suction-flow coefficients for which low wake drags were obtained for the compartmented model were about one-third of those required for configuration 1. The largest Reynolds number at which low drag occurred was also extended by compartmenting the model. Thus, low wake drags might be obtained at Reynolds numbers higher than 7.8×10^6 and at suction-flow coefficients lower than 0.0035 if the chordwise distribution of suction were further controlled.

Total drag.—Throughout the low-drag range the average total-pressure loss in the suction air for the compartmented model corresponded to an average pressure-loss coefficient of about 1.40. This average loss coefficient was used to compute the suction-drag coefficient that could be expected for an airfoil compartmented in the same manner as the model tested. The sum of the wake and suction drags is shown in figure 17 (b). At Reynolds numbers of 3.0×10^6 and 5.9×10^6 the minimum c_{dT} was about 0.0028. At

higher Reynolds numbers the minimum c_{dT} was larger because of the larger C_q necessary for low wake drags. In spite of the larger C_q at a Reynolds number of 7.8×10^6 , the minimum total-drag coefficient, 0.0048, was only slightly greater than the drag coefficient for the smooth solid-surface model of the same contour (reference 9) and somewhat

less than the drag coefficient for the porous bronze model with the skin entirely sealed to suction.

The over-all gain in compartmenting the model may be seen by comparing the total drags in figures 15 (b) and 17 (b) from which it is seen that at a Reynolds number of 5.9×10^6 the total drag of the compartmented model (configuration 2)

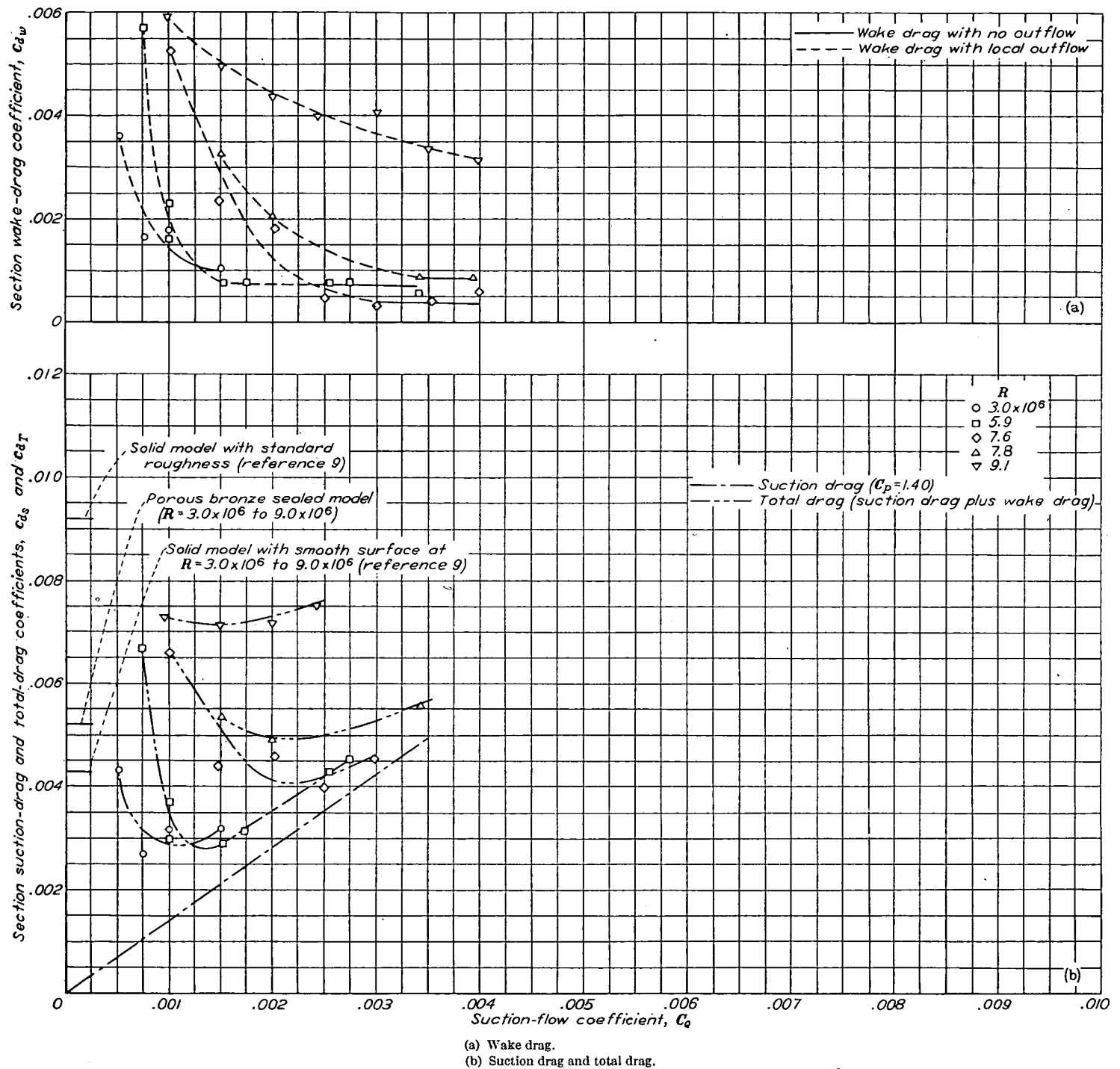


FIGURE 17.—Variation of section drag coefficients with suction-flow coefficient for porous bronze NACA 64A010 airfoil model. $\frac{c_{r0}}{c_l} = 7.2 \times 10^{-4}$; $\alpha_0 = 0^\circ$; configuration 2.

was only 40 percent of that for the uncompartmented model (configuration 1). In an attempt to improve further the chordwise inflow distribution and thus to obtain full-chord laminar flow and reductions in total drag at larger values of the Reynolds number, the model was equipped with a porous sintered-bronze surface of much lower porosity (configuration 3) as described previously in the section entitled "Model."

CONFIGURATION 3

Wake drag.—The leading edge of this model configuration was sealed to the 5-percent-chord station, which is approximately the chordwise station ahead of which no suction is required theoretically to maintain the boundary-layer

Reynolds number less than the critical value for all test Reynolds numbers anticipated (figs. 11 and 14). The section wake-drag coefficients are plotted against suction-flow coefficient in figure 18 (a) for Reynolds numbers from 5.9×10^6 to 19.8×10^6 . The static pressure in the interior of this model arrangement was everywhere the same as it was for the case of configuration 1.

The variation of drag coefficient with suction-flow coefficient C_q is similar for all Reynolds numbers investigated; that is, an abrupt decrease in drag coefficient to a value of about 0.0008 occurs at some critical value of C_q dependent upon the Reynolds number and is followed by a much more gradual decrease with an increase in C_q . Full-chord

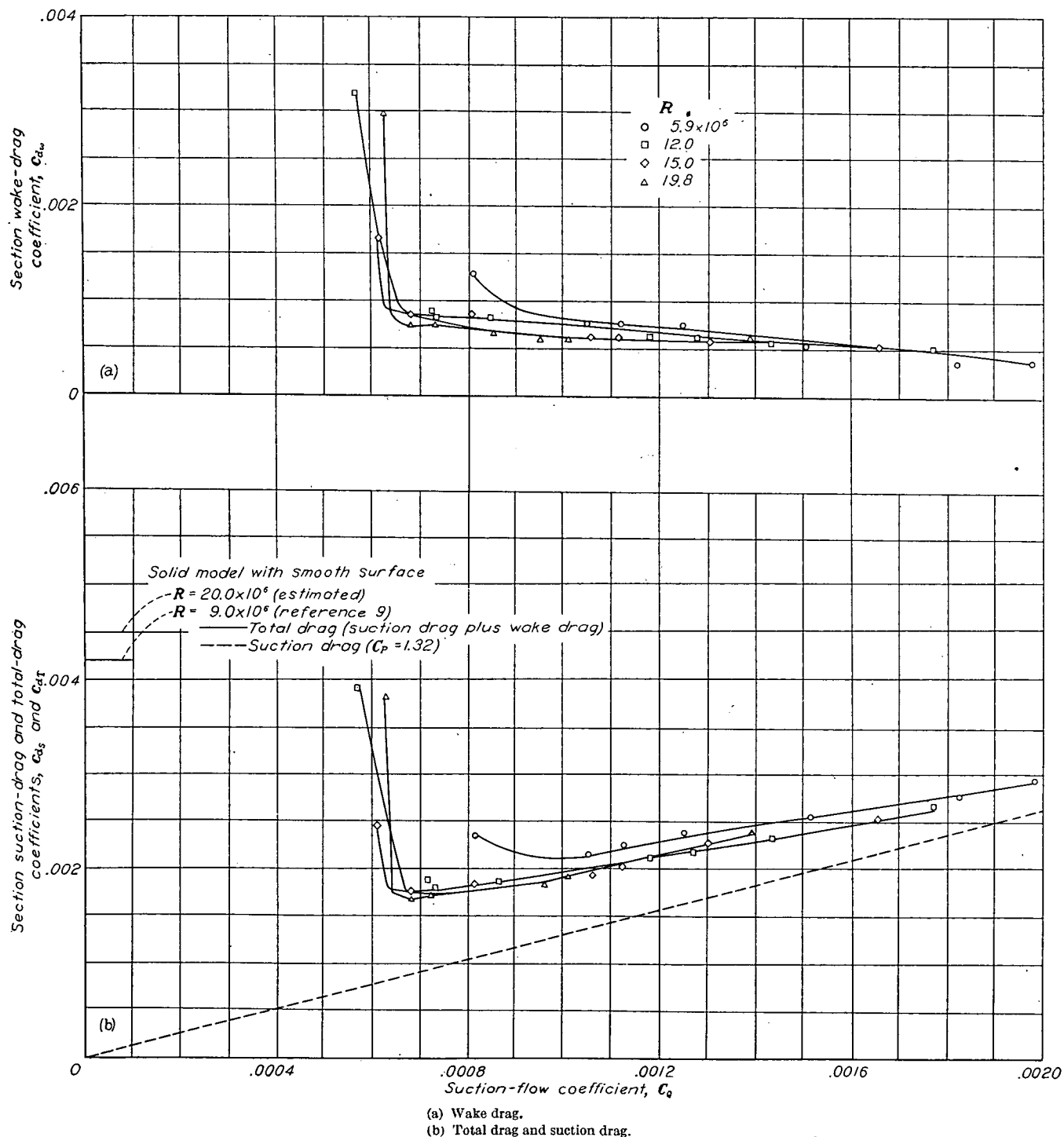


FIGURE 18.—Variation of section drag coefficients with suction-flow coefficient for porous bronze NACA 64A010 airfoil model. $\frac{C_q}{c_l} = 2.33 \times 10^{-10}$; $\alpha_0 = 0^\circ$; configuration 3.

laminar flow is indicated for all values of wake-drag coefficient of about 0.0008 or less, inasmuch as boundary-layer profiles measured during tests of configuration 1 indicated that the boundary layer changed rapidly from turbulent over a large portion of the airfoil to laminar over virtually the full chord whenever a large decrease in drag accompanied a small increase in flow coefficient. The gradual reduction in drag with a further increase in C_Q above the value at which the large drag change occurs corresponds only to a progressive thinning of the full-chord laminar boundary layer. For the tests of configurations 1 and 2, the sudden forward movement in point of transition from laminar to turbulent flow with a small decrease in C_Q was caused by outflow of air through the surface when the static pressure inside the skin was greater than the minimum static pressure on the outside of the airfoil. For the tests of configuration 3, however, the skin was dense enough so that no outflow of air occurred for all values of C_Q investigated at Reynolds numbers up to at least 15×10^6 .

The probable cause of the rapid increase in drag and corresponding forward movement of transition observed in the tests of configuration 3 at Reynolds numbers up to 15×10^6 is revealed by a comparison of the values of C_Q at which the rapid drag rises occurred (fig. 18 (a)) with the minimum theoretical values of C_Q required to keep the laminar boundary layer stable. The experimental values of C_Q at which the rapid drag rises occurred for configuration 3 are plotted in figure 13 for ease of comparison. The minimum experimental values of C_Q for full-chord laminar flow are of the same order of magnitude as the theoretical values and decrease with an increase in Reynolds number in the same manner as the theoretical up to a Reynolds number of 15×10^6 . No outflow of air is known to have existed up to a Reynolds number of 15×10^6 and because of the sealed nose which prevented excessive suction at the leading edge and the especially smooth surface of this model configuration, it is believed that roughness did not cause transition at the low values of C_Q ; therefore, the sudden drag rise probably occurred when the suction became insufficient (at least over a portion of the airfoil chord) to stabilize the laminar boundary layer. At a Reynolds number of 19.8×10^6 outflow of air rather than insufficient suction is likely to have caused the rapid drag rise with a decrease in C_Q . For the porous bronze skin used in the tests of configuration 3, a value of C_P equal to 1.32 was measured at a value of C_Q of about 0.0007 at a Reynolds number of 19.8×10^6 . The minimum local pressure on the airfoil exterior when mounted in the wind tunnel corresponds to a value of pressure coefficient S equal to about 1.31. Inasmuch as C_P decreases with a decrease in C_Q , a value of C_Q slightly lower than 0.0007 would result in outflow of air through the surface and forward movement in transition from laminar to turbulent flow.

The maximum Reynolds number at which recorded data were obtained was 19.8×10^6 . During the tests, however, at a value of C_Q less than 0.001, full-chord laminar flow was maintained up to a Reynolds number of about 24×10^6 as evidenced by a visual observation of the drag manometer which indicated values of the drag coefficient of about 0.0006. Unfortunately, however, a sudden change in the model occurred before any data could be recorded. It was

found that the bronze skin had buckled to such an extent that it was impossible to repeat the full-chord laminar-flow tests at Reynolds numbers somewhat lower than 20×10^6 , although before the skin took the permanent set, repetition of the low-drag results could be made.

Total drag.—The dashed curve of figure 18 (b) represents the variation of c_{d_r} with C_Q for an assumed constant value of C_P equal to 1.32. This value of C_P was chosen in order to present an indication of the minimum suction-drag coefficient required to maintain full-chord laminar flow at any value of the suction-flow coefficient and Reynolds number. As pointed out in the previous section entitled "Wake drag" a value of C_P slightly lower than 1.32 would result in outflow of air through the surface near the 0.40c position and a forward movement in the position of transition. The curves of section total-drag coefficient plotted against C_Q (fig. 18 (b)) are the sums of the suction and wake drags. The minimum section total-drag coefficient decreased with an increase in Reynolds number because of the decrease in C_Q required to obtain low wake drags. At a Reynolds number of 19.8×10^6 the minimum c_{d_r} was 0.0017 as compared with a value of about 0.0045 (estimated from data of reference 14) for an NACA 64A010 airfoil without boundary-layer control at a Reynolds number of 20×10^6 .

COMPARISON BETWEEN THEORY AND EXPERIMENT

The theoretical variations of R_{s^*} , $R_{s^{*cr}}$, and v_0/U_0 with s/c cannot be compared with experiment because local inflow velocities and boundary-layer velocity profiles were not measured. The theoretical total-suction quantities can, however, be compared with the total-suction quantities at the knees of the curves of wake-drag coefficient against suction-flow coefficient.

The following table contains the results of the comparison between theory and experiment:

R	Minimum values of C_Q for laminar stability				
	Theoretical		Experimental		
	Neutral stability	Constant inflow	Configuration 1	Configuration 2	Configuration 3
3.0×10^6	-----	0.00132	0.0028	0.0015	-----
5.9	0.00056	.00096	.0048	.0015	0.00096
7.6	.00051	.00084	-----	.0024	-----
7.8	.00051	.00083	-----	.0034	-----
9.0	.00048	.00077	.0100	-----	-----
12.0	.00044	.00067	-----	-----	.00087
15.0	.00041	.00060	-----	-----	.00063
19.8	.00037	.00054	-----	-----	.00064

Although the chordwise inflow distribution for configuration 3 was more uniform than for either of the previous configurations and although the minimum experimental values of C_Q for laminar stability agree very closely with the theoretical values of C_Q for a constant chordwise inflow, the close agreement cannot be construed as a quantitative check of the theory inasmuch as it is known that the experimental chordwise inflow distribution was not constant.

As shown in the table the minimum experimental values of C_Q for low wake drags for configurations 1 and 2 were appreciably greater than the theoretical values and also increased with an increase in free-stream Reynolds number in contrast to the trend indicated by the theory. This experimental

variation of minimum C_q with Reynolds number for configurations 1 and 2 was shown previously to be a result of outflow of air through the surface and the fact that the minimum C_q for incipient outflow increases with the Reynolds number. (See appendix A.) When it became possible to attain lower values of C_q for incipient outflow at a given Reynolds number by means of the dense skin of configuration 3, the minimum values of C_q became of the same order of magnitude as the theoretical values and had the same trend with Reynolds number as predicted by theory; that is, the minimum C_q decreased with an increase in Reynolds number up to the maximum test Reynolds number at which outflow of air once again prevented a further decrease in C_q for full-chord laminar flow. It appears, therefore, that the theoretical concepts with regard to area suction are valid; the Reynolds number itself, then, should not be a limiting parameter in attainment of full-chord laminar flow provided that the airfoil surfaces are kept sufficiently smooth and fair and provided that outflow of air through the surface is prevented.

Quantitative information on the effects of surface roughness or fairness on the stability of the laminar boundary layer, however, is not yet available although some indications of adverse effects of roughness on the ability to maintain full-chord laminar flow were discussed previously. Another indication of the adverse effects of surface roughness was obtained during a preliminary test run on configuration 1 before the model surfaces were sanded.

For this condition, numerous protuberances existed of sufficient magnitude to cause premature transition of the boundary layer at a Reynolds number as low as 3.0×10^6 . It is possible, however, that the excessive suction at and near the model leading edge may have unduly decreased the boundary-layer thickness relative to the size of the projections; thus, the sensitivity of the boundary layer to the existing disturbances was unnecessarily increased. This explanation, as discussed previously, was prompted by the work of reference 13. Although surface roughness did not appear to cause transition on the sealed-nose configurations (configurations 2 and 3) for the combinations of Reynolds numbers

and flow rates that were tested, it is believed that with other combinations of Reynolds numbers and suction rates (at least for the same model) the boundary layer could become thin enough to allow the surface roughness to cause transition. More information on the effects of area suction on the stability of the laminar boundary layer in the presence of surface disturbances is still required before it can be determined whether area suction can be made practical for the control of the laminar boundary layer.

CONCLUDING REMARKS

Results were presented of a low-turbulence wind-tunnel investigation of an NACA 64A010 airfoil with porous surfaces of sintered bronze. Full-chord laminar flow was maintained by the application of area suction up to a Reynolds number of 19.8×10^6 . At a Reynolds number of 19.8×10^6 , the total-drag coefficient (wake drag plus the drag equivalent of the suction power required) was equal to 0.0017 as compared with an estimated value of 0.0045 for a smooth and fair NACA 64A010 airfoil without boundary-layer control at a Reynolds number of 20×10^6 . The minimum experimental values of suction-flow coefficient for full-chord laminar flow were of the same order of magnitude as the theoretical values and decreased with an increase in Reynolds number in the same manner as the theoretical values. It seems likely from the results that attainment of full-chord laminar flow by means of continuous suction through a porous surface will not be precluded by a further increase in Reynolds number provided that the airfoil surfaces are maintained sufficiently smooth and fair and provided that outflow of air through the surface is prevented. Further research is required, however, to determine quantitatively the effect of finite disturbances on the stability of suction-type velocity profiles.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., March 30, 1951.

APPENDIX A

SINTERED-BRONZE SUCTION REQUIREMENTS FOR THE CASE OF INCIPIENT LOCAL OUTFLOW

Several important conclusions may be drawn from a study of the conditions that cause local outflow through a porous sintered-bronze sheet installed on an airfoil as a boundary-layer suction surface. Outflow occurs through any point on the skin where the local static pressure on the inside of the skin is greater than the local external static pressure despite the existence of a difference in pressures at all other points such as to produce inflow. An internal pressure just low enough to prevent outflow at some critical point produces an average value of area suction-flow coefficient which depends on the porosity characteristics of the material.

For a given type of sintered bronze, the velocity into the surface varies directly as the pressure drop across the surface and inversely as the thickness of the sheet and the absolute viscosity of the fluid; thus,

$$v_0 = -\frac{c_{v_0} \Delta p}{\mu t} \quad (A1)$$

A dimensional analysis will show that the factor c_{v_0} has the dimensions of length². This length is related to the effective diameter of the passages leading through the material. So long as the flow is of the purely viscous type, the value of c_{v_0} for a specific piece of porous material may be expected to be independent of the physical characteristics of the fluid passing through the material—that is, the viscosity and density.

Inasmuch as the suction-drag coefficient increases directly as the suction-flow coefficient C_Q (see appendix B), it is of interest to note the manner in which the porosity relation, equation (A1), affects the minimum C_Q necessary to prevent outflow. The average area suction-flow coefficient for both sides of a two-dimensional airfoil may be written as

$$C_Q = \frac{Q}{U_0 c b} = \frac{-\oint b v_0 ds}{U_0 c b} \quad (A2)$$

where the quantity of suction air is obtained by integrating the inflow velocity v_0 over both upper and lower surfaces as indicated by the line integral sign. The inflow velocity given by equation (A1) may be rewritten as

$$v_0 = -\left(\frac{c_{v_0} q_0}{\mu t}\right) \left[\left(\frac{H_0 - H_i}{q_0}\right) - \left(\frac{H_0 - p}{q_0}\right) \right] \quad (A3)$$

where H_i is equal to the internal static pressure because the internal velocity, and therefore the dynamic pressure, is extremely low. The two relations in equation (A3), $\frac{H_0 - H_i}{q_0}$ and $\frac{H_0 - p}{q_0}$, are the suction pressure-loss coefficient and the external airfoil pressure coefficient, respectively. The inflow velocity at any point is now

$$v_0 = -\frac{c_{v_0} q_0}{\mu t} (C_P - S)$$

or

$$\frac{v_0}{U_0} = -\frac{c_{v_0}}{2ct} R (C_P - S) \quad (A4)$$

The suction-flow coefficient for the whole wing can now be written as

$$C_Q = -\frac{c_{v_0} R}{2ct} \oint (C_P - S) d\frac{s}{c} \quad (A5)$$

It may be seen by equation (A5) that, if conditions of airfoil pressure coefficient and suction pressure-loss coefficient remain fixed for a given porous-skin model wherein c_{v_0} , t , and c are fixed, the value of C_Q to be associated with a given value of $\oint (C_P - S) d\frac{s}{c}$ will vary linearly with R . Thus, for the outflow value of the integral where C_P equals S at some point on the airfoil and is higher at all other points, the value of C_Q corresponding to outflow will increase linearly with increasing R , a result approximately in agreement with experimental results.

In order to have like test conditions for like Reynolds number as ρ , U_0 , c , and μ are varied, it is necessary that c_{v_0}/t vary directly with c . When models of different chords are geometrically similar to the extent that t is proportional to c , then c_{v_0} must vary directly as c^2 . This result would be expected since it means that the effective diameter of the passage must vary directly as the chord; that is, the geometrical similarity must include the detailed geometry of the structure of the porous material, as well as the over-all dimensions of the model.

For like airfoils of different chords covered with the same porous material, equation (A4) indicates that the inflow

distribution at any chordwise position is unchanged if R is proportional to c —that is, if U_0/ν is a constant. Thus, if the same airfoil profile were used throughout the span of a tapered wing and the wing were entirely covered with the same porous material, the inflow distribution would be similar for all sections of the tapered wing for the convenient condition of a constant suction pressure within the wing.

Compartmentation of an area-suction model can be used as a method for improving the chordwise inflow distribution so that excessive suction flows will not occur at any point, but, if this method is used, each compartment must be considered as a source of outflow. The flow removed through each compartment can be decreased by decreasing the pressure drop across the porous surface with a throttling device; and thus the effective C_p based on the static pressure in the compartment becomes less for no outflow than the C_p that would be required at the point of maximum S on the airfoil. (See fig. 2.) The net result of compartmenting the whole airfoil is

that for no outflow the sum of the incremental values of C_q for each compartment will be less than the minimum C_q required for no outflow for the uncompartmented model; furthermore, the greater the number of compartments, the greater is the decrease in the value of total C_q for incipient outflow. The end point of more and more compartments is that the chordwise inflow distribution can be set to any value desired. The greatest difficulty in compartmenting is experienced at and near the nose where even very small compartments must combat a very high chordwise change in pressure drop and thus of inflow distribution because of large variations in the external pressure. Inasmuch as extreme compartmentation may result in an immense construction problem, it may be more desirable to provide a skin which has a chordwise variation of porosity that will produce the desired inflow distribution. Such a skin could be obtained by a chordwise variation of either the thickness or the density of the material.

APPENDIX B

DETERMINATION OF SUCTION-DRAG COEFFICIENT

If it is assumed that the suction air is pumped back to free-stream total pressure by a blower and duct system which together have an efficiency of η_s , then the power required may be written as

$$P = \frac{Q(H_0 - H_t)}{\eta_s}$$

where H_t is the average total pressure of the suction as measured at a point under the surface of the wing and H_0 is the free-stream total pressure.

If the flow quantity is expressed as the suction-flow coefficient

$$C_Q = \frac{Q}{U_0 bc}$$

and the total pressure defect is expressed as a pressure-loss coefficient

$$C_P = \frac{H_0 - H_t}{q_0}$$

then the power may be written as

$$P = \frac{C_Q U_0 bc C_P q_0}{\eta_s} \quad (B1)$$

If this amount of power were to be supplied by the airplane propulsive system of efficiency η_p , then the equivalent drag to be associated with this power could be written as

$$D = \frac{P \eta_p}{U_0}$$

or

$$c_{d_s} bc q_0 = \frac{P \eta_p}{U_0} \quad (B2)$$

where the equivalent drag is expressed in terms of a suction-drag coefficient based on the area of the wing.

Equations (B1) and (B2) permit the suction-drag coefficient to be expressed as

$$c_{d_s} = C_Q C_P \frac{\eta_p}{\eta_s}$$

On the condition that the blower system operates as efficiently as the propulsive system, the suction-drag coefficient reduces to

$$c_{d_s} = C_Q C_P \quad (B3)$$

REFERENCES

1. Lin, C. C.: On the Stability of Two-Dimensional Parallel Flows. Part I.—General Theory. Quarterly Appl. Math., vol. III, no. 2, July 1945, pp. 117–142; Part II.—Stability in an Inviscid Fluid. vol. III, no. 3, Oct. 1945, pp. 218–234; Part III.—Stability in a Viscous Fluid. vol. III, no. 4, Jan. 1946, pp. 277–301.
2. Schubauer, G. B., and Skramstad, H. K.: Laminar-Boundary-Layer Oscillations and Transition on a Flat Plate. NACA Rep. 909, 1948.
3. Schlichting, H.: The Boundary Layer of the Flat Plate under Conditions of Suction and Air Injection. R. T. P. Translation No. 1753, British Ministry of Aircraft Production. (From Luftfahrtforschung, Bd. 19, Lfg. 9, Oct. 20, 1942, pp. 293–301.)
4. Iglisch, Rudolf: Exakte Berechnung der laminaren Grenzschicht an der längsangeströmten ebenen Platte mit homogener Absaugung. Schriften der Deutschen Akademie der Luftfahrtforschung, Bd. 8B, Heft 1, 1944. (Also available as NACA TM 1205, 1949.)
5. Pretsch, J.: Die Leistungersparnis durch Grenzschichtbeeinflussung beim Schleppen einer ebenen Platte. UM Nr. 3048, Deutsche Luftfahrtforschung (Berlin-Adlershof), 1943.
6. Schlichting, H.: Berechnung der laminaren Reibungsschicht mit Absaugung. Forschungsbericht Nr. 1480. Deutsche Luftfahrtforschung (Berlin-Adlershof), 1941.
7. Ulrich, A.: Theoretical Investigation of Drag Reduction in Maintaining the Laminar Boundary Layer by Suction. NACA TM 1121, 1947.
8. Schlichting, H.: An Approximate Method for Calculation of the Laminar Boundary Layer with Suction for Bodies of Arbitrary Shape. NACA TM 1216, 1949.
9. Loftin, Laurence K., Jr.: Theoretical and Experimental Data for a Number of NACA 6A-Series Airfoil Sections. NACA Rep. 903, 1948.
10. Von Doenhoff, Albert E., and Abbott, Frank T., Jr.: The Langley Two-Dimensional Low-Turbulence Pressure Tunnel. NACA TN 1283, 1947.
11. Von Doenhoff, Albert E.: Investigation of the Boundary Layer about a Symmetrical Airfoil in a Wind Tunnel of Low Turbulence. NACA ACR, Aug. 1940.
12. Pretsch, J.: Die Stabilität einer ebenen Laminarströmung bei Druckgefälle und Druckanstieg. Jahrb. 1941 der deutschen Luftfahrtforschung, R. Oldenbourg (Munich), pp. I 158–I 175.
13. Loftin, Laurence K., Jr.: Effect of Specific Types of Surface Roughness on Boundary-Layer Transition. NACA ACR L5J29a, 1946.
14. Loftin, Laurence K., Jr., and Bursnall, William J.: The Effects of Variations in Reynolds Number between 3.0×10^6 and 25.0×10^6 upon the Aerodynamic Characteristics of a Number of NACA 6-Series Airfoil Sections. NACA TN 1773, 1948.

REPORT 1926

NACA INVESTIGATION OF FUEL PERFORMANCE IN PISTON-TYPE ENGINES

By HENRY C. BARNETT

PREFACE

It is generally recognized that the piston-type engine will continue to play an important role in air transportation for an indefinite period in spite of the intensive effort now being devoted to gas-turbine power plants. For this reason, past researches conducted in piston engines are still of interest in the solution of current problems relating to such engines. In order to simplify the task of using the data from previous investigations, an effort has been made to compile a portion of these data into a single reference source.

This particular report is a compilation of many of the pertinent research data acquired by the National Advisory Committee for Aeronautics on fuel performance in piston engines. The original data for this compilation are contained in many separate NACA reports which have in the present report been assembled in logical chapters that summarize the main conclusions of the various investigations. Complete details of each investigation are not included in this summary; however, such details may be found, in the original reports cited at the end of each chapter.

453

Preceding Page Blank

REPORT 1026

TABLE OF CONTENTS

Chapter		Page
I	HIGH-SPEED PHOTOGRAPHIC STUDIES OF KNOCKING COMBUSTION.....	455
II	CORRELATIONS OF KNOCK-LIMITED PERFORMANCE DATA.....	500
III	ANTIKNOCK PERFORMANCE SCALES.....	511
IV	PREIGNITION.....	520
V	HYDROCARBONS AND ETHERS AS ANTIKNOCK BLEND- ING AGENTS.....	532
VI	AROMATIC AMINES AS FUEL ADDITIVES.....	556
VII	TETRAETHYL LEAD AS A FUEL ADDITIVE.....	569
VIII	ANTIKNOCK BLENDING CHARACTERISTICS OF FUELS.....	581
IX	FUEL VOLATILITY.....	599
X	INTERNAL COOLING.....	612
	APPENDIX A—ADDITIONAL DATA ON PERFORM- ANCE OF VARIOUS FUELS.....	621
	APPENDIX B—ADDITIONAL BLENDING CHARTS FOR TERNARY AND QUATERNARY FUEL BLENDS.....	638
454		

REPORT 1026

CHAPTER I

HIGH-SPEED PHOTOGRAPHIC STUDIES OF KNOCKING COMBUSTION

By C. DAVID MILLER*

At the Summer Meeting of the Society of Automotive Engineers in 1946, Mr. C. David Miller of the NACA presented a paper entitled "Roles of Detonation Waves and Autoignition in Spark-Ignition Engine Knock as Shown by Photographs Taken at 40,000 and 200,000 Frames per Second." This paper was a summary of the NACA high-speed photographic studies of combustion in engine cylinders and it is reproduced substantially in its original form as chapter I of this report. The same material, under the title given above, was previously published in the Transactions of the Society of Automotive Engineers in January, 1947.

The National Advisory Committee for Aeronautics has been using high-speed motion-picture photography in research over a period of more than 20 years and has applied this method to the study of combustion within engine cylinders since 1933. By the early part of 1936 it had become apparent that commercially available cameras were not fast enough for the study of engine combustion, particularly for the study of spark-ignition engine knock. For this reason work was begun at Langley Field early in 1936 on the development of a high-speed camera using a newly invented optical system (reference 1). This camera, which was fully described in reference 2, will be referred to throughout this paper as the high-speed camera. It was operated successfully at 40,000 photographs (or frames) per second on its first test late in 1938 and has been used since that time principally in the study of knock in the spark-ignition engine at Langley Field, Va., and at Cleveland, Ohio. It was soon found, however, that even 40,000 frames per second was inadequate for the study of spark-ignition engine knock, and work was begun in 1939 at Langley Field on the development of a still faster camera. This development was based on another optical system (invented early in 1939) entirely unlike that of the high-speed camera developed in the years 1936 to 1938. (See reference 3.) Construction of a camera incorporating the new optical system was completed in 1941. This camera will be referred to throughout this paper as the ultra-high-speed camera. The ultra-high-speed camera could not be operated satisfactorily according to the original design. The difficulties were not directly associated with the optical system of the camera but with incidental mechanical details. Development of the design, continued since 1941, has not yet been completed. One

photograph of the phenomenon of spark-ignition engine knock, however, was secured with the ultra-high-speed camera at Cleveland at the rate of 200,000 frames per second. It is believed that pictures can eventually be obtained with this camera at a speed several times greater than 200,000 frames per second.

Results of the research work done with the high-speed camera and the ultra-high-speed camera have been reported on a restricted basis during the war years except for one paper (reference 4) presenting a small amount of work on diesel combustion. A report on spark-ignition engine combustion printed in 1941 (reference 5) presented preliminary results and showed that the knock reaction occurred in less than 50 microseconds, and that this reaction may be preceded by much slower exothermic end-gas reactions. Reference 6 printed in 1942 included additional preliminary results and included indications that the knock reaction may sometimes originate at a point outside of the end gas and that the knock reaction may be preceded by mild vibrations which become manyfold intensified at the instant of knock. In references 5 and 6, the characteristic reaction occupying less than 50 microseconds as seen in the photographs was more or less assumed to be the knock reaction on the basis of the general appearance of the motion pictures. In reference 7, issued in 1943, this characteristic reaction was definitely shown to begin simultaneously, within 25 microseconds, with the onset of violent knocking gas vibrations as shown by a piezoelectric pressure pickup exposed to the end gas in the combustion chamber. Reference 7 included additional indications that the knock reaction does not necessarily originate in the end gas. Reports released in 1944 showed that the knock reaction instantly rendered a large part of the unreleased energy of combustion chemically unavailable (reference 8), revealed several definite facts about the previously noted preknock vibrations, and showed the development of several types of preknock end-gas autoignition as well as autoignition in a large end-gas volume practically without any resultant knock (reference 9). In 1946 a report was released presenting analyses of some of the high-speed pictures indicating knocking detonation-wave velocities ranging from one to two times the speed of sound in the burned gases (reference 10), and ultra-high-speed photographs taken at 200,000 frames per second have been released confirming the detonation-wave character of knock with a wave speed of 6800 feet per second (reference 11).

*Now with Battelle Memorial Institute.

It is the purpose of the present paper to offer a unified report of the facts concerning spark-ignition engine knock that have been previously reported on a restricted basis and to include as an introduction to the unified report the results of a literature study which the author believes support the new concept of knock that he and his coworkers have obtained from the high-speed pictures. A synopsis of the unified report has also been prepared in the form of a motion-picture film (reference 12). For a general history of the application of photographic methods to engine combustion, see references 4, 10, and 11.

Knock, one of the most serious limitations on the performance of the present-day reciprocating aircraft engine, has been plaguing the designers and users of spark-ignition engines in general at least since 1880, when Clerk suppressed extremely violent knock by the use of water (references 13 and 14). Knock has been the subject of intensive research by groups in various countries for about 25 years.

The past researches on knock have uncovered an immense amount of information, not only concerning the basic nature of knock but also concerning the question of what to do about it. The information available on the basic nature of knock has led most writers, at least in the United States, to accept the autoignition theory in preference to all others. (Though many writers refer to knock as "detonation," they do not mean to imply that they believe knock is caused by a detonation wave.) Only a few dissenters (references 14 to 19) have questioned the adequacy of the autoignition theory.* The photographs obtained with the high-speed camera and the ultra-high-speed camera have forced the author and his coworkers to join the ranks of the dissenters.

The available information on what should be done about knock is outside the scope of this paper and is so well known that it needs no review here. The available information is undoubtedly accurate as far as it goes and is so extensive that many practical workers with engines and fuels even discount the need for definite knowledge as to what knock is.

Aside from the fact that any kind of knowledge concerning any process of nature rarely proves in the end to be of no practical value, some urgent reasons exist for determining exactly what knock is. Probably the most important reason is associated with the fact that little is definitely known even about the harmfulness of knock. As will be shown in this paper, there are probably more than one and perhaps even more than two phenomena that are regarded as knock when they occur in the combustion chamber. In view of the possibility that these phenomena may not all be harmful, it seems urgently desirable to learn which are harmful and how to distinguish between one of the phenomena and another. As was pointed out by Boerlage in 1937 (reference 19), the noise of knock cannot be regarded too seriously until the harm done has been demonstrated to be proportional to the noise. In order to distinguish between the forms of knock and to know which are harmful and which are not, the logical first step appears to be that of learning what the phenomena are and under what conditions the various phenomena occur.

Other reasons for seeking the true explanation of knock are the possible saving of much labor involved in developing and testing ideas based on a possibly false conception of the nature of knock, acquisition of additional fundamental knowledge concerning chemical laws that might prove useful in other fields, and the possibility, however remote, that some new and simpler solution to the knock problem might be suggested.

Next to autoignition, the detonation-wave theory probably is generally regarded as the most plausible of the many theories that have been advanced to explain knock. Study of the photographs taken at 40,000 frames per second over a period of 6 years has convinced the author that a detonation wave, or some phenomenon very much like a detonation wave, actually is involved in the type of knock most frequently encountered in the modern aircraft engine. Autoignition also appears to be often involved in knock. A combined autoignition and detonation-wave theory has been proposed (reference 10) on the basis of a study of the high-speed photographs and the available literature concerning knock.

The autoignition and detonation-wave theories of knock are actually in agreement in many respects. According to either theory knock occurs only after the flame has traveled from the spark plug through most of the fuel-air mixture at speeds ranging from below 50 feet per second to several hundred feet per second, depending on engine speed, fuel-air ratio, and a number of other variables. This speed of 50 to several hundred feet per second is a low speed from the standpoint of tendency to produce shock; it is the normal rate of burning in nonknocking operation. Again according to either theory, the shock known as knock is produced by the sudden inflammation of the end gas, the gas that has not yet been ignited at the time knock occurs by the normal travel of the flame from the spark plug.

If the end gas is considered as being divided into a very large number of extremely small cells or increments, it is clear that no great shock will result from the burning of the individual increments at widely different times, however fast the burning of each increment may be, and it is also clear that shock will not result from the simultaneous burning of all the increments unless each increment goes through the burning process within an extremely small time interval. Shock will result, according to either theory, only if each increment burns within a very small time interval and all increments burn at the same time within a very small limit. If these two conditions are satisfied, then the end gas does not have time to expand during the burning of the increments and a very high pressure is produced in the end gas relative to the gas in the other parts of the chamber. The subsequent expansion of the end gas sets up a violent vibration or system of standing waves throughout the entire contents of the combustion chamber. Such a system of standing waves was shown to be the cause of audible knock, at least under certain conditions, by the researches reported by investigators at the Massachusetts Institute of Technology and at General

*Since the presentation of this paper in June 1946, prior opposition to the autoignition theory by several others has come to the author's attention.

Motors between 1933 and 1939 (references 20 to 23). Slow-motion pictures of these vibrations taken at 40,000 frames per second were first presented at an SAE meeting in March 1940 (reference 23).

The only point of difference between the autoignition and detonation-wave theories is in the means of synchronizing the ignition of the end-gas increments, that is, the mechanism that causes all end-gas increments to burn at the same time within a small enough limit to cause shock.

The argument presenting the synchronizing mechanism of the autoignition theory is as follows: Each end-gas increment will burn explosively when it attains some certain combination of temperature and density (or the equivalent of some particular temperature-density history, as suggested by Leary and Taylor of M. I. T. in reference 24). All end-gas increments are adiabatically compressed at the same time and at the same rate by the expansion of the burning gas behind the flame front. All increments of end gas should, therefore, reach the critical combination of temperature and density at which they will explode at the same time.

In an analysis of the synchronizing mechanism of the autoignition theory, it should be considered that the compression of the end gas by the burning gases is accomplished by an infinite series of sound waves. A given condition of temperature and density should, therefore, be expected to travel through the end gas from the burning zone at the speed of sound. The combination of temperature and density in any end-gas increment may be expressed as some function F , so defined that each end-gas increment will explode when $F = F_{cr} \pm \delta$, the term δ representing an element of uncertainty due to random variation in the behavior of the end-gas increments or to random inhomogeneities. The value of F in each end-gas increment will increase by an amount equal to 2δ in some time interval τ . Now if τ is not greater than the order of the time interval τ' required for a sound wave to pass through the unignited end gas, then it should be expected that autoignition would take place as an explosive reaction traveling through the unignited end gas at least at the speed of sound. It would not take place as a simultaneous reaction throughout the end gas. The explosive reaction would constitute some kind of explosive wave, if not an actual detonation wave. This wave might travel too slowly to produce shock and to be regarded as a true detonation wave. Obviously, however, the less shock the wave produced the less the knocking sound heard outside the engine.

If τ is assumed to be much greater than the order of τ' , then autoignition should be expected to develop homogeneously throughout the end zone. The pressure built up by the combustion of the end gas, however, is relieved also by an infinite series of sound waves. Consequently, if τ is many times greater than the order of a time interval τ'' required for a sound wave to pass through the *autoigniting* end gas, the pressure in the end gas would be relieved many times during the process of autoignition and shock would not occur.

The magnitude of the time interval τ apparently must

lie within a range somewhat greater than τ' but not many times greater than τ'' if knock is to be caused by a homogeneous autoignition of the end gas. Above this range of values for τ no shock can occur; below this range of values the autoignition must occur as something similar to a detonation wave, and becoming more and more like a detonation wave as the knock intensity increases. (Knocks of different intensity can occur with the same end-zone volume according to unpublished NACA photographic records.) The time interval τ'' is a variable for different stages of the homogeneous autoignition process and reaches a value much less than τ' during the later stages of the process. The range of values of τ greater than τ' but not many times greater than τ'' must, therefore, be quite narrow.

Autoignition, either as a detonation wave or as a homogeneous reaction with τ slightly greater than τ' , seems a very plausible synchronizing mechanism. Before it is accepted conclusively, however, the available evidence should be carefully studied as to whether it actually is an adequate synchronizing mechanism. The evidence should also be investigated as to whether autoignition of the individual gas increments proceeds to completion within a short enough time interval to produce shock. A considerable amount of evidence exists against autoignition as the sole cause of the standing waves of knock on both counts, as will be shown in the later parts of this paper.

The synchronizing mechanism postulated by the detonation-wave theory is an intense compressive shock wave that travels through the end gas at supersonic velocity. Each gas increment is ignited probably by the combination of the sudden intense compression occurring in the shock front, the action of chain carriers in the shock front, and the radiation of heat from the shock front. The entire combustion, or some definite stage of the combustion, of each gas increment is presumed to occur in the shock front and to release a large amount of energy immediately behind the shock front. The energy released by the gas increments immediately behind the shock front maintains the high pressure required to propagate the shock front through the charge. Such a phenomenon, being an intense shock wave, would obviously set up vibration of the gases throughout the combustion chamber. If, as the author holds, simple homogeneous autoignition can no longer be accepted as the sole cause of the knocking gas vibrations, the detonation wave appears to be the only remaining plausible explanation that has been suggested. The detonation wave, however, may be only the end result of a comparatively slow homogeneous end-gas autoignition or the detonation wave may act as the initiator of a very rapid homogeneous combustion which in its turn sets up the violent gas vibrations.

A literature-based argument in favor of a combined detonation-wave and autoignition theory of knock will be found in the first part of the section called "Discussion and Analysis," and the unified report of the findings obtained from the high-speed and ultra-high-speed photographs will be found in the second part of the same section.

DEFINITION OF TERMS

Throughout the present paper the following terms are used with the meanings indicated:

1. Knock: Any type of reaction occurring within combustion-chamber contents and producing objectionable noise outside the engine, but not including the phenomenon of early combustion caused by too early spark timing or by early ignition from a hot spot.

2. Explosive knock reaction: A specific reaction observed in NACA photographs of knocking combustion taken at 40,000 frames per second, usually appearing instantaneous when the photographs are projected at the normal rate of 16 frames per second and coinciding chronologically with the onset of gas vibrations as seen in the photographs. (This reaction, being regarded as one form of knock, will sometimes be referred to simply as "knock" when the context makes the meaning clear.)

3. Flame front: The continuously changing surface that separates unflamed parts of the cylinder charge from the burning parts of the charge that have been ignited by the advance of the flame from the spark plug.

4. Autoignition: Spontaneous burning in any part of the cylinder charge not caused by a spark, by contact with a flame front, or by contact with a hot spot, and including not only the initiation of burning but the entire process of burning resulting from the spontaneous ignition.

5. Shock wave: An intense compressive wave traveling through gas at supersonic velocity, the front of such wave constituting an abrupt increase or practical discontinuity in temperature, density, and velocity of the gas.

6. Detonation wave: A type of wave often observed in long tubes consisting of a shock wave traveling through a gas or a gas mixture and causing a reaction of the gas in the shock front, such reaction releasing energy immediately behind the shock front, the energy so released serving to maintain the pressure needed behind the shock front to propagate the wave.

DISCUSSION AND ANALYSIS: LITERATURE-BASED ARGUMENT FOR COMBINED DETONATION-WAVE AND AUTOIGNITION THEORY OF KNOCK

Autoignition theory.—The autoignition theory of knock was suggested by Ricardo in 1919 (reference 25). Midgley in 1920 declined to accept the autoignition theory and held to the detonation-wave theory (reference 14). A year later Woodbury, Lewis, and Canby of the du Pont laboratories (reference 26) presented streak photographs of combustion in a bomb, taken by the method of Mallard and Le Chatelier (reference 27), and drew conclusions favoring the autoignition theory. These du Pont investigators seem to have regarded the detonation-wave theory as the one having had general credence up to that time. From an analysis of their streak photographs and from consideration of various facts reported by previous investigators, they concluded that "the possibility of detonation under such conditions [conditions existing in the engine cylinder] appears exceedingly remote." After mentioning that detonation is set up in a closed cylinder of small dimensions only with great difficulty, they further

stated: "On the other hand, autoignition of the high-density gases ahead of the flame front occurs over a wide range of fuel mixtures and conditions [in their tests] and gives a sudden development of pressure similar, in our opinion, to that characteristic of a knocking explosion. It is possible that this autoignition may set up detonation [a detonation wave] in some cases, thereby acting as an intermediate stage in knocking. Our experiments have not been carried to a definite conclusion, and present data do not warrant presentation of autoignition as a positive explanation for knocking. It is our feeling, however, that information at hand favors more strongly the theory of autoignition of the high-density gases ahead of the flame front than that of detonation [the detonation wave]."

Actual motion pictures of knocking combustion were first published in 1936 by Withrow and Rassweiler of General Motors Corp. (reference 28). These excellent photographs, taken at the rate of 2250 frames per second, showed the development of autoignition in the end gas and greatly increased the already existing confidence in the autoignition theory. They were taken at too low a rate to show a detonation wave, however, even though such a wave might actually have occurred after the autoignition that was photographed.

The autoignition theory, with the additional assumption of preflame chain reactions, has the advantage of explaining and correlating many of the known facts concerning knock. During the period of 1939 to 1945, however, urgent need for a modification of the simple autoignition theory of knock was shown by photographs of knocking combustion taken at the rate of 40,000 frames per second with the NACA high-speed motion-picture camera. The first of these photographs, presented in references 23 and 5, showed a reaction completed in 50 microseconds or less. The authors believed that this reaction was the true knock reaction because they could see in the projected motion pictures that this reaction occurred at the same time as the beginning of the violent vibration of the gases, which by then had come to be regarded as an indication of knock. Later NACA tests (reference 7) showed that this extremely quick reaction did occur simultaneously with the beginning of the vibrations. Serruys had previously (in 1932) concluded that knock generally occupies a time interval less than 100 microseconds in reference 29 and, in 1933, on a basis more in harmony with the standing-wave concept of knock, in reference 30. Considerations presented in the present paper have caused the author to abandon the exclusiveness of the concept "true knock reaction." The reaction will hereinafter be referred to as the "explosive knock reaction."

The need for a modification of the autoignition theory of knock lies in the fact that the evidence available in the literature indicates autoignition requires for its completion a time interval of an entirely higher order than the 50 microseconds involved in the explosive knock reaction, even under conditions of severity approaching those of the modern aircraft engine. The previously mentioned photographs by Withrow and Rassweiler (reference 28) clearly show brightly luminous autoignition occupying a time interval of the order

of 1000 microseconds. NACA high-speed photographs have shown autoignition flames slowly propagating themselves from point to point throughout the end gas before the explosive knock reaction occurs (reference 9); and another NACA high-speed photograph has shown autoignition developing slowly and simultaneously in all parts of the end gas before the occurrence of the explosive knock reaction (reference 5). These two types of autoignition shown in the NACA high-speed photographs, preceding the explosive knock reaction, occupied time intervals ranging from 500 to 1250 microseconds. Streak photographs were published as early as 1911 by Dixon and coworkers (references 31 and 32) showing slow autoignition in glass tubes resulting from quick compression. This autoignition progressed at a rate comparable with the rates of the autoignitions shown by Withrow and Rassweiler and by the NACA investigators.

The evidence showing that autoignition occupies a time interval of a higher order than 50 microseconds is not the only reason for believing simple autoignition to be an inadequate explanation of knock. Many investigations have shown that autoignition can occur without causing marked gas vibrations, which are probably the best known characteristic of knock in the present-day spark-ignition engine. These gas vibrations, if they occur, are visible in streak photographs taken by the method of Mallard and Le Chatelier (reference 27) as a series of bright bands extending across the photograph in a direction perpendicular to the direction of film movement. The gas vibrations also cause oscillations in pressure-time records.

Some excellent streak photographs presented by Withrow and Boyd (reference 33) are examples of nonvibratory autoignition in the engine cylinder. These General Motors investigators stated that both the pressure-time records and the flame traces show that the autoignition required 2° to 5° of crankshaft rotation (400 to 1000 microsec) for its completion. Figures 11 to 16 of their work clearly show the flame front traversing the greater part of the chamber at the normal rate and show the end gas then being consumed at a much higher rate. All of these figures except figure 14, however, reveal not the slightest indication of gas vibrations. It is difficult to conclude from the printed picture of figure 14 whether there is any evidence of vibrations. Moreover, the pressure-time records of figures 11 to 16 of reference 33 show no evidence of gas vibrations. Though audible gas vibrations probably did not occur in those tests, some kind of disturbing noise surely must have occurred, as is discussed in the present paper under "Detonation-Wave and Autoignition Theories Combined."

Withrow and Boyd did not comment on the absence of gas vibrations in their tests. Up to about the time of the writing of their paper (1931), gas vibrations do not seem to have been regarded as a usual feature of knock. The only recognized criterion of knock as seen in pressure-time records appears to have been simply a sharp increase in the rate of pressure rise. In 1932 Rassweiler and Withrow (reference 34) presented streak photographs clearly showing the gas vibrations; and in 1934 (reference 21) they showed that the vibrations as

seen in the photographs coincided, cycle by cycle, with fluctuations shown in the pressure-time records.

Woodbury, Lewis, and Canby in 1921 did not regard the gas vibrations as being associated with knock, for in the previously quoted passage (reference 26) they concluded on the basis of their own experiments that autoignition of the high-density gases ahead of the flame front gives a sudden development of pressure similar, in their opinion, to that characteristic of a knocking explosion. The pressure-time traces presented for the cases of autoignition referred to showed, in general, no gas vibrations but only a sharp increase in rate of pressure rise near the end of combustion. Almost without exception the streak photographs also showed no trace of gas vibrations; the exception was with ether-air mixtures. With initial temperature of 150° C and initial pressure of 65 pounds per square inch, neither the flame trace nor the pressure-time trace for an ether-air mixture showed any sign of gas vibrations, whereas with the same initial temperature and with initial pressure of 75 pounds per square inch, both the flame trace and the pressure-time trace showed the gas vibrations with agreement in frequency. The change that occurred in the phenomena studied in a bomb by these investigators, when passing from 65 to 75 pounds per square inch with ether-air mixture at 150° C, appears to correspond to the change in the recognized criterion of spark-ignition engine knock that developed in the early 1930's.

No particular note appears to have been made in the literature of the change in the recognized criterion of knock that developed in the early 1930's. Sufficient data do not appear to be available to explain the change or to indicate whether it was a real change caused by altered engine design and altered fuels or an apparent change developing with the securing of more extensive data.

In 1939 Boyd (reference 35) compared a streak photograph of autoignition without gas vibrations (fig. 10 of reference 35, same as fig. 16 of reference 33) and a streak photograph of autoignition with gas vibrations (fig. 12 of reference 35, same as fig. 10 of reference 34). He very reasonably regarded the case of his figure 12 as involving a much more violent knock than the case of his figure 10. Examination of his figures 10 and 12, however, discloses that the end zone was of nearly the same size in the two cases at the time the autoignition, or knock, occurred. The comparison therefore indicates that the violence of knock or at least the violence of the gas vibrations is not dependent on the size of the autoigniting end zone. Moreover, NACA high-speed photographs have shown plainly visible gas vibrations in cases where the end zone, if any existed at the time of start of the gas vibrations, was too small even to be seen in the photographs (references 6 and 7).

Other streak photographs showing autoignition without trace of gas vibrations may be found in papers by General Motors investigators (references 36 to 38). The most striking examples of this phenomenon, however, are to be found in the work of Duchene (reference 39). In this work many streak photographs are presented of combustion, with spark ignition, in a bomb equipped with a piston providing compression by

a blow from a heavy pendulum. Many of these flame traces show a sudden darkening extending entirely across the trace, which Duchene considered as indicative of a detonation wave. Only three of the records, however, 21, 35, and 36, show any trace of gas vibrations. In most cases the darkening is quite diffuse instead of practically instantaneous, as it should be if caused by a detonation wave. The records all distinctly show slow autoignition preceding the sudden darkening. The fraction of the total charge involved in the nonvibratory autoignition in the different records covers the entire range from near zero to practically the entire charge. Gas vibrations should not, of course, be expected from simultaneous autoignition of the entire charge at constant volume. Records 23, 28, 29, and 31 of Duchene's work, however, clearly show autoignition of about half the contents of the chamber without any trace of vibrations.

The inadequacy of simple autoignition as an explanation of the phenomenon of knock has been clearly recognized by some investigators. In 1928 Maxwell and Wheeler in reference 15 reported frequently observing autoignition flame, with 50-50 mixtures of pentane and benzene in a bomb, starting from the far end of the cylinder and progressing back to meet the spark flame. They reported that explosions in which this phenomenon occurred were no louder than usual and that the pressure records showed no unusual features. They concluded, in consequence, that such an ignition of unburnt residual mixture is not likely to be the cause of a pinking explosion in an engine cylinder. In reference 16 the same investigators stated: "Our objection to the 'autoignition' theory is that, when such ignitions occur during an explosion in a closed cylinder (for example, figs. 2 and 5), the explosion is no more violent than in their absence. Moreover, what we have termed a 'pink' in our cylinder, because it so closely resembles the pink in an engine cylinder, is obtained most commonly without the occurrence of 'autoignition.'"

In 1935 Egerton, Smith, and Ubbelohde (reference 17), in discussing the work of other investigators, stated: "'Autoignition,' that is, ignition in a region of the gas prior to the arrival of the flame front, was observed both in the knocking zone and elsewhere, but does not necessarily give rise to the knocking type of combustion, though it was supposed that the high rate of combustion in the knocking zone was due to autoignition within it."

In 1936 Boerlage (reference 19), in discussing the results of his own streak photographs, stated: "What surprised us, however, in the results obtained with the test engine, was the relatively slow character of the combustion due to autoignition. The development of the second center of ignition was at all points similar to the progression of the primary flame due to the spark. The 'simultaneous' combustion of the 'end gas,' which we have believed responsible for the knock, thus seems to be reduced to the rather calm development of a secondary center of ignition." He further stated: "* * * the velocity of the secondary flame front is practically equal at each instant to that of the primary flame front. We have never been able to make out any speed equal to the speed of sound, but at most, speeds of 150 meters per second, and

these only in the case of excessive detonation (knock). In the case of slight detonation (knock) the speeds do not attain even half this figure. * * * The pressure diagrams show only moderate pressure rises, and this is still another indirect proof of the fact that the speeds of the flames are relatively low and remain much below the speed of sound. We have not succeeded in demonstrating the existence of extreme local pressures."

The investigations mentioned have shown beyond possible doubt that autoignition can, and in many cases actually does, occur too slowly to cause the gas vibrations characteristic of knock. This fact does not prove that autoignition cannot, under any conditions, occur quickly enough to cause the gas vibrations. It does, however, preclude the possibility of regarding the occurrence or nonoccurrence of autoignition as a criterion for the occurrence or nonoccurrence of the type of knock characterized by gas vibrations. A different criterion must be sought, either the occurrence of autoignition at a rate above some critical value or the occurrence of some other phenomenon.

Indeed the criterion of autoignition at a rate above some critical value seems to be precluded by some of the NACA photographs (references 5 and 9), for in those cases slow autoignition was seen to occur, followed by the much faster reaction that set up the gas vibrations. In this connection it should be noted that some investigators (references 24 and 40) have regarded the apparent autoignition shown in one of the NACA papers (reference 5) as a preflame reaction. The slow apparent autoignitions shown in reference 9, however, are more difficult to explain as preflame reactions because they propagate themselves from point to point in the same manner and at about the same speed as a normal flame.

The available literature, as reviewed in this section, points to the conclusion that some phenomenon other than simple autoignition must be sought as the cause of the gas vibrations associated with knock in the modern spark-ignition engine.

Detonation-wave theory.—Prior to the NACA investigations that will be summarized in the present paper the occurrence of a detonation wave in a bomb or a knocking engine had not been supported by any such abundance of direct experimental evidence as the occurrence of autoignition. This fact is, of course, readily explained by the consideration that the detonation wave, being a many times faster phenomenon than autoignition, requires very much more powerful methods for its detection. A very important consideration in favor of the detonation wave as the explanation of gas vibrations is the unquestionable fact that it would cause gas vibrations if it did occur, whereas it has been shown that simple autoignition does not necessarily cause the vibrations when it occurs.

Many writers have long been strongly opposed to the detonation-wave theory of knock, principally because it is very difficult to set up detonation waves in containers as small as an engine cylinder, or indeed in hydrocarbon-air mixtures at all, and because many variables have unlike effects on the tendency of a combustible charge to knock in an engine and to develop a detonation wave in a tube.

In 1936 the Russian investigators Sokolik and Voinov (reference 18) furnished direct experimental evidence of propagated combustion, as contrasted with the concept of simultaneous autoignition, traveling through the end zone in a knocking engine at the correct speed to be regarded as a detonation wave. This evidence is in the form of streak photographs for which a sufficiently high film speed was used to resolve the slope of the luminosity front developed by the detonation wave. It is unfortunate that this work has not, in the past few years, received more careful consideration. The photographs of Sokolik and Voinov were taken through a narrow window extending across the combustion chamber in the direction of the flame travel. The results show the flame traversing the greater part of the chamber at a mean velocity usually less than 20 meters per second, then traversing the remaining part of the chamber at a velocity of the order of 2000 meters per second.

The photographs of Sokolik and Voinov are, of course, open to the criticism that they show the performance of only a narrow zone in the combustion chamber. For this reason, the illusion of a detonation wave traveling at 2000 meters per second could have been caused by a much slower autoignition traveling through the end gas at a considerable angle to the visible zone. Such an illusion should not be expected to be consistent throughout many records. Sokolik and Voinov, however, do not state how many records they studied.

NACA high-speed motion pictures of knock in references 6, 7, and 11 have suggested that the explosive knock reaction does not necessarily originate in the flame front but that it originates at random anywhere within the normal flame or the autoigniting end gas. For this reason NACA investigators have been slow to accept the results of Sokolik and Voinov as having general validity, suspecting that some difference in test conditions may have caused a type of knocking phenomenon to occur in their work different from any knocking phenomenon that has been found in the NACA investigations.

The new evidence for the occurrence of a detonation wave in the knocking engine, based on the high-speed and ultra-high-speed photographs of references 10 and 11 will be discussed in the second part of the section called "Discussion and Analysis."

Intermediate flame velocity.—Intermediate between the slow autoignition found by various investigators and the detonation-wave velocity determined by the Russian investigators in reference 18 is the finding by Schnauffer (reference 41) of a speed of 265 to 300 meters per second for the travel of a flame through the end zone in knock. Schnauffer made this determination by means of ionization gaps mounted in different parts of the combustion chamber. The ionization current across the successive gaps was amplified and used to light neon bulbs. The time interval between the lighting of the successive bulbs was measured by the record of the bulbs on a photosensitive drum rotating at high speed.

Flame travel at 265 to 300 meters per second through an end zone 2 to 3 centimeters long would be almost fast enough to satisfy the 50-microsecond limitation imposed by the

NACA photographs (reference 5), and such a rate of flame travel might, therefore, very well be regarded as a satisfactory cause of the explosive knock reaction. Note should be made, however, that the speed of 265 to 300 meters per second has not been verified by other investigators. Schnauffer did not indicate how many ionization gaps were used in the actual knocking zone to determine the velocity of 265 to 300 meters per second. Examination of the pattern of the gap locations as shown in the figures of his paper indicates either that the velocity was determined from the time interval between ionization of only two gaps or that the distance over which the velocity was measured was much greater than 2 or 3 centimeters, in which case the 50-microsecond limitation was not satisfied. Measurement of rate of flame travel on the basis of the time interval between ionization of two gaps would not be valid in case of any type of greatly accelerated reaction in the end gas. In such a case the normal flame travel through an indeterminate fraction of the distance between the last two gaps would be erroneously treated as the flame travel across the entire distance; the result would be a meaningless velocity.

In Schnauffer's paper oscillograph records were shown for the ionization currents produced both by the normal flame and the knock reaction in the end zone. The oscillograph records for the two types of combustion look very much alike. Hastings has shown in reference 42, with the vibratory type of knock, that the total time interval throughout which ionization currents are measurable in the end gas is only a small fraction of the time interval throughout which the ionization currents are measurable in the earlier-burned parts of the cylinder charge. The similarity in the oscillograph traces of Schnauffer's work therefore indicates that he was dealing with simple autoignition, not vibratory knock.

Nature of gas vibrations.—Many investigators have shown the occurrence of gas vibrations in bombs and in engine cylinders, both by photography and by pressure-time records. When the vibrations were first observed on indicator records of engine combustion, the question was raised whether they were not natural vibrations of the indicator set up by the blow of knock. Undoubtedly in many cases of simple autoignition this explanation was correct. In this connection the observation by Schnauffer in 1931 (reference 43) is of interest. With the ionization-gap method he found apparently simultaneous ignition of end gas amounting to approximately 50 percent of the entire cylinder charge; the indicator record showed no vibrations but only a sharp increase in rate of pressure rise. Schnauffer commented: "Figures 4 and 5 show that with a pressure indicator sufficiently free of inertia it is very well possible to record the knocking blow without the appearance of pressure oscillations. It is thereby demonstrated that the oscillations are not pressure oscillations." When Withrow and Rassweiler in 1934 (reference 21) showed a precise agreement between the oscillations recorded by an indicator and the bright bands on a streak photograph of the combustion, it was no longer possible to doubt the validity of the vibrations recorded by the best indicators.

Examination of the published records of gas vibrations in bombs and in engine cylinders reveals that with comparatively slow-burning mixtures, such as are used in the spark-ignition engine, these vibrations are generally of two types: With the first type the first cycle of vibration recorded has a larger amplitude than any of the later vibrations and the decay of the vibrations is gradual; with the second type both the buildup and the decay of the vibrations are gradual. A detonation wave, by definition, would cause the first type of vibration but not the second. With auto-ignition ruled out as a cause of the first type of vibration, the detonation wave is probably the only known physical phenomenon that could cause it. This type of vibration will, therefore, be referred to hereinafter, for convenience, as the "detonation wave" type of vibration. The many investigations that have shown the detonation-wave type of gas vibration in bombs or engine cylinders, either photographically or by means of pressure indicators, include some of those previously mentioned (references 5 to 7, 18, 21, 26, 30, 34, 35, and 38), as well as investigations by Souders and Brown (fig. 6 of reference 44), Boerlage, Broeze, van Driel, and Peletier (reference 45), and Rothrock and Spencer (reference 46).

The type of vibration having a gradual buildup obviously requires a gradual feeding in of energy over a period of many cycles. This gradual feeding in of energy could occur only if the vibrations themselves affected the local rates of combustion, or energy release, in such manner as to speed up the combustion in the high-pressure regions relative to the low-pressure regions. Such an effect would cause any slight accidental vibration to become self-amplifying. The cause of an accidental vibration is not hard to find. Ignition at a point in a vessel will unavoidably send forth a pressure wave which, after reflection from the far wall, will return to the point of ignition and may speed up the combustion upon its arrival. Souders and Brown at the University of Michigan (reference 44) found that a very pronounced occurrence of this type of vibration could be eliminated by shortening their spark commutator contact so as to decrease the intensity of the pressure disturbance at ignition. The type of gas vibration having a gradual buildup will be referred to hereinafter as the "vibratory combustion" type of vibration.

The possibility, of course, exists that the inertia and damping characteristics of a pressure indicator might cause it to indicate a gradual buildup of vibrations even though the gas vibrations actually were of the detonation-wave type, particularly in cases where the vibration frequency is nearly the same as the natural frequency of the indicator. The failure of such spurious records to occur in practice, however, is indicated by the fact that all the records to be found in the literature fall very distinctly into the detonation-wave or the vibratory-combustion type; there is apparently no middle ground. A middle ground would be expected when the detonation-wave type of vibration is

modified by a pressure indicator with only slightly too much inertia to produce a faithful record.

The vibratory-combustion type of vibration, as should be expected, generally occurs in fairly long cylindrical bombs, in which the natural frequency is comparatively low and the total time of flame travel is comparatively long. Under such conditions this type of vibration may occur without any evidence of autoignition, hot-spot ignition, or any other type of combustion except the normal flame from the igniting spark, as in the previously mentioned work at the University of Michigan (figs. 5, 7, and 13 of reference 44).

Gas vibrations of the vibratory-combustion type in bombs have also been shown by Hunn and Brown, Kirkby and Wheeler, Lorentzen, Duchene, Wawrziniok, and Köchling in references 47, 48, 49, 39, 50, and 51, respectively. The photographs by Kirkby and Wheeler show how the vibratory combustion requires a bomb of considerable length. Lorentzen pointed out that the vibrations, which he apparently believed were caused by the same phenomenon as knock in the engine cylinder, could not have been caused by detonation because they set in before the attainment of maximum pressure. The vibrations shown by Duchene (records 21, 35, and 36) are of particular interest because they developed long after the charge had been completely inflamed, yet appear to have been built up gradually. The work by Wawrziniok showed gradual buildup not only of the vibrations in a bomb but also of the air vibrations outside a knocking engine. It is possible, however, that the forced vibrations of the engine walls built up gradually even with a detonation-wave type of gas vibration in the combustion chamber. The gradual buildup of the air vibrations in this case was very rapid as compared with the buildup of the gas vibrations in the bomb; in fact, this case seems to be the middle ground that is lacking in indicator records exposed directly to gas vibrations within the combustion chamber.

The work of Maxwell and Wheeler (references 15, 16, and 52) seems unique in the fact that they appear to have encountered both vibratory combustion and the detonation-wave type of vibration in the same explosion, and the one occurring just before the end of the flame travel and the other just after the flame front reached the far wall of the bomb. There seems to be no reason, however, why the two types of vibration should not occur, one after the other in the same combustion cycle. Moreover, two independent vibrations, each of the detonation-wave type, can be set up one after the other in the same combustion cycle, as has been shown in reference 9.

The excellent streak photographs by Payman and Titman (reference 53) are probably not pertinent to the present discussion because they involve only much faster-burning mixtures than are ordinarily used in spark-ignition engines. Inasmuch as pressure-time records are not included with the photographs by Payman and Titman, any discussion of the type of vibration set up by the phenomena shown in those pictures would be only speculation.

Detonation-wave and autoignition theories combined.—

The foregoing discussion clearly indicates a need for some kind of combination of the detonation-wave and autoignition theories of knock, inasmuch as the occurrence of both autoignition and an apparent detonation wave has been demonstrated in the knocking engine. The combined theory proposed herein involves an affirmative decision on the controversial question as to whether afterburning takes place in a volume of gas for a considerable time after the flame front has passed through that volume of gas, or after the entire volume of gas has become inflamed through autoignition. For the purpose of this discussion "afterburning" will be understood to mean continued oxidation of combustible or any other reaction that causes continued spontaneous expansion of gases or pressure increase at constant volume.

If the concept is accepted of a body of end gas inflamed throughout its entire volume by autoignition, then it would seem reasonable that under severe conditions such an inflamed body of gas might be highly susceptible to the propagation of a detonation wave and that a detonation wave traveling through the inflamed body of gas might be the immediate result of autoignition. Such a high susceptibility to the detonation wave might be caused not only by the high temperature within the inflamed gases but by high concentrations of molecular fragments that might be of importance in the propagation of the detonation wave. If the possibility is accepted of a detonation wave traveling through a body of gas previously inflamed by autoignition, it seems almost necessary also to accept the possibility of such a wave traveling through a body of gas in which afterburning is taking place behind the normal flame front. In this manner a detonation wave could develop without autoignition after the entire contents of the combustion chamber had been ignited by the normal flame front. Larger volumes of inflamed gas at any one instant would be expected, however, with autoignition than without autoignition; therefore, a detonation wave should be expected to develop principally in the autoigniting end gas rather than in afterburning gas behind the flame front.

Concerning the possibility of burning after passage of the flame front through a body of gas, Withrow and Rassweiler in reference 36 concluded that the spectrum of the afterglow emitted by such supposedly afterburning gas is the same as that emitted during the CO-O_2 reaction and caused by active CO , O_2 , CO_2 , or O_3 molecules. They suggested that the $\text{H}_2 + \text{CO}_2 \rightleftharpoons \text{CO} + \text{H}_2\text{O}$ reaction is in equilibrium after the flame front has passed and that the afterglow is due to a readjustment of the equilibrium when the pressure and consequently the temperature are increased. They remarked: "The distribution of intensity of the afterglow throughout the combustion chamber accords well with the idea that the emission is by carbon dioxide heated by the increase in pressure brought about by combustion of the rest of the charge."

The suggestion that afterglow is entirely caused by readjustment of equilibrium due to compression does not seem

compatible with the results of Stevens' work at the National Bureau of Standards with a soap-bubble bomb in which no appreciable compression of the earlier-burned gas by the later-burned gas was possible. Some of Stevens' streak photographs show very considerable afterglow (references 54 to 56). On the other hand two of his photographs (references 57 and 58) show only the trace of the luminous flame front without afterglow.

Other sound explanations of the afterglow may exist independent of the concept of afterburning, but the possibility of other explanations only precludes use of the afterglow as support for the afterburning hypothesis; that is, the possibility of such explanations may not be regarded as evidence against afterburning.

Lewis and von Elbe in reference 59 have regarded Stevens' results (references 56 and 57) as evidence against the concept of afterburning, stating "* * * thousands of explosions * * * failed to reveal the slightest indication of further expansion of the burned sphere after the flame had traveled across the entire gas mixture." If close measurements are made on figure 2 of Stevens' 1923 report (reference 54) and figure 2 of his 1930 report (reference 56) it seems questionable whether a positive statement can be made that these figures show not even the slightest continued expansion of the luminous zone after the constant-velocity expansion of the spherical shell of flame had come to an end. (The end of the constant-velocity expansion of the flame shell seems to be the only means of determining from the photographs when the flame "had traveled across the entire gas mixture.") In one of the flame traces of figure 4 of reference 58 in which the afterglow is absent, continued expansion is plainly visible after completion of the constant-velocity expansion of the flame shell. The printed reproductions of photographs in reference 55 show the flame-front trace too indistinctly for judgment on continued expansion after completion of the constant-velocity expansion. Figure 2 of reference 54 shows luminosity fading progressively from the outer edge of the luminous sphere toward the center after some slight expansion has possibly taken place; the progressive fading is probably caused by rapid cooling of the outer shell of hot gases after the combustion is nearly complete. Figure 4 of a report by Randolph and Silsbee (reference 60), obtained with the same Bureau of Standards apparatus as used by Stevens, shows continued expansion most distinctly after completion of the constant-velocity expansion. A consideration that must always be given attention in Stevens' photographs, as well as in all photographs taken by flame radiations, is the fact that these photographs may not represent the true flame front because of low luminosity in the early stages of burning and because of the finite exposure time required to make a record on the photosensitive material.

The experimental work reported and the arguments advanced by Lewis and von Elbe in reference 59 were concerned mainly with the question of whether combustion in a constant-volume bomb is complete at the time peak pressure is reached and not with the question of whether peak pressure is reached at the instant the flame front has passed through

the last increment of gas. The afterburning required by the proposed combined detonation-wave and autoignition theory would cover a time interval of an entirely lower order than that considered in the question of whether combustion is complete at the instant of maximum pressure. The only consideration offered by Lewis and von Elbe, other than the photographs of Stevens, that would have a bearing on the question of afterburning on the smaller time scale is the suggestion that with afterburning the sharp breaks obtained with fast-burning mixtures between the rising pressure curve and the cooling curve would not occur. By the same token it might be suggested that the extremely flat pressure maxima of slower-burning mixtures, such as shown in figure 16 of reference 7, would not occur if there were no afterburning.

Probably the strongest experimental evidence against afterburning is the work presented in papers by General Motors investigators. Tests with a sampling valve (reference 61) showed that free oxygen disappeared from the charge immediately after passage of the flame front, but this evidence is open to the question of whether burning was not completed after the gases were removed from the combustion chamber by the sampling valve. The General Motors investigators also checked flame-front positions as shown by high-speed motion pictures against pressure rise obtained from indicator records. (See references 62 to 64.) The results indicated completion of burning at the flame front, with some exceptions in reference 64. This evidence is open to the previously mentioned objection that the photographs may not show the true flame front. The agreement between flame-front positions as shown by the photographs and as calculated from the pressure records on the assumption of complete combustion in the flame front may be a coincidence or the greater part of the combustion may actually be completed in a very small part of the deep combustion zone.

That the General Motors photographs may not actually have recorded the true flame fronts is indicated by some of the NACA schlieren photographs of reference 7. In this work it was shown that peak pressure was reached, at top center, very nearly at the same time that the schlieren flame pattern completely disappeared from the high-speed motion pictures, or about 10° of crank angle at 500 rpm after the flame front had completely filled the chamber. This finding that peak pressure at constant volume coincides with the final fadeout of the schlieren flame pattern is supported by the previous demonstration of the same fact in a bomb by Lindner (reference 65).

Other evidence in favor of the concept of afterburning has been furnished by various investigators. The ionization records obtained by Hastings (reference 42) showed ionization persisting over 20° to 30° of crank angle at 2000 rpm with normal combustion. With his records of ionization in the end zone during knock, the persistence had only a fraction of that magnitude. He attributed the difference to the much faster combustion in the end zone during knock. It is of interest, in Hastings' records of ionization with normal combustion, that the ionization did not decrease steadily after passage of the flame front but irregularly with even several sharp increases in ionization after the original passage of the flame front.

Souders and Brown (reference 44) with their streak photographs and simultaneous pressure records of combustion in a constant-volume bomb noted an appreciable increase in pressure after the flame front reached the end of the bomb. Marvin and Best (reference 66), observing flame stroboscopically through small windows mounted in a cylinder head, reported pressure rise after complete inflammation of the charge with very low compression ratios. Wawrziniok in reference 55 found maximum pressure developing in his bomb considerably after the flame front had ionized a gap at the end of the bomb. In this case the ionization gap was located at the most distant position in a hemispherical end of the bomb so that error due to curvature of flame front was minimized; yet the lag between ionization of this gap and peak pressure was about 20 percent of the total burning time. Marvin, Caldwell, and Steele (reference 67) observed that total radiation from burning gases increased after inflammation throughout a time interval equivalent to about 20° of crankshaft rotation at 600 rpm.

Bureau of Standards investigators (reference 68), taking streak photographs of combustion in a spherical bomb, suspended fine grains of gunpowder at various points on a diameter of the bomb by means of human hairs. With central ignition, the brilliantly burning grains of gunpowder continued to move toward the center of the bomb for some time after the flame reached the wall of the bomb. This experiment seems to be particularly strong evidence of afterburning in the outer parts of the bomb.

Lewis and von Elbe have done work determining the temperature zones in burner flames in reference 69. Much uncertainty would be involved, however, in applying the results to the much different conditions existing in engine combustion.

In a discussion of combustion in a turbulent stream (reference 70) Shelkin has drawn a model of flame structure that might well apply under the highly turbulent conditions existing during combustion in the engine cylinder. According to this model, the turbulence in the flame front causes the flame to advance in microscopic, or near-microscopic, tongues. The structure behind the flame front is cellular; the cell walls constitute burning gas and the interiors of the cells constitute unignited gas. According to this model, the unignited gas within each cell is gradually consumed as the flame front progresses beyond the cell. With this structure, in the microscopic sense the burning zones might all be very thin; in the microscopic sense a deep afterburning zone would exist beyond the flame front. In any event, the preponderance of experimental evidence available at this time appears to favor the existence of a rather deep zone of combustion behind the flame front in the engine cylinder, though the main part of the combustion may take place only within a small part of this zone. Whether the combustion zone is cellular on the microscopic scale or only on a submicroscopic molecular scale does not seem important in the presentation of the combined theory of knock. In either case there is a possibility that the gases in the combustion zone may be peculiarly susceptible to the propagation of a detonation wave, and the available evidence on this point should be carefully considered.

The concept of autoignition followed by the development of a detonation wave was given passing attention in the previously quoted remarks of Woodbury, Lewis, and Canby in reference 26. Among the streak photographs of autoignition resulting from quick compression of the charge in a glass tube presented by Dixon and his coworkers (references 31 and 32) were included some records of what they believed to be detonation waves. Dixon and his coworkers pointed out the fact that the development of the detonation wave was always preceded by autoignition at some point within the charge. The concept of the development of a detonation wave in autoigniting end gas has also been suggested by Boerlage and van Dyck in reference 71. They pointed out that "simultaneous combustion" at the beginning should be considered as a slow pressure rise in comparison with "true detonation" but that it ultimately may have the same character. The reverse concept, autoignition triggered by a shock wave, has been suggested by Dreyhaupt in reference 72.

The concept of autoignition followed by the development of a detonation wave is consistent with the high-speed motion pictures presented in various NACA reports (references 5 to 9) if the explosive knock reaction is considered to be a detonation wave. In these photographs, in most cases where end gas was visible at the time of the explosive knock reaction, this reaction has been preceded by some form of apparent autoignition. In one case the apparent autoignition developed at definite centers within the end gas and spread out in all directions from those centers to fill the end zone before the explosive knock reaction occurred. (See fig. 10 of reference 9 on preknock vibrations.) In another case the autoignition began at the chamber wall and propagated throughout the end zone before the explosive knock reaction occurred. (See fig. 12 of reference 9.) In this case the visible explosive knock reaction was light. In other cases the autoignition developed uniformly and simultaneously throughout the end zone before the explosive knock reaction occurred. (See fig. 5 of the preliminary report, reference 5.) In yet other cases autoignition was not clearly visible in the photographs but a visible vibration of the gases of the detonation-wave type was set up before the explosive knock reaction occurred (reference 9). The occurrence of a visible vibration before the explosive knock reaction is an effect apparently not frequently encountered. It appears likely that this phenomenon is comparable with the explosive knock reaction in speed and it may, therefore, be a mild detonation wave followed later by the development of a many-times more-powerful detonation wave.

The evidence of the NACA high-speed schlieren photographs of references 5 to 9 is open to the criticism that the end-zone reactions shown before knock may not represent true flame because the schlieren system may reveal reactions much less intense than flame combustion. The same phenomenon has been shown, however, in photographs exposed by direct flame radiation presented by Rothrock and Spencer (reference 46). With 18- and 30-octane fuels at a compression ratio of 7, photographs taken at about 2000 frames per second (fig. 7 of the report by Rothrock and Spencer) showed autoignition in the end gas one frame before the development of the brilliant illumination caused by

knock. In the same paper Rothrock and Spencer showed that this brilliant illumination coincided chronologically with the beginning of the gas vibrations.

The concept of a detonation wave set up in afterburning gases behind the normal flame front has been proposed previously by Maxwell and Wheeler in references 15, 16, and 52. In streak photographs of combustion in a bomb with knocking fuels they found only very faint afterglows behind the flame front during the travel of the flame through the bomb. After the flame had traveled completely through the charge they observed an extremely high-speed travel of a more brilliant glow through the chamber. With non-knocking fuels, however, the afterglow behind the normal flame front was brilliant. They reported invariably a correlation between the pinking tendencies of fuels and the lack of brilliancy in the afterburning, and they reported that the addition of ethyl ether or amyl nitrate to a fuel decreased the brilliance of the afterglow, and that decomposed tetraethyl lead increased the brilliance of the afterglow. These investigators concluded in part that the tendency to knock was dependent on slow afterburning, leaving sufficient energy behind the flame front to maintain a shock wave (detonation wave) set up by collision of the flame front with the chamber wall. Lorentzen in reference 49 found evidence from experiments with a combustion bomb that he believed supported the theory proposed by Maxwell and Wheeler. The finding that knocking fuels show less brilliant afterglows than nonknocking fuels has been verified by Duchene (reference 39) and by Rothrock and Spencer (reference 46). Rothrock and Spencer have also presented in figure 12 of the same paper 2000-frame-per-second motion pictures of combustion of 65-octane gasoline in which the combustion chamber was entirely inflamed before the occurrence of knock, as indicated by very brilliant reillumination of the entire chamber. In figure 4 of one of the NACA reports (reference 6) a knocking reaction is seen to have occurred not only after complete inflammation of the cylinder charge but even so late that the schlieren combustion pattern was almost gone.

The combined detonation-wave and autoignition theory, to be complete, must account for the fact that combustion cycles involving nothing more than simple autoignition have been found by General Motors investigators in references 33, 36, and 37 and have been regarded by those investigators as knocking cycles. It is clear that gas vibrations can cause forced vibrations of the combustion-chamber walls of the same frequency as the gas vibrations and thus cause a high-pitched ping. As gas vibrations apparently did not occur in the combustion cycles reported in those papers, however, the question naturally arises as to the cause of the knock that was heard. The only possible answer appears to be that the knocking sound was due to natural vibrations of engine parts.

The autoignition that occurred in the General Motors investigations has been seen to require a period of approximately one-thousandth part of a second for its completion. The sharp increase of pressure in the combustion chamber within the period of one-thousandth part of a second could set up natural vibrations in some of the stressed engine

parts. The energy imparted to the natural vibrations by the autoignition would, in general, be greater in the case of low-frequency vibration than in the case of high-frequency vibration. The influence of vibration frequency on the energy imparted to the vibration by the autoignition could be determined mathematically only if definite information were available as to the rate at which energy is released by autoignition at each instant throughout the autoignition process. Though no such information is available, the experimental evidence at least indicates that energy is released by the autoignition in such a manner that it does not excite appreciable vibration of the gases. It may, therefore, be reasonably assumed that the autoignition would excite natural vibrations of the stressed engine parts only in such modes as have a natural frequency considerably less than the natural frequency of the vibrating gases.

The suggestion that knock is due to vibration of engine parts caused by autoignition and that pink is caused by gas vibrations had previously been made by Boerlage and his coworkers (references 19, 45, and 71).

Summary of literature-based argument for combined knock theory.—The following facts appear to be supported by the weight of experimental evidence:

1. Autoignition of comparatively large bodies of end gas occurs too slowly under certain conditions to produce audible gas vibrations.
2. Under suitable conditions one or both of two types of gas vibration may occur, the detonation-wave type and the vibratory-combustion type.

3. Either type of gas vibration may occur independently of autoignition, but under some conditions the detonation-wave type of gas vibration tends to occur very soon after slow autoignition has taken place.

4. Under suitable conditions apparent detonation waves can develop in the engine cylinder.

5. Under a wide range of conditions, either combustion continues for a distance sometimes as great as several inches behind the flame front or some adjustment of equilibrium takes place through the same distance, resulting in increased pressure, continued ionization, and continued emission of light.

The foregoing facts, supported by the experimental evidence, suggest the following explanation of knock in the spark-ignition engine:

(a) Knock of a comparatively low pitch is caused by simple autoignition of end gas at a rate too slow to produce audible gas vibrations.

(b) Knock involving both low- and high-pitched tones may be caused by autoignition followed by the development of a detonation wave in the autoignited gases.

(c) Knock of high pitch may be caused by a detonation wave in afterburning gases behind the flame front. This detonation wave, having originated in the afterburning gases behind the flame front, may also pass through unignited end gas.

This explanation of knock harmonizes with the findings of the NACA photographic knock investigations that will be summarized in the second part of this section.

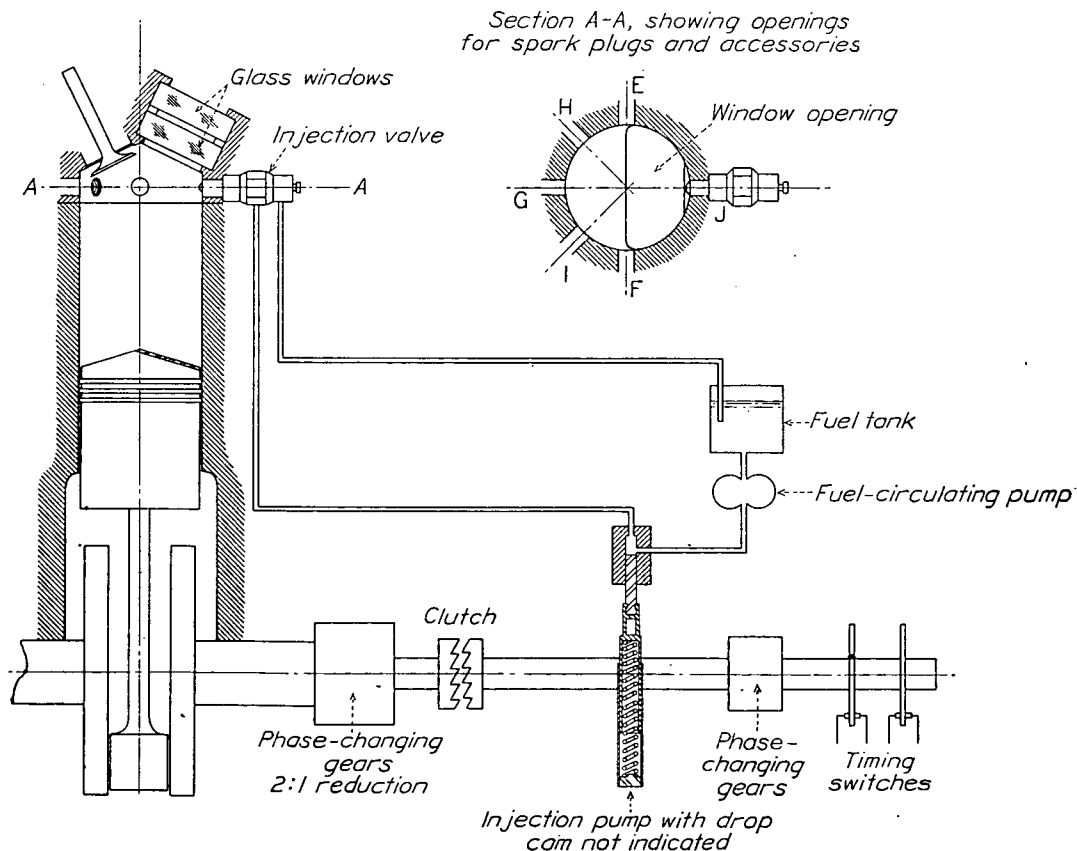


FIGURE I-1.—Old combustion apparatus.

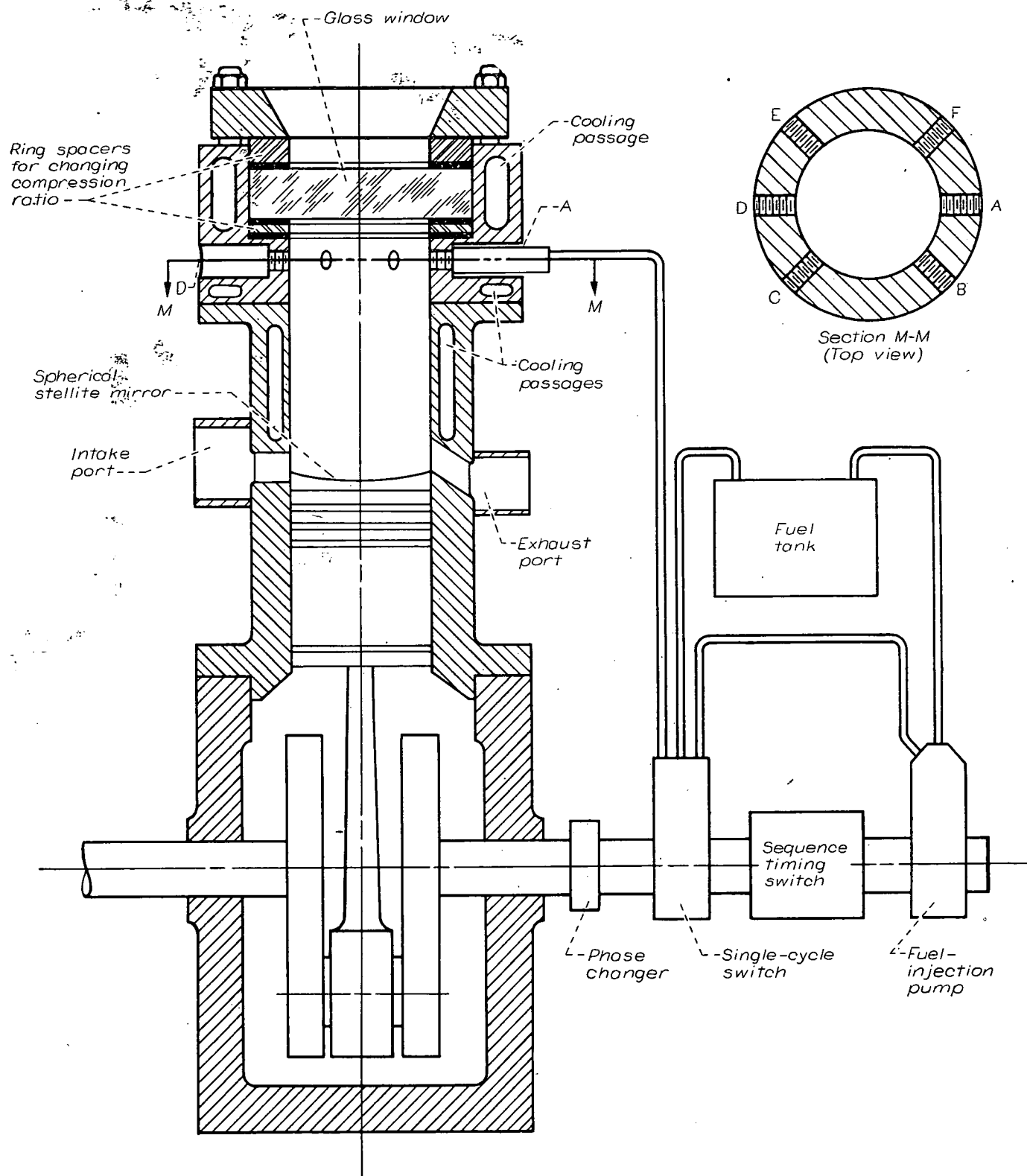


FIGURE I-2.—Full-view combustion apparatus.

REPORT OF FINDINGS OF NACA PHOTOGRAPHIC KNOCK INVESTIGATIONS

Apparatus and operating conditions.—The high-speed motion pictures presented and discussed herein have in part been selected from previously published data (references 5 to 11) obtained with the high-speed and ultra-high-speed motion-picture camera (references 1 to 3) and the NACA combustion apparatus. Most of the work was done with the old combustion apparatus described in references

5 and 46. A small part of the work was done with a newer combustion apparatus. A diagrammatic sketch of the old combustion apparatus is shown in figure I-1; the newer apparatus in figure I-2. The old combustion apparatus is a single-cylinder engine of 5-inch bore and 7-inch stroke, with glass windows in the cylinder head and a glass mirror on the piston top, as shown in the figure. The visible part of the combustion chamber is $4\frac{1}{2}$ inches long, as shown at



FIGURE I-3.—High-speed photographs of normal nonknocking combustion in spark-ignition engine. Fuel, S-1; compression ratio, 7.0; fuel-air ratio, about 0.08; atmospheric intake; one spark plug; spark timing, 20° B. T. C.

frame G-20 in figure I-3. The newer combustion apparatus (fig. I-2) was designed to provide a view of the entire combustion chamber. This engine has a bore of $4\frac{1}{2}$ inches and a 7-inch stroke. Air for combustion is forced into the cylinder through ports uncovered by the piston at the bottom of its stroke and escapes through other ports on the opposite side of the cylinder. The pressure before the start of compression in this engine is controlled by means of the back pressure applied outside the escape ports.

The optical arrangements for schlieren photography (references 5 and 11) will not be redescribed herein, except to state that with each of the combustion apparatus the schlieren photographs were taken by externally supplied light projected into the combustion chamber through the glass windows and then reflected back out through the glass windows to the high-speed camera by the mirror on the piston top.

In all the investigations in which the combustion apparatus has been used, the engine has been driven by an electric motor and operated under its own power for only one combustion cycle in each run; in each case the entire series of photographs was taken during the single combustion cycle. Solid injection of fuel was used and fuel was injected only for the single power cycle. With the old combustion apparatus the fuel was injected on the inlet stroke; with the newer full-view apparatus it was injected early in the compression stroke. The engines were heated prior to the motoring period and kept hot during the motoring period by the circulation of heated glycerin through the cooling passages of cylinder and head. All of the investigations have been made with glycerin temperature approximately at 250°F leaving the cylinder and head. The investigations have all been made with an engine speed of approximately 500 rpm. Spark timings and spark-plug positions were selected to produce knock at top center with the end gas usually well within the field of view. Other engine operating conditions and the fuels used will be stated in the discussions of the individual tests. Atmospheric intake was always used with the old combustion apparatus.

Normal nonknocking combustion.—Figure I-3 of this paper is a reproduction of figure 3 of the preliminary report (reference 5). This figure shows the normal process of smooth nonknocking combustion. Only one spark plug was used, in E position. (See fig. I-1.) The injection valve was in J position. (See fig. I-1.) The fuel was S-1 reference fuel; spark timing, $20^{\circ}\text{B. T. C.}$; compression ratio, 7.0; and fuel-air ratio, approximately 0.08.

Throughout this paper the individual frames of figure I-3 and other figures will be referred to as frame A-1, meaning the first frame of row A, frame D-8, meaning the eighth frame of row D, and so on. The order in which the pictures were taken is from left to right through the first row, frame A-1 through frame A-20, then from left to right through the second row, frame B-1 through B-20, and so on.

In the first few frames of row A of figure I-3 the flame from the igniting spark is just coming into view. A large black spot appears at the upper left-hand corner of each of these frames. This large dark spot is due to an imperfection in the schlieren system and has nothing to do with the combustion. The flame coming into view in these frames is barely visible as a small dark spot at the central upper edge

of each frame. The increase in size of the flame between one frame and the next is extremely small because the time interval involved is only $1/40,000$ second, that is 25 microseconds. The flame very gradually grows larger throughout the first five rows of frames, A to E. In frame E-10 the flame has grown until it covers almost the upper half of the frame as a dark mottled cloud. Although these photographs are actually positive prints, the flame is seen as a dark mottled cloud because of its effect on the externally supplied light. The photographs were taken too fast, and the lens aperture was too small, for the flame to be photographed by the light radiated by it. The burning region appears to scatter the externally supplied light so that it does not get through to the camera lens. The burning region consequently shows up as a dark cloud in contrast with the areas where no combustion is proceeding. The externally supplied light passes through the nonburning areas uninterrupted to the camera lens and causes a fairly uniform white appearance of these areas in the positive prints. For reasons that will be explained it is believed that all of the dark mottling visible in the upper half of frame E-10 is indicative of continuing combustion. In this frame, therefore, the apparent depth of the flame in the direction of flame travel is approximately 2 inches. It is not known to just what extent this apparent depth is real and to what extent it is caused by tonguing of the flame. To explain the apparent depth entirely on the basis of flame tonguing, however, would require a quite illogical assumption that the flame is always much more extensively tongued in the planes that cannot be seen in the photographs than in the plane that is seen in the photographs. (The distance through the combustion chamber in the line of sight from the surface of the mirror on the piston top to the under surface of the inner glass window is approximately 1 in.) The appearance of frame E-10 of figure I-3 is quite typical of this stage of combustion in all photographs taken under the same conditions. The leading edge of the flame as seen in frame E-10 is somewhat irregular but does not show any such pronounced tonguing as would be required to explain the presence of mottling throughout the entire area behind (above) the flame front.

In the frames of rows F, G, and H the flame front slowly advances downward to a position (in frame H-20) about two-thirds of the way across the chamber from the point of ignition. At the same time the dark mottling disappears throughout most of the area behind (above) the flame front, so that in frame H-20 the apparent combustion zone extends only a short distance backward from the flame front. This narrowing of the combustion zone as the flame front travels through the central part of the chamber is also typical of the photographs taken under these conditions. In the frames of rows I to M the flame front slowly completes its travel to the extreme end (lower end) of the combustion chamber. After the flame front reaches the extreme end of the chamber in frame M-20, the mottled combustion zone gradually dissolves in the frames of rows N, O, and P, leaving a clear white field throughout the entire area of the chamber in frame P-20.

It may be noted that the dark spot which was visible in the upper left-hand corners of the early frames of row A is not visible in the frames of row P. This spot, caused by

imperfection of the optical system, moved about because of rocking of the piston at top center and had moved completely out of the field of view in the frames of row P.

From the standpoint of a study of knock, the most important feature of the normal combustion photographs of figure I-3 other than the apparent depth of the combustion zone is the fact that all stages of the combustion process appear slow and smooth. When the photographs are projected on a motion-picture screen the entire process is slowed down so that it resembles the motion of a storm cloud. After the combustion is completed and the mottled zone has faded out the gases in the combustion chamber are very quiescent, a condition contrasting decidedly with the appearance after a knocking combustion.

Nonknocking combustion with preignition from hot spot.—Figure I-4, a reproduction of figure 7 of reference 5, shows a combustion process with the same fuel and the same engine operating conditions as those of figure I-3 but with ignition from a hot spot as well as from the spark plug. The spark plug was again in E position (see fig. I-1), but an electrically heated coil was inserted by means of a special plug at F position (see fig. I-1) and the current through this coil was adjusted to such a value that the coil would ignite the fuel-air mixture at an earlier time than the igniting spark at the plug in E position.

When the camera shutter opened for the shot of figure I-4, in the first few frames of row A, the flame from the hot spot had already come well into the field of view. The flame from the spark plug comes into view in the later frames of row E and the earlier frames of row F, visible at the top of each frame as a whitish spot. In this case the flame shows up white by contrast with the dark spot caused by the imperfection of the schlieren system. When the flame gets well out into the white part of the field of view in row J it again shows up as a dark mottled region as in figure I-3. The flame from the spark plug and the flame from the hot spot merge in the frames of rows J and K and the mottled combustion zone very gradually dissolves in the frames of rows L to P. In this case in the frames of row P the dark spot at the upper left-hand corners of the frames, caused by imperfection of the schlieren system, is still visible. Again it has no significance relative to the combustion process.

From examination of the photographs of figure I-4 as stills, as well as from observation of these photographs as a motion picture projected on the screen, it must be concluded that the entire combustion process is just as smooth and gradual as with the normal nonknocking combustion process of figure I-3. After the mottled combustion zone has faded out the gases again appear very quiescent, in contrast with the appearance after a knocking combustion. The photographs of hot-spot ignition clearly show that this type of ignition is not a direct cause of knock.

Preliminary view of knock.—Figure I-5 of this paper is the same as figure 5 of reference 5 and the same as figure 2 of reference 10. The combustion process shown in this

figure involves one of the most violent knocks ever photographed. The engine operating conditions were the same as those for the normal nonknocking combustion process shown in figure I-3. Instead of S-1 fuel, however, a blend consisting of 50 percent S-1 and 50 percent M-2 reference fuels was used.

The flame from the igniting spark comes into view in figure I-5 in the later frames of row C, visible at the top of each frame as a dark mottled cloud. The flame progresses downward through the field of view in the frames of rows D to K in the same smooth gradual manner as in figure I-3. In the frames of rows J and K the flame front becomes much more irregular than in the case of figure I-3, but this irregularity is considered to be no greater than is occasionally observed with nonknocking combustion and is not thought to be significant relative to the knock phenomenon. In the frames of row L and the earlier frames of row M the end gas, the area ahead of (below) the flame front, gradually turns dark. By the time of exposure of frame M-10 the end gas has become so dark that it can no longer be distinguished from the combustion zone behind (above) the spark-ignited flame front. This darkening of the end gas, which occurs gradually throughout the frames of row L and the earlier frames of row M, is believed to be indicative of at least the early stages of an autoignition process. In frame M-10, where no line of demarcation can be discerned between the end gas and the burning gases ignited by the normal flame travel, the photographs at least indicate that the end gas is burning just as truly as the gases behind the flame front. The explosive knock reaction that sets up the knocking gas vibrations, however, has not yet begun to develop in frame M-10. The first evidence of this explosive reaction appears in frame M-11 as a white streak along the lower right edge of the frame and as a slight blurring of the dark combustion zone. In the next frame, M-12, the explosive knock reaction has progressed through the entire field of view and has given a fairly uniform white appearance to the entire field. (The whitened area below the lower right edge of frame M-12 is due to a pocket of gas $\frac{1}{16}$ inch thick between the glass window and the metal surface of the cylinder head. This pocket, caused by the window gasket, is completely explained in reference 5.)

At the time of writing of reference 5 the reaction occurring during the exposure of frames M-11 and M-12 was confidently believed to be the explosive knock reaction because the projected motion pictures clearly showed that the gas vibrations began at the same time as the whitening of the field of view in frames M-11 and M-12. Moreover, a violent explosion could be observed in the motion pictures in the general vicinity of the end gas at the same time that this whitening occurred. A more rigid proof that blurring of the combustion zone such as occurs in frame M-11 of figure I-3 coincides chronologically with the onset of violent gas vibrations will be discussed later in this paper. A discussion of the nature of the reaction occurring in frames M-11 and M-12 will also appear later in the paper.

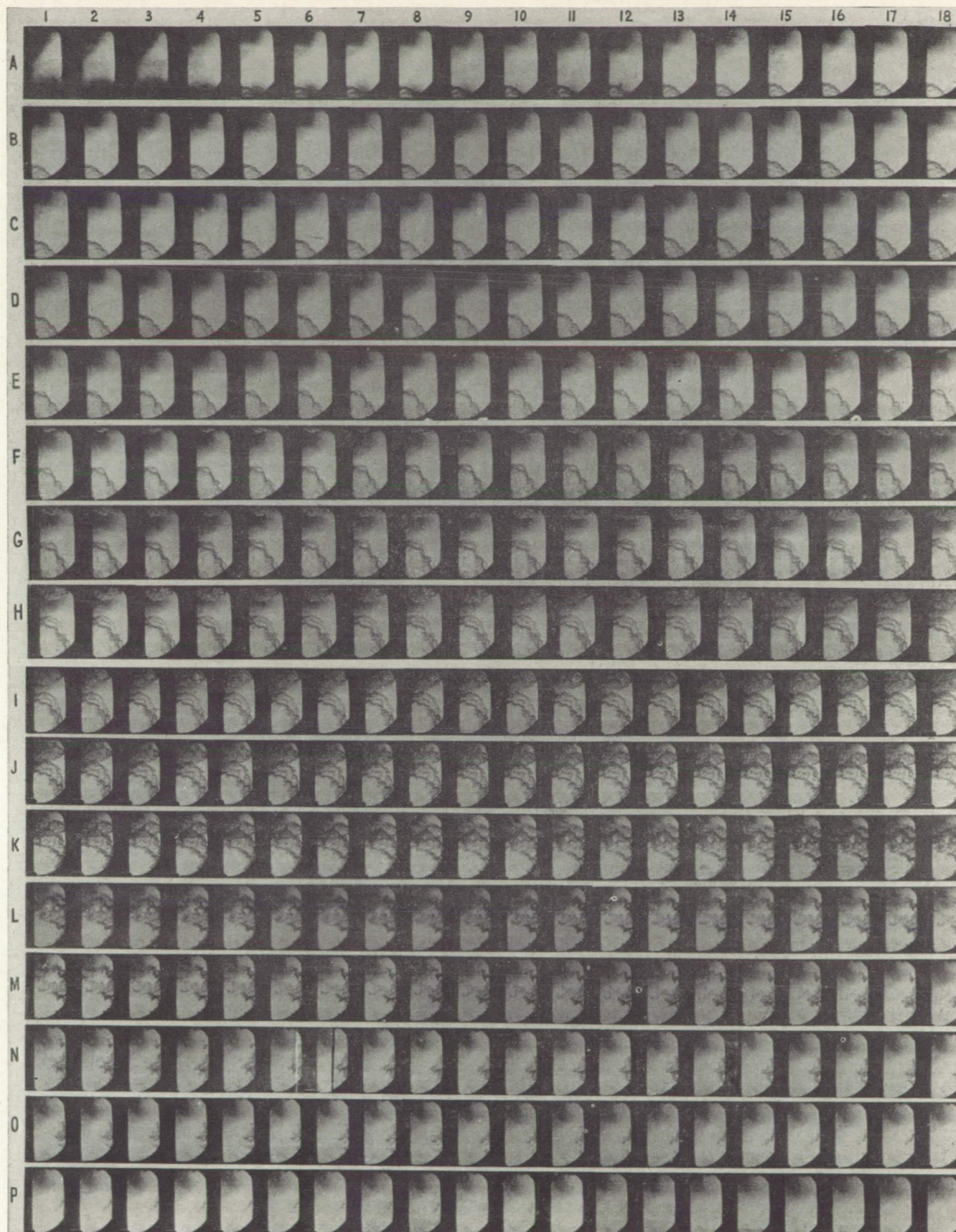


FIGURE I-4.—High-speed photographs of nonknocking combustion in spark-ignition engine with preignition from hot spot. Fuel, S-1; compression ratio, 7.0; fuel-air ratio, about 0.08; atmospheric intake; spark plug at top; hot spot at bottom; spark timing, 20° B. T. C.

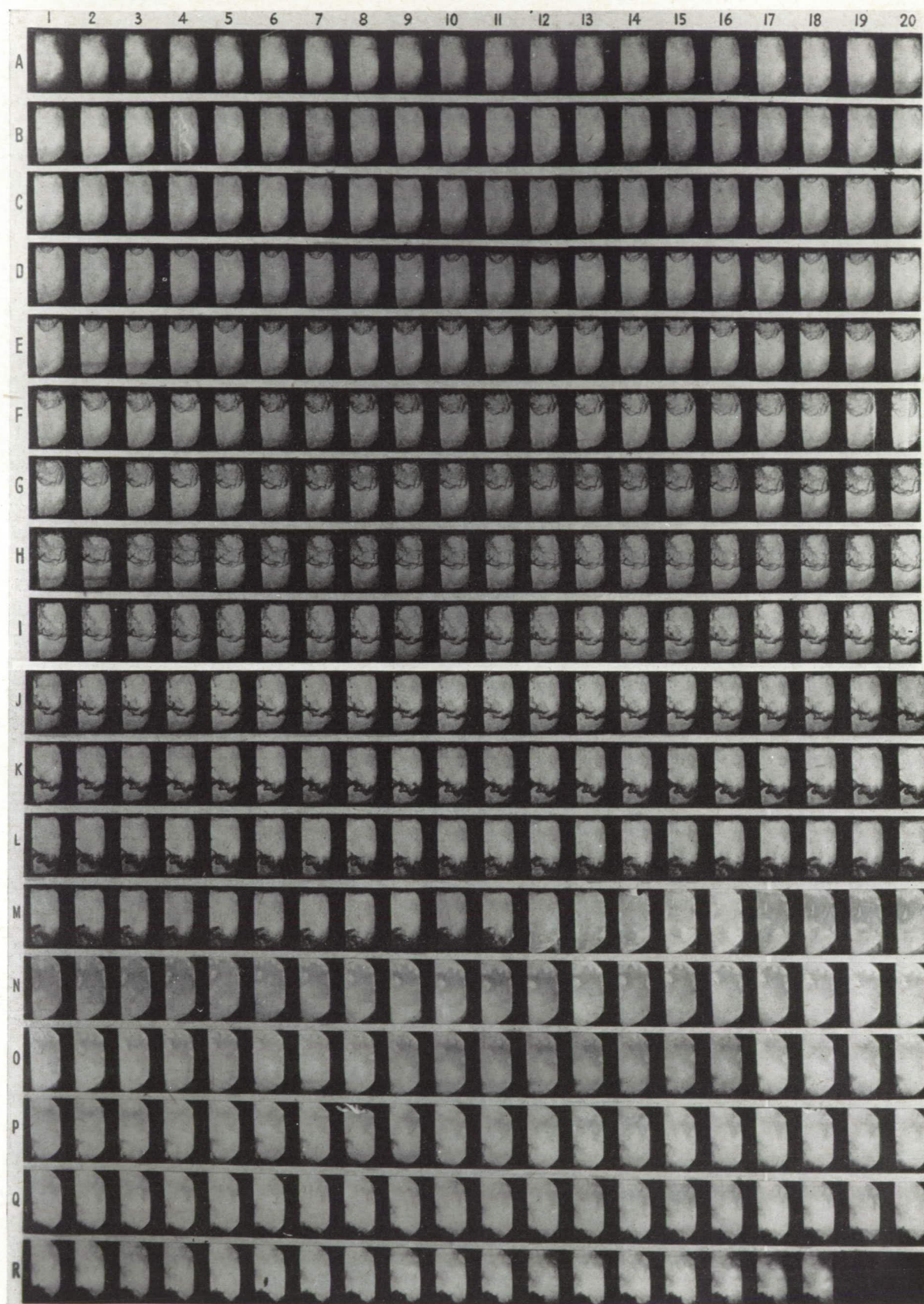


FIGURE I-5.—High-speed photographs of combustion with violent knock in spark-ignition engine. Fuel, 50 percent S-1 with 50 percent M-2; compression ratio, 7.0; fuel-air ratio, about 0.08 atmospheric intake; one spark plug; spark timing, 20° B. T. C.

As previously mentioned, many readers of reference 5 have been entirely unconvinced that the darkening of the end gas prior to the development of the explosive knock reaction in frames M-11 and M-12 ~~actually~~ is autoignition. This author has held the view that this darkening must be regarded as autoignition because there has seemed to be no satisfactory definition of the distinction between preflame and flame reactions in a gradual process occupying a total time interval of less than 1/1000 second (frames L-1 to M-10). In all human experience with flame the preflame reactions have been separated from the flame reactions by a sudden great increase in the rate of reaction, intensity of light radiation, rate of heat release, and so on. The time interval involved in the actual transition from preflame to flame reactions has never been specified and has, in effect, been regarded as instantaneous because of the absence of any method of measuring the interval. The reaction occurring in frames M-11 and M-12 could be regarded as marking the transition from preflame to flame reactions, but so to regard this reaction would be entirely arbitrary and would be only the result of an illogical extrapolation—an extrapolation in the sense that we think because a sharp dividing line appears to exist between preflame and flame reactions when we “see slowly” a sharp dividing line must also be found to exist when we “see fast.”

Qualitatively it could be said that any reaction is proceeding at a very high rate when it produces as much optical change in 1/10,000 second as can be seen between the end gas as it appears in frame M-7 and the end gas as it appears in frame M-10. An investigation is being conducted by G. E. Osterstrom of the NACA technical staff to determine whether the light radiations from the end gas before the explosive knock reaction are sufficient to be measurable. This investigation involves taking simultaneous schlieren photographs and direct flame photographs with a definitely established chronological relationship of the individual frames of one series of photographs to the individual frames of the other series. The preliminary indications are that the radiation from the end gas prior to the explosive knock reaction is sometimes sufficient to photograph at 40,000 frames per second with the high-speed camera. Even if the rate of reaction, intensity of light radiation, and rate of heat release in the end gas could be accurately determined, however, for all of the time intervals covered by frames L-1 to M-10, it would still not be possible to classify the reactions as preflame or flame reactions unless quite arbitrary values were set for these quantities above which the reactions would be regarded as flame reactions.

A possible logical dividing line between preflame and flame reactions on the basis of the thermal theory of autoignition is the condition of more rapid release of heat by the autoigniting gases than normal dissipation of heat from those gases, at constant volume. Another dividing line, based on the chain-reaction theory, is the condition of more rapid formation of chain carriers by chain-branching reactions than destruction of carriers by chain-breaking reactions. Measurements indicate that the normal flame

front did not make any additional progress into the end gas between frames M-1 and M-10 of figure I-5. Furthermore, many photographs have been obtained in which the normal flame front as seen in the projected photographs was actually pushed backward by the end gas a number of frames prior to the development of the explosive knock reaction. Because of this fact, the dividing line on the basis of thermal theory must have been passed at about frame M-1 in figure I-5, so that the end-gas reaction between frames M-1 and M-10 would have to be regarded as flame reaction, unless it is assumed that the end gas is being heated by radiation and conduction from the burning gases more rapidly than the average of the other gases in the chamber including the burning gases themselves. There is no indication, of course, as to where the dividing line based on chain-reaction theory would be drawn relative to the frames of figure I-5.

In view of the necessarily arbitrary nature of any distinction between preflame and flame reactions in frames L-1 to M-10 of figure I-5, the fact that the end gas does sometimes emit sufficient light to photograph by direct flame photography at 40,000 frames per second prior to the explosive knock reaction, and the fact that preknock radiations from the end gas have been photographed at a lower speed by other investigators (reference 46), the reaction visible in the end gas throughout frames L-1 to M-10 of figure I-5 and similar reactions visible in other figures will be referred to throughout this paper as simply “autoignition.” The reaction of frames M-11 and M-12 will be shown later to be a detonation wave and may be regarded as autoignition only in the sense that it develops during a process of autoignition and that its development may be influenced by the autoignition process.

The frames following M-12 in figure I-5 show the development of an extremely brilliant, intermittent luminosity reaching peaks in frames M-16, N-1, N-6, O-1, and so on, and the development of an intense smoke cloud in the lower parts of the frames of rows P, Q, and R. The brilliant luminosity and smoke formation will be discussed further under “Chemical Nature of Explosive Knock Reaction.” The extremely violent vibration, or bouncing, of the gases that is seen when frames M-12 to R-18 are projected on the screen as a motion picture cannot be seen in visual examination of the photographs in the printed figures. This effect is suggestive of a pot of stiff jelly that has been severely jolted and is characteristic of all of the photographs of knocking combustion obtained with the high-speed camera.

Chronological relation between schlieren photographs and records of cylinder pressure.—In the photographs of figure I-5 the occurrence of knock is very definitely visible in frames M-11 and M-12. In cases of much lighter knocks, however, the occurrence of the knock may be detected in the photographs only as a slight blurring of the dark mottled combustion zone. This blurring is often not visible in the printed reproductions of the photographs, because of inevitable loss of detail, but can be seen on careful inspection of

the original negatives. Inasmuch as the slight blurring of the combustion zone could not be assumed without proof to be identical with the occurrence of knock, one of the early investigations (reference 7) was directed toward the establishment of a definite chronological relationship between this blur and the beginning of the violent gas vibrations associated with knock. Incidental to this relation, some investigation was also made into the chronological relation between the final fadeout of the schlieren combustion pattern and the attainment of peak pressure with nonknocking combustion.

One of the high-speed photographic shots used in the establishment of the chronological relationship is reproduced as figure I-6 of this paper (fig. 11 of reference 7). Three spark plugs were used, in E, F, and G positions (see fig. I-1), a quartz piezoelectric pickup was used in opening J, and the injection valve was in opening H. The spark timing at the plugs in E and F positions was 22° B. T. C. The timing at the plug in G position was earlier (29° B. T. C.) because it was desired to focus the end zone at the diaphragm of the piezoelectric pickup in opening J, and to do so required a

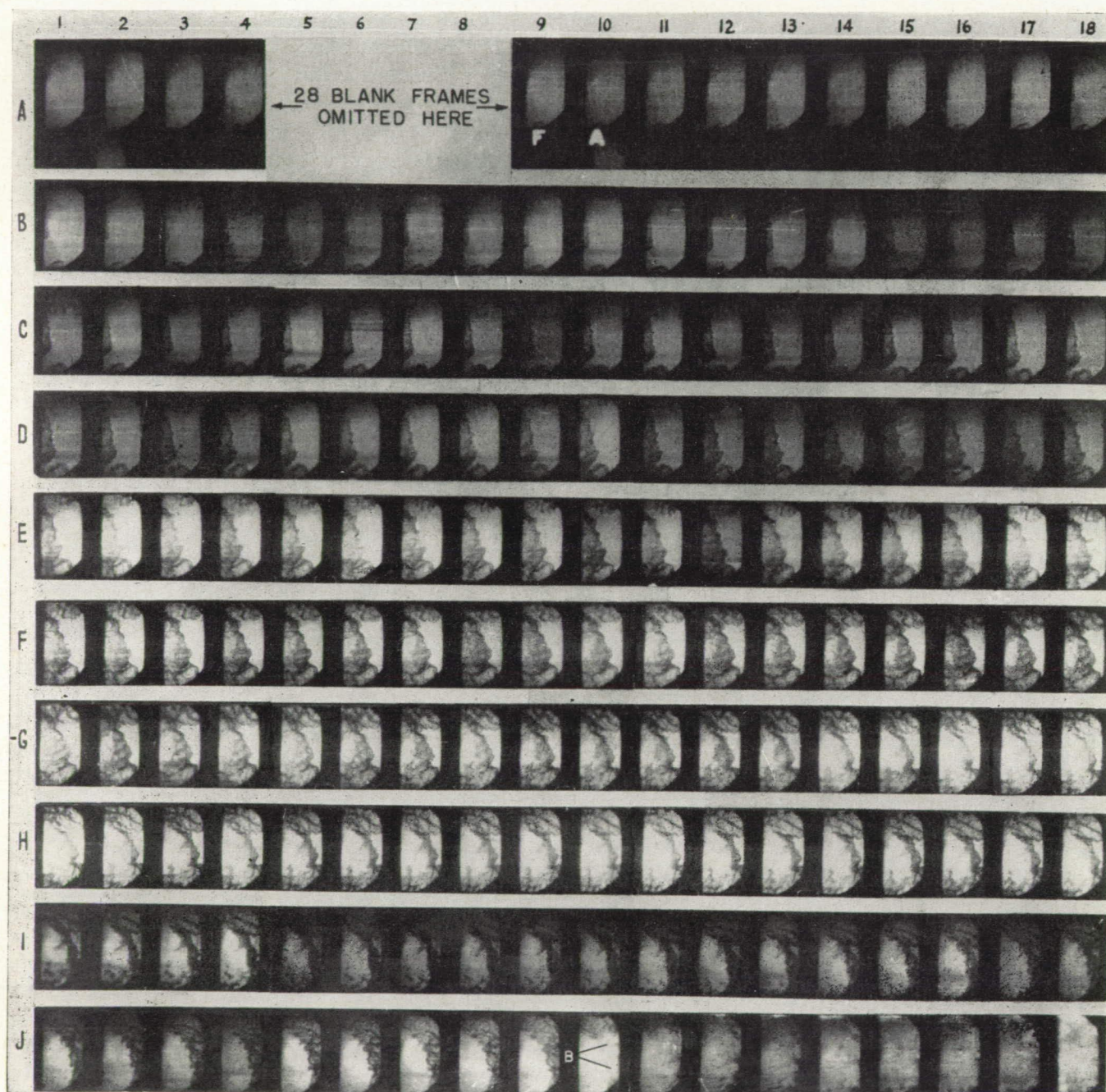


FIGURE I-6.—High-speed photographs of knocking combustion in spark-ignition engine with timing sparks in row A. Fuel, 80 percent S-1 with 20 percent M-2; compression ratio, 7.4; fuel, air ratio, about 0.08; atmospheric intake; three spark plugs; spark timing, for left-hand plug, 29° B. T. C., for other two plugs, 22° B. T. C.; B, blurring caused by knock.

much longer travel for the flame from G position than for the flames from E and F positions. The fuel was a blend of 80 percent S-1 and 20 percent M-2 reference fuels; compression ratio 7.4; and fuel-air ratio, approximately 0.08.

The flames from the three igniting sparks come into view in the frames of rows A to E of figure I-6. By the frames of row I the flames have completely merged and have surrounded a small body of end gas in the immediate vicinity of the piezoelectric pickup diaphragm. In the first nine frames of row J autoignition takes place in the end gas, and in frame J-10 knock occurs, as indicated by the blurring and lightening of the combustion zone in the region designated B.

In row A the film perforations are included in the figure. (The perforations were trimmed from the reproductions used for the other rows.) In the taking of the pictures the film was placed around the inside of a drum in a continuous circle consisting of 372 frames. Of the 372 frames, 168 are omitted from the figure after frame J-18 (or before frame A-1 and 28 are omitted as indicated in row A. A specially designed spark plug in the camera produced a timing mark in the perforation strip at frame A-10 at about the time of the beginning of combustion. The same spark plug produced another timing mark at frame A-2, 339½ frames later, after the combustion and knock were completed. These two timing sparks have been used in the determination of the chronological relationship between the photographs of figure I-6 and the corresponding pressure record of figure I-7.

In figure I-7 the trace designated A is an actual photograph of a trace produced on the screen of a cathode-ray oscilloscope during the exposure of the photographs of figure I-6. The horizontal plates were used, as usual, to produce a time sweep. Two independent voltages were applied to the vertical-plate circuit: a small alternating voltage produced by a 4000-cycle-per-second oscillator, and the voltage produced by the piezoelectric pickup in opening J of the cylinder head. The trace designated B in figure I-7 is an actual photograph of the trace produced on another cathode-ray oscilloscope screen during the exposure of the photographs of figure I-6. In the case of trace B, however, only the voltage from the 4000-cycle-per-second oscillator was applied to the vertical-plate circuit.

The electric conduit that supplied the sparking voltages to the spark plug in the camera was coupled capacitatively with the vertical deflection-plate circuits of both oscilloscopes. This coupling produced the breaks designated F in the two oscillograph traces at the time of the first timing spark (see frame A-10 of fig. I-6) and the breaks designated J at the time of the second timing spark (frame A-2 of fig. I-6). The break designated K in figure I-7 was caused by the first shock on the diaphragm of the piezoelectric pickup at the time of beginning of the gas vibrations in the combustion chamber.

The point in the high-speed camera at which the timing marks were produced and the point in the camera at which

the end gas was photographed are separated by a distance of approximately 34½ frames. Because of this separation, necessitated by mechanical considerations, the breaks F in the oscillograph traces of figure I-7 occurred 196½ (not 162) frames before the development of the knocking blur at B in frame J-10 of figure I-6. Likewise, the breaks J in figure I-7 occurred 143 (not 177½) frames after the development of the knocking blur. In trace B of figure I-7, between breaks F and J, 32.8 oscillator cycles were counted (reference 7); there were therefore $\frac{339.5}{32.8}$ or 10.35 motion-picture frames per oscillator cycle. (The second oscilloscope was provided solely because in many cases the disturbances in trace A caused by the knocking gas vibrations were so violent that it was not possible to count oscillator cycles between breaks F and J in trace A.) Nineteen oscillator cycles were counted (reference 7) on trace A of figure I-7 between the break F and the break K; the knocking break K therefore occurred approximately 196½ frames after the first timing spark. This value agrees exactly with the 196½ frames that separated

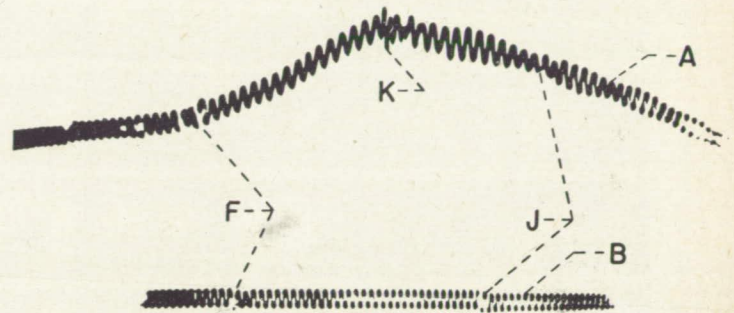


FIGURE I-7.—Composite pressure-time and oscillator trace A and separate oscillator trace B for combustion process of figure I-6. F, breaks caused by first timing spark (frame A-10 of fig. I-6); J, breaks caused by second timing spark (frame A-2 of fig. I-6); K, break caused by start of knock (frame J-10 of fig. I-6); piezoelectric pickup in opening J. (See fig. I-1.)

the first timing spark chronologically from the knocking blur in frame J-10 of figure I-6. (The picture-taking rate is known from camera-acceleration data to be constant within a very small fraction of 1 percent over the period of time involved in the determination, and the frequency of the oscillator output is believed to be similarly constant over the very short time interval involved.)

In eight cases with the end zone in contact with the diaphragm of the piezoelectric pickup the knocking blur was found, by the method used with figures I-6 and I-7, to coincide chronologically with the start of the violent gas vibrations within one-half motion-picture frame. In 16 cases where the end gas was on the opposite side of the chamber from the piezoelectric pickup, the same method indicated that the knocking blur preceded the start of the violent gas vibrations by $4 \pm 1\frac{1}{2}$ motion-picture frames, the value of four motion-picture frames being accounted for by the time required for the knocking disturbance to travel at the speed of sound from the end gas to the diaphragm of the piezoelectric pickup.

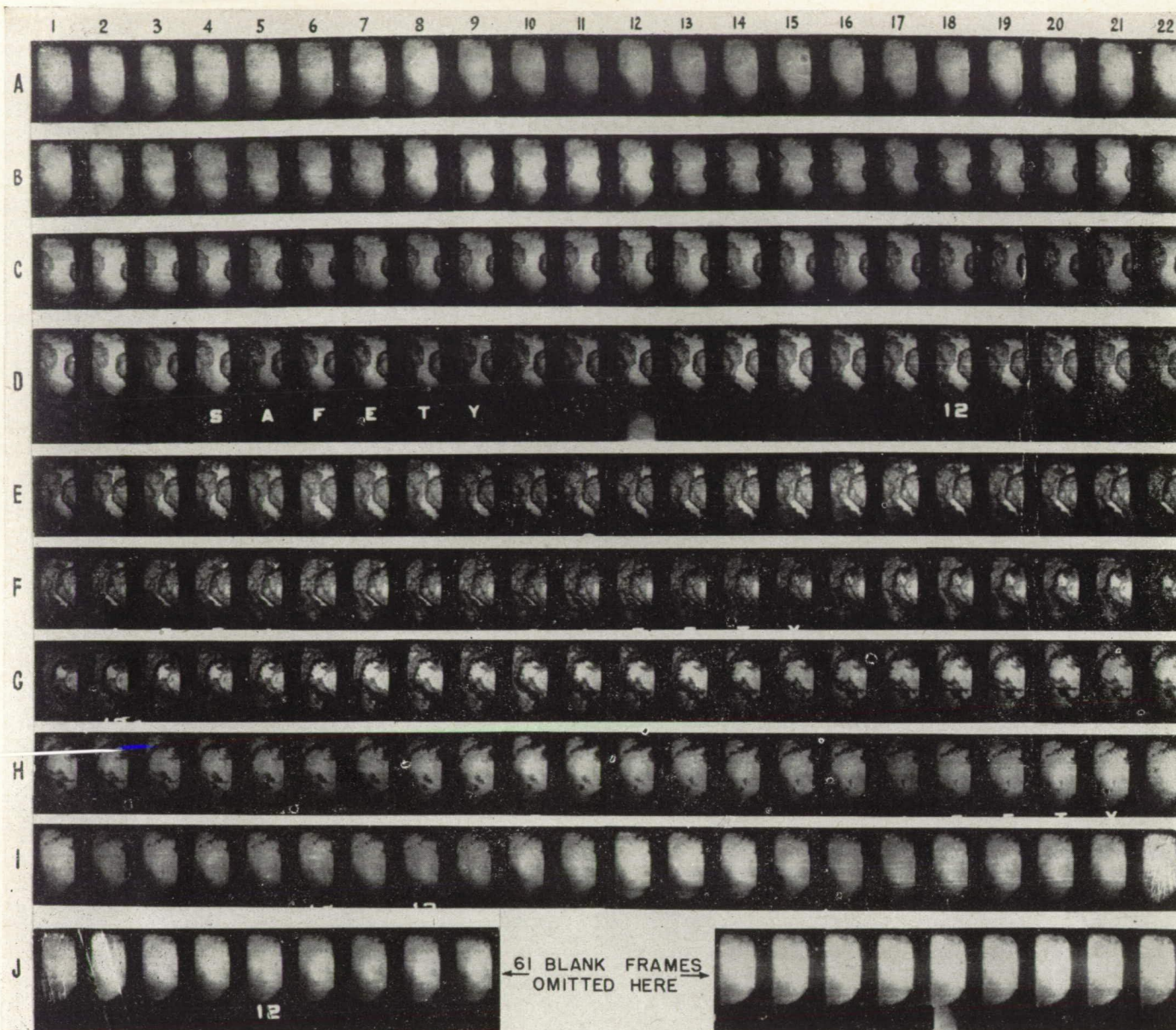


FIGURE I-8.—High-speed photographs of nonknocking combustion in spark-ignition engine with timing sparks in rows D and J. Fuel, S-1; compression ratio, 7.4; fuel-air ratio, about 0.08; atmospheric intake; four spark plugs; spark timing, for left-hand plug, 29° B. T. C., for other three plugs, 22° B. T. C. Frames 1 to 19 selected as point of final fadeout of schlieren pattern.

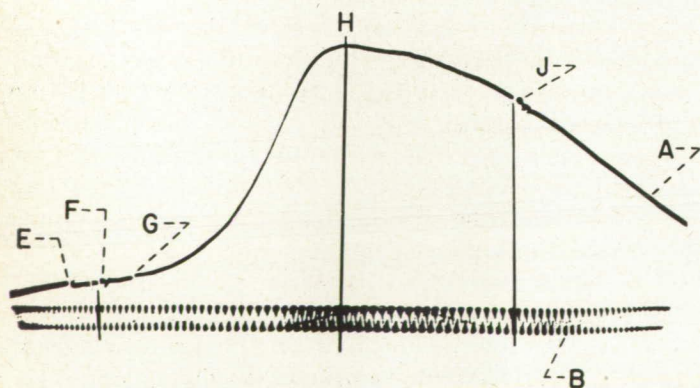


FIGURE I-9.—Pressure-time record A for nonknocking combustion process of figure 8 and separate oscillator trace B. E and G, breaks caused by ignition sparks; F, break caused by first timing spark (frame J-18 of fig. I-8); J, break caused by second timing spark (frame D-12 of fig. I-8); H, point of peak pressure; piezoelectric pickup in opening I. (See fig. I-1.)

A somewhat similar but less accurate method of determining the chronological relationship is illustrated by figures I-8 and I-9 (same as figs. 15 and 16 of reference 7) as applied to the significance of the mottled combustion pattern in the schlieren pictures. Figure I-8 is a shot of normal nonknocking combustion. Engine operating conditions were the same as with the combustion of figure I-6 except that the piezoelectric pickup was placed in opening I (see fig. I-1) and an additional spark plug replaced the piezoelectric pickup at opening J. The fuel was S-1 reference fuel. Ninety-five frames were omitted from figure I-8 after frame J-22 (or before frame A-1), 3 frames were lost at the splice at frame J-2, and 61 frames were omitted as indicated in row J. The first timing spark was exposed before the camera began taking pictures, at frame J-18, and the second timing spark

was exposed 549 frames later, after the camera had ceased taking pictures, at frame D-12.

The first and second timing sparks caused the breaks F and J, respectively, in the pressure-time trace A in figure I-9. With this method the 4000-cycle-per-second oscillator record was not superposed on the pressure-time record. Only one oscilloscope was used. After the pressure-time record was exposed, simultaneously with the exposure of the high-speed photographs, the piezoelectric pickup was immediately disconnected from the vertical-plate circuit of the oscilloscope, the 4000-cycle-per-second oscillator was connected to the vertical-plate circuit, and a second time sweep of the oscilloscope beam was produced, giving the trace B in figure I-9. The trace B was used solely as a measure of the nonlinearity of the time sweep of the oscilloscope. Lines corresponding to constant voltage on the horizontal deflection plates were drawn through trace B from breaks F and J in trace A and also from the point H in trace A, which was considered to be the point of maximum pressure, as shown by trace A. By simple proportionality, the point H in trace A of figure I-9 was found to coincide chronologically with the exposure of frame J-4 of figure I-8 (with due allowance for the $34\frac{1}{2}$ frame correction previously mentioned). Upon careful examination of the original negatives of figure I-8, frame I-19 was selected as the point of final fadeout of the

mottled combustion zone; the difference between times of exposure of frames I-19 and J-4 is 250 microseconds, or only 0.75° of crankshaft rotation.

In five cases similar to that of figures I-8 and I-9 the final fadeout of the schlieren combustion pattern in the photographs was found to precede peak pressure by $0.3 \pm 0.7^\circ$ of crankshaft rotation. As explained in reference 7, peak pressure and the fadeout of the schlieren combustion pattern were so close to top center that no correction was required for piston motion. In 47 cases where the method of figures I-8 and I-9 was applied to knocking combustion, with the end zone on the opposite side of the chamber from the piezoelectric pickup, the knocking blur was found to precede the start of the violent gas vibrations by 8 ± 6 motion-picture frames, as compared with $4 \pm 1\frac{1}{2}$ motion-picture frames in the 16 cases in which the method of figures I-6 and I-7 was used. The method of figures I-8 and I-9 is inherently less accurate than that of figures I-6 and I-7, the principal sources of error probably being lack of reproducibility of the time sweep of the oscilloscope and some slight interaction between the vertical- and horizontal-plate circuits of the oscilloscope. The comparison of the two methods indicates that the method of figures I-8 and I-9 is reproducible within $\pm 0.5^\circ$ of crankshaft rotation and that it has a constant error of about 0.6° of crankshaft rotation. Applying the indi-

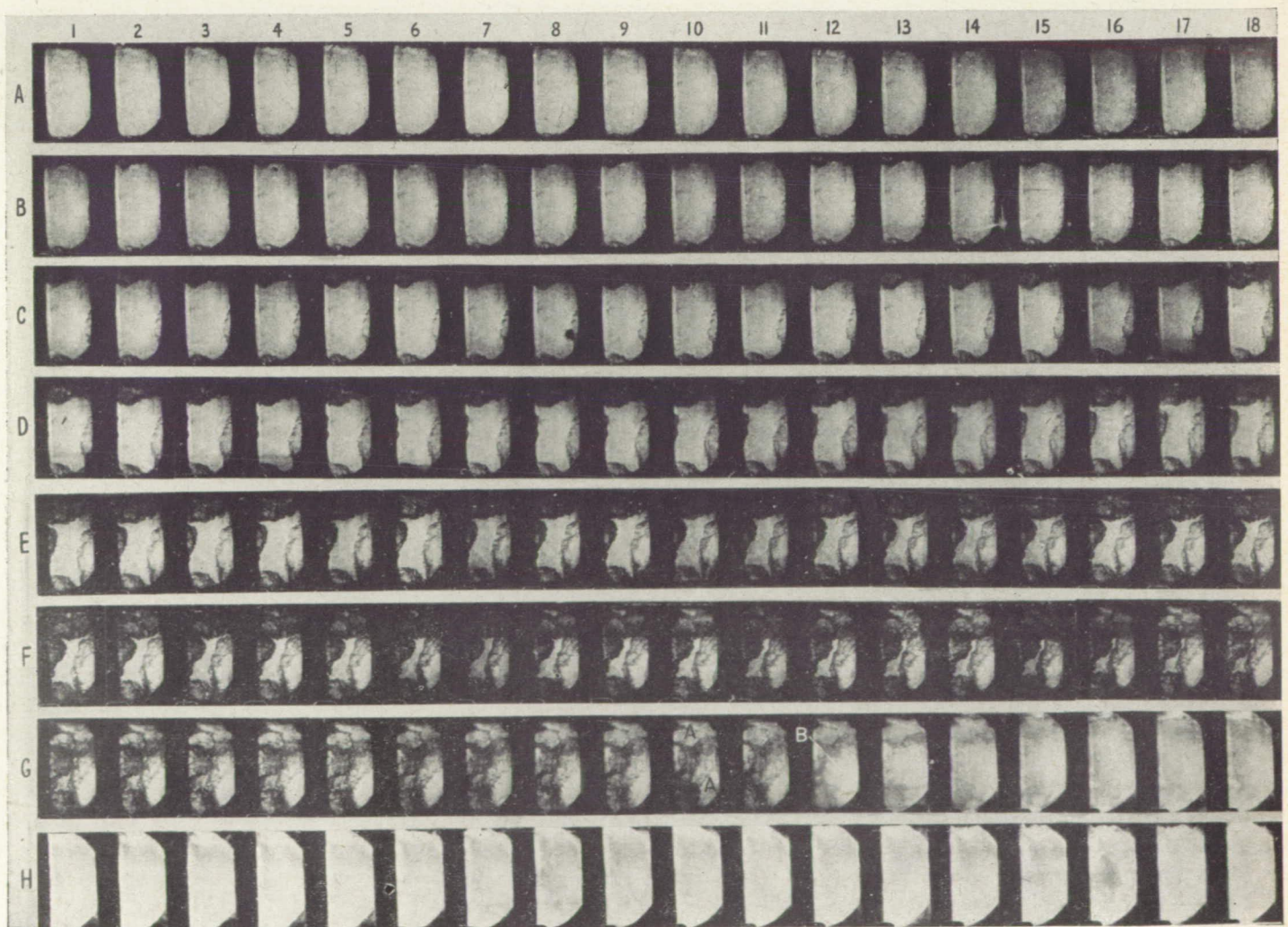


FIGURE I-10.—High-speed photographs of homogeneous end-gas autoignition preceding knock in spark-ignition engine. Fuel, M-2; compression ratio, 7.0; fuel-air ratio, about 0.08; atmospheric intake; four spark plugs; spark timing, for left-hand plug, 27° B. T. C., for other three plugs, 20° B. T. C.; A, regions where burning is complete; B, blurring caused by knock.

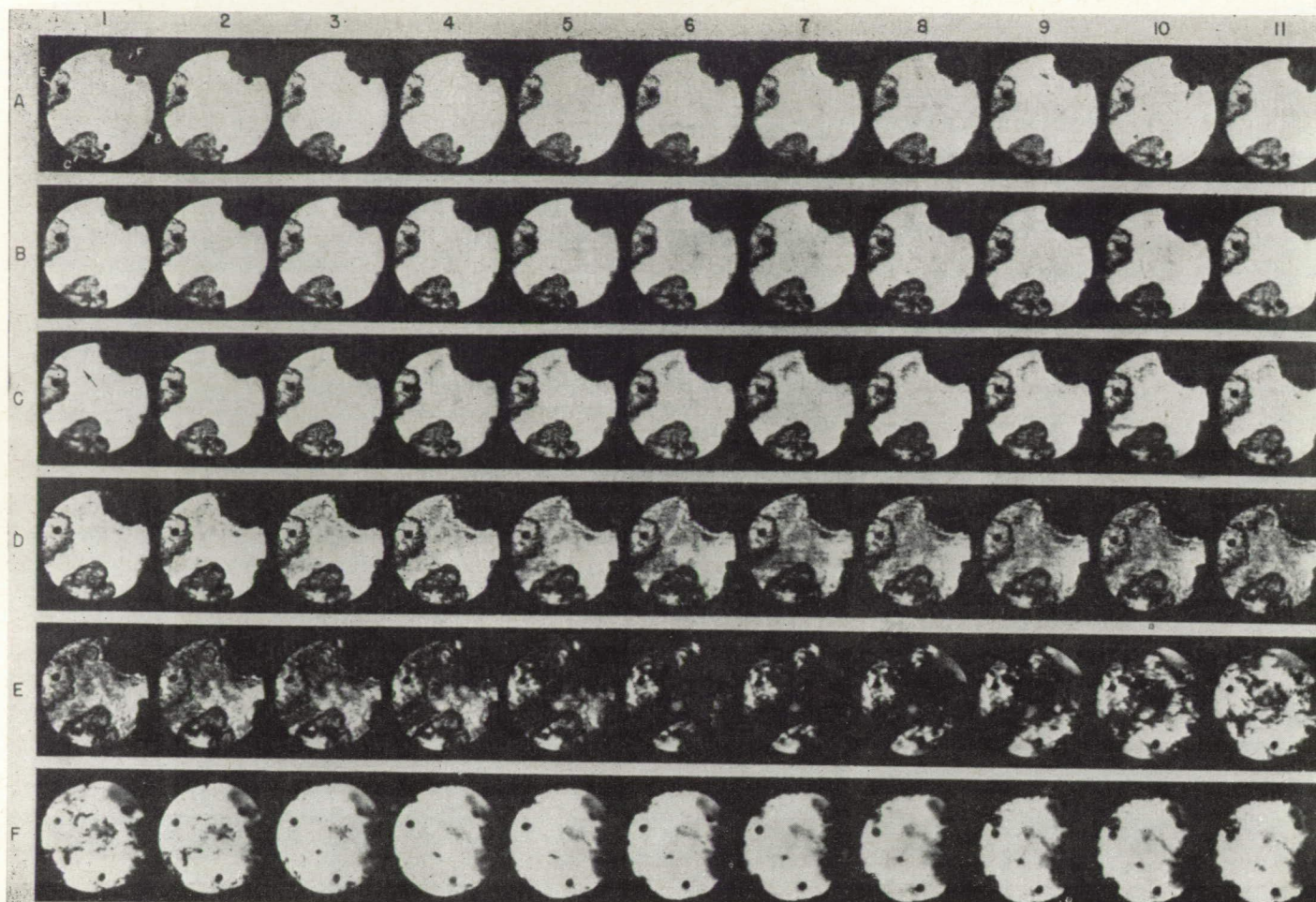


FIGURE I-11.—High-speed photographs of homogeneous autoignition throughout large volume of end gas before knock in spark-ignition engine. Fuel, M-4; compression ratio, 8.7; fuel-air ratio, about 0.17; inlet-air temperature, 445° F; inlet-air pressure, atmospheric; four spark plugs; spark timing, 20° B. T. C.

cated correction for the constant error results in the finding that peak pressure preceded the final fadeout of the mottled combustion zone by $0.3 \pm 0.7^\circ$ of crankshaft rotation. The 0.7° variation from the mean is almost within the demonstrated inaccuracy of the method. It is doubtful whether the point of peak pressure could be selected more accurately than within 0.3° of crankshaft rotation in traces like that of figure I-9.

From the investigation summarized here (reference 7) two conclusions appear justified: first, that the characteristic knocking blur seen in the high-speed photographs does represent the reaction that sets up the violent gas vibrations associated with knock and, second, that the mottled combustion zone does represent continuing combustion, at least so far as the termination of combustion is concerned. The second conclusion as applied to combustion in a constant-volume bomb was previously reached by Lindner (reference 65).

Six types of end-gas autoignition.—The high-speed photographs have revealed six different types of autoignition in the end gas. Some of these types may be interrelated; others seem to be quite distinct. The type of autoignition that will occur under conditions sufficiently severe appears to depend largely upon the fuel used, although the scope of

the investigations has not been sufficient to rule out all other factors entirely as affecting the type of autoignition that will occur. Some other types of autoignition have been followed by the explosive knock reaction in all of the NACA tests where they occurred; other types have been followed by the explosive knock reaction in some cases, in other cases not.

The homogeneous type of autoignition (occurring simultaneously and uniformly throughout the entire body of end gas) is observed in frames L-1 to M-10 of figure I-5 and in frames J-1 to J-9 of figure I-6. This type of autoignition is also well shown in frames G-1 to G-11 of figure I-10, a reproduction of figure 7 of reference 6. (Four spark plugs were used for the combustion process of fig. I-10. The injection valve was in position H of fig. I-1. The fuel was M-2 reference fuel, the spark timing 20° B. T. C. for the plugs in E, F, and J positions and 27° B. T. C. for the plug in G position, compression ratio 7.0, and fuel-air ratio approximately 0.08.) The explosive knock reaction occurs in the area designated B in frame G-12 of figure I-10. The white regions designated A in frame G-10 of the figure are the regions in which combustion is complete; they should not be confused with the end zone which is visible as a white area in frame G-1 but which has become completely dark in frame G-10.

Practically homogeneous autoignition is seen in figure I-11 throughout an end zone very much larger than the end zones of figures I-5, I-6, and I-10. The full-view combustion apparatus was used for the photographs of figure I-11. These photographs were obtained in a recent investigation conducted by H. L. Olsen of the NACA technical staff, as were all photographs of combustion in the full-view apparatus presented later in this paper. Four spark plugs were used in the combustion process for this figure; the flames from three of the plugs made considerable progress through the gases before the first frame of the series was taken, whereas the flame from the fourth spark plug develops very slowly throughout the frames of rows A to D of the figure. The positions of the four spark plugs are shown as B, C, E, and F at frame A-1 of the figure, corresponding to the same lettered positions in figure I-2. The injection valve was in position A (see fig. I-2); the fuel was M-4 reference fuel; spark timing, 20° B. T. C.; compression ratio, approximately 8.7; fuel-air ratio, about 0.17; inlet-air temperature, 445° F; and inlet-air pressure, atmospheric. (Fuel-air ratio was adjusted approximately to maximum knock value; the abnormally high value of 0.17 was required probably because of incomplete fuel vaporization obtained with injection on the compression stroke.)

More than half the total volume of the combustion chamber is involved in the autoignition that developed in the case of figure I-11. The autoignition is first visible as a wisp of mottled gas indicated by the arrow in frame C-1. In all the frames of rows C and D mottling appears throughout the entire area of the end gas, and between frames E-1 and E-7 the entire end gas becomes very dark. The first evidence of the explosive knock reaction appears in frame E-8 and the combustion is completed by frame F-4. The knock in this case was extremely heavy. The pressure-time trace from reference 12 (not reproduced in this paper) indicated a very considerable pressure rise caused by the homogeneous autoignition before the development of the explosive knock reaction.

Homogeneous autoignition has always been followed by the explosive knock reaction in the NACA tests.* In some cases, however, the knock has been quite light in spite of a very large homogeneously igniting end zone, and cases will be presented in which heavy knock has developed in the burning gases alongside of a large end-zone area in which no autoignition has taken place.

A second type of end-gas autoignition, which might be termed "pinpoint" autoignition, is shown in figures I-12 to I-15. Figure I-12 is reproduced from reference 9. In the

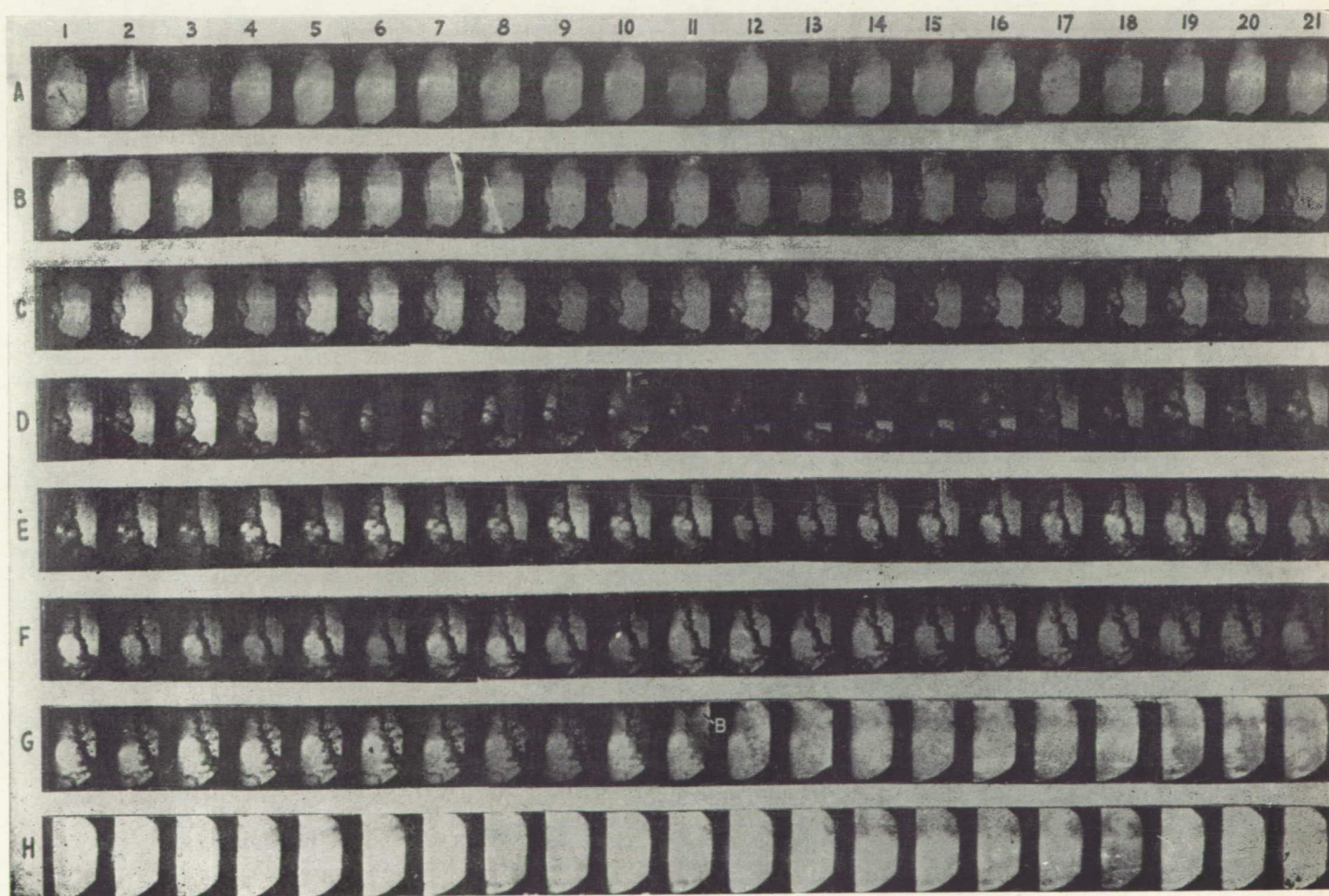


FIGURE I-12.—High-speed photographs of pinpoint end-gas autoignition preceding knock in spark-ignition engine. Fuel, S-1 with 200 ml amyl nitrate per gallon; compression ratio, 7.1; fuel-air ratio, about 0.08; atmospheric intake; two spark plugs; spark timing, at G position (fig. I-1); 27° B. T. C., at F position, 20° B. T. C.; B, luminosity caused by knock.

*Since the writing of this paper a photograph has been obtained by G. E. Osterstrom, NACA, with *n*-heptane fuel, showing homogeneous autoignition throughout a third of the combustion-chamber volume without any evidence of the explosive knock reaction either in the photographs or in the pressure-time record.

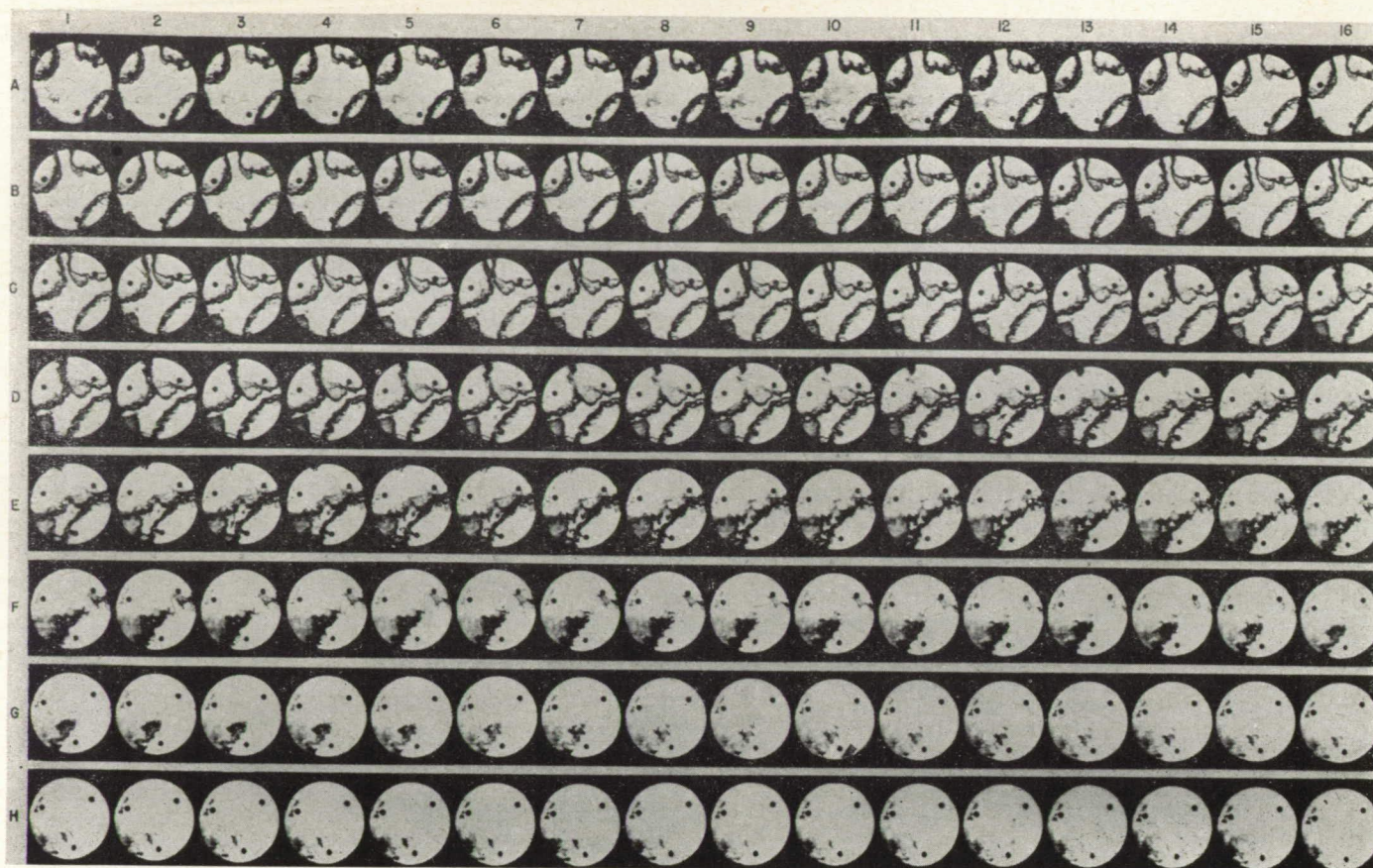


FIGURE I-13.—High-speed photographs of pinpoint end-gas autoignition without knock in spark-ignition engine. Fuel, benzene; compression ratio, 9.0; inlet-air temperature, 425° F; inlet-air pressure, atmospheric; four spark plugs; spark timing, 21° B. T. C.

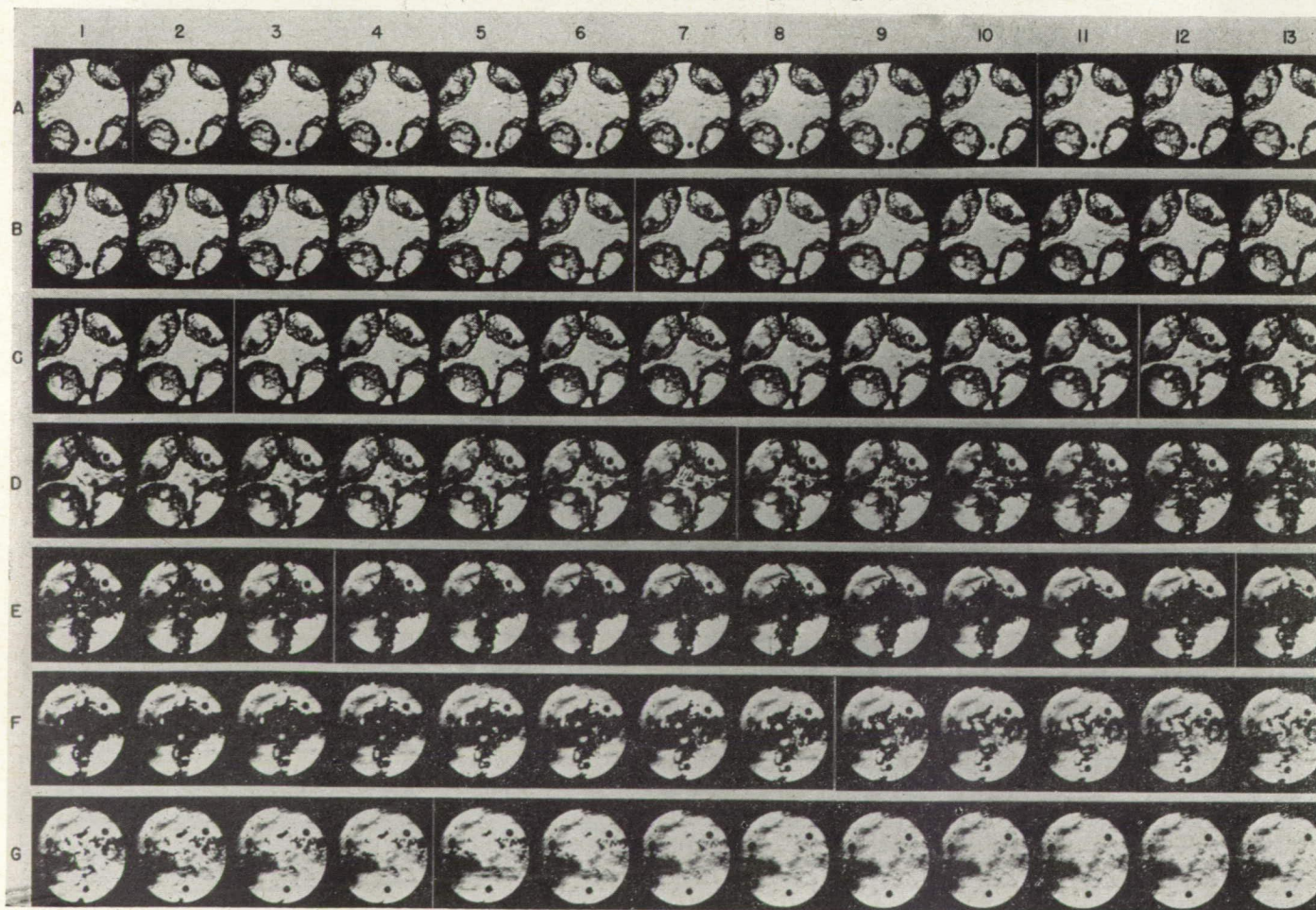


FIGURE I-14.—High-speed photographs of pinpoint end-gas autoignition followed by very light knock in spark-ignition engine. Fuel, triptane; compression ratio, 9.0; fuel-air ratio, about 0.22; inlet-air temperature, 312° F; inlet-air pressure, 20 pounds per square inch; four spark plugs; spark timing, 18° B. T. C.

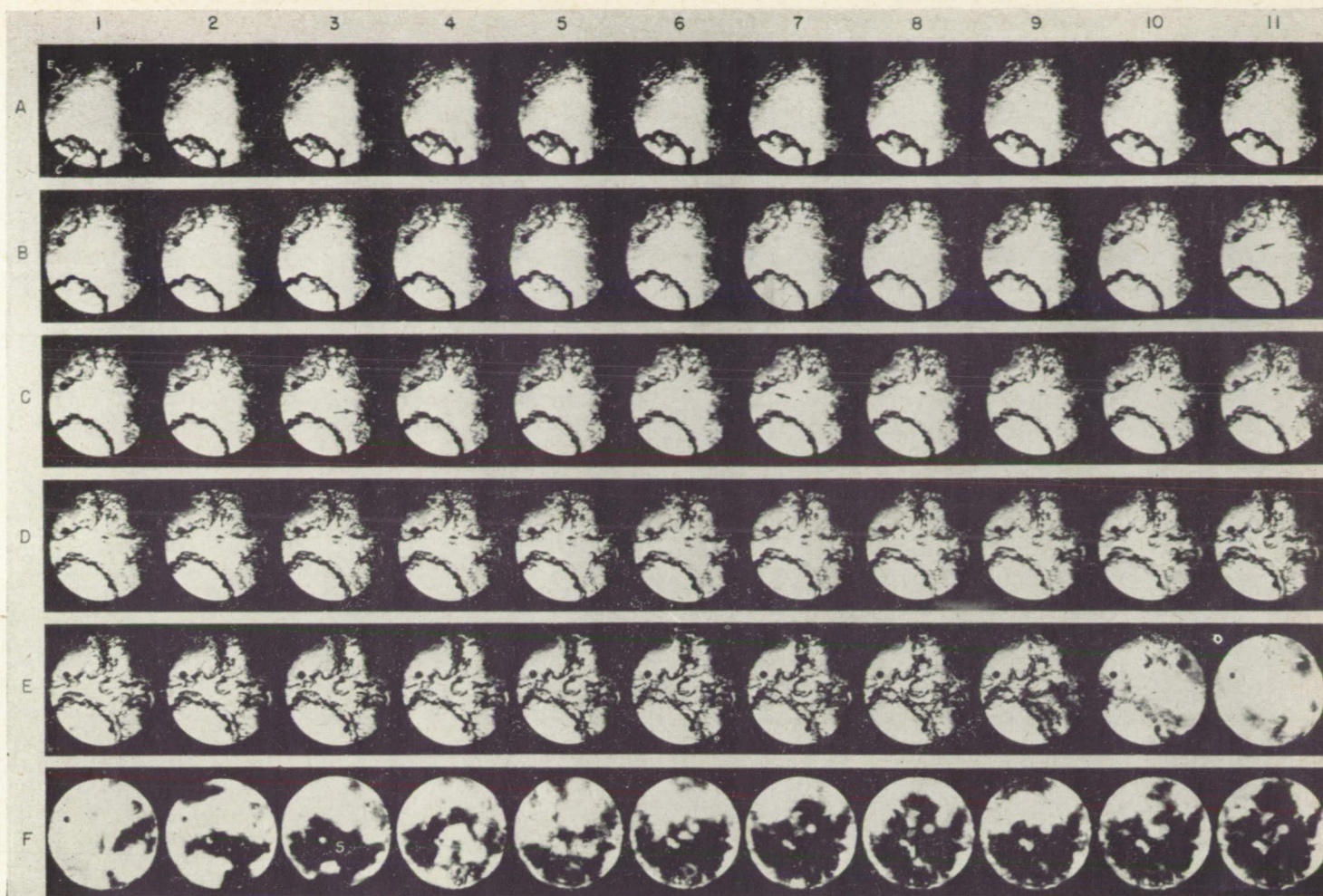
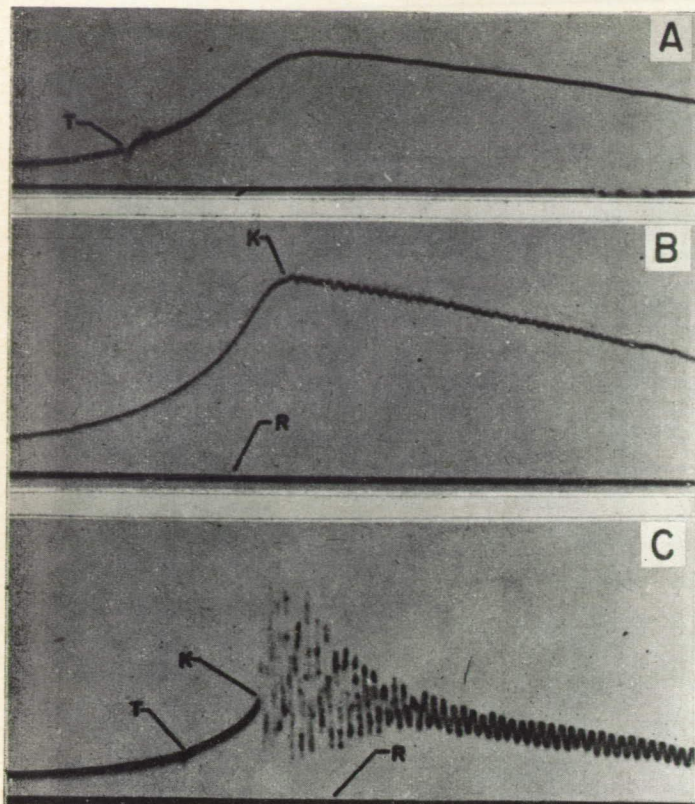


FIGURE I-15.—High-speed photographs of pinpoint end-gas autoignition followed by violent knock in spark-ignition engine. Fuel, S-4; compression ratio, 9.0; fuel-air ratio, about 0.25; inlet-air temperature, 446° F; inlet-air pressure, 18.5 pounds per square inch absolute; four spark plugs; spark timing, 18° B. T. C.

case of figure I-12, taken with the old combustion apparatus, only two spark plugs fired. The spark plug in G position (see fig. I-1) was timed at 27° B. T. C.; that in F position at 20° B. T. C. The injection valve was in H position. The fuel was S-1 reference fuel with admixture of 200 ml amyl nitrate per gallon. The compression ratio was 7.1 and the fuel-air ratio approximately 0.08. The dark frames, D-5 to D-16, in figure I-12 should be disregarded; their appearance was caused by faulty processing of film. The pinpoint autoignition begins to develop in the later frames of row F as very small black dots distributed throughout the end gas. Throughout frames G-1 to G-10 these black dots gradually grow larger until they completely fill the end zone. The explosive knock reaction is first visible at B in frame G-11 and has spread throughout the entire visible part of the chamber in frame G-12. The knock in this case appears to have been violent.

The combustion processes of figures I-13, I-14, and I-15 were fired in the full-view combustion apparatus with four spark plugs timed at 21°, 18°, and 18° B. T. C., respectively. In each case the injection valve was in position A (fig. I-2) and the compression ratio in each case was 9.0. The inlet-air temperatures were 425°, 312°, and 446° F, and the absolute inlet-air pressures atmospheric, 20, and 18.5 pounds per square inch, respectively. The fuel-air ratios for figures I-14 and I-15 were about 0.22 and 0.25, respectively. The

fuel-air ratio for the case of figure I-13 is not known even to an approximation, but it is thought not to be greatly different from the values for the other figures because the flame speeds are of the same order. The fuels for the three cases were benzene, triptane, and S-4 reference fuel, respectively. In each of these three cases the pinpoint autoignition is clearly visible in the end gas (rows D and E of fig. I-13, rows B, C, and D of fig. I-14, and rows B to E of fig. I-15). In many cases the first visibility of the pinpoints is indicated by a small arrow. Careful measurements show that each pinpoint grows at the rate that should be expected if the flame spread out in all directions from a point origin at the same speed as that of the normal flames traveling from the igniting sparks. In the case of benzene (fig. I-13) the pinpoint autoignition does not result in even the slightest gas vibrations. The projected motion pictures appear smooth throughout the entire process. The combustion zone fades out very gradually throughout the frames of rows F, G, and H; this very gradual fadeout is typical of nonknocking combustion and is never seen after occurrence of an explosive knock reaction of any appreciable violence. The pressure-time record (fig. I-16 (a)) shows not the slightest evidence of gas vibration. (All the pressure-time traces of fig. I-16 were obtained from a piezoelectric pickup in position D, fig. I-2.) In the case of triptane (fig. I-14) the development of the pinpoint autoignition and the gradual fadeout



(a) Indicator card for combustion process of figure I-13, showing hot-gas vibrations.
 (b) Indicator card for combustion process of figure I-14, showing light gas vibrations.
 (c) Indicator card for combustion process of figure I-15, showing extremely violent gas vibrations.

Figure I-16.—Pressure-time records.

of the combustion zone is quite similar to the benzene case except that an extremely light knock develops at about frame G-6. The knock at frame G-6 cannot be detected by visual examination of the figure but is seen when the photographs are projected as a motion picture. Also the pressure-time record (fig. I-16 (b)) shows very light gas vibrations starting at the point of peak pressure. In the case of S-4 reference fuel (fig. I-15) the pinpoint autoignition develops in frames B-11 to E-8 and a violent explosive knock reaction develops in frames E-9 and E-10. Upon careful examination unignited end-gas areas may still be seen in frames E-8 and E-9. Some of these unignited end-gas areas still appear white in frame E-10; it appears if the gas in these regions ever burned it did so in a very short interval relative to the photographic exposure time of 25 microseconds. The pressure-time record for the case of S-4 reference fuel (fig. I-16 (c)) shows very violent knock occurring at a time when the heat release from the normal combustion was still rapid.

Figures I-12 to I-16 do not indicate that pinpoint autoignition is in any way related to the explosive knock reaction; this type of autoignition may occur with or without the explosive knock reaction and, as shown by the previous figures, the explosive knock reaction may occur without pinpoint autoignition.

A third type of end-gas autoignition, perhaps fundamentally identical with homogeneous autoignition, is the two-stage type seen in figure I-17. This combustion process was

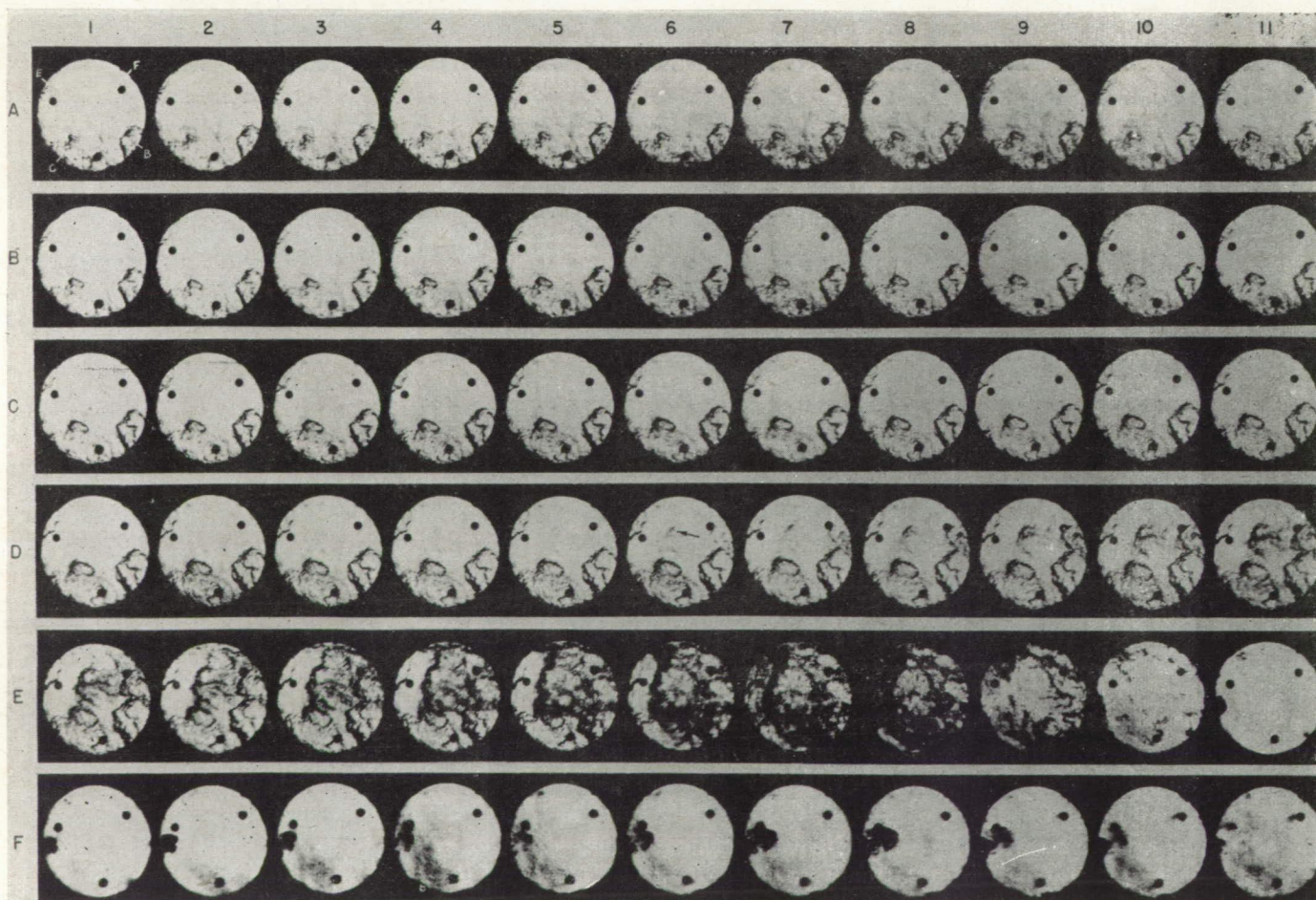


FIGURE I-17.—High-speed photographs of two-stage end-gas autoignition followed by heavy knock in spark-ignition engine. Fuel, M-4; compression ratio, 7.0; inlet air temperature, 398° F; inlet-air pressure, atmospheric; four spark plugs; spark timing, 21° B. T. C.

fired in the full-view combustion apparatus with four spark plugs timed at 21° B. T. C. The injection valve was in position A (fig. I-2); the fuel was M-4 reference fuel; compression ratio, 7.0; inlet-air temperature, 398° F; and inlet-air pressure, atmospheric. The fuel-air ratio is not known but is believed to be quite lean—near the lower limit of flammability. The spark plug at F position (frame A-1) apparently did not fire. The flame from C position is quite indistinct in the frames of rows A and B, but becomes quite sharply defined in rows C and D. The flames from B and E positions are sharply defined throughout rows A to D. The indistinct appearance of the flame from C position in the first two rows of the figure may be an indication that this flame is a borderline case between autoignition and normal propagated flame. A definite case of this type will be presented later in this

section. At about frame D-6 in figure I-17 definite autoignition begins to develop as a wisp of mottled gas, indicated by the arrow in this frame. The wisp indicated in frame D-6 serves as a boundary line in the following frames between homogeneously autoigniting gases and nonautoigniting gases. The gases to the right of this wisp autoignite in frames D-7 to E-5. The gases to the left of the wisp show no evidence whatever of autoignition between the frames D-7 and E-5. The gases to the left of the wisp do autoignite, however, in frames E-6 to E-8. The beginning of a violent explosive knock reaction is apparent in frame E-9.

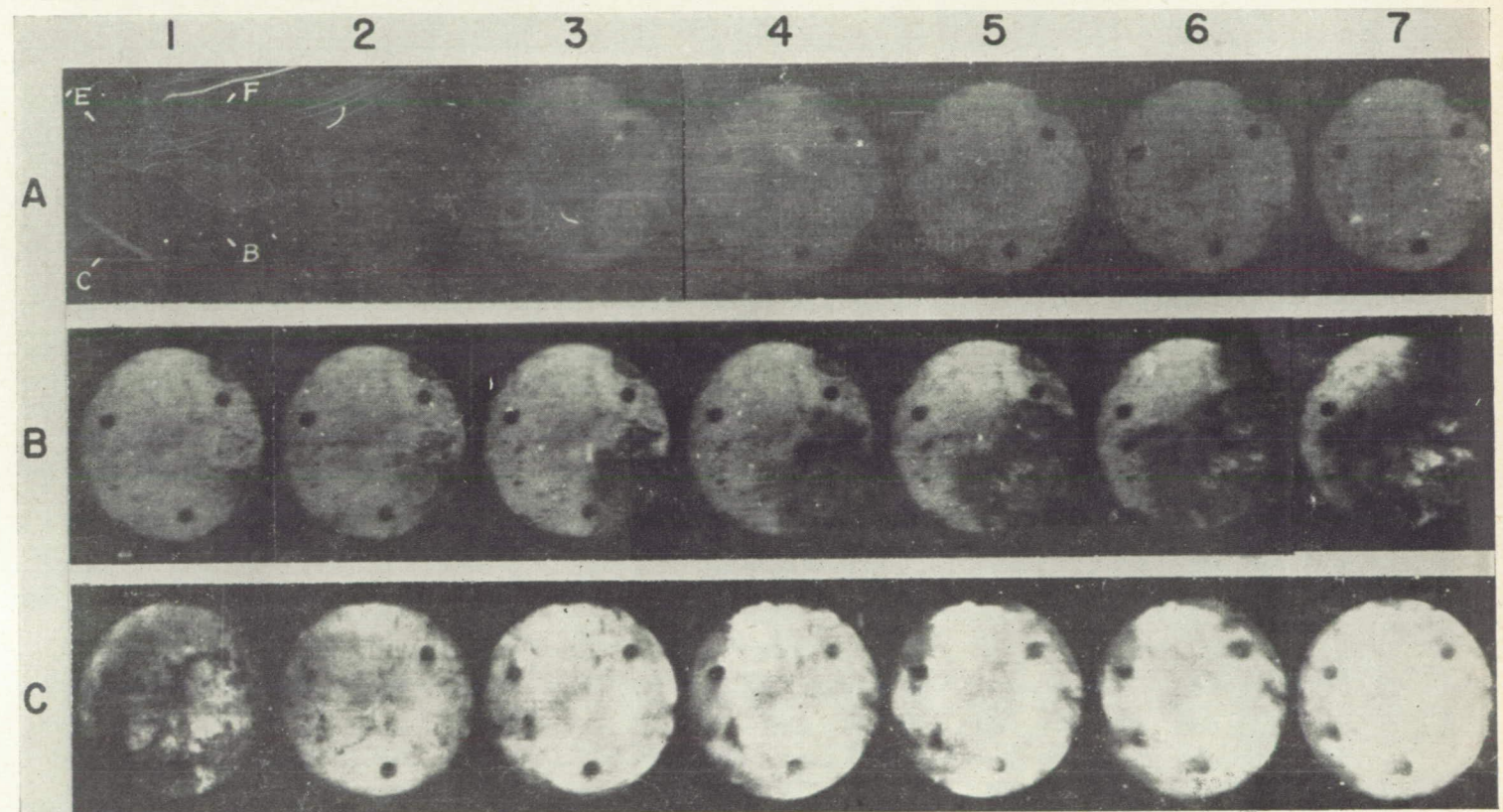


FIGURE I-18.—High-speed photographs of runaway flame preceding violent knock in spark-ignition engine. Fuel, M-4; compression ratio, 9.0; fuel-air ratio, about 0.17; inlet-air temperature, 415° F; inlet-air pressure, 15.5 pounds per square inch absolute; four spark plugs; spark timing, 20° B. T. C.

A fourth type of end-gas autoignition appears in figure I-18 as a wild runaway flame. Because of an accident the photographs of this figure were taken under uniquely severe conditions for the fuel used. No attempt was made to

ratio was 9.0; fuel-air ratio, about 0.17; inlet-air temperature, 415° F; and absolute inlet-air pressure, 15.5 pounds per square inch.

Because of the very severe conditions the charge was apparently ready to autoignite at the time the camera shutter opened for the photographs of figure I-18. The entire combustion process, including the explosive knock reaction, took place within 16 motion-picture frames after the camera shutter started to open. For this reason, the frames do not become well illuminated until about the middle of row B, at which time the camera shutter was open sufficiently to produce fair pictures. The spark-plug positions are indicated at frame A-1. At frame A-7 the flames from B and F positions are fairly visible. The flames from C and E positions cannot be seen in the printed reproduction of

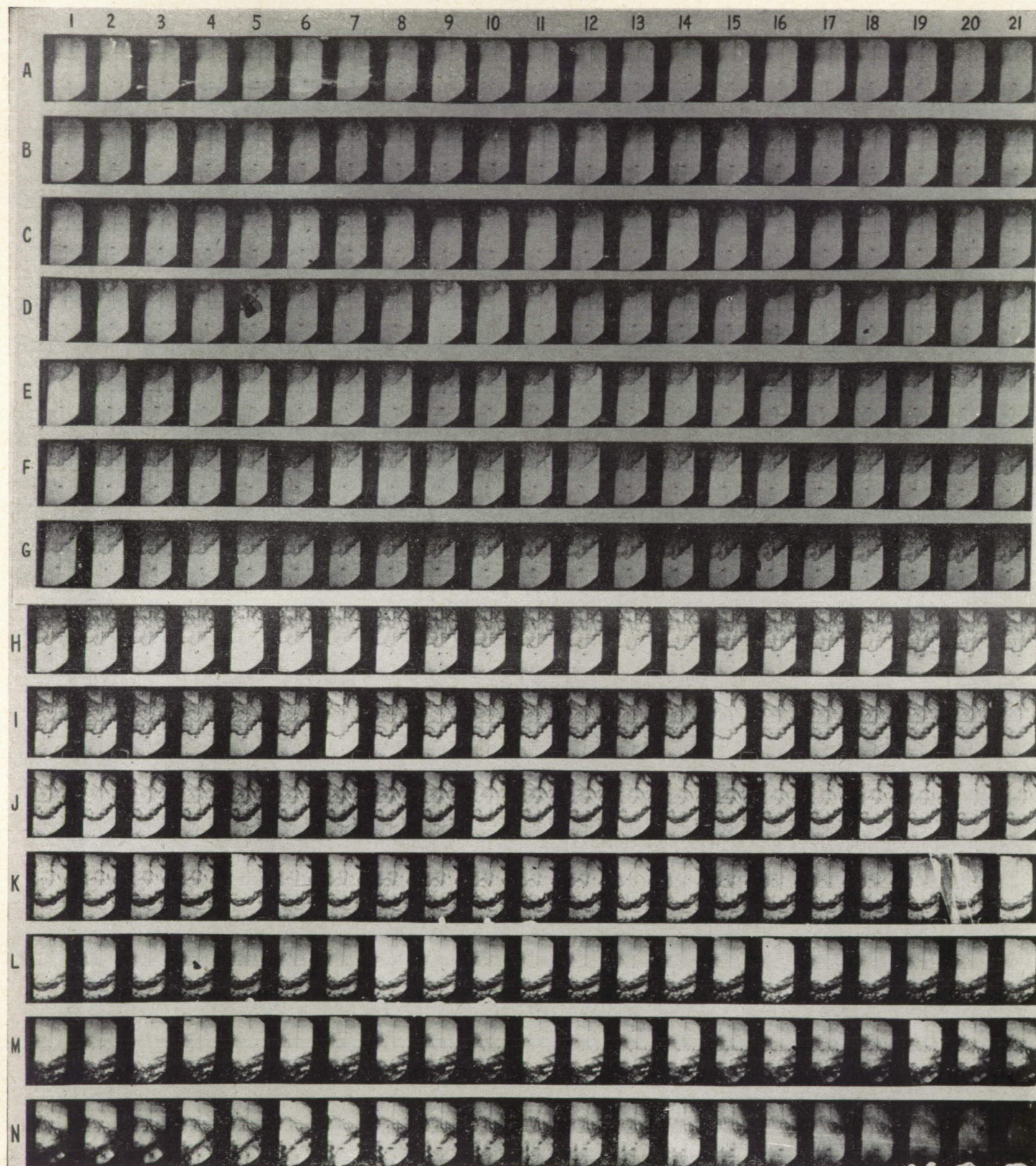


FIGURE I-19. High-speed photographs of development of preknock autoignition flame at far wall of chamber in spark-ignition engine. Fuel, S-2 with 400 ml amyl nitrate per gallon; compression ratio, 7.1; fuel-air ratio, about 0.08; atmospheric intake; one spark plug; spark timing, 20° B. T. C.

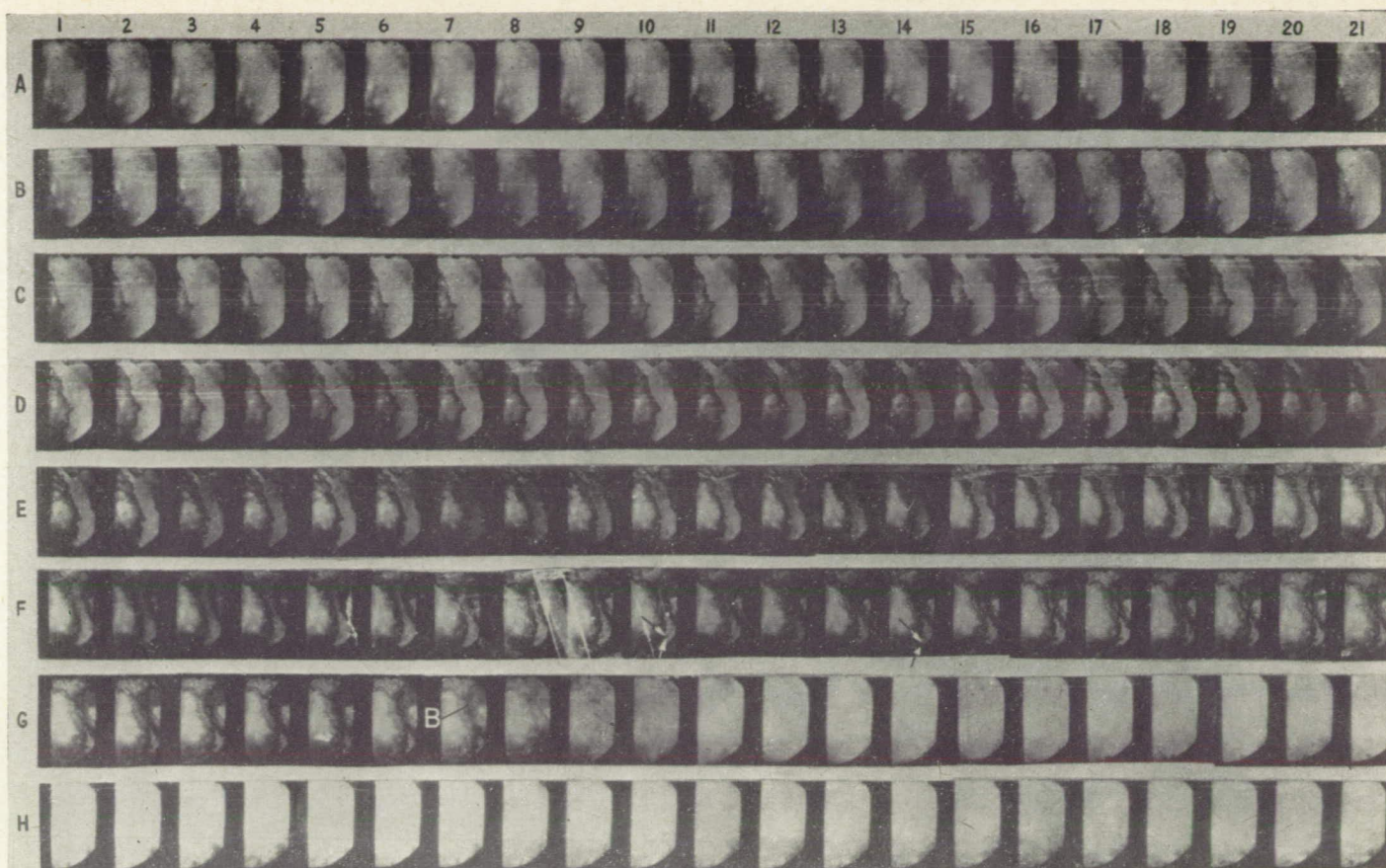


FIGURE I-20.—High-speed photographs of vibratory combustion preceding knock in spark-ignition engine. Fuel, M-2 with 200 ml TEL per gallon; compression ratio, 7.1; fuel-air ratio, about 0.08; atmospheric intake; four spark-plugs; spark timing, for left-hand plug, 27° B. T. C., for other three plugs, 20° B. T. C.; B, blurring caused by knock.

frame A-7, but both of these flames can be seen on careful inspection of the original negative. At frame B-1 the flame from B position breaks loose and it travels all the way across the chamber in frames B-1 to C-1 at a speed of 1900 feet per second. The explosive knock reaction occurs after this flame has completed its travel across the chamber at frame C-2, as shown by the sudden whitening of the entire chamber. In frame B-7, after the runaway flame has nearly completed its travel across the chamber, the flame from E position may be clearly seen still apparently under complete control.

When the photographs of figure I-18 are projected as a motion picture the phenomenon seen in row B of the figure has every appearance of a very fast propagated flame. On close inspection of the still photographs, however, it is found that the leading edge of this flame is never sharply defined and the phenomenon appears to be in fact a multistage autoignition process, similar to the phenomenon of figure I-17 but occurring in many more than two stages. This phenomenon is regarded as a borderline case between homogeneous autoignition and a detonation wave. The occurrence of the explosive knock reaction at frame C-2 of the figure will be further discussed in a later section.

A fifth type of end-gas autoignition is shown in figure I-19, same as figure 13 of reference 9. This combustion process

was fired in the old combustion apparatus with one spark plug in E position (fig. I-1) timed at 20° B. T. C. The injection valve was in H position, the fuel was S-2 admixed with 400 ml amyl nitrate per gallon, compression ratio 7.1, and fuel-air ratio about 0.08. In this case the autoignition occurred as a flame developing at the far wall of the chamber and propagating out from the wall to meet the spark-ignited flame. A slight mottling near the far wall of the chamber (lower edge of the chamber as seen in the photographs) develops in the frames of row K of the figure. Also, in the later frames of row K, a few centers of pinpoint autoignition develop near the far wall of the chamber. In the frames of row L additional pinpoints of autoignition develop and a definite flame propagation begins to proceed out from the far wall. This orderly autoignition-flame propagation proceeds throughout the entire end gas in the frames of row M, and in the frames of row N the mottled combustion zone gradually fades out. When the photographs are observed as motion pictures only a very light explosive knock reaction is seen in the frames of row N; this reaction is so light that it cannot be identified as being associated with any particular motion-picture frame. The type of autoignition shown in figure I-19 may be a special case of pinpoint autoignition.

The sixth type of autoignition, an example of which

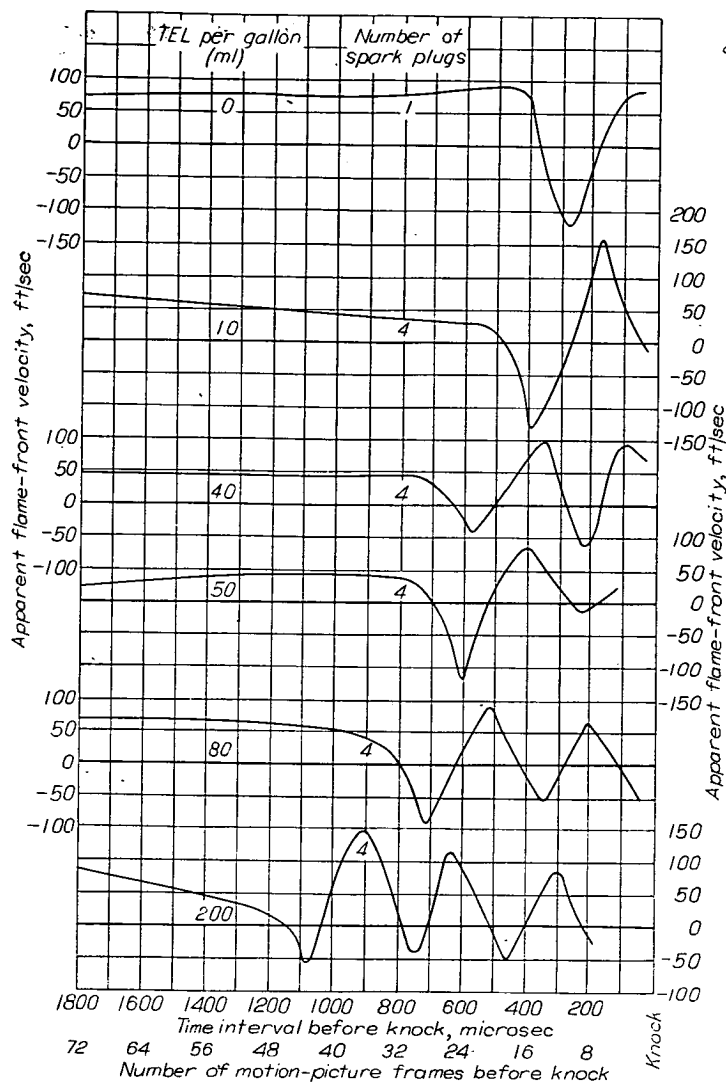


FIGURE I-21.—Apparent flame-front velocities during period of preknock vibration in spark ignition engine.

occurred in the combustion process of figure I-20, is detectable only through the slight preknock vibration which it imparts to the gases. Figure I-20 is a reproduction of figure 2 of reference 9. For this combustion process four spark plugs were used, timed at 20° B. T. C. in E, F, and J positions (fig. I-1) and at 27° B. T. C. in G position. The injection valve was in H position, the compression ratio was 7.1, and the fuel-air ratio about 0.08. The fuel was M-2 reference fuel with an extremely high concentration of tetraethyl lead—200 ml per gallon. Some pinpoint autoignition may be seen in the photographs, originating at centers indicated by the arrows at frames F-10 and F-14. No other visual evidence of autoignition appears, however, and when the explosive knock reaction begins at B in frame G-7 some apparently unignited end gas is still visible at the lower right corner of the chamber. In the original work (reference 9) flame areas were measured on greatly enlarged copies of the individual frames of the figure with a polar planimeter. The results showed that the end gas suddenly expanded at

about frame E-1 of figure I-20, and that between frames E-1 and G-7 at least three cycles of vibration of the gases occurred. The measurements indicated that the amplitude of the vibrations did not increase (in fact, usually diminished)

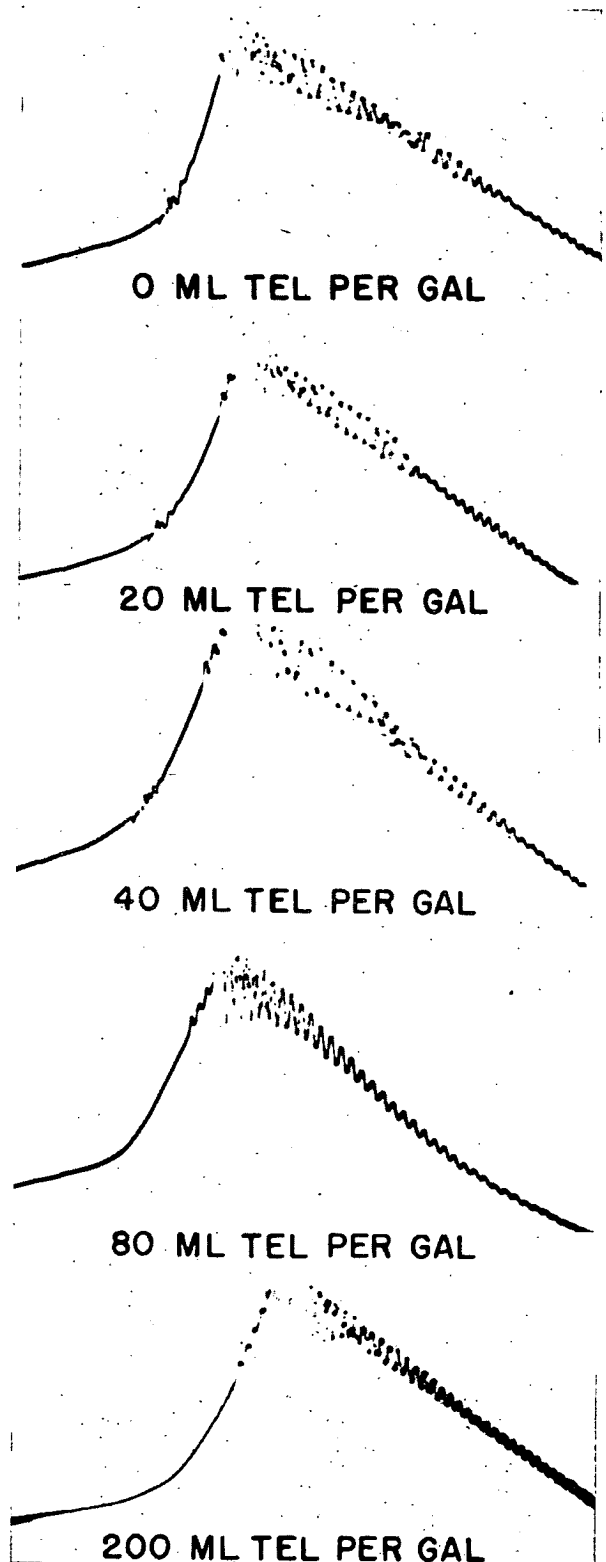


FIGURE I-22.—Actual pressure-time records showing increase in number of cycles of preknock vibration with increasing tetraethyl lead concentration in fuel.

after the first cycle, from which fact it was concluded that a mild explosive reaction occurred in the end gas at about frame E-1. When the photographs of figure I-20 are projected as a motion picture, four cycles of vibration of the gases are easily observed before the explosive knock reaction at frame G-7; they are observed as a backward and forward motion of the flame fronts.

Figure I-21 shows the preknock vibration determined from planimeter measurements of flame areas for different concentrations of tetraethyl lead. A similar variation in number of cycles of preknock vibration with varying tetraethyl lead concentration is shown by the actual pressure-time records of figure I-22, obtained with a piezoelectric pickup in opening I of the cylinder head (fig. I-1). The time interval involved in the preknock vibration was found to increase linearly with the tetraethyl lead concentration up to 200 ml per gallon, the highest concentration used.

Explosive knock reaction without end-gas autoignition.—Many high-speed photographs have been obtained like the ones of figure I-23, in which the end gas appeared to be entirely consumed by the normal flames some time before the occurrence of the explosive knock reaction. This figure is a reproduction of figure 6 of reference 6. Four spark

plugs were used for this combustion, timed at 20° B. T. C. at E, F, and J positions (fig. I-1) and at 27° B. T. C. at G position. The injection valve was at position H, the fuel was a blend of 50 percent 95-octane gasoline with 50 percent M-2 reference fuel, compression ratio 7.0, and fuel-air ratio about 0.08. The last visible end gas disappears, because of normal flame travel, at about frame G-17. The explosive knock reaction occurs at B in frame H-7, 11 frames after the disappearance of the last visible end gas. Other photographs have been obtained in which a mild explosive knock reaction occurred not only after the disappearance of the last visible end gas but even after the entire schlieren combustion pattern had almost faded out (reference 6).

The cases cited are not good proof that the explosive knock reaction can occur without autoignition because of the possible existence of unignited gas pockets in front of or behind the burning gases; such pockets could, of course, not be seen in the photographs. Photographs have been obtained, however, in which unignited end gas may actually be seen at the time the explosive knock reaction occurs and the unignited end gas does not appear to play a part in the explosive knock reaction. Comment has already been made concerning the presence of unignited end gas in one part of

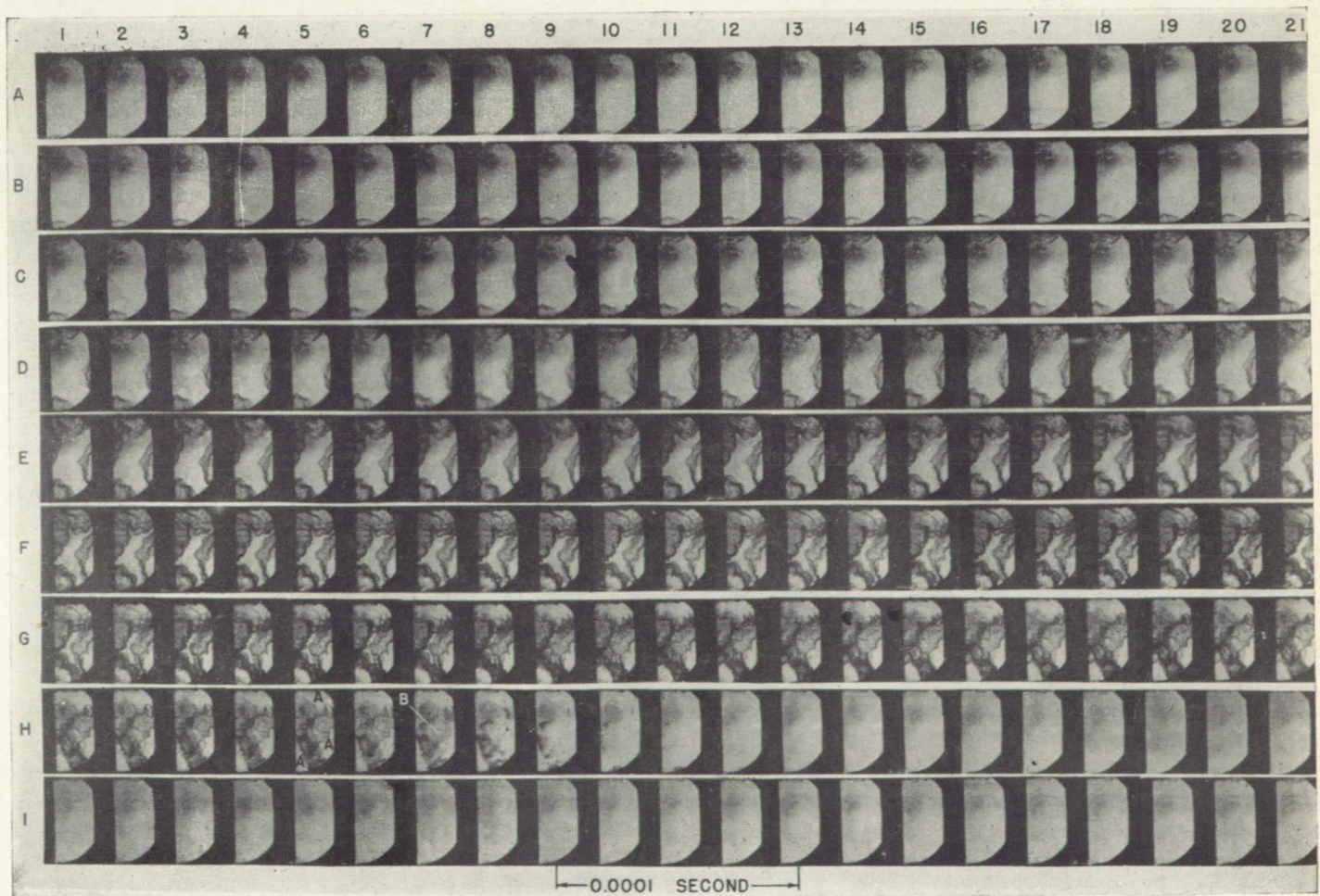


FIGURE I-23.—High-speed photographs showing apparent complete merging of flames before knock in spark-ignition engine. Fuel, 50 percent 95-octane gasoline with 50 percent M-2; compression ratio, 7.0; fuel-air ratio, about 0.08; atmospheric intake; four spark plugs; spark timing, for left-hand plug, 27° B. T. C., for other three plugs, 20° B. T. C.; A, completely burned regions; B, blurring caused by knock.

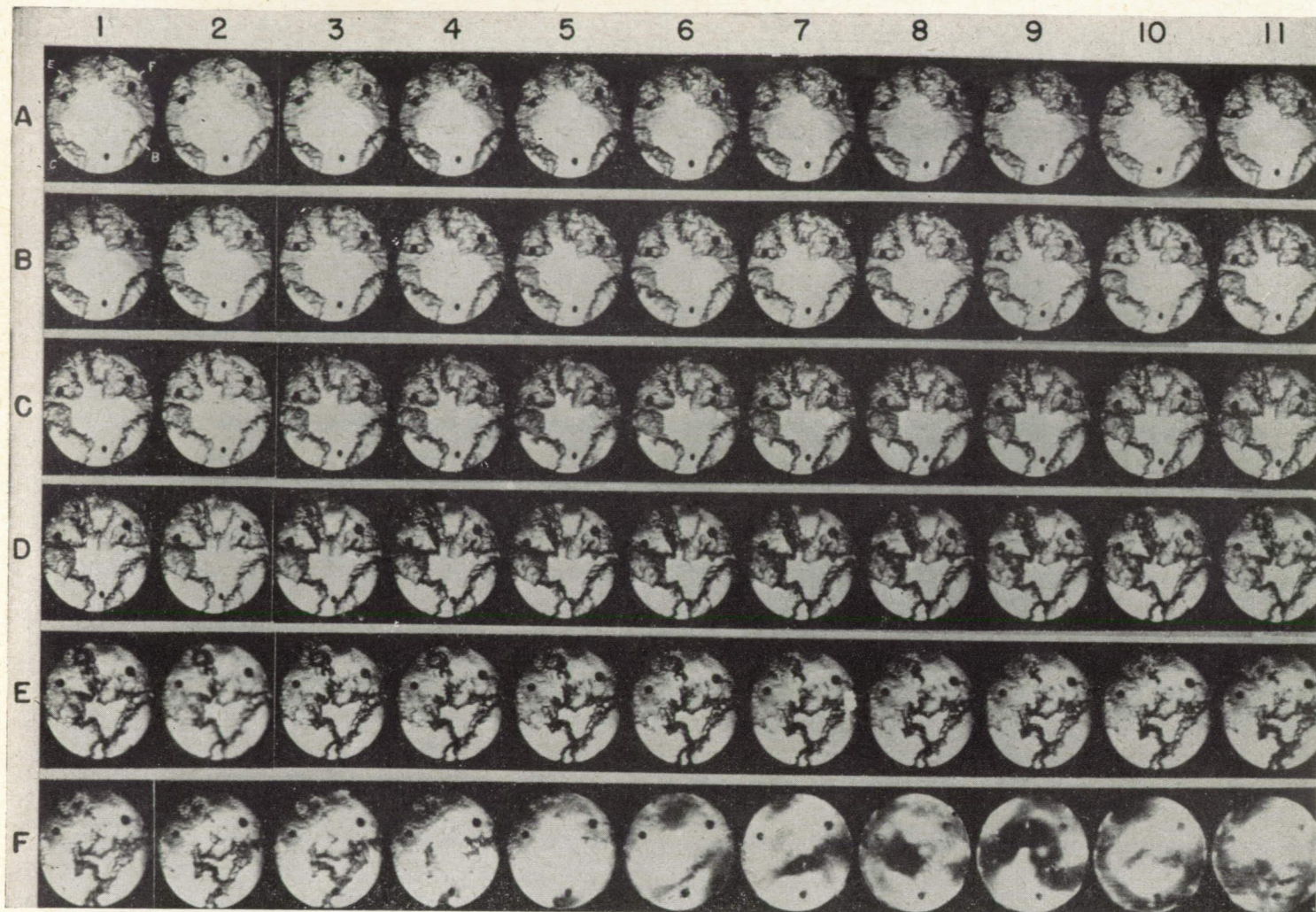


FIGURE I-24.—High-speed photographs of occurrence of violent knock in spark-ignition engine while unignited end gas is still visible. Fuel, S-4; compression ratio, 9.0; fuel-air ratio, about 0.13; inlet-air temperature, 310° F; inlet-air pressure, 15.5 pounds per square inch absolute; four spark plugs; spark timing, 18° B. T. C.

frame G-7 of figure I-20 at the same time that the explosive knock reaction begins in another part of the same frame. A more striking example of this phenomenon appears in figure I-24. Four spark plugs were used in the case of figure I-24, timed at 18° B. T. C. The injection valve was in position A (fig. I-2); the fuel was S-4 reference fuel; compression ratio, 9.0; fuel-air ratio, about 0.13; inlet-air temperature, 310° F; and absolute inlet-air pressure, 15.5 pounds per square inch. In frame F-2 of this figure a fair-sized white unignited end zone is visible, slightly below the center of the chamber, roughly in the shape of a T lying on its side. The explosive knock reaction begins in frame F-3, as indicated by the blurring of the dark mottled combustion zone. This dark mottled combustion zone encroaches somewhat upon the unignited end gas in frame F-3, but most of the end gas visible in frame F-2 remains unignited in frame F-3 in spite of the effect of the explosive knock reaction on the actual combustion zone in frame F-3. Moreover, the end gas that appears white and unignited in frame F-3 still appears white in frames F-4 and F-5, in which the explosive knock reaction is completed. If this body of end gas ever did burn it must have done so in a time interval much shorter than the 25-microsecond exposure time of the individual frames of the figure, or it would have

shown the characteristic dark mottled appearance of burning gas in one of the frames of the figure.

The appearance of frames F-2 to F-5 of figure I-24 indicates that the explosive knock reaction developed in the dark mottled combustion zone and that its effect on the unignited end gas, if it had any such effect, was only incidental. In reference 7 it was concluded that the explosive knock reaction develops only in gases that have been previously ignited either by normal flame travel or by auto-ignition. Photographs were presented in the same paper indicating that the origin of the explosive knock reaction was not necessarily in the same location as the last gas to be ignited.

Physical nature of explosive knock reaction as indicated by high-speed photographs.—From a study of photographs taken at 40,000 frames per second the author of this paper has concluded (reference 10) that the explosive knock reaction is a type of detonation wave traveling through the unburned, or incompletely burned, gases at a speed ranging approximately from one to two times the speed of sound in the burned gases. An example of a detonation wave moving at a speed twice that of sound in the burned gases is found in frames M-11 and M-12 of figure I-5. As previously noted, the explosive knock reaction in this case is first visible as a white streak along the lower right edge of frame M-11

and as a slight blurring of the combustion zone in the same frame. The next frame, M-12, has been rendered white throughout by the knock reaction. As explained in reference 10, it is believed justifiable to disregard the blurring of the mottled combustion zone in frame M-11 and to use the appearances of brilliant luminosity as a measure of the speed of the knock disturbance. On such a basis, the superficial impression obtained from frames M-11 and M-12 of figure I-5 is that the knock disturbance started at the lower right edge of the chamber and spread very rapidly throughout the charge from the point of origin. When the photographs are analyzed, however, with proper allowance for the focal-plane-shutter effect of the camera (reference 1), it is found that the travel of the knock disturbance as shown by frames

tive to the combustion-chamber image as the leading edge of the focal-plane-shutter slit of the camera exposing frame M-12.

Figures I-25 and I-26 are reproductions of portions of a motion-picture animation (reference 12) created for the purpose of demonstrating the manner of exposure of frames M-11 and M-12 of figure I-5. Figure I-25 shows this author's conception of what actually happened during the exposure of the two frames. In this figure the focal-plane shutter is assumed to have been removed from each of the still cameras. The frames of the figure show what would have been seen on the film in each of the two cameras if the film and the image appearing upon it could have been photographed at the rate of about 180,000 frames per second.

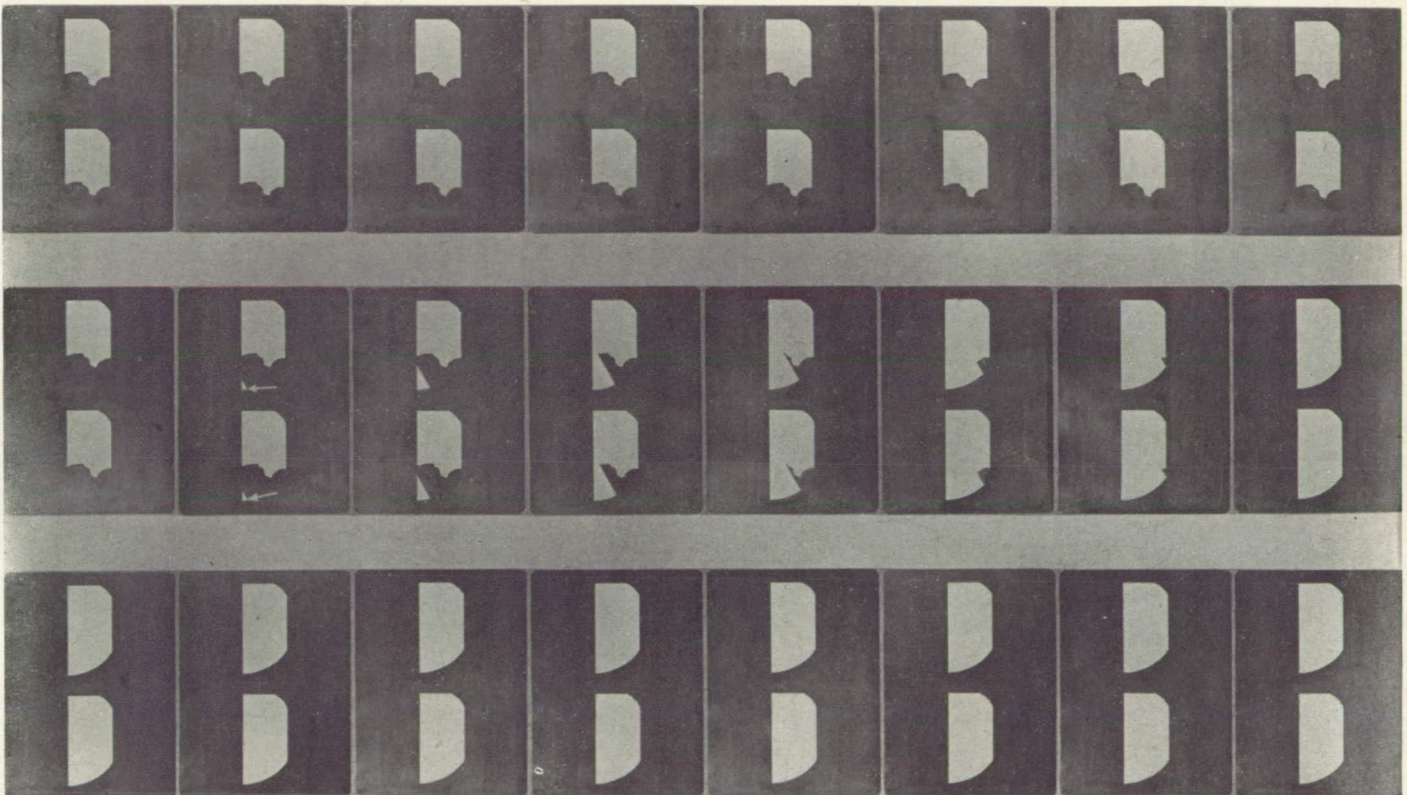


FIGURE I-25.—Animation showing detonation-wave travel as it might appear on film in two still cameras if film and image were photographed at about 180,000 frames per second. Development of detonation wave in frames B-2 to B-8 of this figure corresponds chronologically with exposure of frames M-11 and M-12 of figure I-5.

M-11 and M-12 was actually in a direction opposite to the superficially apparent direction.

An optical effect equivalent to that of the high-speed camera would be obtained if frames M-11 and M-12 of figure I-5 were exposed by two independent still cameras using focal-plane shutters, provided the following three conditions were satisfied: first, the width of the slit aperture in each of the focal-plane shutters must be equal to half the frame spacing, or about $7/10$ the width (not the length) of the combustion chamber as seen in the photographs, second, each focal-plane-shutter slit must travel a distance equal to its own width in $1/40,000$ second in a direction from left to right, as seen in figure I-5 (that is, in the direction away from the previously exposed frames toward the frames yet to be exposed as seen in the figure), and third, the trailing edge of the focal-plane-shutter slit of the camera exposing frame M-11 must at all times be in the same position rela-

The frames are arranged in three rows of eight frames each. Each frame includes two images of the combustion chamber; the lower of the two images is the one seen on the film in the camera exposing frame M-11, the upper of the two images is the one seen on the film in the camera exposing frame M-12. Inasmuch as the focal-plane shutters have been removed from the cameras, and the cameras are still cameras of the most elementary type, the same image appears on the film in both cameras at all times. Throughout the frames of row A and the first frame of row B the film in each camera is being exposed to an image just like that of frame M-10 (fig. I-5). At frame B-2 (fig. I-25) a detonation wave originates at the lower left corner of the visible portion of the chamber as indicated by the two white arrows in this frame. This wave travels across the chamber through the dark mottled combustion zone as seen on the film in each of the two cameras between frames B-2 and B-8. (The

detonation-wave front has been made straight for simplicity of construction of the animation; approximately the same effect would have resulted if the detonation-wave front had been spherical.) The gases behind the detonation-wave front are, of course, incandescent and the entire area that was formerly the dark mottled combustion zone consequently appears brilliant white in all of the frames after the detonation-wave front has completed its travel (frames B-8 to C-8).

Figure I-26 is an animation of the same kind as figure I-25 but shows the motion of the focal-plane shutter in each of the two cameras as well as the progress of the detonation wave. In frame A-1 of figure I-26 the focal-plane-shutter slit that will provide the exposure for the lower photograph (frame M-11 of fig. I-5) may be seen just to the left of the combustion-chamber image as a tall narrow rectangle. Part of the focal-plane-shutter slit of the upper camera may also be seen in the upper part of frame A-1, with its leading edge exactly in line with the trailing edge of the focal-plane-shutter slit of the lower camera. Throughout the frames of row A the two focal-plane-shutter slits may be seen to move from left to right relative to the combustion-chamber images.

The lower focal-plane-shutter slit begins to pass across the lower combustion-chamber image at frame A-5. In this same frame the trailing edge of the lower focal-plane-shutter slit and the leading edge of the upper focal-plane-shutter

slit are each indicated by a white arrow and will be seen to be in line with each other as in all other frames of the figure. The focal-plane shutters are assumed to be constructed of a dark material that does not reflect light well; the combustion-chamber image consequently appears gray in all areas where it falls on the focal-plane-shutter material. In frames A-5 to A-8, however, part of the light that forms the lower combustion-chamber image passes through the focal-plane-shutter slit and falls on the photosensitive film; this film, being a bright material that reflects light well, causes the part of the image formed upon it to appear brilliant white in the animation. Hence, in frames A-5 to A-8 the brilliant white part of the picture shows the portion of the lower image uncovered by the focal-plane-shutter slit at each instant.

The lower focal-plane-shutter slit continues to pass across the lower combustion-chamber image from left to right in frames B-1 to B-8. At frame B-8 this slit has passed all the way across the lower image and the exposure of that image is completed. The upper focal-plane-shutter slit begins to pass across the upper combustion-chamber image at frame B-2, with its leading edge still exactly in line with the trailing edge of the lower focal-plane-shutter slit. The upper slit completes its travel across the upper combustion-chamber image at frame C-5, at which time the exposure of this image is complete.

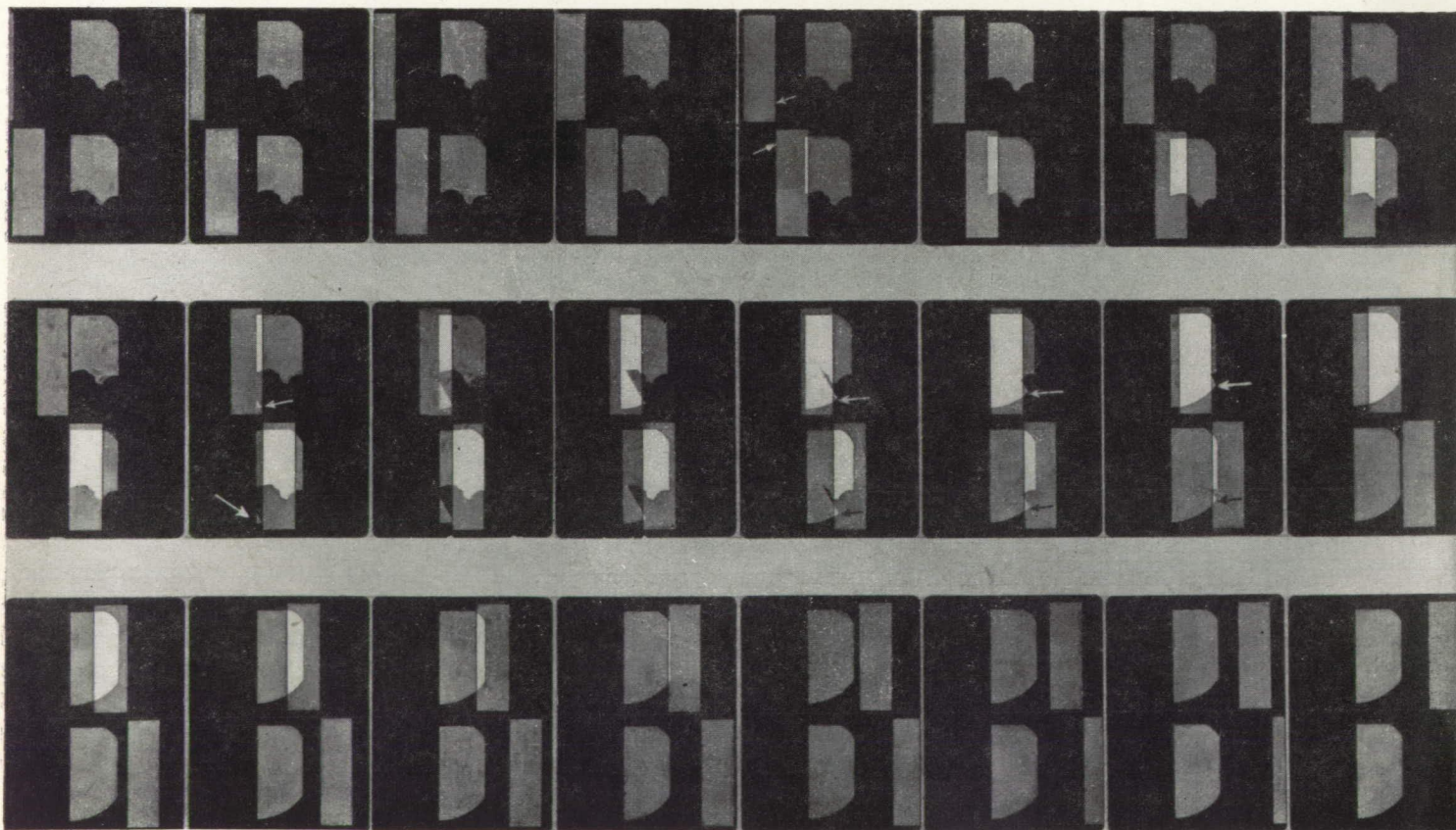


FIGURE I-26.—Animation showing travel of both detonation wave and focal-plane-shutter screen in two still cameras as they might appear if photographed at about 180,000 frames per second. Frames B-2 to B-8 of this figure correspond chronologically with exposure of frames M-11 and M-12 of figure I-5.

In frames B-2 to B-7 of figure I-26 the detonation-wave front may be seen traveling across the upper combustion-chamber image first just a little behind (as indicated by upper white arrow in frame B-2) and later just a little ahead of the leading edge of the upper focal-plane-shutter slit (as indicated by the white arrows in frames B-5, B-6, and B-7), but always quite close to the leading edge of the upper slit. In the same frames (B-2 to B-7) the detonation-wave front may be seen traveling across the lower combustion-chamber image at all times quite close to the trailing edge of the lower focal-plane-shutter slit, first a little behind (as indicated by lower white arrow, frame B-2) and later a little ahead (as indicated by black arrows in frames B-5, B-6, and B-7). As both lower and upper combustion-chamber images (frames M-11 and M-12 of figure I-5) are exposed entirely by light that passes through the upper and lower focal-plane-shutter slits, respectively, during their travel across the images, it is clear from frames B-2 to B-7 of figure I-26 that the upper image (frame M-12 of fig. I-5) when finally developed, will show incandescence over the entire area of the combustion chamber, whereas the lower image (frame M-11 of fig. I-5) will in the main show the same dark mottled combustion zone that is seen in frame M-10 of figure I-5. The only effect the detonation wave will have upon the lower combustion-chamber image (frame M-11 of fig. I-5), as finally developed, will be produced by that part of the detonation wave which travels across the chamber ahead of the trailing edge of the lower focal-plane-shutter slit in frames B-4 to B-7 of figure I-26. Examination of frames B-4 to B-7 shows a very small triangular region of luminosity of gradually increasing size (indicated by black arrows in frames B-5, B-6, and B-7) progressing along the lower right edge of the chamber from left to right. This very small triangular area of luminosity would, in the final developed photograph, produce the white streak along the lower right edge of the chamber, with gradually increasing brilliance toward the right, that is observed in frame M-11 of figure I-5.

In the exposure of frames M-11 and M-12 of figure I-5, the focal-plane-shutter slits moved at a speed of 256 feet per second. The linear dimensions of the actual combustion chamber were 21.5 times as large as those of the combustion-chamber image formed on the film. The detonation wave that produced the effect seen in frames M-11 and M-12 of figure I-5, in the manner illustrated in figures I-25 and I-26, must, therefore, have traveled across the combustion chamber at a speed greater than 6500 feet per second (with due allowance for the fact that the detonation-wave front traveled in a direction at an angle to the direction of motion of the focal-plane shutter.)

If the explosive knock reaction had occurred simultaneously throughout the end gas, as commonly supposed, its luminosity would simply have recorded the relative positions of the trailing edges of the upper and lower focal-plane-shutter slits, consequently the luminosity as seen in frame M-12 of figure I-5 would have extended farther to the left than

the luminosity as seen in frame M-11 by an amount not greater than the width of the focal-plane-shutter slit. The fact that the luminosity visible in frame M-12 extends to the left of the luminosity visible in frame M-11 by an amount greater than the width of the focal-plane-shutter slit precludes the possibility that the luminosity developed simultaneously throughout the chamber. No satisfactory explanation of the appearance of frames M-11 and M-12 of figure I-5 has been advanced other than the explanation illustrated in figures I-25 and I-26. If due consideration is given to the fact that the luminosity visible throughout frame M-12 is quite uniform, and is less than the saturation limit of the photosensitive film, it would seem that each part of frame M-12 must have had about the same exposure to the detonation luminosity and the conditions governing the exposure of the two frames (M-11 and M-12) would, therefore, seem to be mathematically determinate. For these reasons the author believes that the explanation illustrated in figures I-25 and I-26 is the only reasonable explanation of the appearance of these frames.

The effect of the focal-plane shutters of the high-speed camera on apparent velocities of detonation waves is easily shown (reference 10) to comply with the following equation:

$$V = \frac{v V'}{v + V' \cos \alpha}$$

where V is the actual speed of the wave, v the speed of the focal-plane shutter slits, V' the apparent velocity of the wave (progress of the wave between two successive frames, as recorded photographically, divided by nominal time between exposures of the successive frames), and α the angle between the direction of motion of the detonation-wave front and the direction of motion of the focal-plane-shutter slits. (Due regard must be had for the signs of V , V' , and α , as explained in reference 10.) Application of this equation to the case of figure I-27 has given propagation speeds in the neighborhood of 4500 feet per second to the points p_1 , p_2 , . . . p_8 from the point of origin of the knock disturbance (calculated to be at the intersection of line C-C with lines C'-C'). (Fig. I-27 is the same as fig. 7 of reference 10. The complete photographic series and complete treatment are presented in the original paper. The complete photographic series of this combustion process also appears as fig. 8 in reference 5. The engine-operating conditions for this figure were the same as for fig. I-4 of the present paper.)

No correcting equation is required to determine the true detonation-wave velocity from figure I-28, because the front of the detonation wave may be located in three successive frames of this figure at points p_{17} , p_{18} , and p_{19} (wave front position indicated by a bright spot at p_{19}) along a line D-D, which is at right angles to the direction of motion of the focal-plane shutter. The complete treatment of this case (reference 10) yields a value between 3250 and 3400 feet per second for the speed of the detonation wave. (Fig. I-28

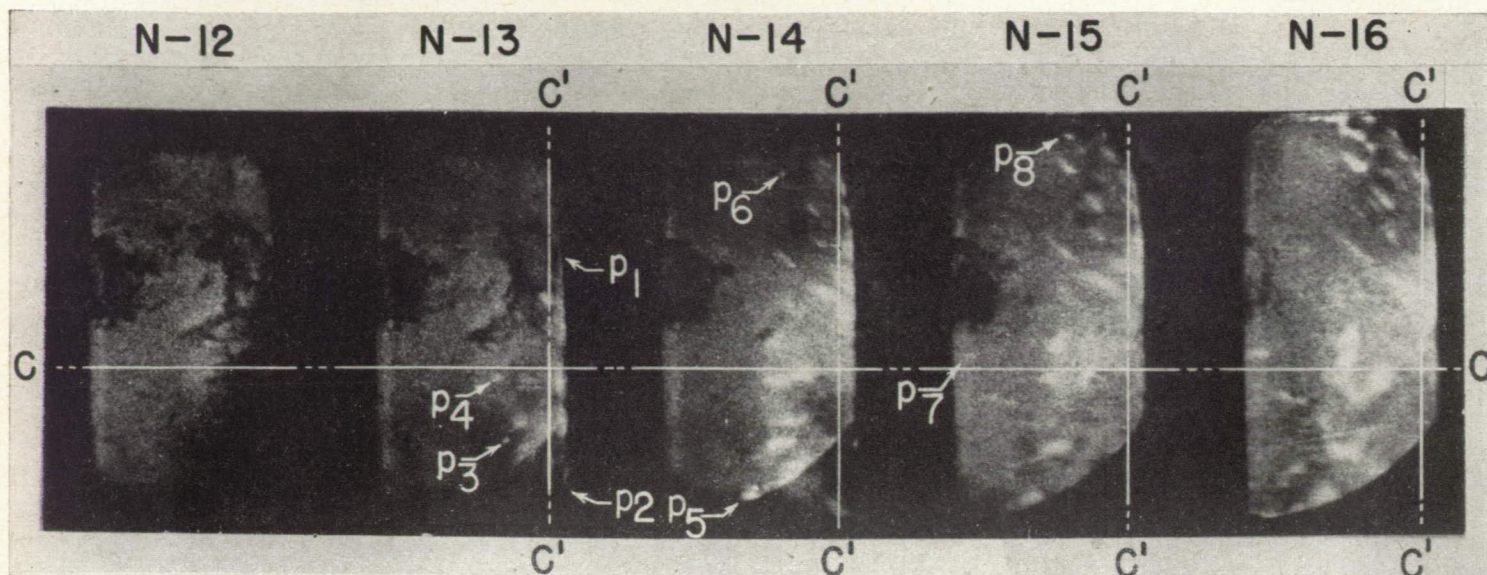


FIGURE I-27.—High-speed photographs of development of incandescent spots caused by detonation wave in knocking spark-ignition engine. Fuel, 50 percent S-1 with 50 percent M-2; compression ratio, 7.0; fuel-air ratio, about 0.08; atmospheric intake; spark plug at top; hot spot at bottom; spark timing, 20° B. T. C.; detonation-wave speed, about 4500 feet per second.

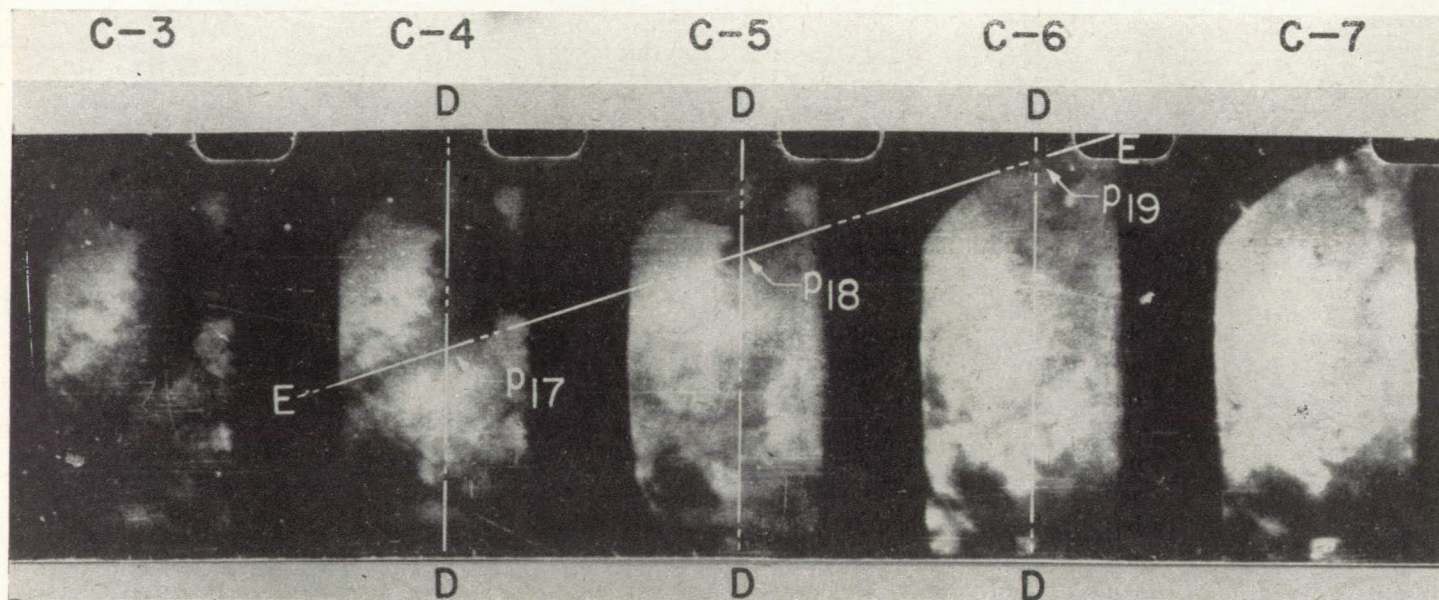


FIGURE I-28.—High-speed photographs of development of luminous area and incandescent spots caused by detonation wave in knocking spark-ignition engine. Fuel, M-1; compression ratio, 7.0; fuel-air ratio, about 0.08; atmospheric intake; four spark plugs; spark timing, for left-hand plug, 27° B. T. C., for other three plugs, 20° B. T. C.; detonation-wave speed, about 3300 feet per second.

is the same as fig. 13 of reference 10. Additional frames of the series are shown in fig. 12 of that paper as well as in fig. 10 of reference 6. The engine-operating conditions were the same as for fig. I-10 of the present paper except that M-1 reference fuel was used.)

Application of the correcting equation to the explosive knock reactions seen in frame G-12 of figure I-10 and frames G-11 and G-12 of figure I-12 has yielded values in the neighborhood of 6000 and 3600 feet per second, respectively (reference 10). The correcting equation, as well as the qualitative treatment illustrated in figures I-25 and I-26,

yields a value of about 7300 feet per second for the reaction that occurred between frames C-1 and C-2 of figure I-18 after the runaway flame had completed its travel across the chamber. Between these two frames the entire combustion chamber changed from a fairly uniform, dark mottled condition to a fairly uniform white condition. (This change was more evident on the original negative than it is in the printed reproduction. The entire series of photographs was badly underexposed and in copying some of the contrast between frames C-1 and C-2 was lost in order that the frames of rows A and B might be brought out more clearly.) A

uniform change throughout the combustion chamber from a dark mottled condition to a white condition between one frame and the next corresponds to an infinite apparent detonation-wave speed, that is, an infinite value of V' . With an infinite value of V' the correcting equation indicates that the actual detonation-wave velocity is equal to the focal-plane shutter velocity, and in the same direction. Consideration of figure I-26 indicates that a uniform change throughout the combustion chamber would be produced if the detonation wave traveled across the chamber in the same direction and at the same speed as the focal-plane-shutter slit rather than at a slight angle to the direction of motion of the slit and at a slightly higher speed. (The white streak along the lower right edge of the chamber in frame M-11 of fig. I-5 would have been absent if the detonation-wave front had not overtaken the trailing edge of the lower focal-plane-shutter slit at about frame C-19 in fig. I-26 and had not traveled across the chamber slightly ahead of this trailing edge in frames C-19 to D-21; the combustion zone would have been uniformly dark in frame M-11 (fig. I-5) and uniformly white in frame M-12.) For the case of figure I-18 the focal-plane-shutter slits traveled at 251 feet per second in a direction approximately opposite to the travel of the runaway flame in frames B-1 to C-1. The actual linear dimensions of the combustion chamber were 29.2 times as great as the dimensions of the combustion-chamber image. The appearance of figure I-18, therefore, indicates that after the runaway flame completed its travel through the end gas in one direction at about 1900 feet per second a detonation wave passed through the gas in the opposite direction at about 7300 feet per second.

Confirmation of detonation-wave aspect of knock by ultra-high-speed photographs.—Only one motion picture of the knock phenomenon taken with the ultra-high-speed camera (reference 3) is available. This one motion picture, however, taken at the rate of 200,000 frames per second, is confirmatory of the conclusions reached from study of the high-speed photographs taken at 40,000 frames per second. It has not been possible to take more than a very few photographic shots with the ultra-high-speed camera because the high-speed rotating part of the camera, operating in a high vacuum, spatters oil on the 94 glass lenses of the camera to such an extent that the lenses become inoperative after only a few runs. Of 5 or 6 shots that were actually taken at 200,000 frames per second, only one happened to be taken at the right time to cover the knock phenomenon. Further developmental work on the ultra-high-speed camera, which is still proceeding, has been directed toward elimination of the oil-spattering problem.

The ultra-high-speed photographs are shown as a series of 20 still pictures in figure I-29. The combustion process for this series of photographs was fired in the old combustion apparatus with one spark plug in G position (fig. I-1). at

27° B. T. C. The injection valve was at opening J (fig. I-1); the fuel was a blend of 70 percent S-3 with 30 percent M-2 reference fuels; compression ratio, 7.0; and fuel-air ratio, about 0.08. Figure I-29 was previously published as figure 4 in reference 11 and was discussed in that paper more extensively than it will be here.

Before frame A-1 of figure I-29 was exposed the flame had traveled all the way across the chamber in the direction of the arrow that has been drawn in this frame. The whitish area in frames A-1 to A-5, designated B in frame A-3, represents the region in which combustion is complete. The dark (nearly black) areas in these frames, designated F in frame A-3, are the dark mottled combustion zone that has become quite familiar in the high-speed photographs of the earlier figures. The series of figure I-29 did not begin early enough to determine whether any autoignition occurred in this combustion process, but at least at the time of exposure of frame A-1 the entire charge was ignited with the exception of possible small pockets in front of or behind the flame. The boundary between the dark and the light areas, designated R in frame A-2 of the figure, is the *rear* edge of the combustion zone—not the flame front.

The ultra-high-speed camera does not have the focal-plane-shutter effect of the high-speed camera, but has in effect a conventional between-the-lens type of shutter. The exposure time of each frame is roughly the same as the time between successive frames, 5 microseconds, although there is a slight overlapping of exposures, as specified in reference 11. The series of figure I-29 is, therefore, truly representative of events as they occurred in the combustion chamber. Examination of frames A-1 to B-1 of the figure reveals no appreciable change of conditions. Frames A-2, A-4, and B-1 are much more sharply defined than frames A-1, A-3, and A-5. This effect, however, is introduced by the camera and has no significance relative to the burning processes. This alternate blurring of frames can be eliminated in further work with the camera. The first significant change in appearance of the photographs occurs in the area designated by the two arrows in frame B-2. This light area is not present in any of the preceding frames. If this area in frame B-2 is compared with the same area in frame B-1, more difference will be noted than in a like comparison of frame B-1 with frame A-1, five frames preceding. In frames B-3 and B-4, the effect of the knock spreads considerably in all directions from the point of origin. The location of the point of origin at the *rear* edge of the combustion zone is strikingly confirmatory of the conclusion reached from study of the 40,000-frame-per-second photographs that the explosive knock reaction apparently originates in the burning gases rather than in the unignited end gas.

Frames B-1 to B-5 of figure I-29 are reproduced in figure I-30. In figure I-30, however, a black line (R_{B-1}) has been drawn in frame B-1 reinforcing the boundary line between

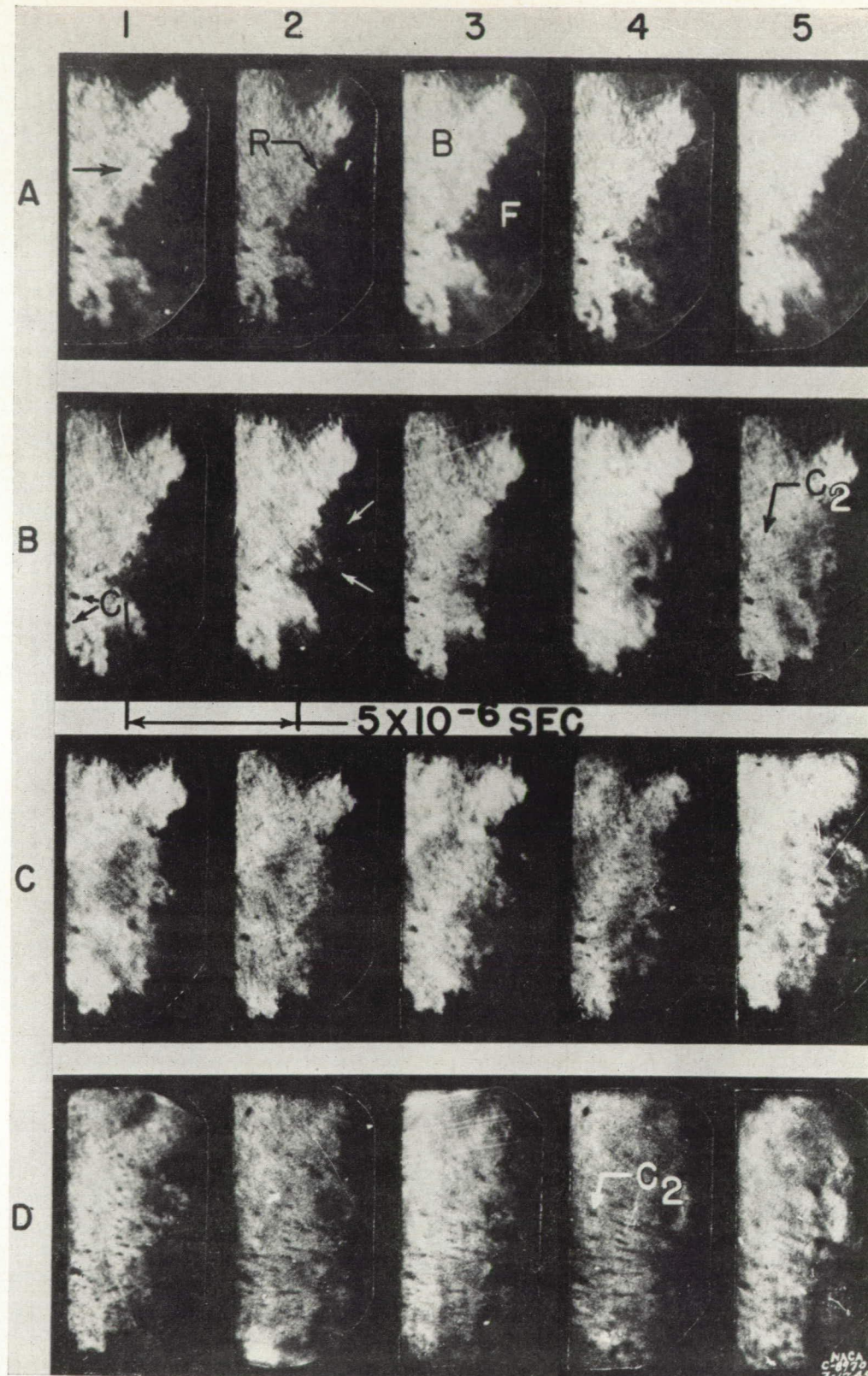


FIGURE I-29.—Ultra-high-speed photographs (200,000 frames per sec) of phenomenon of knock in spark-ignition engine. Fuel, 70 percent S-3 with 30 percent M-2; compression ratio, 7.0; fuel air ratio, about 0.08; atmospheric intake; one spark plug in G position (fig. I-1); spark timing, 27° B. T. C.

the dark combustion zone on the right and the white burned region on the left. In frames B-2 to B-5 of this figure the same black line has been constructed in each frame, not marking the boundary between burned and unburned as it appears in frames B-2 to B-5 but the boundary as it appeared in frame B-1. In each of the frames B-2 to B-5 two horizontal black lines have been drawn marking the upper and lower extents of the knocking reaction in each frame as indicated by the whitening of the combustion zone to the right of the boundary line R_{B-1} . Distances between the horizontal lines have been designated in each frame as l_2 , l_3 , and so on. Treating the difference between successive values of l as representing the combined upward and downward travel of the knocking disturbance between successive frames (l_1 assumed equal to zero), the following values have been obtained for the speed of the knock disturbance according to the equation:

$$V = \frac{1}{2} \left(\frac{l_2 - l_1}{5 \times 10^{-6}} \right)$$

Frames	Velocity of knocking disturbance (ft/sec)
1 to 2-----	9200
2 to 3-----	4200
3 to 4-----	6900
4 to 5-----	1200

The average of the first three values is nearly 6800 feet per second, which is of the same order as the higher speeds

determined from the high-speed photographs and the speed of 2000 meters per second determined by Sokolik and Voinov in reference 18.

After frame B-4 of figure I-29 the ultra-high-speed photographs no longer show a high speed of propagation of the knock reaction. The portion of the black combustion zone that is still visible in frame B-5 gradually disintegrates throughout the frames of rows C and D, leaving a great many black dots and streaks, which will be discussed later. It is thought either that the areas that still appear black in frame B-5 represent gases that were not in a sufficiently advanced state of combustion to be detonable or that they represent a high concentration of carbon particles resulting from the knock reaction.

Chemical nature of explosive knock reaction.—The high-speed and ultra-high-speed photographs, because of their inherent nature, should not be expected to yield much information concerning the chemical nature of the explosive knock reaction. Two indications have been obtained, however, that may be of the most fundamental importance relative to the chemical nature of the reaction; first, the indication that the reaction results in a definite loss of chemical energy that would otherwise be available, and second, the indication that free carbon is released by the explosive knock reaction. Both of these indications forcibly recall the opinion expressed by Midgley in 1920 (reference 14) that the knocking detonation wave burns only the hydrogen in the hydrocarbon molecule and releases free carbon.

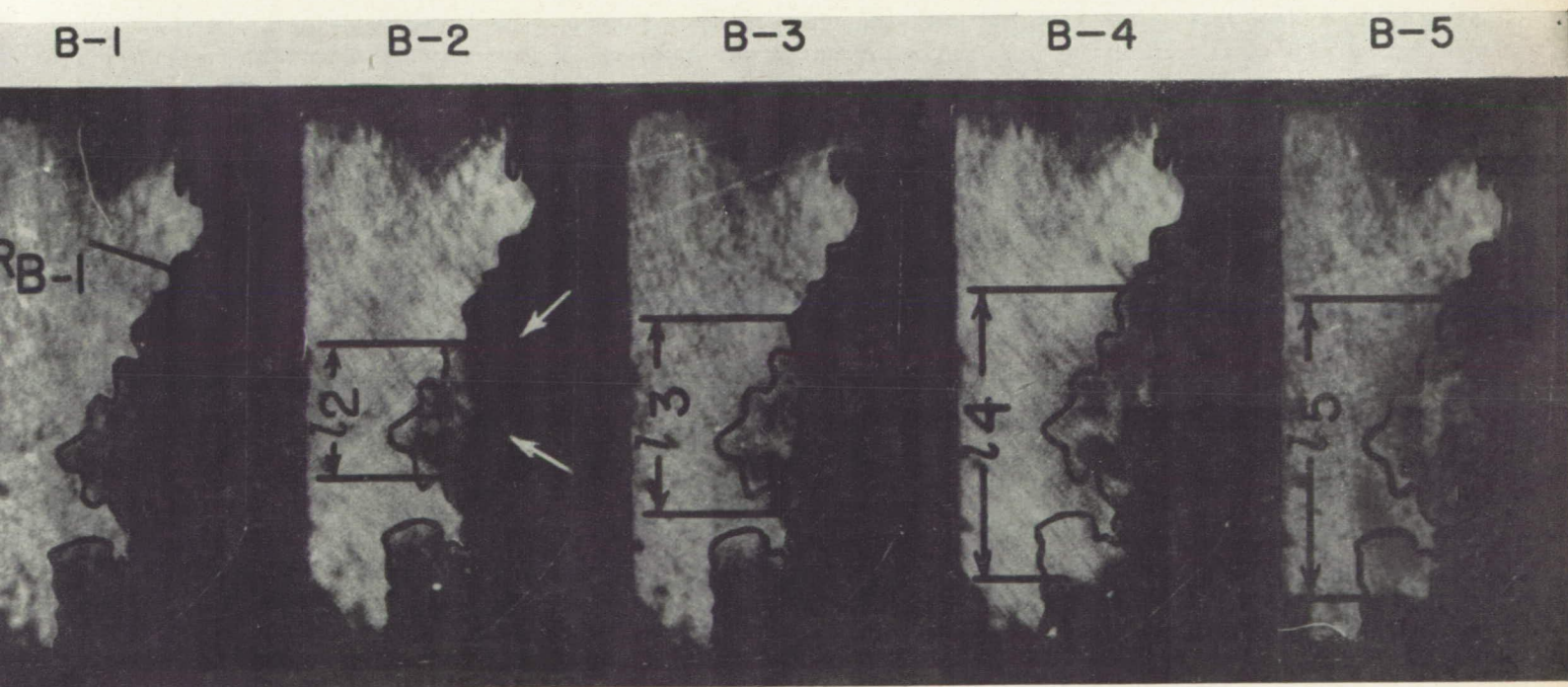


FIGURE I-30.—Frames from figure I-29 outlining actual detonation-wave travel. Detonation-wave speed, about 6800 feet per second both upward and downward from point of origin.

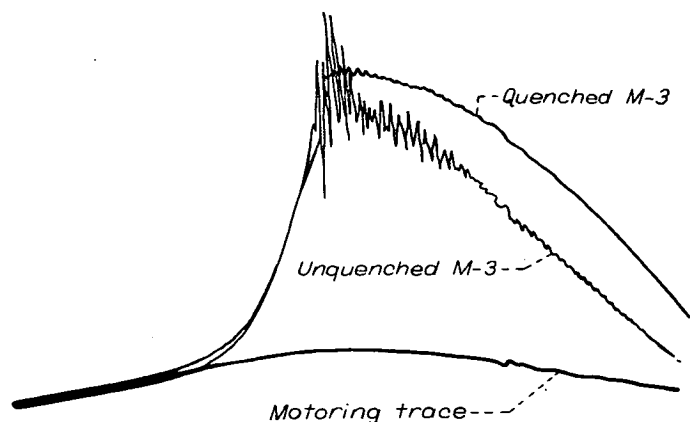


FIGURE I-31.—Pressure-time records for two combustion cycles showing power gain secured by quenching end zone with water to avoid knock.

In figure I-31 are reproduced actual photographs of two pressure-time records obtained with the cathode-ray oscilloscope and a quartz piezoelectric pickup placed at the knocking zone in the combustion apparatus. The manner of obtaining these pressure-time records was fully explained in reference 8. The two traces were taken under identical engine-operating conditions, and with the same fuel (M-3 reference fuel), except that water was injected into the end gas for the "quenched" trace. High-speed photographs were obtained (not reproduced here) showing that the spatial distribution of the injected water coincided fairly well with the remaining combustion zone at the time of knock. As may be observed in the figure the injected water almost eliminated the violent knock that occurred with the unquenched combustion. It is also clear from the figure that the elimination of the knock resulted in a very marked increase in pressure, not only late in the expansion stroke but even immediately after the occurrence of the knock. The comparative traces of figure I-31 are typical of many tests in which the quenched and unquenched combustion cycles followed each other in close succession, and in which the order of taking the quenched and unquenched cycles was repeatedly reversed.

In the ultra-high-speed photographs of figure I-29, a large number of small black spots develop in the frames of row C, both in the dark combustion zone and in the gas that was completely burned before the beginning of the knock development. In the frames of row D these black spots draw out into long narrow streaks in the direction of the mass motion of the gases. Two black spots (C_1 in frame B-1) are visible in the burned gas before the start of knock development, but a very great increase in the number of black spots immediately after the inception of the explosive knock reaction is manifest. Some of the spots, such as C_2 in frame B-5, are visible in the burned gas within 10 microseconds after passage of the knock disturbance through the location of the spots. A number of spots are also visible in the combustion zone in frame B-5 less than 10 microseconds after passage of the knock disturbance.

Black spots and streaks similar to those seen in figure I-29 may be seen in frames J-11 and J-13 of the high-speed photographs presented in figure I-6. The counterpart of the black spots and streaks of frames D-1 to D-5 of figure I-29 and frames J-11 to J-13 of figure I-6 is seen in the

white streaks visible in the frames of figure I-27, which were used to determine the detonation-wave speed in the neighborhood of 4500 feet per second from that figure. The streaks were probably highly incandescent in all cases (figs. I-6, I-27, and I-29). With the ultra-high-speed photographs the exposure time was so short and the lens aperture was sufficiently small so that the incandescence of the streaks did not photograph. The streaks were photographed as black streaks because they were opaque to the externally supplied light by which the photographs were obtained. In the case of figure I-27 the externally supplied light was much less intense and the exposure time was five times greater; consequently, the incandescence of the streaks was photographed and the streaks show up white against the less intense background of externally supplied light. The fact that the spots are black in frames J-11 to J-13 of figure I-6 indicates that these spots were much less incandescent than in the case of figure I-27 obtained under the same photographic conditions. The streaks have probably been lost in many of the photographs because their incandescence has closely matched the brilliance of the externally supplied light.

The facts that approximately circular black spots appeared in the frames of row C in figure I-29 and that these black spots smeared out into streaks in the direction of gas motion in the frames of row D indicate that the material forming these black spots had considerable inertia. Apparently the inertia was sufficient so that the material of the black spots did not go along with the gases in their mass motion: the result was either that the particles were physically smeared out into streaks or the particles caused the precipitation of additional material from the gases flowing by.

Three properties of the material of the black streaks are, therefore, indicated by figures I-27 and I-29; first, the material is incandescent, second, the material is opaque, third, the material is a solid (indicated by its inertia). No substance is known to this author that could be formed from the gases present in the combustion chamber and which would exhibit the three properties stated at the temperature of the gases in the knocking engine except free carbon. It is, of course, well known that excessive free carbon is formed in knocking explosions; the smokiness of the exhaust is often used as a measure of knock. Large quantities of free carbon are deposited on the windows of the combustion apparatuses by a single knocking combustion cycle, as compared with no noticeable carbon deposit by a single nonknocking combustion cycle with the same fuel-air ratio.

The black spots and streaks observed in figures I-6 and I-29, and the corresponding incandescent spots in figure I-27, might be held to be different phenomena. A high-speed photograph has been obtained, however, in which a very large spot appears opaque immediately after the passage of the explosive knock reaction, but later becomes incandescent. Some of the frames of this high-speed photograph appear in figure I-32.* The engine-operating conditions and the fuel for the combustion process of this figure were the same as in the case of figure I-28. At frame A-1 of figure I-32, the normal travel of the flames has passed through most of the chamber; a fair-sized end zone of irregular shape still exists in the central portion of the visible part of the chamber. The end gas is consumed by the normal travel of the flames



FIGURE I-32.—High-speed photographs showing formation of black carbon cloud immediately after knock in spark-ignition engine and later reversal of carbon cloud from black to white. Fuel, M-1; compression ratio, 7.0; fuel-air ratio, about 0.08; atmospheric intake; four spark plugs; spark timing, for left-hand plug, 27° B. T. C., for other three plugs, 20° B. T. C.

between frames A-1 and C-3. The explosive knock reaction begins at frame C-4, as indicated by blurring of parts of the mottled combustion zone. A large black spot is very evident in frames C-7 to C-13 of the figure. Careful inspection of frames C-5 and C-6 reveals that this black spot was beginning to develop in these frames. Between frames D-1 and D-4 the black spot becomes incandescent; in frame D-4 most of the spot is sufficiently incandescent to match the other parts of the combustion chamber. (Non-uniform illumination of the chamber by the externally supplied light is the probable reason why the lowest 20 per cent of the chamber is much less brilliant than the upper parts of the chamber in these frames.) In frames D-5 and D-6 the formerly black spot becomes sufficiently incandescent so that it stands out whiter than the surrounding gases. Between frames D-6 and F-13 parts of this formerly black spot become more and more white as compared with the other gases in the chamber; other parts of the spot apparently cool again and become dark. There is, of course, some gradual change of shape of the spot throughout these frames.

Frames D-6 to about F-8 of figure I-32 have been re-etched

by the engraver to bring out the whiteness of the carbon cloud in contrast with the surrounding gases that are only slightly less white. This re-etching has been done only in order to make the printed reproduction look as nearly as possible like the original untouched photograph in print. Similarly, it was necessary to re-etch several other illustrations to emphasize details that are clearly visible in the photographs but that would have been quite hazy because of the half-tone screen used in the engraving process.

The small spots seen in figures I-27 and I-29, and the large spot in figure I-32, are probably not the only aspect of the free carbon released by the explosive knock reaction. Most of the photographs of heavy knock have shown a very great increase in the general luminosity of the chamber at the time of the explosive knock reaction, amounting in many cases to perhaps a hundredfold increase. This uniform increase in luminosity is probably due to finely divided free carbon. The change is by no means limited to the end gas; it appears to occur throughout the chamber and, if it is concentrated anywhere, it is concentrated at the chamber walls rather than in the end gas. This fact,

together with certain other features of figure I-29 (see reference 11), suggests that the explosive knock reaction may travel through the burned gases in the same manner as the burning gases; such a revolutionary conclusion probably should not be taken very seriously, however, until considerably more evidence is available.

Summary of findings of NACA photographic knock investigations.—The findings of the NACA photographic knock investigations at speeds of 40,000 and 200,000 frames per second over the period from 1939 to 1946 may be summarized as follows:

1. The photographs have shown that normal nonknocking combustion involves an entirely smooth travel of the flames through the combustion chamber and a smooth gradual fadeout of the combustion zones after completion of the flame travel through the chamber.

2. The photographs have indicated that normal combustion involves a zone of continuing combustion behind the flame front with a depth measured in tenths of an inch; the combustion zone, however, may have a cellular structure.

3. The photographs have shown that preignition from a hot spot is not a direct cause of knock and that the flame from a hot spot is similar to the flame from a spark plug.

4. Vibratory knock has been shown to involve an extremely fast reaction, termed the "explosive knock reaction" herein, which develops suddenly after a period of normal burning. This reaction involves a time interval not greater than 50 microseconds.

5. The explosive knock reaction has been shown to begin within 25 microseconds of the same time as the violent knocking vibrations shown by a piezoelectric pickup placed in the end zone of the combustion chamber.

6. The photographs have indicated that the explosive knock reaction originates only in a portion of gas that is already ignited, either by normal flame travel or by autoignition.

7. The photographs have shown a number of different types of end-gas autoignition, some of which appear always to be followed by the explosive knock reaction and some of which may occur without being followed by the explosive knock reaction. The photographs have also shown cases of the explosive knock reaction not preceded by any form of autoignition.

8. Analysis of the photographs taken at 40,000 frames per second has indicated that the explosive knock reaction is a type of detonation wave traveling, under different conditions, at speeds ranging approximately from 3000 to 6500 feet per second or from about one to two times the speed of sound in the burned gases.

9. The propagation speed of the order of 6500 feet per second for the explosive knock reaction has been confirmed by the one series of photographs obtained at 200,000 frames per second.

10. A definite loss of chemical energy that would otherwise have been available has been shown to result from the explosive knock reaction.

11. The photographs have shown that free carbon is released in both the burning and the burned gases within 10 microseconds after passage of the detonation wave associated with the explosive knock reaction.

The indications of the high-speed and ultra-high-speed photographs do not harmonize with the simple autoignition theory of knock or with the simple detonation-wave theory. They appear rather to support the combined detonation-wave and autoignition theory proposed in the literature discussion that forms the first part of this paper.

REFERENCES

1. Miller, Cearcy D.: High-Speed Camera. U. S. Patent Office No. 2,400,885, May 28, 1946.
2. Miller, Cearcy D.: The NACA High-Speed Motion-Picture Camera. Optical Compensation at 40,000 Frames per Second. NACA Rep. 856, 1948.
3. Miller, Cearcy D.: High-Speed Motion-Picture Camera. U. S. Patent Office No. 2,400,887, May 28, 1946.
4. Miller, Cearcy D.: Slow-Motion Study of Injection and Combustion of Fuel in a Diesel Engine. SAE Jour. (Trans.), vol. 53, no. 12, Dec. 1945, pp. 719-733; discussion, pp. 733-735.
5. Rothrock, A. M., Spencer, R. C., and Miller, Cearcy D.: A High-Speed Motion-Picture Study of Normal Combustion, Knock, and Preignition in a Spark-Ignition Engine. NACA Rep. 704, 1941.
6. Miller, Cearcy D.: A Study by High-Speed Photography of Combustion and Knock in a Spark-Ignition Engine. NACA Rep. 727, 1942.
7. Miller, Cearcy D., and Olsen, H. Lowell: Identification of Knock in NACA High-Speed Photographs of Combustion in a Spark-Ignition Engine. NACA Rep. 761, 1943.
8. Brun, Rinaldo J., Olsen, H. Lowell, and Miller, Cearcy D.: End-Zone Water Injection as a Means of Suppressing Knock in a Spark-Ignition Engine. NACA RB E4127, 1944.
9. Miller, Cearcy D., and Logan, Walter O., Jr.: Preknock Vibrations in a Spark-Ignition Engine Cylinder as Revealed by High-Speed Photography. NACA Rep. 785, 1944.
10. Miller, Cearcy D.: Relation between Spark-Ignition Engine Knock, Detonation Waves, and Autoignition as Shown by High-Speed Photography. NACA Rep. 855, 1946.
11. Miller, Cearcy D., Olsen, H. Lowell, Logan, Walter O., Jr., and Osterstrom, Gordon E.: Analysis of Spark-Ignition Engine Knock as Seen in Photographs Taken at 200,000 Frames per Second. NACA Rep. 857, 1946.
12. Miller, Cearcy D.: Relation of Detonation Wave to Spark-Ignition Engine Knock in Ultra-Slow Motion. NACA Tech. Film, 1946.
13. Clerk, Dugald: Cylinder Actions in Gas and Gasoline Engines. SAE Jour., vol. 8, no. 6, June 1921, pp. 523-539.
14. Midgley, Thomas, Jr.: Combustion of Fuels in the Internal-Combustion Engine. SAE Jour., vol. 7, no. 6, Dec. 1920, pp. 489-497.
15. Maxwell, G. B., and Wheeler, R. V.: Some Flame Characteristics of Motor Fuels. Ind. and Eng. Chem., vol. 20, no. 10, Oct. 1928, pp. 1041-1044.
16. Maxwell, G. B., and Wheeler, R. V.: Flame Characteristics of "Pinking" and "Non-Pinking" Fuels. Part II. Jour. Inst. Petroleum Tech., vol. 15, no. 75, Aug. 1929, pp. 408-415.
17. Egerton, A., Smith, and Ubbelohde, A. R.: Estimation of the Combustion Products from the Cylinder of the Petrol Engine and Its Relation to "Knock". Phil. Trans. Roy. Soc. (London), vol. 234, ser. A, July 30, 1935, pp. 433-521.
18. Sokolik, A., and Voinov, A.: Knocking in an Internal-Combustion Engine. NACA TM 928, 1940.
19. Boerlage, G. D.: Detonation and Autoignition. Some Considerations on Methods of Determination. NACA TM 843, 1937.
20. Draper, C. S.: The Physical Effects of Detonation in a Closed Cylindrical Chamber. NACA Rep. 493, 1934.
21. Withrow, Lloyd, and Rassweiler, Gerald M.: Engine Knock. The Auto. Eng., vol. XXIV, no. 322, Aug. 1934, pp. 281-284.
22. Grinstead, C. E.: Sound and Pressure Waves in Detonation. Jour. Aero. Sci., vol. 6, no. 10, Aug. 1939, pp. 412-417.
23. Rothrock, A. M., Miller, Cearcy D., and Spencer, R. C.: Slow Motion Study of Normal Combustion, Preignition, and Knock in a Spark-Ignition Engine. NACA Tech. Film 14, 1940.

24. Leary, W. A., and Taylor, E. S.: The Significance of the Time Concept in Engine Detonation. NACA ARR, Jan. 1943.
25. Ricardo, H. R.: Paraffin as Fuel. The Auto. Eng., vol. IX, no. 2, Jan. 1919, pp. 2-5.
26. Woodbury, C. A., Lewis, H. A., and Canby, A. T.: The Nature of Flame Movement in a Closed Cylinder. SAE Jour., vol. VIII, no. 3, March 1921, pp. 209-218.
27. Mallard, et Le Chatelier: Recherches expérimentales et théoriques sur la combustion des mélanges gazeux explosifs. Deuxieme mémoire sur la vitesse de propagation de la flamme dans les mélanges gazeux. Ann. des Mines, T. IV, Sér. 8, 1883, pp. 296-378.
28. Withrow, Lloyd, and Rassweiler, Gerald M.: Slow Motion Shows Knocking and Non-Knocking Explosions. SAE Jour., vol. 39, no. 2, Aug. 1936, pp. 297-303, 312.
29. Serruys, Max: Mécanique appliquée.—Calcul d'une limite supérieure de la durée de la détonation dans les moteurs à explosion et explication de la présence d'une lacune dans les diagrammes Fournis par Certains manographes électriques. Comptes Rendus, T. 197, Mai 30, 1932, pp. 1894-1896.
30. Serruys, Max: Mécanique appliquée.—Enregistrement des manifestations piezométriques consecutives au cognement dans les moteurs à explosion. Comptes Rendus, T. 197, Nov. 27, 1933, pp. 1296-1298.
31. Dixon, Harold B.: The Initiation and Propagation of Explosions. Jour. Chem. Soc. Trans. (London), vol. XCIX, pt. I, 1911, pp. 588-598.
32. Dixon, Harold B., Bradshaw, Laurence, and Campbell, Colin: The Firing of Gases by Adiabatic Compression. Part 1. Photographic Analysis of the Flame. Jour. Chem. Soc. Trans. (London), vol. CV, pt. II, 1914, pp. 2027-2035.
33. Withrow, Lloyd, and Boyd, T. A.: Photographic Flame Studies in the Gasoline Engine. Ind. and Eng. Chem., vol. 23, no. 5, May 1931, pp. 539-547.
34. Rassweiler, Gerald M., and Withrow, Lloyd: Emission Spectra of Engine Flames. Ind. and Eng. Chem., vol. 24, no. 5, May 1932, pp. 528-538.
35. Boyd, T. A.: Engine Flame Researches. SAE Jour., vol. 45, no. 4, Oct. 1939, pp. 421-432.
36. Withrow, Lloyd, and Rassweiler, Gerald M.: Spectroscopic Studies of Engine Combustion. Ind. and Eng. Chem., vol. 23, no. 7, July 1931, pp. 769-776.
37. Rassweiler, Gerald M., and Withrow, Lloyd: Spectrographic Detection of Formaldehyde in an Engine Prior to Knock. Ind. and Eng. Chem., vol. 25, no. 12, Dec. 1933, pp. 1359-1366.
38. Rassweiler, Gerald M., and Williams, Lloyd: Two Knocks in a Single Explosion. The Auto. Eng., vol. XXIV, no. 324, Oct. 1934, pp. 385-388.
39. Duchene, R.: Combustion of Gaseous Mixtures. NACA TM 694, 1932.
40. Anon.: The Chemical Background for Engine Research, R. E. Burk, and Oliver Grummitt, eds. Vol. 2. Interscience Pub., Inc. (New York), 1943, pp. 165-234.
41. Schnauffer, Kurt: Engine-Cylinder Flame-Propagation Studied by New Methods. SAE Jour., vol. 34, no. 1, Jan. 1934, pp. 17-24.
42. Hastings, Charles E.: Ionization in the Knock Zone of Internal-Combustion Engine. NACA TN 774, 1940.
43. Schnauffer, Kurt: Das Klopfen von Zündermotoren. VDI Zeitschr., Bd. 75, Nr. 15, April 11, 1931, S. 455-456.
44. Souders, Mott, Jr., and Brown, Geo. Granger: Gaseous Explosions. VIII—Effect of Tetraethyl Lead, Hot Surfaces, and Spark Ignition on Flame and Pressure Propagation. Ind. and Eng. Chem., vol. 21, no. 12, Dec. 1929, pp. 1261-1268.
45. Boerlage, G. D., Broeze, J. J., van Driel, H., and Peletier, L. A.: Detonation and Stationary Gas Waves in Petrol Engines. Engineering, vol. CXLIII, no. 3712, March 5, 1937, pp. 254-255.
46. Rothrock, A. M., and Spencer, R. C.: A Photographic Study of Combustion and Knock in a Spark-Ignition Engine. NACA Rep. 622, 1938.
47. Hunn, J. V., and Brown, George Granger: Gaseous Explosions. VI—Flame and Pressure Propagation. Ind. and Eng. Chem., vol. 20, no. 10, Oct. 1928, pp. 1032-1040.
48. Kirkby, William Anthony, and Wheeler, Richard Vernon: Explosives in Closed Cylinders. Part I. Methane-Air Explosions in a Long Cylinder. Part II. The Effect of the Length of Cylinder. Jour. Chem. Soc. Trans. (London), pt. 2, 1928, pp. 3203-3214.
49. Lorentzen, Jörgen: Zur Erklärung des Klopfens in den Vergasermotoren und der Wirkung der Antiklopfmittel. Zeitschr. f. angew. Chem., Jahrg. 44, Nr. 7, Feb. 14, 1931, S. 130-136.
50. Wawrziniok: Pressure Rise, Gas Vibrations and Combustion Noises During the Explosion of Fuels. NACA TM 711, 1933.
51. Köchling, A.: Versuche zur Aufklärung des Klopfvorganges. VDI Zeitschr., Bd. 82, Nr. 39, Sept. 24, 1938, S. 1126-1134.
52. Maxwell, G. B., and Wheeler, R. V.: Flame Characteristics of "Pinking" and "Non-Pinking" Fuels. Jour. Inst. Petroleum Tech., vol. 14, April 1928, pp. 175-189.
53. Payman, W., and Titman, H.: Explosion Waves and Shock Waves. III—Initiation of Detonation in Mixtures of Ethylene and Oxygen and of Carbon Monoxide and Oxygen. Proc. Roy. Soc. (London), vol. 152, no. 876, ser. A, Nov. 1, 1935, pp. 418-445.
54. Stevens, F. W.: Constant Pressure Bomb. NACA Rep. 176, 1923.
55. Stevens, F. W.: The Gaseous Explosive Reaction—The Effect of Inert Gases. NACA Rep. 280, 1927.
56. Stevens, F. W.: The Gaseous Explosive Reaction—The Effect of Pressure on the Rate of Propagation of the Reaction Zone and upon Rate of Molecular Transformation. NACA Rep. 372, 1930.
57. Stevens, F. W.: The Rate of Flame Propagation in Gaseous Explosive Reactions. Jour. Am. Chem. Soc., vol. 48, no. 7, July 1926, pp. 1896-1906.
58. Stevens, F. W.: The Gaseous Explosive Reaction—A Study of the Kinetics of Composite Fuels. NACA Rep. 305, 1929.
59. Lewis, Bernard, and von Elbe, Guenther: On the Question of "Afterburning" in Gas Explosions. Jour. Chem. Phys., vol. 2, no. 10, Oct. 1934, pp. 659-664.
60. Randolph, D. W., and Silsbee, F. B.: Flame Speed and Spark Intensity. NACA Rep. 187, 1924.
61. Withrow, Lloyd, Lovell, W. G., and Boyd, T. A.: Following Combustion in the Gasoline Engine by Chemical Means. Ind. and Eng. Chem., vol. 22, no. 9, Sept. 1930, pp. 945-951.
62. Rassweiler, Gerald M., and Withrow, Lloyd: Motion Pictures of Engine Flames Correlated with Pressure Cards. SAE Jour., vol. 42, no. 5, May 1938, pp. 185-204.
63. Rassweiler, Gerald M., Withrow, Lloyd, and Cornelius, Walter: Engine Combustion and Pressure Development. SAE Jour., vol. 46, no. 1, Jan. 1940, pp. 25-48.
64. Withrow, Lloyd, and Cornelius, Walter: Effectiveness of the Burning Process in Non-Knocking Engine Explosions. SAE Jour., vol. 47, no. 6, Dec. 1940, pp. 526-545.
65. Linder, Werner: Mehrfachfunkenaufnahmen von Explosionsvorgängen nach der Toeplerschen Schlierenmethode. Forschungsarbeiten auf dem Gebiete des Ingenieurwesens, Heft 326, 1930, S. 1-18.
66. Marvin, Charles F., Jr., and Best, Robert D.: Flame Movement and Pressure Development in an Engine Cylinder. NACA Rep. 399, 1931.
67. Marvin, Charles F., Jr., Caldwell, Frank R., and Steele, Sydney: Infrared Radiation from Explosions in a Spark-Ignition Engine. NACA Rep. 486, 1934.
68. Fiock, Ernest F., Marvin, Charles F., Jr., Caldwell, Frank R., and Roeder, Carl H.: Flame Speeds and Energy Considerations for Explosions in a Spherical Bomb. NACA Rep. 682, 1940.
69. Lewis, Bernard, and von Elbe, Guenther: Stability and Structure of Burner Flames. Jour. Chem. Phys., vol. 11, Feb. 1943, pp. 75-97.
70. Shelkin, K. I.: On Combustion in a Turbulent Flow. NACA TM 1110, 1947.
71. Boerlage, G. D., and van Dyck, W. J. D.: Causes of Detonation in Petrol and Diesel Engines. R. A. S. Jour., vol. XXXVIII, no. 288, Dec. 1934, pp. 953-986.
72. Dreyhaupt, Fritz: Up-to-Date Summary of Question of Detonation. The Engineers' Digest, vol. III, no. 12, Dec. 1942, pp. 407-414.

CHAPTER II

CORRELATIONS OF KNOCK-LIMITED PERFORMANCE DATA

If mechanical limitations are disregarded, the power output of an engine can be increased by varying any one of several operating conditions until the knock limit of the fuel is reached. Once this limit is encountered, further efforts to increase the power will result in overheating of the engine and prolonged operation under these knocking conditions may result in damage to the engine. For these reasons, much research has been conducted to extend knowledge of the nature of knock and, through application of this knowledge, to eliminate knock as an obstacle to greater engine power.

Knock itself is fundamentally related to the following events that occur in the engine cylinder after induction of the fuel-air charge:

1. The fuel-air mixture (charge) is taken into the cylinder at a given temperature and pressure.

2. Compression of the charge begins as the piston moves upward. During this compression the volume occupied by the charge decreases, the pressure increases, and the temperature increases. The total temperature increase includes the temperature rise resulting from an amount of heat transferred from the hot cylinder walls.

3. Ignition of the charge takes place at some point before the piston reaches top center:

4. From the point of ignition the burning charge moves across the cylinder and the flame front compresses the unburned portion of the charge, as illustrated in figure II-1. The arcs emanating from the spark represent progressive positions of the flame front.

5. The density and the temperature in the unburned portion of the charge increase as the flame front progresses.

6. For a given fuel, a unique combination of density and temperature may occur and at this point the unburned portion of the charge will spontaneously ignite (ch. I) with considerable violence to produce the phenomenon called knock.

Rothrock and Biermann (reference 1) assume density and temperature to be the physical properties of the combustion gas that determine whether or not the fuel will knock. For each gas density, a gas temperature exists at which the unburned portion of the charge ahead of the flame front will ignite spontaneously. This combination of density and temperature may or may not be reached, depending upon the controlled and uncontrolled factors of engine operation.

These factors are listed in reference 1 as follows:

- (a) Chemical composition of fuel
- (b) Fuel-air ratio
- (c) Exhaust-gas dilution
- (d) Humidity
- (e) Compression ratio
- (f) Inlet-air temperature
- (g) Inlet-air pressure
- (h) Wall temperature of combustion chamber and cylinder
- (i) Spark advance
- (j) Engine speed
- (k) Engine dimensions
- (l) Combustion-chamber form

If, as pointed out in reference 1, many combinations of density and temperature exist at which a fuel will knock, it is impossible to express the knock limit of a fuel adequately by testing that fuel at only one set of engine operating conditions. Each fuel must be tested under a variety of conditions in order to establish the absolute relation between knock-limited density and temperature. This relation will be the same regardless of the engine in which it is determined.

SYMBOLS

The following symbols are used in this chapter:

a, b	coefficients in specific-heat (constant volume) equation for charge
c_v	specific heat of mixture at constant volume, Btu/(lb)/° F
f	fuel-air ratio
F_1, F_2, F_3	functions
H	heat content per pound of mixture, Btu/(lb)
i	intake cycles per minute
J	mechanical equivalent of heat, 778 (ft-lb)/Btu
K	constant
m	molecular weight of charge
n	moles of gas present in cylinder

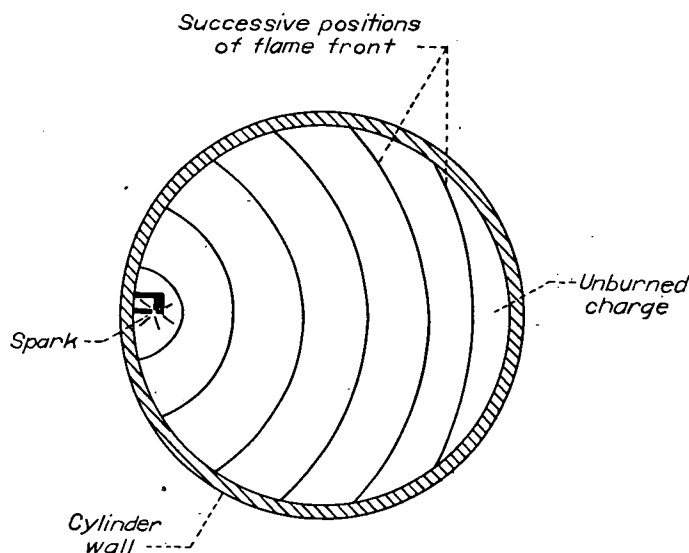


FIGURE II-1.—Schematic diagram of travel of flame front across

P	pressure, (lb)/(sq in.)
R	universal gas constant, (ft-lb)/(mole)(°R)
r	compression ratio
T	temperature, °R
T'	mean temperature throughout combustion chamber just before knock, °R
ΔT	temperature rise of cylinder contents due to constant-volume burning, °F
V_c	clearance volume, (cu in.)
V_d	displacement volume, (cu in.)
V_s	specific volume, (cu ft/lb)
W	gas flow, (lb)/cycle
Z	factor replacing $\frac{\rho_c}{T_c^{\frac{\gamma}{\gamma-1}}}$
γ	ratio of specific heats
η_{com}	combustion efficiency
ρ	density, (lb)/(cu in.)
Subscripts:	
b	burned mixture at top center
c	compression
e	effective
k	in knocking zone (end gas) immediately preceding knock
0	inlet air

CORRELATIONS BASED UPON END-GAS CONDITIONS

The last portion of the charge to burn, which under proper conditions of density and temperature ignites spontaneously, is commonly designated the end gas. The space occupied by this gas is called the end zone. From the conventional thermodynamic equations, expressions are derived in reference 1 for determination of the end-gas density and temperature. The compression of the fresh charge admitted to the cylinder occurs in two steps: (1) the charge is adiabatically compressed by the piston until ignition takes place; and (2) the unburned charge ahead of the flame front is assumed to be adiabatically compressed by both the piston and the flame front until knock occurs, or

Step (1):

$$\frac{T_0}{T_c} = \left(\frac{P_0}{P_c} \right)^{\frac{\gamma-1}{\gamma}}$$

Step (2):

$$\frac{T_c}{T_k} = \left(\frac{P_c}{P_k} \right)^{\frac{\gamma-1}{\gamma}}$$

where

T_0, P_0	inlet-air temperature and pressure, respectively
T_c, P_c	temperature and pressure, respectively, after compression by piston
T_k, P_k	temperature and pressure, respectively, in knocking zone immediately preceding knock
γ	ratio of specific heat at constant pressure to specific heat at constant volume

Combining steps 1 and 2 yields

$$T_k = T_0 \left(\frac{P_k}{P_0} \right)^{\frac{\gamma-1}{\gamma}} \quad (1)$$

This equation expresses the temperature in the knocking zone. The expression for density of the unburned charge ahead of the flame front is given by the equation

$$K \rho_k = \frac{P_0}{T_0} \left(\frac{P_k}{P_0} \right)^{\frac{1}{\gamma}} \quad (2)$$

where

ρ_k gas density in knocking zone immediately preceding knock

K constant

Equations (1) and (2) are the same as those of reference 2 except that density has been used instead of pressure.

These equations show that if an engine is operated at conditions of incipient knock and P_k is measured, the density and the temperature may be calculated. Accurate measurements of P_k , however, are very difficult; therefore, the calculations as well as experimentation can be simplified by expressing the density and the temperature in terms of more easily determined factors. For example, equation (2) can be written

$$K \rho_k = \frac{P_0}{T_0} \left(\frac{P_k}{P_c} \frac{P_c}{P_0} \right)^{\frac{1}{\gamma}}$$

If P_c and T_c are the compression pressure and temperature, respectively, and T' is the mean temperature throughout the combustion chamber just before knock occurs, the density ρ_k is expressed by

$$K \rho_k = \frac{P_0}{T_0} \frac{T'}{T_c} \left(\frac{P_c}{P_0} \right)^{\frac{1}{\gamma}}$$

Because

$$T' = T_c + \frac{H}{c_v} = T_0 r^{\gamma-1} + \frac{H}{c_v}$$

and

$$\left(\frac{P_c}{P_0} \right)^{\frac{1}{\gamma}} = r$$

where

H heat content per pound of mixture

c_v specific heat of mixture at constant volume

r compression ratio of engine

then

$$K \rho_k = \frac{r P_0}{T_0} \left(1 + \frac{H}{c_v T_0 r^{\gamma-1}} \right)^{\frac{1}{\gamma}} \quad (3)$$

Similarly,

$$T_k = T_0 r^{\gamma-1} \left(1 + \frac{H}{c_v T_0 r^{\gamma-1}} \right)^{\frac{\gamma-1}{\gamma}} \quad (4)$$

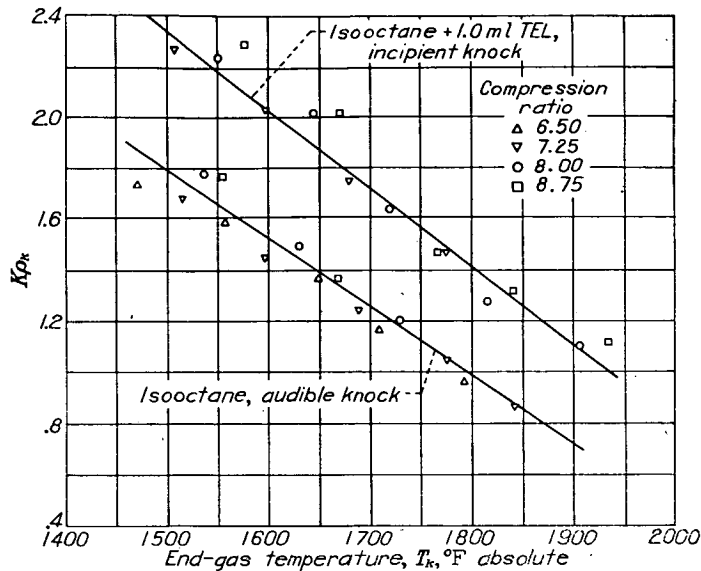


FIGURE II-2.—Effect of estimated end-gas temperature on maximum permissible density factor for leaded and unleaded isooctane in flat-disk combustion chamber. Engine speed, 2500 rpm; improved cooling in cylinder head; fuel-air ratio, 0.081. (Fig. 2 of reference 1.)

In the evaluation (reference 1) of equations (3) and (4), c_p was estimated as 0.25 Btu per pound per °F, γ as 1.29, and H as 1160 Btu per pound of mixture. The quantity H was estimated by the following expression:

$$H = \frac{18,000 \eta_{com} f}{1 + f}$$

where

η_{com} combustion efficiency

f fuel-air ratio

The combustion efficiencies used in reference 1 were obtained from reference 3.

The application of the foregoing relations is illustrated in figure II-2. These data were obtained at an engine speed of 2500 rpm and a fuel-air ratio of 0.081. Although the data represent several compression ratios and several inlet-air temperatures, a single curve is obtained for each fuel. For the range of values examined in reference 1, the value of

$\left(1 + \frac{H}{c_p T_0 r^{\gamma-1}}\right)^{\frac{1}{\gamma}}$ at an inlet-air temperature of 120° F varied from a value of 3.82 at a compression ratio of 6.5 to a value of 3.62 at a compression ratio of 8.75. It was therefore assumed that the density factor (equation (3)) could be expressed by $r P_0 / T_0$. Similarly, the end-gas temperature T_k could be expressed by $T_0 r^{\gamma-1}$. In reference 1, $r^{\gamma-1}$ is also eliminated from the term for T_k because the data justified the omission. Proper consideration of equation (3), however, does not support the elimination of $r^{\gamma-1}$ from equation (4).

The omission of the terms $r^{\gamma-1}$ and $\left(1 + \frac{H}{c_p T_0 r^{\gamma-1}}\right)^{\frac{1}{\gamma}}$ from equation (4) permits the use of T_0 in place of T_k . A comparison of several fuels on this basis is shown in figure II-3. The curves for the fuels tend to converge at high inlet-air tempera-

tures, which indicates that for some temperature it is conceivable that this particular group of fuels might have the same knock-limited performance. This trend is due to the similarity of the fuels used. As will be shown later in this chapter, the curves for various fuels are not ordinarily so similar.

Because the density and temperature relations vary with fuel-air ratio (reference 1), Taylor, Leary, and Diver (reference 4) obtained additional data to improve the accuracy of the expressions in reference 1 and specifically to investigate further the influence of fuel-air ratio. In addition, an effort is made to determine the effect of exhaust-gas dilution on the compression ratio for incipient detonation. As a further refinement of the end-gas conditions, allowance is made for variation of specific heats and chemical equilibrium of the charge before and after combustion with temperature, pressure, and fuel-air ratio. The procedure used for computing end-gas temperatures in reference 4 is as follows:

A point is first selected on the compression stroke of an indicator diagram and the specific volume V_s of the charge is computed. The temperature T_0 is then computed from the following relation:

$$T_0 = \frac{P_0 V_s}{R}$$

where

R gas constant, 1544/m

m molecular weight of charge

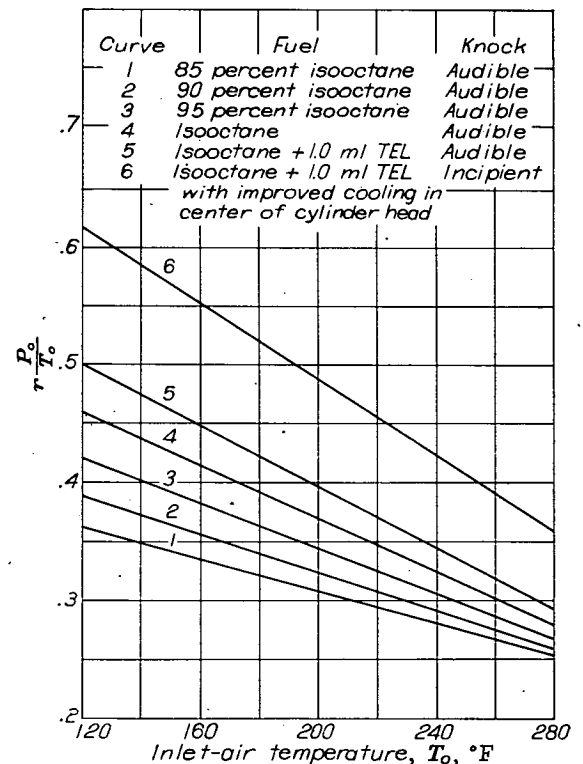


FIGURE II-3.—Effect of inlet-air temperature on maximum permissible density factor for series of fuel blends in flat-disk combustion chamber. Engine speed, 2500 rpm; fuel-air ratio, 0.081. (Fig. 4 of reference 1.)

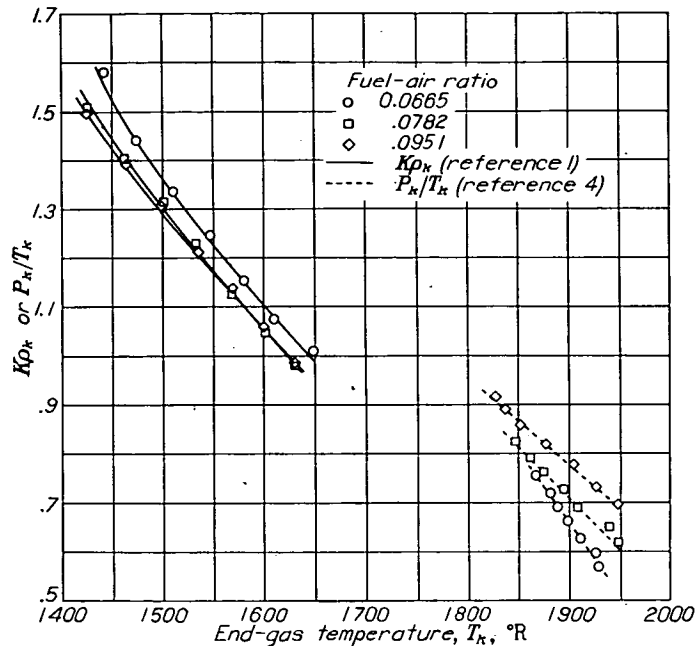


FIGURE II-4.—Comparison of effect of end-gas temperature on maximum permissible end-gas density as determined by methods of references 1 and 4. (Fig. 24 of reference 4.)

The end-gas temperature T_k is calculated from the equation for T_k taken from reference 5.

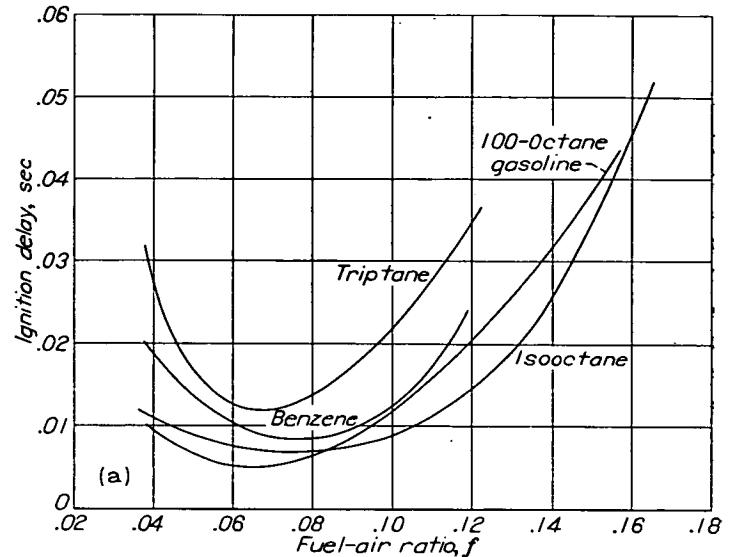
$$\log T_k = \log T_0 + \frac{R}{\left(a + \frac{R}{J}\right) J} \log \frac{P_k}{P_0} - \frac{b}{2.30 \left(a + \frac{R}{J}\right)} (T_k - T_0) \quad (5)$$

where
 a, b coefficients in specific-heat equation at constant volume for charge

J mechanical equivalent of heat, 778(ft-lb)/Btu

A comparison of the end-gas calculations from references 1 and 4 is shown in figure II-4. The results of these references are in agreement as to the variation of end-gas density with end-gas temperature, but the order of variation of end-gas density with fuel-air ratio at any given end-gas temperature is reversed by the two methods of calculation. An additional analysis by Rothrock cited in reference 4 attributes the apparent reversal of the effect of the fuel-air ratio to the difference between the measured maximum pressures of reference 4 and the computed maximum pressures of reference 1.

The studies of reference 4 were made at constant engine speed and therefore do not consider the effect of time or rate of combustion on knock. In a subsequent investigation, Leary and Taylor (reference 6) attempt to explain the significance of time in engine detonation by the use of indicator diagrams obtained during knock tests. As a result of these studies, the investigators conclude that higher maximum permissible pressures can be used in an engine if the rate of compression of the end gas is increased. This conclusion led to the development of an experimental com-

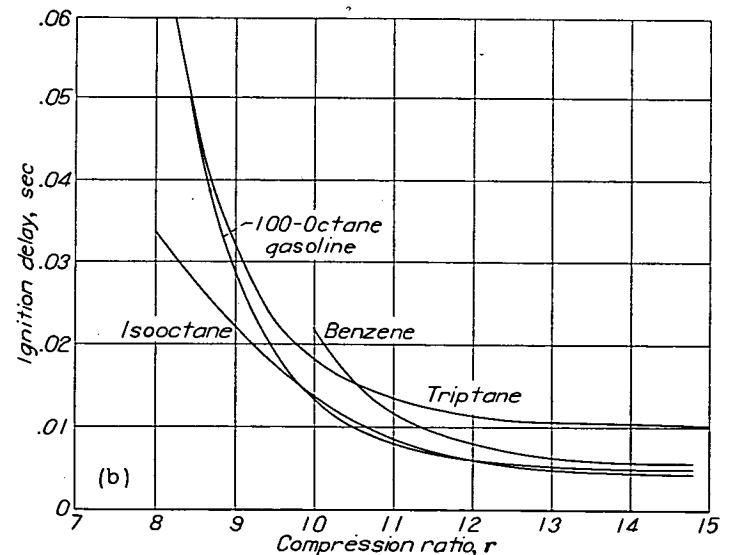


(a) Variable fuel-air ratio. Compression ratio, 11.7. (Fig. 32 of reference 8.)

FIGURE II-5.—Comparison of isooctane, 100-octane gasoline, triptane, and benzene with respect to variation in ignition delay. Initial pressure, 14.7 pounds per square inch absolute; initial temperature, 149° F; compression time, approximately 0.006 second.

pression machine (reference 7) for use in studies of fuel knocking characteristics.

The first results obtained in the compression machine are reported in reference 8 as an explanation of differences in knocking characteristics of fuels. Ignition delay as a function of fuel-air ratio is shown in figure II-5 (a) for four fuels. When the curves for isooctane (2,2,4-trimethylpentane) and 100-octane gasoline are compared, the ignition delay of the gasoline is seen to be greater than that of isooctane at fuel-air ratios greater than 0.085. The reverse is true at fuel-air ratios below 0.085. On the basis of these results, it is suggested (reference 8) that the knock-limited



(b) Variable compression ratio. Fuel-air ratio, stoichiometric. (Fig. 33 of reference 8.)

FIGURE II-5.—Concluded. Comparison of isooctane, 100-octane gasoline, triptane, and benzene with respect to variation in ignition delay. Initial pressure, 14.7 pounds per square inch absolute; initial temperature, 149° F; compression time, approximately 0.006 second.

performance of these fuels may be similar; that is, isooctane would be higher than 100-octane gasoline at low fuel-air ratios but lower than gasoline at high fuel-air ratios. This belief was confirmed by supercharged knock tests. If ignition delay were the only criterion of knock, it could be concluded from figure II-5 (a) that triptane (2,2,3-trimethylbutane) has the highest antiknock performance with benzene next in order; however, the authors of reference 8 contend that this result may not be true inasmuch as the pressure-time curve during the delay period is also of considerable importance.

Ignition delay as a function of compression ratio is shown in figure II-5 (b). The delay period decreases as the compression ratio increases. The ignition delays for triptane are higher than those of isooctane and 100-octane gasoline over the entire range of compression ratios. Benzene has ignition delays greater than those of triptane at compression ratios below 10.5, but over the entire compression-ratio range the pressure-time records of reference 8 for benzene always indicate a smoother reaction than that of triptane. From this relation it may be surmised that the high-antiknock characteristics of benzene are attributable to the slow pressure rise during combustion of the end gas rather than to the long ignition delay periods.

The correlations of end-gas conditions proposed in references 1 and 4 do not account for variations of fuel-air ratio (fig. II-4); consequently, the end-gas density-temperature relations are rederived in reference 9 with this factor taken into consideration. The following equations are proposed in reference 9:

The density of the charge after adiabatic compression by the piston is

$$\rho_c = \frac{W}{V_c} \quad (6)$$

where

ρ_c density after compression
 W weight of gas inducted per cycle
 V_c clearance volume

The temperature of the charge after compression is

$$T_c = T_0 r^{\gamma-1} \quad (7)$$

The state of the gases before burning in the cylinder when the piston is at top center may be expressed by the ideal gas equation,

$$P_c V_c = n R T_c \quad (8)$$

where

P_c compression pressure
 n moles of gas present in cylinder
 R universal gas constant

The state of the gases at top center after burning is

$$P_b V_c = n R T_b \quad (9)$$

where

P_b pressure of burned mixture at top center
 T_b temperature of burned mixture at top center

Combining equations (8) and (9) gives

$$\frac{P_b}{P_c} = \frac{T_b}{T_c} \quad (10)$$

The end gas is adiabatically compressed during constant-volume combustion

$$\frac{P_{k,e}}{P_c} = \left(\frac{T_{k,e}}{T_c} \right)^{\frac{\gamma}{\gamma-1}} \quad (11)$$

and

$$\frac{\rho_{k,e}}{\rho_c} = \left(\frac{T_{k,e}}{T_c} \right)^{\frac{1}{\gamma-1}} \quad (12)$$

where.

$P_{k,e}$ effective end-gas pressure at instant before knock
 $T_{k,e}$ effective end-gas temperature at instant before knock
 $\rho_{k,e}$ effective end-gas density at instant before knock

The term "effective" was used in reference 9 in referring to the end-gas quantities (subscript k,e) inasmuch as these quantities are based upon an idealized air-cycle analysis and are therefore not the same as the actual values.

As the flame front moves across the cylinder, the unburned gas (assumed to be small) is compressed and knock occurs just before completion of combustion of the entire charge.

In equation (11), P_b can be substituted for $P_{k,e}$

$$\frac{P_b}{P_c} = \left(\frac{T_{k,e}}{T_c} \right)^{\frac{\gamma}{\gamma-1}} \quad (13)$$

Combining equations (10) and (13) yields

$$T_{k,e} = T_c \left(\frac{T_b}{T_c} \right)^{\frac{\gamma-1}{\gamma}} \quad (14)$$

however,

$$T_b = T_c + \Delta T \quad (15)$$

where

ΔT temperature rise of cylinder contents due to constant-volume burning

Therefore,

$$T_{k,e} = T_c \left(1 + \frac{\Delta T}{T_c} \right)^{\frac{\gamma-1}{\gamma}} \quad (16)$$

In reference 9, a relation is assumed to exist between effective end-gas density and effective end-gas temperature at the incipient-knock condition; this relation includes the effects of variations of fuel-air ratio, compression ratio, and inlet-air temperature:

$$\rho_{k,e} = F_1(T_{k,e}) \quad (17)$$

Combining equations (12) and (17) yields

$$\rho_c \left(\frac{T_{k,e}}{T_c} \right)^{\frac{1}{\gamma-1}} = F_1(T_{k,e}) \quad (18)$$

or

$$\frac{\rho_c}{T_c^{\frac{1}{\gamma-1}}} = \frac{F_1(T_{k,e})}{T_{k,e}^{\frac{1}{\gamma-1}}} = F_2(T_{k,e}) \quad (19)$$

Replacing $T_{k,e}$ by its equivalent in equation (16) and letting Z equal $\rho_c/T_c^{\frac{1}{\gamma-1}}$ gives

$$Z = F_2 \left[T_c \left(1 + \frac{\Delta T}{T_c} \right)^{\frac{\gamma-1}{\gamma}} \right] \quad (20)$$

The constant-volume temperature rise during burning ΔT is dependent upon factors such as fuel-air ratio and pressure and temperature of the charge before burning. Of these factors, fuel-air ratio is the most important one affecting ΔT ; therefore, ΔT is considered (reference 9) to be a function of fuel-air ratio alone. At some reference fuel-air ratio, ΔT can be taken as a constant. In reference 9, a fuel-air ratio of 0.08 is selected. A representative value (4040° F) of ΔT is chosen from reference 10 to be used in the correlation of the experimental data of reference 9.

The constant-volume temperature rise due to combustion becomes

$$\Delta T = 4040 F_3(f) \quad (21)$$

where f is fuel-air ratio.

Substituting this value of ΔT into equation (20) gives

$$Z = F_2 \left\{ T_c \left[1 + \frac{4040 F_3(f)}{T_c} \right]^{\frac{\gamma-1}{\gamma}} \right\} \quad (22)$$

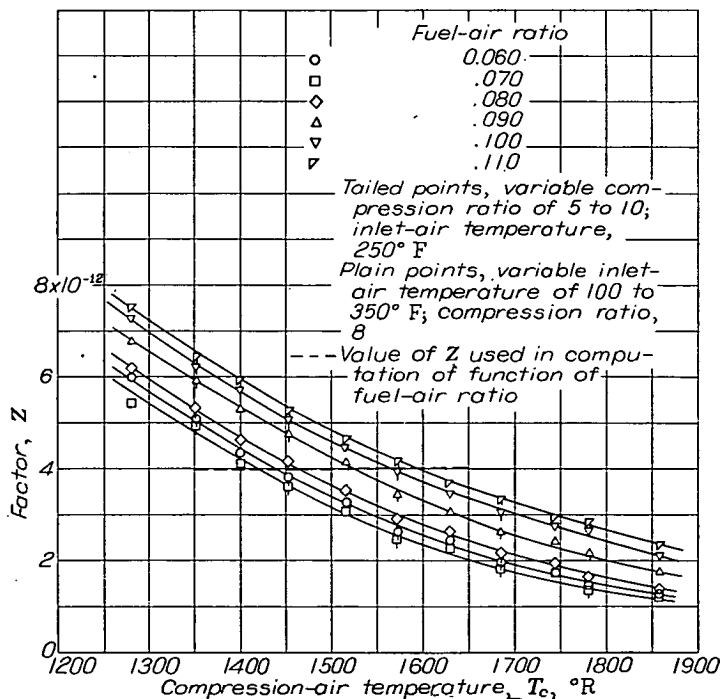


FIGURE II-6.—Effect of compression-air temperature and fuel-air ratio on knock-limited values of factor Z for AN-F-28 fuel in CFR engine. Engine speed, 1800 rpm; coolant temperature, 250° F; spark advance, 30° B. T. C. (Fig. 4 of reference 9; data calculated from reference 10.)

213637—53—33

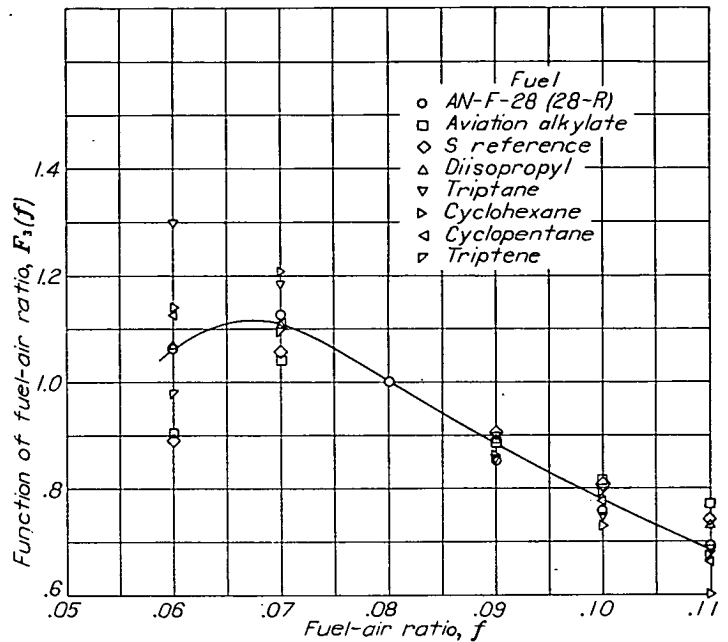


FIGURE II-7.—Effect of fuel-air ratio on function of fuel-air ratio for eight fuels in supercharged CFR engine. (Fig. 5 of reference 9.)

By use of equations (6), (7), (12), (16), and (21), the following expressions may be written for $T_{k,e}$ and $\rho_{k,e}$:

$$T_{k,e} = T_0 r^{\gamma-1} \left[1 + \frac{4040 F_3(f)}{T_0 r^{\gamma-1}} \right]^{\frac{\gamma-1}{\gamma}} \quad (23)$$

$$\rho_{k,e} = \frac{W}{V_{c,r}} \left(\frac{T_{k,e}}{T_0} \right)^{\frac{1}{\gamma-1}} \quad (24)$$

Equations (23) and (24) can be solved for $T_{k,e}$ and $\rho_{k,e}$ if $F_3(f)$ is known. The utility of these equations for correlation of engine variables depends upon whether a single curve results between $T_{k,e}$ and $\rho_{k,e}$, when T_0 , r , or f is varied and also upon the accuracy with which $F_3(f)$ can be evaluated.

In order to determine $F_3(f)$, values of Z are computed (reference 9) from experimental data obtained in a CFR test engine with AN-F-28 (28-R), a 100/130 grade aviation gasoline. The values of Z are computed from $\rho_c/T_c^{\frac{1}{\gamma-1}}$, and a constant γ of 1.4 is used. The computed values are then plotted against T_c as shown in figure II-6. The experimental data upon which this figure is based covered a range of compression ratios from 5 to 10 at a constant inlet-air temperature of 250° F and a range of inlet-air temperatures from 100° to 350° F at a compression ratio of 8. A constant engine speed of 1800 rpm, a coolant temperature of 250° F, and a spark advance of 30° B. T. C., were maintained in all cases.

At constant Z (fig. II-6), a new value of T_c exists for each fuel-air ratio and the right side of equation (22) is constant regardless of fuel-air ratio; consequently, the following equation may be written:

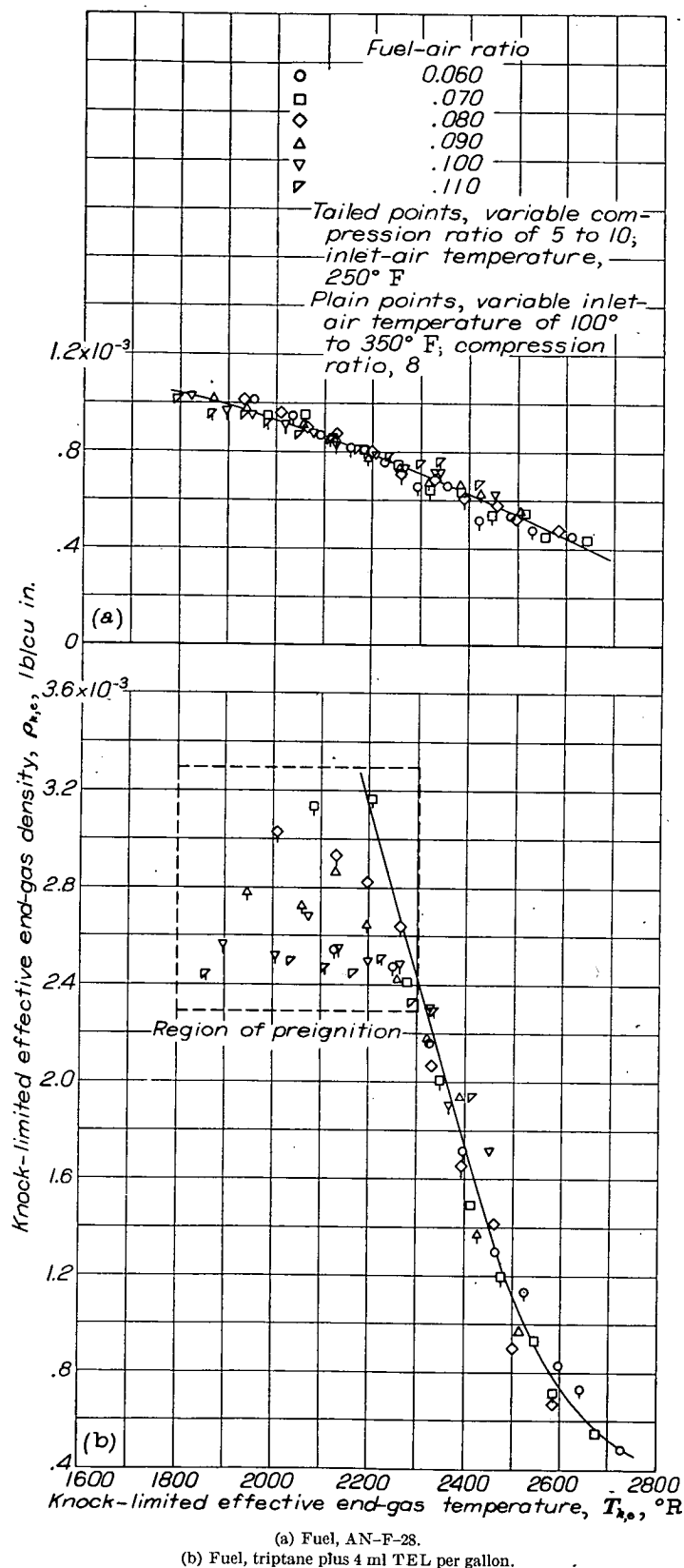


FIGURE II-8.—Variation of knock-limited effective end-gas density with knock-limited effective end-gas temperature for changes in fuel-air ratio, inlet-air temperature, and compression ratio. Inlet-air pressure varied to maintain incipient knock. (Fig. 6 of reference 9; data calculated from reference 12.)

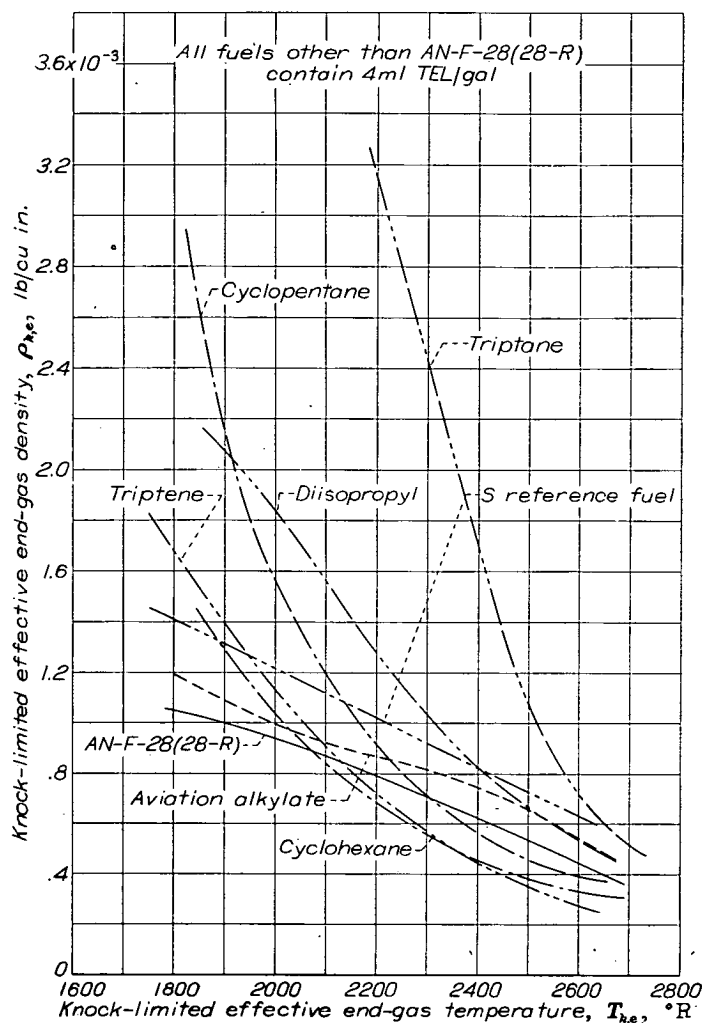


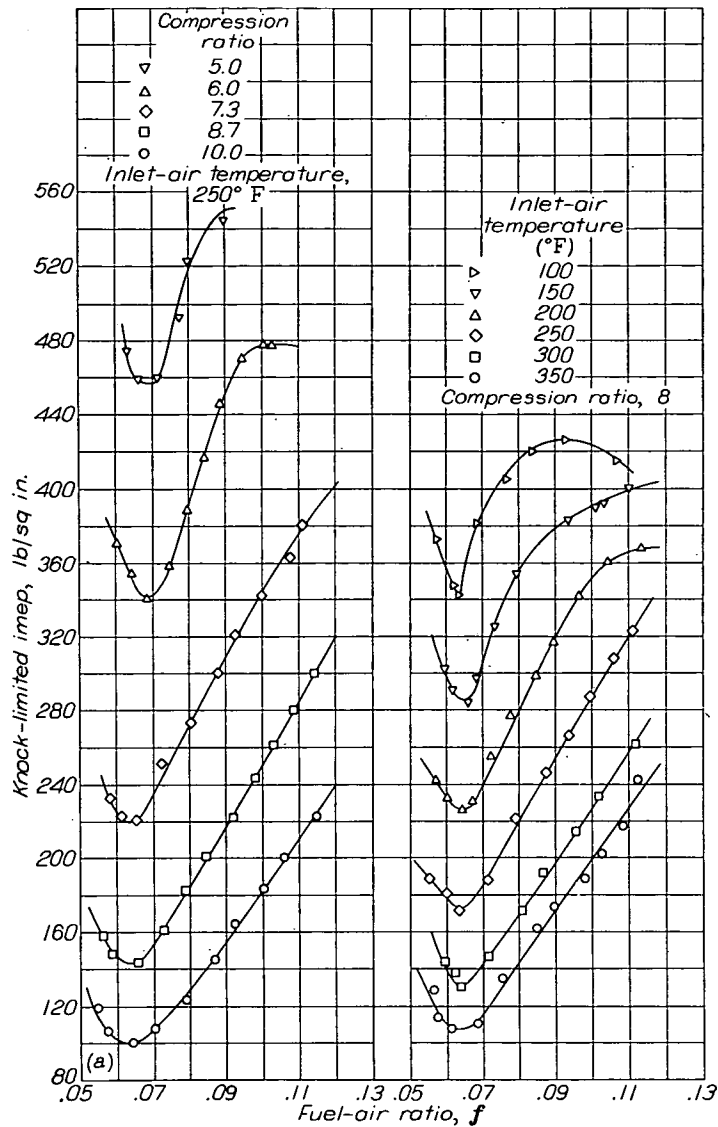
FIGURE II-9.—Effect of change in knock-limited effective end-gas temperature on knock-limited effective end-gas density for eight fuels. (Fig. 8 of reference 9.)

$$T_c \left[1 + \frac{4040 F_3(f)}{T_c} \right]^{\frac{\gamma-1}{\gamma}} = T_{c,0.08} \left(1 + \frac{4040}{T_{c,0.08}} \right)^{\frac{\gamma-1}{\gamma}} \quad (25)$$

The left side of this equation applies for any fuel-air ratio, whereas the right side applies only to the previously mentioned reference fuel-air ratio of 0.08, where $F_3(f)$ has been taken as 1.0. By substitution of different values of T_c from figure II-6 into the left side and the value of T_c for a fuel-air ratio of 0.08 in the right side of equation (25) it is possible to solve for $F_3(f)$.

Values of $F_3(f)$ are determined in reference 9 for seven other fuels and the results are plotted in figure II-7. The curve on this figure represents the mean of the data. Obviously the relation between $F_3(f)$ and f varies with the fuel used and for this reason the use of the mean curve is not recommended unless data for a specific fuel are unavailable.

By means of the $F_3(f)$ curves (fig. II-6) for each fuel, values of $T_{k,e}$ and $\rho_{k,e}$ are calculated (reference 9) from



(a) Knock-limited indicated mean effective pressure. (Fig. 7 of reference 12.)

FIGURE II-10.—Effect of compression ratio and inlet-air temperature on knock-limited performance of diisopropyl plus 4 ml TEL per gallon.

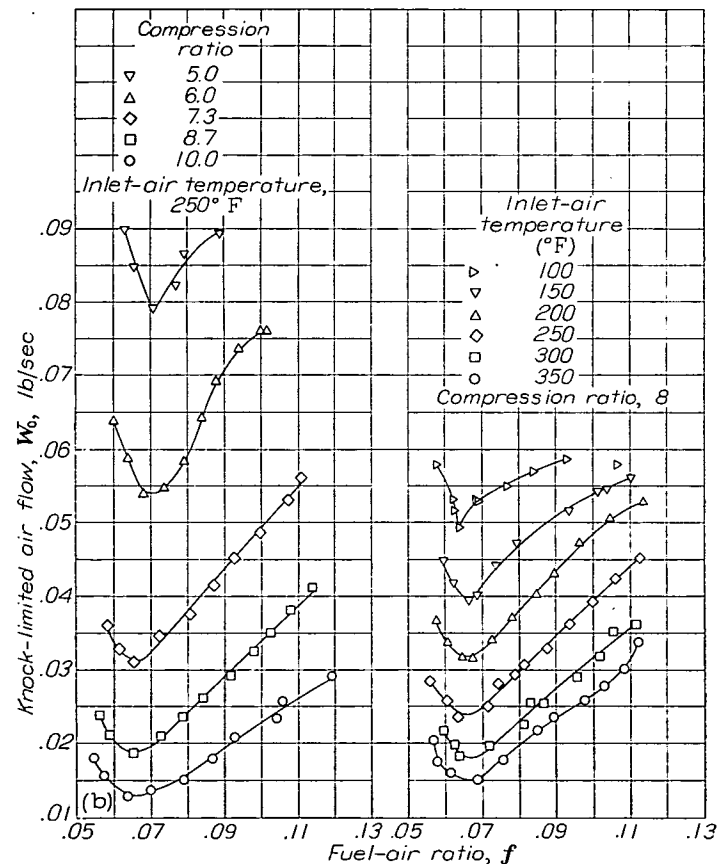
equations (23) and (24). In general, the correlations were quite satisfactory, as illustrated by figure II-8 (a). The poorest correlation was found for triptane (fig. II-8 (b)), particularly at low values of $T_{k,e}$; however, good correlation could not be expected in this case because of the tendency of the fuel to preignite. (See ch. IV.) A comparison of the knock-limited effective density-temperature curves for the fuels investigated is presented in figure II-9.

In order to determine the accuracy of the curves of $T_{k,e}$ against $\rho_{k,e}$, the authors of reference 9 calculated the knock-limited air flow W_a (equation (24)) from $T_{k,e}-\rho_{k,e}$ curves based upon the mean and individual $F_3(f)$ curves of fig-

ure II-7. The results of this comparison are presented in the following table:

Fuel	Mean error in calculated knock-limited charge flow (percent)			
	Variable inlet-air temperature		Variable compression ratio	
	Individual $F_3(f)$	Mean $F_3(f)$	Individual $F_3(f)$	Mean $F_3(f)$
AN-F-28 (28-R).....	3.5	4.2	4.6	5.5
Aviation alkylate.....	2.9	6.4	3.9	5.4
S reference.....	2.7	7.6	3.7	7.0
Diisopropyl.....	3.7	5.0	4.0	3.9
Triptane.....	8.6	24.4	6.6	22.4
Cyclopentane.....	7.3	7.2	7.5	8.6
Cyclohexane.....	5.1	7.4	7.2	9.0
Triptene.....	6.0	8.5	6.2	6.6

As would be expected, the least error in the calculated air flow resulted from use of the curves of $T_{k,e}$ against $\rho_{k,e}$ based upon the individual $F_3(f)$ curves. Diisopropyl at variable compression-ratio conditions and cyclopentane at variable inlet-air-temperature conditions appear to be exceptions, but the difference in mean error is small between the air flows computed from individual or mean $F_3(f)$ curves.



(b) Knock-limited air flow.

FIGURE II-10.—Concluded. Effect of compression ratio and inlet-air temperature on knock-limited performance of diisopropyl plus 4 ml TEL per gallon.

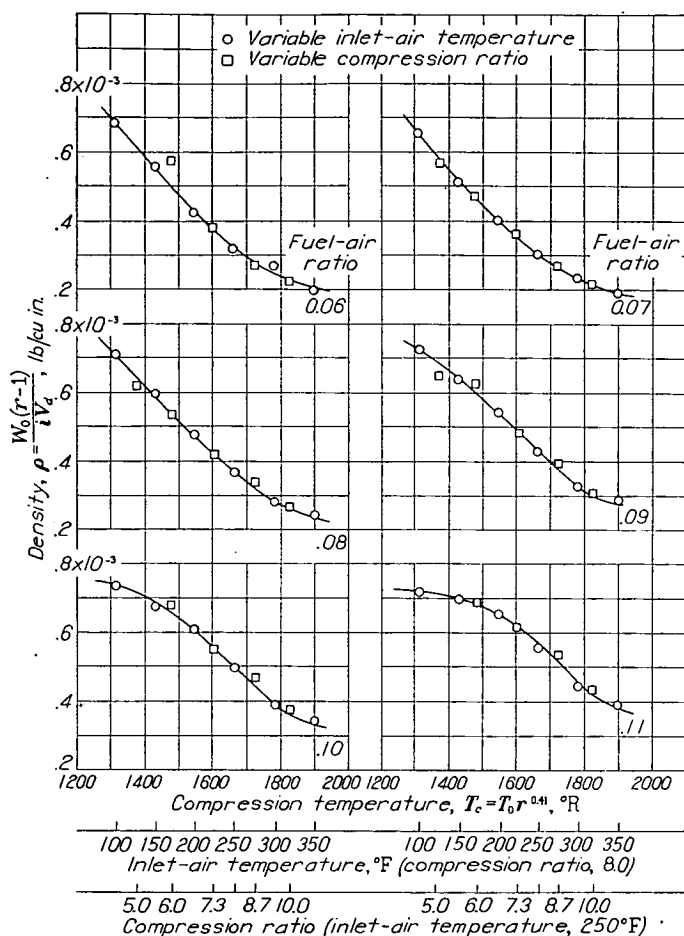


FIGURE II-11.—Effect of compression temperature on knock-limited compression-air density for diisopropyl plus 4 ml TEL per gallon. (Fig. 16 of reference 12.)

CORRELATIONS BASED UPON COMPRESSION CONDITIONS

The calculation of end-gas temperatures and densities described in the foregoing section is sufficiently complicated and uncertain to make it desirable to use some simple method of data correlation. One such method, based upon compression conditions, is proposed in reference 11. The method does not correlate all the variables of engine operation such as spark advance, engine speed, or fuel-air ratio, but is useful in explaining the effects of compression ratio and inlet-air temperature on fuel performance.

In this method, a relation is assumed to exist between compression conditions before ignition occurs and end-gas conditions after ignition occurs; therefore, calculated compression densities and temperatures may be used in place of end-gas densities and temperatures.

The compression density may be calculated as follows:

$$\rho_c = \frac{W_0}{i V_c} = \frac{W_0(r-1)}{i V_d} \quad (26)$$

where

W_0 inlet-air flow, (lb/min)
 i intake cycles per minute

V_c clearance volume, (cu in.)

V_d displacement volume, (cu in.)

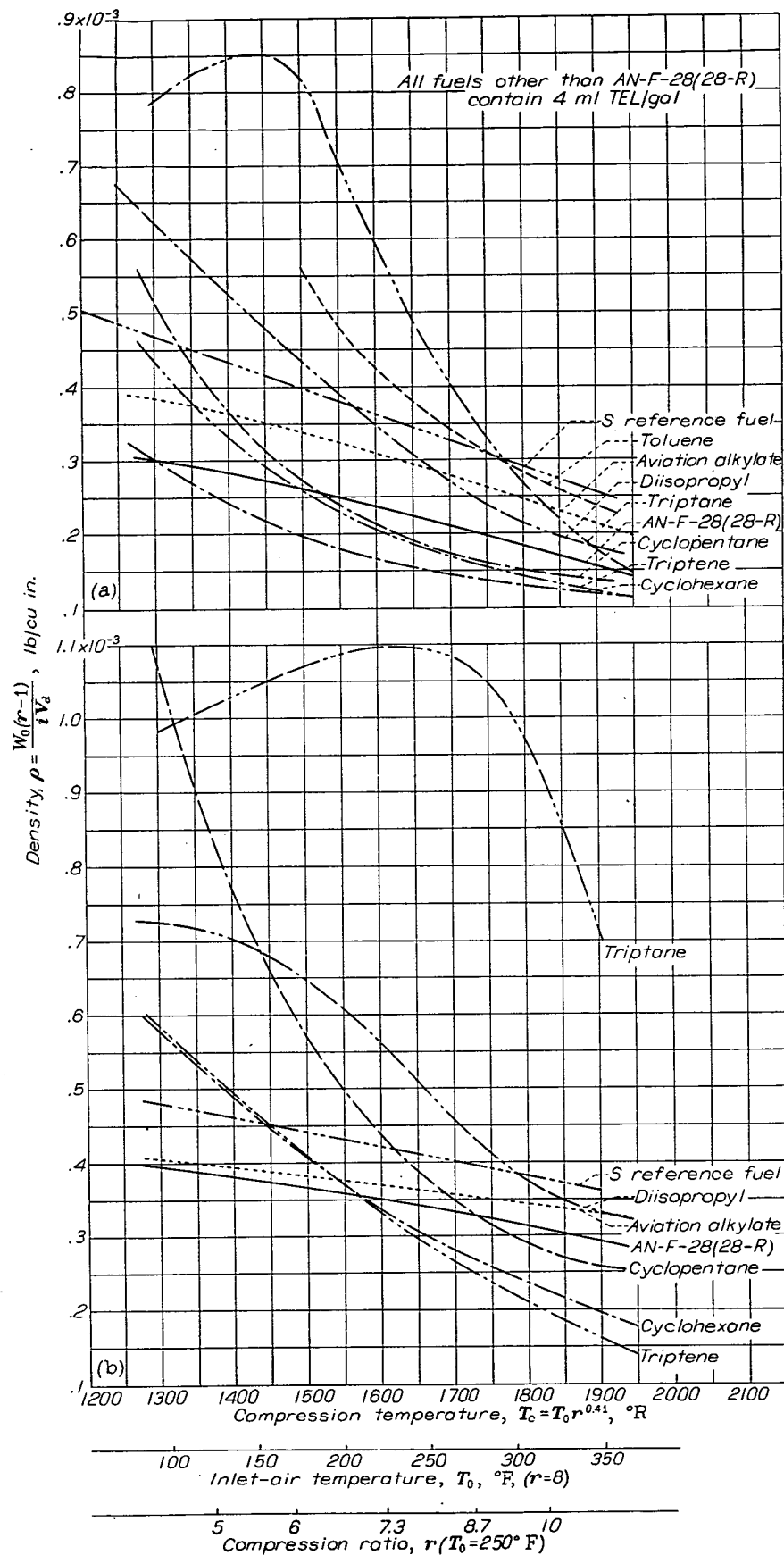
The calculation of compression temperature (reference 11) neglects the effects of fuel vaporization, heat transfer to the cylinder walls, fuel-air ratio, and preflame reactions; however, the correlations obtained with an adiabatic-compression formula justify these omissions.

$$T_c = T_0 r^{\gamma-1} \quad (27)$$

In order to verify this method of correlation, knock-limited performance data were obtained for a number of fuels in references 11 and 12. These data resulted from tests in which all operating conditions except compression ratio were held constant. The tests were then repeated with all variables except inlet-air temperature constant. An illustration of the engine data for a single fuel is presented in figure II-10. Compression densities and temperatures calculated from data in these figures are shown in figure II-11. Comparisons of several fuels (reference 13) are shown in figure II-12.

The curves shown in these figures illustrate the relative sensitivities of fuels to changes of temperature and compression ratio and also may be used to describe the popular terms "mild" and "severe" conditions. For example (fig. II-12 (a)), an increase in compression ratio or inlet-air temperature causes an increase in compression temperature and a decrease in knock-limited performance as indicated by the low compression densities. Any combination of engine operating variables, such as spark advance, inlet-air temperature, compression ratio, and engine speed, that tends to produce exceptionally low knock-limited compression densities, is commonly termed a "severe" condition. Conversely, "mild" conditions tend to produce high knock-limited compression densities.

The type of curve shown in figure II-12 (a) has also been used to express the term "temperature sensitivity." Although no universally accepted definition of temperature sensitivity has been devised, the authors of reference 12 suggest that the compression-density-temperature plot offers a more generally applicable definition than any other yet devised. The slope of the curve of compression density plotted against compression temperature would offer a measurement of temperature sensitivity. Calculations of these slopes for the fuels investigated in reference 12 indicated that the temperature sensitivities of isooctane, aviation gasoline, an aviation alkylate, and diisopropyl (2,3-dimethylbutane) were very nearly independent of engine conditions. The temperature sensitivity of triptane decreased as the severity of the engine conditions decreased; the temperature sensitivities of cyclopentane, cyclohexane, triptene, (2,3,3-trimethylbutene-1), and toluene increased as the severity of the engine conditions decreased.



(a) Fuel-air ratio, 0.065.

(b) Fuel-air ratio, 0.11.

FIGURE II-12.—Temperature-density relations for various fuels in CFR engine. Engine speed, 1800 rpm; coolant temperature, 250° F; spark advance, 30° B. T. C. (Fig. 7 of reference 13.)

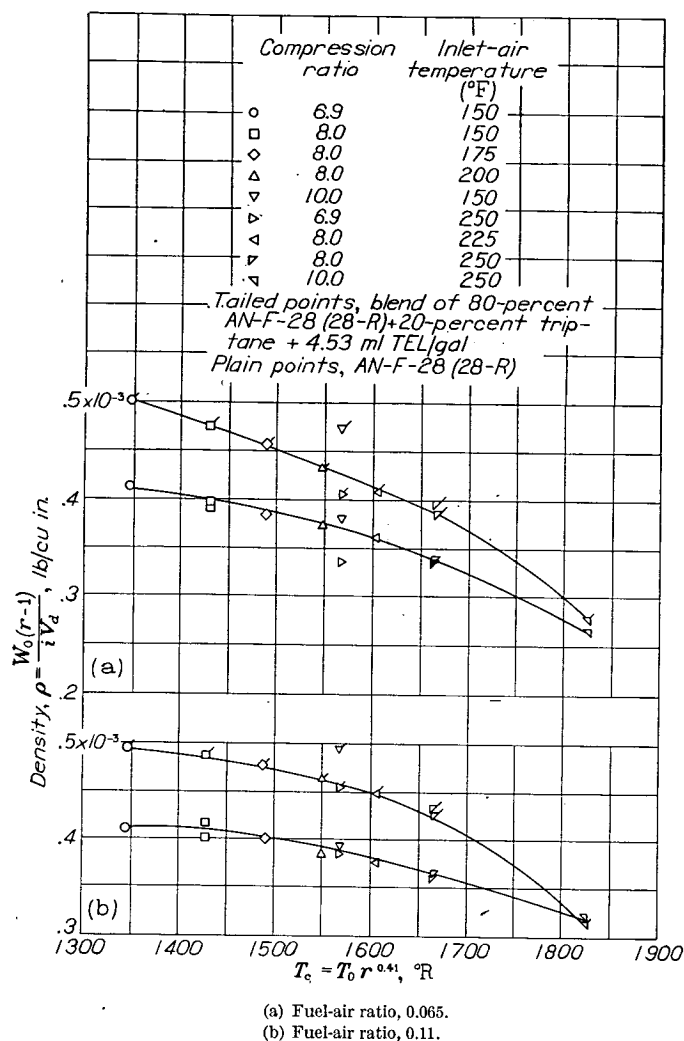


FIGURE II-13.—Temperature-density relations for two fuels in R-2600 cylinder. Engine speed, 2100 rpm; spark advance, 20° B. T. C.; rear spark-plug-bushing temperature, 450° F. (Fig. 8 of reference 13.)

The data presented in figure II-12 were obtained in a CFR engine that is designed to permit variation of compression ratio. In order to determine if the compression-density—temperature correlation would hold for full-scale engines, similar tests were made in an experimental engine employing a single cylinder from an aircraft engine. The compression ratio was varied by using pistons of different displacement in the cylinder. The results of these runs (reference 13) are shown in figure II-13.

Inasmuch as the correlation is satisfactory for full-scale engines (fig. II-13), the experimental evaluation of engines or fuels can be minimized. For example, investigation of the effect of compression ratio on knock-limited performance

in full-scale multicylinder engines is impractical because of the cost, time, and effort involved in changing pistons. Curves like those of figure II-13, however, can be established with relative ease by varying inlet-air temperature. From the curve so established, the effect of compression ratio on knock-limited compression density can be calculated from assumed compression temperatures.

Although the simplicity of the compression-density—temperature correlation is greater than that of the end-gas correlations presented in the preceding section of this chapter, the end-gas method has the obvious advantage of compensating for variations in fuel-air ratio as well as for compression ratio and inlet-air temperature.

REFERENCES

1. Rothrock, A. M., and Biermann, Arnold E.: The Knocking Characteristics of Fuels in Relation to Maximum Permissible Performance of Aircraft Engines. NACA Rep. 655, 1939.
2. Serruys, Max: La combustion detonante dans les moteurs a l'explosion. Pub. No. 103, Pub. Sci. et Tech. du Ministere de l'Air, 1937.
3. Gerrish, Harold C., and Voss, Fred: Mixture Distribution in a Single-Row Radial Engine. NACA TN 583, 1936.
4. Taylor, E. S., Leary, W. A., and Diver, J. R.: Effect of Fuel-Air Ratio, Inlet Temperature, and Exhaust Pressure on Detonation. NACA Rep. 699, 1940.
5. Goodenough, G. A.: Principles of Thermodynamics. Henry Holt and Co., 4th ed., 1931.
6. Leary, W. A., and Taylor, E. S.: The Significance of the Time Concept in Engine Detonation. NACA ARR, Jan. 1943.
7. Leary, W. A., Taylor, E. S., Taylor, C. F., and Jovellanos, J. U.: A Rapid Compression Machine Suitable for Studying Short Ignition Delays. NACA TN 1332, 1948.
8. Leary, W. A., Taylor, E. S., Taylor, C. F., and Jovellanos, J. U.: The Effect of Fuel Composition, Compression Pressure, and Fuel-Air Ratio on the Compression-Ignition Characteristics of Several Fuels. NACA TN 1470, 1948.
9. Tower, Leonard K., and Alquist, Henry E.: Correlation of Effects of Fuel-Air Ratio, Compression Ratio, and Inlet-Air Temperature on Knock Limits of Aviation Fuels. NACA TN 2066, 1950.
10. Hershey, R. L., Eberhardt, J. E., and Hottel, H. C.: Thermodynamic Properties of the Working Fluid in Internal-Combustion Engines. SAE Jour. (Trans.), vol. 39, no. 4, Oct. 1936, pp. 409-424.
11. Evvard, John C., and Branstetter, J. Robert: A Correlation of the Effects of Compression Ratio and Inlet-Air Temperature on the Knock Limits of Aviation Fuels in a CFR Engine—I. NACA ACR E5D20, 1945.
12. Alquist, Henry E., O'Dell, Leon, and Evvard, John C.: A Correlation of the Effects of Compression Ratio and Inlet-Air Temperature on the Knock Limits of Aviation Fuels in a CFR Engine—II. NACA ARR E6E13, 1946.
13. Barnett, Henry C.: An Evaluation of the Knock-Limited Performance of Triptane. NACA MR E6B20, 1946.

CHAPTER III

ANTIKNOCK PERFORMANCE SCALES

For many years the efforts of investigators in the field of fuel research have been directed toward the development of suitable scales for measuring the knocking characteristics of fuels. As a result of these efforts, a number of scales have been studied, but only a few have passed beyond the stage of tentative acceptance. The emphasis in this development has generally been placed upon one objective—that the method permit the assignment of a numerical antiknock value to each fuel and that the rating so determined be reasonably constant regardless of the engine or operating condition used.

The achievement of this objective has been hindered because the many fuels suitable for aviation use vary widely in their response to changes in engine conditions and also because the rating scales currently used are unsatisfactory for rating fuels in the high performance range. As a result of these difficulties, attention has been turned in recent years to the problem of establishing a rating scale that will permit determination of ratings for high-performance fuels. An effort has been made in these studies to select reference fuels that will respond to changes in engine conditions in a manner similar to that of service fuels. The material in this chapter is given in greater detail in reference 1.

DEVELOPMENT OF FUEL-RATING SCALES

Octane scale.—One of the earliest and the most enduring knock-rating scales is the familiar octane-number scale proposed by Edgar (reference 2) in 1927. This scale is based on two reference fuels, isooctane (2,2,4-trimethylpentane) and *n*-heptane, suggested because isooctane knocked less and *n*-heptane more than any other fuels then known. In this system, isooctane is arbitrarily assigned a rating (octane number) of 100 and *n*-heptane, a rating of 0. The characteristic shape of curves representing the octane scale is shown in figure III-1 from data obtained in the A. S. T. M. Supercharge engine (A. S. T. M. designation D909-47T).

The octane scale is restricted to the determination of ratings for fuels producing knock-limited power equal to or less than that of isooctane and equal to or greater than that of *n*-heptane. At the lower end of the scale, the limitation offers no serious obstacle because interest in fuels with greater knocking tendencies than *n*-heptane is purely academic. On the other hand, the upper or isooctane end of the scale offers a very serious obstacle inasmuch as many pure hydrocarbons and commercial blending stocks are known to exceed the knock-limited power output of isooctane.

In addition to being limited in use to fuels having antiknock values between those of *n*-heptane and isooctane, the octane scale fails to provide numerical ratings that remain

constant from engine to engine or from one condition to another. That is, a fuel having an octane number of 80 in one engine may have an octane number of 90 in another engine. Investigation has shown, however, that the ratings of paraffinic fuels, with the exception of certain highly branched isomers, tend to remain more nearly constant from engine to engine than do widely different types of hydrocarbon within the range of performance covered by the octane scale (reference 3). This fact can be attributed to the differences in sensitivities of various fuels to changes in operating conditions. Inasmuch as the rating fuels (isooctane and *n*-heptane) are paraffinic, these fuels would be expected to respond to changes in operating conditions in much the same manner as a paraffinic test fuel. If, on the other hand, the test fuel contains large concentrations of aromatic, cycloparaffinic, or olefinic compounds, the response to variations in operating conditions would differ greatly from the behavior of the paraffinic rating fuels.

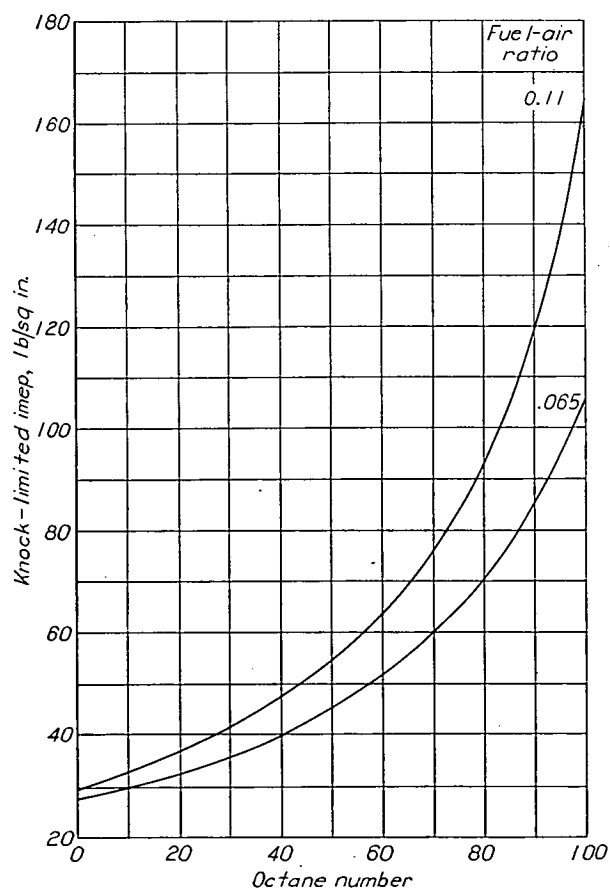


FIGURE III-1.—Octane scale determined on A. S. T. M. Supercharge engine at standard conditions. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 375° F; spark advance, 45° B. T. C. (Fig. 1 of reference 1.)

Lead scale.—When fuels with knock ratings that exceeded the upper limit of the octane scale were first considered for service use, it became apparent that the octane scale was no longer adequate for rating aviation fuels; consequently, the search began for a performance scale that would accommodate the high-performance fuels used in modern aircraft power plants. In order to meet this problem, it was decided that fuels exceeding the performance of isooctane should be matched against isooctane to which had been added a given amount of tetraethyl lead. (See fig. III-2.) The procedure for rating fuels by the lead scale is the same as that used for the octane scale; that is, the knock intensity of the test fuel is determined in a standard engine at standard conditions and the knocking tendencies of blends of isooctane and tetraethyl lead are compared with those of the test fuel until one blend is found to give a knock intensity equal to that of the test fuel.

Performance-number scale.—Neither the octane scale nor the lead scale permitted the assignment of a numerical rating indicative of the power output of a fuel, and in an effort to circumvent this difficulty a scale of performance numbers was adopted. The performance scale (fig. III-3) represents an approximate average of the knock-limited performance as determined in several different engines at differ-

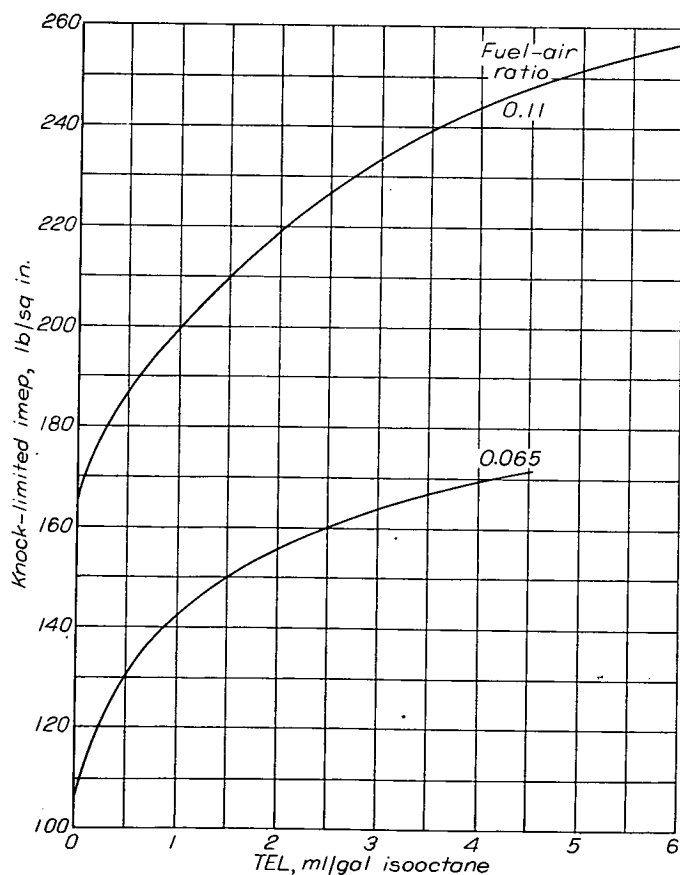


FIGURE III-2.—Lead rating scale determined on A. S. T. M. Supercharge engine at standard conditions. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 375° F; spark advance, 45° B. T. C. (Fig. 2 of reference 1.)

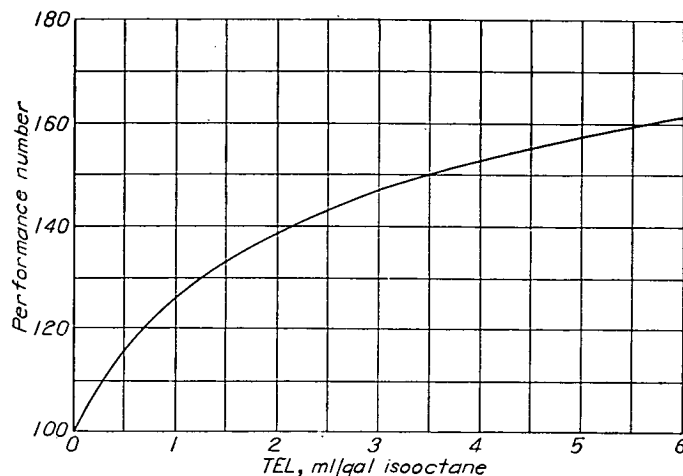


FIGURE III-3.—Performance-number scale. (Fig. 3 of reference 1.)

ent operating conditions for isooctane containing various amounts of tetraethyl lead. Although empirically determined, this scale has proved useful in expressing fuel ratings and in correlating performance of fuels in different engines, but it still does not provide an absolute measure of power output.

The performance-number scale does not alter the procedure for rating fuels by use of leaded isooctane (lead scale) but does change the method of reporting the rating. For example, if a test fuel is found to give knock-limited performance equal to isooctane plus 2.0 ml TEL per gallon in any engine at any condition, its rating is reported as 138 performance number. (See fig. III-3.)

The performance scale as originally adopted was intended for use only with fuels having ratings above an octane number of 100 by the A. S. T. M. Supercharge method. General acceptance of the scale soon led to expression of ratings by the A. S. T. M. Aviation method (A. S. T. M. designation D614-47T) in terms of performance numbers and later resulted in extension of the scale below a performance number of 100.

Proposed rating scales.—One limitation of the octane scale and the lead scale is that the service fuels containing an approximately constant amount of tetraethyl lead (3.0 to 4.6 ml/gal) must be matched against unleaded rating fuels or against rating fuels having variable concentrations of lead. This limitation is inherently related to the problem of fuel sensitivity to changes in operating conditions in that tetraethyl lead affects the sensitivity of the fuel to which it is added.

In order to minimize this effect on fuel ratings, investigators in the field have proposed rating scales in which the reference fuels contain a constant amount of tetraethyl lead in the range of concentrations found in service fuels. Some of these schemes recommend the addition of a third component, such as an aromatic, in the reference fuel blend in order to produce reference fuels having sensitivities more nearly equal to service fuels.

EVALUATION OF PROPOSED FUEL-RATING SCALES

On the basis of studies made by the Coordinating Research Council (reference 4), it was proposed that triptane (2,2,3-trimethylbutane) and *n*-heptane be adopted as the reference fuels for a new rating scale. Both fuels were to contain 3.78 ml TEL per gallon. The studies on which this proposal was based were conducted in A. S. T. M. Aviation and A. S. T. M. Supercharge fuel-rating engines at standard operating conditions. In order to extend the data of the Coordinating Research Council beyond standard operating conditions, an investigation of the proposed rating scale over a range of engine conditions was conducted by the NACA and is reported in reference 1. The results of this investigation of triptane and *n*-heptane were compared with results obtained under similar test conditions for current rating scales and other proposed scales.

The effects of varying engine conditions were simulated by varying inlet-air temperature from 100° to 300° F in the A. S. T. M. Supercharge engine. The combinations of reference fuels investigated were:

1. Isooctane and *n*-heptane
2. Isooctane (containing 0 to 6 ml TEL/gal)
3. Isooctane and *n*-heptane (both containing 3.78 ml TEL/gal).

TABLE III-1.—SERVICE AND SERVICE-TYPE FUELS *

Fuel	Nominal performance grade lean/rich ^b
AN-F-26	76/88
AN-F-28 (28-R)	100/130
Special blend number 1 (SB-1) (23 percent benzene+34 percent virgin base stock+43 percent alkylate+4 ml TEL/gal)	100/140
AN-F-33 (33-R)	115/145
Special blend number 2 (SB-2) (43 percent diisopropyl+12 percent virgin base stock+45 percent alkylate+4 ml TEL/gal)	120/150
Special blend number 3 (SB-3) (34 percent diisopropyl+12.5 percent hot-acid octane+41.5 percent alkylate+12 percent isopentane+4 ml TEL/gal)	130/160
Special blend number 4 (SB-4) (55 percent diisopropyl+8 percent triptane+29 percent alkylate+8 percent isopentane+4 ml TEL/gal)	135/175
Special blend number 5 (SB-5) (62 percent diisopropyl+19 percent triptane+11 percent alkylate+8 percent isopentane+4 ml TEL/gal)	140/200
RAFD-32 (45 percent S reference fuel+45 percent diisopropyl+10 percent isopentane+4.6 ml TEL/gal)	146/175
RAFD-53 (45 percent S reference fuel+45 percent triptane+10 percent isopentane+4.6 ml TEL/gal)	153/192

* Reproduced from reference 1.

^b Lean ratings by A. S. T. M. Aviation method are intended to be indicative of "cruise" performance. Rich ratings by A. S. T. M. Supercharge method are intended to be indicative of take-off performance. (See reference 5.)

4. Triptane and *n*-heptane (both containing 3.78 ml TEL/gal)

5. Triptane and isooctane (both containing 3.78 ml TEL/gal)

In order to compare the merits of the various rating scales, several service and service-type fuels were included in the investigation. (See table III-1.) In choosing these particular fuels and blends, an effort was made to cover a wide range of performance numbers both rich and lean. Inspection properties for these fuels are presented in table III-2.

Although complete knock-limited mixture-response curves were obtained in the A. S. T. M. Supercharge engine, the analysis of results presented (reference 1) is restricted to two fuel-air ratios, 0.065 and 0.11. Examples of data at these two fuel-air ratios are shown in figure III-4.

The points on the curves in these figures are cross-plotted from mixture-response curves.

Antiknock ratings made by the A. S. T. M. Aviation method for the various reference-fuel blends are presented in table III-3.

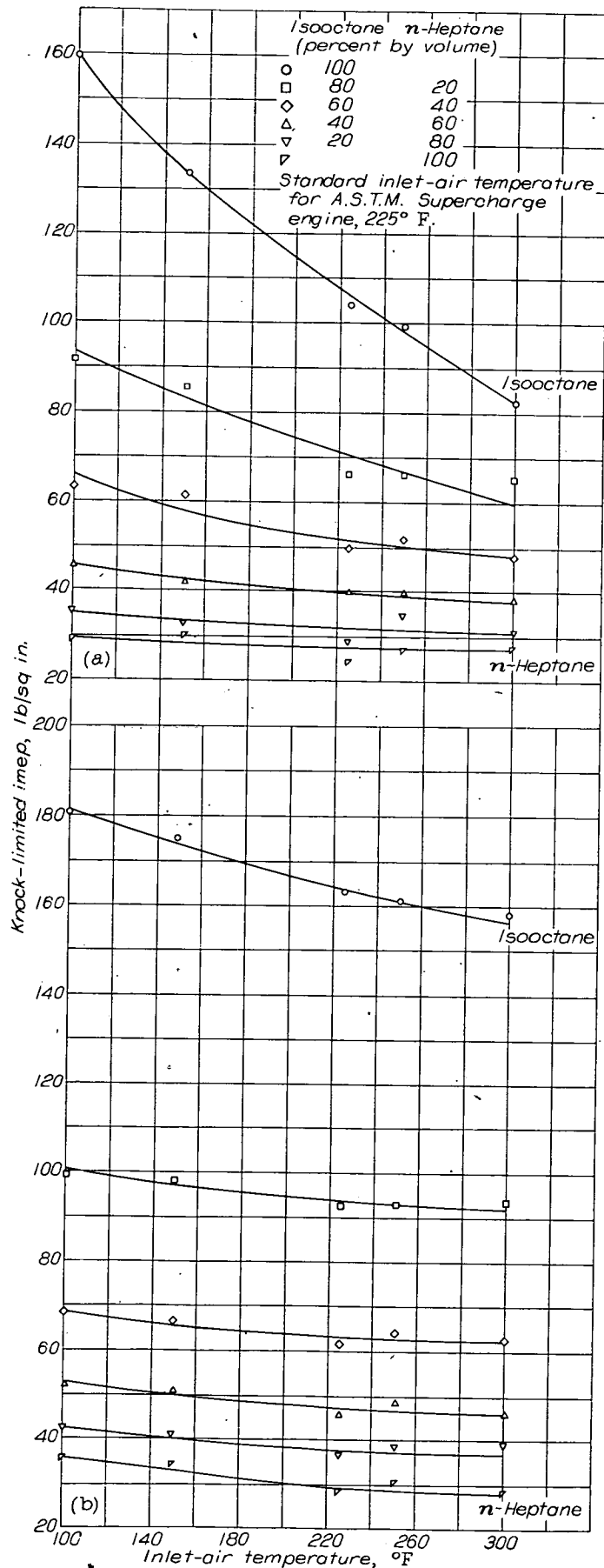
Comparisons among various reference scales.—In figure III-5 the various reference scales are compared at standard A. S. T. M. Aviation and A. S. T. M. Supercharge operating conditions. It is seen in figure III-5 (a) that for A. S. T. M. Aviation conditions the data for leaded blends of isooctane and *n*-heptane fall on a 45° line, which indicates that the A. S. T. M. Aviation (lean) rating of isooctane containing 3.78 ml TEL per gallon is equal to that of triptane containing the same quantity of tetraethyl lead. Conversely, data at A. S. T. M. Supercharge (rich) conditions (fig. III-5 (b)) show that for equal knock-limited indicated mean effective pressures more isooctane is required than triptane in leaded blends with *n*-heptane. In figure III-5 (b) it is shown that between 80 and 84 percent triptane in a blend with *n*-heptane (leaded) would be required to equal the upper limit (161) of the current performance scale.

An additional observation can be made from figure III-5: namely, that for A. S. T. M. Aviation conditions the two-component reference-fuel systems with constant tetraethyl-lead concentrations are related by straight lines, whereas at A. S. T. M. Supercharge conditions definite curvature is obtained.

TABLE III-2.—INSPECTION DATA FOR SERVICE AND SERVICE-TYPE FUELS *

Fuel designation	AN-F-26	AN-F-28 (28-R)	AN-F-33 (33-R)	SB-1	SB-2	SB-3	SB-4	SB-5	RAFD-52	RAFD-53
A. S. T. M. distillation:										
Initial boiling point, (°F).....	136	104	96	122	122	106	124	122	120	140
10 percent evaporated, (°F).....	160	145	134	136	144	134	136	136	142	165
40 percent evaporated, (°F).....	150	192	197	145	160	160	145	144	160	185
50 percent evaporated, (°F).....	190	209	217	152	170	170	152	145	166	190
90 percent evaporated, (°F).....	236	273	274	224	242	240	224	150	204	200
End point, (°F).....	320	338	349	342	350	340	342	290	236	232
Residue, percent.....	0.9	0.9	1.0	0.8	1.0	1.0	1.2	0.9	1.0	0.4
Loss, percent.....	1.1	-----	1.0	1.2	1.3	0.5	1.2	0.6	1.0	0.06
TEL content, (ml/gal).....	4.95	4.36	4.53	4.02	4.17	4.39	4.11	3.96	4.52	4.69
Copper dish gum, (mg/100 ml).....	6.9	2.9	5	10.1	0.8	6.3	1.5	16.6	1.1	3.3
Freezing point, (°F).....	<-76	<-76	<-76	-36.4	<-76	<-76	<-76	<-76	<-76	<-76
Hydrogen-carbon ratio.....	0.177	0.174	0.182	0.156	0.192	0.189	0.194	0.191	0.187	0.188
Net heat of combustion, (Btu/lb).....	19,103	18,700	18,900	18,517	19,196	19,155	19,330	19,236	19,247	19,373

* Table II of reference 1.



(a) Fuel-air ratio, 0.065

(b) Fuel-air ratio, 0.11.

FIGURE III-4.—Variation of knock-limited performance of unleaded blends of isooctane and *n*-heptane with inlet-air temperature. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 375° F; spark advance, 45° B. T. C. (Fig. 5 of reference 1.)

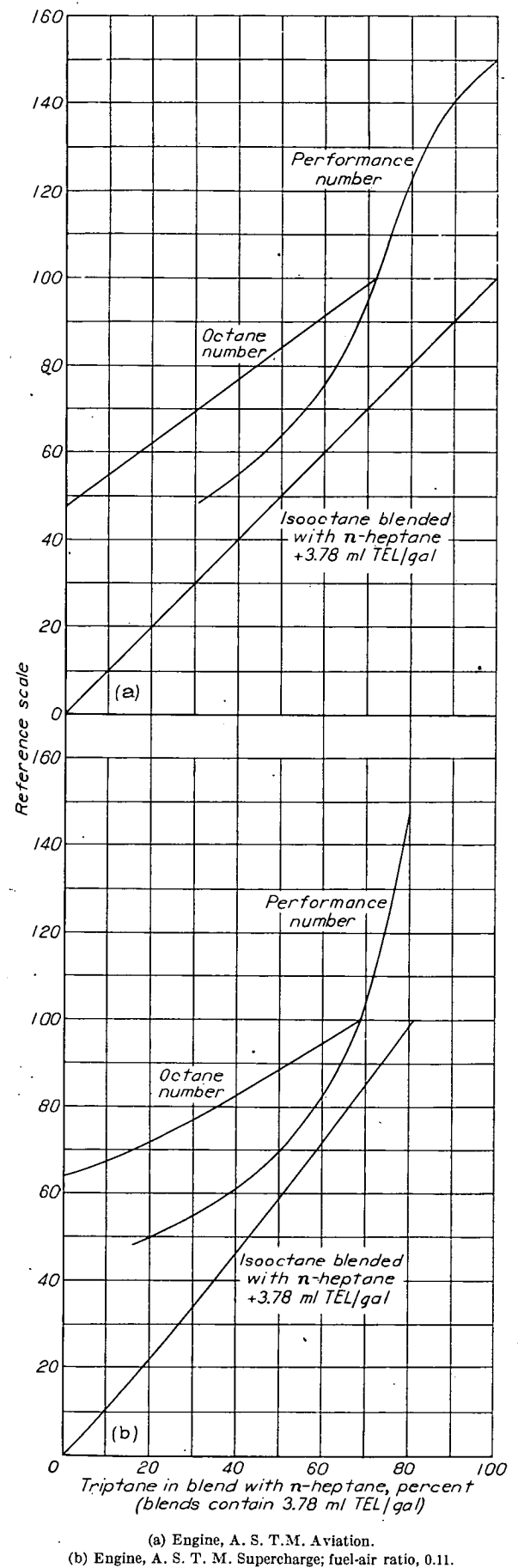
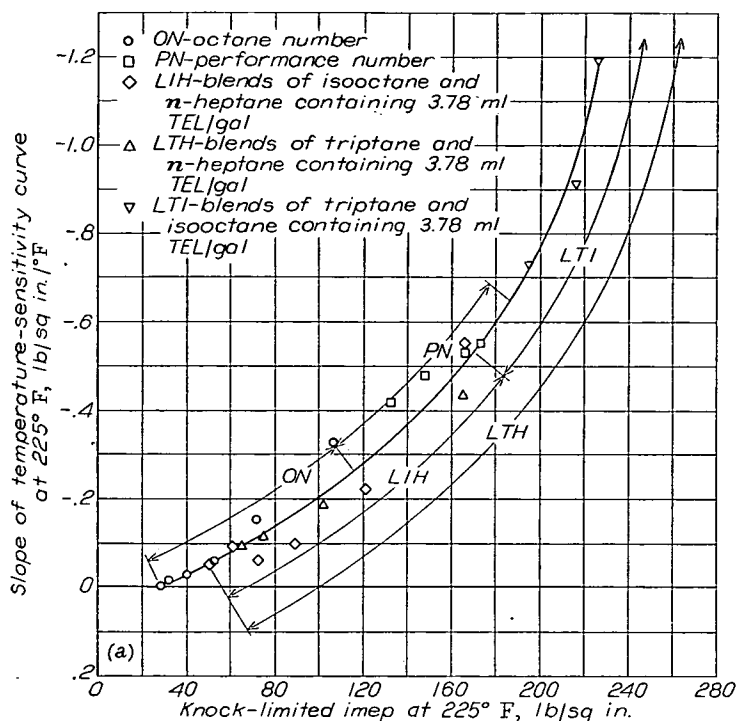
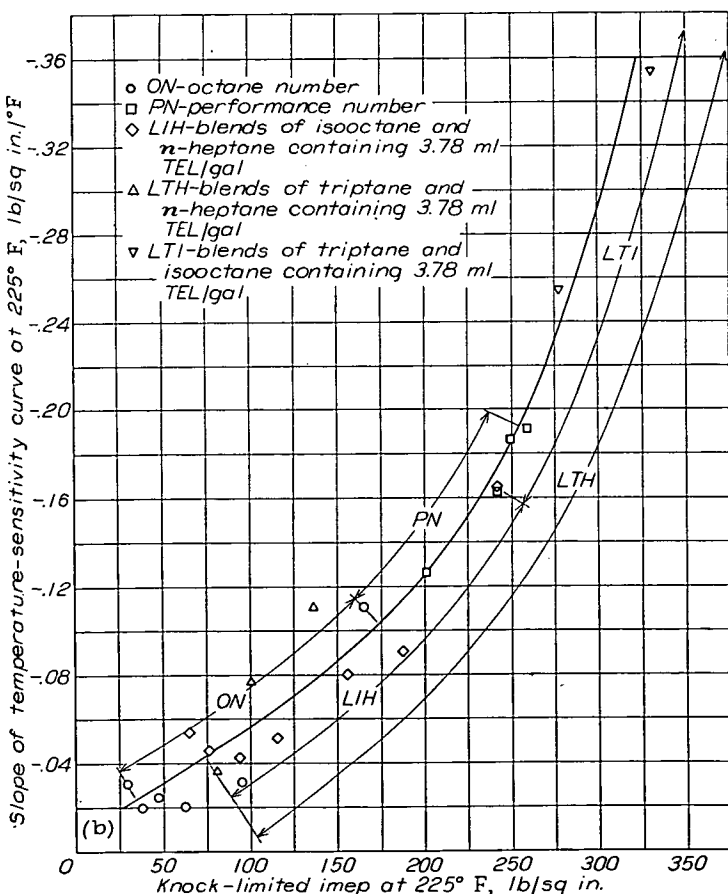


FIGURE III-5.—Relations among various rating scales. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature 375° F; spark advance, 45° B. T. C.; fuel-air ratio, 0.11. (Fig. 15 of reference 1.)



(a) Fuel-air ratio, 0.065.



(b) Fuel-air ratio, 0.11.

FIGURE III-6.—Variation of temperature sensitivity with knock-limited performance for various reference-fuel blends in A. S. T. M. Supercharge engine. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 375° F; spark advance, 45° B. T. C. (Fig. 16 of reference 1.)

TABLE III-3.—A. S. T. M. AVIATION RATINGS FOR REFERENCE-FUEL BLENDS ^a

Composition, percent by volume ^b			A. S. T. M. Aviation Rating	
n-Heptane	Isooctane	Triptane	Octane number or isooctane+TEL	Performance number
100	0	-----	47	-----
80	20	-----	60	-----
60	40	-----	76.3	54
40	60	-----	85.3	65.5
20	80	-----	14	105
0	100	-----	3.78	151.3
-----	80	20	4.0	152.5
-----	60	40	4.0	152.5
-----	40	60	4.0	152.5
-----	20	80	4.4	154.5
-----	0	100	3.6	150.5
80	-----	20	62	-----
60	-----	40	76.8	54.5
40	-----	60	90.8	75.3
20	-----	80	85	123

^a Table III of reference 1.

^b All blends contained 3.78 ml TEL/gal.

Temperature sensitivity of reference fuels.—As a measure of the temperature sensitivity of the reference-fuel combinations, the slopes of curves similar to those in figure III-4 for all the reference-fuel combinations were computed at an inlet-air temperature of 225° F (standard A. S. T. M. Supercharge conditions) and plotted against the knock-limited mean effective pressures at the same temperature. These plots were made for fuel-air ratios of 0.065 and 0.11 and are shown in figure III-6. A single curve was drawn through each set of points (lean and rich) to illustrate the variation of temperature sensitivity with knock-limited performance for the various reference-fuel blends. The different portions of the curve determined by individual reference-fuel combinations are also indicated on figure III-6.

Above the upper limit of the current performance-number scale (fig. III-6), the sensitivity increases rapidly as the knock-limited indicated mean effective pressure increases. If this trend is generally true for service-type fuels, it is apparent from this figure that such fuels having high knock limits should be matched against leaded blends of triptane and isooctane or triptane and n-heptane in order to have reference fuels and service fuels of comparable sensitivity. For lower performance fuels, however, any of the reference scales indicated would be suitable from sensitivity considerations.

Temperature sensitivity of service-type fuels.—The effects of inlet-air temperature on the knock-limited performance of the service-type fuels (table III-1) were also investigated in the A. S. T. M. Supercharge engine. The slopes of the temperature-sensitivity curves (reference 1) were computed at 225° F and plotted in the same manner described for figure III-6. The result is shown in figure III-7 with the curves from figure III-6 reproduced for comparison.

The points representing service fuels (fig. III-7) lie primarily above the curves established for the reference fuels. This divergence is attributed to the fact that the service fuels are blends containing a number of different components; whereas the reference-fuel blends are composed of only two components. Insofar as a perfect rating scale is concerned, the ideal condition would be for the service-fuel data to fall directly on the reference-fuel curve.

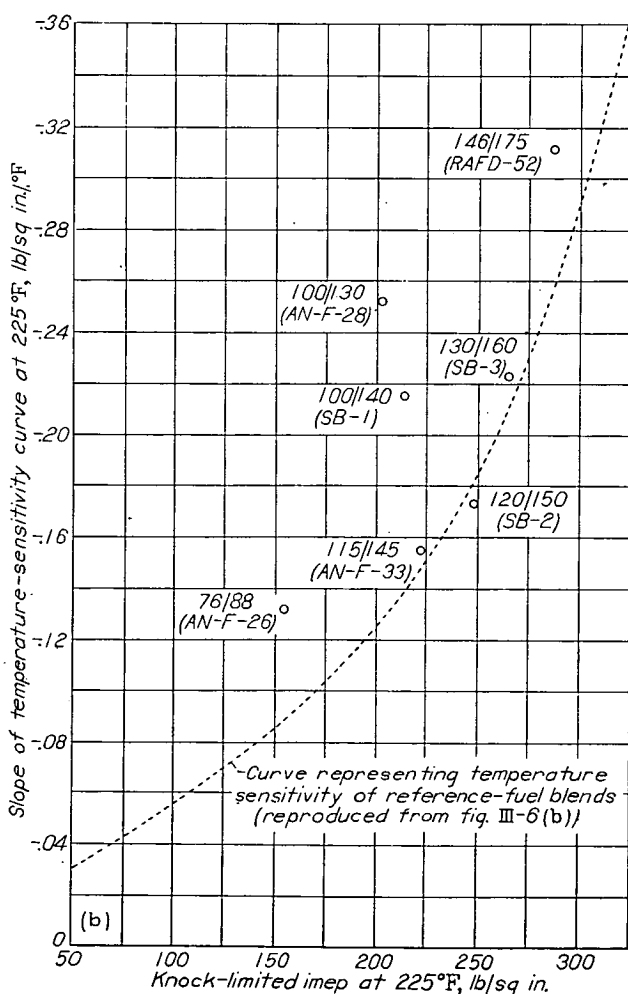
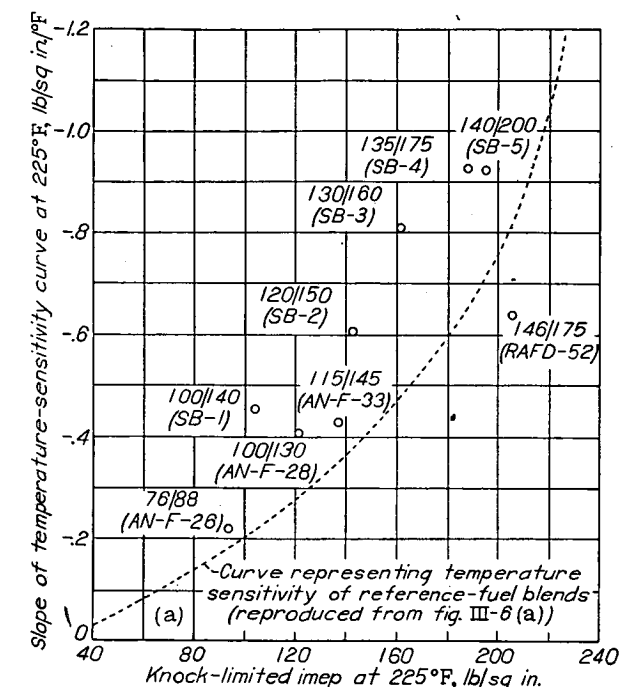


FIGURE III-7.—Comparison among the temperature sensitivities of reference-fuel blends and service-type fuels in A. S. T. M. Supercharge engine. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 375° F; spark advance, 45° B. T. C. (Fig. 18 of reference 1.)

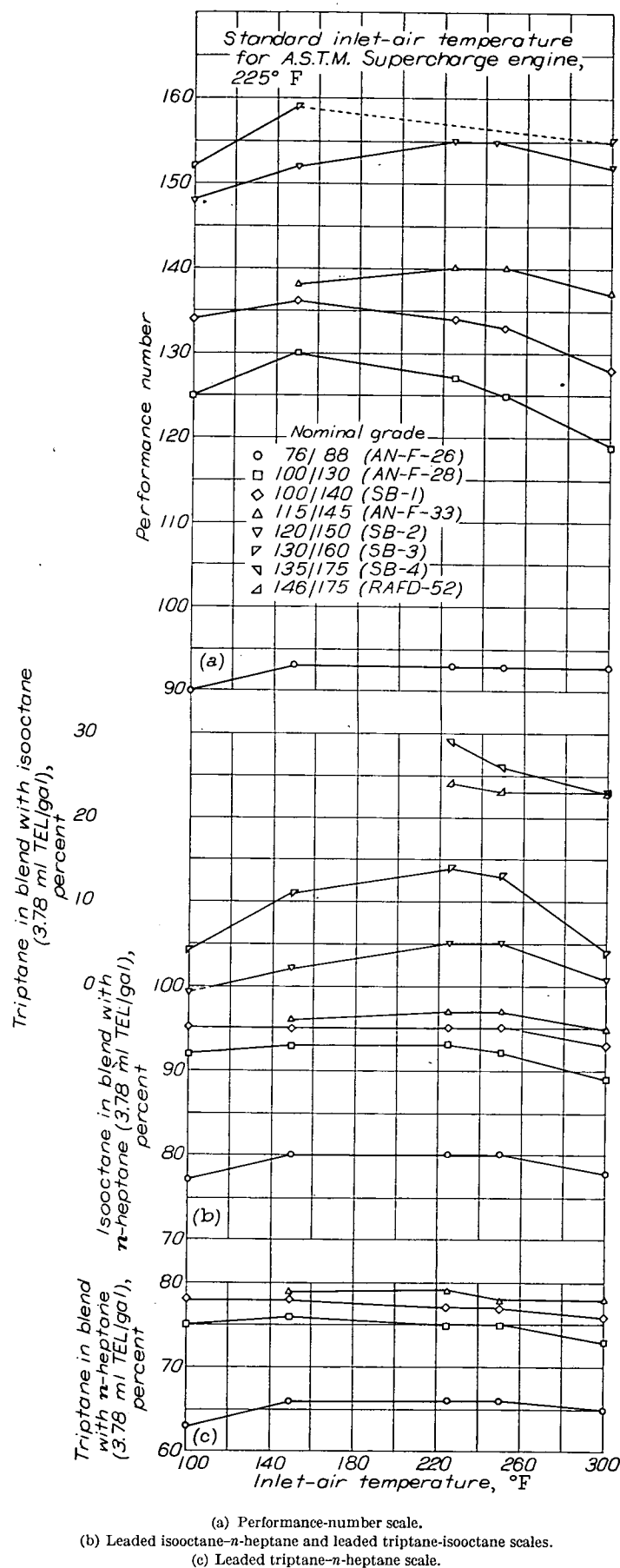


FIGURE III-8.—Effect of inlet-air temperature on antiknock ratings of service-type fuels in A. S. T. M. Supercharge engine at rich fuel-air ratio. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 375° F; spark advance, 45° B. T. C.; fuel-air ratio, 0.11. (Fig. 20 of reference 1.)

TABLE III-4.—A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE RATINGS OF SERVICE AND SERVICE-TYPE FUELS IN TERMS OF SEVERAL REFERENCE-FUEL COMBINATIONS *

Fuel designation and nominal grade	Rating scale ^b	A. S. T. M. Aviation rating ^c	A. S. T. M. Supercharge rating at fuel-air ratio of 0.065					A. S. T. M. Supercharge rating at fuel-air ratio of 0.11				
			Inlet-air temperature, °F					Inlet-air temperature, °F				
			100°	150°	225°	250°	300°	100°	150°	225°	250°	300°
AN-F-26 (76/88)	Imep	---	134	114	94	88	79	159	163	156	152	145
	ON	91	95	94	95	95	97	97	98	98	98	98
	PN	76	85	82	85	85	90	90	93	93	93	93
	LIH	60	72	69	64	61	55	77	80	80	80	78
	LTH	60	60	58	56	55	53	63	66	66	66	65
AN-F-28 (100/130)	LTI	---	---	---	---	---	---	---	---	---	---	---
	Imep	---	180	154	121	111	92	216	214	202	195	180
	ON	---	---	---	---	---	---	---	---	---	---	---
	PN	101	112	110	110	110	109	125	130	127	125	119
	LIH	72	86	84	81	78	68	92	93	93	92	89
SB-1 (100/140)	LTH	72	72	71	68	67	62	75	76	75	75	73
	LTI	---	---	---	---	---	---	---	---	---	---	---
	Imep	---	184	148	104	95	83	231	225	212	207	193
	ON	99.3	---	---	99	99	100	---	---	---	---	---
	PN	98	114	107	97	97	100	134	136	134	133	128
AN-F-33 (115/145)	LIH	71	87	83	72	67	59	95	95	95	95	93
	LTH	71	73	69	61	59	56	78	78	77	77	75
	LTI	---	---	---	---	---	---	---	---	---	---	---
	Imep	---	229	178	137	126	107	---	229	222	218	205
	ON	---	---	---	---	---	---	---	---	---	---	---
SB-2 (120/150)	PN	115	138	126	120	121	122	---	138	140	140	137
	LIH	77	95	91	89	87	81	---	96	97	97	95
	LTH	77	80	76	73	72	70	---	79	79	78	77
	LTI	---	---	---	---	---	---	---	---	---	---	---
	Imep	---	277	212	143	131	109	260	257	249	244	231
SB-3 (130/160)	ON	---	---	---	---	---	---	---	---	---	---	---
	PN	121	157	148	125	125	124	148	152	155	155	152
	LIH	79	---	99	91	89	83	99	---	---	---	---
	LTH	79	---	---	75	74	70	---	---	---	---	---
	LTI	---	7	---	---	---	---	---	2	5	5	1
SB-4 (135/175)	Imep	---	292	235	162	143	118	270	270	265	258	235
	ON	---	---	---	---	---	---	---	---	---	---	---
	PN	134	---	156	147	139	135	152	159	---	---	155
	LIH	86	---	---	99	95	90	---	---	---	---	---
	LTH	86	---	---	79	77	74	---	---	---	---	---
SB-5 (140/200)	LTI	---	16	9	---	---	---	4	11	14	13	4
	Imep	---	---	267	195	173	133	---	---	---	---	299
	ON	---	---	---	---	---	---	---	---	---	---	---
	PN	143	---	---	---	---	155	---	---	---	---	---
	LIH	92	---	---	---	---	100	---	---	---	---	---
RAFD-52 (146/175)	LTH	92	---	---	---	---	80	---	---	---	---	---
	LTI	---	---	27	20	16	2	---	---	---	---	44
	Imep	---	---	247	205	188	137	---	---	287	279	262
	ON	---	---	---	---	---	---	---	---	---	---	---
	PN	147	---	160	---	---	---	---	---	---	---	---
RAFD-53 (153/192)	LIH	96	---	---	---	---	---	---	---	---	---	---
	LTH	96	---	---	---	---	---	---	---	---	---	---
	LTI	---	---	15	29	31	6	---	---	24	23	23
	Imep	---	---	---	246	189	147	---	---	---	---	312
	ON	---	---	---	---	---	---	---	---	---	---	---
	PN	147	---	---	---	---	---	---	---	---	---	---
	LIH	96	---	---	---	---	---	---	---	---	---	---
	LTH	96	---	---	---	---	---	---	---	---	---	---
	LTI	---	---	---	---	32	16	---	---	---	---	50

* Table IV of reference 1.

^b Imep, knock-limited indicated mean effective pressure; ON, octane number; PN, performance number; LIH, blend of iso-octane and *n*-heptane containing 3.78 ml TEL/gal; LTH, blend of triptane and *n*-heptane containing 3.78 ml TEL/gal; LTI, blend of triptane and iso-octane containing 3.78 ml TEL/gal.^c Octane numbers and performance numbers obtained by the standard A. S. T. M. Aviation rating procedure. Ratings in terms of other scales estimated from fig. III-5(a).

Because this situation is improbable, the next best solution must be for reference-fuel blends and service fuels to have sensitivities of approximately the same magnitude at the same knock-limited performance levels. The attainment of this characteristic can be approached by the use of either the joint reference scale, leaded triptane-iso-octane and leaded iso-octane-*n*-heptane, or by the leaded triptane-*n*-heptane scale (fig. III-7).

In evaluating the merits of a rating scale, however, three factors other than temperature sensitivity must be considered, namely that the scale should:

(1) Be a continuous scale

(2) Involve a minimum number of reference fuels

(3) Cover the complete range of knock-limited performance likely to be encountered with a variety of service fuels.

In figures III-6 and III-7, it has been shown that both the leaded triptane-*n*-heptane scale and the joint scale of leaded iso-octane-*n*-heptane and leaded triptane-iso-octane have the last of these characteristics. Only the leaded triptane-*n*-heptane scale, however, possesses the first two characteristics as well; consequently, this scale appears to be the most practical choice. The problem of availability of reference fuels is obviously important but is beyond the scope of the investigation reported in reference 1.

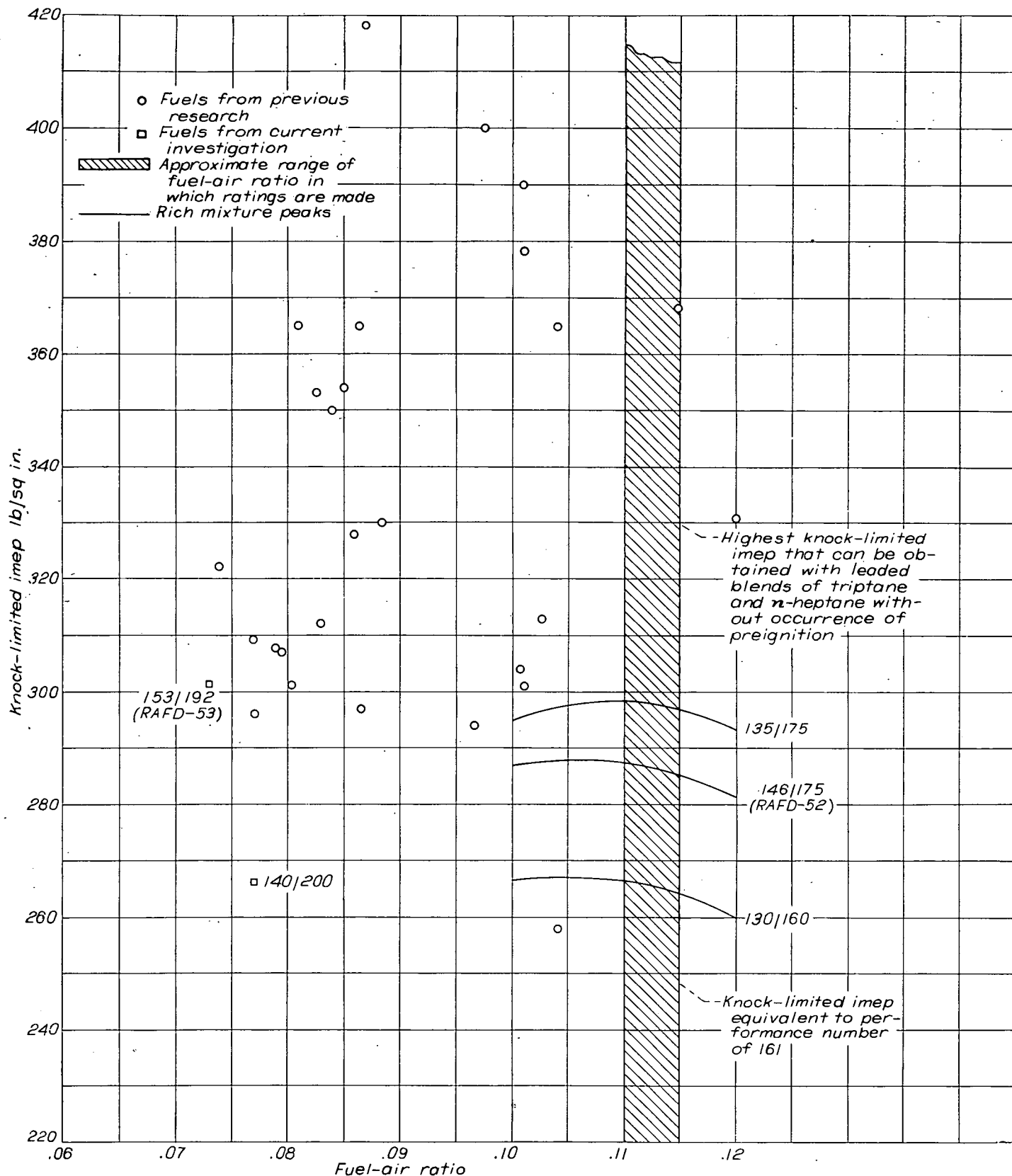


FIGURE III-9.—Maximum knock-limited mean effective pressures attainable for various fuels by A. S. T. M. Supercharge method without encountering preignition. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 375° F; spark advance, 45° B. T. C. (Fig. 21 of reference 1.)

Ratings of service-type fuels.—The 10 service-type fuels were assigned ratings in terms of the 5 reference scales examined in this investigation. These ratings are given in table III-4 for standard A. S. T. M. Aviation conditions and for A. S. T. M. Supercharge conditions at five inlet-air temperatures and two fuel-air ratios. The ratings in this table were determined for comparing the constancy of the assigned ratings by the different rating scales over a range of conditions established by varying inlet-air temperature. This comparison is presented in figure III-8 for the A. S. T. M. Supercharge engine at a fuel-air ratio of 0.11.

These figures alone cannot serve as a basis for choosing the best reference scale because the constancy of an assigned rating from one set of operating conditions to another is largely dependent upon the relative sensitivities of reference fuels and test fuels previously discussed.

Limitations of reference scales.—In the preceding discussion, it has been stated that in many cases knock-limited indicated mean effective pressures were not obtained for some of the higher-performance service-type and reference fuels investigated in the A. S. T. M. Supercharge engine. The attainment of complete data for high-performance fuels was hampered throughout the investigation by the tendency of these fuels to preignite in the A. S. T. M. Supercharge engine. The engine operator at all times obtained as much of the mixture-response curve as possible before the start of preignition (see ch. IV); for some fuels, however, points richer than a fuel-air ratio of about 0.08 could not be taken. Inasmuch as the peaks of the curves were not obtained, ratings could not be made in the fuel-air-ratio range characteristic of the standard A. S. T. M. Supercharge method.

Preignition occurred with the high-performance fuels regardless of the condition of the engine; changing the type of spark plug failed to prevent preignition.

Preignition was encountered so frequently that early data obtained in the NACA laboratories were examined in order to define the range of indicated mean effective pressure in which preignition is likely to occur in the A. S. T. M. Supercharge engine. The results of this study are presented in figure III-9, together with data from the present investigation. Each point on this plot represents the richest point of a mixture-response curve that could be obtained before the occurrence of preignition. It is seen in figure III-9 that with several fuels it was possible to reach preignition-free performance at indicated mean effective pressures greater than 400 pounds per square inch; however, a number of fuels preignited at levels below 400 down to about 258 pounds per square inch. In most cases, the limiting fuel-air ratios were leaner than the fuel-air-ratio range in which A. S. T. M. Supercharge ratings are usually made.

Inasmuch as preignition imposes a very real limit on an A. S. T. M. Supercharge rating scale at a rich fuel-air ratio, the maximum indicated mean effective pressures that can be obtained with current and proposed reference scales are indicated in figure III-9. The difference between scales is about 75 pounds per square inch indicated mean effective pressure; the adoption of either reference scale utilizing triptane as one of the components will thus extend the range in which "direct-match" ratings can be made by about 75 pounds per square inch. Even then there will be some fuels (as indicated by preignition test points in fig. III-9) that cannot be rated in this range. In fact, only three of the five service-type fuels exceeding a performance number of 161 could be rated (shown as solid lines between fuel-air ratios of 0.10 and 0.12 in fig. III-9). The remaining two fuels having performance numbers over 161 preignited at fuel-air ratios leaner than 0.08 (indicated by the square data points on fig. III-9).

If the A. S. T. M. Aviation and A. S. T. M. Supercharge engines are to continue in use as the standard fuel-rating engines, the advantages to be gained by the adoption of new reference-fuel systems utilizing triptane are clearly questionable, as pointed out by the authors of reference 1. The present scale of performance numbers has been shown to permit ratings for fuels up to a performance number of 161 in the A. S. T. M. Aviation engine; triptane plus 3.78 ml TEL per gallon, which represents the maximum performance of either of the proposed reference-fuel combinations, permits ratings only up to a performance number of 151. Moreover, in the A. S. T. M. Supercharge engine the tendencies of many high-performance service-type fuels and high-performance reference-fuel blends to preignite makes the advantage of extending the range in which ratings can be made in this engine somewhat uncertain.

REFERENCES

1. Barnett, Henry C., and Clarke, Thomas G.: An Evaluation of Proposed Reference Fuel Scales for Knock Rating. NACA TN 1619, 1948.
2. Edgar, Graham: Measurement of Knock Characteristics of Gasoline in Terms of a Standard Fuel. *Ind. and Eng. Chem.*, vol. 19, no. 1, Jan. 1927, pp. 145-146.
3. Wear, Jerrold D., and Sanders, Newell D.: Experimental Studies of Knock-Limited Blending Characteristics of Aviation Fuels. II—Investigation of Leaded Paraffinic Fuels in an Air-Cooled Cylinder. NACA TN 1374, 1947.
4. Brooks, Donald B.: Development of Reference Fuel Scales for Knock Rating. *SAE Jour. (Trans.)*, vol. 54, no. 8, Aug. 1946, pp. 394-402.
5. Anon.: ASTM Manual of Engine Test Methods for Rating Fuels. A. S. T. M. (Philadelphia), 1948.

CHAPTER IV

PREIGNITION

Over a period of years research directed toward increasing engine power has been paralleled by research in engine cooling. This pattern of investigation arises from the occurrence of mechanical failures in engine cylinders when certain established temperature limits are exceeded. Opinions vary as to the principal danger of high cylinder temperatures; however, final cylinder failure obviously results from high temperatures, high pressures, or both (reference 1). The initial cause of failure may be knock, preignition, or failure of sealing devices, such as piston rings and valves, but the sequence in which the events leading up to failure occur will vary with operating conditions, design features, and fuel.

In an effort to clarify the exact roles played by these factors in service operation, this chapter presents a discussion of NACA investigations in which the characteristics of preignition are studied and the relation of preignition and knock to engine temperatures is analyzed.

CHARACTERISTICS OF PREIGNITION

Preignition and knock.—A hot spot in an engine cylinder may cause ignition of the fuel-air charge in the cylinder prior to ignition by the spark plug. This phenomenon is known as preignition. Preignition and knock are separate phenomena, as indicated by Rothrock and Biermann (reference 2). If, as suggested by this investigation, the occurrence of knock depends upon the density and the temperature of the last portion of the charge to burn, it is reasonable to assume that preignition which causes the end gas to burn nearer top center also tends to promote fuel knock, whereas preignition which causes the end gas to burn well before top center may lessen the possibility of fuel knock. In reference 1, this theory is thus concluded to provide one explanation of why preignition is often accompanied by knock and why, in other cases, in which preignition may be severe enough to cause immediate engine failure, fuel knock may not be present. A specific case of this nature is reported in reference 3.

Run-away and stable preignition.—Run-away and stable preignition are defined by Hundere and Bert (reference 4) in terms of the rate of increase of the hot-spot temperature and the required temperature for surface ignition. If, for example, the hot-spot temperature increases much more rapidly than the required surface temperature for ignition, run-away preignition occurs. When this situation exists, the ignition of the fuel-air charge may advance more and more until ignition takes place in the intake duct; a backfire then occurs.

On the other hand, stable preignition is obtained when the required surface temperature for ignition increases more rapidly than the actual hot-spot temperature. A point is then reached where the required temperature and the actual temperature are equal, and stable ignition occurs. In this case, the ignition system may be inoperative and the engine will operate normally; ignition is then accomplished solely by the hot spot in the cylinder.

Preignition characteristics vary with fuel type; for some fuels, such as benzene, preignition may become severe enough to wreck an engine cylinder so quickly that no appreciable increase in temperature or decrease in engine output can be observed.

EFFECTS OF ENGINE VARIABLES ON PREIGNITION

An investigation reported by Corrington and Fisher (reference 5) was undertaken to obtain information on the behavior of an engine during preignition operation. Spark plugs of various heat ranges were used as sources of preignition in most of the tests of reference 5. In some cases, however, preignition was excited by exhaust valves. A valve having a badly corroded head was installed in place of the Nichrome coated valves normally employed; cold-operating spark plugs were used.

Preignition was excited at various fuel-air ratios, power levels, engine speeds, and mixture temperatures.

The investigation reported in reference 5 was made in a single, liquid-cooled engine cylinder. The setup consisted of a multicylinder engine block mounted in such a manner that any one cylinder could be used. Details of the installation and the methods of instrumentation are described in references 5 and 6.

Reproducibility of preignition runs.—Cylinder design and the location and heat capacity of the preignition source are factors that influence the behavior of an engine during preignition operation. Reproducibility of preignition data will be imperfect, however, even if these factors and all operating conditions are held constant. For the studies in which spark plugs initiated preignition (reference 5), the cause of irreproducibility was the time required for the preignition to advance from normal spark timing to a point about 60° B. T. C. On occasional cycles, a run would start with ignition a few degrees early. The ignition time became earlier as the cycles with early ignition occurred more frequently. Cylinder temperature rose slowly during this period of operation. When the ignition time had advanced to about 60° B. T. C., the preignition process was greatly accelerated and successive runs were fairly reproducible. The time required was 30 to 60 seconds for preignition to become advanced to this critical value (60° B. T. C.). It is suggested in reference 5 that this variation in time to reach the critical value may result either from changes in condition of the spark plug due to deposits and corrosion or errors in setting the engine conditions.

Typical preignition runs.—Data from typical preignition runs are shown in figure IV-1. The location of the zero point on the abscissa has no significance inasmuch as the curves were adjusted on the time scale so that the points coincided where preignition became rapidly accelerated. This arrangement eliminates the period of poor reproducibility from consideration.

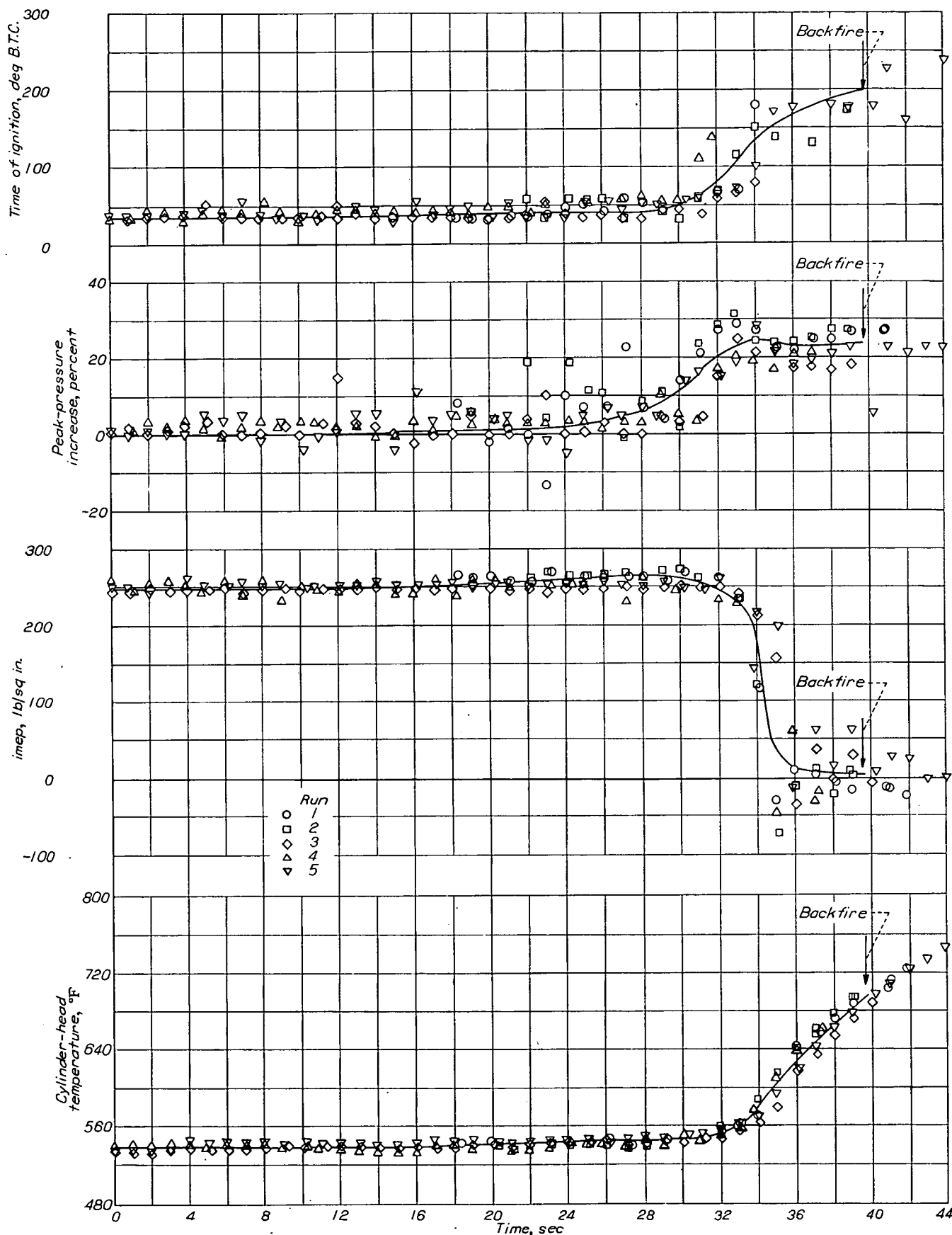


FIGURE IV-1.—Typical preignition runs indicating reproducibility of data in full-scale single-cylinder engine. Compression ratio, 6.65; engine speed, 3000 rpm; mixture temperature, 175° F. coolant temperature, 250° F; spark advance: inlet, 28° B. T. C.; outlet, 34° B. T. C.; fuel-air ratio, 0.070; preignition source, spark plug in exhaust side. (Fig. 4 of reference 5.)

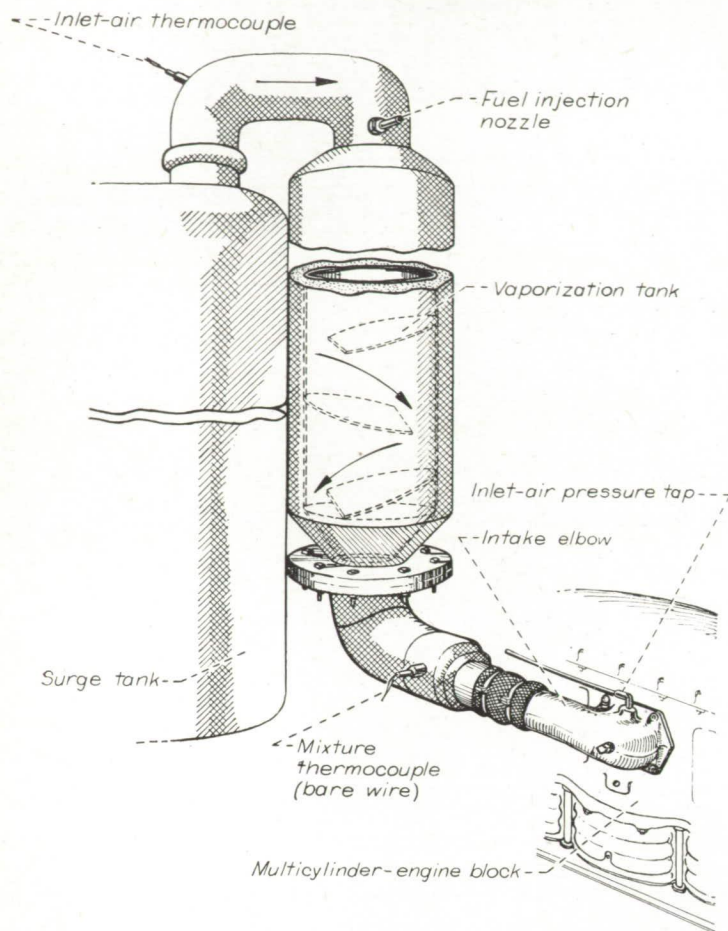


FIGURE IV-2.—Full-scale single-cylinder test-engine induction system. (Fig. 2 of reference 5.)

The curves in figure IV-1 represent the averages of the individual runs and may be considered typical of engine operation during preignition at a given set of conditions. All curves discussed in succeeding paragraphs have been obtained in the same manner from several runs at each operating condition.

The preignition run in figure IV-1 was terminated by backfiring, as was true for most of the runs conducted in reference 5. Backfiring occurs when the time of ignition is considerably earlier than that at which the intake valves close (118° B. T. C.). The flame cannot pass into the induction system unless a pressure rise occurs sufficient to slow down or to reverse the flow through the intake valves.

After a backfire, the fuel-air charge in the vaporization tank (fig. IV-2) burned and the resulting pressure rise caused reversal of the air flow at the tank entrance. Because the fuel was injected at this point, the flame burned out quickly. Combustion in the cylinder was reestablished after about 1 second, but the preignition source was sufficiently hot to cause early ignition again. Backfiring continued at intervals until a change in the operating conditions cooled the preignition source.

At high power output and fuel-air ratios leaner than 0.095, preignition caused backfiring. Preignition was stable at richer mixtures and ignition occurred near bottom center.

Effect of fuel-air ratio on preignition.—Fuel-air ratio, because of its influence on combustion temperatures and rate

of burning, is one of the important variables to be considered during preignition operation. The effect of fuel-air ratio on preignition is illustrated in figure IV-3. As seen in figure IV-3, the time of ignition at the lean fuel-air ratio (0.070) advanced to about 220° B. T. C. (40° B. B. C.) about 7 seconds after the start of the rapidly advancing preignition period. The time of ignition at the start of this period was about 48° B. T. C. At the rich fuel-air ratio (0.099), the time of ignition advanced less rapidly than at the lean fuel-air ratio and became stable at about 150° to 160° B. T. C. No backfire occurred at the rich fuel-air ratio.

The increase in peak pressure was about 20 percent at the lean fuel-air ratio and 25 percent at the rich fuel-air ratio; according to reference 5, however, these increases are in doubt because of scatter in the experimental data.

The power output (fig. IV-3) dropped sharply during the period of preignition. During the same period the cylinder-head temperatures increased sharply. As pointed out in reference 5, the piston temperature might be expected to follow curves similar to those shown for the cylinder-head temperature with the possibility that the temperature rise might be even greater.

Effect of power output on preignition.—The effect of engine power level on preignition is shown in figure IV-4. In order to obtain these data, it was necessary (reference 5) to use spark plugs of three different heat ranges. The results of these tests indicate that the time before backfiring increased as the power level decreased. This result is consistent with the decrease in cyclic temperatures as the power decreased, as indicated by the cylinder-head temperatures (fig. IV-4). The percentage increase in peak cylinder pressures increased as the power level decreased.

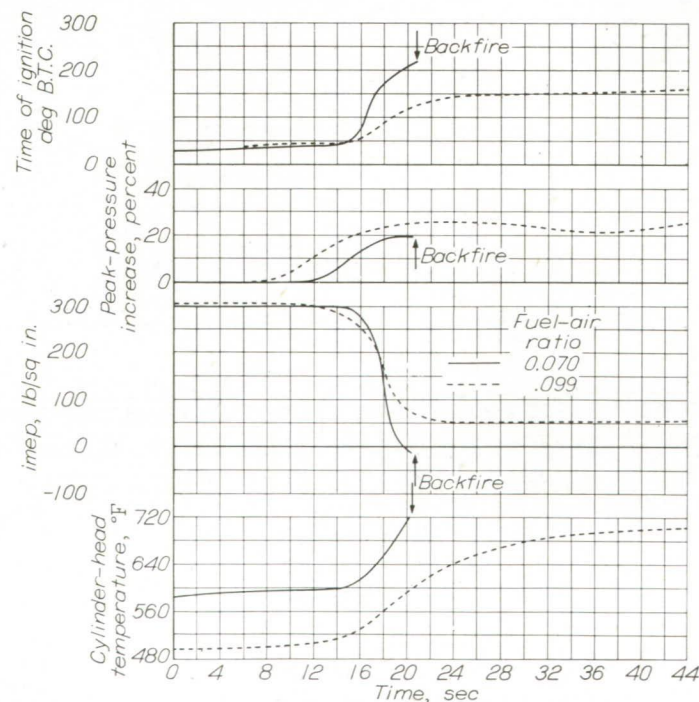


FIGURE IV-3.—Effect of fuel-air ratio on engine behavior during preignition. Compression ratio, 6.65; engine speed, 3000 rpm; mixture temperature, 175° F; coolant temperature, 250° F; spark advance: inlet, 28° B. T. C.; outlet, 34° B. T. C.; preignition source, spark plug in exhaust side. (Fig. 5 of reference 5.)

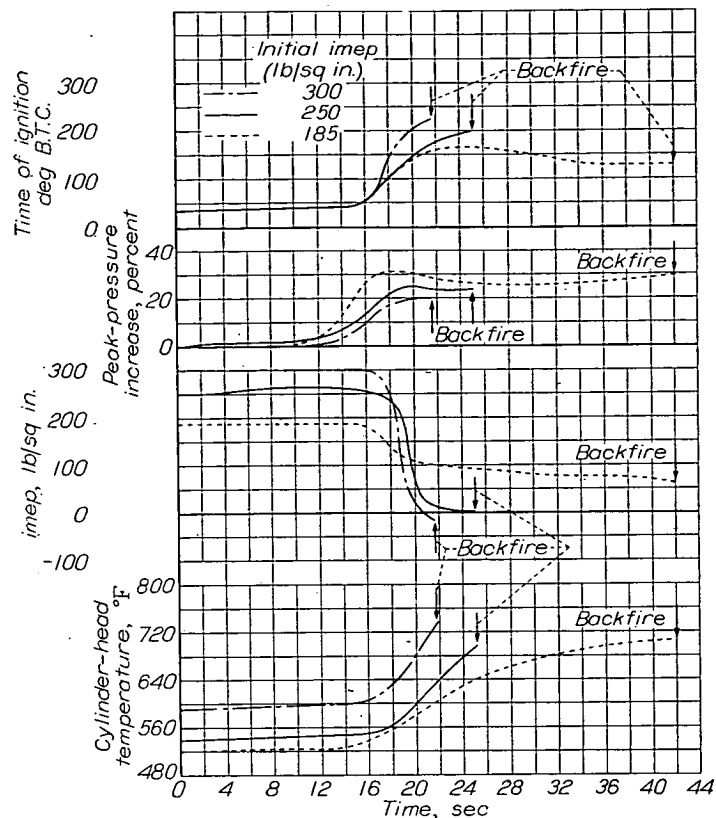


FIGURE IV-4.—Effect of power output on engine behavior during preignition. Compression ratio, 6.65; engine speed, 3000 rpm; mixture temperature, 175° F; coolant temperature, 250° F; spark advance: inlet, 28° B. T. C.; outlet, 34° B. T. C.; fuel-air ratio, 0.070; pre-ignition source, spark plug in exhaust side. (Fig. 6 of reference 5.)

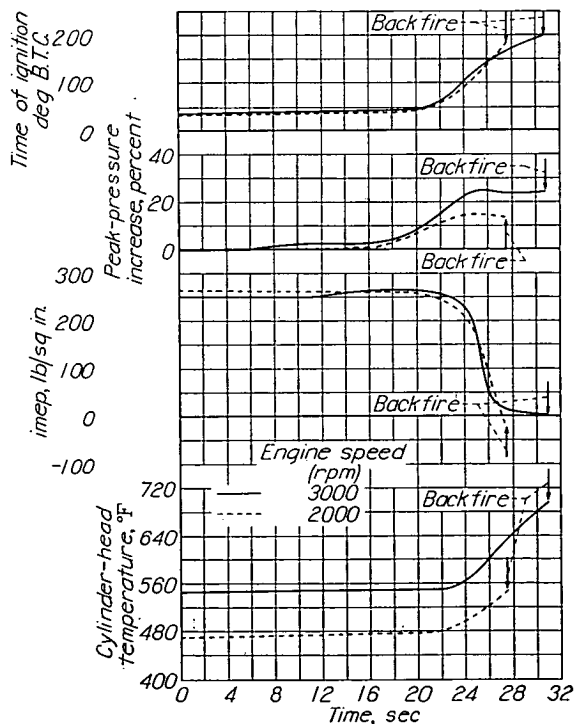


FIGURE IV-5.—Effect of engine speed on engine behavior during preignition. Compression ratio, 6.65; mixture temperature, 175° F; coolant temperature, 250° F; spark advance: inlet, 28° B. T. C.; outlet, 34° B. T. C.; fuel-air ratio, 0.070; pre-ignition source, spark plug in exhaust side. (Fig. 8 of reference 5.)

Effect of engine speed on preignition.—Preignition data obtained at two engine speeds are shown in figure IV-5.

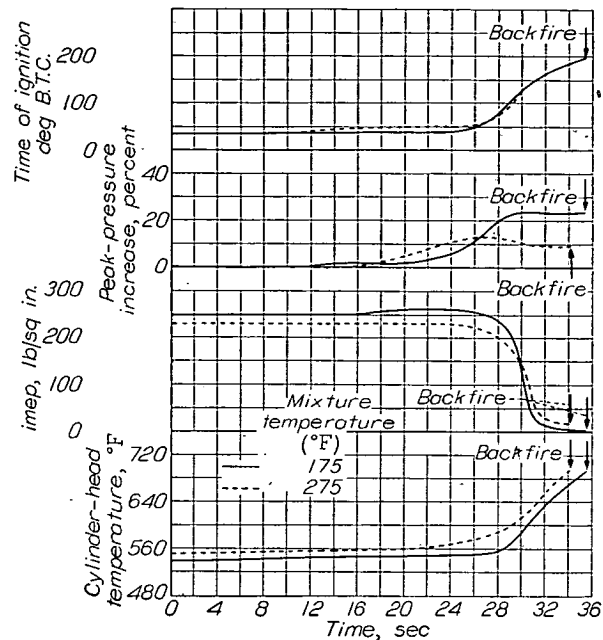


FIGURE IV-6.—Effect of mixture temperature on engine behavior during preignition. Compression ratio, 6.65; engine speed, 3000 rpm; coolant temperature, 250° F; spark advance: inlet, 28° B. T. C.; outlet, 34° B. T. C.; fuel-air ratio, 0.070; preignition source, spark plug in exhaust side. (Fig. 9 of reference 5.)

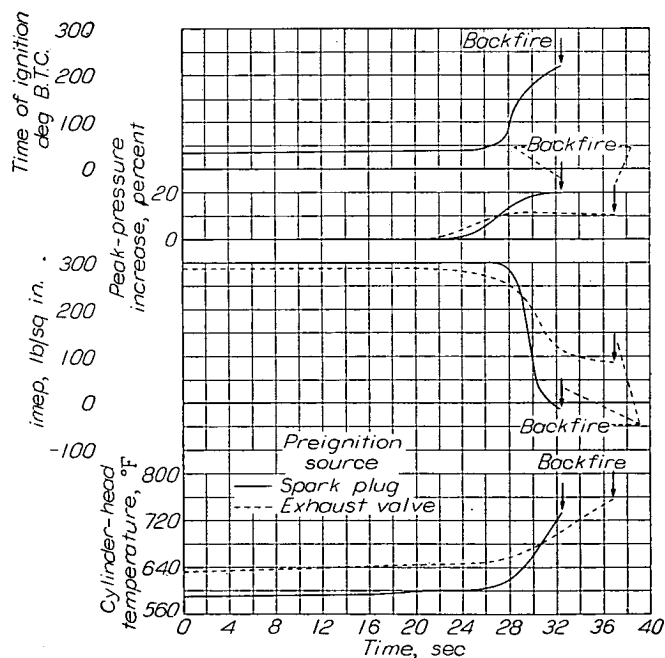


FIGURE IV-7.—Effect of preignition source on engine behavior during preignition. Compression ratio, 6.65; engine speed, 3000 rpm; mixture temperature, 175° F; coolant temperature, 250° F; spark advance: inlet, 28° B. T. C.; outlet, 34° B. T. C.; fuel-air ratio, 0.070. (Fig. 10 of reference 5.)

These two runs were made at about the same power level. Backfires were encountered in both cases at about the same time of ignition (195° B. T. C.).

The peak cylinder pressure increased about 15 and 25 percent at engine speeds of 2000 and 3000 rpm, respectively. This difference can be partly explained by the increase in the heat transferred per cycle at the lower engine speed. (See reference 5.) The cylinder-head-temperature rise (fig. IV-5) was less at the lower speed because backfiring occurred sooner.

Effect of mixture temperature on preignition.—Little difference was found in the preignition behavior of the engine that could be attributed to the influence of mixture temperature (fig. IV-6). At a mixture temperature of 175°F , the peak pressures were higher than at 275°F . The time-of-ignition curve at 275°F is incomplete because of difficulties with instrumentation. (See reference 5.)



(a) Piston failure. Clearance between top of piston skirt and cylinder barrel, 0.004 inch less than normal; imep, 271 pounds per square inch; fuel-air ratio, 0.098.

FIGURE IV-8.—Exhaust-side view of pistons after preignition runs. (Fig. 12 of reference 5.)



(b) Piston failure. Clearance between top of piston skirt and cylinder barrel, normal (0.021 in.); imep, 364 pounds per square inch; fuel-air ratio, 0.094.

FIGURE IV-8.—Continued. Exhaust-side view of pistons after preignition runs. (Fig. 12 of reference 5.)



(c) Piston failure. Clearance between top of piston skirt and cylinder barrel, 0.005 inch greater than normal; imep, 364 pounds per square inch; fuel-air ratio, 0.094.

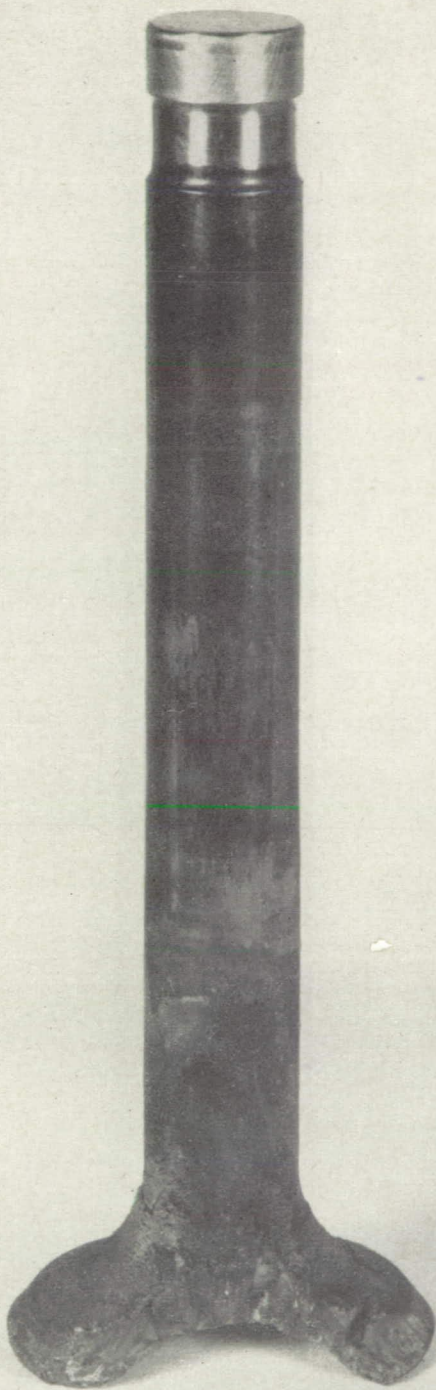
FIGURE IV-8.—Continued. Exhaust-side view of pistons after preignition runs. (Fig. 12 of reference 5.)



(d) Undamaged piston. Clearance between top of piston skirt and cylinder barrel, 0.013 inch greater than normal; imep, 392 pounds per square inch.

FIGURE IV-8.—Concluded. Exhaust-side view of pistons after preignition runs. (Fig. 12 of reference 5.)

Effect of ignition source on preignition.—In order to evaluate the effect of ignition source on preignition, two runs were made that employed a hot spark plug in one case and a hot exhaust valve in the other. The resulting data are shown in figure IV-7. Preignition advanced more slowly with the exhaust valve as the source; however, backfire was



(a) Exhaust valve after run of figure IV-8 (b). (Fig. 14 of reference 5.)

FIGURE IV-9.—Damage to engine components during piston failures in preignition runs.

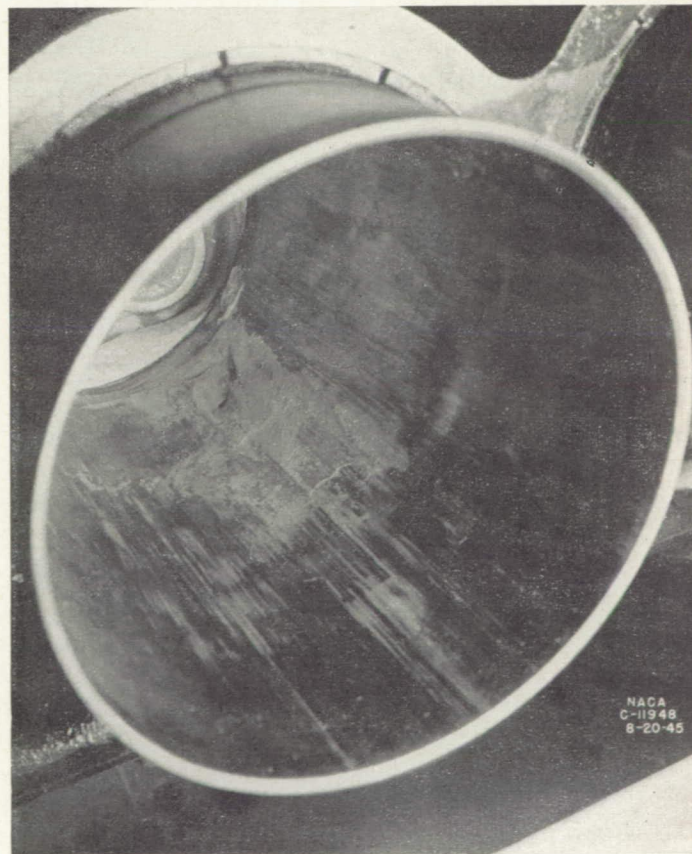
encountered sooner when the spark plug initiated the preignition. The peak-pressure increase and the power decrease were greater when the spark plug served as the source of preignition.

Types of failure caused by preignition.—Typical examples of piston failures are shown in figure IV-8. These failures occurred in runs with reduced piston cooling and at generally

higher power levels than those of the runs described in the preceding paragraphs.

The first failure (fig. IV-8 (a)) resulted from a run in which the clearance between piston and cylinder was about 0.004 inch less than normal (0.021 in.) for the engine. In this case the piston seized and the side of the piston melted. With normal clearance (fig. IV-8 (b)) a higher power level was necessary to cause failure. This failure differed from the failure shown in figure IV-8 (a) in that local melting took place in the center of the piston crown. The failure shown in figure IV-8 (c) resulted from a run with the clearance 0.005 inch greater than normal. More local melting in the center of the piston crown was found with this clearance than was found in the run with normal clearance (fig. IV-8 (b)). In the last run (fig. IV-8 (d)), the clearance was increased 0.013 inch greater than normal and seizure and subsequent failure were not experienced at the power level investigated.

Examples of failures to other cylinder parts are shown in figure IV-9. The exhaust valve (fig. IV-9 (a)) burned during the run in which the piston failure in figure IV-8 (b) was encountered. The valve failure is attributed (reference 5) to particles of aluminum or other substance lodging between the valve and the valve seat. The cylinder barrel (fig. IV-9 (b)) was damaged when the piston shown in figure IV-8 (c) failed. The hot gases caused burning through the barrel into the coolant passage.



(b) Cylinder barrel after run of figure IV-8 (c). (Fig. 15 of reference 5.)

FIGURE IV-9.—Concluded. Damage to engine components during piston failures in preignition runs.

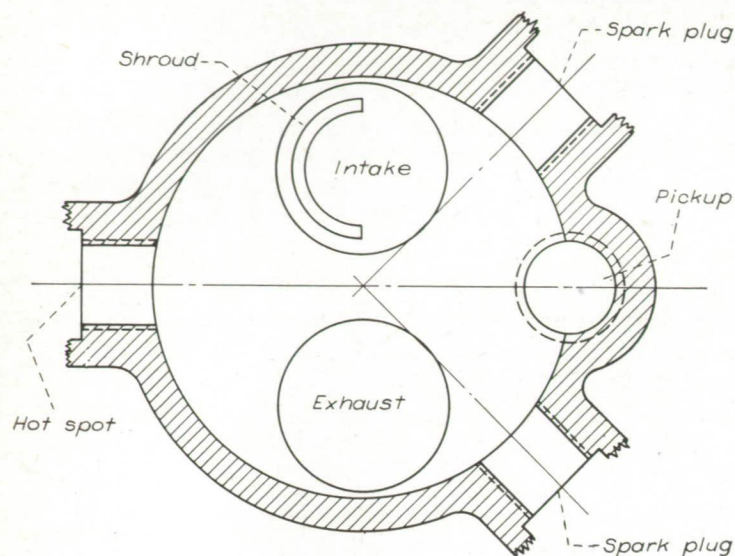
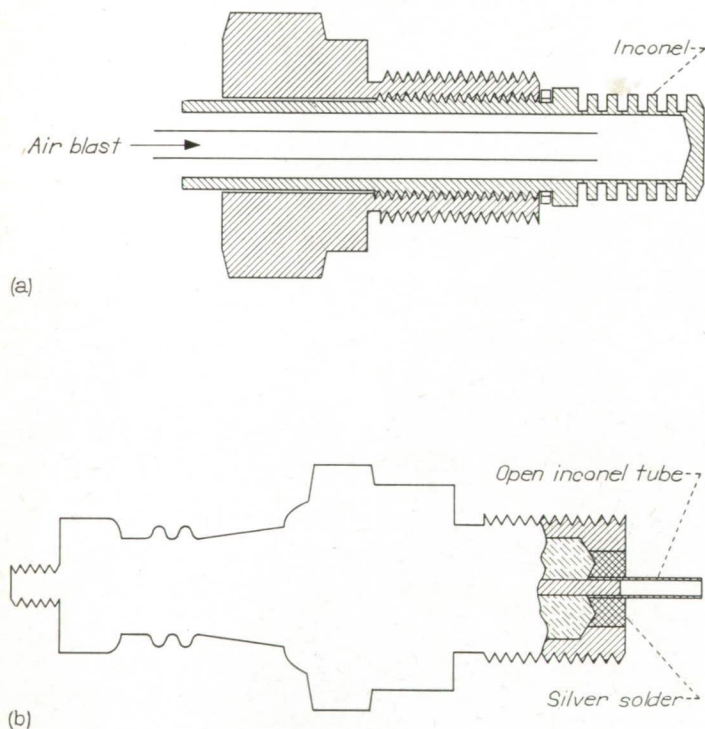


FIGURE IV-10.—Diagram of CFR cylinder used in small-scale engine studies of fuel preignition limits. (Fig. 1 of reference 10.)



(a) Finned hot spot. (Fig. 2 of reference 10.)

(b) Open-tube hot spot. (Fig. 3 of reference 10.)

FIGURE IV-11.—Hot spots employed in preignition studies.

PREIGNITION CHARACTERISTICS OF FUELS

The nature of preignition and the behavior of an engine during preignition operation has been discussed; the influence of fuels on preignition is now considered. The fundamental relations that govern the preignition of fuels are studied in references 7 to 9. The results as cited by Male and Evvard (reference 10) indicate that the hot-spot "threshold" temperature required to produce preignition is relatively insensitive to fuel composition. The ability of a fuel to heat an engine hot spot to the preignition temperature by normal or surface combustion, however, was found dependent upon

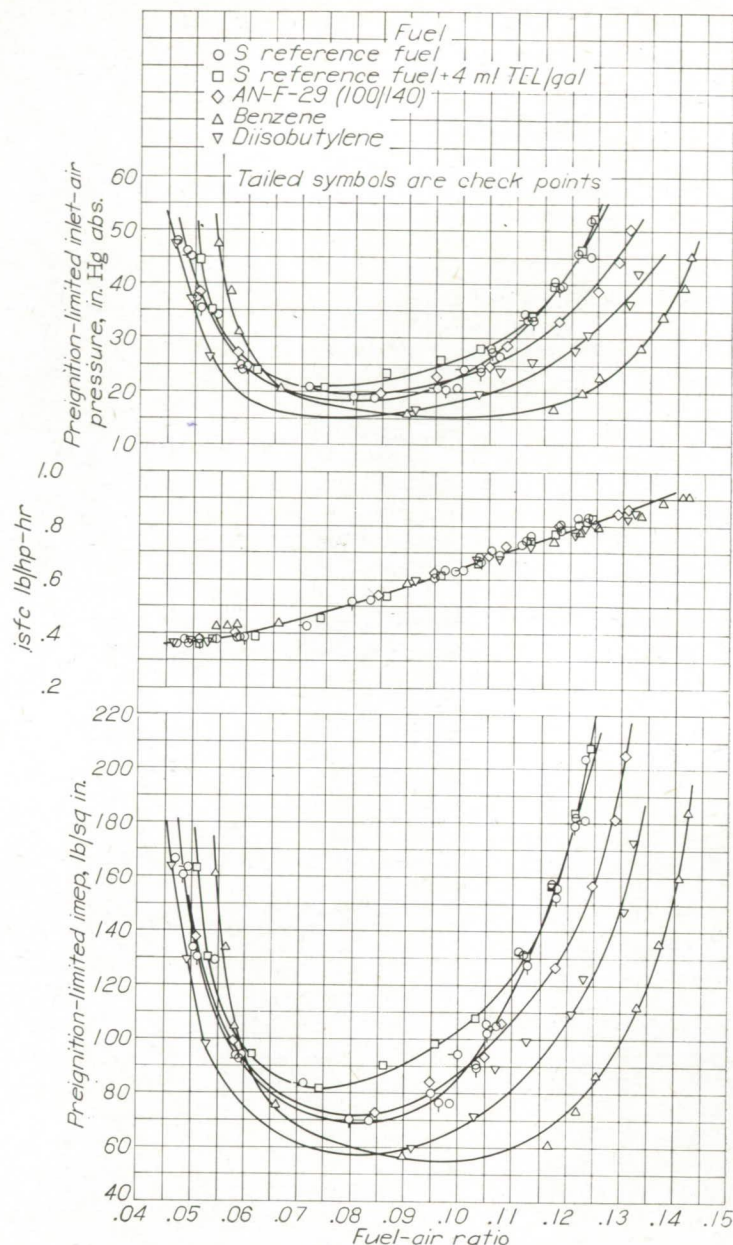


FIGURE IV-12.—Preignition limits of fuels in supercharged CFR engine. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 100° F; coolant temperature, 250° F; spark advance, 32° B. T. C.; preignition source, finned hot spot. (Fig. 4 of reference 10.)

fuel composition, operating conditions, and design of engine and hot spot.

The investigation reported in reference 10 was intended to demonstrate the preignition-limited performance of several fuels. A later investigation reported by Male (reference 11) illustrates how the preignition-limited performance of fuels is affected by engine operating conditions. Both these investigations were conducted on small-scale engines.

Preignition limits of several fuels.—For the study of preignition-limited performance of fuels (reference 10), a supercharged CFR engine with an aluminum piston and a sodium-cooled exhaust valve were used. The intake valve was shrouded (180° shroud) and was installed as shown in figure IV-10. According to reference 10, the shrouded valve aids in isolating the effects of preignition from the effects of knock.

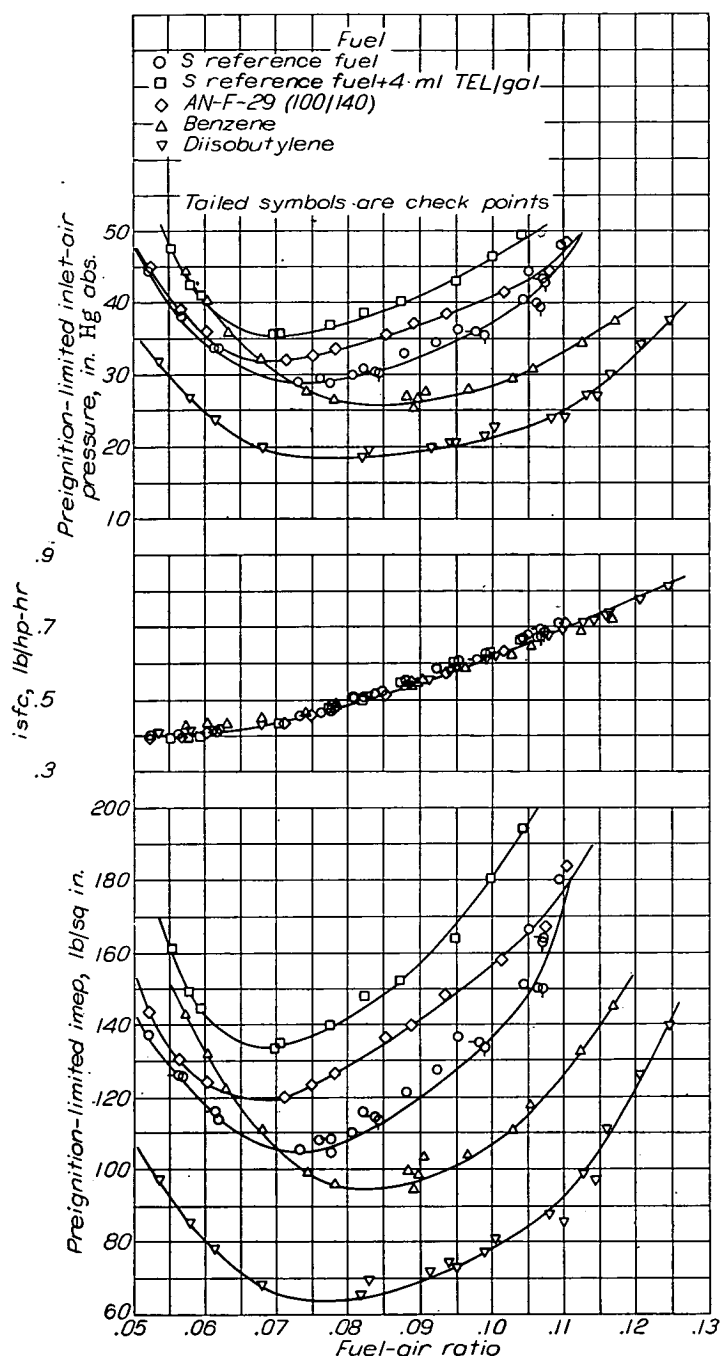


FIGURE IV-13.—Preignition limits of fuels in supercharged CFR engine. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 250° F; spark advance, 20° B. T. C.; preignition source, open-tube hot spot. (Fig. 5 of reference 10.)

This fact is substantiated by unpublished data showing that a shrouded intake valve decreases the sensitivity of thermal-plug temperatures to knock. In order to follow the changing pressure diagram during preignition and to detect knock, a magnetostriction pickup unit was used in conjunction with a cathode-ray oscilloscope. The position of this pickup unit is shown in figure IV-10. (See reference 10.)

The two types of hot spot used in reference 10 are shown in figure IV-11. For tests in which the finned hot spot (fig. IV-11 (a)) was used, the inlet-air temperature was 100° F and the spark advance was 32° B. T. C. The inlet-air

temperature was 225° F and the spark advance 20° B. T. C. for the tests in which the open-tube hot spot (fig. IV-11 (b)) was used.

Preignition-limited performance data for five fuels at two sets of engine conditions are shown in figures IV-12 and IV-13. In general, the curves of preignition-limited indicated mean effective pressure have similar shapes with the minimum power points occurring at fuel-air ratios richer than stoichiometric. Between fuel-air ratios of about 0.070 to 0.085 the relative performance of the fuels in order of decreasing performance was

S reference fuel+4 ml TEL/gal
AN-F-29 (100/140 grade)
S reference fuel
Benzene
Diisobutylene

The over-all spread in indicated mean effective pressure for the five fuels was greater in figure IV-13 than in figure IV-12. This difference and the change in order of benzene and diisobutylene at mixtures richer than 0.085 is attributed (reference 10) to differences in the hot spots, or conditions, or both.

In figures IV-12 and IV-13, the reproducibility of preignition data is demonstrated by check points on the curve for unleaded S reference fuel. The deviations of the data points from the faired curve are about the same as those found in determinations of knock-limited performance.

At constant inlet-air pressures, the maximum thermal-plug temperatures were determined for the fuels (reference 10). This step was considered necessary because the ability of a fuel to increase general engine temperatures might affect the preignition characteristics of the fuel. Preignition-limited fuel-air ratios (fig. IV-12) and thermal-plug temperatures at corresponding inlet-air pressures are shown in the following table:

Fuel	Inlet-air pressure, in. Hg abs					
	20		30		40	
	Thermal-plug temperature * (° F)	Preignition-limited fuel-air ratio ^b	Thermal-plug temperature * (° F)	Preignition-limited fuel-air ratio ^b	Thermal-plug temperature * (° F)	Preignition-limited fuel-air ratio ^b
Benzene.....	561	0.123	669	0.135	761	0.141
Diisobutylene.....	541	.106	649	.123	735	.133
S reference fuel.....	523	.093	631	.110	715	.118
AN-F-29.....	537	.089	641	.114	728	.125
S reference fuel+4 ml TEL.....	520	-----	631	.108	715	.118

* Observed at fuel-air ratio for maximum thermal-plug temperature.
^b Data taken from fig. IV-12.

It was concluded (reference 10) from these data that thermal-plug temperatures do not offer a dependable basis for establishing fuel preignition ratings. This conclusion was drawn from the investigations of S reference fuel in which it was found that the same thermal-plug temperatures were obtained with and without tetraethyl lead, although the preignition-limited-performance curves were different (fig. IV-12).

The presence of tetraethyl lead in S reference fuel appears (fig. IV-12) to increase the preignition-limited performance. A similar investigation in which triptane was used showed the same trend (fig. IV-14). Also shown in figure IV-14 is a curve representing the preignition-limited performance of AN-F-28 (28-R) aviation gasoline. The preignition-limited performance of this fuel is very nearly the same as that of triptane containing 4 ml TEL per gallon. This result is of interest inasmuch as the knock-limited performance of triptane is considerably greater than that of AN-F-28 fuel under most operating conditions. (See ch. II.) A curve for S reference fuel is included in figure IV-14 for comparison. This curve was determined at the time of the triptane and AN-F-28 runs and is slightly different from the S reference fuel curves in figures IV-12 and IV-13.

Preignition limits of aromatic amines.—The effects of six aromatic amines on the preignition-limited performance of AN-F-28 (28-R) fuel were investigated in reference 12.

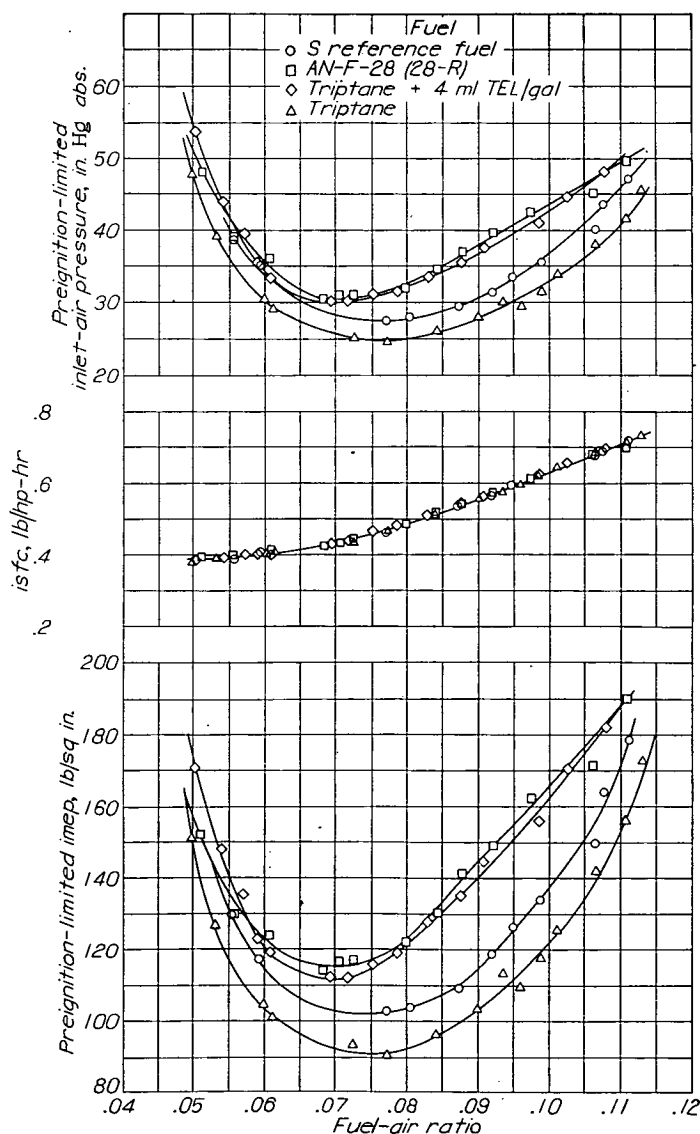


FIGURE IV-14.—Comparison of preignition limits of triptane (leaded and unleaded) with preignition limits of other fuels in supercharged CFR engine. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 250° F; spark advance, 20° B. T. C.; preignition source, open-tube hot spot. (Fig. 6 of reference 10.)

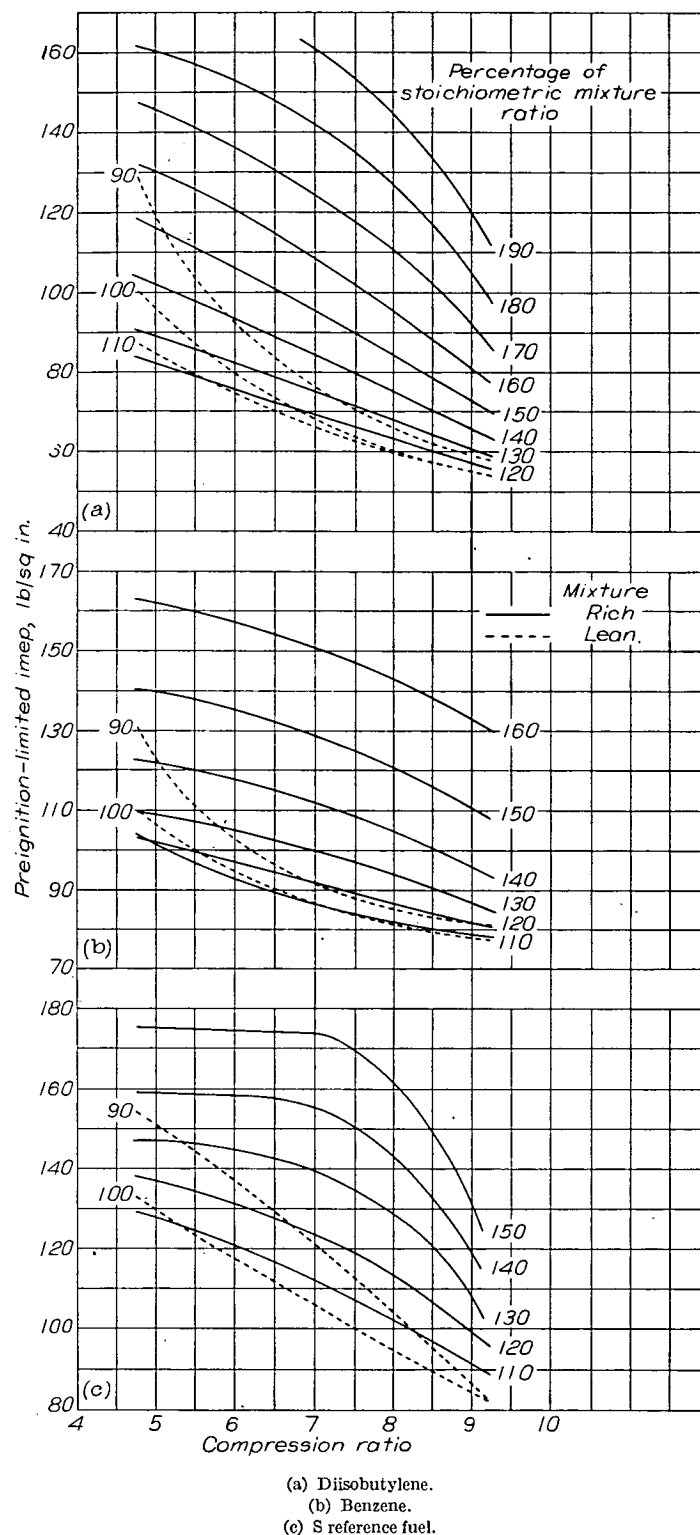


FIGURE IV-15.—Effect of compression ratio on preignition limits in supercharged CFR engine. Engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 250° F; spark advance, 20° B. T. C.; preignition source, open-tube hot spot. (Fig. 14 of reference 11.)

In this study, blends of AN-F-28 (28-R) fuel containing 2 percent by weight of the following amines were prepared and tested: xylidines, cumidines, N-methylxylidines, N-methylcumidines, N-methylaniline, and N-methyltoluidines. A CFR engine was used for these experiments. Pertinent details of the setup have been mentioned in connection with the runs shown in figures IV-12 to IV-14. A more complete description is presented in reference 11.

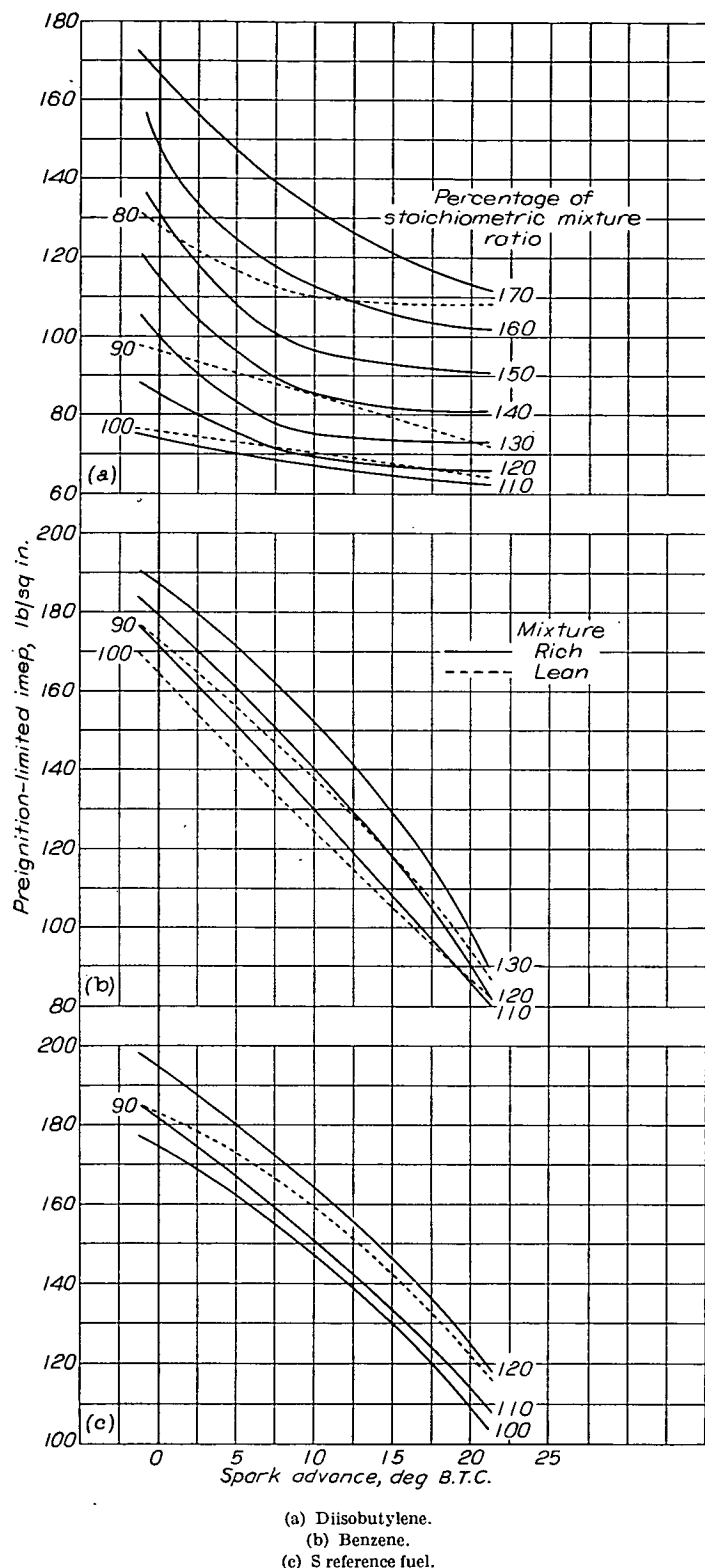


FIGURE IV-16.—Effect of spark advance on preignition limits in supercharged CFR engine. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; coolant temperature, 250° F; preignition source, open-tube hot spot. (Fig. 15 of reference 11.)

The results of the investigation of reference 12 are presented in the following table in the form of ratios of preignition-limited indicated mean effective pressures of the six blends to that of AN-F-28 (28-R) fuel. This basis of comparison was chosen in order to eliminate effects of day-to-day variations in performance of the engine as well as differences arising from changes in the hot spot.

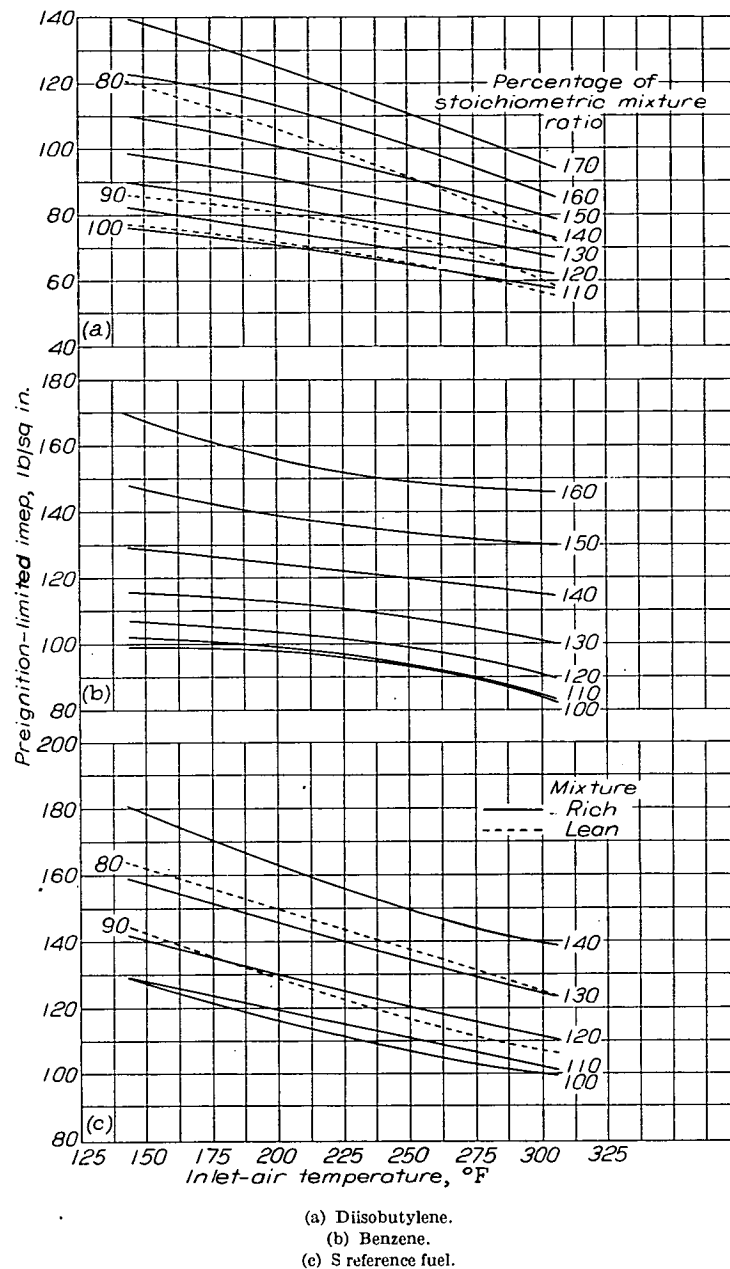


FIGURE IV-17.—Effect of inlet-air temperature on preignition limits in supercharged CFR engine. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 250° F; spark advance, 20° B. T. C.; preignition source, open-tube hot spot. (Fig. 16 of reference 11.)

Aromatic amine added to AN-F-28 (28-R) (2 percent by weight)	Fuel-air ratio				
	0.06	0.07	0.08	0.09	0.10
	imep ratio *				
None.....	1.00	1.00	1.00	1.00	1.00
N-methylaniline.....	1.02	.98	.99	.99	1.01
N-methyltoluidines.....	.91	.92	.94	.97	.99
Xylidines.....	.96	.95	.96	1.00	1.02
N-methylxylidines.....	.98	.98	.98	.99	1.00
Cumidines.....	.95	.91	.91	.91	.91
N-methylcumidines.....	.92	.90	.96	.97	.99

* Ratio of imep of blend to imep of 28-R fuel.

Of the aromatic amines examined in reference 12, N-methylaniline and N-methylxylidines had the highest preignition limits; however, within the experimental accuracy of these tests, the preignition limits of these two aromatic amines appear to be equal to the preignition limit

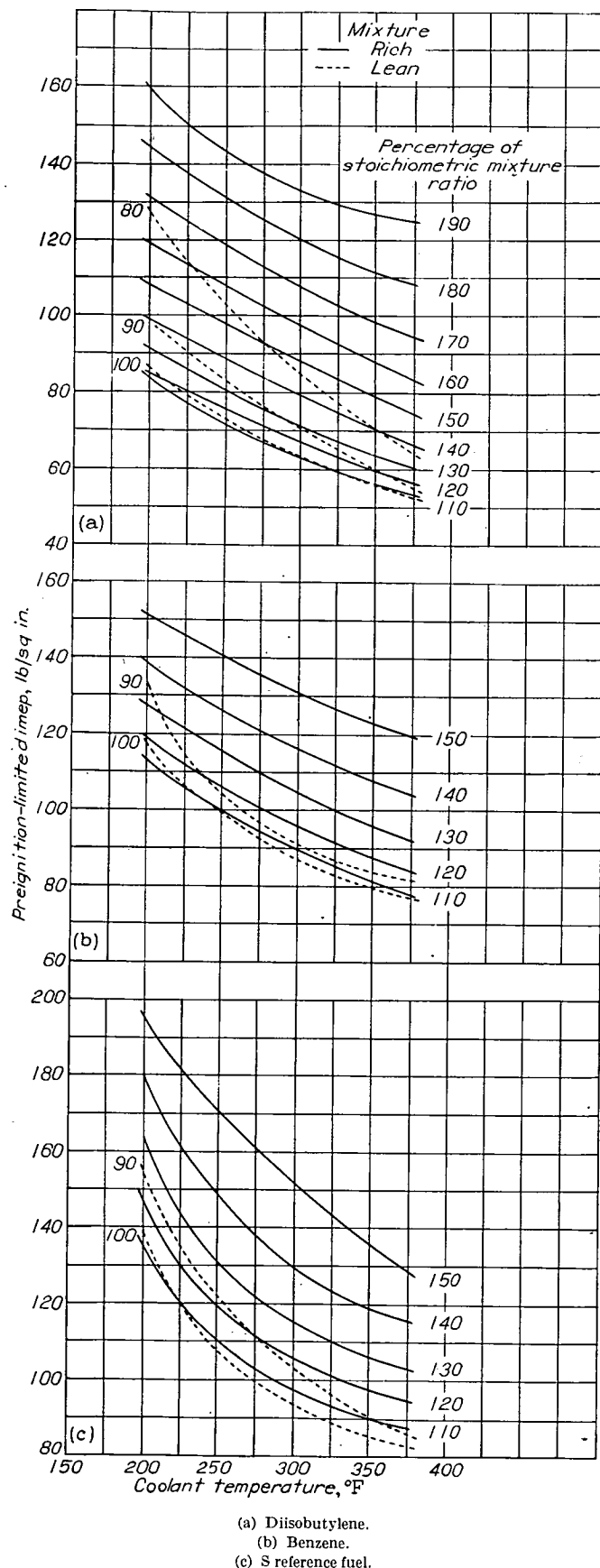


FIGURE IV-18.—Effect of coolant temperature on preignition limits in supercharged CFR engine. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 225° F; spark advance, 20° B. T. C.; preignition source, open-tube hot spot. (Fig. 17 of reference 11.)

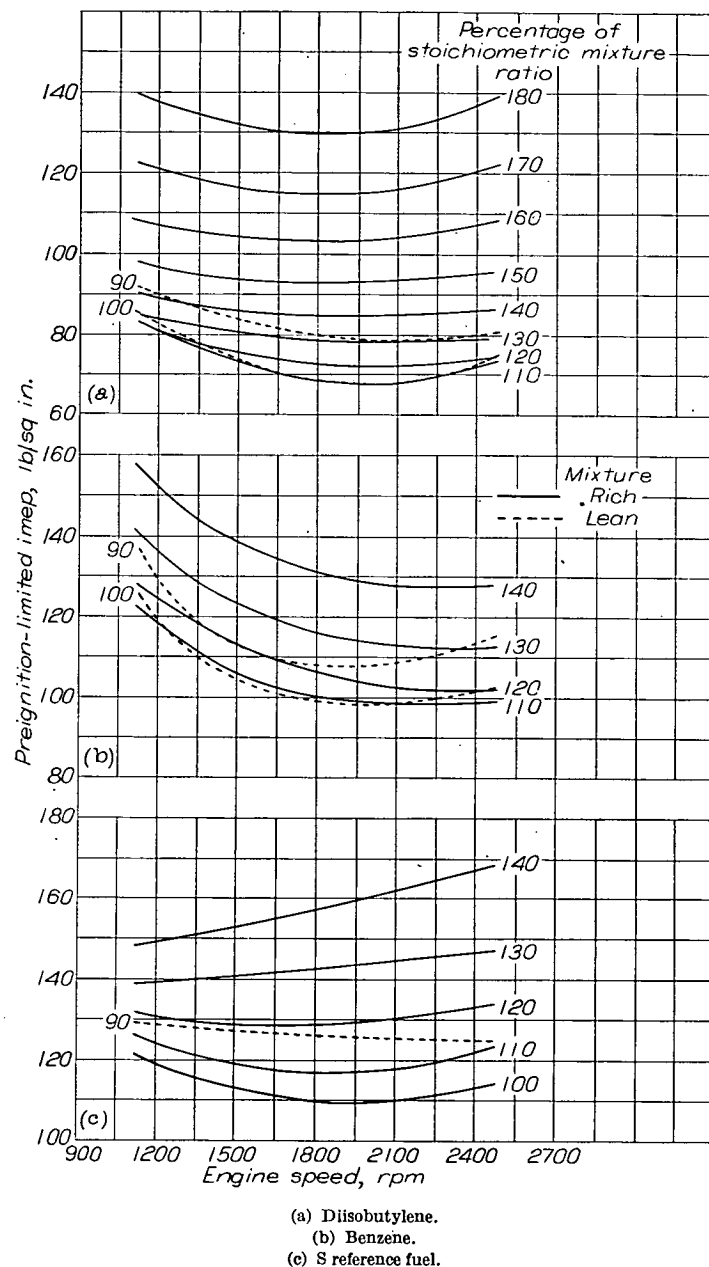


FIGURE IV-19.—Effect of engine speed on preignition limits in supercharged CFR engine. Compression ratio, 7.0; inlet-air temperature, 225° F; coolant temperature, 250° F; spark advance, 20° B. T. C.; preignition source, open-tube hot spot. (Fig. 18 of reference 11.)

of the base fuel, AN-F-28 (28-R). At lean fuel-air ratios, the remaining aromatic amines lowered the preignition limit of AN-F-28 (28-R) fuel about 2 to 10 percent. At rich fuel-air ratios, the cumidines lowered the preignition limit of AN-F-28 (28-R) fuel about 9 percent, whereas the other aromatic amines had little or no effect on the base fuel.

EFFECT OF ENGINE VARIABLES ON PREIGNITION-LIMITED PERFORMANCE

The influence of engine operating variables on preignition-limited performance is illustrated in reference 11. The study was made in a supercharged CFR engine; the following three fuels were used: S reference fuel, diisobutylene, and benzene. The hot spot was an open-tube type. (See fig. IV-11 (b).)

Effect of compression ratio on preignition limit.—The variation of preignition-limited indicated mean effective pressure with compression ratio is shown in figure IV-15. For the three fuels examined, the preignition-limited performance decreased as the compression ratio increased. At the stoichiometric fuel-air ratio, this change for the fuels varied between approximately 7 and 12 pounds per square inch per unit change in compression ratio. The decrease in preignition limits may be attributed to the increase in cyclic temperatures as the compression ratio is increased.

Effect of spark advance on preignition limit.—In figure IV-16, the preignition-limited indicated mean effective pressure is seen to decrease as the spark was advanced. Among the fuels this decrease was about 0.5 to 4 pounds per square inch per degree of spark advance at stoichiometric mixtures. The magnitude of this spread in indicated mean effective pressure among the fuels results from the fact that the sensitivity of diisobutylene to changes of spark advance is so different from the sensitivity of the other fuels (fig. IV-16).

Effect of inlet-air temperature on preignition limit.—The effect of inlet-air temperature on preignition-limited performance is illustrated in figure IV-17. The slopes of the curves for the three fuels are very similar. At stoichiometric mixtures, the decrease in preignition-limited indicated mean effective pressure in these experiments was about 0.1 to 0.3 pound per square inch per °F. Here again the decrease in preignition limits can be attributed to increased cyclic temperatures.

Effect of coolant temperature on preignition limit.—The effect of coolant temperature on preignition limits of the three fuels was determined by using three different coolants in an evaporative cooling system. The coolants used were water, ethylene glycol, and a mixture of water and ethylene glycol. The results of these tests are shown in figure IV-18.

As shown in figure IV-18, the preignition-limited performance decreased as the coolant temperature was increased; however, as emphasized in reference 11, the effect is not solely a temperature effect but includes also the effect of

differences in heat-transfer characteristics of the three coolants.

Effect of engine speed on preignition limit.—The results of runs in which the effect of engine speed on preignition-limited performance were investigated are shown in figure IV-19. Many of these curves, especially those near stoichiometric mixtures, pass through a minimum. At the stoichiometric mixture, minimum values of preignition-limited indicated mean effective pressure were found at engine speeds between 1500 and 2100 rpm.

REFERENCES

1. Biermann, Arnold E., and Corrington, Lester C.: Relation of Preignition and Knock to Allowable Engine Temperatures. NACA ARR 3G14, 1943.
2. Rothrock, A. M., and Biermann, Arnold E.: The Knocking Characteristics of Fuels in Relation to Maximum Permissible Performance of Aircraft Engines. NACA Rep. 655, 1939.
3. MacCoull, Neil: Power Loss Accompanying Detonation. SAE Jour., vol. 44, no. 4, April 1939, pp. 154-160.
4. Hundere, A., and Bert, J. A.: Preignition and Its Deleterious Effects in Aircraft Engines. SAE Quart. Trans., vol. 2, no. 4, Oct. 1948, pp. 546-562.
5. Corrington, Lester C., and Fisher, William F.: Effect of Preignition on Cylinder Temperatures, Pressures, Power Output, and Piston Failures. NACA TN 1637, 1948.
6. Waldron, C. D., and Biermann, A. E.: Method of Mounting Cylinder Blocks of In-Line Engines on CUE Crankcases. NACA RB E4G27, 1944.
7. Serruys, Max: Experimental Study of Ignition by Hot Spot in Internal Combustion Engines. NACA TM 873, 1938.
8. Spencer, R. C.: Preignition Characteristics of Several Fuels under Simulated Engine Conditions. NACA Rep. 710, 1941.
9. Alquist, Henry E., and Male, Donald W.: Trends in Surface-Ignition Temperatures. NACA ARR E4I25, 1944.
10. Male, Donald W., and Evvard, John C.: Preignition-Limited Performance of Several Fuels. NACA Rep. 811, 1945.
11. Male, Donald W.: The Effect of Engine Variables on the Preignition-Limited Performance of Three Fuels. NACA TN 1131, 1946.
12. Male, Donald W.: The Effect of Six Aromatic Amines on the Preignition-Limited Performance of 28-R Aviation Fuel in a CFR Engine. NACA MR E5E12a, 1945.

CHAPTER V

HYDROCARBONS AND ETHERS AS ANTIKNOCK BLENDING AGENTS

Improvements in aircraft power plants during the past 30 years have resulted in demands for fuels of increasingly high antiknock performance. This trend has necessitated a thorough investigation of possible high-antiknock compounds that may or may not occur naturally in petroleum. The task of surveying an endless procession of possible fuel-blending agents has fallen to the petroleum industry and interested research groups. Through the combined efforts of the organizations concerned, a large quantity of data has been amassed. These data permit an accurate appraisal of the merits of many chemical compounds heretofore given little more than cursory consideration as fuel-blending agents.

As a participant in this field of research, the NACA in 1937 sponsored a project by the National Bureau of Standards for the preparation of 1-liter quantities of selected paraffins and olefins. The engine evaluation of the antiknock qualities of these compounds was first conducted under the sponsorship of the American Petroleum Institute (API) and the results of this work have been reported by Lovell (reference 1). In addition, the API has sponsored a synthesis program conducted at the laboratories of Ohio State University. All these programs have been continued up to the present and were augmented during 1942-47 by additional synthesis and engine evaluation at the NACA Lewis laboratory.

The synthesis project at the National Bureau of Standards has been devoted to compounds in the paraffinic and olefinic classes; the synthesis project at the NACA Lewis laboratory has been devoted to compounds in the aromatic and ether classes; and the synthesis program at Ohio State University has been devoted to compounds in these and other classes.

The engine evaluation of pure compounds sponsored by the API was conducted in laboratories of the General Motors Corp. and the Ethyl Corp. The engine evaluation of blends reported in this chapter was conducted at the NACA Lewis laboratory.

Results of the NACA study of paraffins, olefins, aromatics, and ethers are published in a number of reports (references 2 to 14); each report contains data for several compounds on factors such as blending characteristics, temperature sensitivity, lead response, and relation between molecular structure and antiknock ratings. In the succeeding sections of this chapter the effects of molecular structure on these factors of performance are discussed.

ENGINES AND EXPERIMENTAL CONDITIONS

The engine evaluation of the antiknock characteristics of organic compounds was conducted in four test engines: (1) a CFR engine conforming to specifications for the A. S. T. M. Aviation method (D614-47T) for rating fuels; (2) CFR engine conforming to specifications for the A. S. T. M. Supercharge method (D909-47T) for rating fuels; (3) an engine having a displacement of 17.6 cubic inches (about half that of a CFR engine) and popularly known as the 17.6

engine; and (4) a full-scale air-cooled aircraft cylinder mounted on a CUE crankcase.

The 17.6 and A. S. T. M. Supercharge engines were equipped with dual fuel systems, one line for the "warm-up fuel" and one for the test fuel. Knocking was detected in both engines by means of a cathode-ray oscilloscope in conjunction with a magnetostriction pickup unit.

The full-scale single-cylinder test engine was fitted with baffles and cooling air was directed toward the cylinder in order to simulate cooling conditions in flight. Further details of the full-scale installation are given in reference 2.

Pertinent operating conditions for the various engines are presented in table V-1. The 17.6 engine was operated at two inlet-air temperatures, 100° and 250° F, in order to obtain an indication of the sensitivity of fuels to changes in temperature. When the inlet-air temperature was varied, all other conditions were held the same as shown in table V-1.

TABLE V-1.—ENGINE OPERATING CONDITIONS

Condition	Engine				
	17.6	A. S. T. M. Aviation	A. S. T. M. Supercharge	Full-scale single cylinder	
				Simulated take-off	Simulated cruise
Compression ratio.....	7.0	Variable	7.9	7.3	7.3
Inlet-air temperature, °F ..	100 250	125	225	250	210
Inlet-mixture temperature, °F	-----	220	-----	-----	-----
Inlet-air pressure.....	Variable	Atmospheric	Variable	Variable	Variable
Fuel-air ratio.....	Variable	* 0.07	Variable	Variable	Variable
Speed, rpm.....	1800	1200	1800	2500	2000
Spark advance, deg B. T. C. .	30	35	45	20/20	20/20
Coolant temperature, °F.....	212	374	375	-----	-----
Cooling-air temperature, ^b °F	-----	-----	-----	85	85

* Approximate.

^b Cooling-air flow was determined by running engine at brake mean effective pressure of 140 lb/sq in. and fuel-air ratio of 0.10 and by adjusting air flow until temperature of rear spark-plug bushing was 365° F.

The conditions shown in table V-1 for the A. S. T. M. Aviation and A. S. T. M. Supercharge engines are standard for these engines when antiknock ratings are being determined. As indicated in table V-1, the A. S. T. M. Aviation engine is a nonsupercharged engine in which the compression ratio is varied in order to determine the knock limit of a given fuel at a lean fuel-air ratio with all conditions other than compression ratio held reasonably constant. On the other hand, the A. S. T. M. Supercharge engine is operated with all conditions except inlet-air pressure and fuel-air ratio held constant. Knock limits are determined by varying the manifold pressure until knocking occurs. Although the fuel-air ratio can be varied for this engine, antiknock ratings are made at a rich fuel-air ratio, usually about 0.11. The A. S. T. M. Aviation method (lean ratings) may thus be indicative of fuel performance at cruise conditions; whereas the A. S. T. M. Supercharge method (rich ratings) may be indicative of take-off performance.

The full-scale engine conditions were proposed by the Coordinating Research Council in an effort to standardize full-scale single-cylinder experimental engine operation throughout the country. During the early stages of the NACA investigation, fuels were investigated in the full-scale single-cylinder engine (quantity permitting), but these methods were later abandoned when it became apparent that the small-scale engine adequately described the fuel performance.

COMPOUNDS INVESTIGATED

The compounds investigated included 13 branched paraffins, 5 branched olefins, 27 aromatics, and 22 ethers. The paraffins and olefins examined were in the C_5 to C_9 molecular-weight range; the aromatics were in the C_6 to C_{12} range; and the ethers were in the C_4 to C_{11} range.

The individual compounds, together with physical properties determined by the National Bureau of Standards or the NACA Lewis Laboratory, are listed in table V-2.

TABLE V-2.—PHYSICAL PROPERTIES

(a) Paraffins and olefins.*

Paraffins and olefins	Formula	Freezing point (°C)	Boiling point		Density at 20° C (gram/ml)	Refractive index n ²⁰ _D
			(°F)	(°C)		
Paraffins						
2-Methylbutane.....	C ₅ H ₁₂	−159.890	82.14	27.854	0.61967	1.35373
2,2-Dimethylbutane.....	C ₆ H ₁₄	−99.73	121.54	49.743	0.64917	1.36876
2,3-Dimethylbutane.....		−128.41	136.38	57.990	.66164	1.37495
2,2,3-Trimethylbutane.....	C ₇ H ₁₆	−24.96	177.57	80.871	0.69002	1.38946
2,3-Dimethylpentane.....			193.62	89.79	.69512	1.39200
2,2,3-Trimethylpentane.....	C ₈ H ₁₈	−112.27	229.72	109.844	0.71605	1.40295
2,3,3-Trimethylpentane.....		−100.70	238.57	114.763	.72620	1.40752
2,3,4-Trimethylpentane.....		−109.210	236.25	113.470	.71905	1.40422
2,2,3,3-Tetramethylpentane.....	C ₉ H ₂₀	−9.9	284.41	140.23	0.7566	1.4234
2,2,3,4-Tetramethylpentane.....		−121.6	271.42	133.01	.7390	1.4146
2,2,4,4-Tetramethylpentane.....		−66.54	252.10	122.28	.7196	1.4068
2,3,3,4-Tetramethylpentane.....		−102.1	286.77	141.54	.7547	1.4220
2,4-Dimethyl-3-ethylpentane.....			278.11	136.73	.7379	1.4137
Olefins						
2,3-Dimethyl-2-pentene.....	C ₇ H ₁₄	−119	207	97	0.728	1.421
2,3,4-Trimethyl-2-pentene.....	C ₈ H ₁₆		241.27	116.26	0.7434	1.4275
2,4,4-Trimethyl-1-pentene.....		−93.5	214.59	101.44	.7150	1.4086
2,4,4-Trimethyl-2-pentene.....		−106.4	220.84	104.91	.7212	1.4160
3,4,4-Trimethyl-2-pentene.....			234	112	.739	1.423

*Data from reference 15.

(b) Aromatics.

Aromatic	Formula	Freezing point (°C)	Boiling point		Density at 20° C (gram/ml)	Refractive index n_D^{20}
			(°F)	(°C)		
Benzene.....	C_6H_6	5.49	176.2	80.1	0.8789	1.5012
Methylbenzene.....	C_7H_8	-95.014	231.1	110.6	0.8670	1.4967
Ethylbenzene.....	C_8H_{10}	-95.025	276.8	136.0	0.8672	1.4960
1,2-Dimethylbenzene.....		-25.34	291.9	144.4	.8799	1.5052
1,3-Dimethylbenzene.....		-48.31	282.4	139.1	.8642	1.4971
1,4-Dimethylbenzene.....		13.25	281.1	138.4	.8610	1.4960
<i>n</i> -Propylbenzene.....	C_9H_{12}	-99.61	318.7	159.3	0.8620	1.4920
Isopropylbenzene.....		-96.16	306.3	152.4	.8621	1.4913
1-Methyl-2-ethylbenzene.....		-80.94	329.2	165.1	.8807	1.5045
1-Methyl-3-ethylbenzene.....		-95.62	322.3	161.3	.8645	1.4965
1-Methyl-4-ethylbenzene.....		-63.60	323.6	162.0	.8611	1.4951
1,2,3-Trimethylbenzene.....		-25.97	349.0	176.1	.8945	1.5137
1,2,4-Trimethylbenzene.....		-44.23	336.7	169.3	.8758	1.5048
1,3,5-Trimethylbenzene.....		-44.85	328.8	164.9	.8650	1.4990
<i>n</i> -Butylbenzene.....		-88.19	361.8	183.2	0.8603	1.4898
Isobutylbenzene.....		-51.87	342.0	172.2	.8527	1.4860
<i>sec</i> -Butylbenzene.....	$C_{10}H_{14}$	-75.73	343.9	173.3	.8620	1.4900
<i>tert</i> -Butylbenzene.....		-57.96	336.6	169.2	.8665	1.4925
1-Methyl-4-isopropylbenzene.....		-68.39	351.0	177.2	.8568	1.4906
1,2-Diethylbenzene.....		-32.05	361.8	183.2	.8797	1.5032
1,3-Diethylbenzene.....		-84.64	358.9	181.6	.8643	1.4955
1,4-Diethylbenzene.....		-43.31	262.7	183.7	.8621	1.4948
1,3-Dimethyl-5-ethylbenzene.....		-84.43	362.5	183.6	.8647	1.4980
1-Methyl-3- <i>tert</i> -butylbenzene.....		-41.53	372.6	189.2	0.8658	1.4945
1-Methyl-4- <i>tert</i> -butylbenzene.....		-52.73	378.7	192.6	.8612	1.4919
1-Methyl-3,5-diethylbenzene.....		-74.01	393.1	200.6	.8633	1.4969
1,3,5-Triethylbenzene.....	$C_{12}H_{18}$	-66.44	420.6	215.9	0.8620	1.4957

TABLE V-2.—PHYSICAL PROPERTIES—Concluded

(c) Ethers.

Ether	Formula	Freezing point (°C)	Boiling point		Density at 20° C (gram/ml)	Refractive index n_D^{20}
			(°F)	(°C)		
Methyl <i>tert</i> -butyl ether	C ₅ H ₁₂ O	-109.00	130.3	54.63	0.7403	1.3689
Ethyl <i>tert</i> -butyl ether	C ₈ H ₁₈ O	-94.44	161.5	71.93	.7395	1.3756
Isopropyl <i>tert</i> -butyl ether	C ₇ H ₁₆ O	-88.10	189.4	87.42	.7413	1.3800
Methyl phenyl ether (anisole)	C ₇ H ₈ O	-37.16	308.5	153.63	.9939	1.5170
Ethyl phenyl ether (phenetole)	C ₈ H ₁₀ O	-29.49	337.9	169.95	.9651	1.5075
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole)	C ₈ H ₁₀ O	-32.20	350.0	176.69	.9701	1.5123
<i>o</i> -Methylanisole	C ₈ H ₁₀ O	-34.21	341.3	171.81	.9796	1.5178
<i>m</i> -Methylanisole	C ₈ H ₁₀ O	-56.05	349.8	176.53	.9716	1.5137
<i>p</i> - <i>tert</i> -Butylanisole	C ₁₁ H ₁₆ O	19.11	453.7	223.18	.9383	1.5030
<i>n</i> -Propyl phenyl ether	C ₉ H ₁₂ O	-27.09	372.8	189.31	.9475	1.5012
Isopropyl phenyl ether	C ₉ H ₁₂ O	-33.05	350.1	176.73	.9405	1.4975
<i>tert</i> -Butyl phenyl ether	C ₁₀ H ₁₄ O	-18.38	369	187	.9247	1.4880
Methyl benzyl ether	C ₉ H ₁₂ O	-53.11	337.8	169.9	.9630	1.5019
Isopropyl benzyl ether	C ₁₀ H ₁₄ O	-67.18	379	193	.9214	1.4859
Methyl methallyl ether	C ₅ H ₁₀ O	-113.15	152.3	66.86	.7772	1.3941
Isopropyl methallyl ether	C ₇ H ₁₄ O	-----	217.8	103.20	.7753	1.4012
<i>tert</i> -Butyl methallyl ether	C ₈ H ₁₆ O	-85.69	237	114	.7853	1.4083
Dimethallyl ether	C ₈ H ₁₆ O	-57.72	273.9	134.40	.8131	1.4285
Phenyl methallyl ether	C ₁₀ H ₁₂ O	-33.32	210	120	.9634	1.5157
Methyl cyclopropyl ether	C ₄ H ₈ O	-----	109.8	43.20	.7839	1.3799
Methyl cyclopentyl ether	C ₆ H ₁₂ O	-135.03	221.7	105.39	.8625	1.4205
Methyl cyclohexyl ether	C ₇ H ₁₄ O	-74.39	272.0	133.35	.8756	1.4346

* Approximate value (decomposed on atmospheric boiling).

BASE FUELS

Inasmuch as limited quantities of the compounds were available, all tests were conducted on blends rather than on the pure compound. By this procedure, considerable information could be obtained with a relatively small quantity of a given compound. The pure fuels were investigated in blends with two base fuels, one of which was S reference fuel. The other was a blend of 85 percent (by volume) S reference fuel and 15 percent M reference fuel. This blend contained 4.0 ml TEL per gallon. For all practical purposes, S reference fuel is pure isooctane and M reference fuel is a straight-run stock of about 20 octane number (A. S. T. M. Motor method). Use of this base blend was discontinued during the investigation and a blend of 87½ percent S reference fuel and 12½ percent *n*-heptane was substituted. This blend, too, contained 4.0 ml TEL per gallon.

The performance rating of the leaded blend of S and M reference fuels was about 113/108, whereas the rating of the leaded blend of S reference fuel and *n*-heptane was about 120/112.

PRESENTATION OF DATA

The antiknock performance data for all blends and base fuels are presented in appendix A, tables A-1 to A-8. In many cases the performance values have been adjusted to compensate for differences in the base blend used. Where these adjustments have been made, the values will obviously disagree with values reported in references 2 to 14; however, for the purposes herein, the data as a whole have been placed on a more uniform basis.

The previously mentioned adjustments, in effect, permit treatment of the data as if only two base fuels had been used, namely, isooctane (leaded and unleaded) and a leaded blend of isooctane and *n*-heptane.

RELATION BETWEEN MOLECULAR STRUCTURE AND ANTIKNOCK CHARACTERISTICS

A large part of past research relating to molecular structure and antiknock behavior has been summarized by Lovell (reference 1) and by Lovell and Campbell (reference 16).

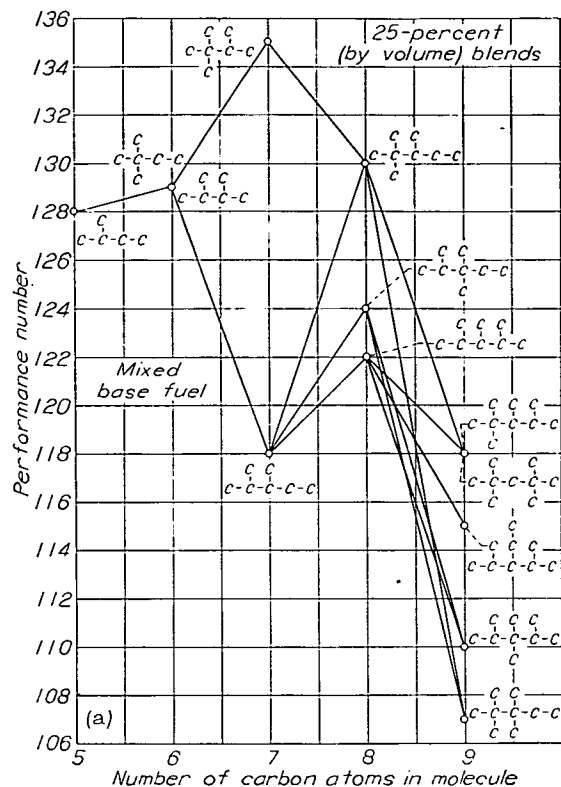
In both these investigations, an attempt was made to secure generalizations that would assist in the prediction of relative antiknock values from molecular structures. The past studies have on the whole been very successful in this respect. As this particular phase of fuel research has progressed, however, the basic knowledge of engine performance has advanced; consequently, exceptions to these generalizations can and do exist by virtue of differences in engines and engine operating conditions. That is, the relative antiknock characteristics of a given group of fuels can be changed considerably by altering the engine or experimental conditions.

As a result, the concept of "severe" and "mild" engine conditions has been devised as an aid in evaluating the merits of different fuels. A severe condition is one in which controlled conditions such as inlet-air temperature, coolant temperature, compression ratio, spark advance, and engine speed combine in their effects to make a fuel knock more readily. (See ch. II.) In reference 14, the various engine operating conditions used in the NACA investigation of ethers are alined into a relative order of severity. This same order of severity is used in the present discussion and is presented in table V-3.

TABLE V-3.—DEGREE OF SEVERITY OF VARIOUS OPERATING CONDITIONS

Engine	Mixture condition	Degree of severity
A. S. T. M. Aviation	Lean	Severe.
Full-scale take-off	do	Moderate to severe.
A. S. T. M. Supercharge	Rich	Moderate.
Full-scale take-off	do	Do.
Full-scale cruise	Lean	Do.
17.6 (inlet-air temperature, 250° F)	do	Do.
Full-scale cruise	Rich	Moderate to mild.
17.6 (inlet-air temperature, 250° F)	do	Do.
17.6 (inlet-air temperature, 100° F)	Lean	Do.
17.6 (inlet-air temperature, 100° F)	Rich	Mild.

Because of this so-called severity concept, any statement to the effect that one fuel performs better than another fuel has little significance unless it is true for all operating conditions or restricted to one operating condition. For this reason, the emphasis in an investigation of the type reported



(a) Engine, A. S. T. M. Aviation.

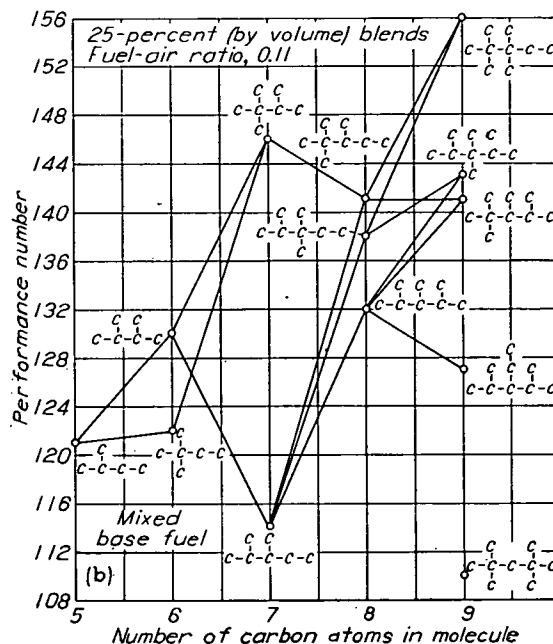
FIGURE V-1.—Knock-limited performance of paraffins in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

herein must be placed upon the trends in the relation between structure and knock rating that appear to apply under most conditions.

Paraffins.—Data were obtained for 13 paraffinic hydrocarbons in leaded blends with the mixed base fuel. Inasmuch as the quantities of hydrocarbon were somewhat limited, all the paraffins were compared only at the 25-percent (by volume) concentration level and only at standard A. S. T. M. Aviation and A. S. T. M. Supercharge conditions (appendix A, table A-1 (a)). The data for these blends are shown in figure V-1.

This figure illustrates the relation between molecular structure and antiknock performance for the paraffins investigated. The lines joining the various data points are shown merely to define the paths followed by compounds in an homologous series. An increase of one carbon atom on the abscissa of these figures is equivalent to a molecular-weight increase equal to the molecular weight of a CH_2 group.

At the A. S. T. M. Aviation conditions (fig. V-1 (a)),

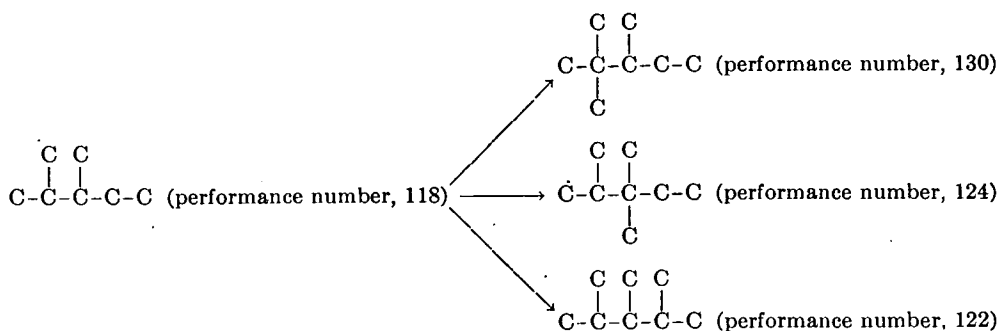


(b) Engine, A. S. T. M. Supercharge.

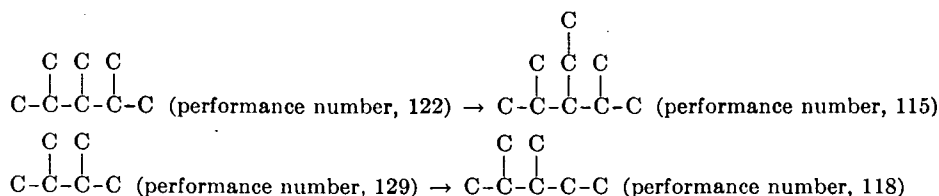
FIGURE V-1.—Concluded. Knock-limited performance of paraffins in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

seven of the paraffinic hydrocarbons raised the knock-limited performance of the base fuel. The increases varied between 2 and 15 performance numbers with 2,2,3-trimethylbutane (triptane) having the highest rating. This result indicates that under severe conditions, represented by the A. S. T. M. Aviation (lean) method, triptane has outstanding antiknock characteristics.

Insofar as the effect of molecular structure on antiknock characteristics is concerned, three trends have been emphasized (references 1 and 16). The first trend is concerned with centralization of the molecule. For example, 2,2,3,3-tetramethylbutane is a more centralized or compact molecule than 2,2,3-trimethylpentane and should therefore have a higher antiknock rating. The second trend shows the effect of adding methyl (CH_3) groups to a molecule in order to form successive members of an homologous series. The addition of a methyl group to increase the branching tends to produce a compound having a higher antiknock rating; however, the position in which the group is added to the molecule will influence the rating of the new compound. This effect, based on A. S. T. M. Aviation antiknock ratings for the blends examined in the present investigation, is illustrated as follows:



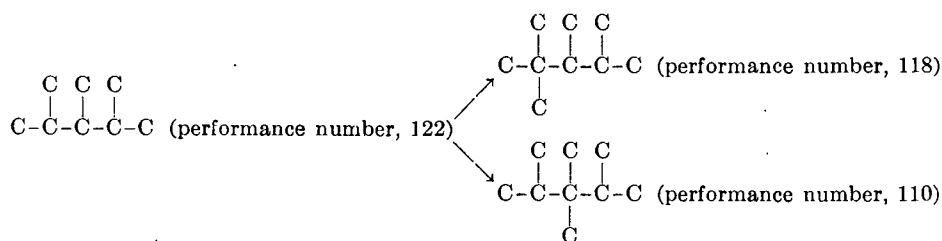
The third trend is concerned with the increase in length of a carbon side chain or the primary carbon chain of a molecule. The effect of such an addition is to decrease the antiknock rating as illustrated by the following examples:



In general, the trends reported in references 1 and 16 and discussed in the preceding paragraph (fig. V-1 (a)) appear to be valid at mild or moderate engine operating conditions. At severe operating conditions, however, exceptions do occur as regards centralization of the molecule or increased branching in the molecule.

At the A. S. T. M. Supercharge conditions, which, as indicated in table V-3, are of moderate severity, the NACA data (fig. V-1 (b)) agree substantially with the results found by Lovell (reference 1). In this case (fig. V-1 (b)), 12 of the 13 paraffinic hydrocarbons investigated raised the knock-limited performance of the base fuel; the increases were in the range of 2 to 44 performance numbers. The antiknock rating of the blend containing 2,2,3,3-tetramethylpentane was the highest obtained and the triptane blend was next.

In order to illustrate the fact that increased centralization of the molecule does not always result in high antiknock values, the A. S. T. M. Aviation ratings are plotted against the A. S. T. M. Supercharge ratings for five nonanes blended with the mixed base fuel in figure V-2. If, in this figure, 2,2,3,3-tetramethylpentane is considered the most compact molecule and 2,2,4,4-tetramethylpentane the least compact, then it is apparent (because the correlating line has a negative slope) that increasing compactness may improve antiknock performance under one set of conditions and depreciate antiknock performance at other conditions. As previously mentioned, the addition of methyl groups, that is, increased branching, does not always result in improved performance. This fact is illustrated by the following A. S. T. M. Aviation ratings:



It is emphasized, however, that these exceptions appear to exist at severe operating conditions as exemplified by the A. S. T. M. Aviation engine.

Olefins.—Five olefins were examined in leaded blends with the mixed base fuel at standard A. S. T. M. Aviation and A. S. T. M. Supercharge conditions (appendix A, table A-1 (a)). The concentration of olefin in each blend was 25 percent by volume.

The data obtained are somewhat limited insofar as the relation between molecular structure and antiknock value is concerned; however, comparisons can be made with references 1 and 16 to determine further the consistency of trends noted by previous investigators. Lovell (reference 1) found that for branched aliphatic compounds if the parent paraffin hydrocarbon had a high antiknock value the introduction of a double bond would decrease the antiknock value. This

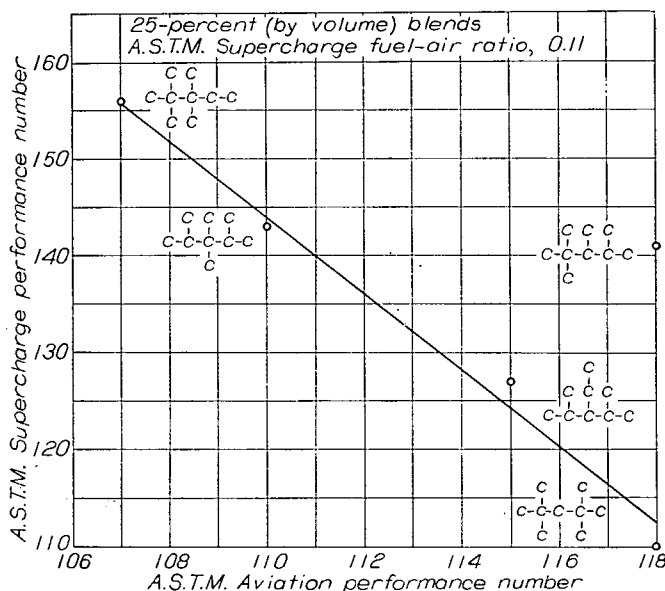


FIGURE V-2.—Relation between A. S. T. M. Supercharge and A. S. T. M. Aviation performance numbers of nonanes in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

trend is supported by the following data from the present investigation (appendix A, table A-1 (a)):

Paraffin	Performance number of 25-percent blend *		Olefin	Performance number of 25-percent blend *	
	A. S. T. M. Aviation	A. S. T. M. Supercharge		A. S. T. M. Aviation	A. S. T. M. Supercharge
$\begin{array}{c} \text{C} \quad \text{C} \\ \quad \\ \text{C}-\text{C}-\text{C}-\text{C}-\text{C} \end{array}$	118	114	$\begin{array}{c} \text{C} \quad \text{C} \\ \quad \\ \text{C}-\text{C}=\text{C}-\text{C}-\text{C} \end{array}$	100	117
$\begin{array}{c} \text{C} \quad \text{C} \quad \text{C} \\ \quad \quad \\ \text{C}-\text{C}-\text{C}-\text{C}-\text{C} \end{array}$	122	132	$\begin{array}{c} \text{C} \quad \text{C} \quad \text{C} \\ \quad \quad \\ \text{C}-\text{C}=\text{C}-\text{C}-\text{C} \end{array}$	101	104
$\begin{array}{c} \text{C} \quad \text{C} \\ \quad \\ \text{C}-\text{C}-\text{C}-\text{C}-\text{C} \\ \\ \text{C} \end{array}$	130	141	$\begin{array}{c} \text{C} \quad \text{C} \\ \quad \\ \text{C}-\text{C}=\text{C}-\text{C}-\text{C} \\ \\ \text{C} \end{array}$	106	108

* All blends were leaded to 4 ml TEL/gal.

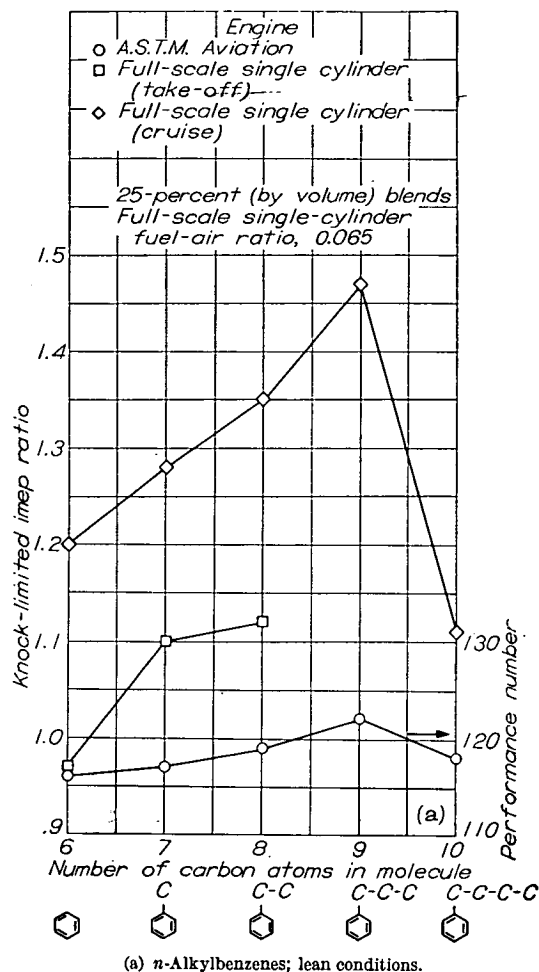


FIGURE V-3.—Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

In the foregoing examples, the double bond in the olefin appeared in the 2 position and, with one exception, the ratings for the olefins are lower than those of the corresponding paraffins. The one exception is shown for the A. S. T. M. Supercharge ratings of 2,3-dimethylpentane and 2,3-dimethyl-2-pentene where the olefin has an antiknock rating three performance numbers higher than the paraffin.

Of the five olefins investigated, only two, 2,4,4-trimethyl-1-pentene and 2,4,4-trimethyl-2-pentene, indicate the effect of the position of the double bond on antiknock performance. For the engines and the conditions examined (appendix A, tables A-1 (a) and A-5 (a)), the ratings of these two compounds appear to be the same at the more severe conditions. At milder conditions, the 2,4,4-trimethyl-2-pentene has lower ratings than 2,4,4-trimethyl-1-pentene. This trend is contrary to the trend found for straight-chain olefins but is in agreement with data for branched-chain olefins (reference 1).

Aromatics.—The most complete set of antiknock performance data obtained in the present investigation resulted from engine studies made with 27 aromatic hydrocarbons in blends with selected base fuels. On the basis of these data, the relations between molecular structure and antiknock value and the influence of engine operating conditions on these relations for the aromatics can be readily seen.

The relation between structure and antiknock performance for a series of *n*-alkylbenzenes at a lean fuel-air ratio

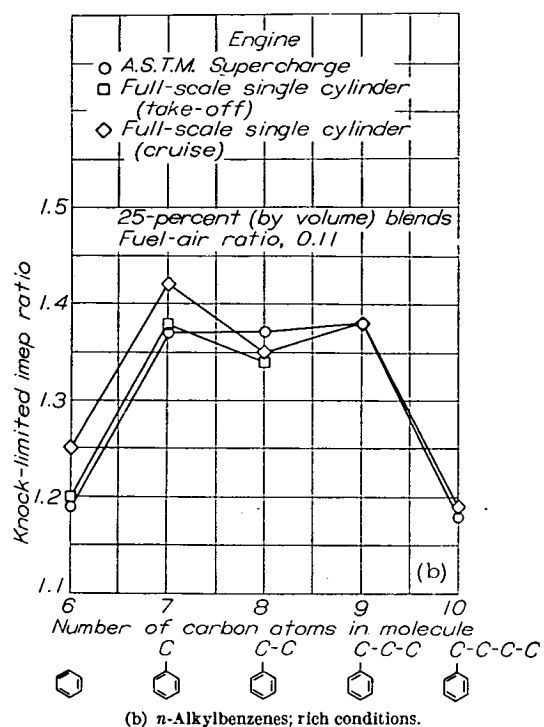


FIGURE V-3.—Continued. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

is shown in figure V-3 (a). In this figure it was necessary to use performance numbers for the A. S. T. M. Aviation engine, inasmuch as knock-limited indicated mean effective pressures are not measured on this engine. The first three carbon atoms added to the side chains of the aromatic compounds successively increased the blend knock limits. The addition of a fourth carbon atom to the side chain caused a sharp drop in performance at the full-scale single-cylinder cruise condition and a slight drop in the A. S. T. M. Aviation engine.

More specifically, the data in figure V-3 (a) indicate that, for the full-scale single-cylinder cruise condition, the 25-percent benzene blend has a knock limit 20 percent higher than the base fuel; toluene is 28 percent higher; ethylbenzene, 35 percent higher; *n*-propylbenzene, 47 percent higher; whereas, *n*-butylbenzene is only 11 percent better than the base fuel. At the other experimental conditions (fig. V-3 (a)), the trends are the same but the magnitude of the increases is less. In fact, under simulated full-scale take-off conditions the benzene blend is lower in performance than the base fuel, which is represented by the ratio 1.0. In the A. S. T. M. Aviation engine, the base fuel has a performance number of 120 and, with the exception of *n*-propylbenzene, all the aromatic blends have performance numbers lower than 120. This depreciation in performance is characteristic of aromatics at conditions as severe as those encountered in the A. S. T. M. Aviation engine.

Figure V-3 (b) is similar to figure V-3 (a) except that the fuel-air ratio is rich and the A. S. T. M. Supercharge (rich) rating method has replaced the A. S. T. M. Aviation (lean) rating method. The trends shown are somewhat different from those in figure V-3 (a), but the similarity between the A. S. T. M. Supercharge data and the full-scale data is apparent. At the conditions investigated, the first addition

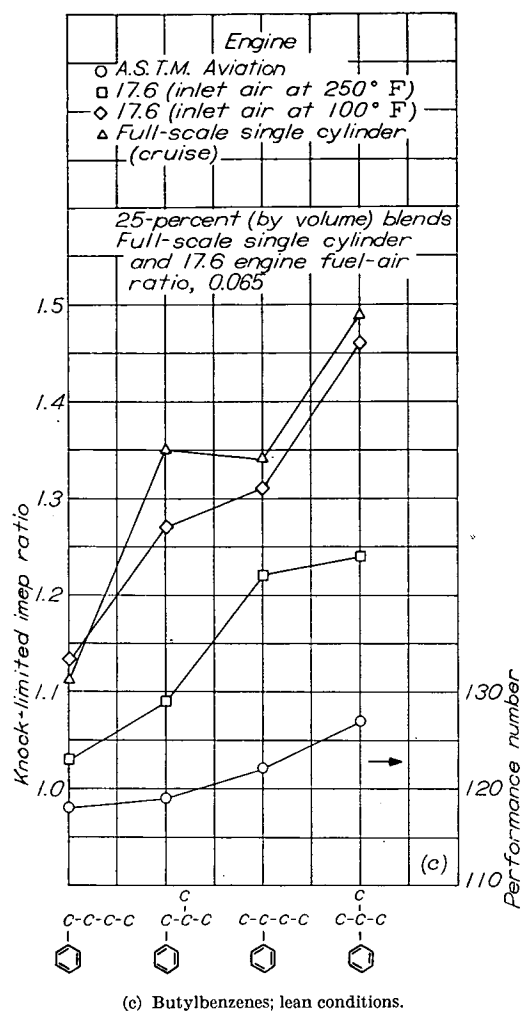


FIGURE V-3.—Continued. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

of a carbon atom to the benzene ring produces a sharp improvement in performance; the next addition results in a decrease except for the A. S. T. M. Supercharge data, which are unchanged; the next addition slightly increases the performance; and the addition of the fourth carbon atom to the side chain results in a very sharp decrease in knock limit, as found at the lean conditions (fig. V-3 (a)).

The change in performance accompanying changes in molecular weight in an homologous series is illustrated in figures V-3 (a) and V-3 (b). The effect of different isomeric structures on performance when the molecular weight is unchanged is shown in figure V-3 (c). For this example, the four butylbenzenes, *n*-butylbenzene, isobutylbenzene, *sec*-butylbenzene, and *tert*-butylbenzene, were chosen. At the two 17.6 engine conditions and the A. S. T. M. Aviation condition, changing from the normal to the iso, the secondary, and the tertiary structures progressively improves the performance. Under simulated full-scale cruise conditions, the isobutylbenzene is slightly better than the *sec*-butylbenzene, but the small difference in antiknock value is probably insignificant.

Data for the four butylbenzenes at a rich fuel-air ratio are

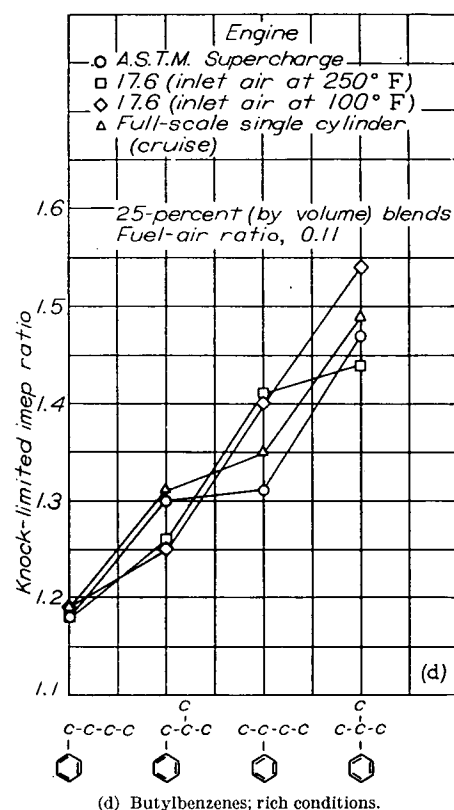


FIGURE V-3.—Continued. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

presented in figure V-3 (d). The trends shown in this figure are similar to those found in figure V-3 (c).

Generally speaking, in figures V-3 (a) to V-3 (d), the trends in performance of the aromatic blends in the standard A. S. T. M. Aviation and A. S. T. M. Supercharge engines were similar to those in the other engines. This similarity among engines, however, is not always observed over wide ranges of operating conditions. Nevertheless, the comparison of performance characteristics of the organic compounds throughout the remainder of this chapter will be based primarily upon the A. S. T. M. Aviation and A. S. T. M. Supercharge engine data because these data were obtained in engines currently accepted as standards for rating fuels.

The knock-limited performance of dimethylbenzenes (xylenes) is illustrated in figure V-3 (e). In both engines, the 1,3-dimethylbenzene blend gave higher performance than either 1,2- or 1,4-dimethylbenzene. The 1,4-dimethylbenzene has an antiknock rating only slightly less than that of 1,3-dimethylbenzene but still considerably higher than that of 1,2-dimethylbenzene.

The trends shown in figure V-3 (f) for the methylethylbenzenes are the same as those shown in figure V-3 (e) for the dimethylbenzenes; that is, 1-methyl-3-ethylbenzene is appreciably better than 1-methyl-2-ethylbenzene and slightly better than the 1-methyl-4-ethylbenzene. A similar result was obtained for the diethylbenzenes (fig. V-3 (g)).

The antiknock performance of disubstituted compounds is illustrated in figures V-3 (e) to V-3 (g). Figure V-3 (h) illustrates antiknock trends for trisubstituted compounds.

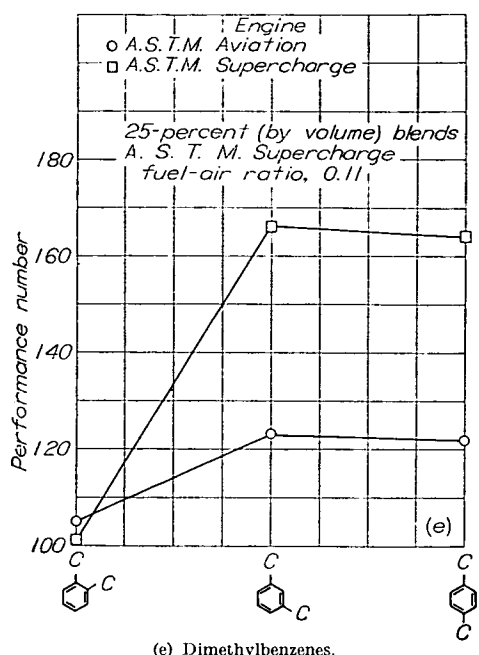


FIGURE V-3.—Continued. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

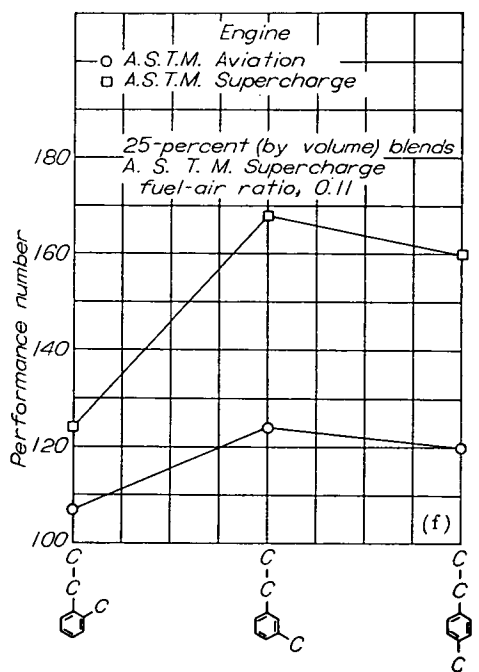


FIGURE V-3.—Continued. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

The 1,2,4-trimethylbenzene blend has a slightly higher knock limit than the 1,2,3-trimethylbenzene blend in the A. S. T. M. Supercharge engine but has a slightly lower knock limit in the A. S. T. M. Aviation engine. The 1,3,5-trimethylbenzene is considerably better than either of the other trimethylbenzenes.

The relative antiknock characteristics of all the aromatic hydrocarbons examined are presented in figure V-3 (i) at

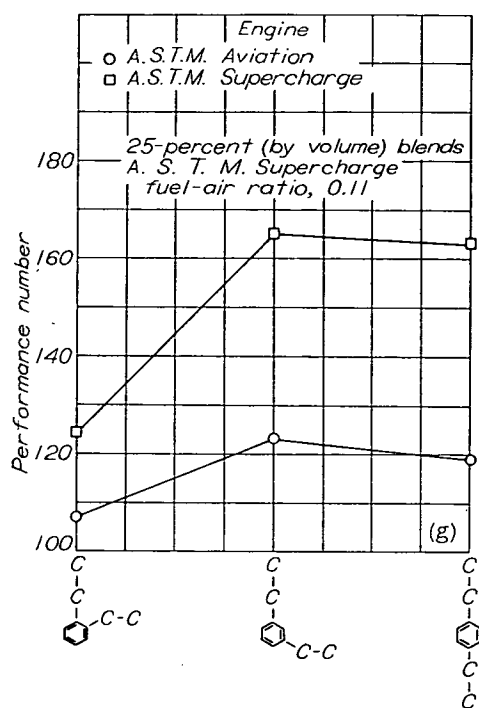


FIGURE V-3.—Continued. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

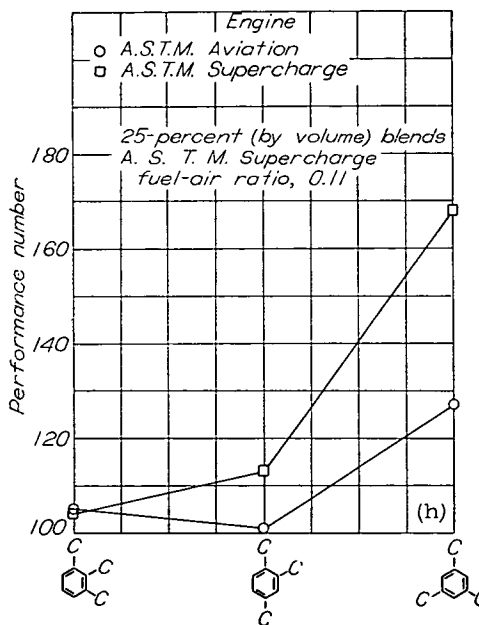
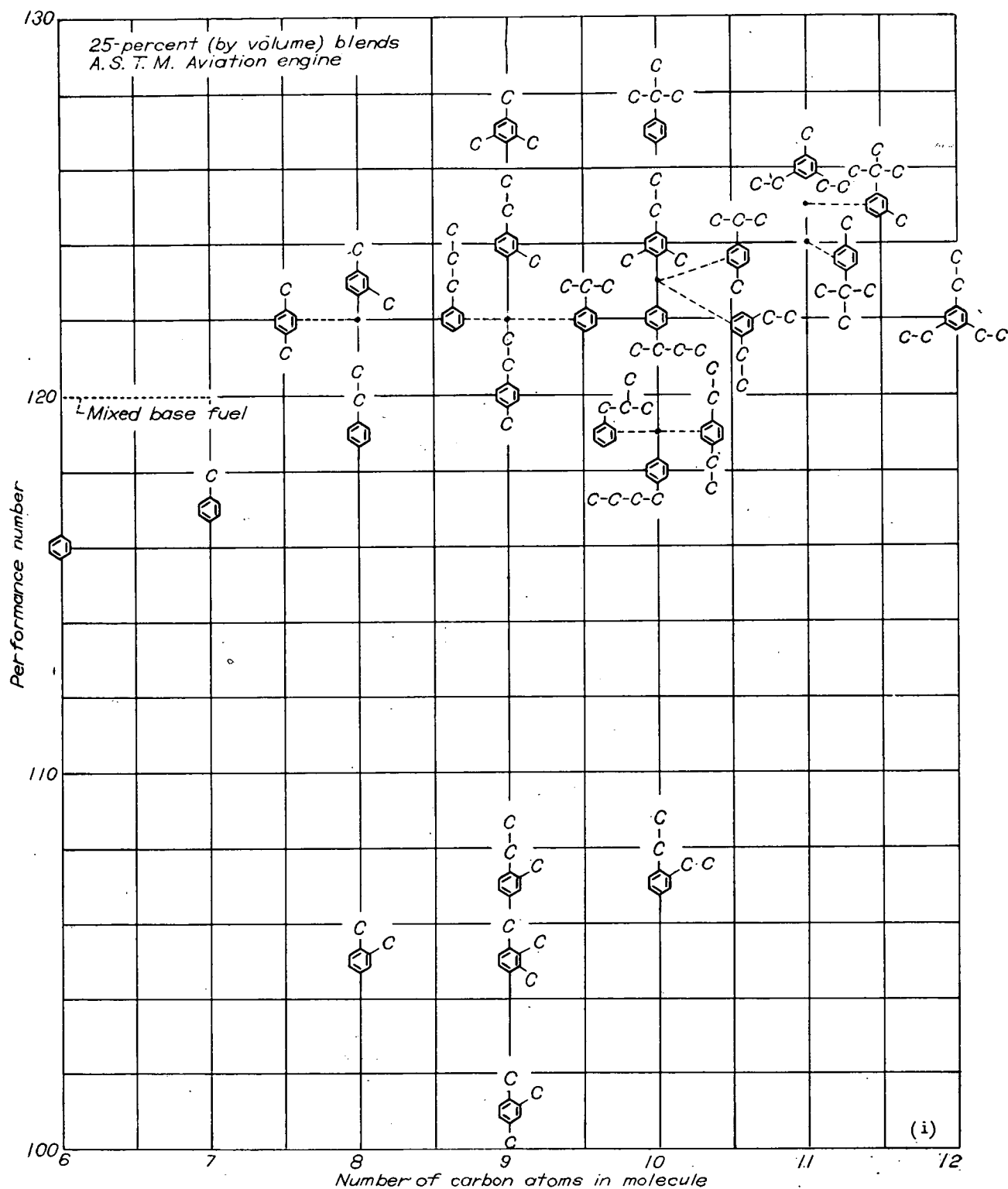


FIGURE V-3.—Continued. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

A. S. T. M. Aviation lean conditions. About 15 aromatics improved the knock-limited performance of the base fuel. These particular blends fall within a range about seven performance numbers above the base fuel. From these data at lean conditions, 1,3,5-trimethylbenzene and *tert*-butylbenzene appear to be the most desirable aromatics in the 25-percent blends investigated.

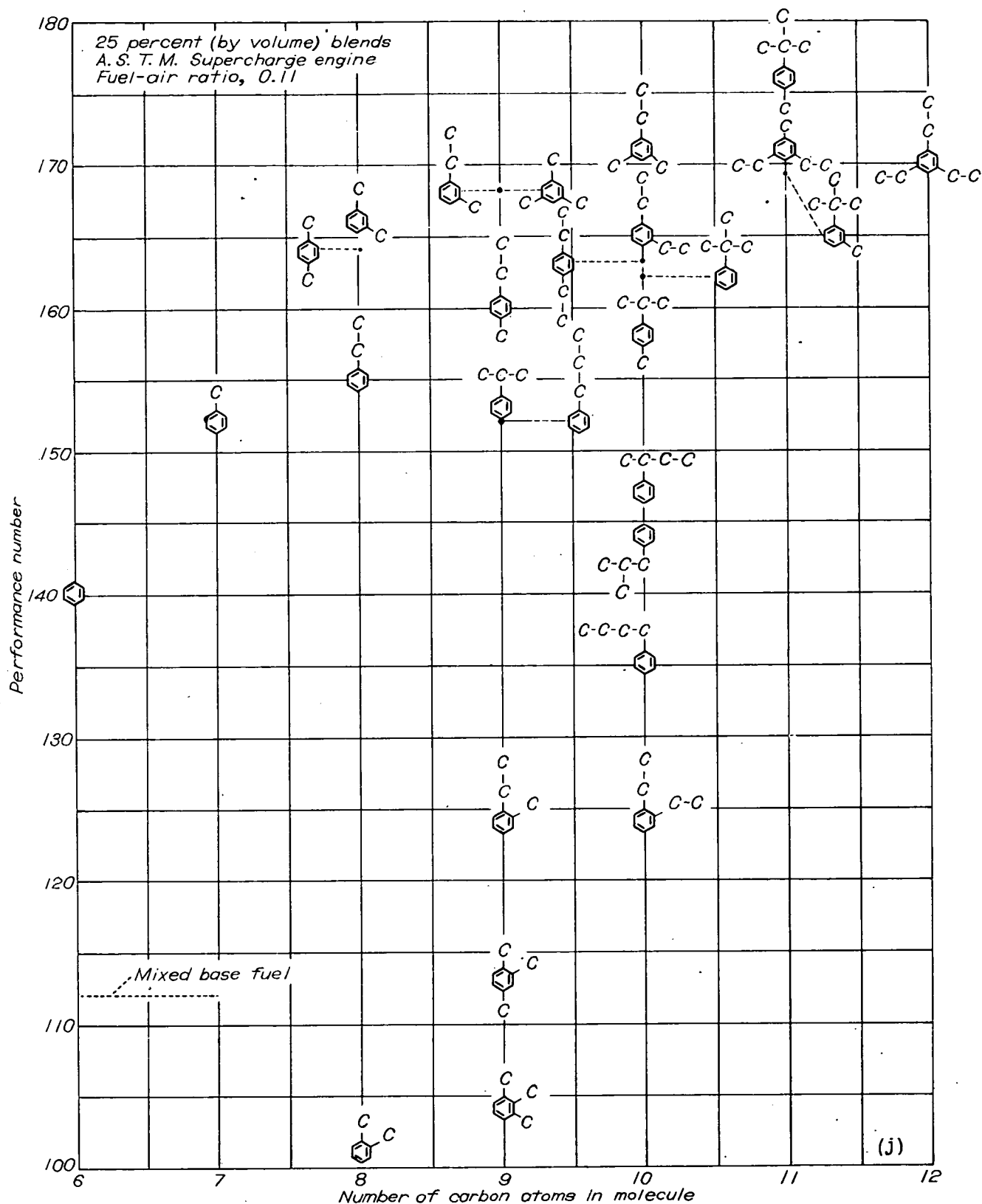


(i) Aromatics; lean conditions.

FIGURE V-3.—Continued. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

The aromatic blends are compared at A. S. T. M. Supercharge rich conditions in figure V-3 (j). In contrast to the A. S. T. M. Aviation data (fig. V-3 (i)), the 25-percent addi-

tions of aromatics to the base fuel caused considerable improvement in A. S. T. M. Supercharge performance, from a performance number of 112 for the base fuel to about 176



(j) Aromatics; rich conditions.

FIGURE V-3.—Concluded. Knock-limited performance of aromatics in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

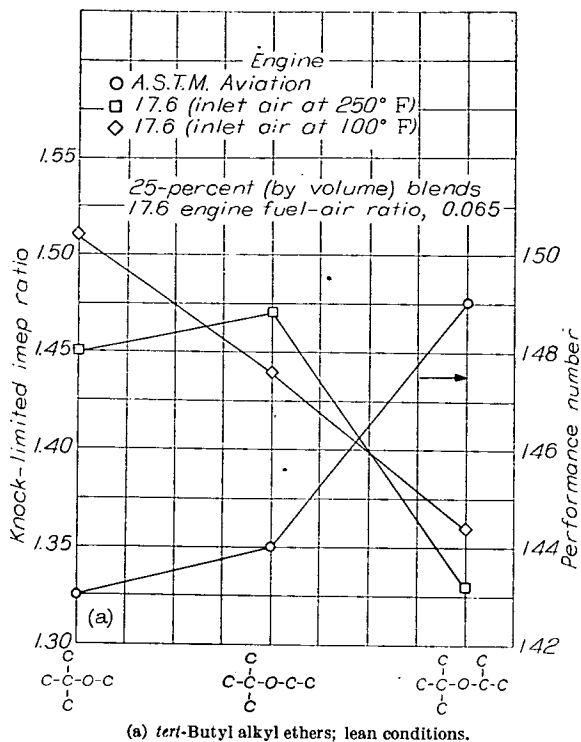


FIGURE V-4.—Knock-limited performance of ethers in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

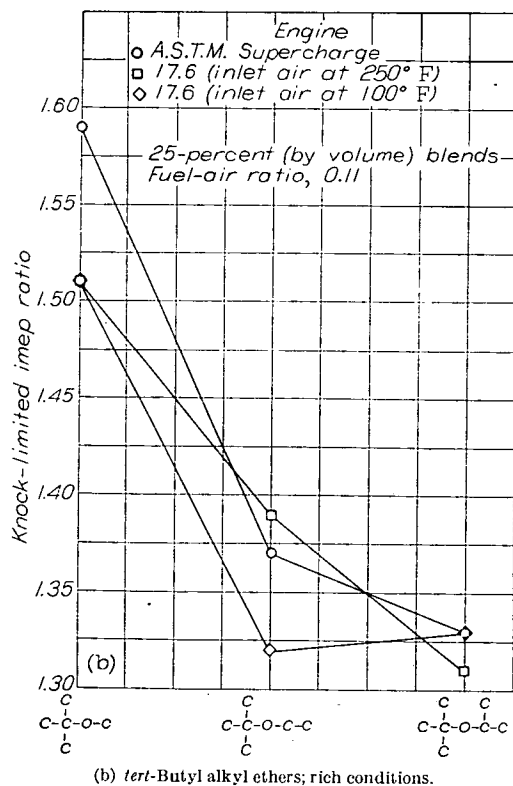


FIGURE V-4.—Continued. Knock-limited performance of ethers in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

for the best aromatic. These results are consistent with results obtained by other investigators in that aromatics in fuel blends generally offer considerable advantage at rich fuel-air ratios but only moderate improvement or even depreciation at lean fuel-air ratios under severe operating conditions. The 1,3,5-trimethylbenzene and *tert*-butylbenzene blends, which have good antiknock characteristics at A. S. T. M.

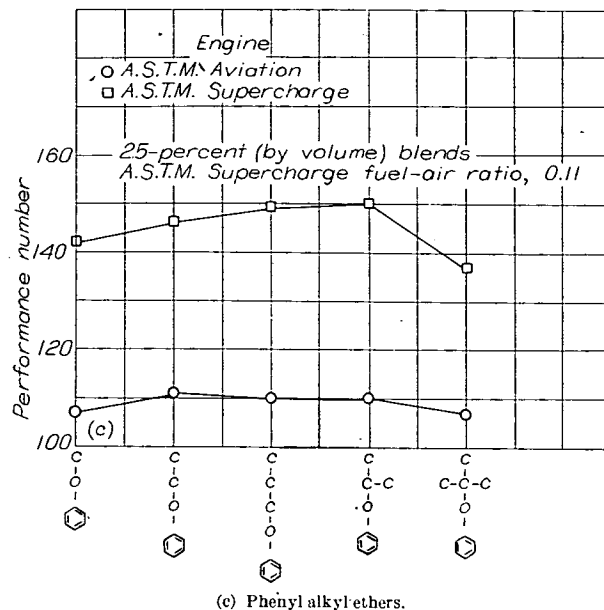


FIGURE V-4.—Continued. Knock-limited performance of ethers in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

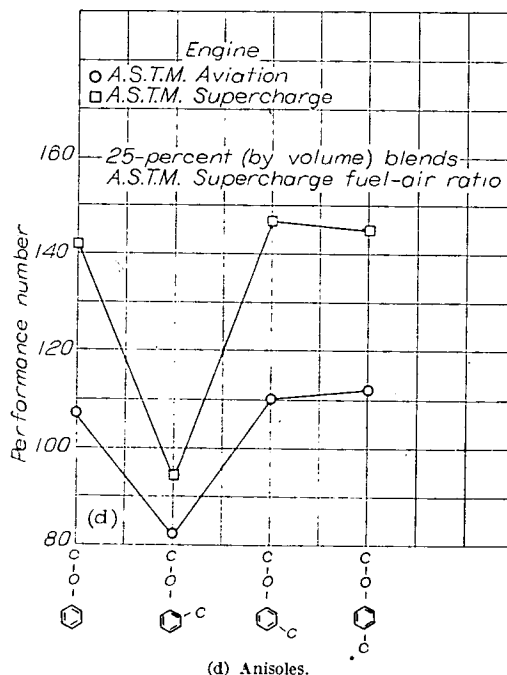
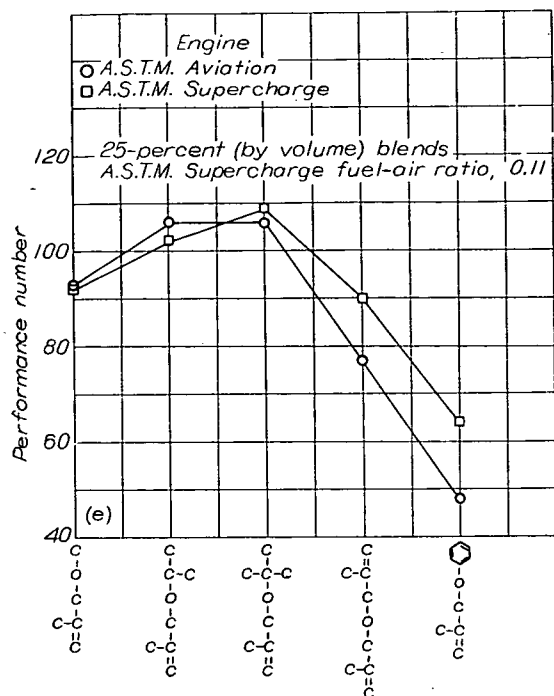


FIGURE V-4.—Continued. Knock-limited performance of ethers in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

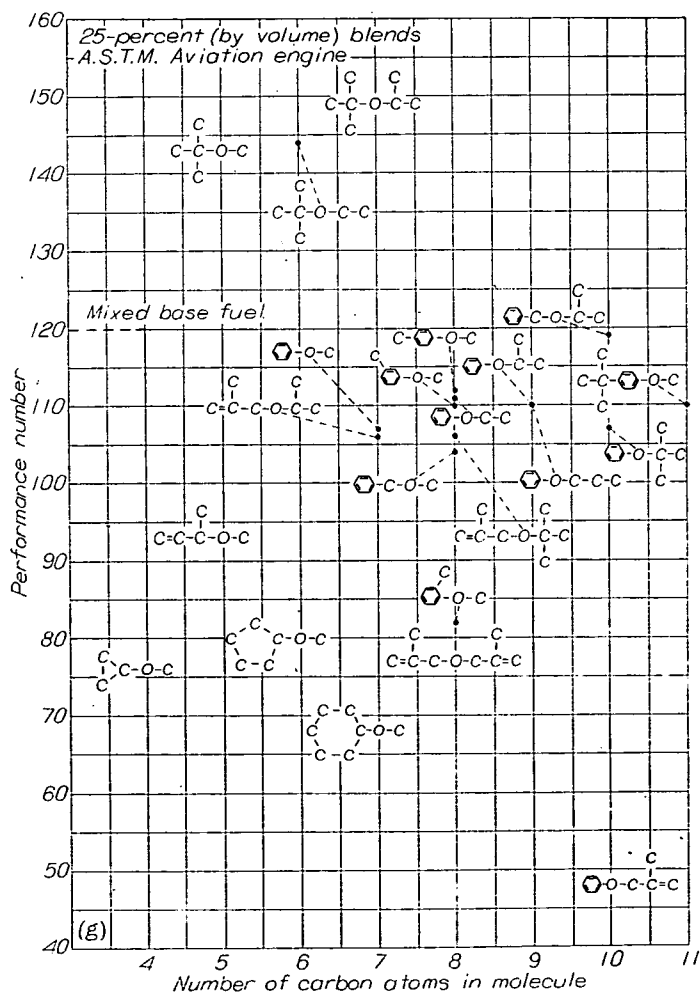
Aviation conditions (fig. V-3 (i)), were still relatively high in performance at rich conditions (fig. V-3 (j)) but were exceeded by other aromatics. Among these high-performance aromatics were 1,3-dimethyl-5-ethylbenzene, 1-methyl-3,5-diethylbenzene, 1-methyl-4-*tert*-butylbenzene, and 1,3,5-triethylbenzene.

In the aromatic data just discussed, only one trend appears worthy of mention, namely, that meta structural arrangements are equal to or slightly better than para arrangements in antiknock performance and both arrangements are considerably better than the ortho structural arrangement. In one case (fig. V-3 (j)), however, the para arrangement was better than the meta arrangement as shown by comparison of 1-methyl-3-*tert*-butylbenzene and 1-methyl-4-*tert*-butylbenzene. Essentially the same trend



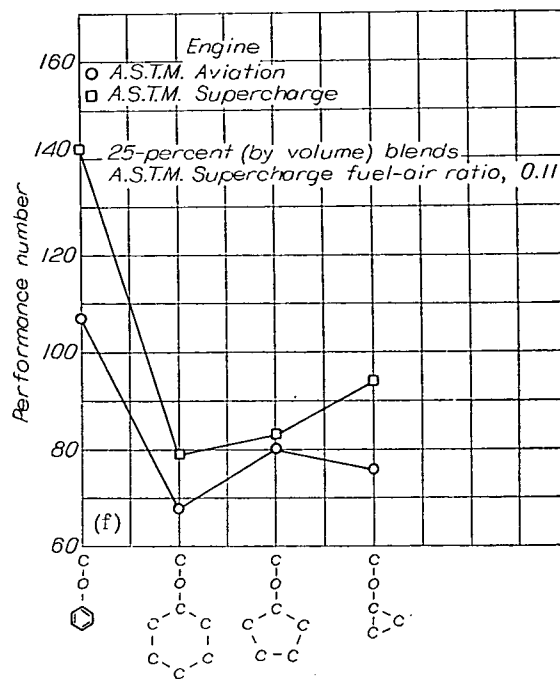
(e) Methallyl ethers.

FIGURE V-4.—Continued. Knock-limited performance of ethers in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.



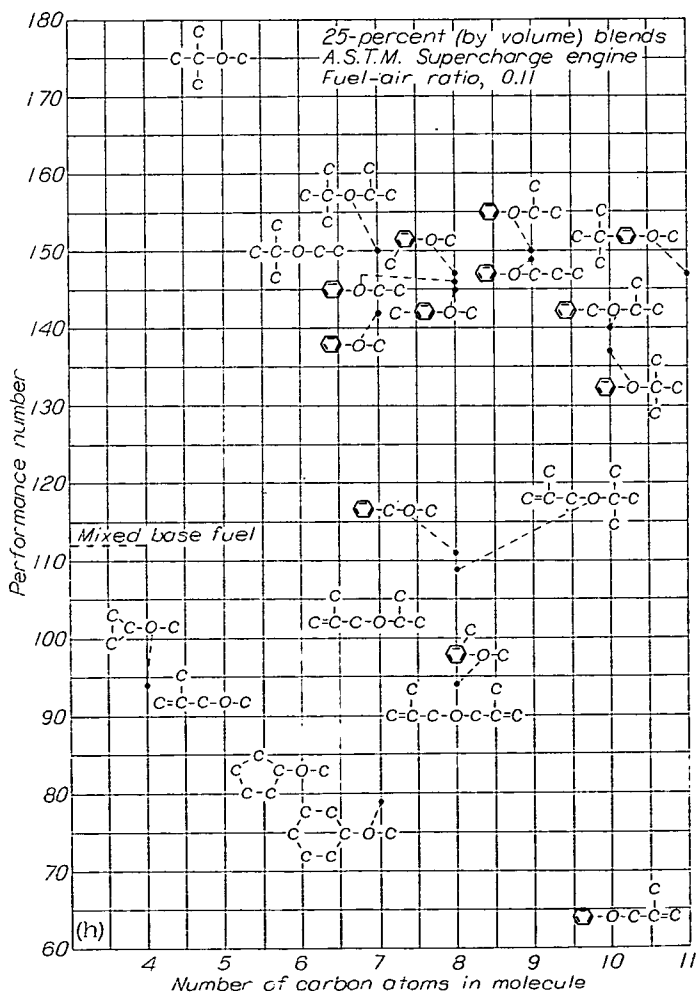
(g) Ethers; lean conditions.

FIGURE V-4.—Continued. Knock-limited performance of ethers in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.



(f) Anisole and three methyl cycloalkyl ethers.

FIGURE V-4.—Continued. Knock-limited performance of ethers in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.



(h) Ethers; rich conditions.

FIGURE V-4.—Concluded. Knock-limited performance of ethers in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

is reported in reference 1 for the relation among ortho, meta, and para compounds.

For the paraffins (fig. V-1), increasing the length of the primary carbon chain resulted in a decrease in the antiknock performance; however, for the aromatics (fig. V-3 (a) and V-3 (b)), an increase in length of a carbon side chain is beneficial up to a certain point, but further additions to the side chain are detrimental to the antiknock performance.

Ethers.—The antiknock characteristics of three alkyl ethers are illustrated in figures V-4 (a) and V-4 (b) for lean and rich fuel-air ratios, respectively. At lean conditions (fig. V-4 (a)) in the A. S. T. M. Aviation engine, isopropyl *tert*-butyl ether was appreciably higher in antiknock value than either methyl or ethyl *tert*-butyl ether. Ethyl *tert*-butyl ether appears to be slightly higher than methyl *tert*-butyl ether in this engine. In the 17.6 engine (fig. V-4 (a)) at both conditions, the results obtained for the three alkyl ethers were directly opposite to those found in the A. S. T. M. Aviation engine. Methyl *tert*-butyl ether was equal to or better than ethyl *tert*-butyl ether and both were appreciably better than isopropyl *tert*-butyl ether. This trend was found also at the rich conditions shown in figure V-4 (b).

The antiknock characteristics of five phenyl alkyl ethers are shown in figure V-4 (c). In both engines methyl phenyl ether and *tert*-butyl phenyl ether gave the lowest performance numbers. The remaining three ethers were about equal in performance in both engines. A comparison of figures V-4 (a) and V-4 (c) shows that the phenyl alkyl ethers investigated have considerably poorer antiknock characteristics than do the *tert*-butyl alkyl ethers at A. S. T. M. Aviation conditions.

The effects of ortho, meta, and para structural arrangements on the antiknock performance of phenyl alkyl ethers are illustrated in figure V-4 (d). The basic ether for this particular example is methyl phenyl ether (anisole), which is shown on the left side of the figure. The addition of a carbon atom to the benzene ring to form *o*-methylanisole caused a decrease in performance. Adding a carbon atom in the meta or para position to form *m*-methylanisole and *p*-methylanisole slightly increased the antiknock performance. In each engine, *m*-methylanisole and *p*-methylanisole were about equal in performance number and both were considerably better than *o*-methylanisole. This result was similar to that obtained for the aromatics (figs. V-3 (e) to V-3 (g)).

Several ethers containing olefinic radicals are shown in figure V-4 (e). Isopropyl methallyl ether and *tert*-butyl methallyl ether blends had the highest performance numbers of this group of compounds and phenyl methallyl ether the lowest. At A. S. T. M. Aviation and A. S. T. M. Supercharge conditions, phenyl methallyl ether was the poorest of the 22 ethers examined.

Hydrogenating the benzene nucleus of anisole to give methyl cyclohexyl ether is shown in figure V-4 (f) to produce a large drop in performance number. Of the three methyl cycloalkyl ethers shown, all of which were relatively low, methyl cyclopropyl ether was the highest at A. S. T. M. Supercharge conditions and methyl cyclopentyl ether was highest at A. S. T. M. Aviation conditions.

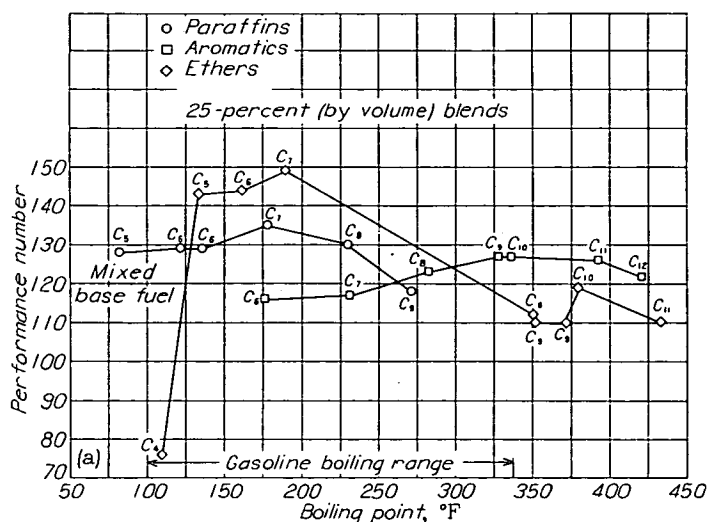
The relative antiknock characteristics of all the ethers investigated are presented in figure V-4 (g) at A. S. T. M. Aviation (lean) conditions. Under these conditions only the three *tert*-butyl alkyl ethers raised the knock limit of the base fuels. The maximum improvement in performance number was 29 and was obtained with isopropyl *tert*-butyl ether.

The antiknock characteristics of all the ethers investigated are compared in figure V-4 (h) at A. S. T. M. Supercharge (rich) conditions. Twelve of the ethers improved the performance of the base fuel; the greatest increase in knock-limited performance, about 63 performance numbers, was obtained with methyl *tert*-butyl ether. Comparison of figures V-4 (g) and V-4 (h) clearly shows that nine of the phenyl alkyl ethers have much better antiknock characteristics at rich mixtures than at lean mixtures. It is also apparent that the methyl cycloalkyl ethers show little promise as antiknock blending agents at the A. S. T. M. Aviation and A. S. T. M. Supercharge conditions.

Comparison of classes of compounds.—As a matter of interest, the isomers having the highest antiknock ratings in figures V-1, V-3 (i), V-3 (j), V-4 (g), and V-4 (h) have been plotted in figure V-5. The performance numbers have been plotted against boiling points in order to illustrate the most promising antiknock compounds in the boiling range of commercial gasolines. Comparison of the curves in figure V-5 is not strictly valid, inasmuch as all the isomers in a given group of compounds have not been studied. Within the limitations of the investigation, however, these two figures do illustrate how the antiknock characteristics of the better paraffins, aromatics, and ethers compare.

When the boiling range of aviation gasoline is assumed to be 100° to 338° F, it is seen (fig. V-5 (a)) that for A. S. T. M. Aviation lean conditions the C₅ and C₆ paraffins have the highest performance numbers in the boiling range from 80° to 120° F. In the boiling range between 130° and about 300° F, the ethers have the highest performance numbers. Above 300° F the highest performance numbers were obtained with the aromatic blends.

At A. S. T. M. Supercharge conditions (fig. V-5 (b)), the paraffin blends had the highest performance numbers in the range of boiling temperatures from 80° to 120° F. Above 120° F the ethers had the highest antiknock ratings up to a boiling temperature of 220° F. At higher boiling temperatures the aromatics exhibited superior antiknock characteristics.



(a) Engine, A. S. T. M. Aviation.

FIGURE V-5.—Comparison of isomers having highest antiknock values in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

BLENDING CHARACTERISTICS

In the preceding section, the discussion of structural trends was based on studies in which 25 percent of a given compound was blended with a selected base fuel. On the basis of such studies, it can be concluded that one compound is better than another or that all compounds align themselves in an order of antiknock performance that is influenced by engine operating conditions. This situation is complicated, however, in that the relative order of antiknock value of a series of compounds at a fixed engine condition is influenced by the concentration of the compound in the blends upon which such an investigation is based. In other words, one compound could be better than another if both were compared in 25-percent blends but the reverse could be true if both were compared in 50-percent blends.

Blending characteristics of various potential aviation-fuel blending agents have been the subject of considerable investigation. A portion of the more recent findings in such studies is reported in references 17 to 20. The results of these investigations show conclusively that compounds differ radically in their blending behavior as regards antiknock performance.

Paraffins.—The blending characteristics of paraffinic fuels at rich fuel-air ratios may be expressed by the following equation (see chapter VIII):

$$\frac{1}{P_b} = \frac{N_1}{P_1} + \frac{N_2}{P_2} + \frac{N_3}{P_3} + \dots \quad (1)$$

where

$N_1, N_2, N_3, \dots$ mass fractions of components 1, 2, 3, $\dots$, respectively, in blend

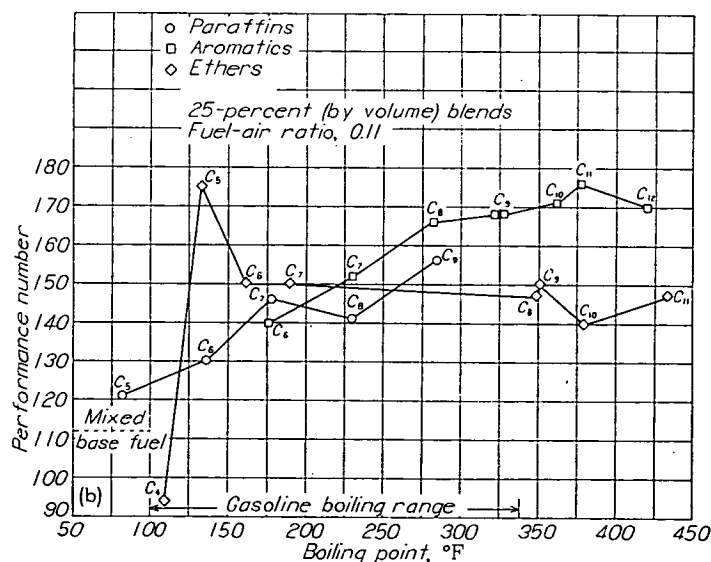
P_b knock-limited indicated mean effective pressure of blend

$P_1, P_2, P_3, \dots$ knock-limited indicated mean effective pressures of components 1, 2, 3, $\dots$, respectively

The application of this equation to data in the present investigation is illustrated in figure V-6 (a) for the A. S. T. M. Supercharge engine. The ordinate of this figure is a reciprocal scale and the abscissa is linear. For the fuels shown, 2,2,3,4-tetramethylpentane, 2,3,3,4-tetramethylpentane, and 2,2,3-trimethylbutane, the blending relation with the base fuel is linear up to a concentration of 50 percent added paraffin. Knock-limited indicated mean effective pressures (fig. V-6 (a)) for 2,2,3,4-tetramethylpentane and 2,3,3,4-tetramethylpentane are from reference 12. Similar data for 2,2,3-trimethylbutane are from reference 11.

Although data for these fuels at lean fuel-air ratios are not shown herein, an examination of such data indicated that the blending relation is nonlinear. The authors of reference 17 attribute this fact to the variation of the end-gas temperature from one blend to another. That is, for a system in which a paraffinic blending agent is blended with a paraffinic base stock, the relation between the reciprocal of the knock-limited performance and the composition will be linear if the end-gas temperature, or a wall temperature closely related to the end-gas temperature, is held constant for each blend tested.

Olefins.—Blending data for two olefins (reference 12) are shown in figure V-6 (b) for the A. S. T. M. Supercharge engine operating at a rich fuel-air ratio. In this case, olefinic



(b) Engine, A. S. T. M. Supercharge.

FIGURE V-5.—Concluded. Comparison of isomers having highest antiknock values in blend with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

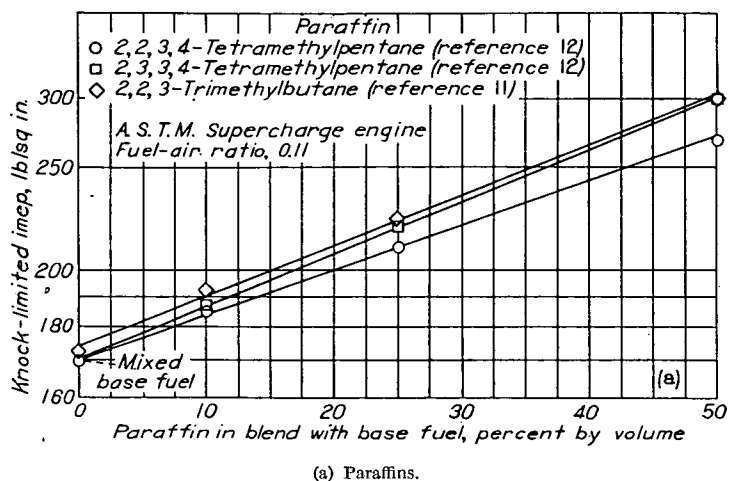


FIGURE V-6.—Knock-limited performance of blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

blending agents are blended with a paraffinic base fuel and the resulting relation between the reciprocal of the knock-limited performance and composition is nonlinear. The blending equation (1) is based upon one assumption, that for the equation to apply the blends should be tested at a constant percentage of excess of fuel or air. The differences between stoichiometric fuel-air ratios for olefins and paraffins, however, do not appear sufficiently great to explain the non-linearity of this blending curve.

Aromatics.—The blending relations for the aromatic hydrocarbons (fig. V-6 (c)), like those of the olefins (fig. V-6 (b)), were found to be nonlinear in the A. S. T. M. Supercharge engine at rich mixtures. With the exception of 1,2-dimethylbenzene and 1,2,4-trimethylbenzene, all the aromatics increased the knock-limited performance of the base fuel at the concentration investigated.

It has previously been mentioned that the concentration level at which compounds are examined may have considerable effect on the relative order of antiknock rating, as shown

in figure V-6 (c) for isopropylbenzene. For example, a blend of 50 percent by volume of isopropylbenzene has the second highest antiknock rating of the aromatics investigated; at concentrations below 35 percent by volume, however, the performance of isopropylbenzene is exceeded by that of 1,3-dimethylbenzene, 1,3-diethylbenzene, 1-ethyl-4-methylbenzene, and *n*-propylbenzene.

This result can perhaps be seen a little more clearly in figure V-7 (b), in which the blending data for the A. S. T. M. Supercharge engine are illustrated by a bar chart. The hydrocarbons are listed on this chart in order of decreasing antiknock rating, as determined by the 50-percent blends. At lower concentrations, however, the bars indicate a different order of rating.

At A. S. T. M. Aviation conditions (fig. V-7 (a)), the variation of knock-limited performance with composition was found to be different from that obtained at A. S. T. M. Supercharge conditions (figs. V-6 (c) and V-7 (b)). For example, the data presented in figure V-7 (a) indicate that the knock-limited performance of the base fuel is decreased as the concentration of aromatic is increased. Moreover, in figure V-7 (a) the aromatics do not rate in the same order at all concentrations.

Ethers.—Blending data for six ethers determined at A. S. T. M. Supercharge conditions are shown in figure V-8 (b). Methyl *tert*-butyl ether and ethyl *tert*-butyl ether have the highest antiknock characteristics of the six ethers at all concentrations. Isopropyl *tert*-butyl ether is also better than the three aromatic ethers at a concentration of 50 percent; however, at concentrations below about 20 percent, isopropyl *tert*-butyl ether is lower than any of the other ethers.

The ethers shown in figure V-8 (b), like the olefins and aromatics, do not follow the reciprocal blending relation defined by equation (1).

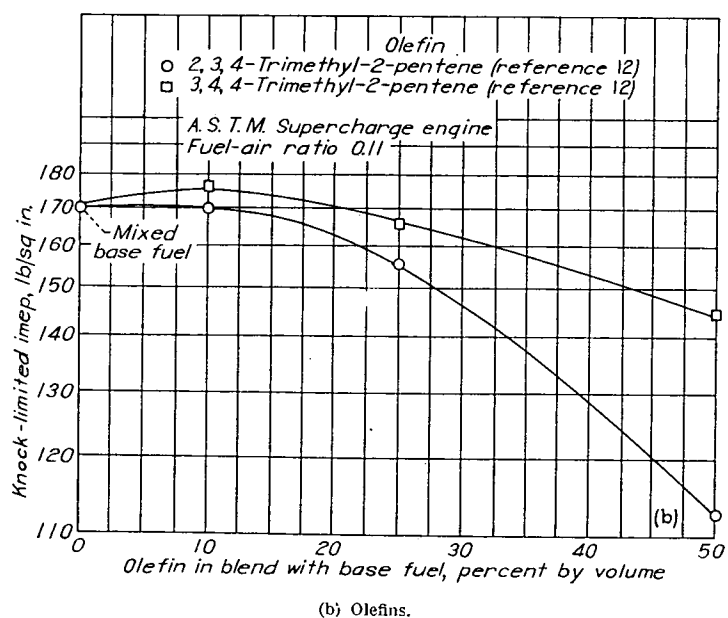


FIGURE V-6.—Continued. Knock-limited performance of blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

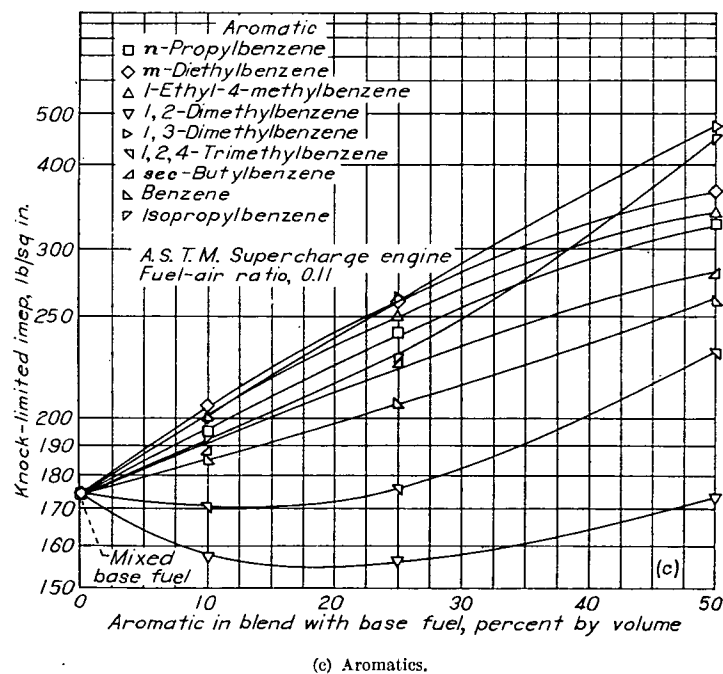
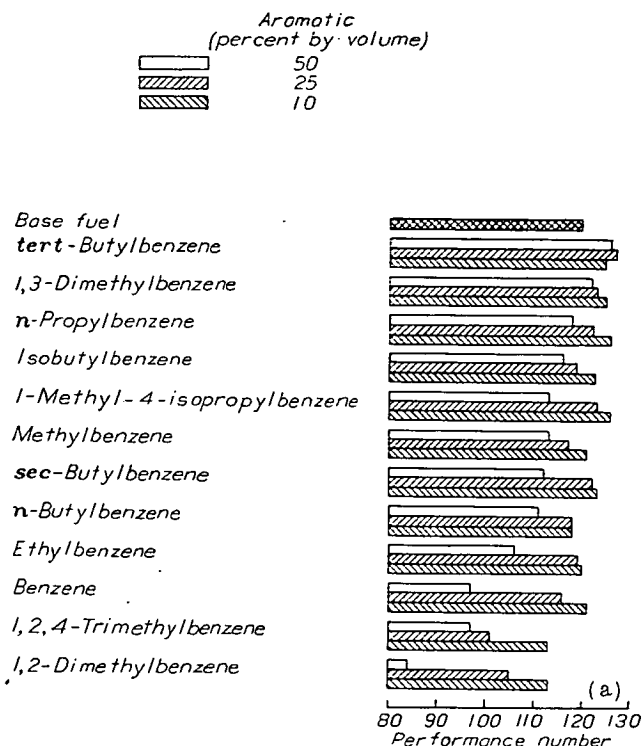
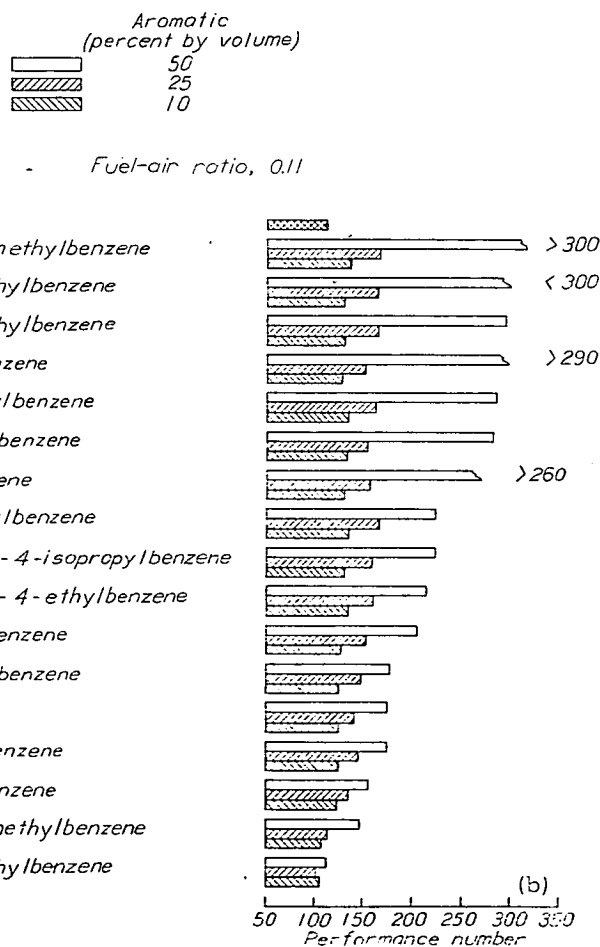


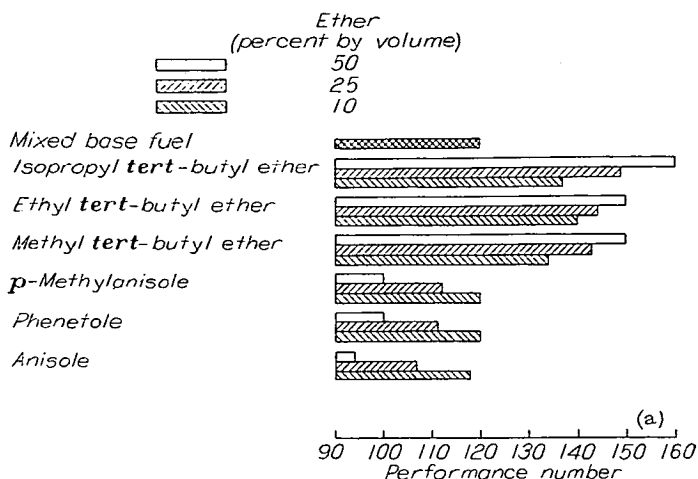
FIGURE V-6.—Concluded. Knock-limited performance of blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.



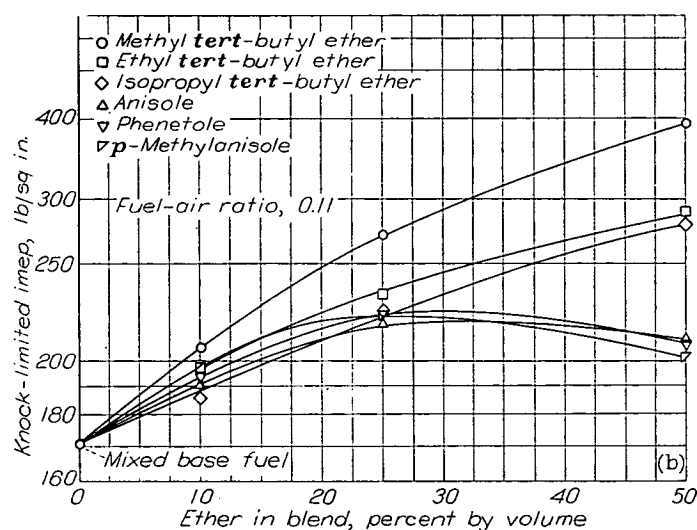
(a) Engine, A. S. T. M. Aviation.

FIGURE V-7.—Comparison of knock-limited performance of aromatic blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

(b) Engine, A. S. T. M. Supercharge.

FIGURE V-7.—Concluded. Comparison of knock-limited performance of aromatic blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

(a) Engine, A. S. T. M. Aviation.

FIGURE V-8.—Comparison of knock-limited performance of ether blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

(b) Engine, A. S. T. M. Supercharge.

FIGURE V-8.—Concluded. Comparison of knock-limited performance of ether blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon.

The blending relations for the ethers in figure V-8 (b) were investigated at A. S. T. M. Aviation conditions and the results obtained are presented in figure V-8 (a). At these conditions, the three *tert*-butyl alkyl ethers all improved the knock-limited performance of the base fuel; the improvement became greater as concentration was increased. On the other hand, the three aromatic ethers decreased the performance of the base fuel; the decrease became greater as the concentration was increased.

TEMPERATURE SENSITIVITY

In order to determine the effects of changes of inlet-air temperature on knock-limited performance, most of the hydrocarbons and ethers were evaluated in the 17.6 engine at inlet-air temperatures of 100° and 250° F. These tests were made with each compound in 20-percent-by-volume blends with isooctane. The final blends were evaluated at both temperatures in the unleaded state and with 4 ml TEL per gallon. (See appendix A, tables A-3 and A-4, respectively.) The greatest portion of the temperature-sensitivity studies of this investigation was conducted on blends with

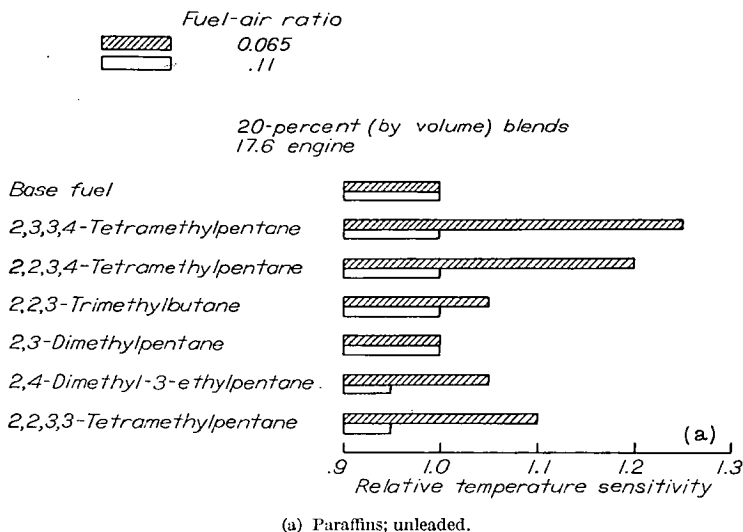
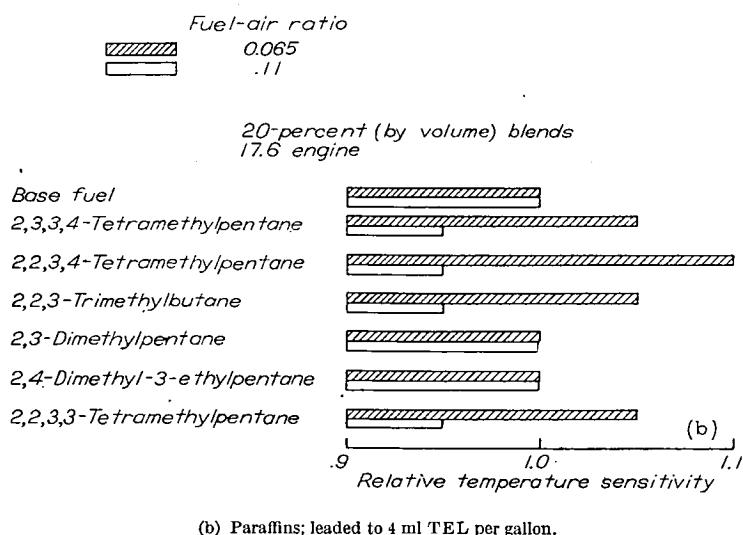


FIGURE V-9.—Temperature sensitivity of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.

isooctane. A few experiments, however, were made in which the compounds were blended with the mixed base fuel. (See appendix A, table A-5.)

The term "temperature sensitivity" has been given several definitions by investigators in the field of fuel research; however, none of these definitions has been wholly satisfactory. Perhaps the data offering the most scientific approach to such a definition are reported in references 21 to 24, but the emphasis in these references is placed upon engine severity rather than the more restricted idea of temperature sensitivity; that is, engine severity is a more inclusive term that considers other factors of engine performance such as compression ratio, spark advance, engine speed, and cooling, as well as inlet-air temperature.

Considerable experimental data are required in order to evaluate fully the engine severity as described in references 21 to 24 and in most cases during the present investigation the available quantities of the pure fuels were too small for



(b) Paraffins; leaded to 4 ml TEL per gallon.

FIGURE V-9.—Continued. Temperature sensitivity of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.

extensive studies. For this reason, the sensitivity studies of these fuels to changes of engine conditions were restricted merely to measurements of the effect of inlet-air temperature on knock-limited performance. In so doing it was necessary to establish arbitrarily a definition for temperature sensitivity. This term is defined by the following equation:

$$\text{Relative temperature sensitivity} = \frac{\text{knock-limited imep of blend (inlet air at 100° F)}}{\text{knock-limited imep of base fuel (inlet air at 100° F)}} \div \frac{\text{knock-limited imep of blend (inlet air at 250° F)}}{\text{knock-limited imep of base fuel (inlet air at 250° F)}}$$

The term "relative" is used in this definition as the equation essentially describes the temperature sensitivity of the blend relative to that of the base fuel. This definition is the same as that used in references 5 to 11 and 13. The base fuels used in this study were paraffins and do not show high temperature sensitivity.

Temperature sensitivities computed by this equation for all the compounds in the present investigation are presented in appendix A, table A-6. In the discussion of temperature sensitivity in the following paragraphs and in the subsequent discussion of lead susceptibility, it should be remembered that the data were obtained over a long period of time and reproducibility errors therefore exist. Although no extensive reproducibility data were obtained, a few such runs indicated that relative temperature sensitivities computed by the equation and relative lead susceptibilities computed by a similar equation may be in error by ± 0.05 .

Paraffins.—The temperature sensitivities of unleaded and leaded paraffinic fuel blends in the 17.6 engine at two fuel-air ratios are compared in figures V-9 (a) and V-9 (b). Of the paraffinic blending agents investigated (references 12 and 13), the three nonanes, 2,3,3,4-tetramethylpentane, 2,2,3,4-tetramethylpentane, and 2,2,3,3-tetramethylpentane, appear to be most sensitive to changes of inlet-air temperature at the lean fuel-air ratio in unleaded blends (fig. V-9 (a)). At the rich fuel-air ratio, however, the differences in temperature sensitivities among the paraffins are small.

In figures V-9 (a) and V-9 (b), the paraffins are listed in the same order. Inspection of these plots illustrates that tetraethyl lead affects temperature sensitivity. For example, in figures V-9 (a) and V-9 (b) the order of temperature sensitivities of the various paraffins is obviously different at both fuel-air ratios.

As previously mentioned, a few of the compounds in this investigation were examined in blends with the mixed base fuel. In the investigation of reference 13, paraffinic and olefinic blending agents in blends with the mixed base fuel were subjected to variations of compression ratio. When these data are computed in the manner explained in references 23 and 24, it is possible to compare over a reasonably wide range the influence of engine severity on knock-limited performance. This effect is determined by computation of compression-air densities and compression temperatures at the knock limit; the main assumption is that these factors are related in some manner to end-gas densities and temperatures that cannot be directly measured (reference 21). The

Fuel-air ratio
 0.065
 .11

20-percent (by volume) blends
 17.6 engine

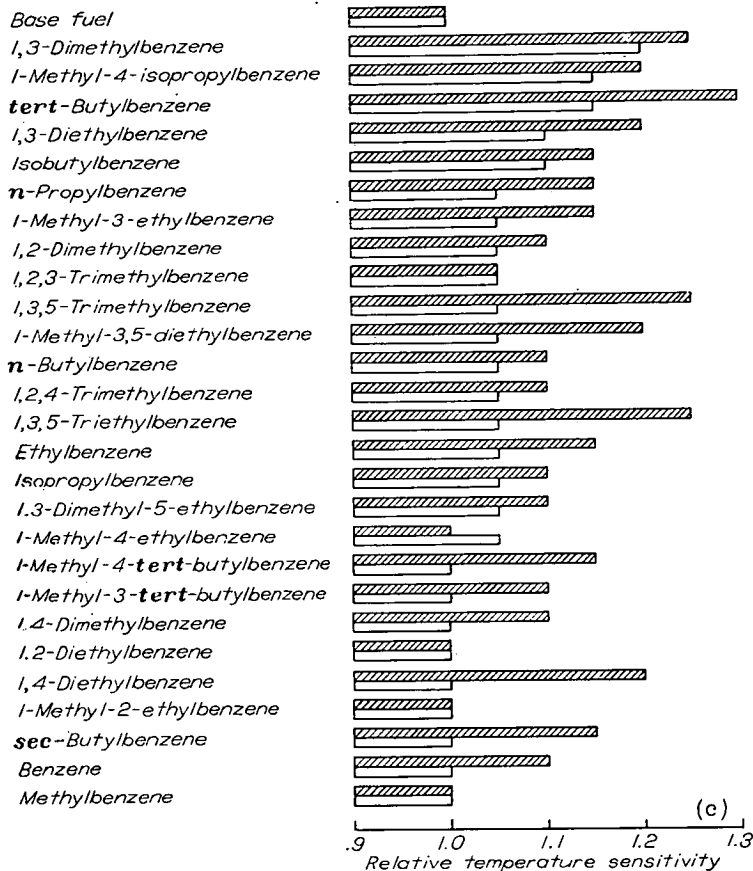


FIGURE V-9.—Continued. Temperature sensitivity of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.

Fuel-air ratio
 0.065
 .11

20-percent (by volume) blends
 17.6 engine

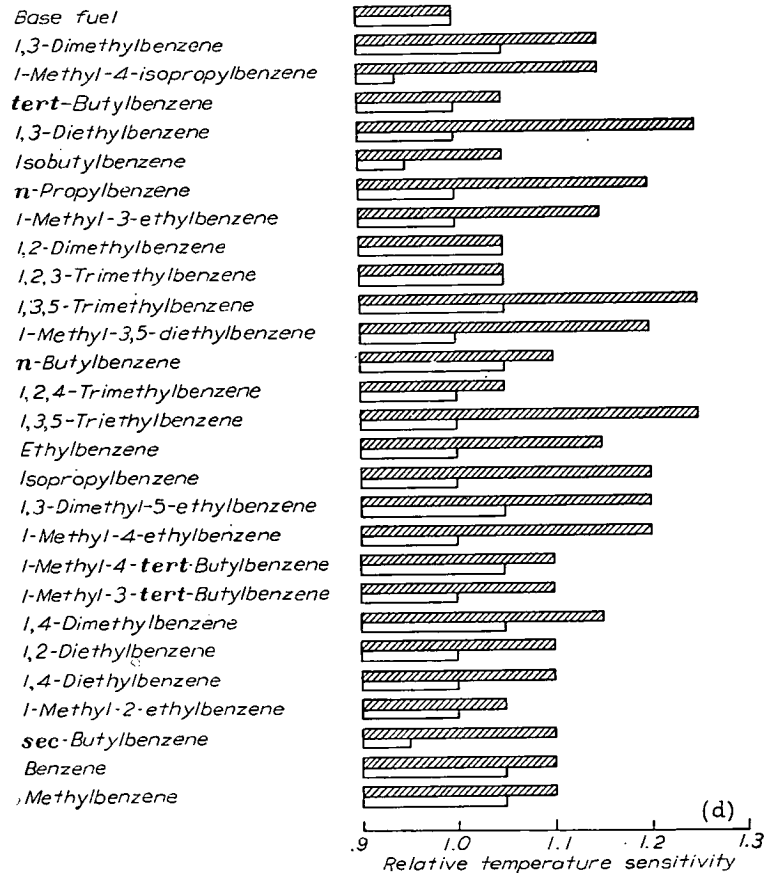


FIGURE V-9.—Continued. Temperature sensitivity of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.

Fuel-air ratio
 0.065
 .11

20-percent (by volume) blends
 17.6 engine

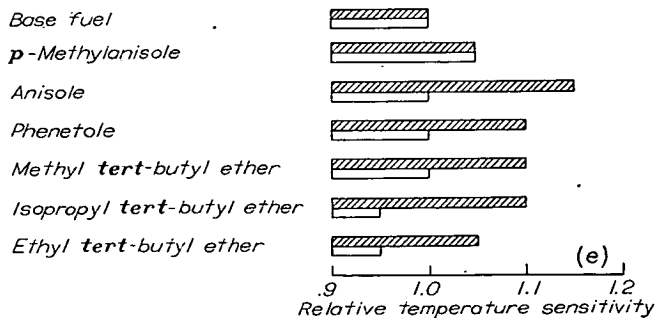


FIGURE V-9.—Continued. Temperature sensitivity of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.

Fuel-air ratio
 0.065
 .11

20-percent (by volume) blends
 17.6 engine

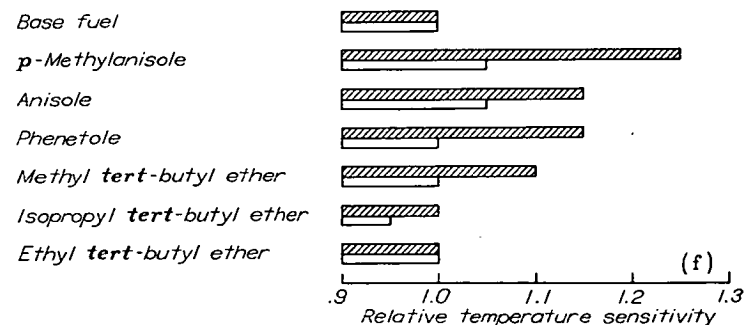


FIGURE V-9.—Concluded. Temperature sensitivity of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.

compression-air densities and temperatures are calculated by the following equations:

$$\rho_c = \frac{W_0 (r-1)}{i V_d} \quad (2)$$

$$T_c = T_0 r^{\gamma-1} \quad (3)$$

where

ρ_c compression-air density, pounds per cubic inch

W_0 intake-air flow, pounds per minute

r compression ratio

i intake cycles per minute

V_d engine displacement volume, cubic inches

T_c compression-air temperature, °R

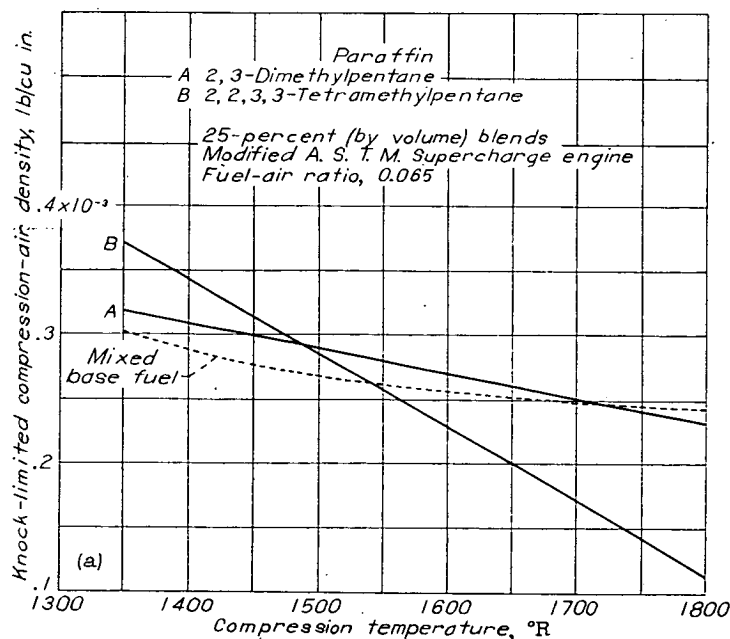
T_0 intake-air temperature, °R

γ ratio of specific heat of charge at constant pressure to that at constant volume (assumed to be 1.4)

Although the data in reference 13 were determined by varying the compression ratio, it is apparent from the equation of compression temperature that the effect of varying the compression ratio is equivalent to that of varying the intake-air temperature.

The sensitivities of two paraffinic fuels (reference 13) are shown in figures V-10 (a) and V-10 (b) at two fuel-air ratios in a modified A. S. T. M. Supercharge engine. The two paraffin blends are more sensitive than the base fuel to changes of compression ratio or intake-air temperature, as indicated by the slopes of the curves in figures V-10 (a) and V-10 (b). The two paraffin blends had lower knock limits than the base fuel at severe conditions (high compression temperatures), but higher limits at mild conditions (low compression temperatures).

Olefins.—Plots similar to those in figures V-10 (a) and V-10 (b) are shown in figures V-10 (c) and V-10 (d) for



(a) Paraffins; lean conditions.

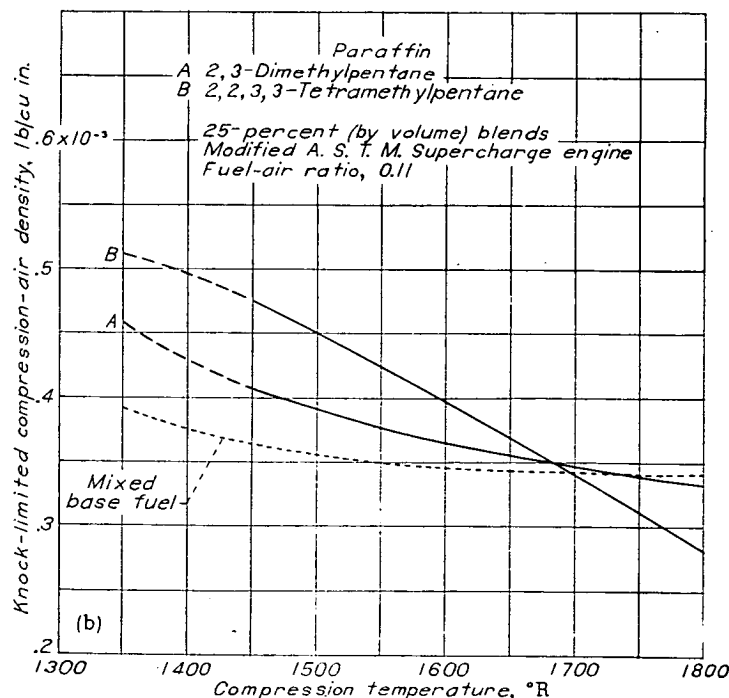
FIGURE V-10.—Effect of compression temperature on compression-air density for blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon. Compression ratio, variable; engine speed, 1800 rpm; inlet-air temperature, 250° F.; coolant temperature, 250° F.; spark advance, 30° B. T. C.

three olefins in blends with the mixed base fuel (reference 13). At both fuel-air ratios, the three olefin blends were more sensitive to a change of engine severity than the base fuel. At the severe conditions the three olefin blends had lower knock limits than did the base fuel, but at milder conditions the olefin blends had higher knock limits.

Aromatics.—The temperature sensitivities of aromatic blends determined in the 17.6 engine are shown in figures V-9 (c) and V-9 (d). The aromatics are listed in figure V-9 (c) in the order of decreasing sensitivity at the rich fuel-air ratio. As in the case of paraffins (figs. V-9 (a) and V-9 (b)), the sensitivities were inconsistent from one fuel-air ratio to another. Moreover, the sensitivities were influenced by tetraethyl lead.

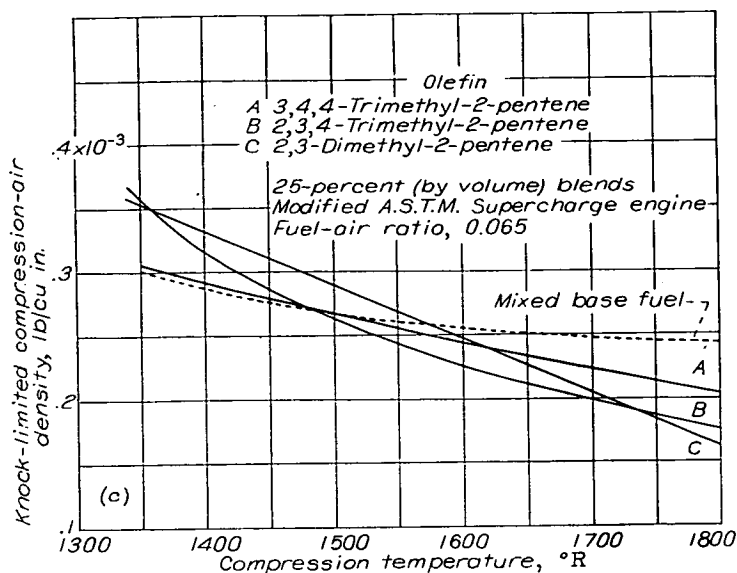
The most sensitive aromatics at the rich fuel-air ratio (fig. V-9 (c)) were 1,3-dimethylbenzene, 1-methyl-4-isopropylbenzene, and *tert*-butylbenzene; whereas at the lean fuel-air ratio, a number of aromatics had high sensitivities. In leaded blends (fig. V-9 (d)), the differences in relative temperature sensitivity among the aromatics were not great at the rich fuel-air ratio, but at a lean fuel-air ratio appreciable differences occurred. At the lean fuel-air ratio, a number of the aromatics had sensitivities 20 to 25 percent greater than the sensitivity of the base fuel.

It has been shown herein that 1,3,5-trimethylbenzene and *tert*-butylbenzene had higher performance numbers than the other aromatics investigated at the lean condition of the A. S. T. M. Aviation method (fig. V-3 (i)). For this reason the temperature sensitivities of these two aromatics are of particular interest. These two aromatics in unleaded blends have temperature sensitivities equal to or greater than sensitivities of the other aromatics investigated at the lean fuel-



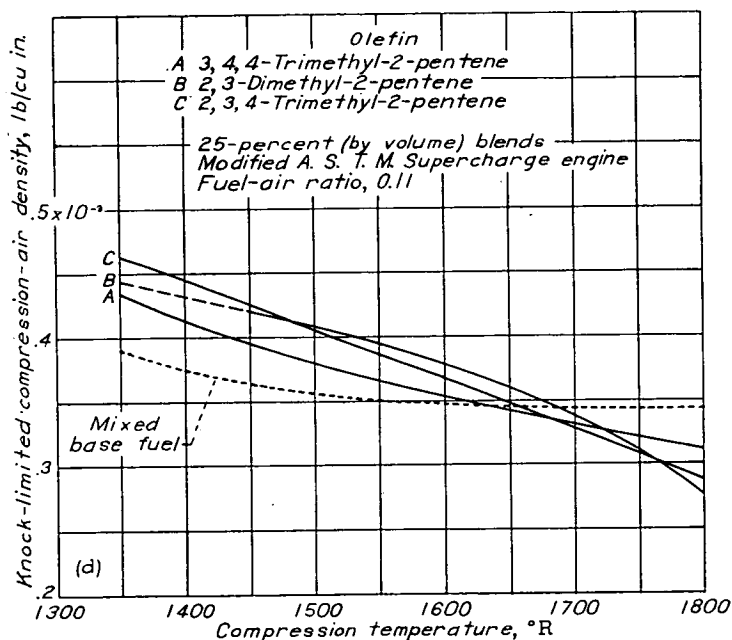
(b) Paraffins; rich conditions.

FIGURE V-10.—Continued. Effect of compression temperature on compression-air density for blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon. Compression ratio, variable; engine speed, 1800 rpm; inlet-air temperature, 250° F.; coolant temperature, 250° F.; spark advance, 30° B. T. C.



(c) Olefins; lean conditions.

FIGURE V-10.—Continued. Effect of compression temperature on compression-air density for blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon. Compression ratio, variable; engine speed, 1800 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F; spark advance, 30° B. T. C.

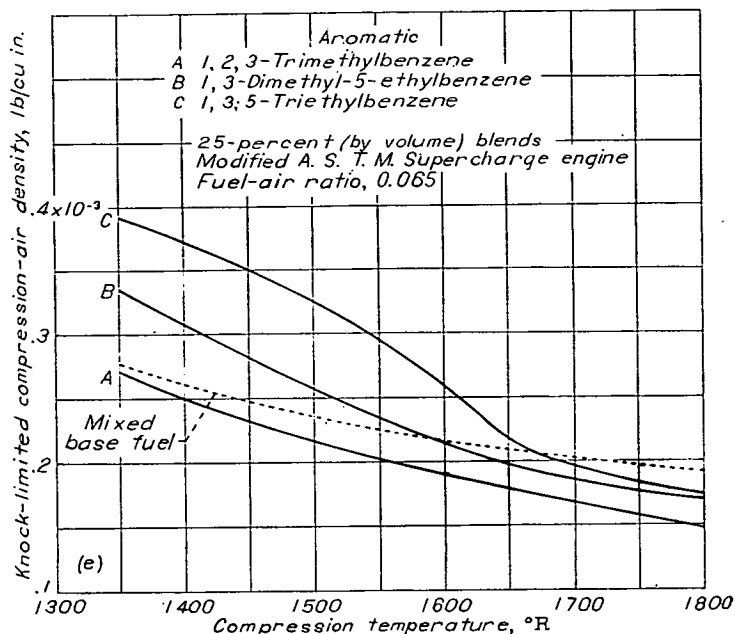


(d) Olefins; rich conditions.

FIGURE V-10.—Continued. Effect of compression temperature on compression-air density for blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon. Compression ratio, variable; engine speed, 1800 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F; spark advance, 30° B. T. C.

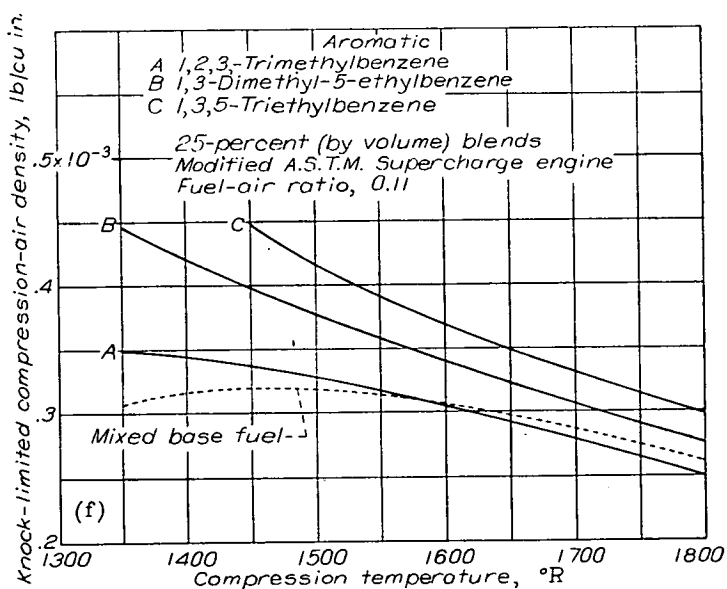
air ratio (fig. V-9 (c)). On the other hand, the leaded blends shown in figure V-9 (d) indicate that the temperature sensitivity of *tert*-butylbenzene is reduced considerably, whereas 1,3,5-trimethylbenzene is still quite sensitive.

Similarly, among the better aromatics at A. S. T. M. Supercharge conditions (fig. V-3 (j)) were 1,3-dimethyl-5-ethylbenzene, 1-methyl-3,5-diethylbenzene, 1-methyl-4-*tert*-



(e) Aromatics; lean conditions.

FIGURE V-10.—Continued. Effect of compression temperature on compression-air density for blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon. Compression ratio, variable; engine speed, 1800 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F; spark advance, 30° B. T. C.



(f) Aromatics; rich conditions.

FIGURE V-10.—Concluded. Effect of compression temperature on compression-air density for blends with mixed base fuel consisting of 87.5 percent isooctane and 12.5 percent *n*-heptane+4 ml TEL per gallon. Compression ratio, variable; engine speed, 1800 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F; spark advance, 30° B. T. C.

butylbenzene, and 1,3,5-triethylbenzene. As indicated in figure V-9 (c) for unleaded blends at a rich fuel-air ratio, these four aromatics show only moderate temperature sensitivity varying between 1.0 and 1.05. In leaded blends (fig. V-9 (d)) and at a rich fuel-air ratio, the four aromatics still exhibited only moderate temperature sensitivity varying between 1.0 and 1.05.

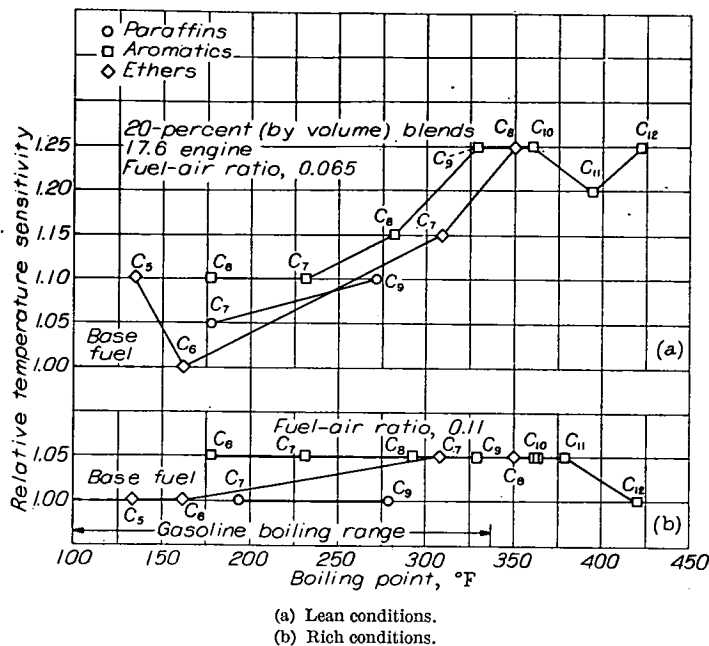


FIGURE V-11.—Comparison of isomers having highest temperature sensitivities in blend with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.

Compression-air density-temperature relations were determined for several aromatics and are reported in reference 10. The relation obtained for three of the aromatics is presented in figures V-10 (e) and V-10 (f) in order to illustrate the nature of the results. As indicated by the slopes of the curves in these figures, the sensitivities of the aromatic blends are somewhat greater than the sensitivity of the base fuel.

Ethers.—Temperature sensitivities determined for six ethers are shown in figures V-9 (e) and V-9 (f). The ethers (unleaded blends) are listed in figure V-9 (e) in the order of decreasing sensitivity at the rich fuel-air ratio (0.11); at this fuel-air ratio the three aromatic ethers appear to be more

sensitive to temperature changes than do the *tert*-butyl alkyl ethers, with the possible exception of methyl *tert*-butyl ether. At the lean fuel-air ratio (0.065), anisole appears to be the most sensitive of the ethers; however, with consideration for the estimated reproducibility of these data there may be little real difference in the sensitivities of the six ethers shown.

In leaded blends (fig. V-9 (f)), the aromatic ethers are perhaps more temperature-sensitive than the *tert*-butyl alkyl ethers with the possible exception of methyl *tert*-butyl ether at the lean fuel-air ratio. At the rich fuel-air ratio, anisole and *p*-methylanisole show the highest sensitivities; however, the experimental accuracy may minimize the apparent differences shown on the figures.

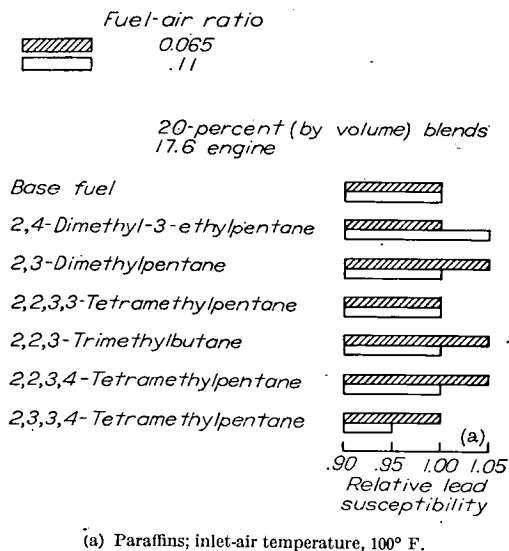
Comparison of classes of compounds.—The temperature sensitivities of the various classes of compounds are compared in figure V-11. The procedure used in preparing these plots was the same as that used for figure V-5.

In figure V-11 at two fuel-air ratios, the low-boiling ethers have the greatest temperature sensitivities in the boiling range of 100° to 175° F. Above 175° F the aromatics are more sensitive than the other classes examined. In the boiling range from 300° to 350° F, however, the ethers have temperature sensitivities comparable to those of the aromatics.

LEAD SUSCEPTIBILITY

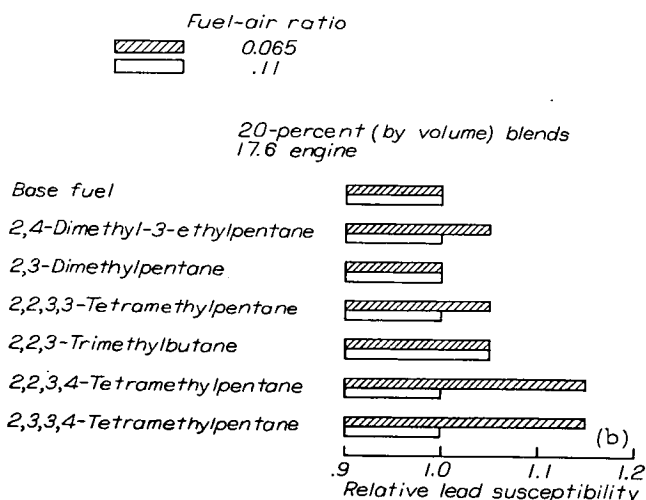
Lead susceptibilities of the various organic compounds investigated were determined in the 17.6 engine by comparing unleaded blends (20 percent by volume) with blends containing 4 ml TEL per gallon. Data were obtained at two inlet-air temperatures, 100° and 250° F. (See appendix A, table A-7.)

Lead susceptibility, or lead response, is usually defined as the increase in octane number or power output resulting from the addition of a given quantity of tetraethyl lead to a fuel. For the present investigation, however, lead susceptibility is



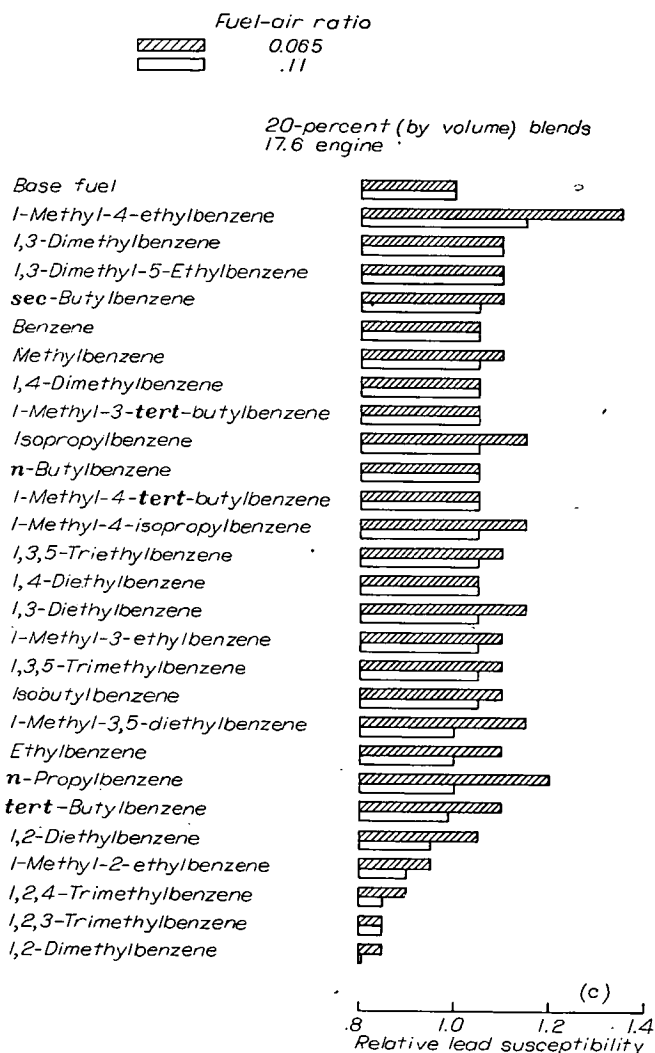
(a) Paraffins; inlet-air temperature, 100° F.

FIGURE V-12.—Lead susceptibility (4 ml TEL/gal) of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.



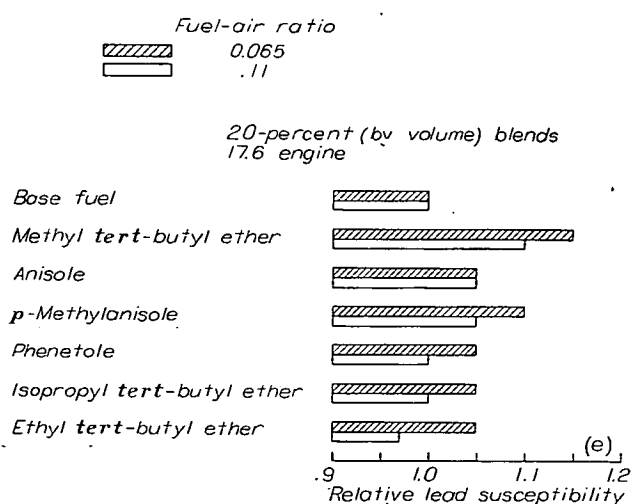
(b) Paraffins; inlet-air temperature, 250° F.

FIGURE V-12.—Continued. Lead susceptibility (4 ml TEL/gal) of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.



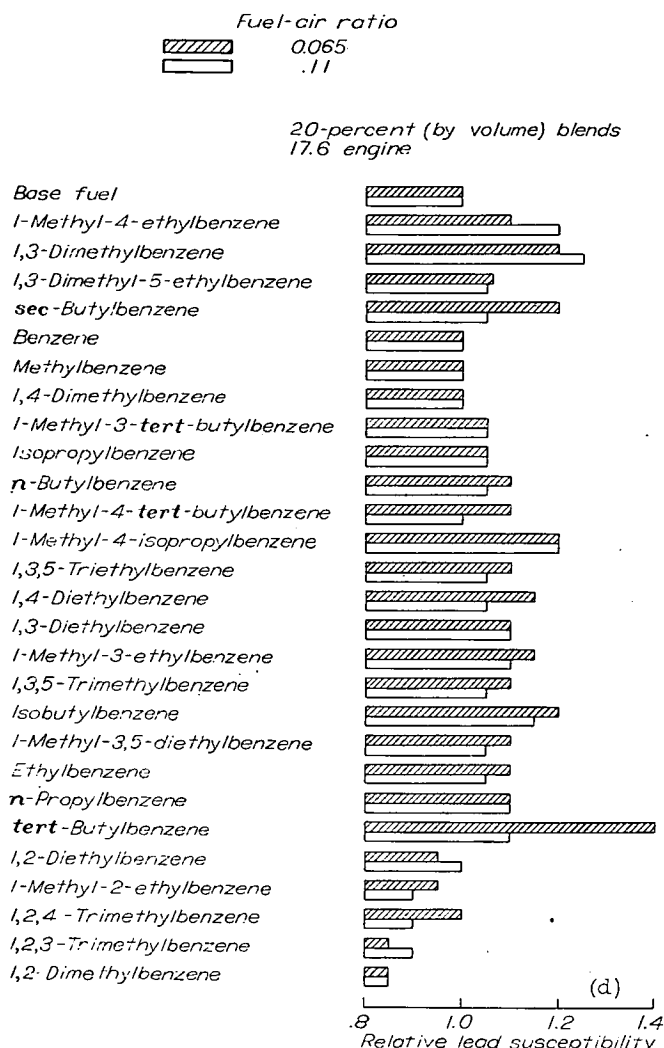
(c) Aromatics; inlet-air temperature, 100° F.

FIGURE V-12.—Continued. Lead susceptibility (4 ml TEL/gal) of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.



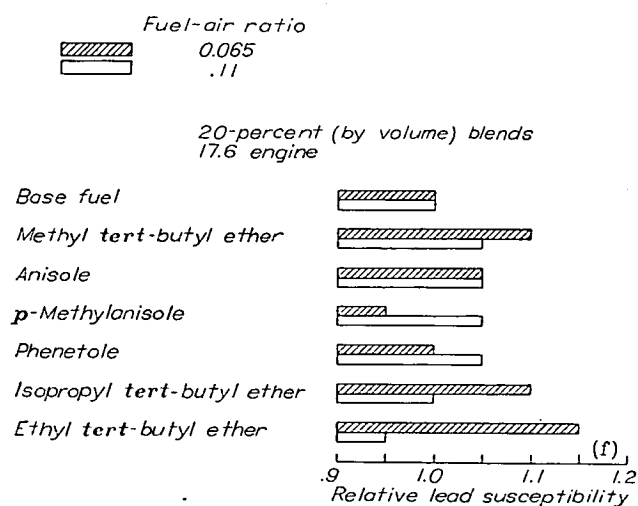
(e) Ethers; inlet-air temperature, 100° F.

FIGURE V-12.—Continued. Lead susceptibility (4 ml TEL/gal) of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.



(d) Aromatics; inlet-air temperature, 250° F.

FIGURE V-12.—Continued. Lead susceptibility (4 ml TEL/gal) of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.



(f) Ethers; inlet-air temperature, 250° F.

FIGURE V-12.—Concluded. Lead susceptibility (4 ml TEL/gal) of blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.

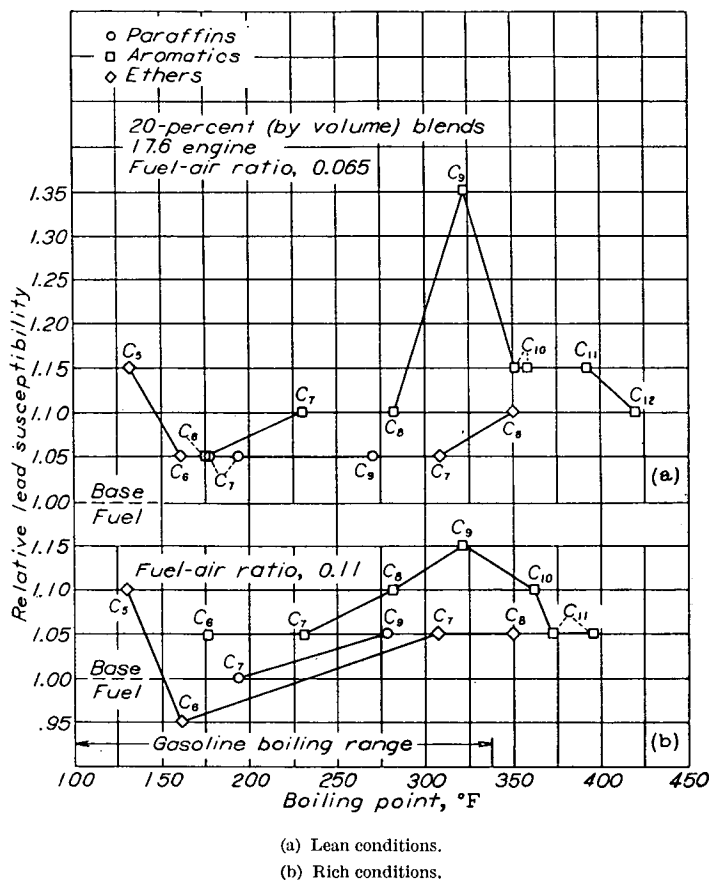


FIGURE V-13.—Comparison of isomers having highest lead susceptibility in blends with isooctane. Compression ratio, 7.0; engine speed, 1800 rpm; inlet-air temperature, 100° F; coolant temperature, 212° F; spark advance, 30° B. T. C.

expressed in a manner similar to that used for temperature sensitivity:

$$\text{Relative lead susceptibility} = \frac{\left\{ \begin{array}{l} \text{knock-limited imep of blend} + 4 \text{ ml TEL/gal} \\ \text{knock-limited imep of base fuel} + 4 \text{ ml TEL/gal} \end{array} \right\}}{\left\{ \begin{array}{l} \text{knock-limited imep of blend} + 0 \text{ ml TEL/gal} \\ \text{knock-limited imep of base fuel} + 0 \text{ ml TEL/gal} \end{array} \right\}}$$

As in the foregoing discussion of temperature sensitivity, the estimated accuracy of these ratios is about ± 0.05 .

Paraffins.—The lead susceptibilities of six paraffinic blends are shown in figures V-12 (a) and V-12 (b). In figure V-12 (a) (inlet-air temperature, 100° F), the fuels are arranged in order of decreasing response at the rich mixture. At this condition, 2,4-dimethyl-3-ethylpentane exhibits the greatest susceptibility to tetraethyl lead, but at the lean fuel-air ratio, 2,3-dimethylpentane, 2,2,3-trimethylbutane, and 2,2,3,4-tetramethylpentane have the best response. The lead susceptibility is appreciably influenced by fuel-air ratio.

In figure V-12 (b) (inlet-air temperature, 250° F), the fuels are listed in the same order as that of figure V-12 (a), but little or no difference in lead susceptibility is apparent at the rich fuel-air ratio except in the case of 2,2,3-trimethylbutane. At the lean fuel-air ratio, 2,2,3,4-tetramethylpentane and 2,3,3,4-tetramethylpentane had the highest lead susceptibilities.

Olefins.—A limited amount of data was obtained in the 17.6 engine to show the lead susceptibility of olefins in 20-

percent-by-volume blends with isooctane. (See appendix A, table A-7 (a).) For convenience, a portion of these data is summarized in the following table:

Olefin	Lead susceptibility of 20-per- cent olefinic blends relative to isooctane			
	Inlet-air temperature (°F)			
	250		100	
	Fuel-air ratio			
	0.065	0.11	0.065	0.11
	2,3-dimethyl-2-pentene.....	0.95	1.00	1.00
2,3,4-trimethyl-2-pentene.....	1.05	1.05	1.05	1.05
3,4,4-trimethyl-2-pentene.....	1.00	1.00	1.05	1.00

Aromatics.—In figures V-12 (c) and V-12 (d), the lead susceptibilities of aromatic blends are shown. The blends in figure V-12 (c) are listed in order of decreasing response at the rich fuel-air ratio. At this ratio, the data indicate that 1-methyl-4-ethylbenzene is the aromatic most susceptible to additions of tetraethyl lead. This particular aromatic also had the greatest response at the lean fuel-air ratio. From figures V-12 (c) and V-12 (d), lead susceptibility is obviously affected by fuel-air ratio.

At the higher inlet-air temperature (fig. V-12 (d)), the trend in lead susceptibility differs from that observed at 100° F (fig. V-12 (c)) for the aromatics. For the rich fuel-air ratio (fig. V-12 (d)), three of the aromatics, 1-methyl-4-ethylbenzene, 1,3-dimethylbenzene, and 1-methyl-4-isopropylbenzene, appear to be the most susceptible. At the lean fuel-air ratio, however, *tert*-butylbenzene is considerably more susceptible than the other aromatics.

Ethers.—Lead susceptibilities of the ether blends are presented in figures V-12 (e) and V-12 (f). At an inlet-air temperature of 100° F (fig. V-12 (e)), methyl *tert*-butyl ether and *p*-methylanisole have the greatest lead susceptibilities at the lean fuel-air ratio. At the rich fuel-air ratio, methyl *tert*-butyl ether has the highest susceptibility with anisole and *p*-methylanisole next.

At an inlet-air temperature of 250° F (fig. V-12 (f)), the three *tert*-butyl alkyl ethers have the highest susceptibilities at the lean fuel-air ratio. The three aromatic ethers and methyl *tert*-butyl ether exhibit the highest susceptibilities at the rich fuel-air ratio.

Comparison of classes of compounds.—In figure V-13, the lead susceptibilities are plotted against boiling points for the isomers having highest lead susceptibilities in each class of compounds. At both lean (fig. V-13 (a)) and rich (fig. V-13 (b)) fuel-air ratios, the low-boiling ethers appear to be most susceptible to tetraethyl lead in the boiling range from 125° to 160° F. Above 160° F, the aromatics show the greatest lead response.

CONCLUDING REMARKS

On the basis of an investigation of the type reported herein, it is difficult to draw any specific conclusions, inasmuch as antiknock characteristics are influenced by many

factors. The relative order of antiknock ratings of a series of compounds is influenced by engine conditions, by the tetraethyl lead content, and by the concentration of blending agent in the base fuel with which a comparison is made. With consideration for these factors, *tert*-butylbenzene, methyl and ethyl *tert*-butyl ethers, 2,2,3-trimethylbutane, and several nonanes were among the best compounds in their respective organic classes. This selection was based upon temperature sensitivity and lead susceptibility as well as antiknock value.

In an effort to generalize the data obtained in this investigation, the subsequent conclusions are expressed in terms of the relation of various performance factors to the gasoline boiling range as influenced by the classes of organic compounds investigated. Furthermore, these conclusions must necessarily be restricted to the limitations of this investigation and therefore cannot be applied without exception.

Antiknock ratings.—In the low-boiling gasoline range, the highest antiknock ratings are among the more volatile paraffins and ethers. In the intermediate gasoline range, the ethers excel; in the high-boiling range the aromatics have the highest antiknock ratings.

Temperature sensitivity.—In the low-boiling gasoline range, the data are incomplete as regards temperature sensitivity, but there are indications that the volatile ethers are more sensitive to temperature changes than are the paraffins or aromatics. In the intermediate and high-boiling ranges of gasoline, the aromatics are more sensitive to temperature than the paraffins and the ethers. Moreover, the aromatics that have the highest antiknock ratings are also sensitive to temperature.

Lead susceptibility.—In the low-boiling gasoline range, the data are incomplete as regards lead susceptibility, but there are indications that the more volatile ethers are more susceptible to additions of tetraethyl lead than are the paraffins and the aromatics. In the intermediate and high-boiling ranges of gasoline, the aromatics show greater lead susceptibility than either the paraffins or the ethers.

REFERENCES

1. Lovell, Wheeler G.: Knocking Characteristics of Hydrocarbons. *Ind. and Eng. Chem.*, vol. 40, no. 12, Dec. 1948, pp. 2388-2438.
2. Jones, Anthony W., and Bull, Arthur W.: Knock-Limited Performance of Pure Hydrocarbons Blended with a Base Fuel in a Full-Scale Aircraft-Engine Cylinder. I—Eight Paraffins, Two Olefins. NACA ARR E4E25, 1944.
3. Bull, Arthur W., and Jones, Anthony W.: Knock-Limited Performance of Pure Hydrocarbons Blended with a Base Fuel in a Full-Scale Aircraft-Engine Cylinder. II—Twelve Aromatics. NACA ARR E4I09, 1944.
4. Jones, Anthony W., Bull, Arthur W., and Jonash, Edmund R.: Knock-Limited Performance of Pure Hydrocarbons Blended with a Base Fuel in a Full-Scale Aircraft-Engine Cylinder. III—Four Aromatics, Six Ethers. NACA ARR E6B14, 1946.
5. Meyer, Carl L., and Branstetter, J. Robert: The Knock-Limited Performance of Fuel Blends Containing Aromatics. I—Toluene, Ethylbenzene, and *p*-Xylene. NACA ARR E4J05, 1944.
6. Branstetter, J. Robert, and Meyer, Carl L.: The Knock-Limited Performance of Fuel Blends Containing Aromatics. II—Iso-propylbenzene, Benzene, and *o*-Xylene. NACA ARR E5A20, 1945.
7. Meyer, Carl L., and Branstetter, J. Robert: The Knock-Limited Performance of Fuel Blends Containing Aromatics. III—1,3,5-Trimethylbenzene, *tert*-Butylbenzene, and 1,2,4-Trimethylbenzene. NACA ARR E5D16, 1945.
8. Meyer, Carl L., and Branstetter, J. Robert: The Knock-Limited Performance of Fuel Blends Containing Aromatics. IV—Data for *m*-Diethylbenzene, 1-Ethyl-4-methylbenzene, and *sec*-Butylbenzene Together with a Summarization of Data for 12 Aromatic Hydrocarbons. NACA ARR E5D16a, 1945.
9. Meyer, Carl L., and Branstetter, J. Robert: The Knock-Limited Performance of Fuel Blends Containing Aromatics. V—*n*-Propylbenzene, *n*-Butylbenzene, Isobutylbenzene, *m*-Xylene, and 1-Isopropyl-4-methylbenzene. NACA ARR E6C05, 1946.
10. Drell, I. L., and Alquist, H. E.: Knock-Limited Performance of Fuel Blends Containing Aromatics. VI—10 Alkylbenzenes. NACA TN 1994, 1949.
11. Branstetter, J. Robert: Comparison of the Knock-Limited Performance of Triptane with 23 Other Purified Hydrocarbons. NACA MR E5E15, 1945.
12. Jonash, Edmund R., Meyer, Carl L., and Branstetter, J. Robert: Knock-Limited Performance Tests of 2,2,3,4-Tetramethylpentane, 2,3,3,4-Tetramethylpentane, 3,4,4-Trimethyl-2-pentene, and 2,3,4-Trimethyl-2-pentene in Small-Scale and Full-Scale Cylinders. NACA ARR E6C04, 1946.
13. Genco, R. S., and Drell, I. L.: Knock-Limited Performance of Several Branched Paraffins and Olefins. NACA TN 1616, 1948.
14. Drell, I. L., and Branstetter, J. R.: Knock-Limited Performance of Blends Containing Ethers. NACA TN 2070, 1950.
15. Rossini, Frederick K., Pitzer, Kenneth S., Taylor, William J., Ebert, Joan P., Kilpatrick, John E., Beckett, Charles W., Williams, Mary G., and Werner, Helene G.: Selected Values of Properties of Hydrocarbons. Circular C461, NBS, Nov. 1947.
16. Lovell, Wheeler G., and Campbell, John M.: Knocking Characteristics and Molecular Structure of Hydrocarbons. *The Science of Petroleum*, vol. IV, Oxford Univ. Press (London), 1938, pp. 3004-3023.
17. Sanders, Newell D., Hensley, Reece V., and Breitwieser, Roland: Experimental Studies of the Knock-Limited Blending Characteristics of Aviation Fuels. I—Preliminary Tests in an Air-Cooled Cylinder. NACA ARR E4I28, 1944.
18. Wear, Jerrold D., and Sanders, Newell D.: Experimental Studies of the Knock-Limited Blending Characteristics of Aviation Fuels. II—Investigation of Leaded Paraffinic Fuels in an Air-Cooled Cylinder. NACA TN 1374, 1947.
19. Drell, I. L., and Wear, J. D.: Experimental Studies of the Knock-Limited Blending Characteristics of Aviation Fuel. III—Aromatics and Cycloparaffins. NACA TN 1416, 1947.
20. Hensley, Reece V., and Breitwieser, Roland: Knock-Limited Blending Characteristics of Benzene, Toluene, Mixed Xylenes, and Cumene in an Air-Cooled Cylinder. NACA MR E5B03, 1945.
21. Rothrock, A. M., and Biermann, Arnold E.: The Knocking Characteristics of Fuels in Relation to Maximum Permissible Performance of Aircraft Engines. NACA Rep. 655, 1939.
22. Taylor, E. S., Leary, W. A., and Diver, J. R.: Effect of Fuel-Air Ratio, Inlet Temperature, and Exhaust Pressure on Detonation. NACA Rep. 699, 1940.
23. Evvard, John C., and Branstetter, J. Robert: A Correlation of the Effects of Compression Ratio and Inlet-Air Temperature on the Knock Limits of Aviation Fuels in a CFR Engine—I. NACA ACR E5D20, 1945.
24. Alquist, Henry E., O'Dell, Leon, and Evvard, John C.: A Correlation of the Effects of Compression Ratio and Inlet-Air Temperature on the Knock Limits of Aviation Fuels in a CFR Engine—II. NACA ARR E6E13, 1946.

CHAPTER VI

AROMATIC AMINES AS FUEL ADDITIVES

In addition to those fuel components that may be classified in the broad category of blending agents, there are other compounds known as fuel additives. These compounds are generally distinguished by their chemical dissimilarity to the constituents normally found in petroleum and by their pronounced effect, when present in even small concentrations, on certain fuel characteristics. An optimum concentration of additive for over-all fuel performance can usually be found that best satisfies all requirements of the fuel; that is, the concentration of additive that will yield maximum improvement in a given fuel characteristic may have a deleterious effect on another equally important characteristic so that a compromise must be made.

By far the most important of the fuel additives are the so-called antiknock *dopes*. Tetraethyl lead (ch. VII), one of the most widely publicized additives in use today, is known primarily for its knock-suppressing qualities. The critical shortages of tetraethyl lead and the high-antiknock blending agents during World War II stimulated the search for other compounds that might be of value as antiknock agents. The NACA participated in this search for new antiknock additives, and a summarization of that survey of the aromatic amines is presented in this chapter.

Although the studies of the NACA placed greatest emphasis on the antiknock qualities of the aromatic amines, other properties were investigated to determine the useful concentrations that could be employed. These properties included low-temperature solubility measurements and determinations of the gasoline-water distribution coefficients. In addition, new methods of analysis were devised to determine the quantities of amines present in prepared fuels. Such analytical methods are necessary for purposes of fuel inspection because the use of additives is controlled by specifications.

ANTIKNOCK EVALUATION OF AROMATIC AMINES

Engines and test conditions.—Two small-scale engines and one full-scale single-cylinder test engine were used in the evaluation of the antiknock characteristics of aromatic amines. One small-scale engine was a CFR engine that conformed to the A. S. T. M. Supercharge method for knock rating except for the fuel system and the method of knock detection. The fuel system was arranged so that fuel was circulated through a primary pump, a fuel cooler, and back into the injection pump gallery. Knock was detected by a magnetostriction pickup unit in conjunction with a cathode-ray oscilloscope. Incipient detonation was taken as the criterion of knock.

As pointed out in chapter II, antiknock behavior at one condition of engine operation does not afford a satisfactory basis for estimation of performance at another condition or in another engine. For this reason, the following three sets of conditions were chosen for the small-scale engine evaluation of the aromatic amines.

	Inlet-air temperature (° F)	Spark advance (° B. T. C.)	Coolant temperature (° F)
A. S. T. M. Supercharge standard conditions	225	45	375
Condition A	250	30	250
Condition B	150	30	250

The engine speed of 1800 rpm and compression ratio of 7.0 were held constant throughout the investigation.

Of these three sets of conditions, the A. S. T. M. Supercharge condition is considered to be the most severe by virtue of the advanced spark and high coolant temperature; condition A is somewhat milder; and condition B is the mildest of the three. Additional runs were made in the second CFR engine, which was equipped to conform to specifications of the A. S. T. M. Aviation method for knock rating.

At each of these conditions, 2-percent (by weight) blends of the aromatic amines were examined with AN-F-28 (28-R) fuel as the base fuel. In order to eliminate reproducibility errors, the straight base fuel and the base fuel containing amine were compared on the same day. The selection of 2 percent amine as the only concentration to be investigated was determined primarily by the quantities of amines available.

The full-scale engine investigation was conducted in an air-cooled aircraft cylinder mounted on a Cooperative Universal Engine (CUE) crankcase. The auxiliary apparatus used in these tests was similar to that described in reference 1 except that a heat exchanger was installed in the cooling-air line to control the cooling-air temperature; the exhaust system was so modified that the engine could be operated either at atmospheric or reduced exhaust pressure.

The cooling-air flow was determined for each run by operating the engine at a brake mean effective pressure of 140 pounds per square inch and a fuel-air ratio of 0.10 and by adjusting the damper valve in the cooling-air line until a rear-spark-plug-bushing temperature of 365° F was reached. The cooling-air pressure drop across the cylinder was maintained constant for each run.

Mixture-response curves were determined at two operating conditions: (1) simulated cruise conditions recommended by the Coordinating Research Council (CRC), which specify an engine speed of 2000 rpm, an inlet-air temperature of 210° F, a spark advance of 20° B. T. C., and atmospheric exhaust pressure; and (2) a modification of these CRC conditions that consisted of an advance spark setting of 30° B. T. C. and a reduced exhaust pressure of 15 inches of mercury absolute. The exhaust pressure of 15 inches of mercury was chosen in view of earlier test results (reference 2) in which a critical relation was shown to exist between manifold and exhaust pressures and knock-limited power in the lean region where the manifold pressure is within 10 and -5 inches

TABLE VI-1.—ANTIKNOCK EFFECTIVENESS OF AROMATIC AMINE ADDITIONS TO AN-F-28 (28-R) FUEL IN SMALL-SCALE (CFR) ENGINE CYLINDER *

Aromatic amine (2-percent addition to 28-R fuel)	Relative power= $\frac{\text{imep of aromatic amine plus 28-R}}{\text{imep of 28-R}}$												A. S. T. M. Aviation ratings ^b
	Fuel-air ratio												
	0.062			0.07			0.09			0.11			
	Condition			Condition			Condition			Condition			
	A. S. T. M. Supercharge	A	B	A. S. T. M. Supercharge	A	B	A. S. T. M. Supercharge	A	B	A. S. T. M. Supercharge	A	B	
1 Base fuel (28-R).....	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	100
2 Aniline.....	1.00	1.03	1.13	.97	1.07	1.15	1.03	1.12	1.15	1.09	1.11	1.14	105
3 N-Methylaniline.....	.93	1.11	1.10	.99	1.15	1.13	1.05	1.15	1.11	1.14	1.12	1.10	-----
4 N-Ethylaniline.....	.91	.99	1.04	.97	1.04	1.04	1.04	1.05	1.05	1.01	1.04	1.04	102
5 N-Propylaniline.....	1.03	1.00	1.03	.98	1.06	1.06	.96	1.03	1.04	1.02	1.02	1.06	100
6 N-Isopropylaniline.....	.96	.96	1.01	.95	.95	.99	.96	1.00	1.01	1.01	1.04	1.01	-----
7 N-Butylaniline.....	.91	1.00	1.03	.90	.94	1.01	.91	1.00	1.01	1.01	1.01	1.01	-----
8 N-tert-Butylaniline.....	.97	1.02	1.01	.84	1.00	1.01	.97	1.00	1.01	1.02	.99	1.02	100
9 N,N-Dimethylaniline.....	-----	.99	1.01	-----	.93	1.00	-----	.98	.99	-----	.99	1.00	-----
10 N,N-Diethylaniline.....	-----	1.00	1.01	-----	.98	.98	.99	1.00	.99	1.00	1.03	1.01	-----
11 o-Toluidine.....	.88	1.03	1.01	.92	1.03	1.04	.99	1.08	1.10	1.07	1.09	1.15	-----
12 m-Toluidine.....	.98	1.15	1.09	.96	1.16	1.09	1.05	1.08	1.10	1.09	1.06	1.11	-----
13 p-Toluidine.....	1.03	1.08	1.02	1.06	1.13	1.11	1.09	1.13	1.12	1.11	1.14	1.11	-----
14 o-Ethylaniline.....	1.00	1.01	1.04	1.00	1.03	1.01	.92	1.05	1.04	1.00	1.04	1.07	100
15 p-Ethylaniline.....	.97	1.10	1.12	.97	1.13	1.12	1.04	1.12	1.14	1.09	1.13	1.12	101
16 o-Isopropylaniline.....	.92	.99	1.00	.93	.98	1.03	.90	1.01	1.06	.97	1.01	1.05	100
17 p-Isopropylaniline.....	1.00	1.09	1.16	1.00	1.08	1.16	1.02	1.11	1.13	1.11	1.16	1.12	-----
18 p-tert-Butylaniline.....	1.06	1.02	1.09	1.03	1.02	1.11	1.08	1.07	1.08	1.11	1.12	1.10	104
19 2,4-Xyldine.....	.93	1.04	1.13	.92	1.03	1.14	.99	1.12	1.13	1.05	1.12	1.12	102
20 2,5-Xyldine.....	.98	1.01	1.12	.81	1.07	1.12	.97	1.10	1.09	1.10	1.07	1.11	100
21 2,6-Xyldine.....	.95	1.04	1.07	1.00	1.03	1.11	1.06	1.10	1.10	1.07	1.15	1.12	-----
22 2,4-Diethylaniline.....	.87	.98	1.04	.92	.92	1.04	.93	1.03	1.03	.98	1.06	1.03	99.5
23 2-Methyl-5-isopropylaniline.....	.86	1.10	1.11	.86	1.00	1.12	.95	1.07	1.12	1.11	1.11	1.12	-----
24 2,4,6-Trimethylaniline.....	.88	1.16	1.06	.92	1.07	1.05	.98	1.08	1.08	1.05	1.10	1.11	97.5
25 N-Methyl-o-toluidine.....	.97	1.07	1.08	.95	1.07	1.07	.95	1.10	1.09	1.02	1.11	1.14	100
26 N-Methyl-p-toluidine.....	.98	1.13	1.16	.91	1.07	1.17	1.05	1.20	1.19	1.19	1.18	1.15	-----
27 N-Methyl-p-ethylaniline.....	.97	1.14	1.13	.98	1.18	1.16	1.06	1.14	1.12	1.14	1.15	1.13	100
28 N-Methyl-p-isopropylaniline.....	.95	1.15	1.11	1.05	1.17	1.14	1.08	1.14	1.12	1.15	1.11	1.14	102
29 N-Methyl-p-tert-Butylaniline.....	1.00	1.11	1.12	.91	1.13	1.14	1.02	1.16	1.10	1.14	1.12	1.10	100
30 N-Ethyl-p-toluidine.....	1.00	1.04	1.07	.98	1.06	1.06	.99	1.04	1.07	1.02	1.04	1.06	102
31 N-Isopropyl-p-toluidine.....	.89	.99	1.00	.87	.99	1.00	.88	1.01	1.03	.96	1.02	1.03	-----
32 N-Isopropyl-p-isopropylaniline.....	1.00	1.00	1.00	.99	1.02	1.03	1.00	1.01	1.02	1.00	1.03	1.03	104
33 N-Methyl-2,4-xytidine.....	.98	1.07	1.08	1.00	1.08	1.09	1.00	1.09	1.06	1.04	1.15	1.04	101
34 N,N-Dimethyl-2-methyl-5-isopropyl-aniline.....	.97	.97	1.01	.95	.97	.98	.94	.98	.99	.97	.99	1.00	96
35 N,N-Dimethyl-2,4,6-trimethylaniline.....	.97	1.04	.98	.95	.98	.98	.88	.96	.97	.94	.97	.98	93
36 N,N-Dimethyl-p-phenylenediamine.....	.97	1.12	-----	.89	1.13	-----	.99	1.18	-----	1.16	1.16	-----	-----
37 N,N'-Dimethyl-p-phenylenediamine.....	-----	1.15	-----	-----	1.13	-----	-----	1.22	-----	-----	-----	-----	-----
38 N,N-Diethyl-p-phenylenediamine.....	.98	1.11	1.10	.89	1.11	1.19	1.10	1.17	1.19	1.16	1.17	1.14	99
39 Diphenylamine.....	1.10	1.11	1.10	1.07	1.08	1.08	1.09	1.10	1.11	1.11	1.15	1.10	120
40 Methyl-diphenylamine.....	1.00	1.06	1.16	1.10	1.01	1.04	1.04	1.01	1.00	1.01	1.01	1.07	101
41 o-Methoxyaniline.....	.76	1.00	1.00	.74	.91	1.01	.80	1.01	1.03	.99	1.03	1.04	-----

* Data from references 2 to 6.

^b Performance numbers.^c 1.76 percent amine added.

mercury of the exhaust pressure. The spark advance of 30° B. T. C. was chosen because of the interest in aircraft-engine operation at advanced spark under cruise conditions.

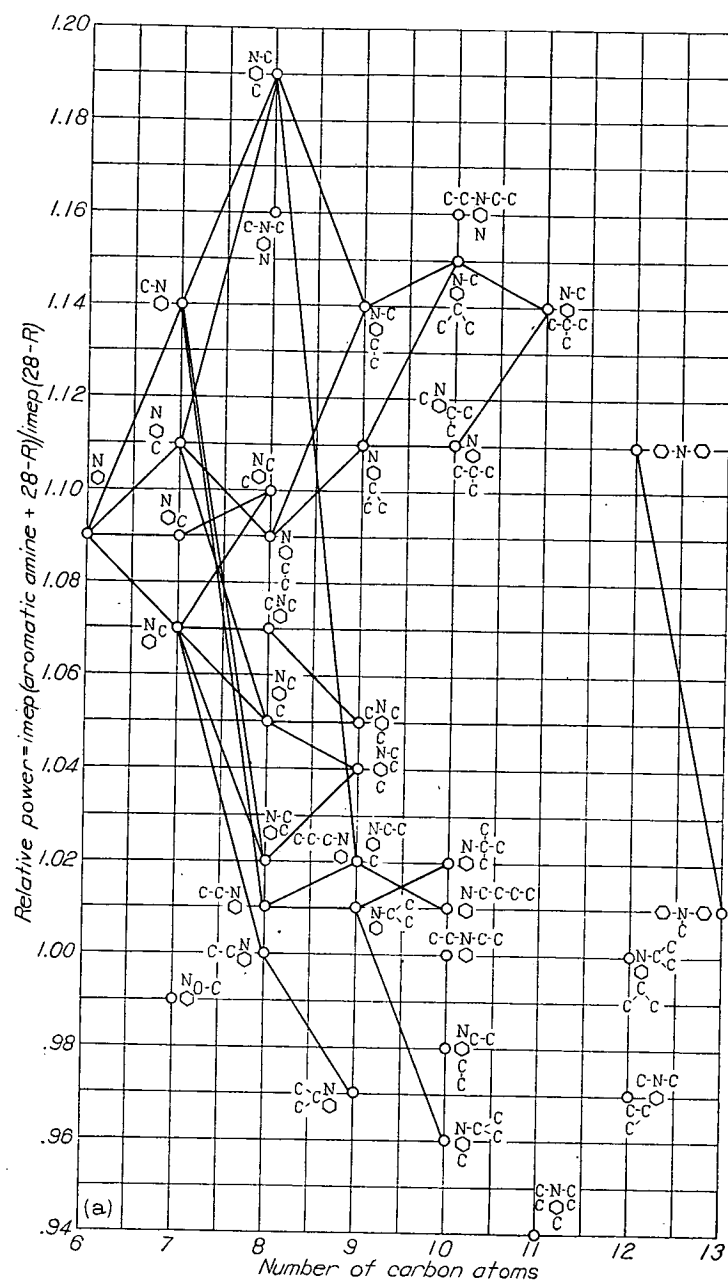
In the full-scale engine studies, 2-percent (by weight) amine blends with AN-F-28 fuel were also used.

Knock-limited performance.—In order to present the most reliable comparison of the many amines examined, the anti-knock ratings of all blends are expressed as power ratios (references 3 to 7). The small-scale engine results shown in table VI-1 (except for A. S. T. M. Aviation results) are therefore expressed as the quotient of the knock-limited indicated mean effective pressure (imep) of the amine blend divided by the knock-limited indicated mean effective pressure of the base fuel (AN-F-28). By this method the

influence of day-to-day variations in engine reproducibility is reduced to a minimum because each blend was examined on the same day as the base fuel.

The data from table VI-1 have been plotted in figure VI-1 to illustrate the relations that exist between antiknock effectiveness and molecular structure. Because the A. S. T. M. Supercharge method is a rich-mixture rating method, only the data at a fuel-air ratio of 0.11 have been considered in this analysis.

Fuel sensitivity (ch. II) is an obstacle to the development of any rigid generalizations between chemical structure and performance. For example, in figure VI-1 (a) at A. S. T. M. Supercharge conditions, it is seen that 15 aromatic amines have antiknock values equal to or greater than aniline. At condition A (fig. VI-1 (b)), 17 aromatic amines are equal to

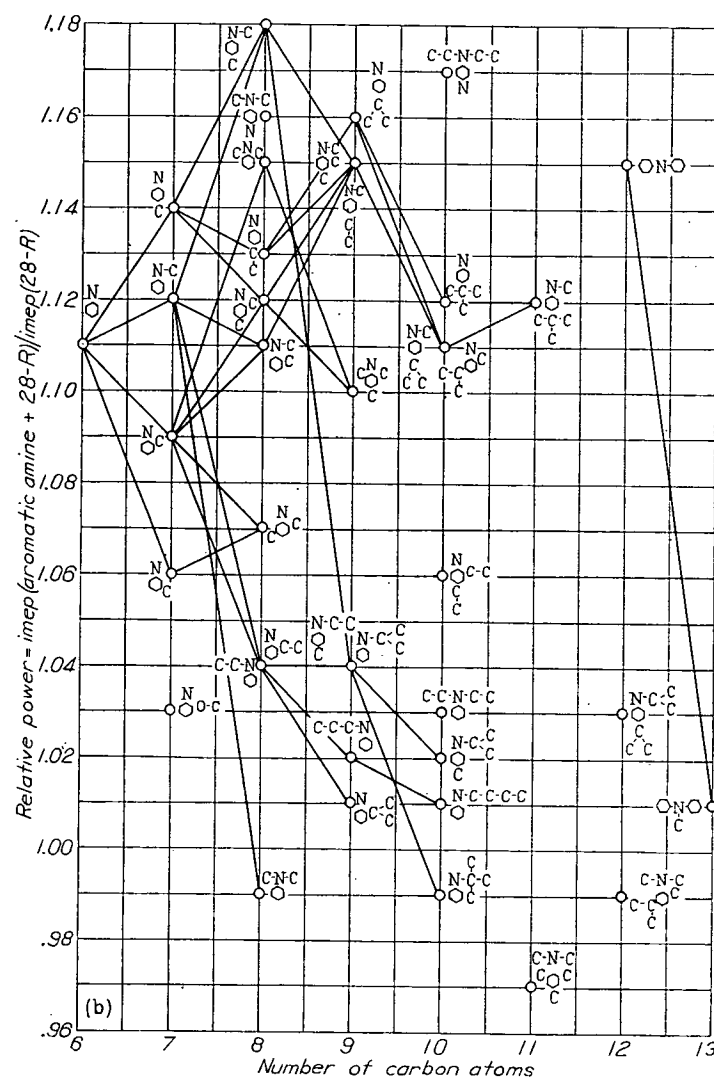


(a) A. S. T. M. Supercharge standard conditions.

FIGURE VI-1.—Antiknock effectiveness of 2-percent aromatic amine blends with AN-F-28 (28-R) fuel. Fuel-air ratio, 0.11.

or greater than aniline, but at condition B (fig. VI-1 (c)), only 5 compounds are equal to or greater than aniline. Purely from consideration of nominal engine operating conditions it will be recalled that the A. S. T. M. Supercharge condition is probably the most severe, with condition B the mildest, and condition A intermediate. It is obvious, then, that some risk is involved in a statement that any single amine is always better than aniline. In fact, N-methyl-*p*-toluidine is the only compound that did exceed aniline in antiknock value at all three conditions, and at condition B the margin of superiority was, for all practical purposes, negligible.

On the basis of the data contained in figures VI-1, however, two generalizations can be made with regard to

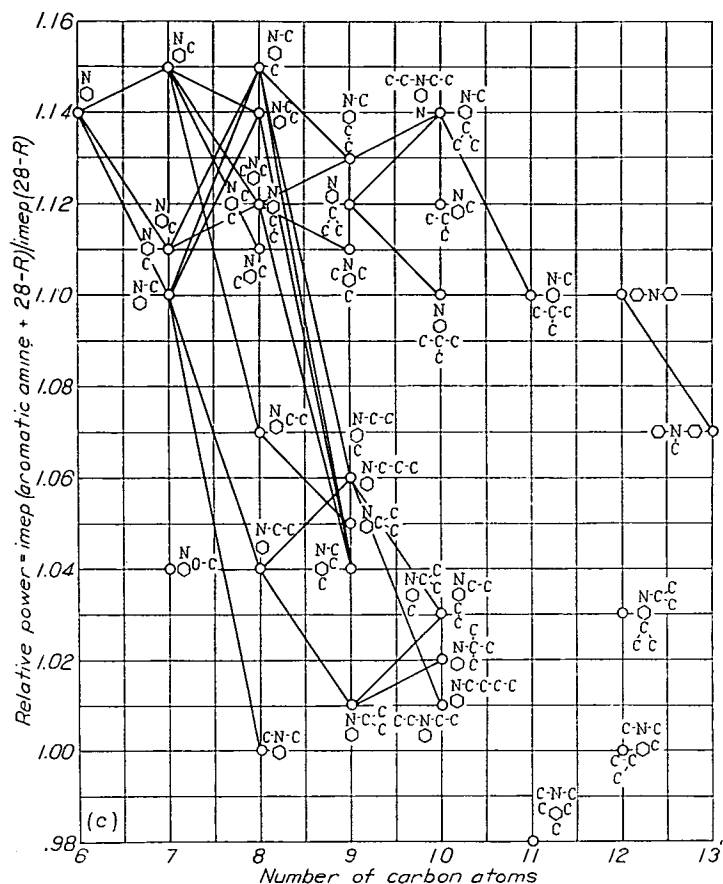


(b) A. S. T. M. Supercharge modified A conditions.

FIGURE VI-1.—Continued. Antiknock effectiveness of 2-percent aromatic amine blends with AN-F-28 (28-R) fuel. Fuel-air ratio, 0.11.

structural trends. The first of these is as follows: When one hydrogen atom in the $-NH_2$ group of any given primary aromatic amine is replaced by an alkyl group, the greatest increase or least decrease in antiknock value from that of the primary amine will result when the alkyl substituent is a methyl radical ($-CH_3$). This statement is supported by the following data:

Aromatic amine	Relative Power		
	A. S. T. M. Supercharge	Condition A	Condition B
Aniline.....	1.09	1.11	1.14
N-Methylaniline.....	1.14	1.12	1.10
N-Ethylaniline.....	1.01	1.04	1.04
N-Propylaniline.....	1.02	1.02	1.06
N-Butylaniline.....	1.01	1.01	1.01
N-Isopropylaniline.....	1.01	1.04	1.01
N-tert-Butylaniline.....	1.02	.99	1.02
<i>p</i> -Toluidine.....	1.11	1.14	1.11
N-Methyl- <i>p</i> -toluidine.....	1.19	1.18	1.15
N-Ethyl- <i>p</i> -toluidine.....	1.02	1.04	1.06
N-Isopropyl- <i>p</i> -toluidine.....	.96	1.02	1.03
<i>p</i> -Isopropylaniline.....	1.11	1.16	1.12
N-Methyl- <i>p</i> -isopropylaniline.....	1.15	1.11	1.14
N-Isopropyl- <i>p</i> -isopropylaniline.....	1.00	1.03	1.03



(c) A. S. T. M. Supercharge modified B conditions.

FIGURE VI-1.—Concluded. Antiknock effectiveness of 2-percent aromatic amine blends with AN-F-28 (28-R) fuel. Fuel-air ratio, 0.11.

For each of these three series of compounds, the antiknock values of the three primary amines considered as bases (aniline, *p*-toluidine, and *p*-isopropylaniline) were increased more by the substitution of a methyl radical for one of the hydrogen atoms attached to the nitrogen than by substitution of any other radical. For the two cases where the performance decreased (N-methylaniline at condition B and N-methyl-*p*-isopropylaniline at condition A), the decrease in performance was less with the methyl radical than with any of the other radicals examined.

The influence of replacing a hydrogen atom attached to the nitrogen with an aromatic radical was investigated for only one compound; however, in this one case the aromatic radical appeared approximately equal in effectiveness to a methyl radical, as shown by the following table:

Aromatic amine	Relative Power		
	A. S. T. M. Supercharge	Condition A	Condition B
Aniline.....	1.09	1.11	1.14
N-Methylaniline.....	1.14	1.12	1.10
Diphenylamine.....	1.11	1.15	1.10

The second generalization of the data can be made with reference to hydrogen substitutions on the aromatic ring: *The addition of an alkyl radical to the aromatic ring of an aromatic amine is more effective insofar as antiknock value is concerned when the addition is made in the para position rather*

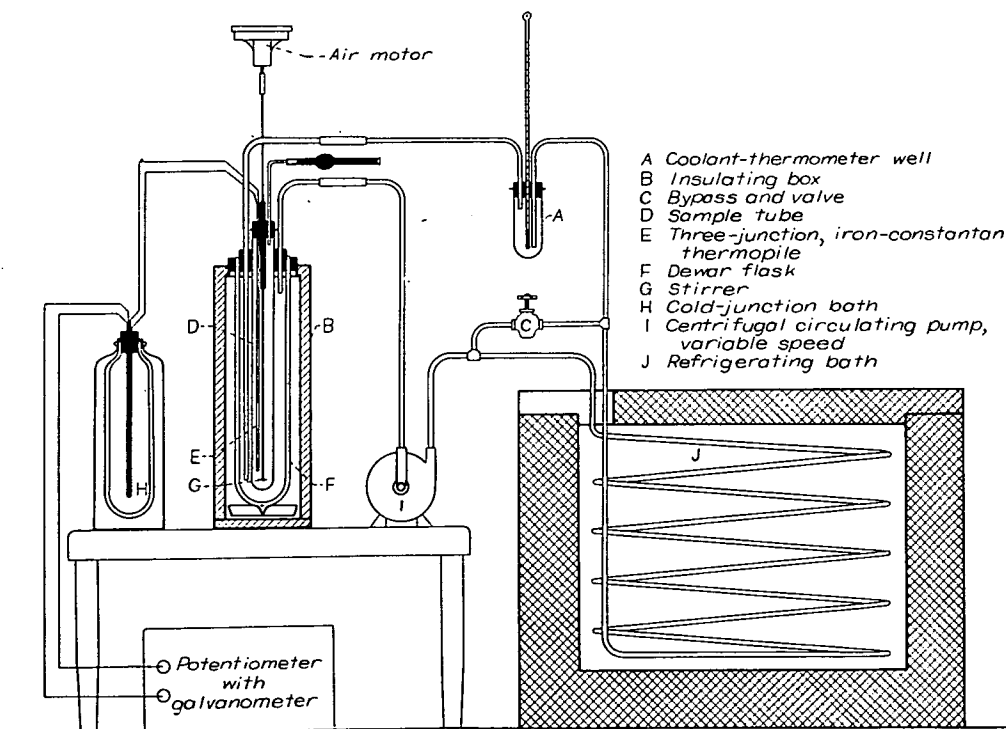


FIGURE VI-2.—Apparatus for determination of cloud point. (Fig. 1 of reference 9.)

TABLE VI-2.—ANTIKNOCK EFFECTIVENESS OF AROMATIC-AMINE ADDITIONS TO AN-F-28 (28-R) FUEL IN A FULL-SCALE AIR-COOLED AIRCRAFT ENGINE CYLINDER *

For each compound there are two rows of values. The first row is imep, lb/sq in.; the second is imep ratio, which is the ratio of the imep of 98 percent 28-R fuel plus 2 percent aromatic amine to the imep of 28-R.]

Aromatic amine (2-percent addition to 28-R)	Relative power (engine speed, 2000 rpm; inlet-air temperature, 210° F; compression ratio, 7.3)									
	Spark advance, 20° B. T. C.; exhaust pressure, 29±0.5 in. Hg abs.					Spark advance, 30° B. T. C.; exhaust pressure, 15 in. Hg abs.				
	Fuel-air ratio					Fuel-air ratio				
	0.065	0.07	0.08	0.09	0.10	0.065	0.07	0.08	0.09	0.10
Base fuel (28-R).....	141 1.00	155 1.00	209 1.00	235 1.00	253 1.00	166 1.00	179 1.00	203 1.00	226 1.00	236 1.00
N-Methylxylidines (mixed isomers).....	164 1.16	193 1.25	222 1.06	256 1.09	283 1.12	173 1.04	184 1.03	209 1.03	232 1.03	249 1.06
N-Methyleumidines (mixed isomers).....	195 1.38	208 1.34	245 1.17	272 1.16	290 1.15	182 1.10	194 1.08	220 1.08	245 1.08	259 1.10
N-Methyltoluidines (75 percent <i>p</i> -, 25 percent <i>o</i> -).....	195 1.38	216 1.39	244 1.17	275 1.17	288 1.14	178 1.07	187 1.05	204 1.01	239 1.06	260 1.10
Xylidines (mixed isomers).....	172 1.22	192 1.24	230 1.10	257 1.09	280 1.11	173 1.04	184 1.03	214 1.05	237 1.05	257 1.09
Cumidines (from refinery cumene—mixed isomers).....	184 1.30	200 1.29	237 1.13	257 1.09	282 1.11	173 1.04	188 1.05	215 1.06	232 1.03	250 1.06
N-Methylaniline.....	192 1.36	210 1.36	240 1.15	269 1.14	290 1.15	181 1.09	190 1.06	218 1.07	240 1.06	262 1.11

* Reference 8.

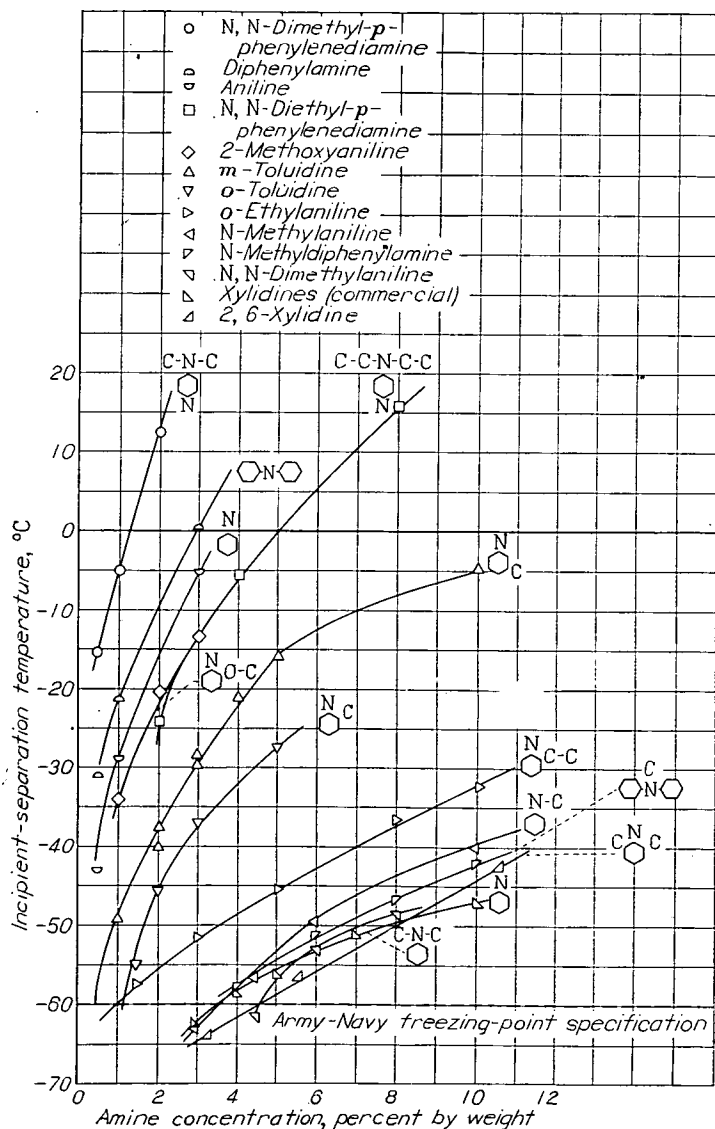


FIGURE VI-3.—Solubility of aromatic amines in grade 65 base stock with aromatic hydrocarbons extracted. (Fig. 2 of reference 9.)

than in the ortho or meta positions. This statement is based on the following data:

Aromatic amine	Relative Power		
	A. S. T. M. Supercharge	Condition A	Condition B
Aniline.....	1.09	1.11	1.14
<i>o</i> -Toluidine.....	1.07	1.09	1.15
<i>m</i> -Toluidine.....	1.09	1.06	1.11
<i>p</i> -Toluidine.....	1.11	1.14	1.11
N-Methylaniline.....	1.14	1.12	1.10
N-Methyl- <i>o</i> -toluidine.....	1.02	1.11	1.14
N-Methyl- <i>p</i> -toluidine.....	1.19	1.18	1.15
Aniline.....	1.09	1.11	1.14
<i>o</i> -Ethylaniline.....	1.00	1.04	2.07
<i>p</i> -Ethylaniline.....	1.09	1.13	1.12
Aniline.....	1.09	1.11	1.14
<i>o</i> -Isopropylaniline.....	.97	1.01	1.95
<i>p</i> -Isopropylaniline.....	1.11	1.16	1.12

The small number of A. S. T. M. Aviation ratings (table VI-1) does not justify inclusion as supporting data for the foregoing generalizations. It is interesting to note, however, that the addition of diphenylamine improved the performance of the base fuel by 15 performance numbers more than any of the other aromatic amines for which A. S. T. M. Aviation ratings were obtained.

One additional observation to be made from the data of table VI-1 concerns the replacement by alkyl radicals of both hydrogen atoms attached to the nitrogen, that is,



As shown by the following comparisons, the replacement of one hydrogen atom by an alkyl radical is more effective for maintaining or improving the antiknock value of the starting compound than replacement of both hydrogen atoms:

Aromatic amine	Relative Power		
	A. S. T. M. Supercharge	Condition A	Condition B
Aniline.....	1.09	1.11	1.14
N-Methylaniline.....	1.14	1.12	1.10
N,N-Dimethylaniline.....	---	.99	1.00
Aniline.....	1.09	1.11	1.14
N-Ethylaniline.....	1.01	1.04	1.04
N,N-Diethylaniline.....	1.00	1.03	1.01

Suitable quantities of the pure aromatic amines used in the small-scale engine studies were not available for the full-scale engine investigation (reference 8); therefore, a few isomeric mixtures were examined. The results of these tests are presented in table VI-2.

The data show that all the aromatic amines increased the knock-limited performance of the base fuel at both full-scale engine conditions throughout the fuel-air ratio range. At both sets of engine conditions and lean and rich fuel-air ratios, the blends of N-methylcumidines, N-methyltoluidines, and N-methylaniline were about equal in knock-limited performance, and all three were superior to the other blends tested.

When the severity of the test conditions was changed by advancing the spark and reducing the exhaust pressure, contrary effects were noted for the several blends. In every case the rich-mixture (fuel-air ratio, 0.10) knock-limited performance of the several fuels was decreased by the change of conditions. The lean-mixture (fuel-air ratio, 0.065) performance of 28-R fuel and the blend containing N-methylxylidines was increased, whereas the blend containing xylidines remained about the same. The percentage improvement in the knock-limited performance of the base fuel resulting from addition of the amines was less at the condition of advanced spark and reduced exhaust pressure than at the other condition.

PHYSICAL PROPERTIES OF AROMATIC AMINES

Of the physical properties examined in the NACA investigation of aromatic amines, low-temperature solubility and gasoline-water distribution coefficients received the greatest attention. The low-temperature solubility characteristics are of primary importance because of specification requirements that the additives must remain in solution at the lowest temperatures encountered in service. Current fuel specifications limit the freezing point to a maximum of -76°F (-60°C). Gasoline-water distribution coefficients are of interest because of the wide usage of water-distribution storage systems. In such systems the additive may be extracted from the fuel by diffusion when fuel and water are in intimate contact.

Physical properties, other than low-temperature solubility and gasoline-water distribution coefficients, were determined for the amines examined and are presented in table VI-3.

Low-temperature solubility.—Solubilities of 42 aromatic amines in blends with gasoline were measured at temperatures as low as -85°F (-65°C) and at concentrations as high as 10 percent by weight (reference 9). The solubilities of aromatic amines are appreciably affected by the compo-

TABLE VI-3.—PHYSICAL PROPERTIES OF AROMATIC AMINES *

Aromatic amine	Boiling range ^b (°C)	Index of refraction _D ²⁰	Density (grams/ml)
Aniline.....	184-184.5	1.5853	1.0220
N-Methylaniline.....	195-196	1.5704	.9860
N-Ethylaniline.....	203-204	1.5538	.9607
N-Propylaniline.....	220.5-223.5	1.5425	.9448
N-Butylaniline.....	240.0-240.5	1.5339	.9323
N-Isopropylaniline.....	206.5-209	1.5404	.9374
N-tert-Butylaniline.....	95 at 16 mm	1.5270	.9244
N,N-Dimethylaniline.....	192.5-193.5	1.5580	.9564
N,N-Diethylaniline.....	215-217	1.5418	.9347
o-Toluidine.....	198.5-201.5	1.5718	.9989
m-Toluidine.....	202.5-203.5	1.5674	.9893
p-Toluidine.....	c 44.0-44.4	---	---
N-Methyl-p-toluidine.....	209-211	1.5570	.9610
N-Methyl-o-toluidine.....	206.5-207.5	1.5646	.9763
N-Methyltoluidines (60 percent p-, 40 percent o-)	208.5-215	1.5600	.9668
N-Methyltoluidines (80 percent p-, 20 percent o-)	210-213	1.5590	.9638
N-Ethyl-p-toluidine.....	217-220	1.5439	.9441
N-Isopropyl-p-toluidine.....	222-223	1.5319	.9238
o-Ethylaniline.....	211	1.5602	.9810
p-Ethylaniline.....	216	1.5547	.9672
N-Methyl-p-ethylaniline.....	227.5	1.5485	.9485
N-Methylethylaniline, mixed isomers (from chloroethylbenzenes)	222.5-230.5	1.5493	.9503
p-tert-Butylaniline.....	96.5-98.0 at 5-6 mm	1.5388	.9446
o-Isopropylaniline.....	219-220	1.5484	.9643
p-Isopropylaniline.....	225.5-226.5	1.5432	.9514
N-Methyl-p-isopropylaniline.....	240	1.5390	.9347
N-Isopropyl-p-isopropylaniline.....	246-247	1.5209	.9075
2,4,6-Trimethylaniline.....	110.0 at 15 mm	1.5502	.9615
Cumidines (from synthetic cumenes)	225-226	1.5448	.9536
Cumidines (from refinery cumenes)	220-241	1.5434	.9531
N-Methylcumidines (from bromocumenes)	237.5-241.5	1.5390	.9366
N-Methyl-p-tert-butylaniline.....	245.5-249.5	1.5348	.9305
o-Methoxyaniline.....	224-225	1.5750	1.0931
Xylidines (commercial).....	216-219.5	1.5601	.9771
2,4-Xylidine.....	215.0-215.5	1.5591	.9751
2,5-Xylidine.....	216	1.5596	.9755
2,6-Xylidine.....	216-217	1.5616	.9768
N-Methyl-2,4-xylidine.....	221-222	1.5542	.9582
N-Methylxylidines (from bromoxylenes)	220-227	1.5540	.9586
2,4-Diethylaniline.....	241-242	1.5433	.9511
2-Methyl-5-isopropylaniline.....	240-242	1.5408	.9436
N,N-Dimethyl-2-methyl-5-isopropylaniline.....	84 at 5 mm	1.5124	.9028
N,N-Dimethyl-2,4,6-trimethylaniline.....	213.5	1.5116	.9066
Pseudocumidine (technical).....	225-241	1.5568	.9720
Diphenylamine.....	c 52.9-53.6	---	---
p-Phenylenediamine.....	c 140.0-142.0	---	---
N-Methyl-p-phenylenediamine.....	121 at 5 mm	1.621	---
N,N-Dimethyl-p-phenylenediamine.....	108-111 at 4-5 mm	---	---
N,N-Diethyl-p-phenylenediamine.....	117 at 2.5 mm	---	---
N,N'-Dimethyl-p-phenylenediamine.....	117 at 1 mm	---	---
N-Methyldiphenylamine.....	295-296	1.6224	1.0527

* Table I of reference 9.

^b Boiling range at 760 mm Hg except where noted.

c Melting point measured for this solid rather than boiling range.

sition of gasoline to which the addition is made; therefore, three different base gasolines were used:

1. Grade 65 gasoline from which aromatic hydrocarbons were successively extracted with 10-percent fuming sulfuric acid and silica gel.

2. Extracted grade 65 gasoline to which was added 15 percent by volume of an aromatic mixture of 5 parts xylene, 2 parts cumene, and 1 part toluene.

3. Different batches of typical current aviation gasoline, AN-F-28 (Amendment 2) fuel containing 12 to 20 percent aromatic hydrocarbons by volume.

The apparatus for determination of low-temperature solubilities of the amines is shown in figure VI-2. Briefly, the procedure used in reference 9 to determine the solubility consisted in slowly cooling and stirring the gasoline-amine sample in the sample tube until the amine separated as a cloud from the gasoline; the temperature at which the cloud formed was recorded. The sample was then slowly warmed until the amine went into solution and the cloud disappeared; the temperature was again recorded. These two temperatures were averaged to give the incipient-separation temperature or cloud point. Cloud points were reproducible to within $\pm 1.5^{\circ}\text{C}$.

TABLE VI-4.—SOLUBILITY OF AROMATIC AMINES IN THREE AVIATION FUELS AT -60°C ^a

[Percentages by weight]

Aromatic amine	Aromatic-free grade 65	Aromatic-free grade 65 plus 15 percent by volume aromatics ^b	AN-F-28
Aniline	<0.5	0.5	>10
N-Methylaniline	3.6	12(-62° C)	>10
N-Ethylaniline	>10	>10	>10
N-Propylaniline	>10	>10	>10
N-Isopropylaniline	>10	>10	>10
N-tert-Butylaniline	>10	>10	>10
N,N-Dimethylaniline	4.7	9.8	8.3
N,N-Diethylaniline	>10	>10	>10
o-Toluidine	1.3	4.2	4.4
m-Toluidine	<0.5	2.6	3.6
N-Methyl-p-toluidine	>10	>10	>10
N-Methyl-o-toluidine	>10	>10	>10
N-Methyltoluidines (from chlorotoluenes)	>10	>10	>10
N-Ethyl-p-toluidine	>10	>10	>10
N-Isopropyl-p-toluidine	>10	>10	>10
o-Ethylaniline	0.9	>10	>10
N-Methyl-p-ethylaniline	>10	>10	>10
N-Methylethylaniline, mixed isomers (from chloroethylbenzenes)	>10	>10	>10
o-Isopropylaniline	>10	>10	>10
p-Isopropylaniline	>10	>10	>10
N-Methyl-p-isopropylaniline	>10	>10	>10
N-Isopropyl-p-isopropylaniline	>10	>10	>10
Cumidines (from synthetic cumenes)	>10	>10	>10
Cumidines (from refinery cumenes)	>10	>10	>10
N-Methylcumidines (from bromocumenes)	>10	>10	>10
N-Methyl-p-tert-butylaniline	>10	>10	>10
2-Methoxyaniline	<0.5	<0.5	<0.5
Xylidines (commercial) (reference 10)	3.7	>10	>10
2,6-Xylidine	4.6	11.1	9.1
N-Methyl-2,4-xylidine	>10	>10	>10
N-Methylxylidines (from bromoxylenes)	>10	>10	>10
2,4-Diethylaniline	>10	>10	>10
2-Methyl-5-isopropylaniline	>10	>10	>10
N,N-Dimethyl-2-methyl-5-isopropylaniline	>10	>10	>10
N,N-Dimethyl-2,4,6-trimethylaniline	>10	>10	>10
Pseudocumidine (technical)	>10	>10	>10
Diphenylamine	<0.5	<0.5	<0.5
p-Phenylenediamine	<0.5	<0.5	<0.5
N-Methyl-p-phenylenediamine	<0.5	<0.5	<0.5
N,N-Dimethyl-p-phenylenediamine	<0.5	<0.5	<0.5
N,N-Diethyl-p-phenylenediamine	<0.5	<0.5	<0.5
N,N'-Dimethyl-p-phenylenediamine	<0.5	<0.5	<0.5
N-Methyldiphenylamine	3.3	>10	>10

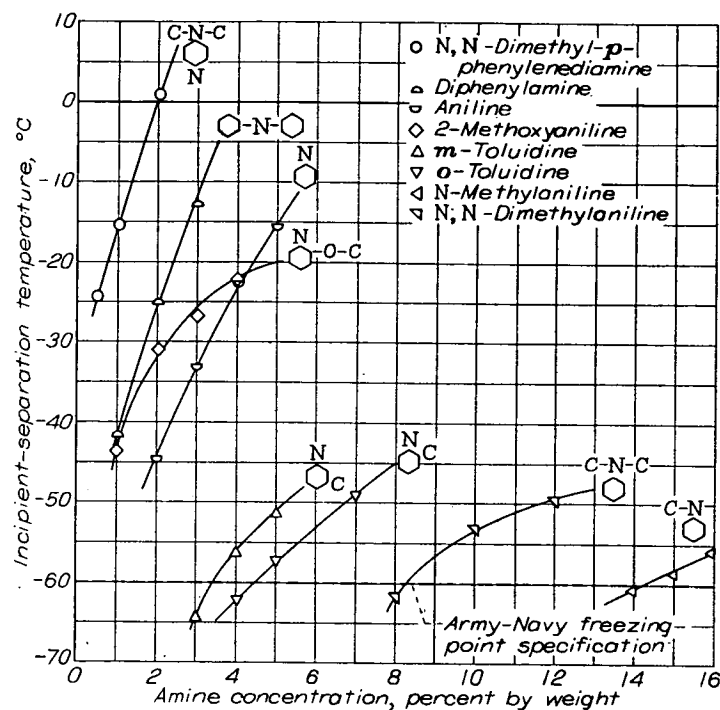
^a Table II of reference 9.^b Aromatic mixture consisted of five parts xylene, two parts cumene, and one part toluene.^c Solubility at room temperature.

FIGURE VI-5.—Solubility of aromatic amines in AN-F-28, Amendment-2, fuel. (Fig. 4 of reference 9.)

The solubilities of the amines in the aromatic-free gasoline, in the gasoline of 15-percent aromatic content, and in the AN-F-28 fuel are presented in figures VI-3, VI-4, and VI-5, respectively. The amines that were soluble to at least 10 percent by weight at -76°F (-60°C) in each of the gasolines are

- N-Ethylaniline
- N-Propylaniline
- N-Isopropylaniline
- N-tert-Butylaniline
- N,N-Diethylaniline
- N-Methyl-p-toluidine
- N-Methyl-o-toluidine
- N-Methyltoluidines (from chlorotoluenes)
- N-Ethyl-p-toluidine
- N-Isopropyl-p-toluidine
- N-Methyl-p-ethylaniline
- N-Methylethylaniline, mixed isomers (from chloroethylbenzenes)
- o-Isopropylaniline
- p-Isopropylaniline
- N-Methyl-p-isopropylaniline
- N-Isopropyl-p-isopropylaniline
- Cumidines (from synthetic cumenes)
- Cumidines (from refinery cumenes)
- N-Methylcumidines (from bromocumenes)
- N-Methyl-p-tert-butylaniline
- N-Methyl-2,4-xylidine
- N-Methylxylidines (from bromoxylenes)
- 2,4-Diethylaniline
- 2-Methyl-5-isopropylaniline
- N,N-Dimethyl-2-methyl-5-isopropylaniline
- N,N-Diethyl-2,4,6-trimethylaniline
- Pseudocumidine (technical)

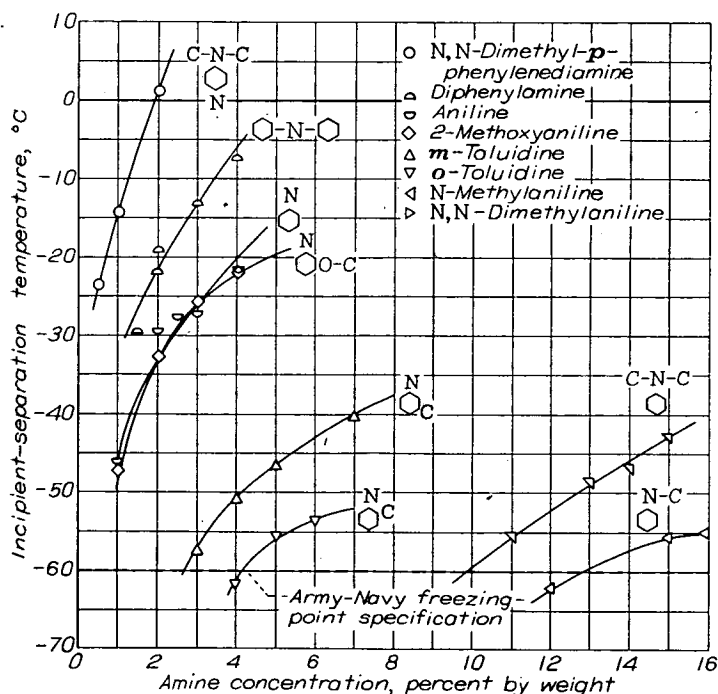


FIGURE VI-4.—Solubility of aromatic amines in blend of 85 percent extracted grade 65 base stock and 15 percent (by volume) of aromatic mixture consisting of 5 parts xylene, 2 parts cumene, and 1 part toluene. (Fig. 3 of reference 9.)

At room temperature, *N*-methyl-*p*-phenylenediamine and *p*-phenylenediamine were less than 0.5 percent (by weight) soluble in the test gasoline; no additional solubility data were taken for these compounds. At room temperature, *N,N'*-dimethyl-*p*-phenylenediamine was soluble to the extent of 1 to 2 percent by weight but was too unstable to permit accurate measurement of solubility by the method employed. *N,N*-Diethyl-*p*-phenylenediamine was tested only in the aromatic-free gasoline.

When figures VI-3 and VI-4 are compared, it is seen that composition of the base fuel greatly influenced the solubilities of the amines. The addition of 15 percent aromatics to the aromatic-free gasoline approximately doubled or tripled the amine solubility. Solubilities in AN-F-28 fuel (fig. VI-5) were about the same as those in the gasoline containing 15 percent aromatics. Representative samples of AN-F-28 fuel contained 12 to 20 percent (by volume) aromatics.

A summary of the solubilities of the amines at -76°F (-60°C) in the different test gasolines is presented in table VI-4. The results were obtained by interpolating or extrapolating the experimental data. The data obtained for commercial xylydines (reference 10) are included for comparison.

The solubility of an aromatic amine in the aromatic-free gasoline at -76°F (-60°C) may be taken as an indication of the maximum concentration in which the amine could be added to current aviation fuels on the basis of solubility alone. The aromatic hydrocarbons present in most of the wartime aviation fuels provided a margin of safety in preventing this concentration of amine from separating at -76°F (-60°C).

Gasoline-water distribution coefficients.—If a fuel containing an additive such as an aromatic amine is stored in contact with water for an extended period of time, a certain amount of the additive will be extracted by the water. An analysis of a fuel-storage system in which this extraction might occur was made by Olson and Tischler (reference 11) in order to develop an expression from which the loss of amine to water could be approximated. Storage systems of this type have been variously termed "overwater storage systems, water-displacement storage systems, and aqua storage systems."

As pointed out in reference 11, it is unlikely that a single expression can be written for use under all conditions encountered in practice, but certain assumptions permit a mathematical derivation of an equation that covers a wide range of situations. In the operation of a tank utilizing the water-displacement principle, fuel is removed from the tank by adding water at the bottom of the tank. Fuel is added to the tank by removal of storage water from the tank. Two phases, one of gasoline and one of water, exist in the tank, which is full of liquid at all times.

The following assumptions are made in reference 11 in determining the expressions for concentration of additive in fuel stored over water at any volume of fuel:

1. When equilibrium conditions are reached, the distribution law applies to the additive; namely, the ratio between the concentrations of the additive in the two phases of the gasoline-water system is constant at constant temperature.

The ratio is described as the distribution coefficient and defined as

$$K = \frac{\text{additive concentration in gasoline phase}}{\text{additive concentration in water phase}}$$

This constant must be experimentally determined for each additive. It varies with temperature of the fuel and water and with the nature of the fuel. Any variation in the distribution coefficient caused by additive concentration may be assumed negligible in the range of concentrations produced by the distribution process.

2. The volume of the additive itself, when present to the extent of 2 or 3 percent or less, is negligible by comparison with the volumes of gasoline and water. The equilibrium concentration is the minimum concentration of additive in the fuel at the volume in question.

The following symbols are used in the analysis of reference 11:

K	gasoline-water distribution coefficient (experimentally determined)
X	concentration of additive in gasoline before addition or removal of portion of fuel
X'	concentration of additive in gasoline after addition or removal of portion of fuel (at equilibrium)
V	volume of storage tank
V_g	volume of fuel in tank before addition or removal of portion of fuel
V_g'	volume of fuel in tank after addition or removal of portion of fuel
V_w	volume of water in tank before addition or removal of portion of fuel
V_w'	volume of water in tank after addition or removal of portion of fuel
V_f	volume of fuel added or withdrawn ($V_f = V_g' - V_g$)
Y	concentration of additive in water added or withdrawn
Z	concentration of additive in fuel added or withdrawn

Because the manner in which an overwater-storage system is operated is one of the most important variables influencing the additive concentration in the stored fuel, two cases representing two very different operating procedures have been studied:

Case 1.—The first case applies to successive additions or withdrawals of fuel stored in contact with water when an appreciable period of time has elapsed between each addition or withdrawal.

In order to approximate this situation it is assumed that during the actual addition or withdrawal of fuel, no additive is transferred between the gasoline and water phases but that equilibrium distribution of additive between the two phases is attained during the period between successive additions or withdrawals of fuel. This assumption is based on the slowness of the rate of diffusion of additive between the two phases. The following equation will agree more closely with the actual situation the longer the period between additions or withdrawals of fuel:

$$(X V_g + Z V_f) + \left(\frac{X V_w}{K} - Y V_f \right) = X' (V_g + V_f) + \frac{X'}{K} (V_w - V_f) \quad (1)$$

This equation may be expressed in words as follows:

(Quantity of additive in gasoline after addition or withdrawal of fuel but before equilibrium) + (quantity of additive in water after addition or withdrawal of fuel but before equilibrium) = (quantity of additive in gasoline after equilibrium) + (quantity of additive in water after equilibrium).

Expanding and collecting terms gives

$$X' = \frac{KXV_g + KZV_f + XV_w - KYV_f}{KV_g + KV_f + V_w - V_f} \quad (2)$$

If fuel is displaced from the tank by fresh water, $Z=X$, $Y=0$, $V_g' = V_g + V_f$ and $V_w' = V_w - V_f$. Then

$$X' = X \left(\frac{V_w + KV_g'}{V_w' + KV_g'} \right) \quad (3)$$

Case 2.—The second case applies to successive small withdrawals of fuel over an appreciable length of time. For this approximation, equilibrium distribution of additive is assumed to exist between gasoline and water at all times. An equation of this sort permits evaluation of additive concentration under circumstances where withdrawals of fuel have been small, compared with the tank capacity, but very numerous. Equation (4) may thus be considered as the result of applying the equations of case 1 to an infinite number of steps, with equilibrium established between each step.

Let $\Delta X = X' - X$, $\Delta V_g = V_g' - V_g$, and $\Delta V_w = V_w' - V_w$. Then,

$$X'V_g' + \frac{X'V_w'}{K} = X' - \Delta X(V_g' - \Delta V_g) + \left(\frac{X' - \Delta X}{K} \right) (V_w' - \Delta V_w) + (X' - \Delta X)\Delta V_g \quad (4)$$

or, in words,

(Quantity of additive in fuel stored over water) + (quantity of additive in water) = (quantity of additive in stored fuel before removal of small increment) + (quantity of additive in water before addition of increment of fresh water) + (quantity of additive removed in small increment of fuel).

By removing second-order differentials, equation (4) can be expanded and simplified:

$$-X'\Delta V_w = (KV_g' + V_w')\Delta X \quad (5)$$

But $-\Delta V_w = \Delta V_g$ and $V_w' = V - V_g'$; therefore,

$$\frac{\Delta X}{X'} = \frac{\Delta V_g}{V_g'(K-1) + V} \quad (6)$$

By integration from any concentration X and volume V_g to X' and V_g' ,

$$X' = X \left[\frac{V_g'(K-1) + V}{V_g(K-1) + V} \right]^{\frac{1}{K-1}} \quad (7)$$

TABLE VI-5.—STORAGE-SYSTEM INVESTIGATION

Sample taken *	May 27	June 7	June 19	July 3	July 13 1943
Fuel in tank before removal of batch, gal.	8105	8105	6105	2105	1105
Water in tank before removal of batch, gal.	16,895	16,895	18,895	22,895	23,895
Fuel removed, gal.-----	(fuel placed in storage system)	2000	4000	1000	1105
Xylidines, grams/100 ml (electrometrically)	1.05 .98 Mean 1.01	(b)	{ 0.94 .826 .826 .82	0.825 .83	0.815 .82 (c)
Equilibrium concentration, X' (equation (2))		^d 0.900	^e 0.886	0.825	0.806

* Analysis indicated under each date is prior to removal from the storage tank of the quantity of fuel indicated under that date.

^b Sample was inadvertently mixed with some xylidine-free fuel.

^c The fuel stored over water gave a negligible decrease in knock rating when compared with the original batch of fuel.

^d Sample calculation for run on June 7: Because the tank contained no fuel prior to the addition of the 8105 gallons on May 27, $V_g=0$, $X=0$, and $Y=0$; therefore, from equation (2)

$$X' = \frac{(17)(1.01)(8105)}{(17)(8105) + 25,000 - 8105}$$

$$= 0.900 \text{ (equilibrium concentration attained on standing).}$$

^e Sample calculation for run on June 19: In this case, $Z=X$ and $Y=0$. Equation (2) therefore becomes equation (3) and

$$X' = 0.900 \left(\frac{(16,895 + (17)(6105))}{(18,895 + (17)(6105))} \right) = 0.886$$

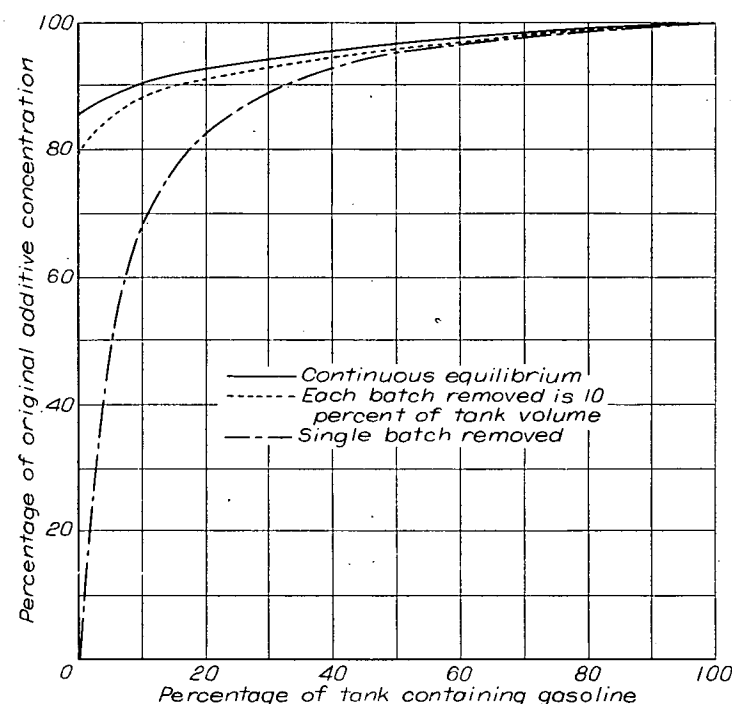


FIGURE VI-6.—Additive concentration in gasoline stored over water for distribution coefficient of 20. (Fig. 1 of reference 11.)

In full-scale, overwater-storage tests of xylidine-blended fuel, 8105 gallons of aviation fuel were stored in a tank of 25,000-gallon capacity and immediately sampled and analyzed. The stored fuel (containing xylidines) was sampled and analyzed periodically, and part of the fuel was removed. This process is a batch process; therefore, equation (2) is applicable. This test provides an evaluation of the equation for the particular conditions.

With the assumption of an average temperature of 65° F for the tank contents and fuel displacement with fresh water,

a volume distribution coefficient of 17 was selected for xylidines (reference 12). The results of the full-scale, water-system test and the equilibrium concentration calculated from equation (2) are compared in table VI-5. The calculated results agree well with the experimental data for this particular test.

Normally anticipated additive concentration in fuel stored over water is shown in figure VI-6 for three different types of tank operation. A volume distribution coefficient of 20 is assumed, inasmuch as experimental values range from 13 to 26 for xylidines (reference 12).

The topmost curve of figure VI-6 represents fuel displaced

from a full tank in successive small increments over a period of time sufficiently long for equilibrium to exist continuously. Equation (7) applies for this case. The second curve illustrates the variation in concentration to be expected if fuel is displaced from a full tank in successive quantities with equilibrium existing between withdrawals. Equation (2) applies for this case. The lower curve illustrates the concentration to be expected if fuel is displaced from a full tank in a single, fairly rapid operation with equilibrium established only after the withdrawal; it also illustrates the change in concentration resulting from the addition of a batch of fuel to a tank containing additive-free water.

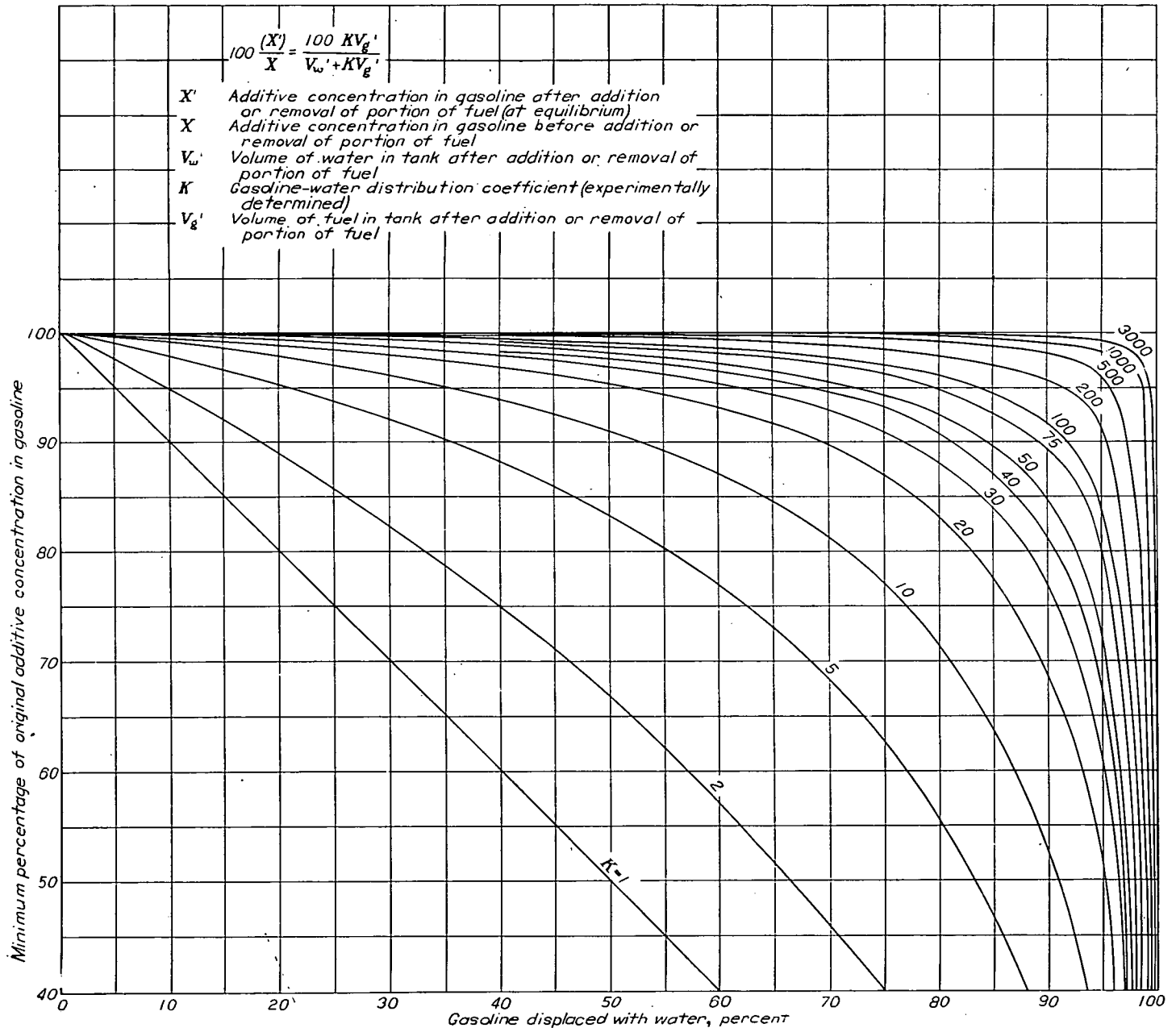


FIGURE VI-7.—Maximum possible loss of additive from fuel stored over water. A value for distribution coefficient is required for particular system, but plot is independent of original concentration of additive, nature of fluids, or temperature. (Fig. 2 of reference 13.)

Generally speaking, the loss of an additive, such as xylidines, with a distribution coefficient of about 20 will not be severe. The greatest loss will occur when the tank contains a ratio of xylidine-free water to fuel greater than about 4:1 (reference 11).

It is emphasized that the equations presented in the foregoing paragraphs apply only to the extent that the assumptions hold for a particular application. In practice, factors such as temperature, varying gasoline-water interface area, and frequency and quantity of additions or withdrawals will influence the results.

As previously stated, the evaluation of loss of additive to water in a water-displacement fuel-storage system is dependent on the distribution coefficient K for the particular additive under consideration. Olson, Tischler, and Goodman (reference 13), and Goodman and Howard (reference 14) have determined distribution coefficients for 45 aromatic amines.

The base gasoline used in determining the distribution coefficients was an aromatic-extracted grade 65 gasoline to which was added 15 percent (by volume) of an aromatic-hydrocarbon mixture consisting of five parts xylene, two parts cumene, and one part toluene. Blends of 1, 3, and 6 percent (by weight) aromatic amine in this base fuel were examined. Earlier data on xylidines (reference 12) show that temperature is the most significant variable affecting the distribution coefficient; consequently, measurements were made at 40° and 100° F. The distribution coefficients for the amines are shown in table VI-6.

The effect of the distribution coefficient in loss of additive to water in a water-displacement system is illustrated in figure VI-7 (reference 13). This figure was prepared by use of the previously discussed equations (reference 11) and is independent of the original additive concentration, the nature of the two fluids, or the temperature. Figure VI-7 shows the minimum possible additive concentration in the fuel remaining in the tank after part of the fuel is displaced from a full tank with amine-free water. The greatest amine loss will occur if the fuel is removed in a single batch; however, this seldom occurs in practice. A lower loss of additive than indicated in figure VI-7 will result if removal of the stored fuel is stepwise.

Methods of analysis for aromatic amines.—For a period of about 2 years the NACA used spectrophotometric measurements of the ultraviolet absorption of aromatic amines in the analysis for presence of amines in gasoline and water. This method of analysis has been reported in detail by

Tischler and Howard (references 15 and 16). The spectrophotometric method is more accurate than titration methods and has proved to be applicable to quantitative determination in hydrocarbon solutions of all monoaryl amines tested in the NACA laboratories.

REFERENCES

1. Jones, Anthony W., and Bull, Arthur W.: Knock-Limited Performance of Pure Hydrocarbons Blended with a Base Fuel in a Full-Scale Aircraft-Engine Cylinder. I—Eight Paraffins, Two Olefins. NACA ARR E4E25, 1944.
2. Cook, Harvey A., Held, Louis F., and Pritchard, Ernest I.: The Influence of Exhaust Pressure on Knock-Limited Performance. NACA MR E5A05, 1945.
3. Branstetter, J. Robert: Knock-Limited Performance of Blends of AN-F-28 Fuel Containing 2 Percent Aromatic Amines—I. NACA MR, April 1944.
4. Alquist, Henry E., and Tower, Leonard K.: Knock-Limited Performance of Blends of AN-F-28 Fuel Containing 2 Percent Aromatic Amines—II. NACA MR, June 1944.
5. Alquist, Henry E., and Tower, Leonard K.: Knock-Limited Performance of Blends of AN-F-28 Fuel Containing 2 Percent Aromatic Amines—III. NACA MR, August 1944.
6. Alquist, Henry E., and Tower, Leonard K.: Knock-Limited Performance of Blends of AN-F-28 Fuel Containing 2 Percent Aromatic Amines—IV. NACA MR E4L21, 1944.
7. Alquist, Henry E., and Tower, Leonard K.: Knock-Limited Performance of Blends of AN-F-28 Fuel Containing 2 Percent Aromatic Amines—V. NACA MR E5H06, 1945.
8. Jones, Anthony W., Bull, Arthur W., and Jonash, Edmund R.: Knock-Limited Performance of Six Aromatic Amines Blended with a Base Fuel in a Full-Scale Aircraft-Engine Cylinder. NACA MR E5D04, 1945.
9. Kelly, Richard L.: The Low-Temperature Solubility of 42 Aromatic Amines in Aviation Gasoline. NACA MR E5K09, 1945.
10. Olson, Walter T.: The Low-Temperature Solubility of Technical Xylidines in Aviation Gasoline. NACA MR, June 1943.
11. Olson, Walter T., and Tischler, Adelbert O.: Loss of Xylidines in Overwater Storage of Xylidine-Blended Fuel. NACA MR, March 1944.
12. Tischler, Adelbert O., Slabey, Vernon A., and Olson, Walter T.: Gasoline-Water Distribution Coefficients of Xylidines. NACA MR, June 1943.
13. Olson, Walter T., Tischler, Adelbert O., and Goodman, Irving A.: Gasoline-Water Distribution Coefficients of 27 Aromatic Amines. NACA MR, Aug. 1944.
14. Goodman, Irving A., and Howard, J. Nelson: Suitability of 18 Aromatic Amines for Overwater Storage When Blended with Aviation Gasoline. NACA MR E5F20a, 1945.
15. Tischler, Adelbert O.: Quantitative Analysis for Aromatic Amines in Aviation Fuels by Ultraviolet Spectrophotometry. NACA ARR E5H27, 1945.
16. Tischler, Adelbert O., and Howard, J. Nelson: Ultraviolet Absorption Spectra of Aromatic Amines in Isooctane and in Water. NACA ARR E5H27a, 1945.

TABLE VI-6.—DISTRIBUTION COEFFICIENTS OF AROMATIC AMINES *

Aromatic amine	Original concentration in fuel (percent by weight)	Temperature, ° F					
		40			100		
		Concentration in water at equilibrium (percent by weight)	Distribution coefficient on weight basis K_{wt}	Distribution coefficient on volume basis K_{vol}	Concentration in water at equilibrium (percent by weight)	Distribution coefficient on weight basis K_{wt}	Distribution coefficient on volume basis K_{vol}
Aniline.....	1 3 6	0.405 1.200 2.178	1.48 1.53 1.84	1.08 1.12 1.35	0.287 .850 1.557	2.50 2.57 2.94	1.76 1.81 2.07
N-Methylaniline.....	1 3 6	0.041 .112 .195	23 26 30	17 19 22	0.029 .076 .141	34 39 41	24 27 29
N-Ethylaniline.....	1 3 6	0.014 .043 .074	70 69 80	51 51 59	0.009 .026 .050	110 115 120	72 80 85
N-Propylaniline.....	1 3 6	0.003 .009 .019	300 330 320	220 240 230	0.002 .007 .013	500 400 460	350 250 320
N-Butylaniline.....	1 3 6	0.002 .005 .009	500 600 700	370 440 500	0.001 .004 .007	1000 800 900	700 550 630
N-Isopropylaniline.....	1 3 6	0.016 .048 .099	62 62 60	45 45 44	0.012 .034 .064	82 87 93	52 61 65
N-tert-Butylaniline.....	1 3 6	0.010 .025 .043	100 120 140	73 88 105	0.005 .014 .023	200 210 260	140 150 170
N, N-Dimethylaniline.....	1 3 6	0.003 .009 .017	300 330 350	220 240 260	0.003 .003 .014	300 370 430	210 260 300
N, N-Diethylaniline.....	1 3 6	0.0005 .001 .0015	2000 3000 4000	1500 2000 3000	0.0003 .0006 .0003	3000 5000 8000	2000 4000 6000
o-Toluidine.....	1 3 6	0.150 .415 .705	5.68 6.24 7.62	4.16 4.53 5.58	0.096 .267 .450	9.4 10.3 11.6	6.6 7.2 8.2
m-Toluidine.....	1 3 6	0.154 .410 .717	5.51 6.37 7.47	4.04 4.66 5.47	0.104 .282 .505	8.6 9.7 11.0	5.7 6.8 7.7
p-Toluidine.....	1 3 6	0.196 .571 -----	4.12 4.30 -----	3.02 3.15 -----	0.123 .352 .569	7.2 7.6 9.7	5.1 5.4 6.8
N-Methyl-p-toluidine.....	1 3 6	0.020 .055 .099	49 54 60	36 40 44	0.013 .036 .066	76 82 90	54 58 63
N-Methyl-o-toluidine.....	1 3 6	0.021 .045 .078	47 66 76	34 45 56	0.011 .029 .051	94 105 115	66 74 80
N-Methyltoluidines (60 percent p-, 40 percent o-).....	1 3 6	0.029 .074 .129	34 40 46	25 29 34	0.020 .051 .091	49 55 65	35 41 46
N-Methyltoluidines (80 percent p-, 20 percent o-).....	1 3 6	0.022 .053 .091	45 56 65	33 41 48	0.015 .038 .064	65 79 93	46 56 65
N-Ethyl-p-toluidine.....	1 3 6	0.004 .010 .017	250 300 350	180 220 260	0.003 .008 .015	300 370 400	210 260 280
N-Isopropyl-p-toluidine.....	1 3 6	0.007 .020 .035	140 150 170	105 110 125	0.005 .014 .022	200 210 270	140 150 190
o-Ethylaniline.....	1 3 6	0.054 .146 .244	18 20 24	13 15 18	0.032 .038 .152	30 33 39	21 23 27
p-Ethylaniline.....	1 3 6	0.067 .171 .238	14 17 20	10 12 15	0.040 .105 .180	24 25 33	17 20 23

TABLE VI-6.—DISTRIBUTION COEFFICIENTS OF AROMATIC AMINES—Continued

Aromatic amine	Original concentration in fuel (percent by weight)	Temperature, °F					
		40			100		
		Concentration in water at equilibrium (percent by weight)	Distribution coefficient on weight basis K_{w1}	Distribution coefficient on volume basis K_{v1} ^a	Concentration in water at equilibrium (percent by weight)	Distribution coefficient on weight basis K_{w1}	Distribution coefficient on volume basis K_{v1} ^a
N-Methylethylanilines.....	1 3 6	0.006 .017 .030	155 180 200	115 130 145	0.005 .011 .025	200 270 240	140 190 170
N-Methyl- <i>p</i> -ethylaniline.....	1 3 6	0.006 .015 .028	160 200 210	115 145 155	0.003 .010 .018	300 300 330	210 210 230
<i>p</i> - <i>tert</i> -Butylaniline.....	1 3 6	0.008 .021 .035	120 140 170	88 105 125	0.005 .014 .027	200 210 220	140 150 155
<i>o</i> -Isopropylaniline.....	1 3 6	0.021 .053 .090	47 56 66	34 41 48	0.010 .030 .053	100 100 115	70 70 80
<i>p</i> -Isopropylaniline.....	1 3 6	0.021 .048 .077	47 62 77	34 45 56	0.012 .033 .059	82 90 100	58 63 70
N-Methyl- <i>p</i> -isopropylaniline.....	1 3 6	0.003 .006 .009	300 500 700	220 370 500	0.001 .004 .008	1000 800 800	700 550 550
2,4,6-Trimethylaniline.....	1 3 6	0.013 .036 .065	76 82 91	56 60 67	0.007 .021 .040	140 140 150	100 100 105
Cumidines (from synthetic cumene).....	1 3 6	0.021 .059 .100	47 51 59	34 37 43	0.013 .037 .066	76 80 90	54 56 63
N-Methylcumidines.....	1 3 6	0.009 .020 .035	110 150 170	81 110 125	0.003 .010 .019	300 300 320	210 210 230
2-Methoxyaniline.....	1 3 6	0.183 .491 .846	4.48 5.15 6.21	3.28 3.77 4.54	0.129 .350 .632	6.8 7.6 8.6	4.8 5.4 6.1
Xylidines (technical).....	1 3 6	0.050 .143 .260	19 20 22	14 15 16	0.033 .083 .157	29 35 37	20 25 26
2,4-Xylidine.....	1 3 6	0.053 .142 .257	18 20 22	13 15 16	0.031 .089 .170	31 32 34	22 23 24
2,5-Xylidine.....	1 3 6	0.048 .132 .225	20 22 26	15 16 19	0.029 .085 .149	33 34 39	23 24 27
2,6-Xylidine.....	1 3 6	0.063 .173 .290	15 16 20	11 12 15	0.040 .109 .190	24 27 31	17 19 22
N-Methyl-2,4-xylidine.....	1 3 6	0.009 .018 .028	110 165 210	81 120 155	0.005 .012 .017	200 250 350	140 175 250
N-Methylxylidines.....	1 3 6	0.010 .021 .039	100 140 150	73 105 110	0.005 .013 .024	200 230 250	140 160 175
2,4-Diethylaniline.....	1 3 6	0.005 .014 .026	200 210 230	145 155 170	0.003 .008 .014	300 370 430	210 260 300
2-Methyl-5-isopropylaniline.....	1 3 6	0.008 .023 .037	120 130 160	88 95 115	0.005 .014 .026	200 210 230	140 150 160
Pseudocumidine (technical).....	1 3 6	0.036 .103 .172	27 28 34	20 21 25	0.022 .060 .108	45 49 55	32 34 39

Standard water solutions of the following amines could not be prepared because of the very low water solubility of the amine:

N-Methyl-*p*-*tert*-butylaniline
N-Isopropyl-*p*-isopropylaniline
N, N-Dimethyl-2-methyl-5-isopropylaniline
N, N-Dimethyl-2,4,6-trimethylaniline
Diphenylamine
N-Methyl-diphenylamine.

^a Data from references 13 and 14.

^b Density of gasoline solution at 40° F, 0.732 gram/ml.

^c Values for K_{v1} were computed by multiplying corresponding K_{w1} values by density of gasoline solution.

^d Density of gasoline solution at 100° F, 0.704 gram/ml.

CHAPTER VII

TETRAETHYL LEAD AS A FUEL ADDITIVE

For years tetraethyl lead $\text{Pb}(\text{C}_2\text{H}_5)_4$ has been considered the most effective practical additive for suppressing knock in conventional spark-ignition engines. In comparison with the additive-type compounds discussed in chapter VI, tetraethyl lead is considerably more effective. An investigation reported by Calingaert (reference 1) indicates that the knock-suppressing tendency of tetraethyl lead is 118 times greater than that of aniline.

On the basis of cost and antiknock effectiveness alone, tetraethyl lead is far more desirable than any known blending agent, because so little is required in a fuel to equal the antiknock effectiveness of a much larger volume of blending agent. Deleterious effects may, however, occur in engines when fuels containing high concentrations of tetraethyl lead are used. These harmful effects have necessitated strict regulations on the quantity that can be tolerated in fuels for various types of service.

In general, fuels used in commercial aviation contain smaller quantities of tetraethyl lead than those used in military aircraft. At the outbreak of World War II, Army-Navy aviation-fuel specifications permitted tetraethyl lead in quantities less than 3.0 ml per gallon. This limit was later raised to 4.0 ml per gallon; as critical shortages of more desirable blending agents occurred, the limit was again raised to 4.6 ml per gallon. Upon one occasion during the war, continued shortages of fuel stocks and the need for much higher performance fuels resulted in considerable experimentation with fuels containing 6.0 ml per gallon.

PROCESS OF LEAD-DEPOSIT ACCUMULATIONS ON AIRCRAFT-ENGINE SPARK PLUGS

As noted in reference 2, one of the disadvantages of tetraethyl lead as an antiknock agent for aircraft fuels is its characteristic of depositing, either as metallic lead or as a compound, on combustion-chamber surfaces during engine operation. Such deposits on spark plugs cause ignition failures both by bridging the electrode gaps and by forming conducting paths across the insulator surfaces and are suspected of lowering preignition ratings of spark plugs (ch. IV) and contributing to erosion of spark-plug electrodes. For these reasons lead deposition on spark plugs constitutes a flying hazard, increases time and cost of engine maintenance, and limits the use of tetraethyl lead in fuels.

The investigation described in reference 2 illustrates the characteristics of the progressive build-up of spark-plug deposits. In order to determine the progressive accumulation of deposit on spark plugs, each spark plug was weighed periodically throughout a series of test runs in a CFR engine at constant operating conditions. Additional runs were made in a single cylinder of a full-scale liquid-cooled aircraft engine. Three different types of spark plug, designated A, B, and C, were used.

Variation of deposition with time.—The results of a deposit test in the CFR engine for a type A spark plug are shown in figure VII-1. The condition of large total deposit was simulated without the necessity of excessive operating periods by increasing the tetraethyl lead content of the fuel to 34 ml per gallon.

The accumulation of deposit increases with operating time (fig. VII-1), but the rate of deposition decreases. After completion of this test, an effort was made to operate the engine with the coated spark plug, but the test was unsuccessful because of short circuiting.

Effect of power level on deposition.—Operating conditions affect the accumulation of deposits on spark plugs, as illustrated in figure VII-2. Failure of the spark plug during the low-power run occurred shortly after 14 hours of operation because of the formation of a small bead of lead deposit

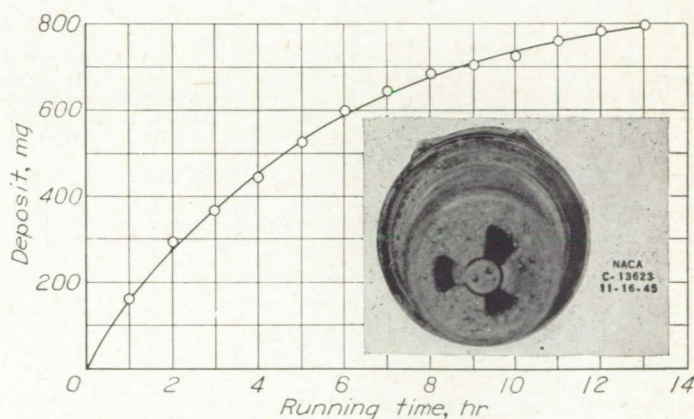


FIGURE VII-1.—Lead deposition for type A spark plug in CFR engine with AN-F-28 fuel leaded to 34 ml TEL per gallon. Compression ratio, 6.65; engine speed, 1800 rpm; imep, 82 pounds per square inch; fuel-air ratio, 0.10; inlet-air temperature, 85° F; coolant temperature, 212° F; spark advance, 45° B. T. C. (Fig. 1 of reference 2.)

on an electrode gap. A photograph of the spark plug in this case is shown in figure VII-3.

Accumulation of deposit masses.—Spark-plug deposition for a type B spark plug in the full-scale single-cylinder test engine is shown in figure VII-4 (reference 2). The spark plug was located in the exhaust position and the engine was operated under simulated cruise conditions. The fuel contained 18 ml TEL per gallon. During this test the condition of the plug was photographically recorded at various time intervals (fig. VII-5). As shown in these photographs, the deposit existed during operation in the form of molten globules, which changed position. The changes in position are attributed to the influence of gravitation and molecular forces and forces arising from gas flow in the combustion chamber (reference 2). The concentration of deposit near the lowest point of the spark-plug cavity became more pronounced as the deposit increased.

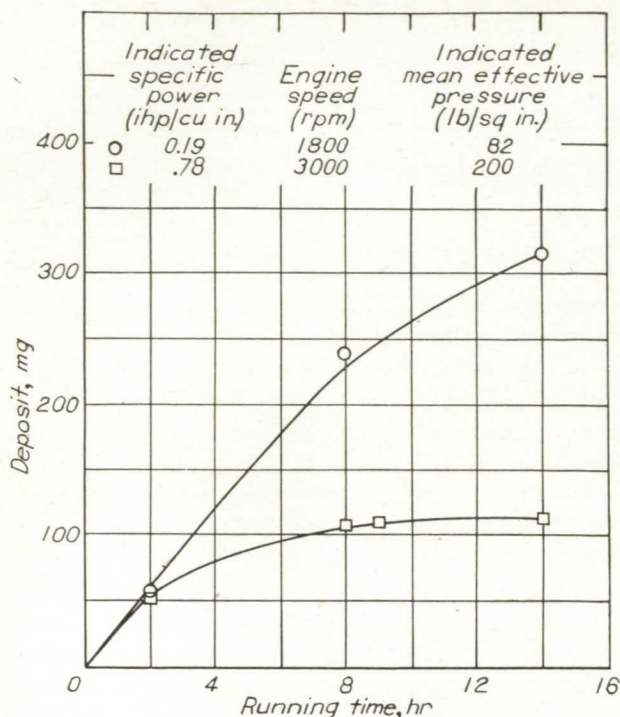


FIGURE VII-2.—Effect of power level on lead deposition for type C spark plug in CFR engine with AN-F-28 fuel leaded to 4.6 ml TEL per gallon. Compression ratio, 6.65; fuel-air ratio, 0.10; inlet-air temperature, 90° F; coolant temperature, 212° F; spark advance, 45° B. T. C. (Fig. 2 of reference 2.)



FIGURE VII-3.—Type C spark plug with lead in left gap. Failure of plug occurred after 14 hours operation at low power in CFR engine. (Fig. 3 of reference 2.)

As pointed out in reference 2, observation of deposits on many spark plugs indicates that these deposits may vary considerably in characteristics and appearance under different conditions. Consequently, no particular deposit can be considered typical. Deposits similar to those in figure VII-5 have been observed on badly fouled spark plugs from service engines.

Deposit losses.—From the data discussed in the foregoing paragraphs, it was concluded (reference 2) that the manner in which the rate of deposition varies (figs. VII-1, VII-2, and VII-4) suggests that the net rate is the result of deposition (possibly at constant rate) and loss of deposit at a rate

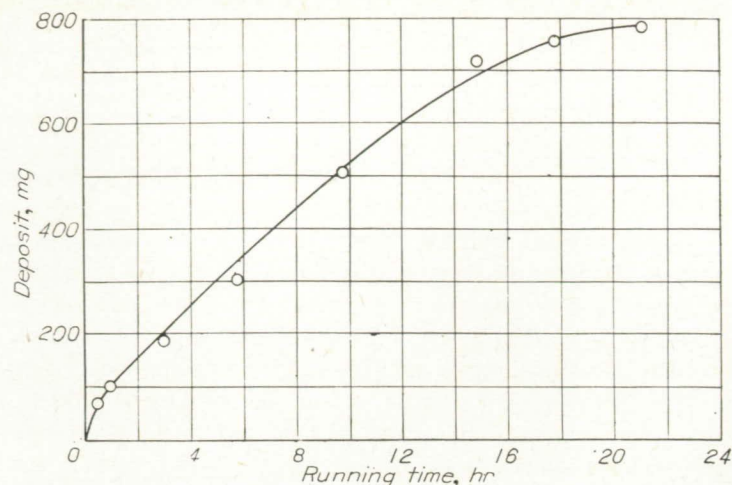


FIGURE VII-4.—Lead deposition for type B spark plug in exhaust position of liquid-cooled aircraft cylinder with AN-F-28 fuel leaded to 18 ml TEL per gallon. Engine speed, 2000 rpm; inlet-air pressure, 30 inches of mercury absolute; fuel-air ratio, 0.10; inlet-air temperature, 85° F; coolant temperature, 250° F. (Fig. 4 of reference 2.)

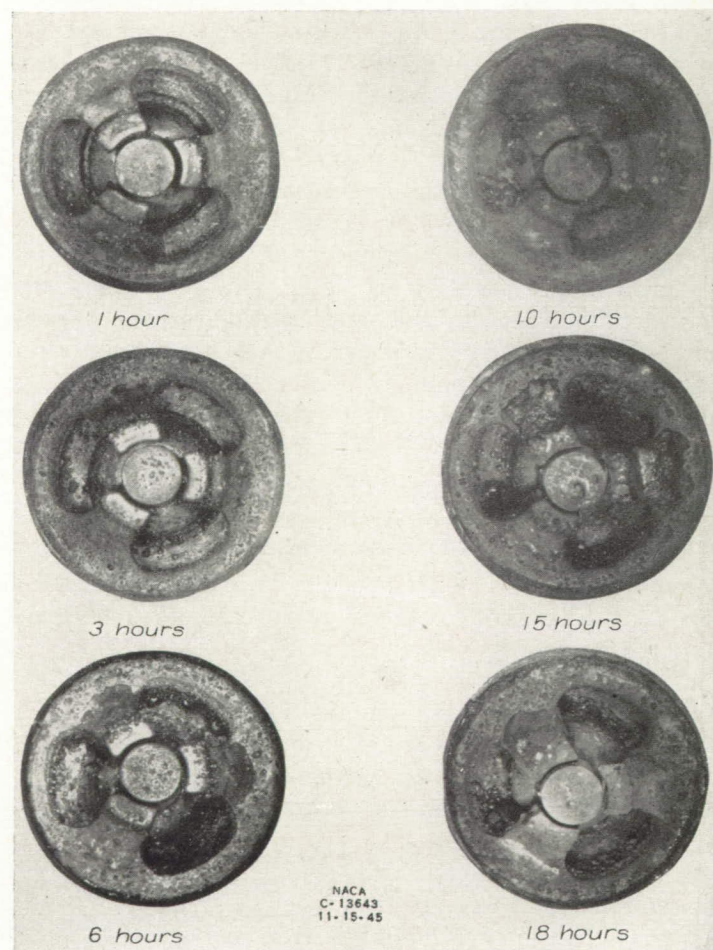


FIGURE VII-5.—Accumulation of lead deposits on type B spark plug in exhaust position of liquid-cooled aircraft cylinder. (Fig. 5 of reference 2.)

that increases as the total deposit increases. In order to test this conclusion, a spark plug was operated in the CFR engine on S reference fuel containing 6 ml TEL per gallon.

After $7\frac{1}{2}$ hours of operation, the deposit weight was found to be 159 milligrams. The spark plug was then replaced without cleaning and operated for 1 hour under the same conditions but with unleaded S reference fuel. At the end of 1 hour the deposit weight had decreased to 112 milligrams, a decrease of 30 percent. A similar test on another spark plug resulted in a 35-percent weight loss.

Significance of deposit weights.—A spark plug may fail electrically when a deposit of sufficient electric conductance to prevent sparking is formed between the spark-plug shell, or a ground electrode, and the center electrode. A very small quantity of deposit may cause failure if the deposit directly bridges the electrode gap. (See fig. VII-3.) There is evidence, however, that gap bridging is not the most common type of failure of spark plugs incorporating large nickel or tungsten electrodes. In the more usual case of failure by conduction along deposits on the insulator, relatively large quantities of deposit are necessary before short circuiting occurs. The amount, distribution, and conductivity of the deposit determine the path of short circuiting. From the photographic evidence in figure VII-5 it was concluded (reference 2) that, aside from certain concentrating influences such as the force of gravity and large temperature differences, the deposit distribution is primarily a matter of chance. The amount of deposit and the chemical or physical properties of the deposit can then be considered as factors determining the probability of failure. The actual failure, however, is dependent upon the distribution of deposit.

ETHYLENE DIBROMIDE IN TETRAETHYL LEAD FLUID

One of the principal causes of exhaust-valve failure in reciprocating engines has been found to be corrosion and collapse of the valve crown at excessively high operating temperatures (reference 3). Analysis of deposits on exhaust valves has shown conclusively that all the lead present in the fuel has not been properly scavenged from the cylinder. This fact supports the conclusion of Banks (reference 4) and Hives and Smith (reference 5) that hot corrosion of exhaust valves is a result of the action of lead oxide deposited on the valve head during operation with leaded fuel.

In order to minimize the undesirable lead deposits, ethylene dibromide is added to the tetraethyl lead before addition to the fuel. The ethylene dibromide reacts with lead in the cylinder during combustion to form volatile compounds that may be easily scavenged with the exhaust gases. The mixture of ethylene dibromide and tetraethyl lead (and small quantities of dye and kerosene) prior to addition to the fuel is called tetraethyl lead fluid. The fluid currently used in aircraft fuels contains 0.96 gram of ethylene dibromide per ml TEL. This quantity of ethylene dibromide is the amount theoretically required to combine with all the lead in the

mixture. Such a mixture of the dibromide and tetraethyl lead is popularly known as a 1-T (one-theory) mix.

An investigation (reference 6) was conducted to determine the influence of excess ethylene dibromide on the degree of valve-crown corrosion occurring during engine operation. The effects resulting from use of 1-T and 2-T mixes were compared. Because the use of excess ethylene dibromide could conceivably reduce the knock-limited performance of the leaded fuel, studies of knock-limited performance were included.

Effect on exhaust-valve corrosion.—Results of valve-corrosion tests in a full-scale air-cooled aircraft cylinder are summarized in the following table (reference 6):

Fuel *		Length of run (hr)	Weight loss of valve		
TEL (ml/gal)	Mix		Total (grams)	Rate (gram/hr)	Rate (percentage of loss with 1-T mix)
4.53	1-T	22	5.18	0.235	100
	2-T	16½	1.97	.119	51
6.0	1-T	18	3.17	0.176	100
	2-T	18	.53	.029	16

*AN-F-28 (28-R) was used as base fuel. This fuel contained about 4.53 ml TEL/gal.

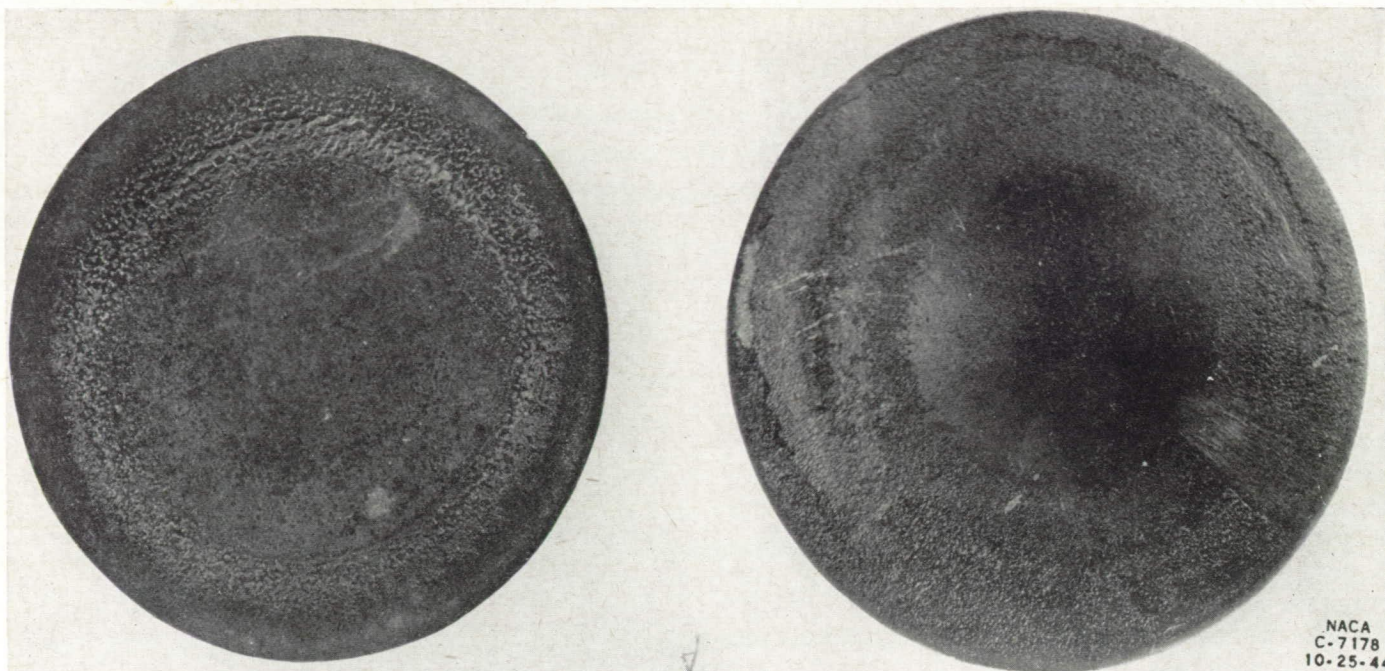
For the tests in which the concentration of tetraethyl lead was 4.53 ml per gallon, the exhaust-valve temperature was held at approximately 1200° F. When the concentration was increased to 6.0 ml per gallon, however, the combustion-air inlet temperature was reduced to 95° F as compared with 150° F for the test with the fuel of lower tetraethyl lead concentration. This change caused a lower exhaust-valve temperature and probably accounts for the reduction in amount of valve corrosion in the test of fuel with 6.0 ml TEL per gallon.

Comparative photographs of the valves from the tests just described are shown in figures VII-6 and VII-7.

Effect on lead deposition.—Because of corrosion of the valve crown it was difficult to determine quantitatively the effect of ethylene dibromide on deposits. (See reference 6.) For this reason, the measurement of deposits was confined to the piston crown and the spark plugs. In the test with the fuel containing 4.53 ml TEL per gallon, the rate of lead deposition on the piston crown with the 2-T mix was about 73 percent of the rate during the test with the 1-T mix.

The effect of ethylene dibromide on spark-plug deposits was also determined for fuel containing 6.0 ml TEL per gallon; these results are shown in the following table:

Spark plug	Mix	Weight of deposit (gram)	Reduction in deposit (percent)
Front.....	1-T 2-T	0.083 .021	0 75
Rear.....	1-T 2-T	0.109 .014	0 87



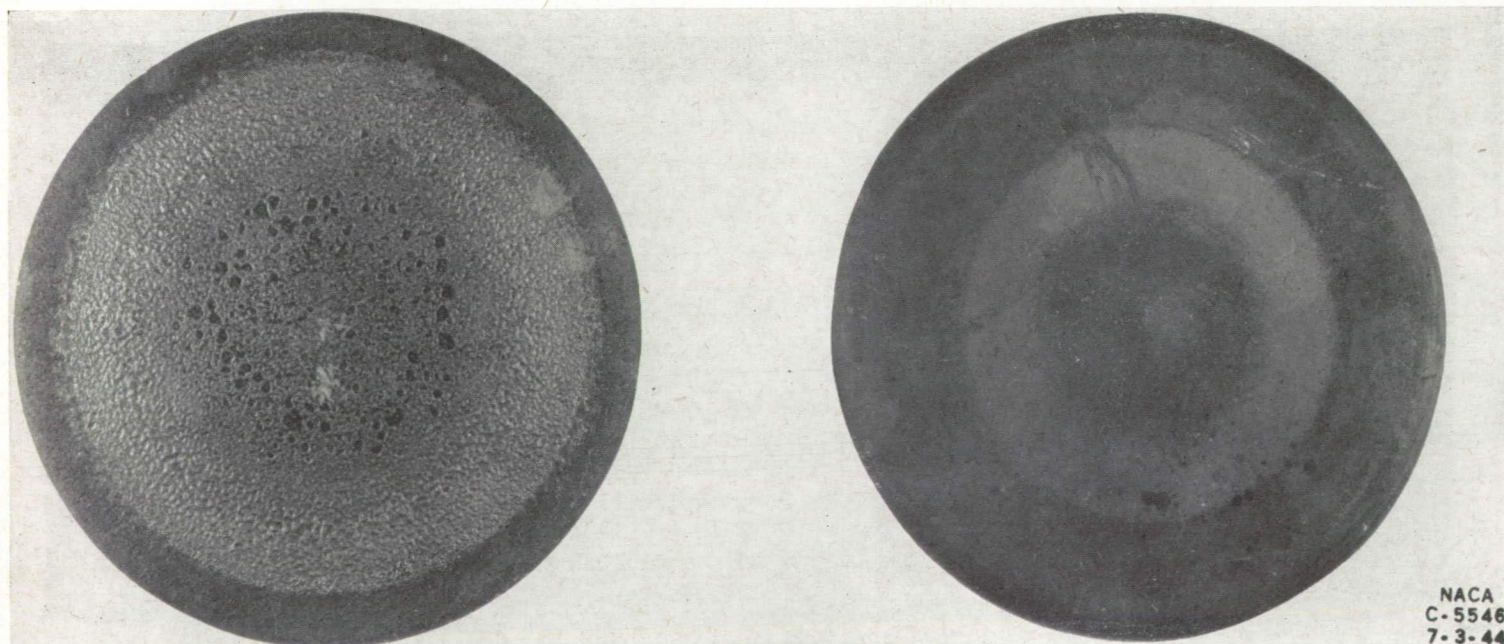
(a) Valve after 22 hours of operation with 1-T mix.

(b) Valve after 16½ hours of operation with 2-T mix.

FIGURE VII-6.—Effect of excess ethylene dibromide on exhaust valves in air-cooled aircraft cylinder with fuel leaded to 4.53 ml TEL per gallon. (Fig. 1 of reference 6.)

Effect on knock-limited power.—When the ethylene dibromide concentration was increased to the equivalent of a 2-T mix, there was apparently no effect on the knock limit over the range of conditions investigated in the full-scale air-cooled cylinder (fig. VII-8). (See references 6 and 7.)

Other tests were made to evaluate the effects of ethylene dibromide on knock-limited power in a CFR test engine operating at conditions specified for the A. S. T. M. Supercharge method; the results are presented in table VII-1.



(a) Valve after 18 hours operation with 1-T mix.

(b) Valve after 18 hours operation with 2-T mix.

FIGURE VII-7.—Effect of excess ethylene dibromide on exhaust valves in air-cooled aircraft cylinder with fuel leaded to 6.0 ml TEL per gallon. (Fig. 2 of reference 6.)

TABLE VII-1.—EFFECT OF ETHYLENE DIBROMIDE IN LEADED AND UNLEADED S REFERENCE FUEL ON KNOCK-LIMITED POWER

[Where two values are given in the table for the same determination, the second value was obtained from a check run.]

Leaded S reference fuel				
Ethylene dibromide in S reference fuel +6 ml TEL (ml/gal)	Relative power ^a			
	Fuel-air ratio			
	0.062	0.070	0.090	0.110
2.67 (1-T).....	1.00	1.00	1.00	1.00
0 (0-T).....	0.98	1.00	1.01 1.01	1.00
4.01 (1.5-T).....	1.01 .99	1.01 1.00	1.01 .98	1.01 1.00
5.34 (2-T).....	0.92	0.91	0.94 .95	0.95 .93
Unleaded S reference fuel				
Ethylene dibromide in unleaded S reference fuel (ml/gal)	Relative power ^b			
	Fuel-air ratio			
	0.062	0.070	0.090	0.110
0.....	1.00	1.00	1.00	1.00
2.67.....	0.98	0.96	0.94 .96	0.97 .98
5.34.....	0.97	0.92	0.92 .94	0.97 .96

^a imep (S+6 ml TEL+test concentration of ethylene dibromide)

imep (S+6 ml TEL+2.67 ml ethylene dibromide (1-T))

^b imep (S+test concentration of ethylene dibromide)

imep (S)

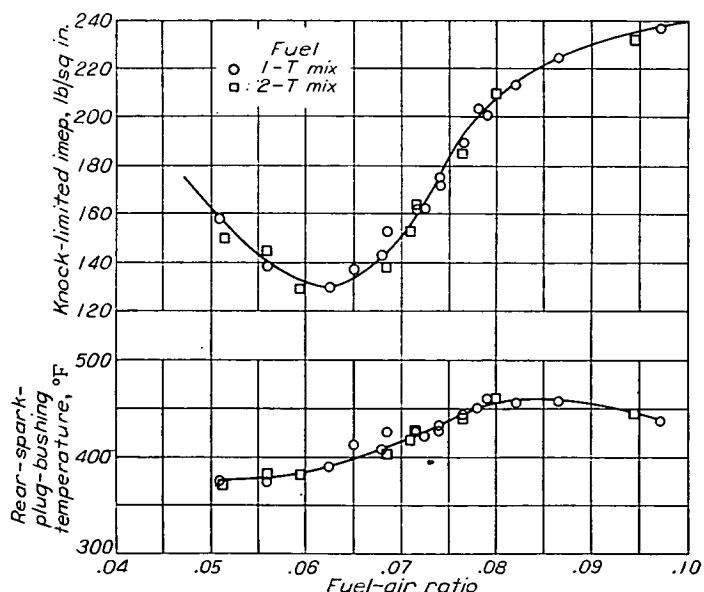


FIGURE VII-8.—Effect of excess ethylene dibromide on knock-limited performance in air-cooled aircraft cylinder. Engine speed, 2200 rpm; inlet-air temperature, 250° F; spark advance, 20° B. T. C. (Fig. 3 of reference 6.)

Increasing the ethylene dibromide concentration from 0 to 5.34 ml per gallon decreased the knock-limited power of the leaded and unleaded S reference fuel about the same amount, 6 to 9 percent depending upon the fuel-air ratio. Changing the concentration of ethylene dibromide from 0 to 2.67 to 4.01 ml per gallon had no effect on the knock-limited performance of the leaded fuel; however, a change of 0 to 2.67 ml per gallon decreased the knock-limited power of the unleaded fuel a maximum of about 6 percent.

LEAD SUSCEPTIBILITY

The lead susceptibility or lead response of a fuel is defined as the increase in knock-limited performance of the fuel resulting from the addition of a given quantity of tetraethyl lead. In 1943 the NACA conducted an analysis of existing data on pure hydrocarbon fuels to determine the manner in which the lead susceptibility is influenced by fuel type, engine type, and engine operating conditions. This analysis (references 8 and 9) was based on data obtained from the American Petroleum Institute (API) Hydrocarbon Research Project.

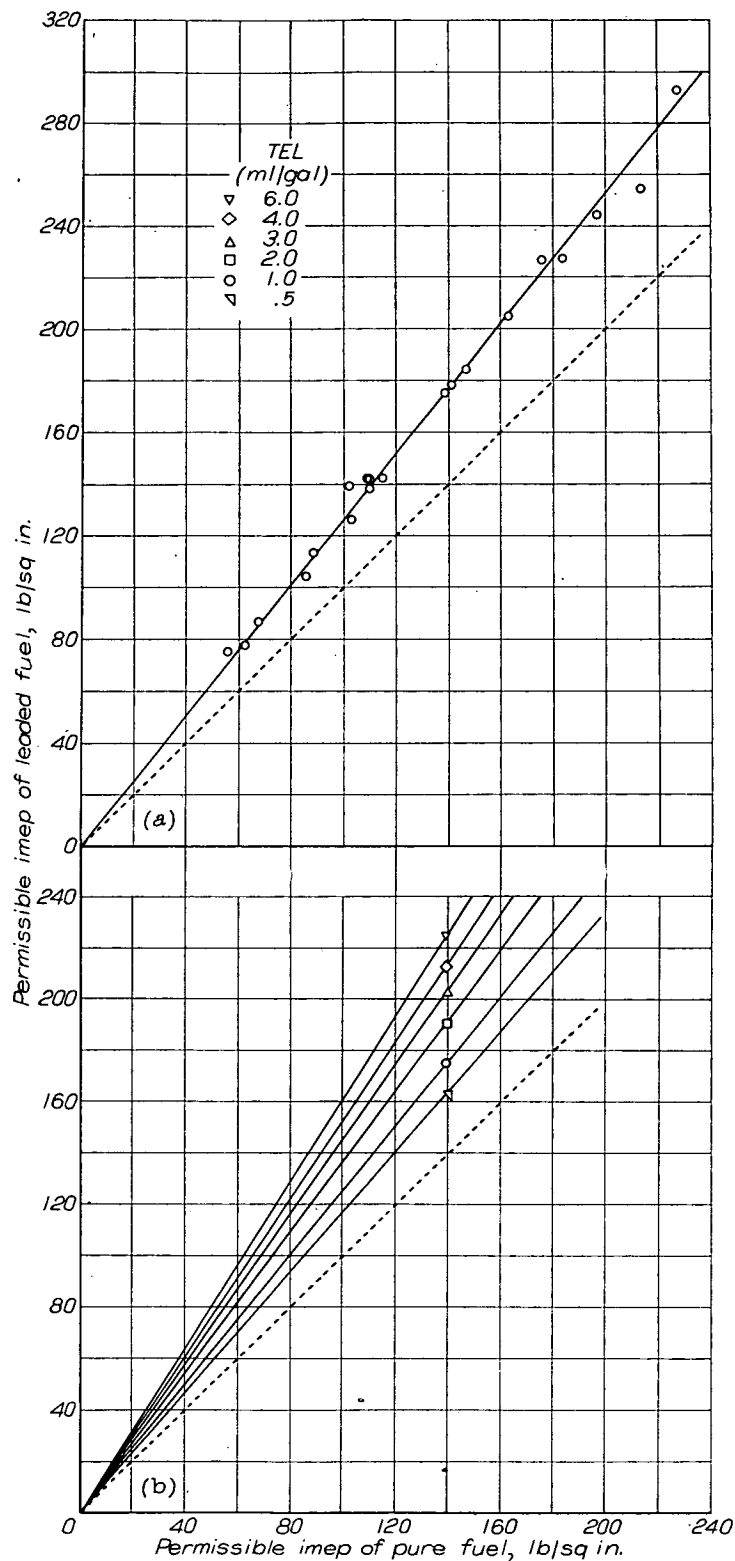
DATA FROM SUPERCHARGED ENGINES

In cooperation with the API, the Ethyl Corporation investigated the lead susceptibility of many pure compounds. These data were determined in a small-scale engine having a displacement of 17.6 cubic inches.

Paraffins.—Examination of the Ethyl Corp. data demonstrated that the lead susceptibility could be represented as a straight line by plotting the knock-limited indicated mean effective pressures of the pure fuels against the knock-limited indicated mean effective pressures of the leaded fuels (reference 8). By this method, data for paraffinic fuels were plotted as shown in figure VII-9 (a). It is obvious in this figure that the fuel which has the highest permissible indicated mean effective pressure in the pure state has the greatest lead response. The percentage increase in indicated mean effective pressure for 1.0 ml TEL, however, is constant regardless of the permissible indicated mean effective pressure. Heron and Beatty (reference 10) found this fact to be true for 6.0-ml additions of tetraethyl lead.

The susceptibility of isooctane to various quantities of tetraethyl lead is shown in figure VII-9 (b). Because it was believed that the lead susceptibility of all paraffins might be represented by a chart similar to figure VII-9 (b), a later investigation (reference 9) was made with blends of two paraffinic fuels (S and M reference fuels) containing up to 8.0 ml TEL per gallon. This study was made in a supercharged CFR engine modified as described in reference 11.

Data from reference 9 are plotted in figure VII-10. These plots are based on results reported by Heron and Beatty (reference 10) which indicate that for supercharged tests a linear relation exists between the reciprocal of the knock-limited indicated mean effective pressures and blend composition. (See chapter VIII.)



(a) Paraffins. (Fig. 10 of reference 8.)
(b) Isooctane (2,2,4-trimethylpentane). (Fig. 11 of reference 8.)

FIGURE VII-9.—Lead susceptibility of various fuels in 17.6 engine. Compression ratio, 5.6; engine speed, 900 rpm; approximate fuel-air ratio, 0.07; inlet-air temperature, 225° F; coolant temperature, 300° F.

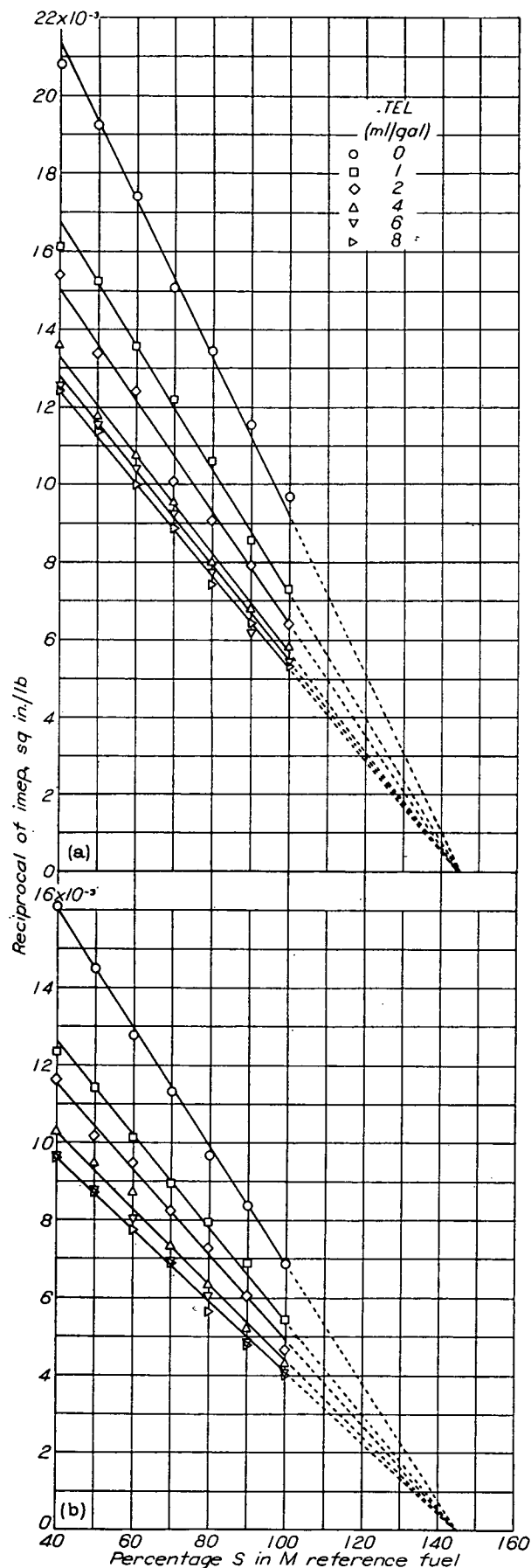
The lines of constant tetraethyl lead concentration in these figures can be represented by the equation of an equilateral hyperbola:

$$P(A-N)=B \quad (1)$$

where

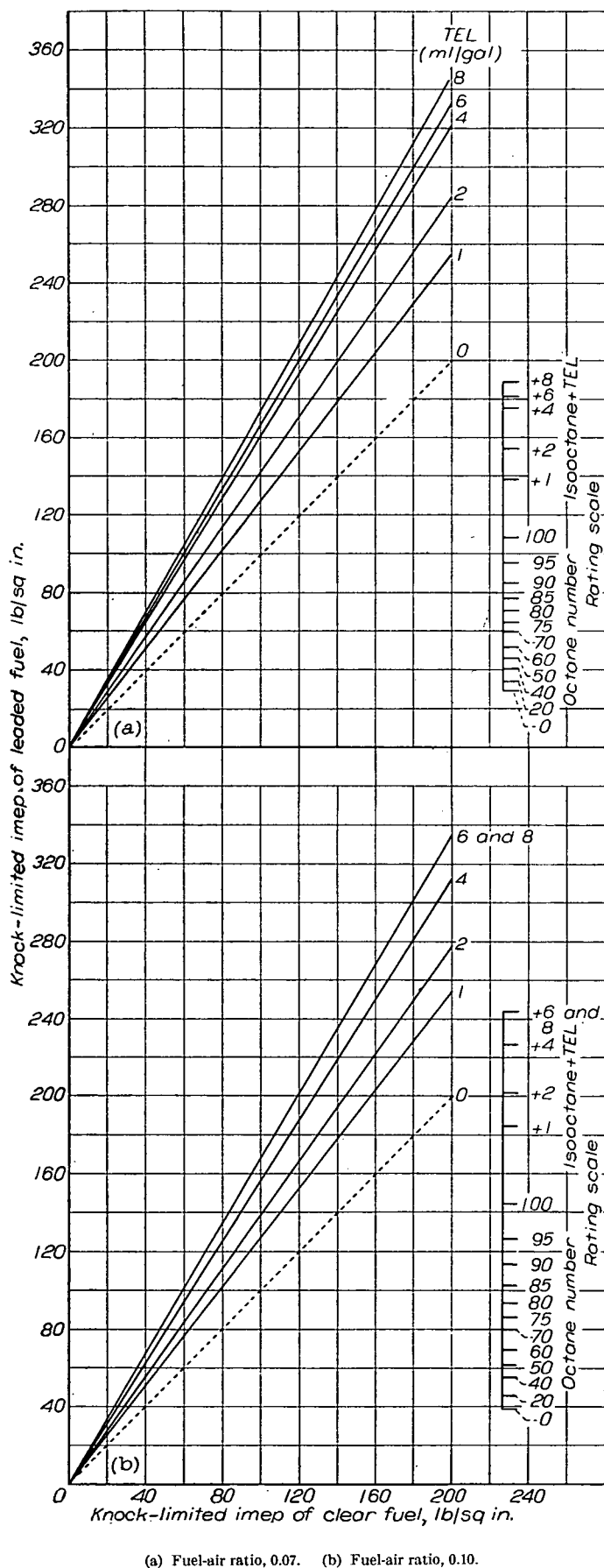
P knock-limited indicated mean effective pressure

A constant



(a) Fuel-air ratio, 0.07. (b) Fuel-air ratio, 0.10.

FIGURE VII-10.—Lead susceptibility of blends of S and M reference fuels in CFR engine. Compression ratio, 7.0; engine speed, 2000 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F; spark advance, 35° B. T. C. (Fig. 6 of reference 9.)



(a) Fuel-air ratio, 0.07. (b) Fuel-air ratio, 0.10.

FIGURE VII-11.—Lead susceptibility of paraffinic fuels in CFR engine. Compression ratio, 7.0; engine speed, 2000 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F; spark advance, 35° B. T. C. (Fig. 7 of reference 9.)

N volume percentage of S reference fuel in M reference fuel
 B slope

The asymptotes of the hyperbola represented by equation (1) are zero and A . The constant A is independent of tetraethyl lead concentration and fuel-air ratio and is equal to 145 (the intersection of the curves with the abscissa).

In figures VII-11, the data from figure VII-10 have been converted to plots of the knock-limited indicated mean effective pressure of the clear fuel against the knock-limited indicated mean effective pressure of the leaded fuel. This type of plot has an advantage over that of figure VII-10 in that the lead susceptibility can be represented by a linear relation without the use of a reciprocal scale.

A scale representing antiknock ratings in terms of octane numbers and leaded isooctane has been added to figure VII-11. This scale corresponds to the ordinate values of knock-limited indicated mean effective pressure; that is, a 90-octane fuel whether clear or leaded (fig. VII-11(a)) has a knock-limited indicated mean effective pressure of 85 pounds per square inch under the conditions presented. With the addition of this scale, the chart may be used to obtain either the knock-limited indicated mean effective pressure or the octane rating of a paraffinic fuel containing various quantities of tetraethyl lead. Only the unleaded fuel need be tested in order to estimate the lead response for any concentration of tetraethyl lead.

The estimation of lead susceptibility according to the foregoing procedure is of value in the study of paraffinic fuels in which the quantity of fuel available for testing is limited. In such cases the rating of the clear fuel can be determined by a standard rating method such as the A. S. T. M. Supercharge rating method, and the rating of the test fuel plus various amounts of tetraethyl lead can be approximated from a chart previously established for the selected rating method.

Cycloparaffins and olefins.—Insufficient data exist to establish definitely the susceptibilities of the cycloparaffins (fig. VII-12 (a)) and olefins (fig. VII-12 (b)). When figures VII-9, VII-12 (a), and VII-12 (b) are compared, the order of lead response for the paraffins, cycloparaffins, and olefins is found to be as follows:

Hydrocarbons	Percentage increase imep					
	(ml TEL/gal)					
	1.0	2.0	3.0	4.0	5.0	6.0
Paraffins.....	26	36	46	51	57	61
Cycloparaffins.....	23					
Olefins.....	14					

* Percentage increase in imep for olefins was estimated from diisobutylene.

DATA FROM NONSUPERCHARGED ENGINES

The relation between critical compression ratio and A. S. T. M. Motor method octane number can be represented by the equation for a hyperbola. (See fig. VII-13.) The curve in this figure was calculated from figure 1 of reference 12, which shows the relation between height of compression chamber and A. S. T. M. Motor method octane number.

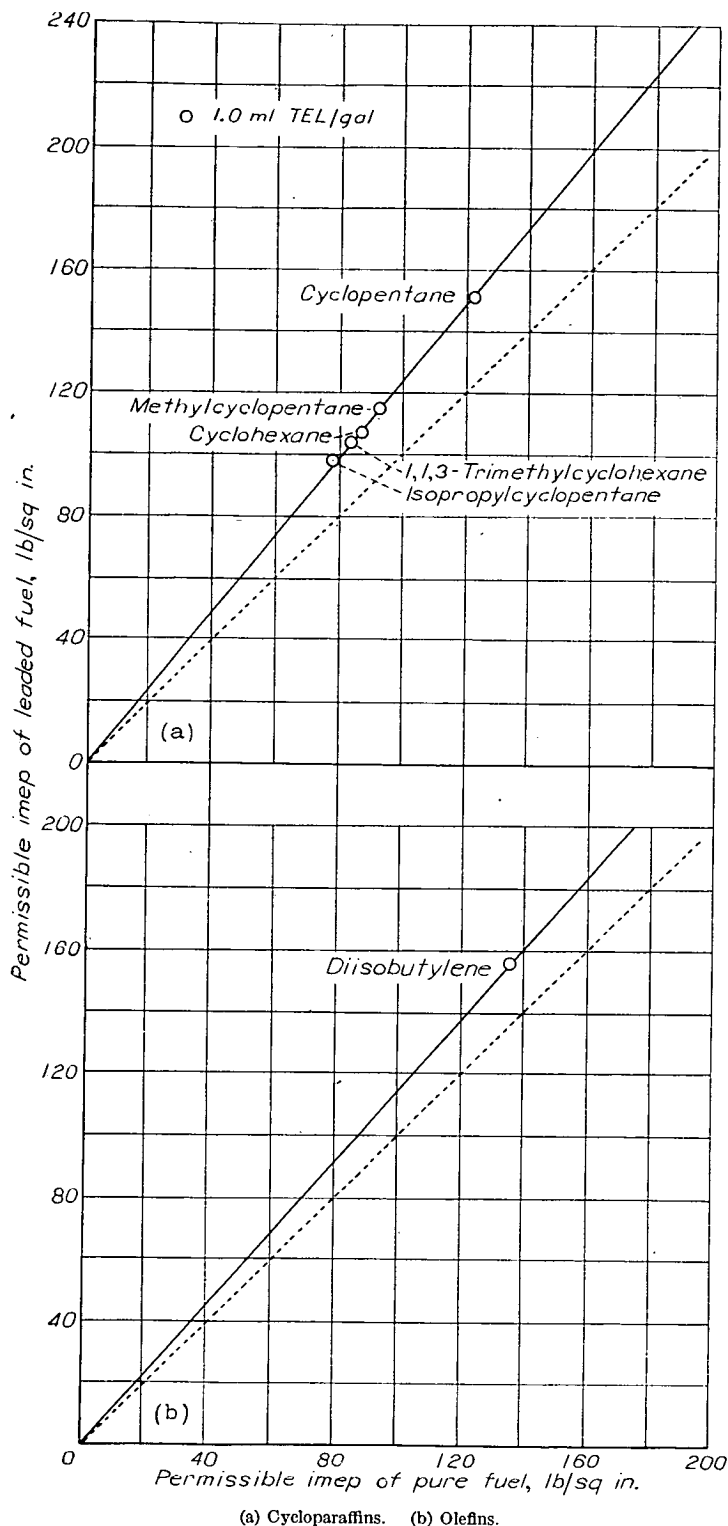


FIGURE VII-12.—Lead susceptibility of various fuels in 17.6 engine. Compression ratio, 5.6; engine speed, 900 rpm; approximate fuel-air ratio, 0.07; inlet-air temperature, 225° F; coolant temperature, 300° F. (Fig. 13 of reference 8.)

The equation for the curve in figure VII-13 is

$$(R-3.3)(125-N_1)=125 \quad (2)$$

where

R critical compression ratio

N_1 A. S. T. M. Motor method octane number

The asymptotes for the curve are critical compression ratio, 3.3 and A. S. T. M. Motor method octane number, 125. As

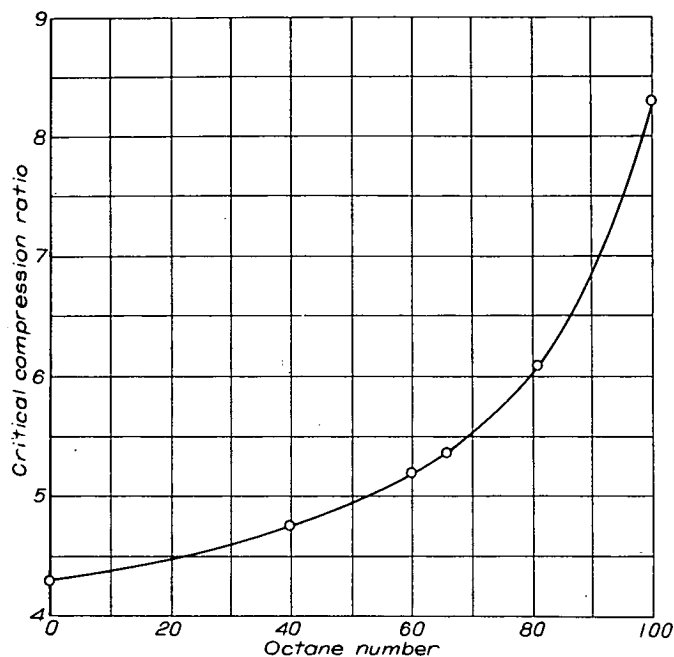


FIGURE VII-13.—Variation of critical compression ratio with octane number for A. S. T. M. Motor method. (Fig. 3 of reference 8.)

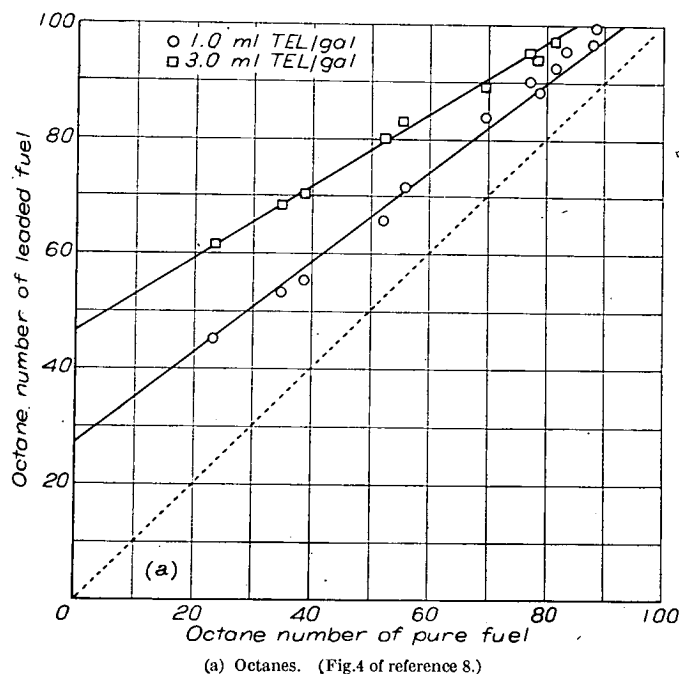
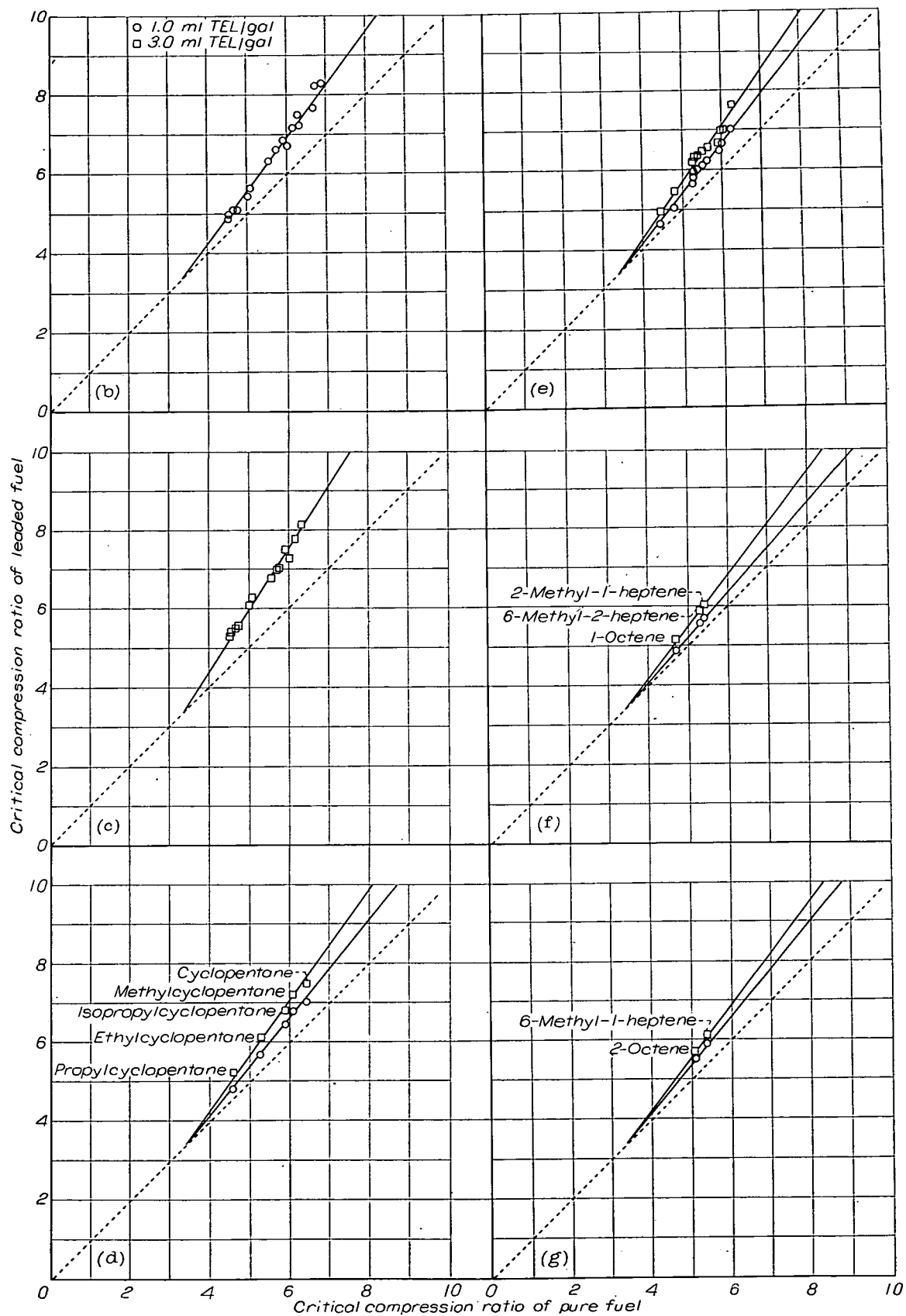


FIGURE VII-14.—Lead susceptibility of various fuels based on A. S. T. M. Motor ratings.

pointed out in reference 8, these asymptotes will vary with engine operating conditions.

The curve just discussed is related to a linear method used by Heron and Beatty (reference 10) to express lead susceptibility (gain of octane number per ml TEL). The method is illustrated in figure VII-14 (a) for data obtained by the API. The relation between figures VII-13 and VII-14 (a) may be shown by extending the lines in figure VII-14 (a) until they intersect. This intersection occurs at an octane number of 125. If both scales of figure VII-14 (a) were converted to critical compression ratios, the point of intersection of the curves would be at a compression ratio of 3.3. These two points of intersection are the asymptotes of figure VII-13.



(b) Paraffins. Tetraethyl lead content, 1.0 ml per gallon. (Fig. 19 of reference 8.)

(c) Paraffins. Tetraethyl lead content, 3.0 ml per gallon. (Fig. 20 of reference 8.)

(d) Cyclopentane derivatives. (Fig. 21 of reference 8.)

(e) Cyclohexane derivatives. (Fig. 22 of reference 8.)

(f) Octenes. (Fig. 23 of reference 8.)

(g) Octenes. (Fig. 24 of reference 8.)

Figure VII-14.—Concluded. Lead susceptibility of various fuels based on A. S. T. M. Motor method ratings.

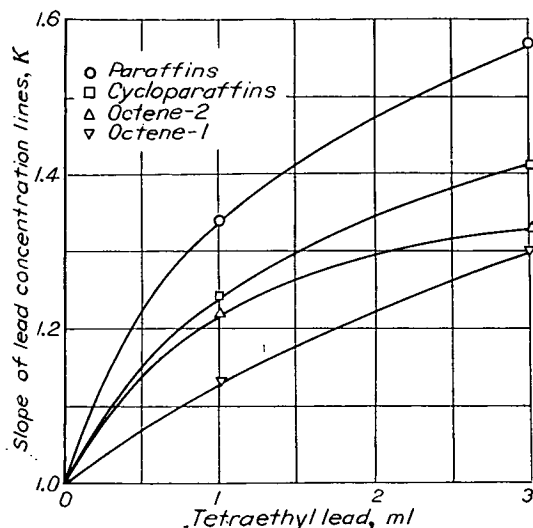
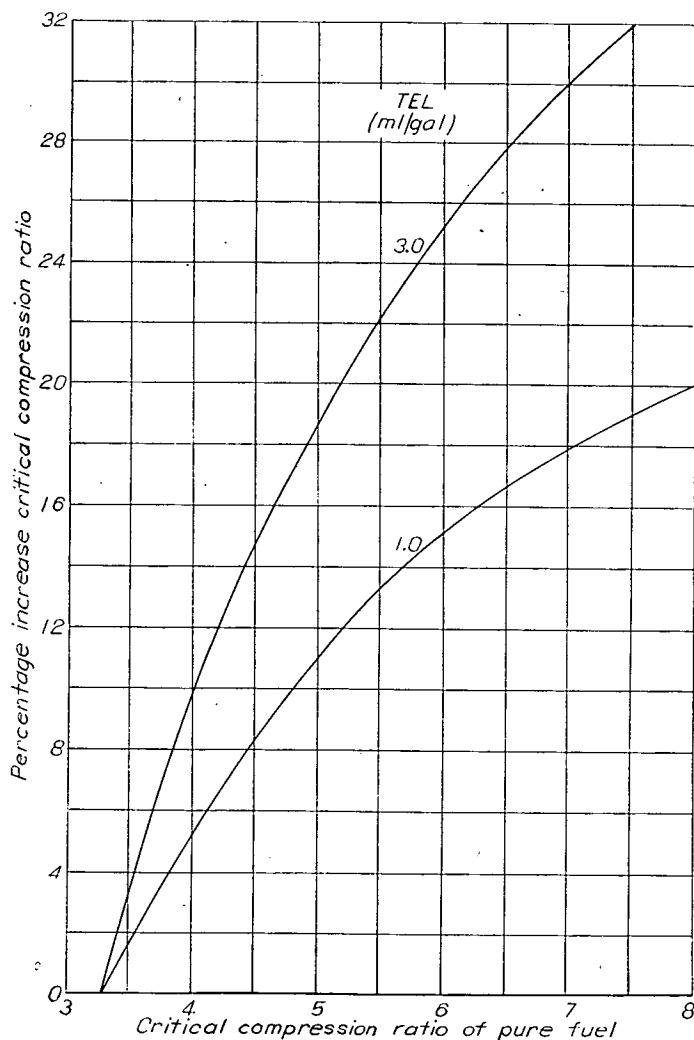
FIGURE VII-15.—Variation of K with tetraethyl lead concentration. (Fig. 25 of reference 8.)

FIGURE VII-16.—Variation of compression ratio with tetraethyl lead concentration. (Fig. 26 of reference 8.)

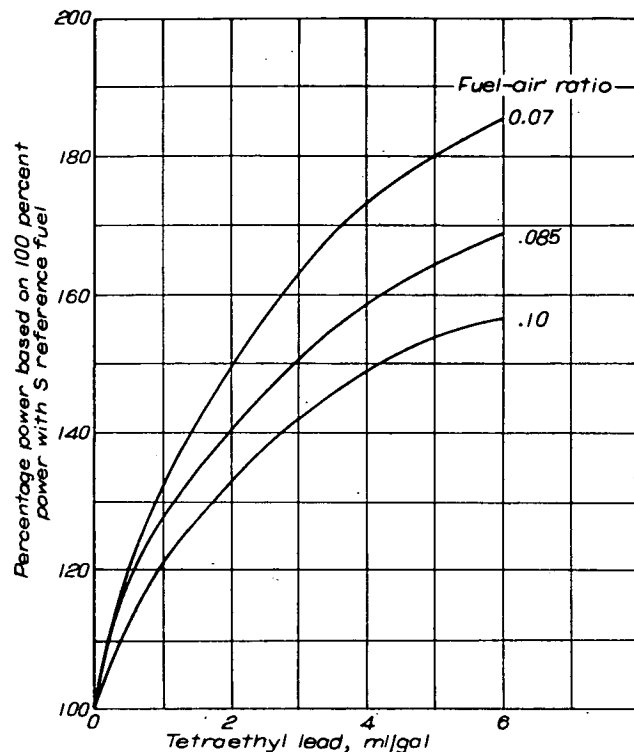


FIGURE VII-17.—Variation of power increase with tetraethyl lead concentration for reference fuel as determined by A. S. T. M. Supercharge method. (Fig. 10 of reference 9.)

By the use of critical compression ratios, an analysis similar to that used for supercharged test engine data has been made (reference 8). As shown in figures VII-14 (b) and VII-14 (c), the lead response of the paraffins may be represented by single straight lines for 1.0 and 3.0 ml TEL, respectively. The equation for these straight lines of constant tetraethyl lead concentration is

$$R_1 = KR - 3.3 (K - 1) \quad (3)$$

where

R_1 critical compression ratio of leaded fuel

R critical compression ratio of pure fuel

K slope of line of constant tetraethyl lead concentration

The cycloparaffins (figs. VII-14 (d) and VII-14 (e)) may also be represented by single straight lines. The lead susceptibility of the olefins (figs. VII-14 (f) and VII-14 (g)) is apparently affected by the position of the double bond and therefore a single line cannot be used to represent the class as a whole.

For each class of hydrocarbons a different value of K will be obtained (equation (3)). The values of K determined for the data presented in this analysis are as follows:

Hydrocarbon	Slopes of lead susceptibility curves	
	For 1.0 ml	For 3.0 ml
Paraffins.....	1.34	1.57
Cycloparaffins.....	1.24	1.41
Octene-2.....	1.22	1.33
Octene-1.....	1.13	1.30

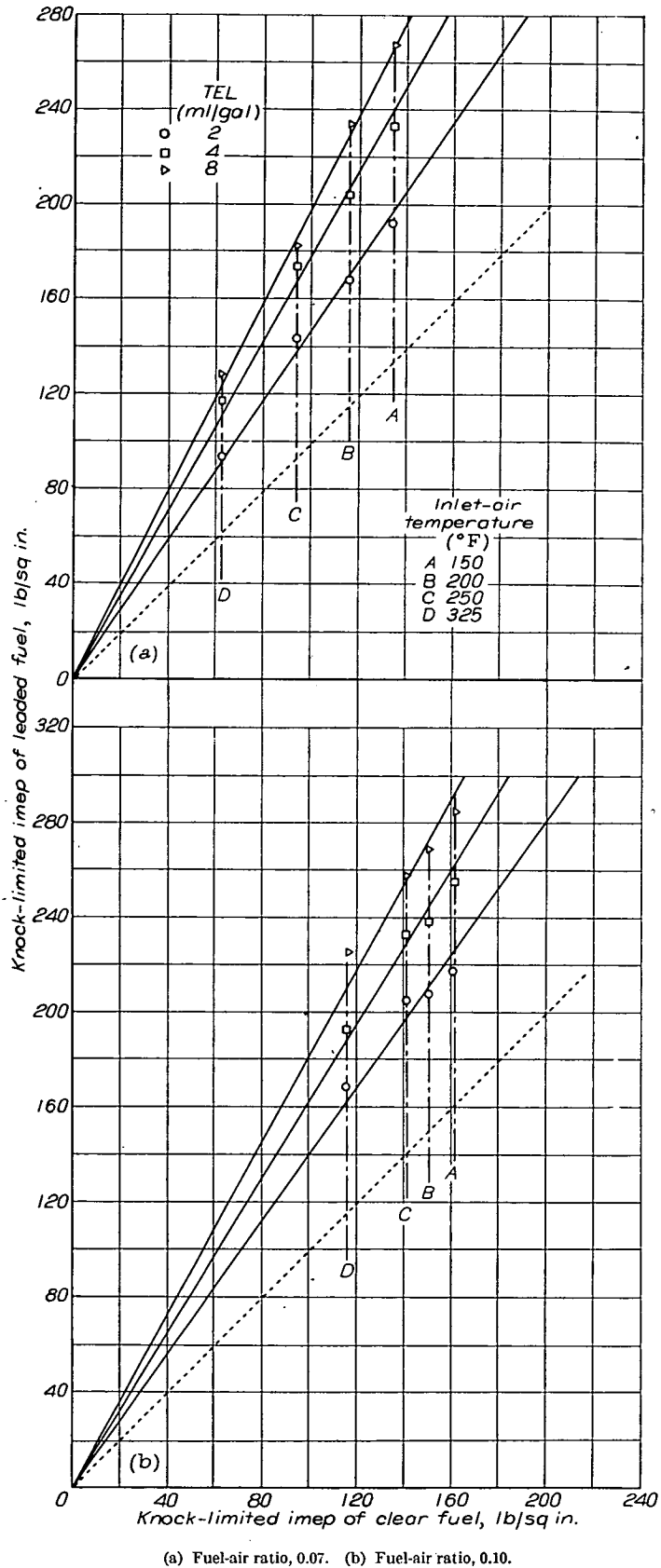


FIGURE VII-18.—Effect of inlet-air temperature on lead susceptibility of S reference fuel in CFR engine. Compression ratio, 7.0; engine speed, 2000 rpm; coolant temperature, 250° F; spark advance, 35° B. T. C. (Fig. 11 of reference 9.)

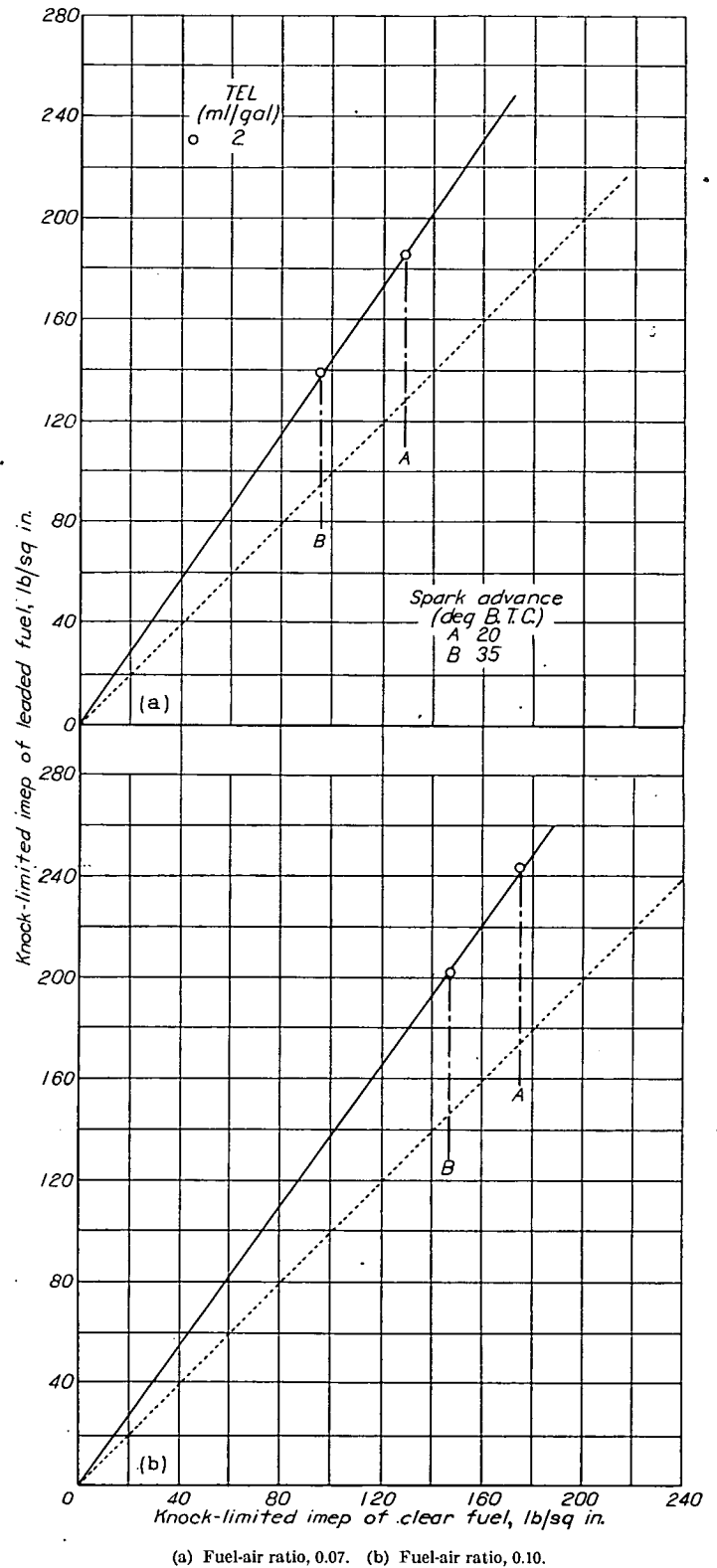


FIGURE VII-19.—Effect of spark advance on lead susceptibility of S reference fuel in CFR engine. Compression ratio, 7.0; engine speed, 2000 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F. (Fig. 12 of reference 9.)

Because K is the slope of the line of constant tetraethyl lead concentration, it is necessary to determine the lead response of only one compound in a particular hydrocarbon classification in order to estimate the lead response of other compounds of the same type from their pure ratings. The variation of K with tetraethyl lead concentration is shown in figure VII-15.

Unlike supercharged data in which the percentage increase in power is constant for a given quantity of tetraethyl lead regardless of the pure fuel rating, the percentage increase in critical compression ratio for a given quantity of tetraethyl lead varies with the pure fuel ratings. This variation is illustrated for the paraffins by the curves in figure VII-16.

EFFECTS OF ENGINE CONDITIONS ON LEAD SUSCEPTIBILITY

Fuel-air ratio.—The effects of fuel-air ratio on lead susceptibility are readily seen in figure VII-17. Curves from the standard reference fuel framework for the A. S. T. M. Supercharge rating engine were used in preparing this plot. It is emphasized that the effects shown will vary considerably from one engine or condition to another.

Inlet-air temperature.—The lead susceptibility from tests at different inlet-air temperatures is presented in figure VII-18 at two different fuel-air ratios. For the range of inlet-air temperatures examined, the lead susceptibility of S reference fuel was constant; that is, the percentage increase of knock-limited indicated mean effective pressure caused by the addition of a given amount of tetraethyl lead was the same regardless of the inlet-air temperature.

Spark advance.—Tests made on S reference fuel clear and with 2 ml TEL per gallon at spark advances of 20° and 35°

B. T. C. are shown in figure VII-19. These data indicate that the lead susceptibility of S reference fuel is independent of variations of spark advance.

REFERENCES

1. Calingaert, George: Anti-Knock Compounds. Vol. IV, pt. IV, sec. 42 of *The Science of Petroleum*. Univ. Press (Oxford), 1938, pp. 3024-3029.
2. Sloop, J. L., Kinney, George R., and Rowe, William H.: Process of Lead-Deposit Accumulations on Aircraft-Engine Spark Plugs. NACA MR E5K27, 1945.
3. Mulcahy, B. A., and Zipkin, M. A.: Tests of Improvements in Exhaust-Valve Performance Resulting from Changes in Exhaust-Valve and Port Design. NACA ARR E5G26, 1945.
4. Banks, F. R.: Some Problems of Modern High-Duty Aero Engines and Their Fuels. *Jour. Inst. Petroleum Technologists*, vol. 23, no. 160, Feb. 1937, pp. 63-137; discussion, pp. 138-177.
5. Hives, E. W., and Smith, F. L.: High-Output Aircraft Engines. *SAE Jour. (Trans.)*, vol. 46, no. 3, March 1940, pp. 106-117.
6. Mulcahy, B. A., and Zipkin, M. A.: The Effects of an Increase in the Concentration of Ethylene Dibromide in a Leaded Fuel on Lead Deposition, Corrosion of Exhaust Valves, and Knock-Limited Power. NACA ARR E5E04a, 1945.
7. Kinney, George R., and Niemi, Richard O.: The Effect of Ethylene Dibromide on the Knock-Limited Performance of Leaded and Nonleaded S Reference Fuel. NACA MR E6B12, 1946.
8. Barnett, Henry C.: Lead Susceptibility of Paraffins, Cycloparaffins, and Olefins. NACA ARR 3E26, 1943.
9. Barnett, Henry C., and Imming, Harry S.: The Effect of Engine Conditions on the Lead Susceptibility of Paraffinic Fuels. NACA ARR E4J02, 1944.
10. Heron, S. D., and Beatty, Harold A.: Aircraft Fuels. *Jour. Aero. Sci.*, vol. 5, no. 12, Oct. 1938, pp. 463-479.
11. Imming, Harry S.: The Effect of Piston-Head Temperature on Knock-Limited Power. NACA ARR E4G13, 1944.
12. Smittenberg, J., Hoog, H., Moerbeek, B. H., and v. d. Zijden, M. J.: Octane Ratings of a Number of Pure Hydrocarbons and Some of Their Binary Mixtures. *Jour. Inst. Petroleum*, vol. 26, no. 200, June 1940, pp. 294-300.

CHAPTER VIII

ANTIKNOCK BLENDING CHARACTERISTICS OF FUELS

For many years it has been known that all hydrocarbons do not exhibit the same antiknock blending characteristics. Lovell and Campbell (reference 1) illustrated this fact, and Smittenberg *et al* (reference 2) likewise clearly demonstrated that the antiknock characteristics of fuel blends depend upon the nature of the components in the blend.

An investigation was initiated in 1943 by Sanders (reference 3) to ascertain the possibility of developing a method for predicting the knock limits of fuel blends under supercharged and nonsupercharged engine operating conditions. In the original phases of this analysis, results of critical-compression-ratio (nonsupercharged) rating methods rather than results of supercharged methods were emphasized because more complete data were available and the analysis required was more detailed.

SYMBOLS

The following symbols are used in this chapter:

A, B	constants characteristic of fuels
a, b, c	constants
F	$= N_1(B_1R_1 - B_2R_2) + B_2R_2$
G	$= N_1(B_1 - B_2) + B_2 = N_1B_1 + (1 - N_1)B_2$
H	height of compression chamber, inches
k	quantity of knock-producing agent generated per unit mass
N	mass fraction
P	knock-limited indicated mean effective pressure
R	compression ratio
W, X, Y	constants in hyperbolic blending relation
β	blending constant
ρ	density
Subscripts:	
a	value of subscript 1 at asymptote
b	blend
cr	critical
d	pertaining to knock agent produced under conditions at which blend will knock
i	isooctane
m	pertaining to knock-producing agent at condition of incipient knock
t	total
1, 2, 3,	pertaining to components 1, 2, 3... in fuel blend

DERIVATIONS OF BLENDING EQUATIONS

The derivations developed in reference 3 require a clear understanding of the principles of knock testing by supercharged and nonsupercharged methods. In supercharged engine tests, all conditions except inlet-air pressure and fuel-air ratio are held constant. Fuel-air ratio may or may not be constant depending upon the operating technique employed. Inlet-air pressure is increased until a predetermined

standard knock intensity is observed. The knock-limited indicated mean effective pressure or the density of air in the cylinder charge may be used as a measure of the knock limit of the fuel being tested.

In nonsupercharged engine tests, all conditions except compression ratio are held constant. The compression ratio is increased until the standard knock intensity is reached. The compression ratio at this intensity is the measure of the knock limit and is generally termed critical compression ratio. Changes of inlet-air pressure do not affect any of the cyclic temperatures; however, an increase of compression ratio increases the temperature at the end of compression, the combustion temperature, and the end-gas temperature. The primary difference between nonsupercharged and supercharged engine methods thus lies in the fact that knock limits in nonsupercharged engines are measured at varying end-gas temperatures, whereas in supercharged engines knock limits are determined at the same end-gas temperature. (See reference 4.)

In order to derive the desired blending equations, certain assumptions were made in reference 3 regarding the mechanism of knock:

(1) Knock results from the reaction of some intermediate products during combustion, and the nature of these products as well as the reaction between them is the same regardless of the fuel used. This assumption is introduced in order that the knocking properties of fuels may be treated as additive properties.

(2) Knock occurs when the mass of knock-producing agents per unit volume reaches a given value at any end-gas temperature.

(3) At any one end-gas temperature and for any one fuel component, the mass per unit volume of the knock-producing agent evolved by that component is directly proportional to the mass of the component per unit volume; or, as stated in reference 3, a unit mass of a particular component will generate a certain mass of knock-producing agent regardless of the other components in the blend.

(4) In order to consider the differences in knock limits of the blend components, it is assumed that for any one end-gas temperature, the mass per unit volume of the knock-producing agents is a function of the molecular structure of the fuel.

(5) During combustion the temperature increase for all fuels or blends under consideration is the same. On the basis of this assumption it is possible to relate fuel knock limits to engine compression density and temperature instead of end-gas density and temperature. Many hydrocarbons of interest fulfill this condition, but some classes of fuels, such as alcohols and ethers, do not (reference 3).

The initial step in the derivation of the desired equation is justified by assumptions (1) and (2).

where

$$\rho_m = \rho_1 + \rho_2 + \rho_3 + \dots \quad (1)$$

ρ_m mass per unit volume of knock-producing agent at condition of incipient knock

$\rho_1, \rho_2, \rho_3, \dots$ mass per unit volume of knock agent produced by each component under conditions at which the blend will knock

It has been stated previously that the end-gas temperatures are constant for supercharged-engine tests but vary with the critical compression ratio of the fuel in nonsupercharged-engine tests; therefore, from assumptions (3) and (4) the following relations hold for supercharged tests:

$$\rho_1 = k_1 \rho_t N_1$$

$$\rho_2 = k_2 \rho_t N_2$$

$$\rho_3 = k_3 \rho_t N_3$$

where

N_1, N_2, N_3 mass fractions of components 1, 2, 3, respectively, in fuel blend

ρ_t total mass of fuel per unit volume

k_1, k_2, k_3 quantity of knock-producing agent generated per unit mass by components 1, 2, 3, respectively

When the pure compound 1 is tested, the knock-limited density of fuel in the charge is $\rho_{t,1}$ and

$$\rho_m = \rho_1 = k_1 \rho_{t,1} \quad (2)$$

Therefore,

$$k_1 = \frac{\rho_m}{\rho_{t,1}}$$

and

$$\rho_1 = \frac{\rho_m}{\rho_{t,1}} \rho_t N_1$$

Similarly,

$$\rho_2 = \frac{\rho_m}{\rho_{t,2}} \rho_t N_2$$

and

$$\rho_3 = \frac{\rho_m}{\rho_{t,3}} \rho_t N_3$$

Substituting these values of $\rho_1, \rho_2, \rho_3, \dots$ in equation (1) yields

$$\rho_m = \frac{\rho_m}{\rho_{t,1}} \rho_t N_1 + \frac{\rho_m}{\rho_{t,2}} \rho_t N_2 + \frac{\rho_m}{\rho_{t,3}} \rho_t N_3 + \dots$$

and

$$\frac{1}{\rho_t} = \frac{N_1}{\rho_{t,1}} + \frac{N_2}{\rho_{t,2}} + \frac{N_3}{\rho_{t,3}} + \dots \quad (3)$$

As limited by the assumptions, equation (3) is a blending equation applicable to supercharged-engine data. Because the compression ratio and fuel-air ratio must be the same for the various components in a given application of this equation, knock-limited inlet-air densities may be used instead of the fuel densities for values of $\rho_t, \rho_{t,1}, \rho_{t,2}, \rho_{t,3}, \dots$. With this substitution, equation (3) may be expressed in words as

follows: The reciprocal of the knock-limited inlet-air density of a fuel blend tested by a supercharged-engine method is the weighted average of the reciprocals of the knock-limited inlet-air densities of the pure components.

Indicated mean effective pressure is proportional to inlet-air density; therefore, it can be substituted in equation (3) for density:

$$\frac{1}{P_b} = \frac{N_1}{P_1} + \frac{N_2}{P_2} + \frac{N_3}{P_3} + \dots \quad (3b)$$

where

P_b knock-limited indicated mean effective pressure of fuel blend

P_1, P_2, P_3 knock-limited indicated mean effective pressures of components 1, 2, 3, respectively, when tested individually

For nonsupercharged tests, the charge density at the end of compression is proportional to the compression ratio. By use of an equation similar to equation (2), ρ_1 was related to the compression ratio instead of the mass of fuel per unit volume (reference 3). The value of k_1 , however, varies with compression ratio because the compression temperature varies with compression ratio. In order to account for the variation of k_1 , the following relation was assumed (reference 3) between ρ_1 and compression ratio R :

$$\rho_1 = (A_1 + B_1 R) N_1 \quad (4)$$

where

A_1, B_1 constants characteristic of fuel 1

Similarly,

$$\rho_2 = (A_2 + B_2 R) N_2$$

and

$$\rho_3 = (A_3 + B_3 R) N_3$$

where

A_2, B_2, A_3, B_3 constants characteristic of fuels 2 and 3

Substituting these values in equation (1) gives

$$\rho_m = N_1 (A_1 + B_1 R) + N_2 (A_2 + B_2 R) + N_3 (A_3 + B_3 R) + \dots \quad (5)$$

The value of A_1 may be determined by letting $N_1=1, N_2=0, N_3=0$, and $R=R_{1,cr}$ where $R_{1,cr}$ is the critical compression ratio of fuel 1 when tested individually.

$$\rho_m = A_1 + B_1 R_{1,cr}$$

$$A_1 = \rho_m - B_1 R_{1,cr}$$

Likewise,

$$A_2 = \rho_m - B_2 R_{2,cr}$$

and

$$A_3 = \rho_m - B_3 R_{3,cr}$$

Substituting in equation (5) gives

$$R = \frac{N_1 B_1 R_{1,cr} + N_2 B_2 R_{2,cr} + N_3 B_3 R_{3,cr} + \dots}{N_1 B_1 + N_2 B_2 + N_3 B_3 + \dots} \quad (6)$$

As limited by the assumptions, equation (6) is the blending equation applicable to nonsupercharged-engine data.

In this equation $B_1, B_2, B_3 \dots$ are defined as blending constants and each fuel has a blending constant that is independent of the other fuels in the blend and is determined by the critical compression ratio of the fuel and the rate of change of knock limit with inlet-air temperature.

For two-component blends, $N_2 = 1 - N_1$ and equation (6) may be written as

$$R = \frac{N_1(B_1 R_{1,cr} - B_2 R_{2,cr}) + B_2 R_{2,cr}}{N_1(B_1 - B_2) + B_2} \quad (7)$$

Equation (7) is the equation of an equilateral hyperbola asymptotic to

$$N = \frac{B_2}{B_2 - B_1} = N_{a,1}$$

and

$$R = \frac{B_1 R_1 - B_2 R_2}{B_1 - B_2} = R_a$$

where

$N_{a,1}$ value of N_1 at asymptote

R_a value of R at asymptote

The asymptotic form of equation (7) is

$$(R - R_a)(N_a - N_1) = N_a(R_2 - R_a) \quad (8)$$

According to reference 3, it is unnecessary to know the absolute values of the constants in equations (6) and (7), but the relative values must be known; that is, one component may be assigned an arbitrary value of the constant, thereby fixing the values of the constants for all other components tested at the same conditions. Values of B for other components may be found by determining the critical compression ratio of the pure component and of one blend of the component with another component for which the blending constant is known. With these values the equation can be solved for the value of the unknown blending constant. When the relative values of B have been determined for all compounds, the knock limits of all blends may be computed from the blending equation.

GRAPHICAL SOLUTION OF BLENDING EQUATION FOR NONSUPERCHARGED-ENGINE DATA

A graphical method was suggested in reference 3 for determination of the blending characteristics of fuels when tested in nonsupercharged engines. Equation (7) may be put into the following form:

$$F = RG \quad (9)$$

where

$$F = N_1(B_1 R_1 - B_2 R_2) + B_2 R_2$$

and

$$G = N_1(B_1 - B_2) + B_2 = N_1 B_1 + (1 - N_1) B_2$$

If R is held constant, F is proportional to G . In figure VIII-1, the value of F has been plotted against the value of G for values of R between 2 and 15. Inasmuch as G is actually the weighted average value of B_1 and B_2 , the abscissa of

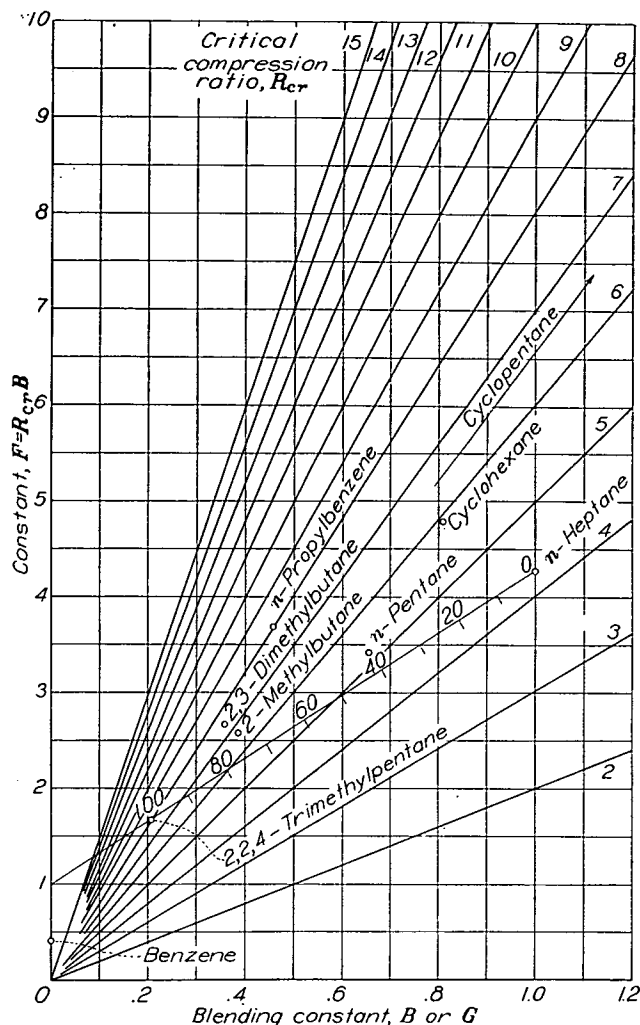


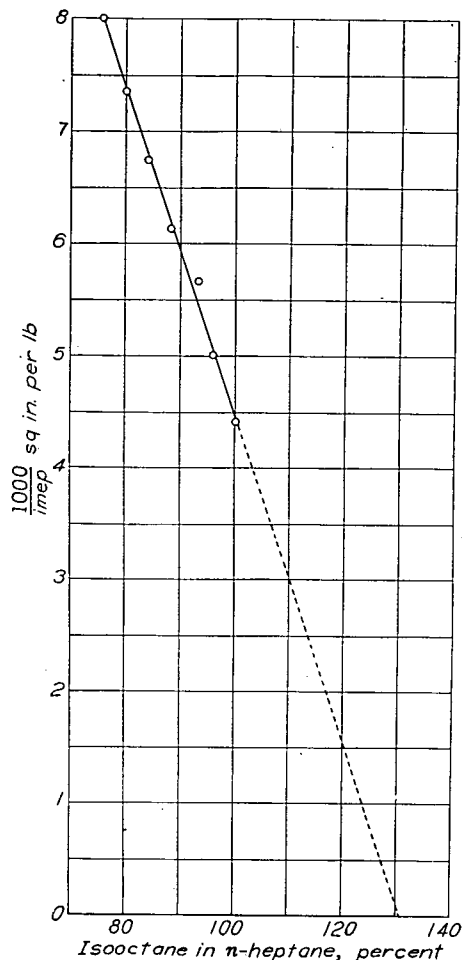
FIGURE VIII-1.—Blending chart for A. S. T. M. Motor method. (Fig. 1 of reference 3.)

figure VIII-1 has been marked G or B . All potential fuel components whose critical compression ratios and blending constants are known may be plotted on this chart.

A straight line joining the points for any two components represents all blends of the two components. This relation is true because F and G are linear functions of N_1 and, consequently, F is a linear function of G . A point representing any blend of the two components divides the line in the same ratio as that in which the components exist in the blend. For example, a blend containing 60 percent 2,2,4-trimethylpentane (isooctane) and 40 percent n -heptane will be on the line joining the points for the two components and will be 60 percent of the distance from n -heptane to 2,2,4-trimethylpentane.

EXPERIMENTAL VERIFICATION OF BLENDING EQUATIONS

Heron and Beatty (reference 5) have shown that if the reciprocal of the knock-limited indicated mean effective pressures of blends of isooctane and n -heptane is plotted against the percentage of isooctane in the blends, a straight line will result. This fact substantiates equation (3) for supercharged-engine data. The data obtained in reference 5 are shown in figure VIII-2 (a).



(a) 17.6 engine. Compression ratio, 5.6; engine speed, 900 rpm; inlet-air temperature, 225° F; coolant temperature, 300° F; spark advance, 20° B. T. C. (Fig. 2 of reference 3.)

FIGURE VIII-2.—Effect of blend composition on knock-limited performance.

Further verification of this relation is presented in figure VIII-2 (b), which shows data for S and M reference fuel blends investigated in reference 3.

Data obtained in references 2 and 6 were used in reference 3 for verification of the blending equation developed for nonsupercharged engine tests. The blending equation for *n*-heptane and isooctane under A. S. T. M. Motor method conditions is shown in reference 2 to be

$$N_i = \left(\frac{1.369 - H}{1.954 - H} \right) 179 \quad (10)$$

where

N_i isooctane in blend, percent

H height of compression chamber, inches

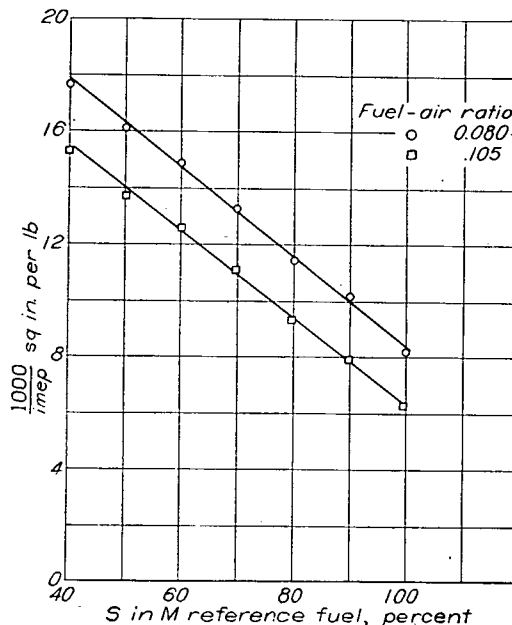
The value of N_i is given on a volumetric basis in reference 2 but, since the densities of the two components are about the same, N may be used as the mass fraction. The length of the engine stroke was 4.5 inches; therefore, the relation between R and H can be expressed as follows:

$$R = \frac{4.5 + H}{H} \quad (11)$$

Combining equations (10) and (11) gives

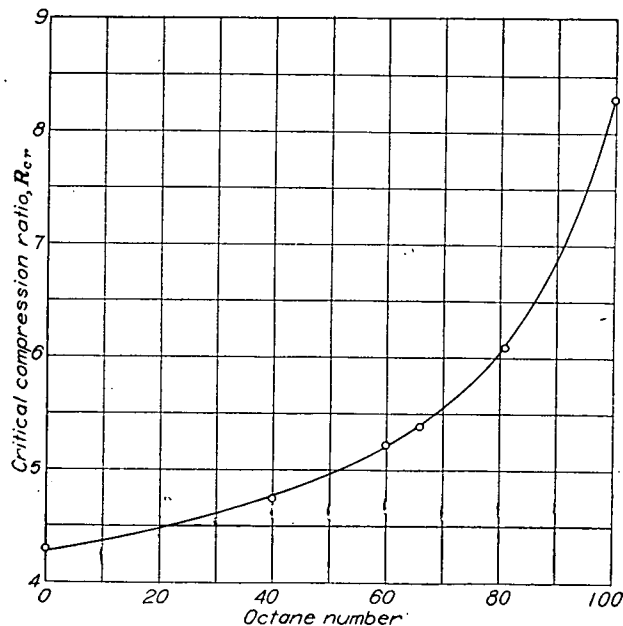
$$(R - 3.3)(125 - N_i) = 125 \quad (12)$$

The equilateral hyperbola represented by equation (12) is asymptotic to $R = 3.3$ and $N_i = 125$. The relation between R and N_i is shown in figure VIII-3(a). Inasmuch as the



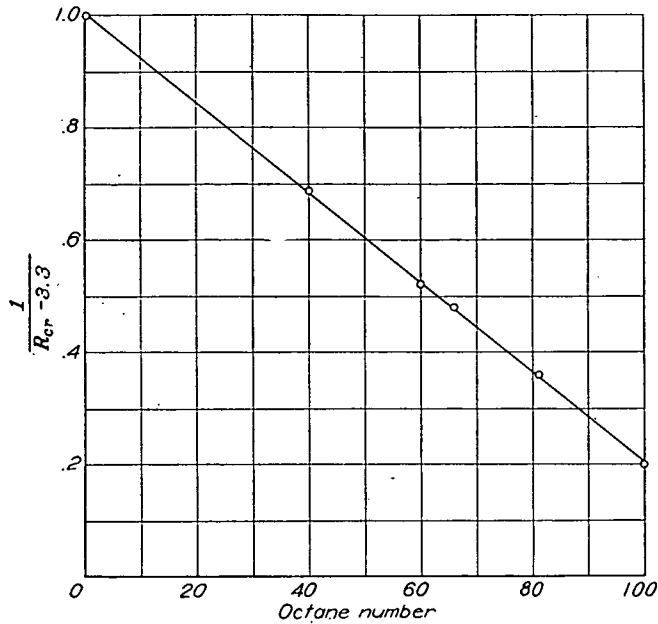
(b) CFR engine. Compression ratio, 7.0; engine speed, 2000 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F; spark advance, 35° B. T. C. (Fig. 3 of reference 3.)

FIGURE VIII-2.—Concluded. Effect of blend composition on knock-limited performance.

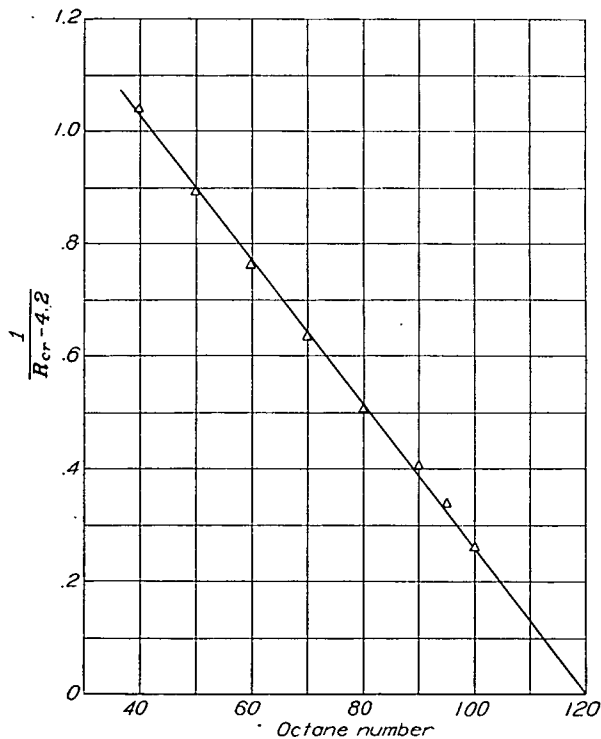


(a) A. S. T. M. Motor method. (Fig. 4 of reference 3.)

FIGURE VIII-3.—Blending characteristics of 2,2,4-trimethylpentane and *n*-heptane.



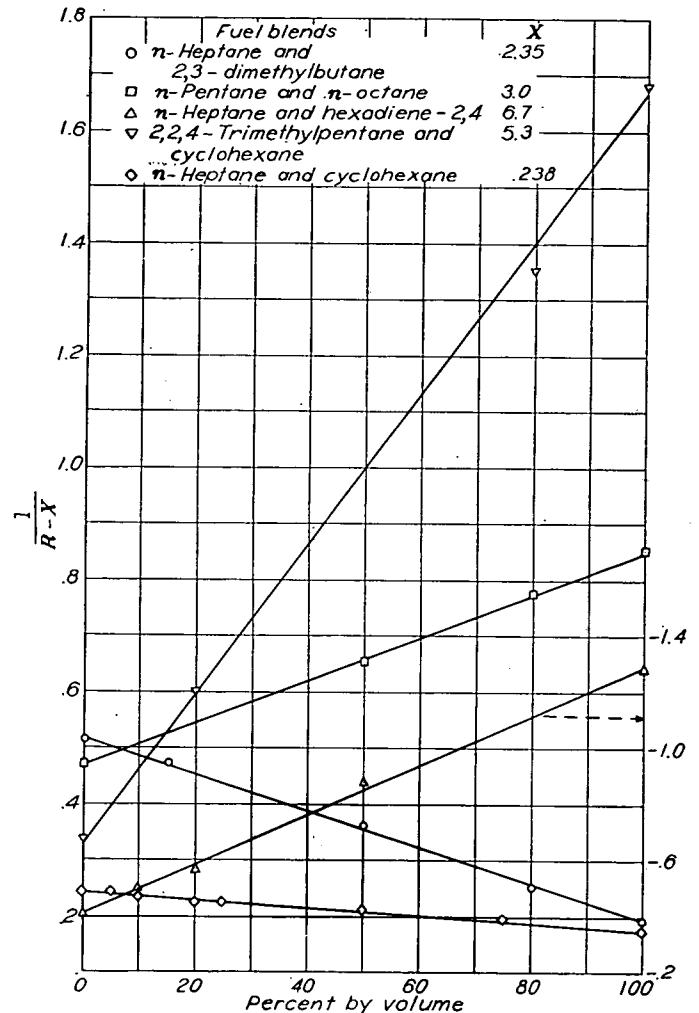
(b) A. S. T. M. Motor method; reciprocal plot. (Fig. 5 of reference 3.)

FIGURE VIII-3.—Continued. Blending characteristics of 2,2,4-trimethylpentane and *n*-heptane.

(c) A. S. T. M. Research method; reciprocal plot. (Fig. 6 of reference 3.)

FIGURE VIII-3.—Concluded. Blending characteristics of 2,2,4-trimethylpentane and *n*-heptane.

components of this system are *n*-heptane and isooctane, *N* is equivalent to octane number. (See ch. III.) The curve in figure VIII-3(a) is drawn from equation (12), whereas the data points shown are taken directly from reference 2. The data from figure VIII-3(a) were replotted (reference 3)

FIGURE VIII-4.—Verification of hyperbolic blending relation for knock ratings of fuel blends by A. S. T. M. Motor method. (Fig. 7 of reference 3.) Equation of hyperbola: $(R-X)(Y-N)=W$.

as shown in figure VIII-3(b) as confirmation of the hyperbolic blending relation.

Similar data for the A. S. T. M. Research method were reported by Brooks (reference 6) and are plotted in figure VIII-3 (c). Here again the relation is linear, but the intercepts are 120 and 1.53 as compared with 125 and 1.0 for the A. S. T. M. Motor method shown in figure VIII-3 (b). The equation for the A. S. T. M. Research method data is

$$(R-4.2)(120-N)=78.3 \quad (13)$$

From the foregoing comparison of the A. S. T. M. Motor and Research methods, it was concluded (reference 3) that the asymptotes of the blending hyperbola change with changing engine conditions.

Blending relations for other two-component blending systems are shown in figure VIII-4. The data for these systems were taken from reference 2. Three two-component systems investigated in reference 2 did not follow the hyperbolic relation. These systems are shown in figure VIII-5.

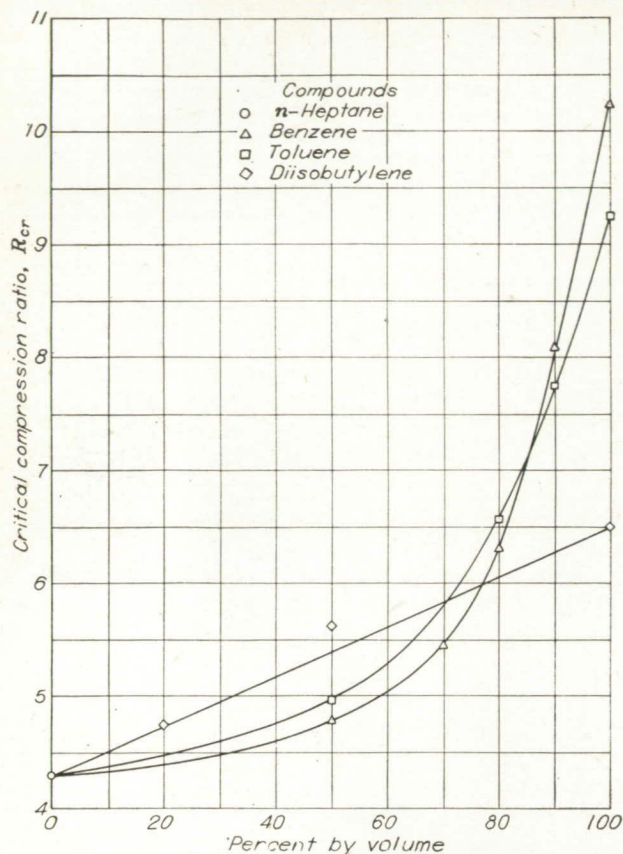


FIGURE VIII-5.—Blending characteristics of diisobutylene, benzene, and toluene in *n*-heptane by A. S. T. M. Motor method. (Fig. 6 of reference 3.)

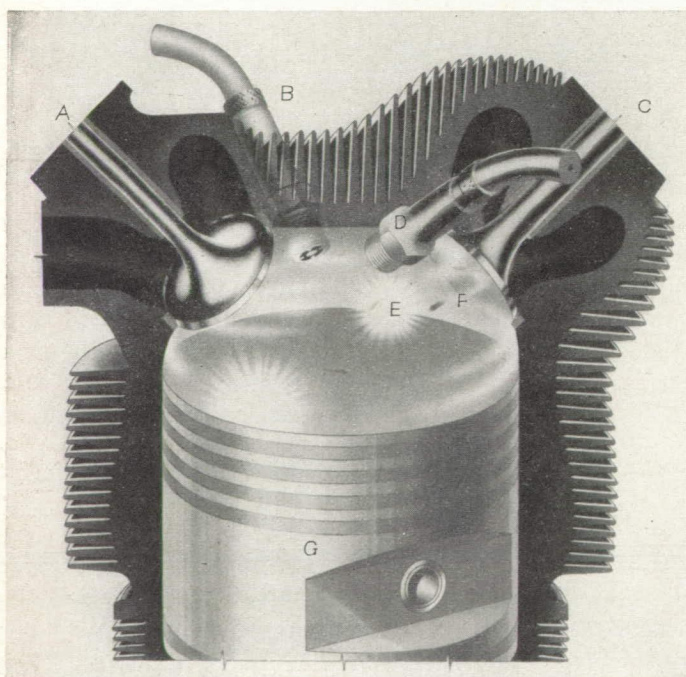


FIGURE VIII-6.—View of combustion chamber showing location of thermocouple in end zone. (Fig. 1 of reference 8.)

In connection with the system of blends of *n*-heptane and benzene (fig. VIII-5), it was found that up to a concentration of 90 percent benzene a hyperbolic relation is valid, but the value for pure benzene does not follow the relation (reference 3). In explanation of this discrepancy it should be noted that the authors of reference 2 expressed the belief that the knock rating of pure benzene might be in error.

In order to illustrate the invariance of the blending constant, a series of calculations was made involving several hydrocarbons (reference 3). The constants evolved in these calculations have been plotted on figure VIII-1. Values for cyclopentane and *n*-propylbenzene were computed from data reported in reference 7.

APPLICABILITY OF EQUATIONS TO DIFFERENT BLENDING AGENTS AND ADDITIVES

The preliminary data used in support of the derivation developed in the foregoing pages were admittedly meager; consequently, an extensive investigation was initiated to determine the general applicability of the blending equation. In the course of this investigation a number of fuels of various types were studied under different engine operating conditions in supercharged engines. The results of this investigation are reported in the succeeding paragraphs.

Paraffins.—The base fuels used in this phase of the study were an alkylate blending agent and a virgin base gasoline. The blending agents examined were:

- Triptane (2,2,3-trimethylbutane)
- Isopentane (2-methylbutane)
- Diisopropyl (2,3-dimethylbutane)
- Neohexane (2,2-dimethylbutane)
- Hot-acid octane

All blending agents and base stocks contained 4.0 ml TEL per gallon. Inspection data for these fuels are included in appendix A, table A-9. (See references 8 and 9.)

The experimental studies of these fuels were made in a single air-cooled aircraft cylinder mounted on a CUE crankcase. This apparatus is described in detail in reference 9 with the exception of a special thermocouple embedded in the cylinder head about $\frac{1}{16}$ inch from the combustion-chamber wall at a position estimated to be very near the exhaust end zone (fig. VIII-6). The temperature at this thermocouple was held constant by means of an automatic potentiometer regulator that controlled cooling-air flow across the cylinder. This instrumentation is consistent with the assumption made in reference 3 that for the blending equation to be applicable to supercharged engine data, the cyclic and end-gas temperatures must be held constant.

In figure VIII-7, a portion of the data determined in the investigation of reference 8 is shown. The ordinate scales of these figures are inverted reciprocal scales. When plotted in this manner, the data should fall on a straight line if the results are applicable to equation (3b). It is apparent in these figures that the paraffinic fuels examined, with the

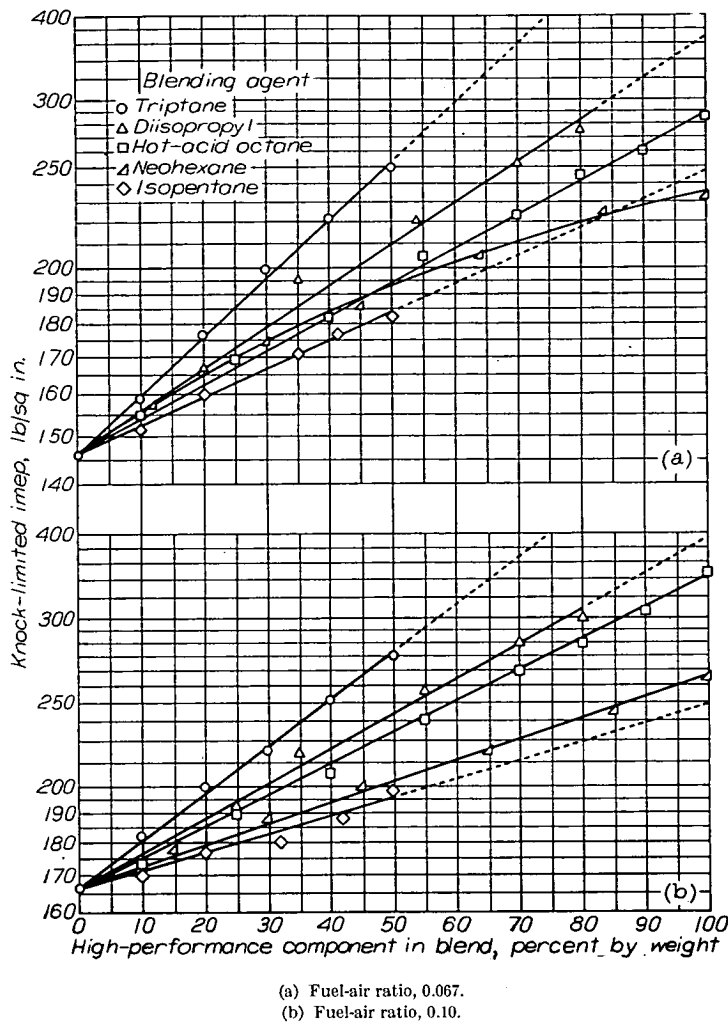
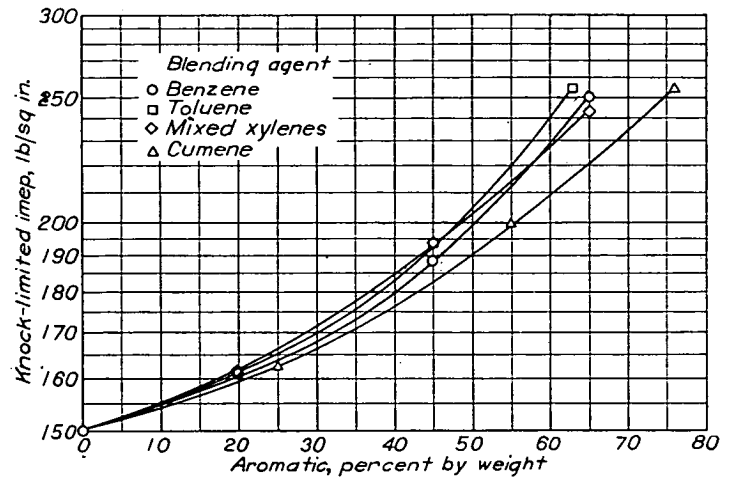


FIGURE VIII-7.—Blending characteristics of paraffins in virgin base stock leaded to 4 ml TEL per gallon in full-scale air-cooled aircraft cylinder. Compression ratio, 7.7; engine speed, 2000 rpm; spark advance, 20° B. T. C.; inlet mixture temperature, 350° F; exhaust back pressure, atmospheric. (Data taken from reference 8.)

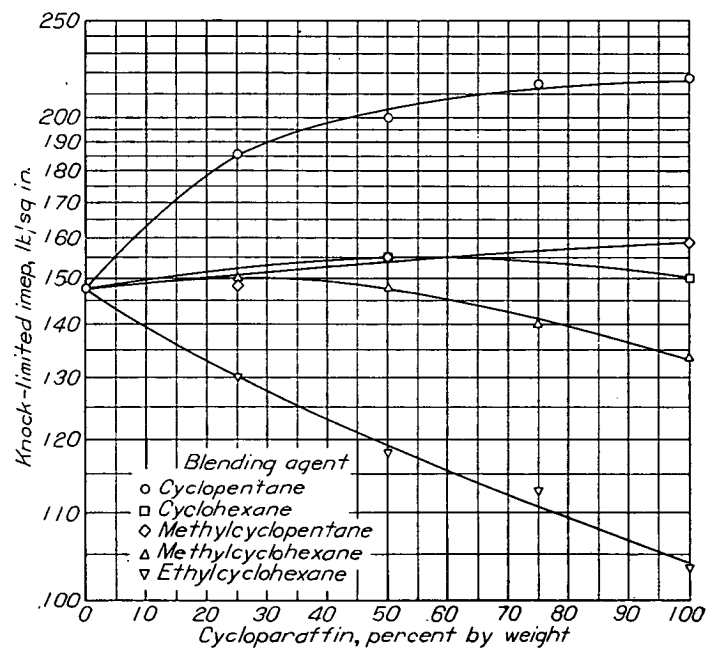
exception of neohexane at the lean fuel-air ratio, are in accord with the blending equation for supercharged-engine data.

These data were obtained at a spark advance of 20° B. T. C., a condition considered to be relatively mild; consequently, additional tests were made at a more severe condition (spark advance, 30° B. T. C.). The results of these advanced spark studies are essentially the same as those shown in figure VIII-7; that is, the relation between the reciprocal indicated mean effective pressure and composition was linear. In this case, however, data for neohexane followed the relation at both lean and rich mixtures. (See reference 8.)

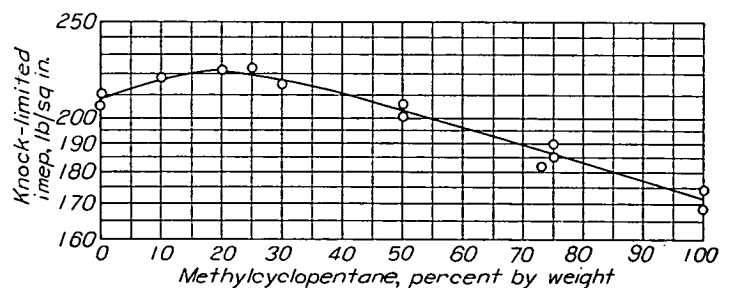
Aromatics.—Experimental verification of the blending relation for aromatic fuels proved to be difficult inasmuch as pure aromatics have exceptionally high knock limits and are prone to preignite. (See reference 10.) Despite these obstacles, however, a reasonable quantity of data was obtained to determine the limitations of this class of compounds. In pursuing this phase of the investigation, no effort was made to attain the knock limit of the pure aromatics. Instead, each aromatic was blended with a paraffinic base stock and the concentration was increased until the power level was con-



(a) Aromatics in virgin base stock.



(b) Cycloparaffins in virgin base stock.



(c) Methylcyclopentane in aviation alkylate.

FIGURE VIII-8.—Blending characteristics of various fuels leaded to 4 ml TEL per gallon in full-scale air-cooled aircraft cylinder. Compression ratio, 7.3; engine speed, 2250 rpm; spark advance, 20° B. T. C.; inlet mixture temperature, 240° F; end-zone temperature, 400° F; exhaust back pressure, 15 inches of mercury absolute; fuel-air ratio, 0.10. (Data taken from reference 10.)

sidered too high to warrant further increases. Results of these tests are shown in figure VIII-8 (a). Here it is seen that the relation between composition and reciprocal indicated mean effective pressure is nonlinear.

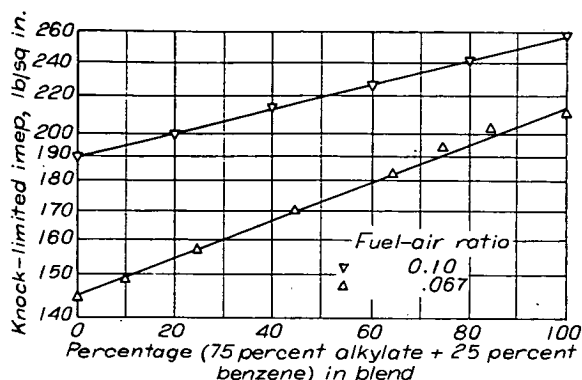


FIGURE VIII-9.—Blending characteristics of benzene, virgin base stock, and aviation alkylate in full-scale air-cooled aircraft cylinder. Concentration of benzene in all blends, 25 percent by volume; compression ratio, 7.7; engine speed, 2000 rpm; spark advance, 20° B. T. C.; inlet mixture temperature, 240° F; end-zone temperature, 350° F. (Fig. 3 (b) of reference 11.)

A modification of equation (3b) is proposed in reference 9 in order to account for fuel components that do not follow the linear blending relations.

$$P_o = \frac{P_1 N_1 B_1 + P_2 N_2 B_2 + P_3 N_3 B_3 + \dots}{N_1 B_1 + N_2 B_2 + N_3 B_3 + \dots} \quad (14)$$

Another form of equation (14) is (reference 9)

$$\frac{N_1 \beta_1 + N_2 \beta_2 + N_3 \beta_3 + \dots}{P_o} = \frac{N_1 \beta_1}{P_1} + \frac{N_2 \beta_2}{P_2} + \frac{N_3 \beta_3}{P_3} + \dots \quad (15)$$

$\beta_1, \beta_2, \beta_3, \dots$ blending constants related to constants $B_1, B_2, B_3, \dots$ by the following equations:

$$\beta_1 = P_1 B_1$$

$$\beta_2 = P_2 B_2$$

$$\beta_3 = P_3 B_3$$

Both equations (14) and (15) are equal to equation (3b) when $\beta_1 = \beta_2 = \beta_3$ or $P_1 B_1 = P_2 B_2 = P_3 B_3$.

Performance data for aromatic fuels were employed (reference 10) to determine the validity of equation (15). From the analysis it was concluded that the equation does not apply over the entire range of composition; that is, the blending constant β for a given aromatic varies somewhat.

From a practical point of view, however, marketed aviation fuels seldom contain concentrations of aromatics in excess of 25 percent (generally much less), and for a limited range of concentration it is believed that equation (15) may be useful. Evidence to support this belief is shown in figure VIII-9.

It is seen in this figure that the blending relations are linear for a 25-percent concentration of benzene in blends of alkylate and virgin-base stock. Similar results were obtained with toluene, mixed xylenes, and cumene (reference 11). On the basis of figure V-6 (c) described in chapter V it is doubtful if the relations would be linear for concentrations much greater than 25 percent.

Cycloparaffins.—The results of tests with cycloparaffinic fuels are shown in figure VIII-8 (b). As in the case of the

aromatic fuels, the cycloparaffinic fuels examined failed to follow the blending relation in the form of either equation (3b) or equation (15).

One result of unusual interest was found in the cycloparaffinic investigation of reference 10. (See fig. VIII-8 (c).) In this figure it is seen that methylcyclopentane, when blended with alkylate, produced intermediate blends having higher knock limits than either of the pure components. This was also true for other cycloparaffins in blends with different base stocks. (See reference 10.)

Special additives.—The influence of special additives (for example, tetraethyl lead and xylydines) on the applicability of the blending equation is discussed in reference 9. In brief, the effect of such additives is essentially the same as the effect of low concentrations of aromatics mentioned in connection with figure VIII-9. That is, the blending equation applies so long as the concentration of such additives in the major blend components is constant.

PREPARATION OF BLENDING CHARTS

In the preceding sections of this chapter, a blending equation has been described and its limits of applicability have been shown by experimental data. The remainder of this chapter is devoted to an analysis that will illustrate the practical use of this equation. Specifically, the purpose of this analysis is to illustrate the preparation of performance charts from which blends of any desired lean and rich antiknock ratings can be selected. These charts are based entirely upon the A. S. T. M. Aviation (lean) and A. S. T. M. Supercharge (rich) knock rating methods. Experimental data upon which the charts are based may be found in appendix A, table A-10. (See reference 12.)

Performance numbers.—The scale of performance numbers for fuel knock rating terminates at 161. (See ch. III.) It was therefore necessary to extrapolate the scale in order to evaluate blends that would have antiknock ratings in excess of 161 by the A. S. T. M. Supercharge method. This extrapolation was made by plotting the performance numbers against knock-limited indicated mean effective pressures from a reference-fuel framework given in reference 13. The rating scale thus developed is shown in figure VIII-10. Although a definite break in this curve occurs at a performance number of 130, the curve appears to be linear between 130 and 160. On the assumption that this linear relation is true, a straight line was drawn through the points at 130 and 161 and extended to a performance number of 200.

Ternary blends.—As an example of the preparation of a performance chart, consider first that it is desired to know the A. S. T. M. Aviation and the A. S. T. M. Supercharge performance numbers of all possible ternary blends of hot-acid octane, an aviation alkylate, and a virgin base stock. A plot of composition against the reciprocal of the knock-limited indicated mean effective pressure for binary blends of any two of these fuels will result in a straight line. The three binary combinations of these materials are shown in figure VIII-11.

In the next operation, lines of constant performance number are drawn on the plot (shown as dashed lines on fig. VIII-11). These lines are established by reading values of indicated mean effective pressure at equal increments of performance number in figure VIII-10. The data as shown in figure VIII-11 comprise the basic information needed to establish A. S. T. M. Supercharge rating lines on the final chart for multicomponent blends.

For convenience in relating composition and knock-limited performance of ternary fuel blends, all performance charts in this investigation were prepared on triangular coordinate paper. The equation of a straight line on triangular coordinate paper is of the form

$$a = bN_1 + cN_2 + N_3$$

where

a, b, c constants

N_1, N_2, N_3 concentration of components 1, 2, and 3 in

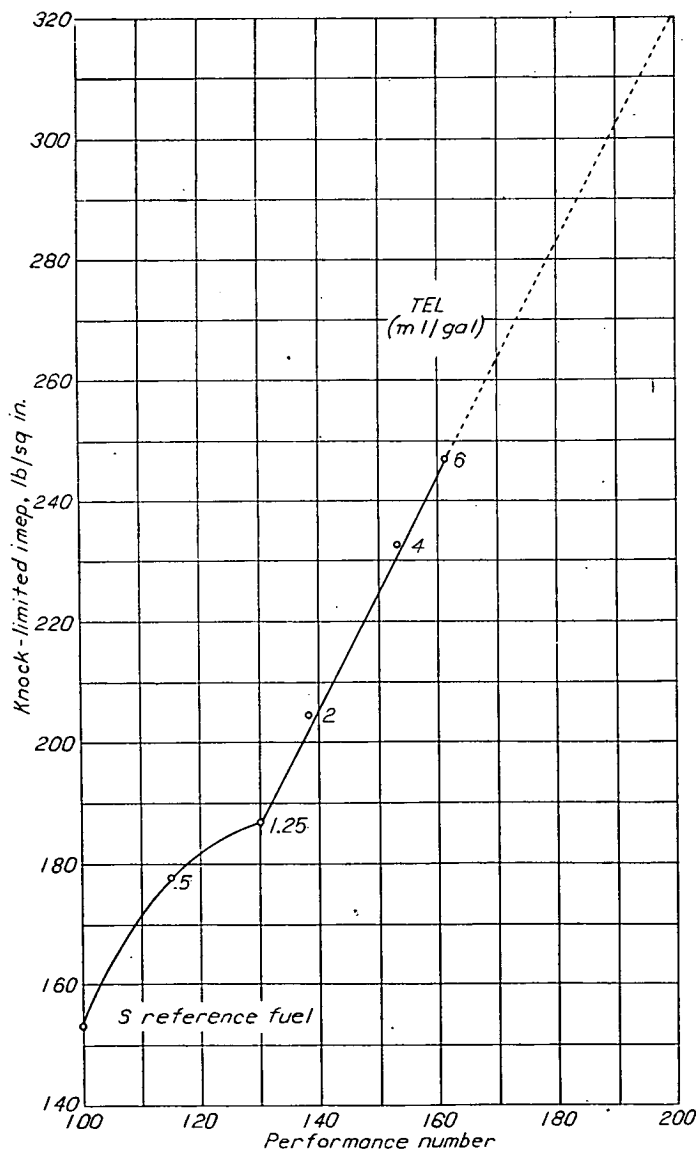


FIGURE VIII-10.—Relation between performance number and knock-limited indicated mean effective pressure by A. S. T. M. Supercharge method. Fuel-air ratio, 0.11. (Fig. 1 of reference 12.)

ternary blend. Any equation relating knock-limited performance and blend composition that can be reduced to this form can be represented by a straight line of constant performance on triangular coordinate paper. Equation (3b) can be reduced to this form by multiplying through by any one of the performance values P_1, P_2 , or P_3 .

The performance chart for the system of hot-acid octane, aviation alkylate, and virgin base stock is shown in figure VIII-12 (a). Lines of constant performance number in this figure were determined by noting the intersections of the constant performance lines (fig. VIII-11) with the blending lines. For example, the 150-performance-number line in figure VIII-11 intersects the blending line of hot-acid octane and aviation alkylate at a composition of 32 percent hot-acid octane and 68 percent alkylate and intersects the blending line of hot-acid octane and virgin base stock at a composition

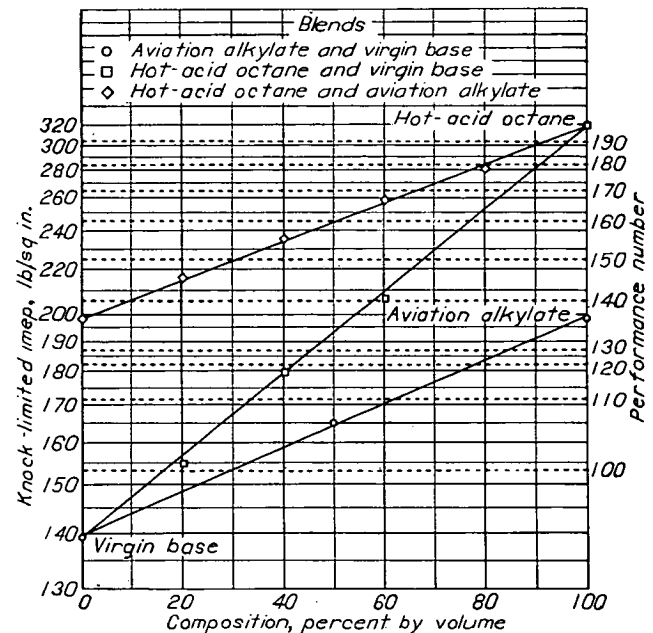
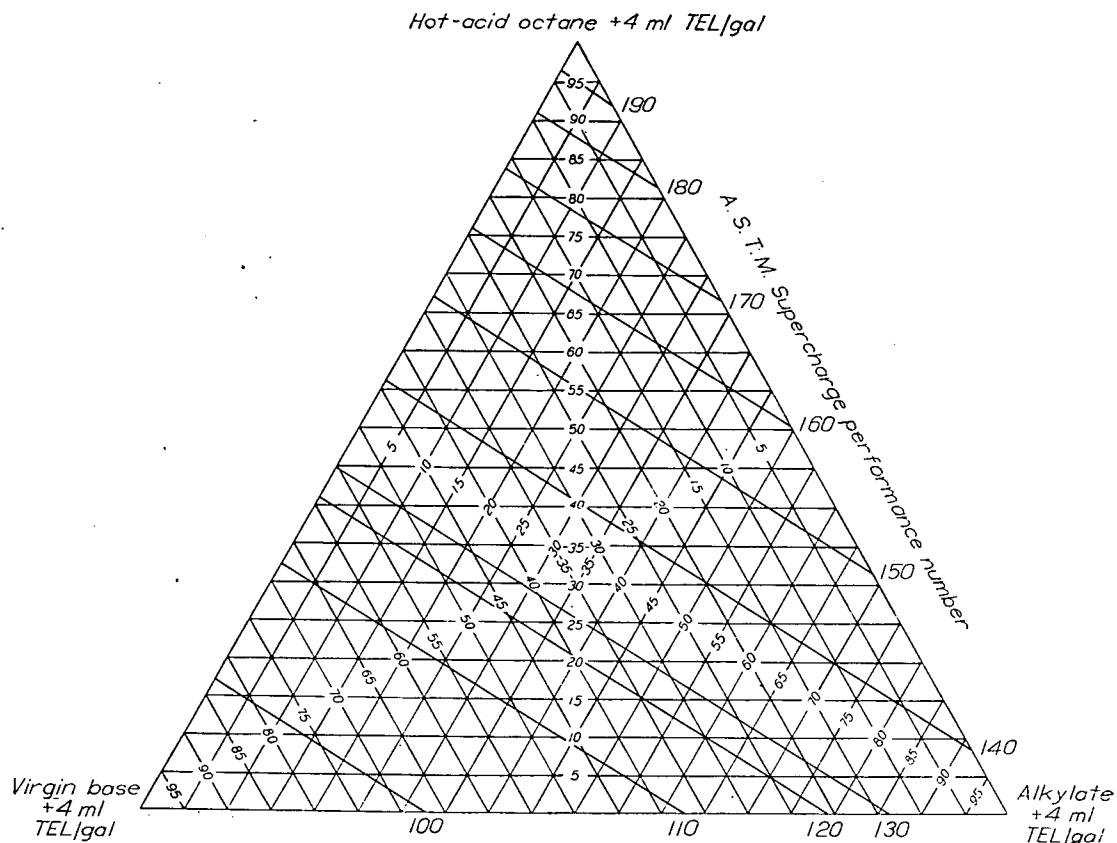


FIGURE VIII-11.—A. S. T. M. Supercharge blending characteristics of hot-acid octane, aviation alkylate, and virgin base stock loaded to 4 ml TEL per gallon. Fuel-air ratio, 0.11. (Fig. 2 of reference 12.)

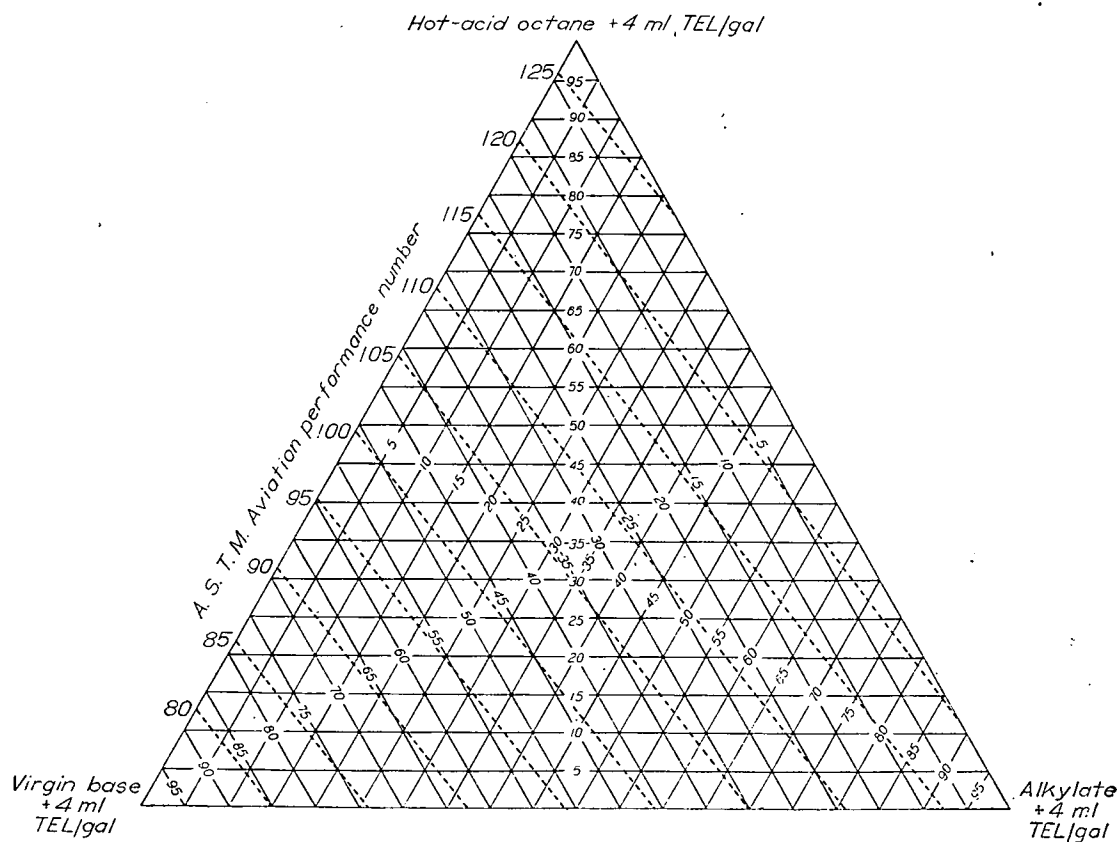
of 67 percent hot-acid octane and 33 percent virgin base stock. These two compositions were plotted on figure VIII-12 (a) and joined by a straight line. Any point on this line represents a blend of hot-acid octane, alkylate, and virgin base stock that will give a performance number of 150 in the A. S. T. M. Supercharge engine at a fuel-air ratio of 0.11. All other lines in figure VIII-12 (a) were established in the same manner.

The lines in figure VIII-12 (a) are parallel, which is true when curves of the type shown in figure VIII-11 are linear. On the basis of data presented earlier in this chapter it appears that most paraffinic fuels do blend linearly. Even though certain constituents, such as the aromatics or ethers, do not blend linearly in combination with paraffinic base fuels, the procedure just outlined for the preparation of performance-number charts is not altered. In such cases,



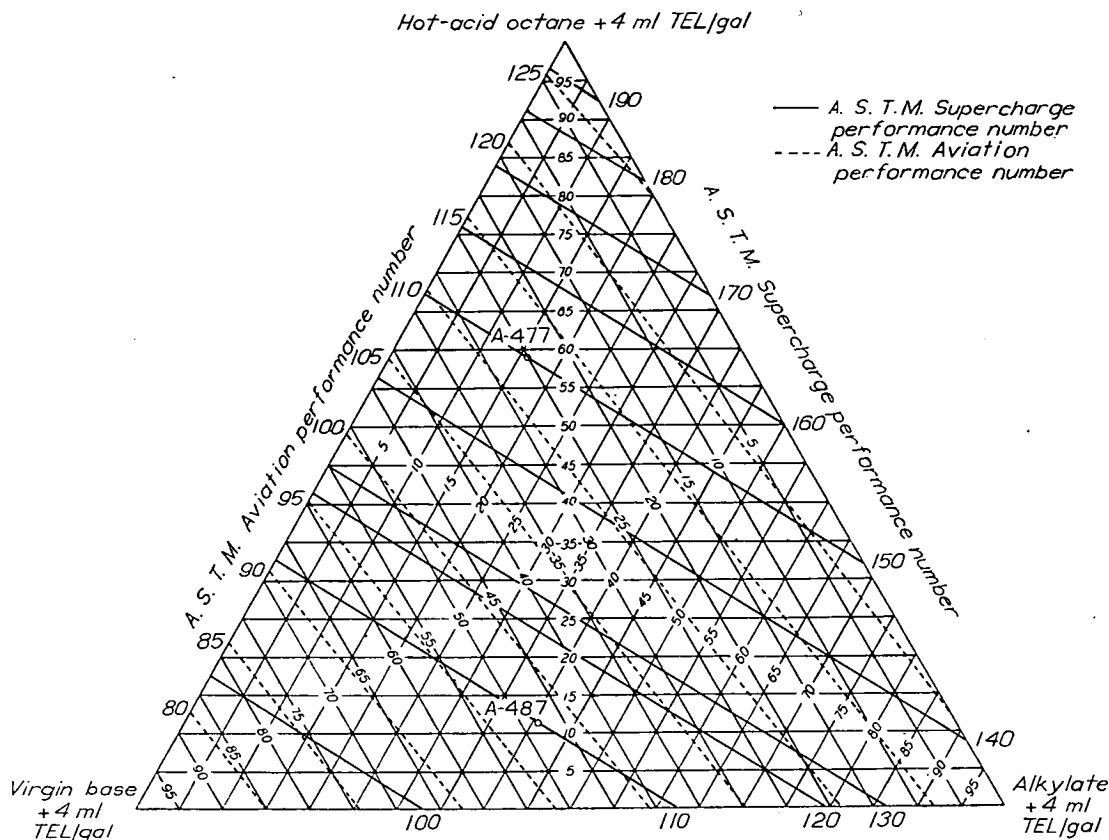
(a) A. S. T. M. Supercharge method. (Fig. 3 of reference 12.)

FIGURE VIII-12.—Blending charts for hot-acid octane, aviation alkylate, and virgin base stock.



(b) A. S. T. M. Aviation method. (Fig. 5 of reference 12.)

FIGURE VIII-12.—Continued. Blending charts for hot-acid octane, aviation alkylate, and virgin base stock.



(c) A. S. T. M. Supercharge and A. S. T. M. Aviation methods. (Fig. 6 of reference 12.)

FIGURE VIII-12.—Concluded. Blending charts for hot-acid octane, aviation alkylate, and virgin base stock.

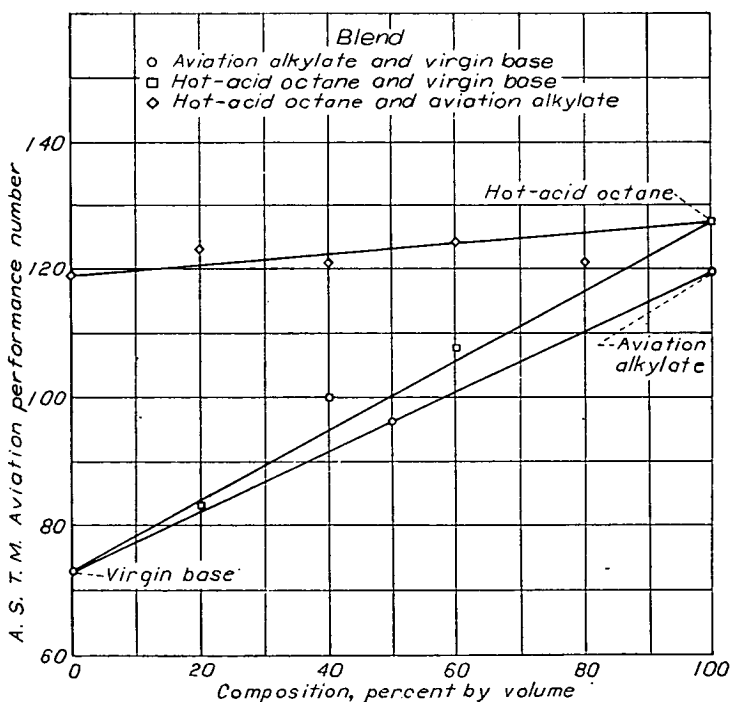


FIGURE VIII-13. A. S. T. M. Aviation blending characteristics of hot-acid octane, aviation alkylate, and virgin base stock leaded to 4 ml TEL per gallon. (Fig. 4 of reference 12.)

however, the charts prepared are subject to some error that arises from the assumption that curves of constant performance number on triangular coordinate paper are linear. Moreover, a nonlinear relation in a plot of the type shown in figure VIII-11 results in a variation of slope with performance number on the final triangular plot.

The procedure used for determining the lines of constant A. S. T. M. Aviation performance numbers for blends of the same fuels used in preparing figure VIII-12 (a) differs from that used for A. S. T. M. performance in that performance numbers are plotted directly against composition on linear coordinate paper and an estimated "best" curve is drawn through the data points to determine the binary blending relations shown in figure VIII-13. There is nothing to justify the use of this empirical method for dealing with A. S. T. M. Aviation ratings other than the fact that the end result agrees reasonably well with the experimental results. One or two exceptions to this method will be pointed out later.

The composition at the intersections of a given constant performance line with the blending lines (fig. VIII-13) were plotted on triangular coordinate paper and joined by straight lines. The resulting performance lines are shown in figure VIII-12 (b). The final chart (fig. VIII-12 (c)) was obtained by superimposing figure VIII-12 (b) on figure VIII-12 (a). Additional charts are shown in appendix B, figures B-1 and B-2 for other blending agents and base stocks. (All blends contained 4 ml TEL/gal.)

In appendix B, figure B-1 (e) represents the system of blends for cumene, virgin base, and alkylate; the lines showing A. S. T. M. Supercharge performance numbers were determined by plotting peak knock-limited-power values rather than power values at a fuel-air ratio of 0.11. This deviation from the procedure used for all other charts was necessitated by the fact that most of the mixture-response

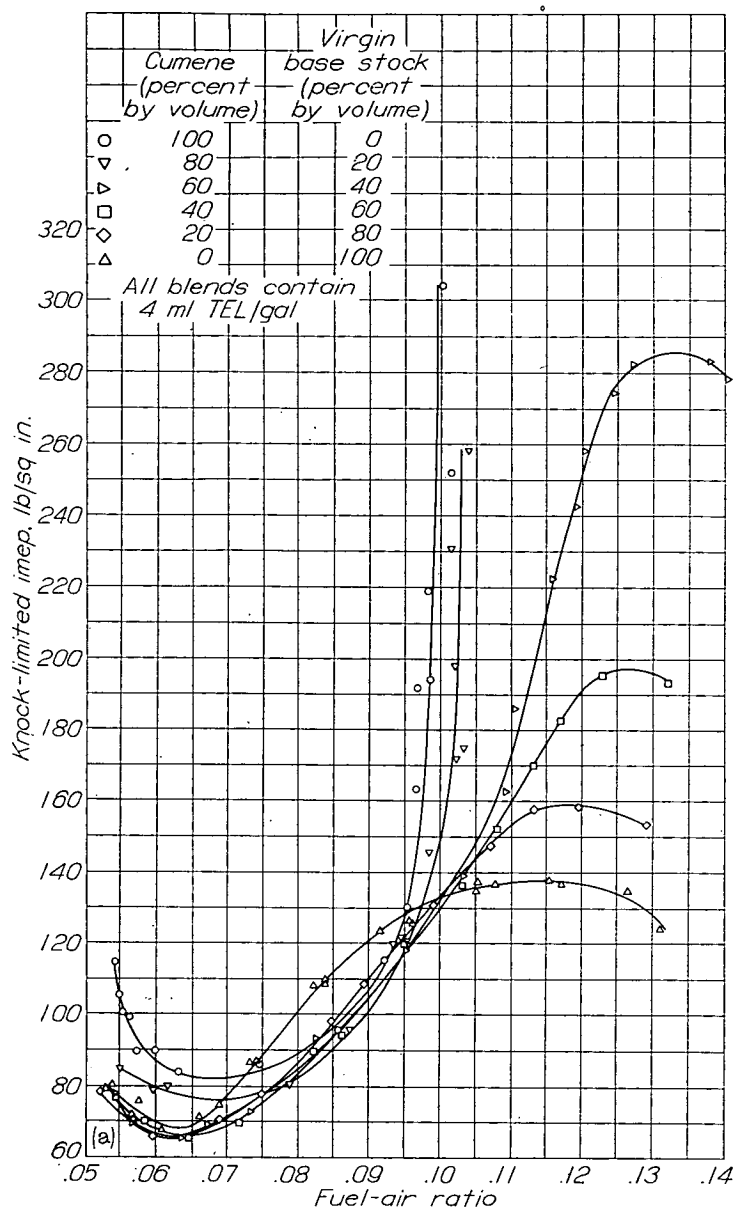


FIGURE VIII-14.—A. S. T. M. Supercharge mixture-response curves for blends of cumene and virgin base stock leaded to 4 ml TEL per gallon. (Fig. 9 (a) of reference 12.)

curves for the cumene blends intersected at fuel-air ratios between 0.10 and 0.11. (See fig. VIII-14.) The fuel-air ratio for peak-knock-limited power varied between 0.115 and 0.132 for the cumene blends. No chart was prepared for blends of cumene, aviation alkylate, and one-pass catalytic stock because rich-mixture peaks were not obtained for a sufficient number of the binary blends of cumene and one-pass catalytic stock.

For the charts representing blends of xylenes (appendix B, figs. B-1 (f) and B-2 (g)), A. S. T. M. Aviation performance lines were omitted. This omission arises from the fact that the curve of composition against performance number for binary blends of xylenes and aviation alkylate passed through a minimum. (See fig. VIII-15.)

In general, data obtained on the A. S. T. M. Aviation engine for the aromatic blends could not be handled with complete satisfaction by the empirical procedure previously

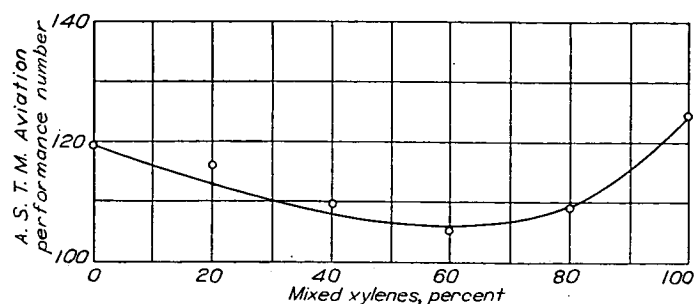


FIGURE VIII-15.—A. S. T. M. Aviation blending characteristics of mixed xylenes and aviation alkylate. (Fig. 10 of reference 12.)

explained. For this reason the accuracy of the lines of constant A. S. T. M. Aviation performance shown for the aromatic-paraffinic systems is questionable.

Quaternary blends.—The performance charts presented in the preceding section are of interest primarily from consideration of maximum knock-limited performance attainable with various combinations of fuel blending agents and base stocks. On the other hand, producers of aviation fuel are interested in the maximum knock-free power attainable with a finished blend that meets physical-property specifications for aviation fuels. Accordingly, several charts have been prepared to illustrate how physical properties might be taken into consideration. In these examples the only physical property considered is the Reid vapor pressure. Current aviation fuel specifications preclude the use of fuels having vapor pressures in excess of 7 pounds per square inch.

In order to prepare these charts, adjustments in Reid vapor pressures of the main components were made by adding isopentane, which in the pure state has a vapor pressure of about 20 pounds per square inch. The addition of isopentane to adjust the vapor pressure of the components in a system such as that shown in figure VIII-16 will necessarily affect the maximum knock-free power attainable because of the performance rating of isopentane relative to the ratings of the other components in the system, as shown in table VIII-1.

TABLE VIII-1.—A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE PERFORMANCE RATINGS AND REID VAPOR PRESSURES FOR VARIOUS AVIATION-FUEL COMPONENTS

Blending agent	Reid vapor pressure (lb/sq in.)	Performance number ^a	
		A. S. T. M. Aviation	A. S. T. M. Supercharge ^b
Isopentane.....	19.6	^c 133	^c 144
Neohexane.....	9.7	161	159
Methyl <i>tert</i> -butyl ether.....	8.8	> 161	> 200
Diisopropyl.....	7.4	142	202
Virgin base stock.....	5.9	73	94
Alkylate.....	4.7	119	137
Benzene.....	3.5	^d 68	> 200
Triptane.....	3.0	149	> 200
Hot-acid octane.....	2.7	127	197
Toluene.....	1.1	118	> 200
Mixed xylenes.....	.5	124	> 200
Cumene.....	.3	85	> 200

^a Performance numbers are for pure blending agent containing 4 ml TEL/gal.

^b Performance numbers over 161 are extrapolated (fig. VIII-10). Ratings are for fuel-air ratio of 0.11.

^c Extrapolated from experimental data for blends containing up to 60 percent isopentane.

^d Assumed to be the same as the rating for unleaded benzene.

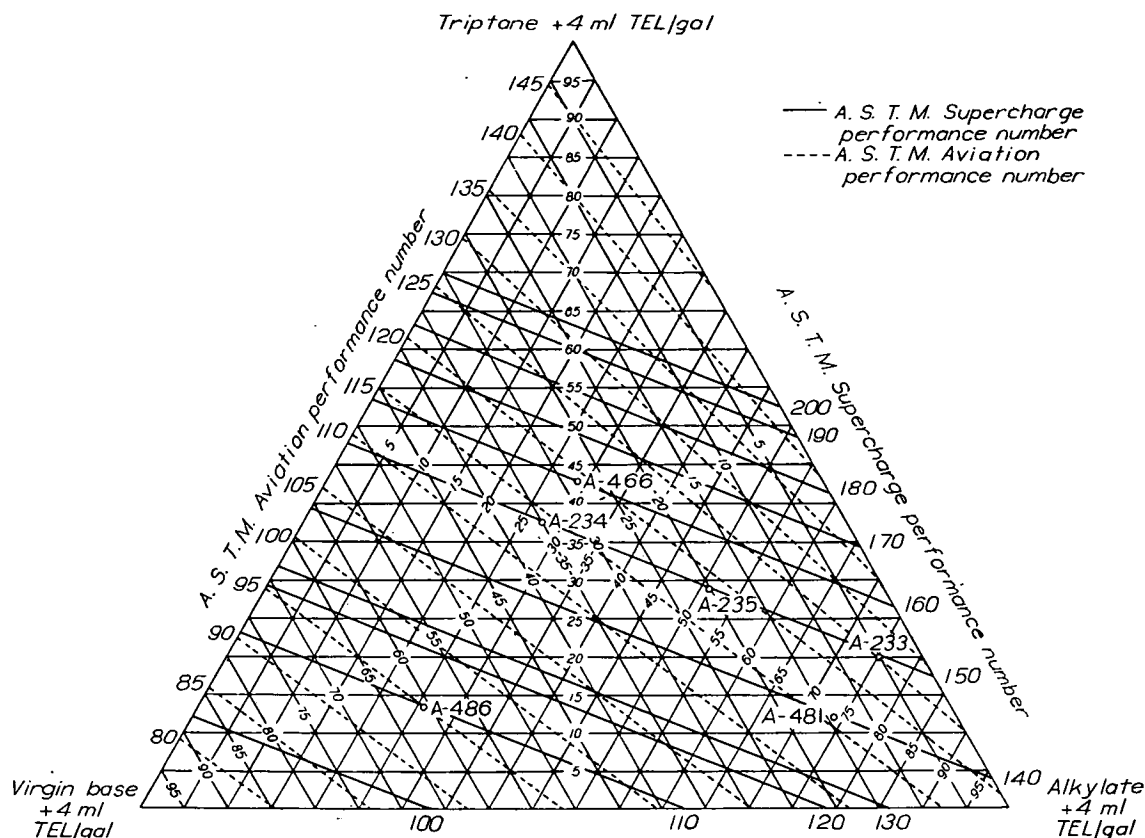


FIGURE VIII-16.—Blending chart for triptane, aviation alkylate, and virgin base stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. (Fig. 7 (a) of reference 12.) A. S. T. M. Supercharge fuel-air ratio, 0.11.

In figure VIII-16, for example, a blend of 58.5 percent triptane, 30.5 percent alkylate, and 11 percent virgin base stock has a lean-rich performance-number rating of 135/200 and a Reid vapor pressure of approximately 3.5 pounds per square inch. In order to obtain the same performance (135/200) with a blend of triptane, alkylate, and virgin base stock that has been isopentane to a Reid vapor pressure of 7 pounds per square inch (maximum permitted), a blend of 55 percent triptane, 17 percent alkylate, 7 percent virgin base stock, and 21 percent isopentane could be used. The addition of isopentane has thus effectively decreased the quantity of triptane needed to obtain the 135/200 performance rating. Isopentane has better performance characteristics than the alkylate or the virgin base stock used, as well as a higher Reid vapor pressure than the other three constituents in the blend. (See table VIII-1.)

It must be emphasized that the preceding example is given merely as an illustration of a fuel characteristic other than antiknock quality that must be considered for a finished fuel blend. This example is not intended to imply that the preparation of fuel blends with Reid vapor pressures adjusted to meet specifications will necessarily produce blends that will meet all other pertinent specifications.

Several performance charts for quaternary blends containing isopentane were prepared for comparison with the

charts previously described for ternary blends. In each of the quaternary systems the vapor pressure was adjusted to 7 pounds per square inch. Three assumptions were made in the preparation of these charts:

- (1) The relation between composition and Reid vapor pressure is linear for binary blends of isopentane with another paraffinic fuel.
- (2) The relation between composition and the reciprocal of the A. S. T. M. Supercharge (rich) knock-limited indicated mean effective pressure is linear for binary blends of isopentane with another paraffinic fuel.
- (3) The relation between composition and the A. S. T. M. Aviation performance number is linear for binary blends of isopentane with another paraffinic fuel.

On the basis of the available data, assumption (3) appears to be valid for only a few cases. For this reason the A. S. T. M. Aviation performance lines on the charts for quaternary blends may be in error.

As an example of the preparation of the performance chart for a quaternary system, assume that it is desired to isopentane the blends represented by figure VIII-16. The first step in this problem is to determine the amount of isopentane to be added to each of the pure components to obtain a Reid vapor pressure of 7 pounds per square inch and to determine the resultant lean and rich performance-

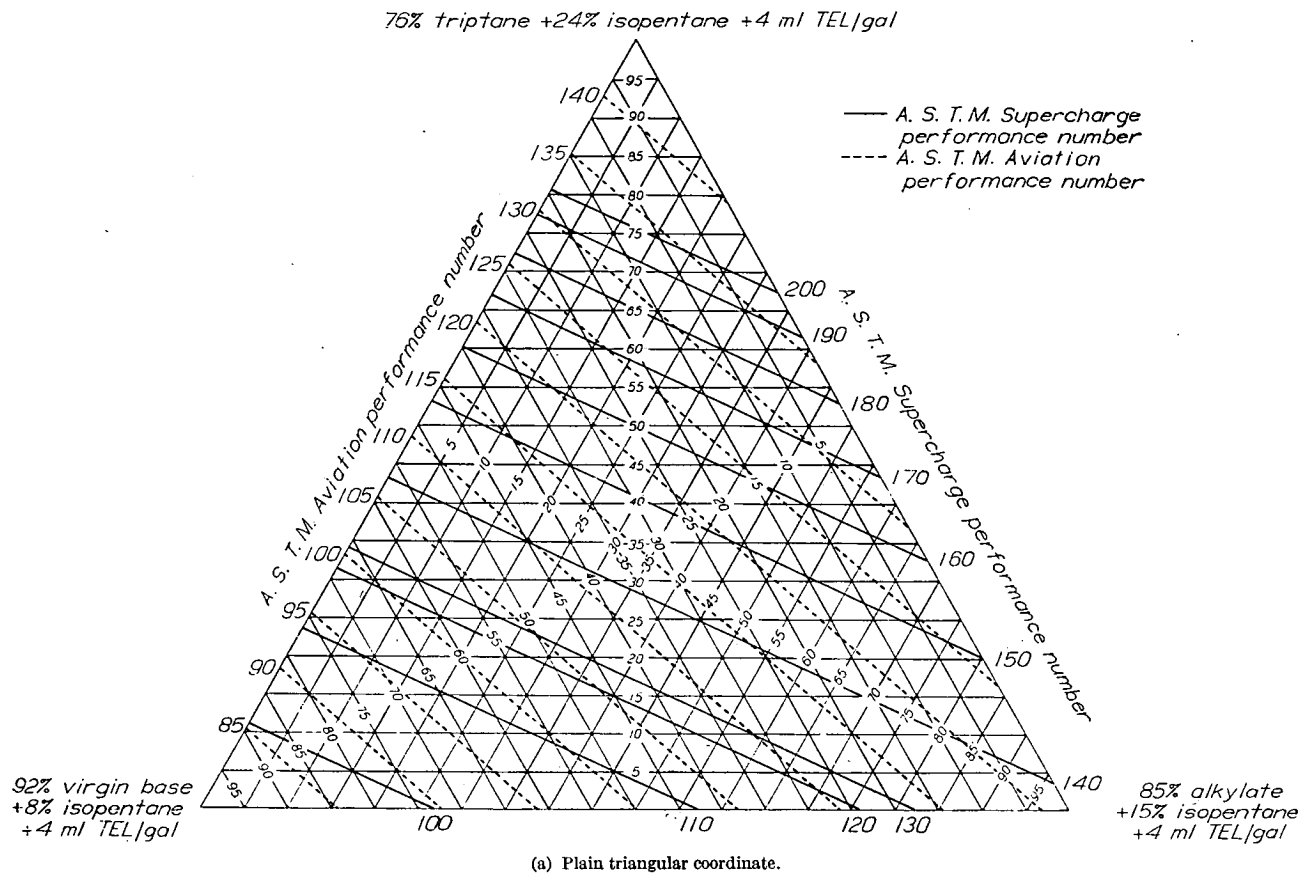


FIGURE VIII-17.—Blending charts for triptane, aviation alkylate, virgin base stock, and isopentane by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. Reid vapor pressure, approximately 7 pounds per square inch; A. S. T. M. Supercharge fuel-air ratio, 0.11. (Fig. 11 of reference 12.)

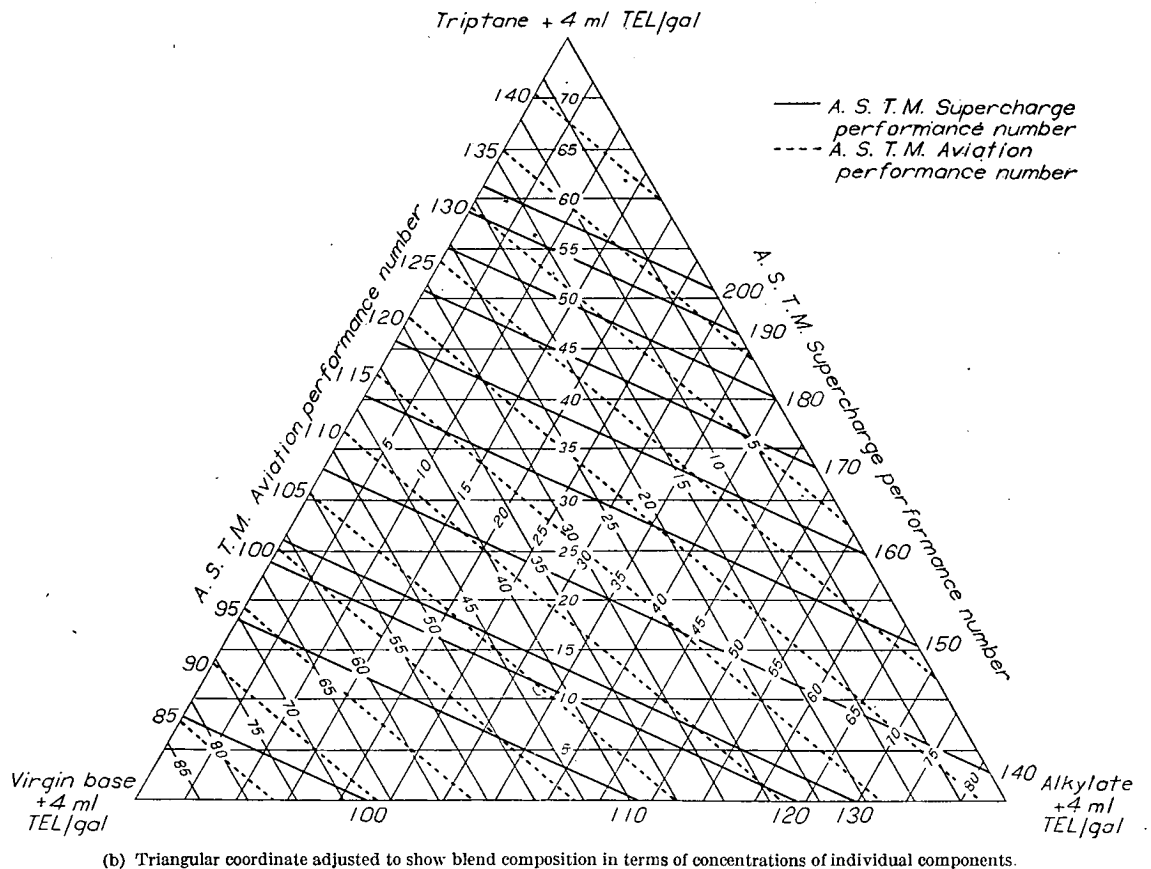


FIGURE VIII-17.—Concluded. Blending charts for triptane, aviation alkylate, virgin base stock, and isopentane by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. Reid vapor pressure, approximately 7 pounds per square inch; A. S. T. M. Supercharge fuel-air ratio, 0.11. (Fig. 11 of reference 12.) Percentage isopentane can be determined by subtracting sum of percentages of other components from 100.

number ratings for these blends. This information as determined from the foregoing assumptions and table VIII-1 is presented in the following table:

Fuel blend	A. S. T. M. Aviation performance number	A. S. T. M. Supercharge indicated mean effective pressure (lb/sq in.)
76 percent triptane + 24 percent isopentane + 4 ml TEL/gal	145	455
85 percent alkylate + 15 percent isopentane + 4 ml TEL/gal	121	200
92 percent virgin base + 8 percent isopentane + 4 ml TEL/gal	78	142

The triangular chart shown in figure VIII-17 (a) was obtained by treating these three blends (all of which have Reid vapor pressures of 7 lb/sq in.) as separate components by the procedure used in preparing figure VIII-16. Any point on figure VIII-17 (a) represents the A. S. T. M. Aviation and the A. S. T. M. Supercharge performance number of a quaternary blend. The actual quantity of each component in the blend, however, cannot be readily determined from the chart because the percentages given on the altitudes of the triangle show only the amounts of the binary blends at the vertexes. For this reason, the grid of the chart was so adjusted, as shown in figure VIII-17 (b), that the quantity of any one of the four components in the blend could be read directly from the chart.

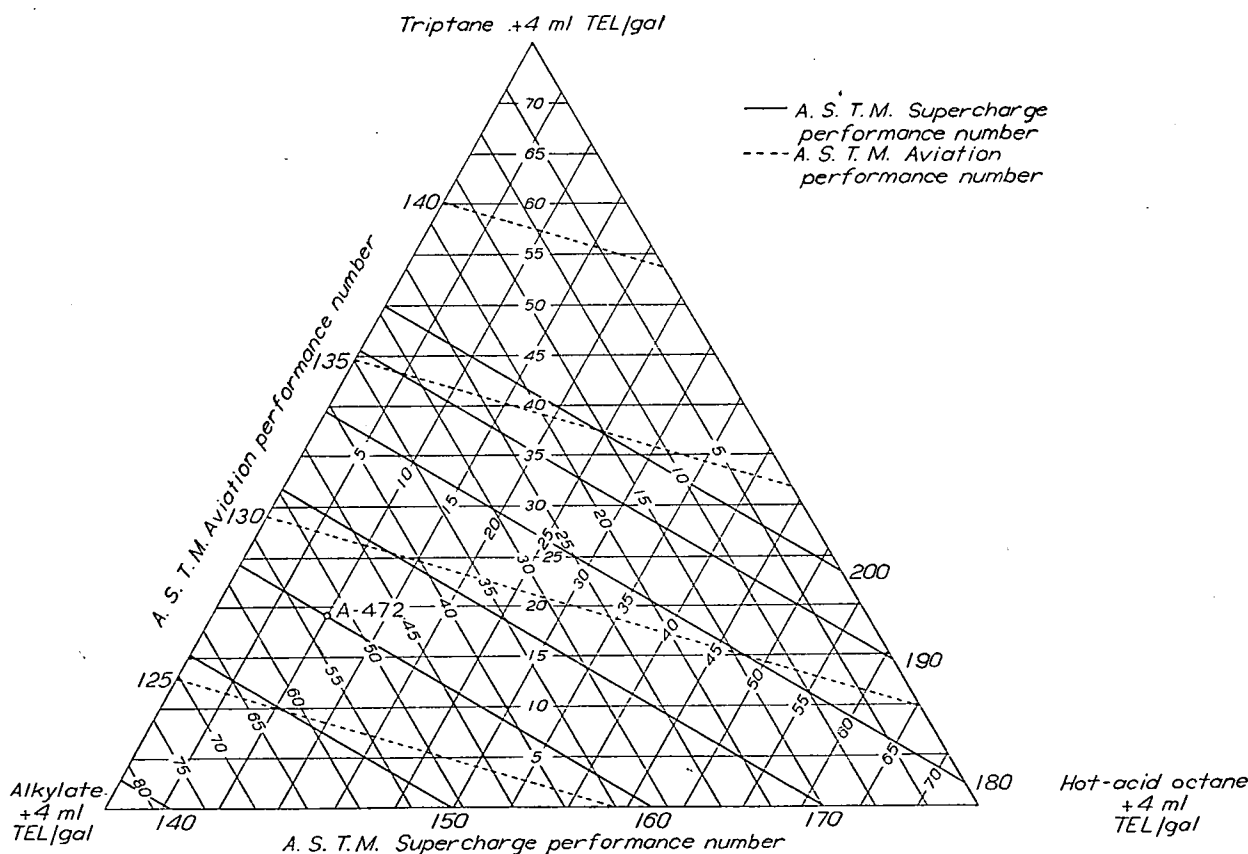
As an illustration of the method for determining the com-

position of a fuel in figure VIII-17 (b), suppose that it is desired to prepare a blend of triptane, aviation alkylate, virgin base stock, and isopentane having a lean-rich rating of 130/180. The concentrations of triptane, alkylate, and virgin base in the blend having the desired rating can be read directly from the altitudes of the triangle in the manner used in previous charts. These concentrations are 48, 19, and 13 percent, respectively. The concentration of isopentane can be determined by subtracting the sum of the percentages of the other components from 100.

Performance charts for other quaternary systems are presented in figure VIII-18. The experimental value for the Reid vapor pressure of diisopropyl (table VIII-1) is 7.4; however, in preparing figures VIII-18 (b) to VIII-18 (d), a value of 7.0 was assumed for this fuel.

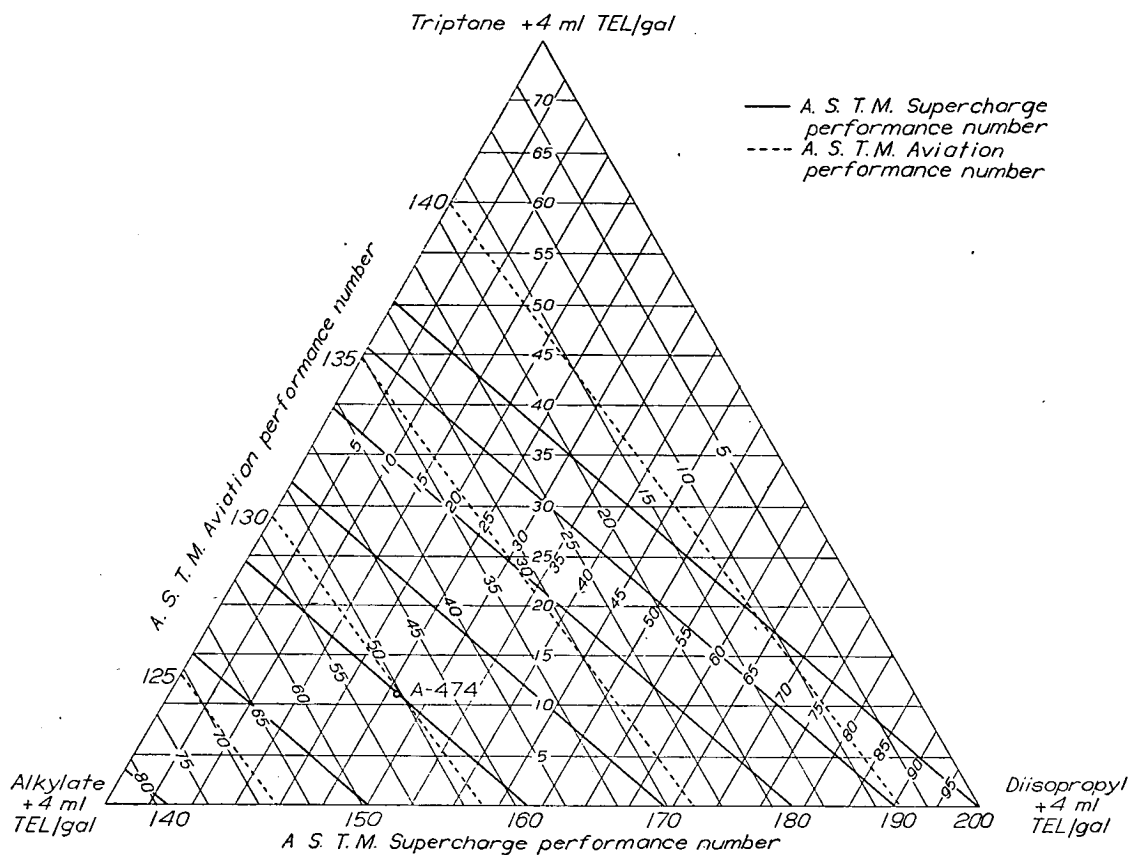
Accuracy of performance charts.—The accuracy of the charts was determined by selecting ternary or quaternary blends from the various charts and testing these blends by the standard A. S. T. M. Aviation and A. S. T. M. Supercharge procedures. Inasmuch as the rich ratings shown on the charts were estimated at a fuel-air ratio of 0.11, the check ratings were determined at this same fuel-air ratio.

The check blends tested are shown with their ratings in table VIII-2. These blends are also shown on the various charts (appendix B and figs. VIII-12 (c), VIII-16, and VIII-18 (a) to VIII-18 (c)) by the symbols. The fuel numbers shown adjacent to each of the symbols on the charts correspond to the fuel numbers given in this table.



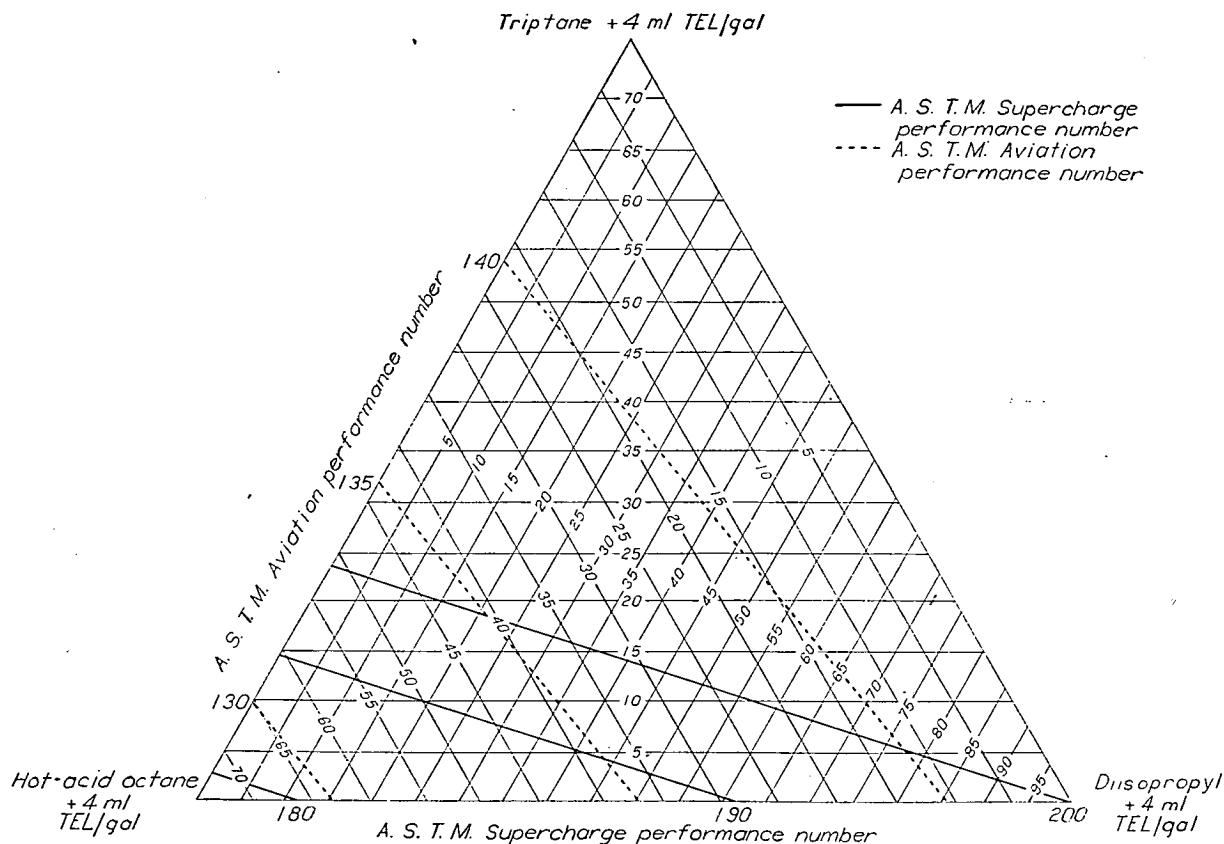
(a) Blends of triptane, aviation alkylate, hot-acid octane, and isopentane.

FIGURE VIII-18.—Blending charts for quaternary blends by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. Reid vapor pressure, approximately 7 pounds per square inch; A. S. T. M. Supercharge fuel-air ratio, 0.11. (Fig. 12 of reference 12.) Percentage isopentane can be determined by subtracting sum of percentages of other components from 100.



(b) Blends of triptane, aviation alkylate, diisopropyl, and isopentane.

FIGURE VIII-18.—Continued. Blending charts for quaternary blends by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. Reid vapor pressure, approximately 7 pounds per square inch; A. S. T. M. Supercharge fuel-air ratio, 0.11. (Fig. 12 of reference 12.) Percentage isopentane can be determined by subtracting sum of percentages of other components from 100.



(c) Blends of triptane, hot-acid octane, diisopropyl, and isopentane.

FIGURE VIII-18.—Continued. Blending charts for quaternary blends by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. Reid vapor pressure, approximately 7 pounds per square inch; A. S. T. M. Supercharge fuel-air ratio, 0.11. (Fig. 12 of reference 12.) Percentage isopentane can be determined by subtracting sum of percentages of other components from 100.

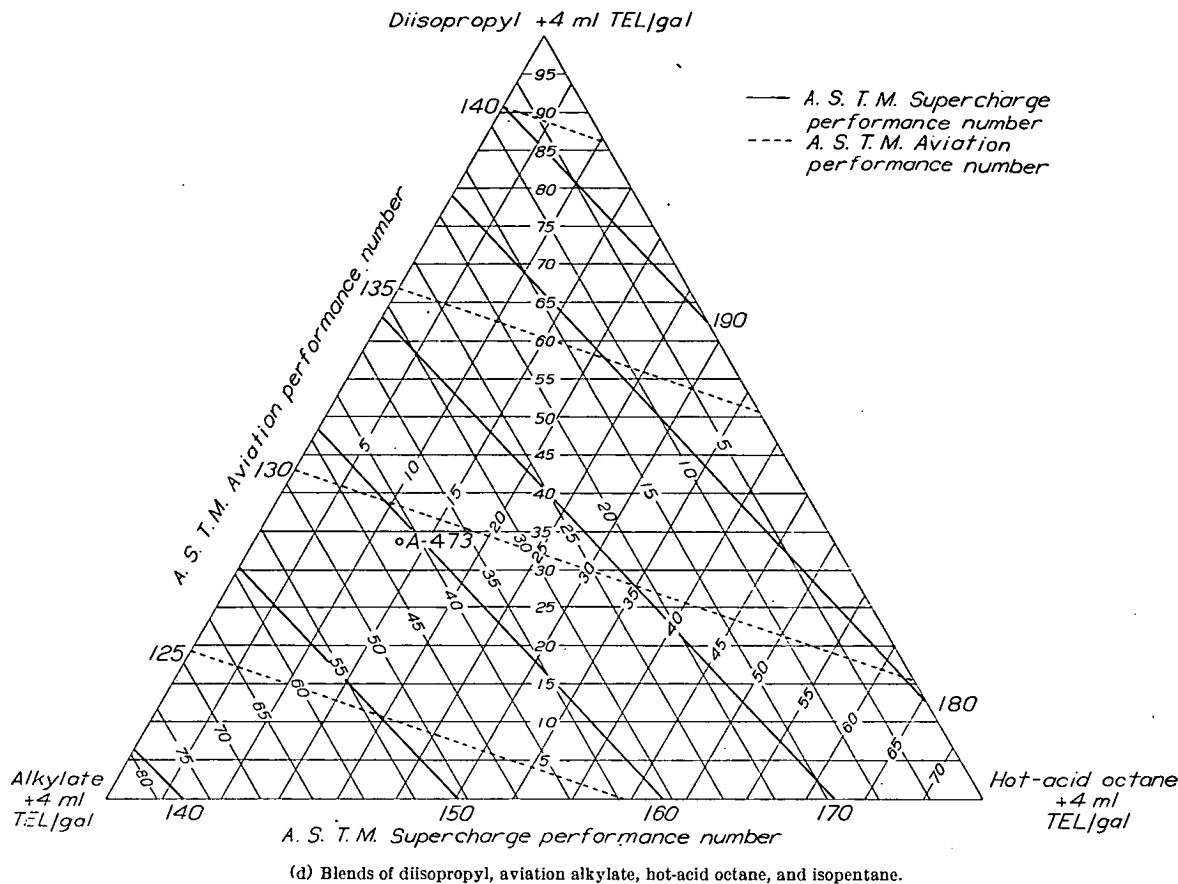


FIGURE VIII-18.—Concluded. Blending charts for quaternary blends by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. Reid vapor pressure, approximately 7 pounds per square inch; A. S. T. M. Supercharge fuel-air ratio, 0.11. (Fig. 12 of reference 12.) Percentage isopentane can be determined by subtracting sum of percentages of other components from 100.

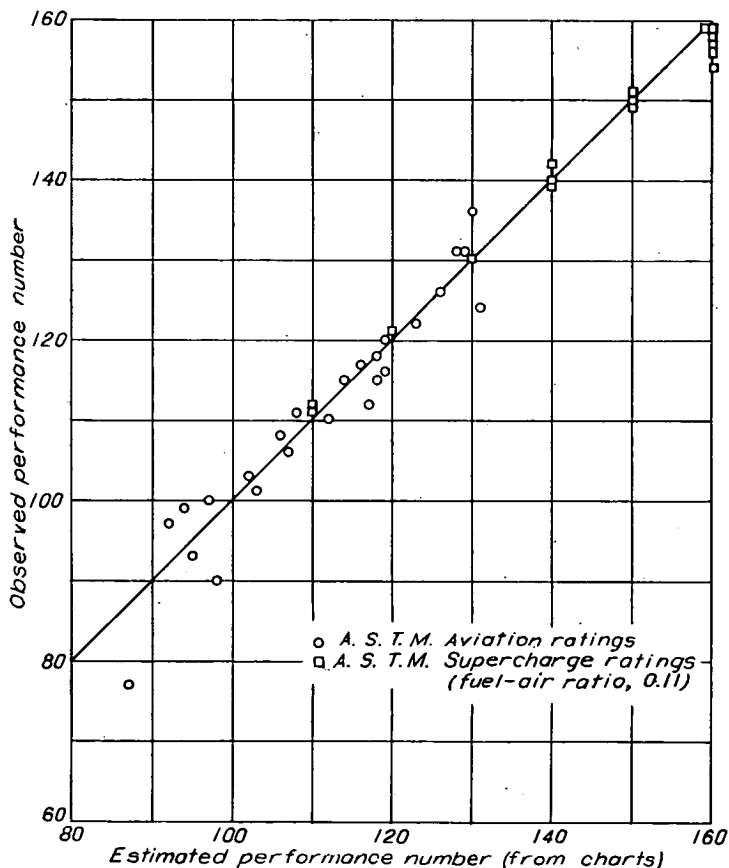


FIGURE VIII-19.—Comparison of estimated and observed knock-limited performance ratings determined by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. A. S. T. M. Supercharge fuel-air ratio, 0.11. (Fig. 13 of reference 12.)

All the data in table VIII-2 are presented in figure VIII-19 to show the relation between estimated and observed performance numbers. For the 25 blends shown in figure VIII-19, the average deviation from the match line was 3.1 performance numbers for the lean ratings and 1.5 for the rich ratings.

In consideration of the accuracy of the charts it must be emphasized that the previously mentioned discrepancies noted in the lean ratings of binary blends containing aromatics are responsible for some of the large deviations in table VIII-2.

Discussion of performance charts.—The performance charts prepared in connection with this chapter can be used for certain general comparisons of paraffins, aromatics, and ethers. In figure VIII-16, for example, at the point representing a blend of 80 percent aviation alkylate, 20 percent virgin base stock, and 4 ml TEL per gallon, the lean-rich rating is 110/122. If a straight line is followed from this point toward the triptane vertex until 20 percent triptane has been added, the rating becomes 118/145. The addition of 20 percent triptane to the base blend has thus increased the lean rating of the base blend by 8 performance numbers and the rich rating by 23. If, however, the chart for benzene (appendix B, fig. B-1 (d)) with the same base stocks is examined, it is seen that a 20-percent addition of benzene changes the rating of the starting blend from 110/122 to 106/146. The addition of 20 percent benzene has therefore decreased the lean rating by 4 performance numbers, whereas

TABLE VIII-2.—A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE PERFORMANCE RATINGS OF TERNARY AND QUATERNARY FUEL BLENDS

[The following abbreviations are used throughout the table: VBS for virgin base stock; alkylate for aviation alkylate; one-pass stock for one-pass catalytic stock; and MTB ether for methyl *tert*-butyl ether]

Fuel	Fuel composition ^a	Performance numbers			
		A. S. T. M. Aviation ratings		A. S. T. M. Supercharge ratings ^b	
		Estimated	Observed	Estimated	Observed
Ternary blends					
A-477..	59% hot-acid octane+25% VBS+16% alkylate.....	112	110	150	149
A-487..	11% hot-acid octane+48% VBS+41% alkylate.....	98	90	110	111
A-233..	20% triptane+5% VBS+75% alkylate.....	126	126	150	151
A-235..	20% triptane+20% VBS+51% alkylate.....	119	120	150	151
A-234..	38% triptane+35% VBS+27% alkylate.....	114	115	150	150
A-466..	43% triptane+28% VBS+29% alkylate.....	119	116	160	158
A-481..	12% triptane+14% VBS+74% alkylate.....	116	117	140	142
A-486..	13% triptane+61% VBS+26% alkylate.....	95	93	110	112
A-478..	43% diisopropyl+12% VBS+45% alkylate.....	123	122	150	150
A-524..	34% diisopropyl+52% VBS+14% alkylate.....	103	101	120	121
A-483..	56% neohexane+14% VBS+30% alkylate.....	131	124	140	140
A-523..	12% neohexane+43% VBS+45% alkylate.....	102	103	110	111
A-482..	23% benzene+34% VBS+43% alkylate.....	97	100	140	139
A-522..	47% benzene+41% VBS+12% alkylate.....	87	77	160	154
A-484..	14% toluene+54% VBS+32% alkylate.....	92	97	130	130
A-521..	23% toluene+17% VBS+60% alkylate.....	107	106	160	156
A-520..	33% MTB ether+55% VBS+12% alkylate.....	106	108	160	154
A-539..	6% MTB ether+59% VBS+35% alkylate.....	94	99	110	111
A-470..	55% hot-acid octane+13% one-pass stock+32% alkylate.....	118	118	160	159
A-471..	35% triptane+45% one-pass stock+20% alkylate.....	108	111	160	159
A-480..	20% triptane+16% one-pass stock+64% alkylate.....	117	112	150	150
A-555..	39% diisopropyl+24% one-pass stock+37% alkylate.....	118	115	150	150
Quaternary blends					
A-472..	19% triptane+10% hot-acid octane+52.5% alkylate+18.5% isopentane.....	128	131	160	157
A-474..	11.5% triptane+25.5% diisopropyl+50.5% alkylate+12.5% isopentane.....	130	136	160	159
A-473..	34% diisopropyl+12.5% hot-acid octane+41.5% alkylate+12% isopentane..	129	131	159	159

^a Each fuel contains approximately 4 ml TEL/gal.

^b Ratings made at fuel-air ratio of 0.11.

the rich rating has been increased by 24. From this comparison it follows that in this example the aromatic (benzene) and the paraffin (triptane) are equally effective for increas-

ing the A. S. T. M. Supercharge (rich) performance but that triptane is more effective than benzene for improving lean performance.

REFERENCES

1. Lovell, Wheeler G., and Campbell, John M.: Knocking Characteristics and Molecular Structure of Hydrocarbons. The Science of Petroleum, vol. IV, Oxford Univ. Press (London), 1938, pp. 3004-3023.
2. Smittenberg, J., Hoog, H., Moerbeek, B. H., and v. d. Zijden, M. J.: Octane Ratings of a Number of Pure Hydrocarbons and Some of Their Binary Mixtures. Jour. Inst. Petroleum, vol. 26, no. 200, June 1940, pp. 294-300.
3. Sanders, Newell D.: A Method of Estimating the Knock Rating of Hydrocarbon Fuel Blends. NACA Rep. 760, 1943.
4. Rothrock, A. M.: Fuel Rating—Its Relation to Engine Performance. SAE Jour. (Trans.), vol. 48, no. 2, Feb. 1941, pp. 51-65.
5. Heron, S. D., and Beatty, Harold A.: Aircraft Fuels. Jour. Aero. Sci., vol. 5, no. 12, Oct. 1938, pp. 463-479.
6. Brooks, Donald B.: Effect of Altitude on Knock Rating of CFR Engines. Res. Paper 1475, Nat. Bur. of Standards Jour. Res., vol. 28, no. 6, June 1942, pp. 713-734.
7. Doss, M. P.: Physical Constants of the Principal Hydrocarbons. The Texas Company, 3d ed., 1942.
8. Wear, Jerrold D., and Sanders, Newell D.: Experimental Studies of the Knock-Limited Blending Characteristics of Aviation Fuels. II—Investigation of Leaded Paraffinic Fuels in an Air-Cooled Cylinder. NACA TN 1374, 1947.
9. Sanders, Newell D., Hensley, Reece V., and Breitwieser, Roland: Experimental Studies of the Knock-Limited Blending Characteristics of Aviation Fuels. I—Preliminary Tests in an Air-Cooled Cylinder. NACA ARR E4I28, 1944.
10. Drell, I. L., and Wear, J. D.: Experimental Studies of the Knock-Limited Blending Characteristics of Aviation Fuels. III—Aromatics and Cycloparaffins. NACA TN 1416, 1947.
11. Hensley, Reece V., and Breitwieser, Roland: Knock-Limited Blending Characteristics of Benzene, Toluene, Mixed Xylene and Cumene in an Air-Cooled Cylinder. NACA MR E5B03, 1945.
12. Barnett, Henry C.: Estimation of F-3 and F-4 Knock-Limited Performance Ratings for Ternary and Quaternary Blends Containing Triptane or Other High-Antiknock Aviation-Fuel Blending Agents. NACA Rep. 904, 1948. (Formerly MR E5A29.)
13. Imming, Harry S., Barnett, Henry C., and Genco, Russell S.: F-3 and F-4 Engine Tests of Several High-Antiknock Components of Aviation Fuel. NACA MR E4K27, 1944.

CHAPTER IX

FUEL VOLATILITY

A number of problems arise from the influence of fuel volatility on engine and aircraft performance. Among these are vapor lock, mixture distribution, ease of starting, cold weather operation, and vaporization loss. In order to permit reasonable control of these factors, fuel specifications include provisions for regulation of volatility. The particular limits established on the volatility characteristics of fuels are based on numerous laboratory investigations and considerable service experience. Some of the laboratory researches on volatility characteristics of fuel are described in this chapter.

MIXTURE DISTRIBUTION

In multicylinder engines an important requisite of good performance is uniform distribution of the fuel-air charge to the individual cylinders. However, design factors relating to engine induction systems (for example, intake manifolds, carburetors, and superchargers) tend to make the problem of distribution one of varying significance with each particular engine. Aside from the undesirable effect on general engine operation, nonuniform distribution increases fuel consumption and cooling requirements and decreases over-all knock limits. For a given engine and a fuel of given volatility, the mixture distribution is affected by over-all fuel-air ratio of the engine, engine speed, supercharger gear ratio, charge weight flow or engine power, throttle setting, and combustion-air inlet temperature. The investigation of reference 1 illustrates the degree to which these engine variables may affect mixture distribution.

Distribution of fuel-air mixture to the various cylinders was determined by exhaust-gas analysis (reference 2) by an improved sampling technique reported by Cook and Olson (reference 3). The engine used was an 18-cylinder, double-row, radial air-cooled engine installed on a test stand and fitted with a flight cowling. (See fig. IX-1.) The engine was equipped with a single-stage supercharger with low and high gear ratios of 7.6 and 9.45, respectively. Additional installation details may be found in reference 1.

The carburetor-deck pressure was maintained atmospheric and the throttle setting was varied to maintain fixed power for the tests in which the over-all fuel-air ratio, the engine speed, the power, or the intake-air temperature was varied. (See reference 1.) The results therefore show the effect of both the variable under investigation and the associated variation of throttle setting and are representative of normal sea-level performance. When the engine power and speed were varied, one of the most important influences on mixture-distribution changes was the variation in throttle angle resulting from atmospheric carburetor-deck pressure. During the tests of varying throttle setting, the carburetor-deck pressure was changed to maintain constant power; operation at altitude was thereby simulated.

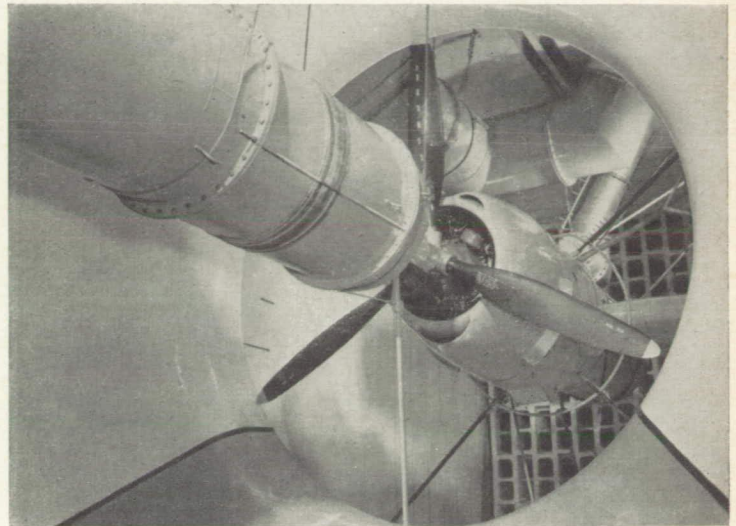


FIGURE IX-1.—Installation of test engine and cowling. (Fig. 1 of reference 1.)

Effect of over-all fuel-air ratio on mixture distribution.—In figure IX-2 the influence of over-all fuel-air ratio on distribution is illustrated. It is readily seen in this figure that the distribution is best at lean fuel-air ratios. The difference between the richest and the leanest cylinders at an over-all fuel-air ratio of 0.101 is reduced from 0.032 to 0.003 at an over-all fuel-air ratio of 0.059. This general trend was found to be true at all operating conditions, although the magnitude of the differences varied. The trend is attributed to the fact that more nearly complete fuel evaporation occurs at low fuel flows where the irregularities of distribution caused by the concentration of fuel droplets are reduced. (See reference 1.)

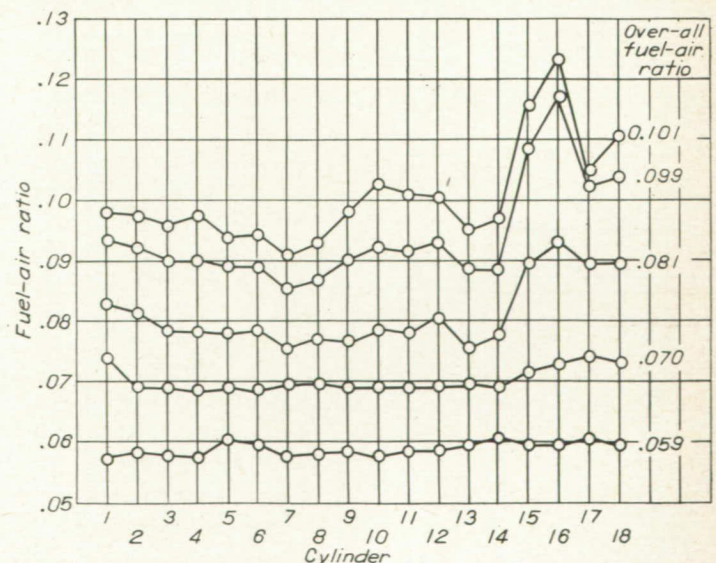
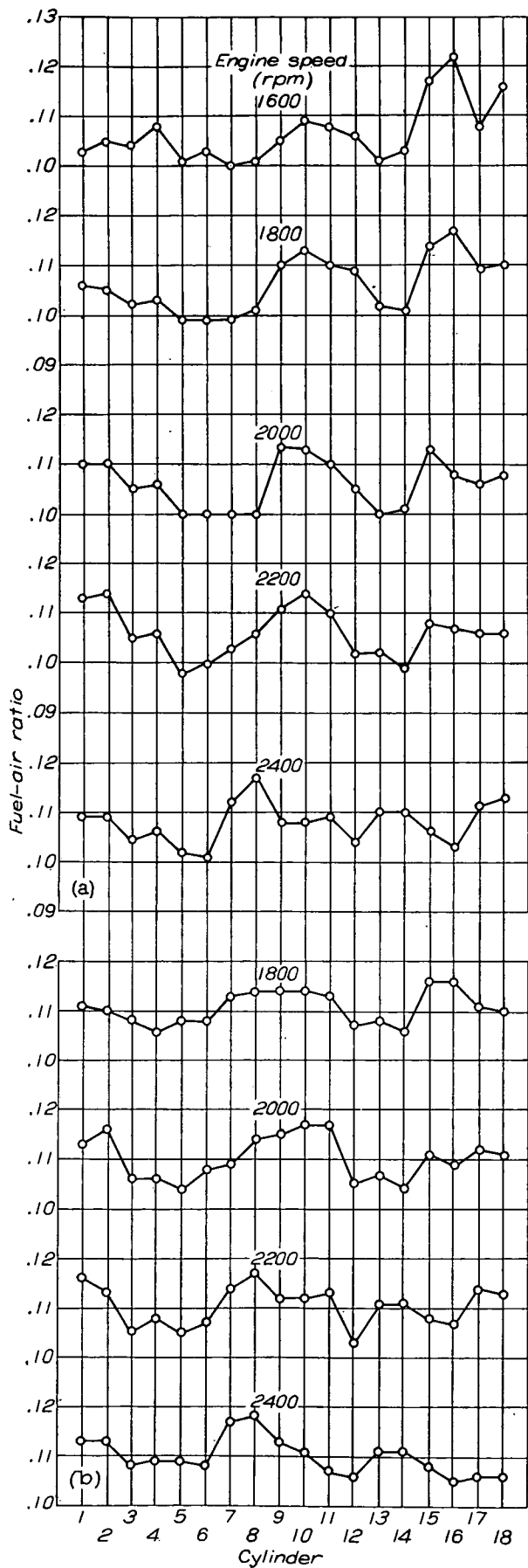


FIGURE IX-2.—Variation of mixture distribution with over-all fuel-air ratio. Engine speed, 1600 rpm; engine power, 800 brake horsepower; low supercharger gear ratio. (Fig. 7 (a) of reference 1.)



(a) Engine power, 800 brake horsepower; over-all fuel-air ratio, 0.106.
(b) Engine power, 1000 brake horsepower; over-all fuel-air ratio, 0.110.
FIGURE IX-3.—Variation of mixture distribution with engine speed. (Fig. 9 of reference 1.)

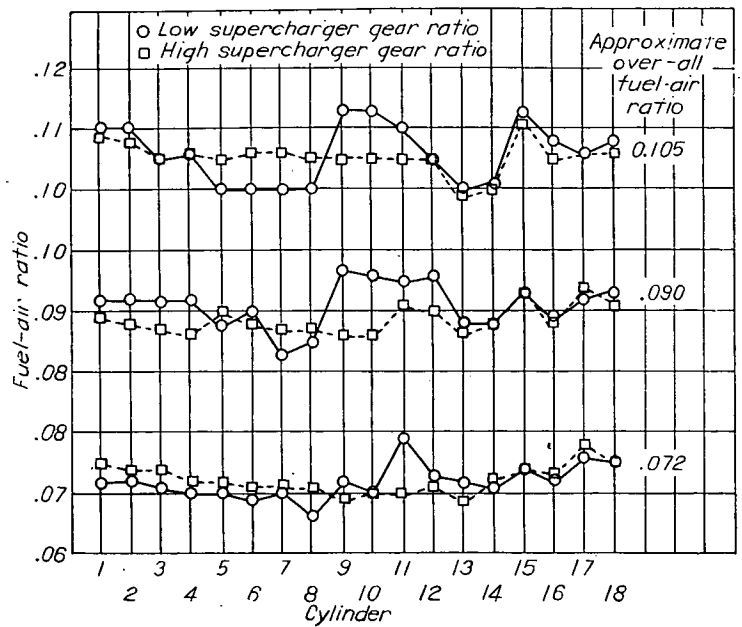


Figure IX-4.—Variation of mixture distribution with supercharger gear ratio. Engine speed, 2000 rpm; engine power, 800 brake horsepower. (Fig. 10 (a) of reference 1.)

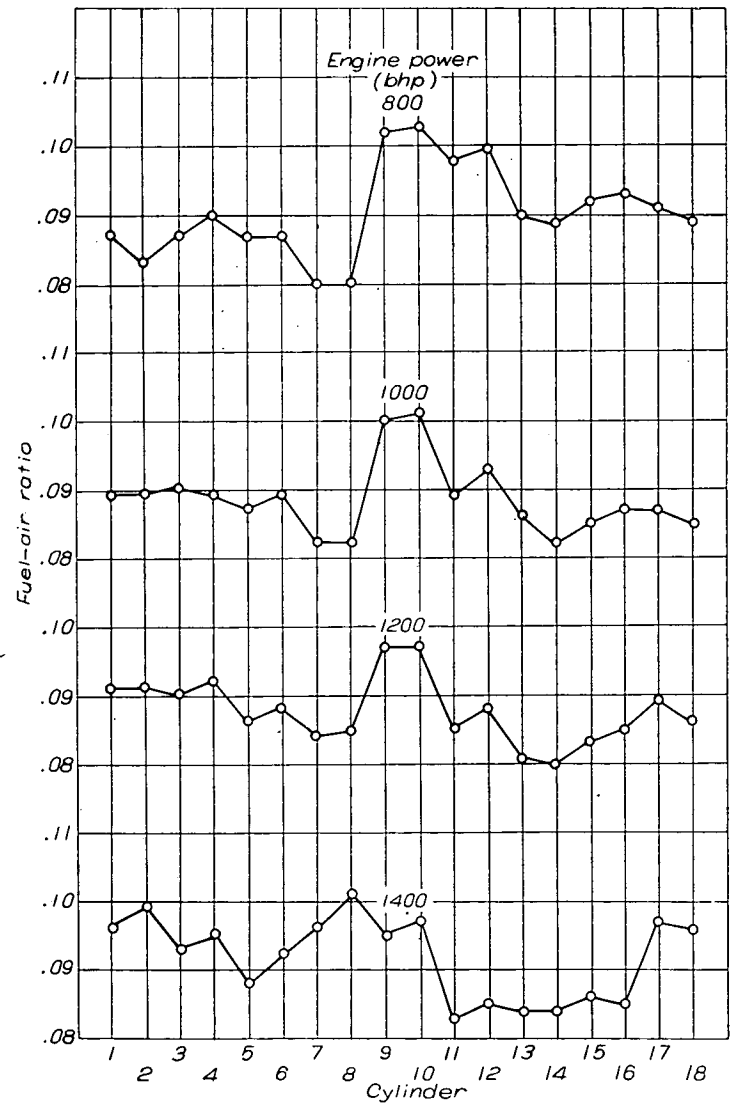


FIGURE IX-5.—Variation of mixture distribution with engine power. Engine speed, 2200 rpm; over-all fuel-air ratio, 0.089; low supercharger gear ratio. (Fig. 11 (b) of reference 1.)

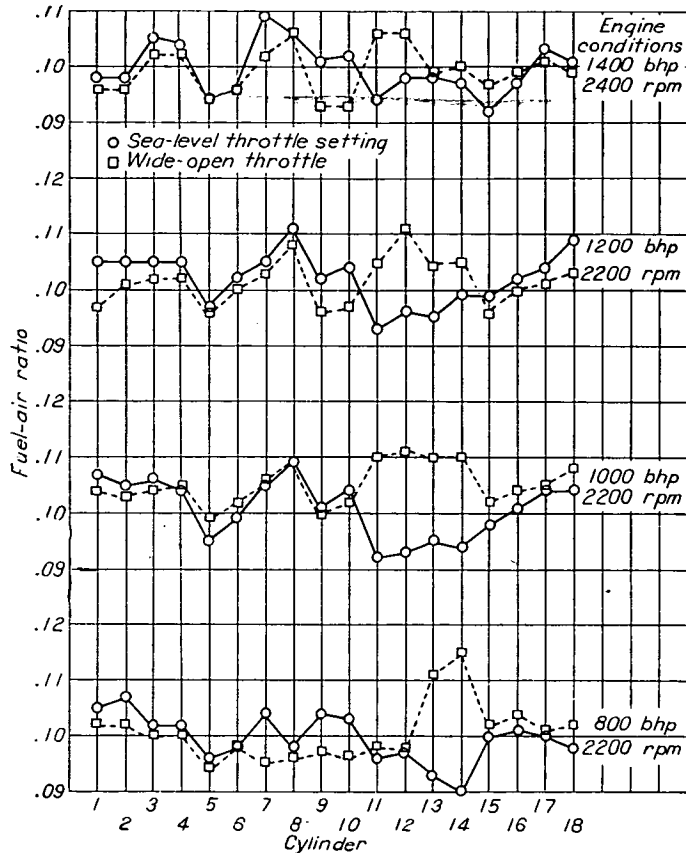


FIGURE IX-6.—Variation of mixture distribution with throttle setting. Over-all fuel-air ratio, 0.100; low supercharger gear ratio. (Fig. 12 (a) of reference 1.)

Effect of engine speed on mixture distribution.—The results of tests at low supercharger gear ratio indicated that the variation of mixture strength between the richest and the leanest cylinders is not appreciably affected by changes of engine speed between 1600 and 2400 rpm (fig. IX-3) despite the accompanying variation in throttle angles. The general shape of the distribution patterns does vary somewhat. For example, in figure IX-3 two peaks may be seen occurring in the neighborhood of cylinders 10 and 16 at low engine speed. The peak near cylinder 16 is diminished whereas that near cylinder 10 becomes more prominent as the speed increases. At high speeds each of the maximum points tends to move to the adjacent cylinder in a direction opposite that of the impeller rotation, that is, from cylinders 10 and 16 to cylinders 8 and 14, respectively.

Effect of supercharger gear ratio on mixture distribution.—The effect of supercharger gear ratio on mixture distribution is shown in figure IX-4. At the high supercharger gear ratio, the mixture distribution was considerably improved over the distribution at low gear ratio. It was suggested in reference 1 that the more nearly uniform mixture distribution at the higher impeller speeds was due to the higher combustion-air temperature and the resulting better evaporation of the fuel passing through the supercharger as well as from the more thorough mixing at the diffuser entrance. It should be recalled, however, that in the preceding discussion no effect on mixture distribution was found when the impeller speed was increased by increasing engine speed at constant supercharger gear ratio.

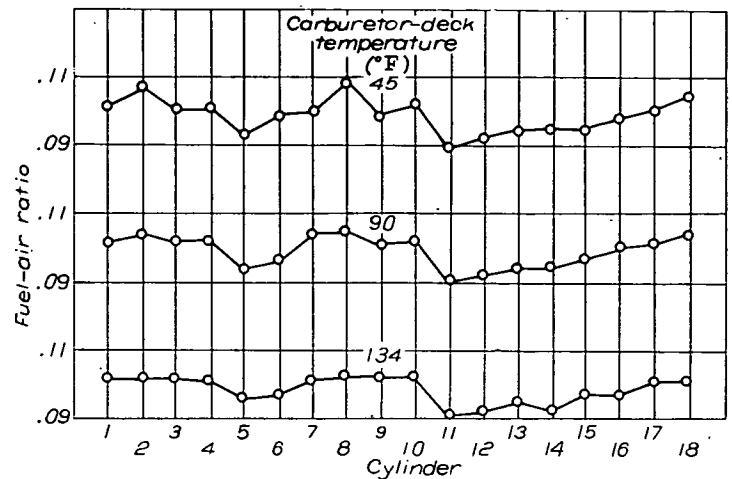


FIGURE IX-7.—Variation of mixture distribution with carburetor-deck temperature. Engine speed, 2200 rpm; engine power, 1000 brake horsepower; approximate over-all fuel-air ratio, 0.100; low supercharger gear ratio. (Fig. 14 of reference 1.)

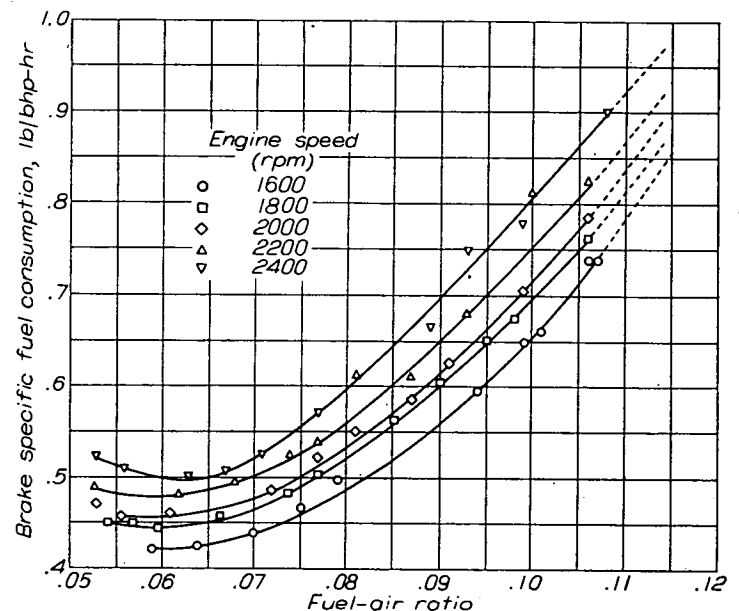


FIGURE IX-8.—Effect of fuel-air ratio on fuel consumption at various engine speeds. Engine power, 800 brake horsepower; low supercharger gear ratio. (Fig. 17 of reference 1.)

Effect of variation of power on mixture distribution.—Mixture-distribution curves for various values of brake horsepower are shown in figure IX-5 for an over-all fuel-air ratio of 0.089. The differences in mixture strength between the richest and leanest cylinder are not greatly affected by power level. Similar results were obtained at over-all fuel-air ratios leaner and richer than 0.089.

Effect of throttle setting on mixture distribution.—Tests were conducted at different throttle settings to determine the change in mixture distribution. The results obtained at one fuel-air ratio are shown in figure IX-6. The most significant differences in the distribution pattern were observed at the low power conditions. At 800 brake horsepower, for example, cylinder 14, the leanest cylinder at sea-level throttle setting, became the richest cylinder at wide-open throttle.

Although the mixture distribution patterns were different at the various power levels and throttle settings, no trend of improvement in distribution was apparent.

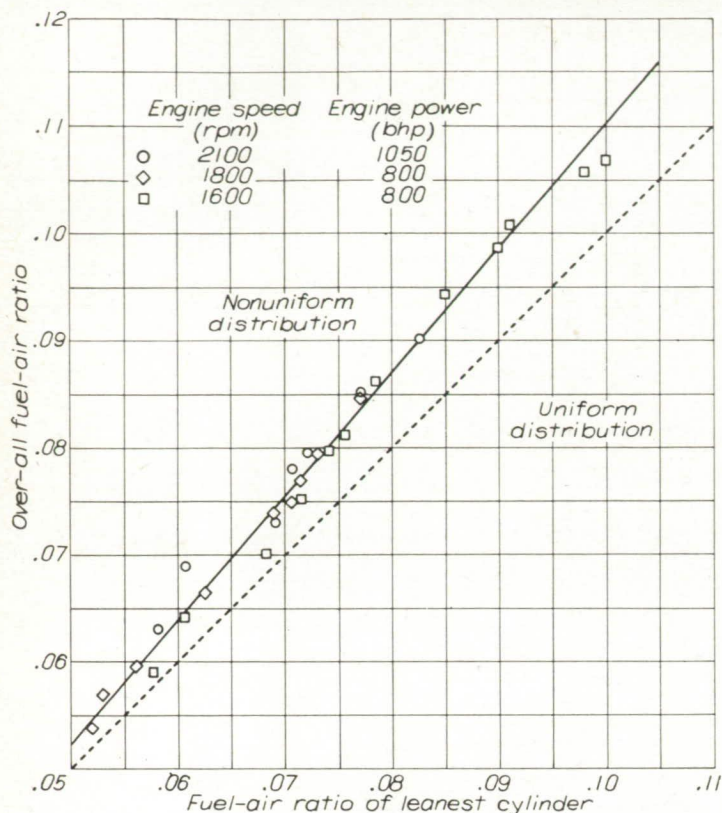


FIGURE IX-9.—Relation between over-all fuel-air ratio and fuel-air ratio of leanest cylinder. (Fig. 16 of reference 1.)

Effect of combustion-air temperature on mixture distribution.—Mixture distribution patterns for carburetor-deck temperatures of 45°, 90°, and 134° F are shown in figure IX-7. The general shapes of the curves in this figure are similar, but the differences between the richest and leanest cylinders decrease with increasing carburetor-deck temperature. This trend may be attributed to increased fuel evaporation at the higher temperatures and the resulting better mixing of fuel and air.

Effect of mixture distribution on fuel consumption.—In a multicylinder engine, operation with nonuniform distribution results in higher fuel consumption. The relation between brake specific fuel consumption and fuel-air ratio (fig. IX-8) is such that the over-all brake specific fuel consumption for nonuniform mixture distribution is necessarily greater than that for uniform distribution. Calculations (reference 1) indicated, however, that the difference is more pronounced when the poor distribution is obtained at a lean over-all fuel-air ratio. If mixture distribution is uniform, the brake specific fuel consumption of the entire engine is the same as that of each cylinder inasmuch as the fuel-air ratio for each cylinder is equal to that of the engine average.

If the extent to which the over-all fuel-air ratio may be reduced is (because of detonation or cooling) determined by the leanest cylinder, the effect of nonuniform mixture distribution is important. The relation between the fuel-air ratio of the leanest cylinder and the over-all fuel-air ratio is presented in figure IX-9. Also shown is the curve representing uniform distribution. At any given value of the abscissa

the difference between the two curves represents the increase in over-all fuel-air ratio required by nonuniform distribution to attain a given value of fuel-air ratio for the leanest cylinder.

Inasmuch as the mixture distribution is more nearly uniform at low over-all fuel-air ratios (fig. IX-2), the differences between the over-all fuel-air ratio and that of the leanest cylinder (fig. IX-9) decrease with decreasing mixture strength.

IMPROVEMENT OF MIXTURE DISTRIBUTION BY MECHANICAL CHANGES

One approach to the problem of elimination of nonuniform distribution is through the use of improved induction devices to permit introduction of more nearly vaporized fuel to the intake manifold. An investigation (reference 4) conducted by the NACA during World War II was concerned with the development of an injection-type impeller that would improve the uniformity of fuel-air mixture and thus promote more uniform distribution to the engine cylinders.

Injection impeller.—The variation in fuel-air ratio among the engine cylinders is influenced by centrifugal and gravitational separation of fuel droplets from the air as well as by coarse, nonuniform injection of fuel into the air stream. Consequently, an improvement in distribution can be achieved by injecting the fuel at a point where the fuel droplets are least likely to separate from the combustion air and at the same time by eliminating the coarse droplets present in the charge-air stream.

The impeller developed in reference 4 was designed to avoid the causes of nonuniform distribution by injecting fuel near the impeller outlet, where elbow and carburetor disturbances are at a minimum and where high velocity and turbulent conditions might be utilized to reduce the effect

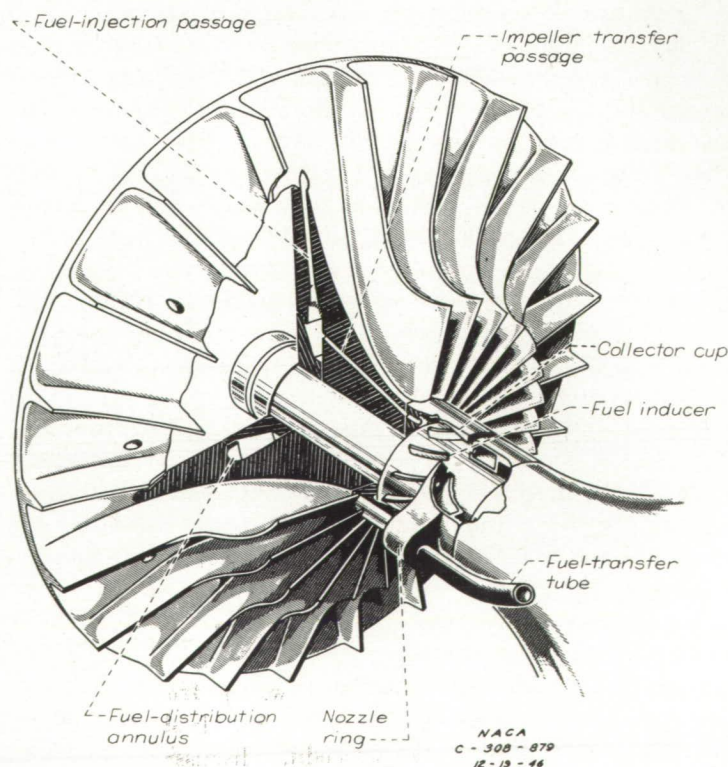


FIGURE IX-10.—Injection impeller. (Fig. 2 of reference 5.)

of gravitational forces and to provide better mixing and fuel evaporation. The impeller (fig. IX-10) is a centrifugal impeller modified to act as a fuel distributor as well as a supercharger. The fuel passages discharge from a centrally located supply chamber into the air passages at a point sufficiently near the impeller tip to avoid fuel impingement on the stationary shroud. Fuel is discharged with a centrifugal force that exceeds the gravitational forces; consequently, a uniform peripheral fuel discharge from the impeller is attained.

Metered fuel passes from the carburetor to a stationary nozzle ring, instead of feeding to the conventional carburetor spray bar, and is delivered from the ring into a collector cup that rotates with the impeller. Between the nozzle ring and the collector cup is an air gap which eliminates surging in the system. The fuel flows by centrifugal action through the collector cup and the impeller transfer passages to the fuel-distribution annulus. From the annulus the fuel is thrown through the radial holes into the air stream.

A comparison of the mixture distribution resulting from

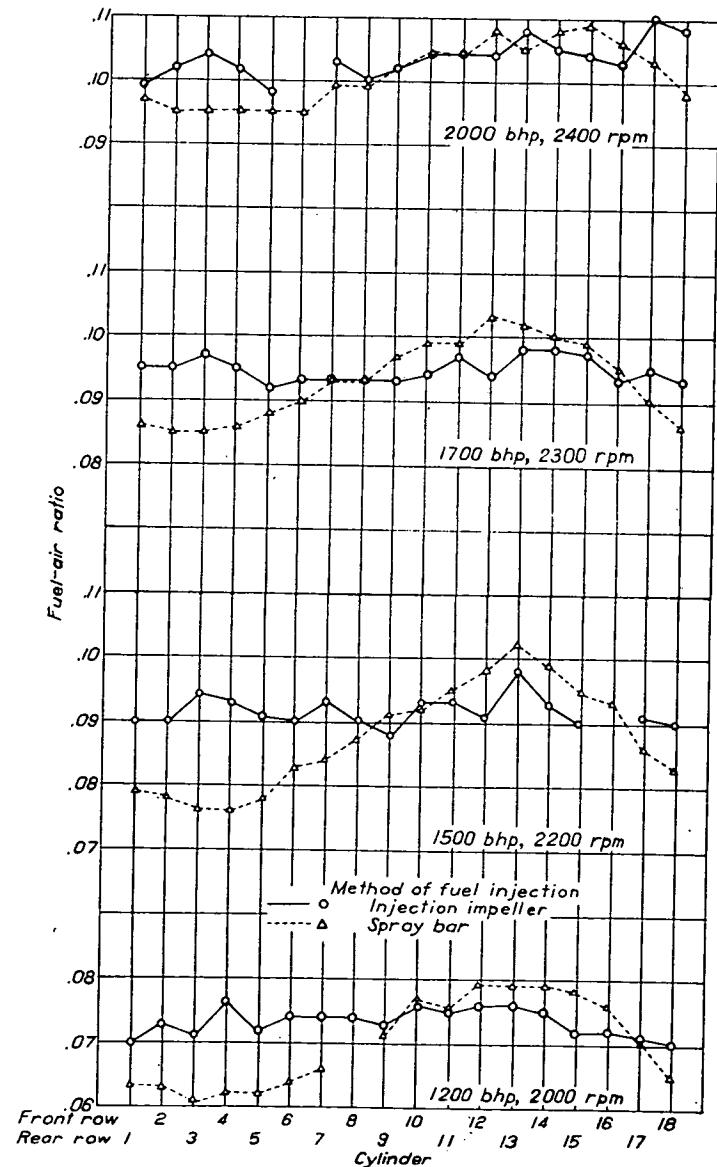


FIGURE IX-11.—Effect of injection impeller on mixture distribution of engine at various engine powers and speeds. (Fig. 6 of reference 4.)

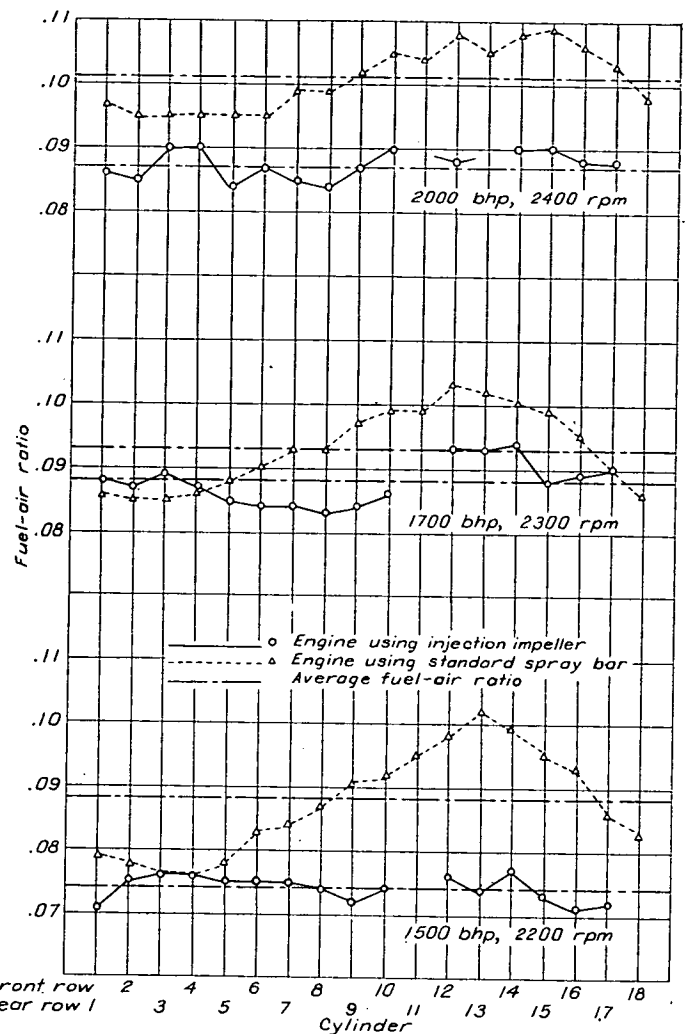


FIGURE IX-12.—Effect of injection impeller on mixture distribution of engine at reduced fuel flows. (Fig. 7 of reference 4.)

a standard carburetor spray bar with that from the injection impeller is presented in figure IX-11. A summary of the data from this figure is given in table IX-1.

At 1200, 1500, and 1700 brake horsepower (table IX-1), the mixture spread was considerably less with the injection impeller than with the standard carburetor spray bar; however, at 2000 brake horsepower the improvement was less noticeable. The poorer showing at 2000 horsepower is attributed in reference 4 to the fact that the impeller does not operate as satisfactorily at the higher fuel flows, whereas

TABLE IX-1.—EFFECT OF INJECTION IMPELLER ON MIXTURE DISTRIBUTION *

Brake horsepower	Fuel-air ratio							
	1200		1500		1700		2000	
Method of fuel injection	Stand-ard spray bar	Injection impeller	Stand-ard spray bar	Injection impeller	Stand-ard spray bar	Injection impeller	Stand-ard spray bar	Injection impeller
Maximum	0.079	0.076	0.102	0.098	0.103	0.098	0.109	0.110
Minimum	0.061	0.070	0.076	0.088	0.085	0.092	0.095	0.098
Spread	0.018	0.006	0.026	0.010	0.018	0.006	0.014	0.012

* Reproduced from reference 4.

the spray bar apparently provides better mixture distribution at the higher flows.

A comparison of the mixture distribution at reduced fuel flows for the injection impeller with the mixture distribution at higher fuel flows for the standard spray bar is shown in figure IX-12. A summary of the data from this figure is given in table IX-2.

For 1500 and 2000 brake horsepower (table IX-2), the variations in mixture distribution with the impeller at reduced fuel flows are much smaller than the variations at higher fuel flows (table IX-1). At 1700 brake horsepower the variations are greater. On the basis of these data it was suggested (reference 4) that the ability of the injection impeller to overcome the effects of gravitational and centrifugal fuel separation is inhibited at high powers and fuel flows

TABLE IX-2.—EFFECT OF INJECTION IMPELLER ON MIXTURE DISTRIBUTION AT REDUCED FUEL FLOWS *

Fuel-air ratio						
Brake horsepower	1500		1700		2000	
Method of fuel injection	Standard spray bar	Injection impeller	Standard spray bar	Injection impeller	Standard spray bar	Injection impeller
Maximum.....	0.102	0.077	0.103	0.094	0.109	0.090
Minimum.....	0.076	0.071	0.085	0.083	0.095	0.084
Spread.....	0.026	0.006	0.018	0.011	0.014	0.006

* Reproduced from reference 4.

by some design factor, the effects of which are eliminated at reduced values of fuel flow.

EFFECTS OF THREE FUEL-INJECTION METHODS ON MIXTURE DISTRIBUTION WITH LOW-VOLATILITY FUEL

In order to compare the effects of three methods of fuel injection on engine performance, an investigation (reference 5) was conducted utilizing low-volatility fuel (so-called safety fuel). Low-volatility fuel has long been considered as a substitute fuel for conventional aviation gasoline to reduce the fire hazard in aircraft. Conventional gasoline, because of its high vapor pressure (Reid vapor pressure, 7 lb/sq in.), is much more easily vaporized in an induction system than is low-volatility fuel (Reid vapor pressure, 0.1 lb/sq in.); consequently, safety fuel offers a more rigid test of injection methods. Physical properties for gasoline and low-volatility fuel (reference 5) are compared in the following table:

Property	Gasoline	Low-volatility fuel
Grade.....	100/130	106/139
Tetraethyl lead content, ml/gal.....	4.55	4.63
Flash point, close cup, °F.....	Below -30	122
Distillation range, °F:		
Initial boiling point.....	108	320
10-percent evaporated.....	141	334
50-percent evaporated.....	205	346
90-percent evaporated.....	255	362
Final boiling point.....	332	384
Freezing point, °F.....	Below -76	Below -76
Reid vapor pressure, lb/sq in.....	7	0.1
Residue, copper dish, mg/100 ml.....	2.0	2.0
Accelerated gum content, mg/100 ml.....	2.3	2.0
Accelerated gum precipitate, mg/100 ml.....	0.2	0.6
Sulfur content, percent.....	0.012	0.0001
Heating value, Btu/lb.....	18740	18650
Hydrogen-carbon ratio.....	0.166	0.165
Specific gravity at 60° F.....	0.719	0.782

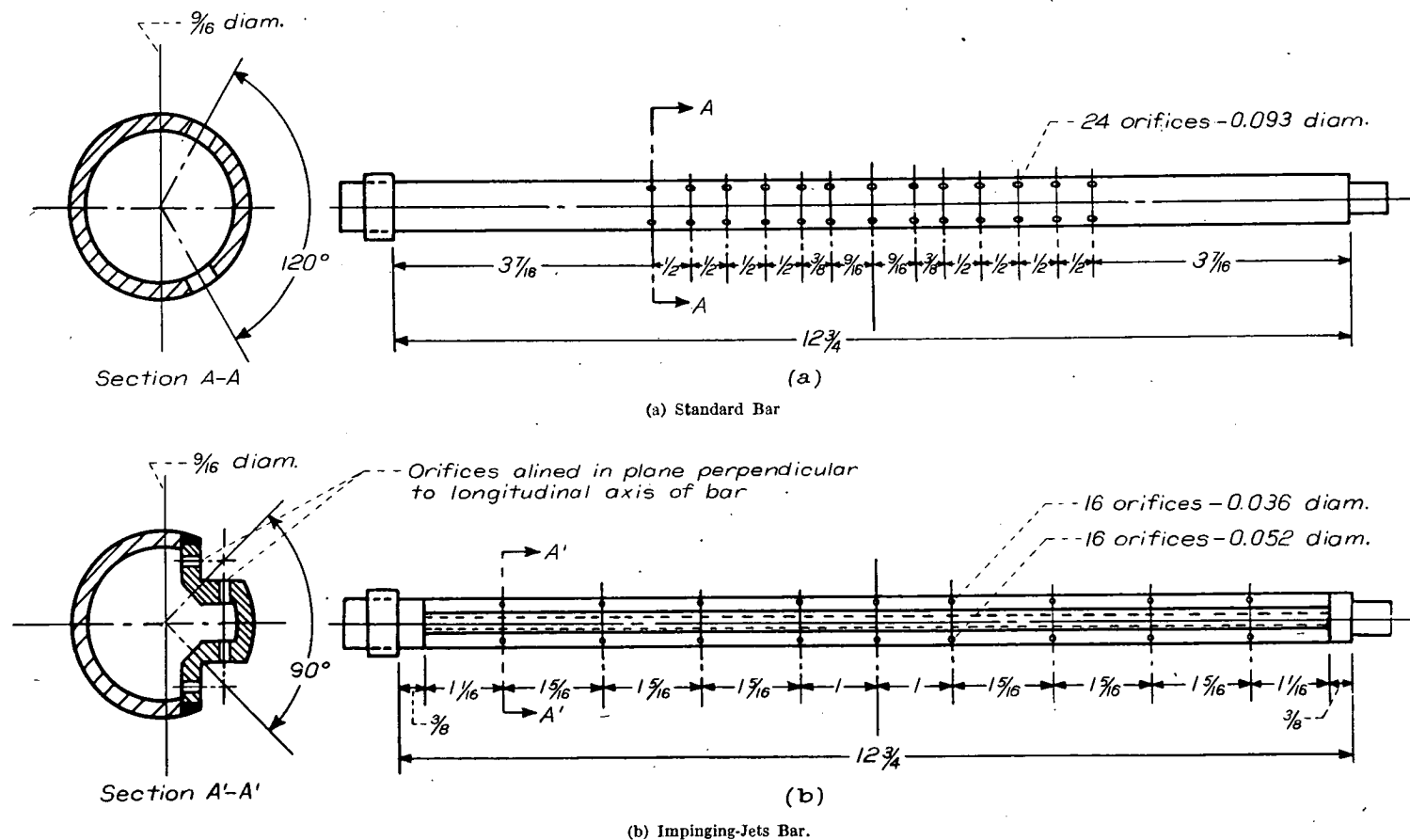
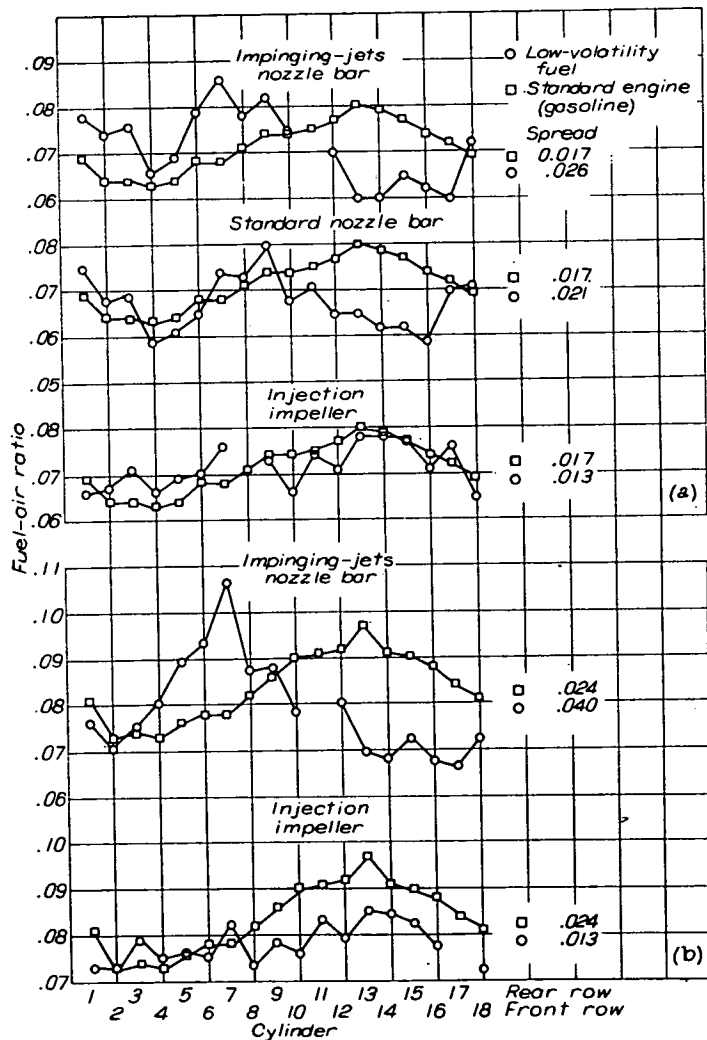


FIGURE IX-13.—Comparison of standard nozzle bar with impinging-jets nozzle bar. (All dimensions given in inches.) (Fig. 3 of reference 5.)

Previous investigations (references 6 and 7) have shown that low-volatility fuel can be utilized successfully in reciprocating engines by direct injection into the cylinders. However, the complexities of a direct injection system can be avoided by better atomization of fuel and injection into the manifold. Aside from the injection impeller discussed in the preceding paragraphs, the NACA has investigated the impinging-jets nozzle bar as a means of achieving better fuel atomization. With the injection impeller, finely dispersed fuel enters the combustion-air stream near the impeller exit, whereas the impinging-jets nozzle bar introduces the finely atomized fuel into the air stream immediately after the carburetor. A standard nozzle bar and the impinging-jets bar are compared in figure IX-13. In the standard bar are located 24 orifices as compared to 32 for the impinging-jets bar. The orifices in the standard bar are larger and the fuel is discharged in less than half the length of the bar. These factors result in poorer atomization and only partial coverage of the combustion-air duct. The orifices in the impinging-jets bar are at right angles to each other and the sprays impinge to form finely atomized fuel.



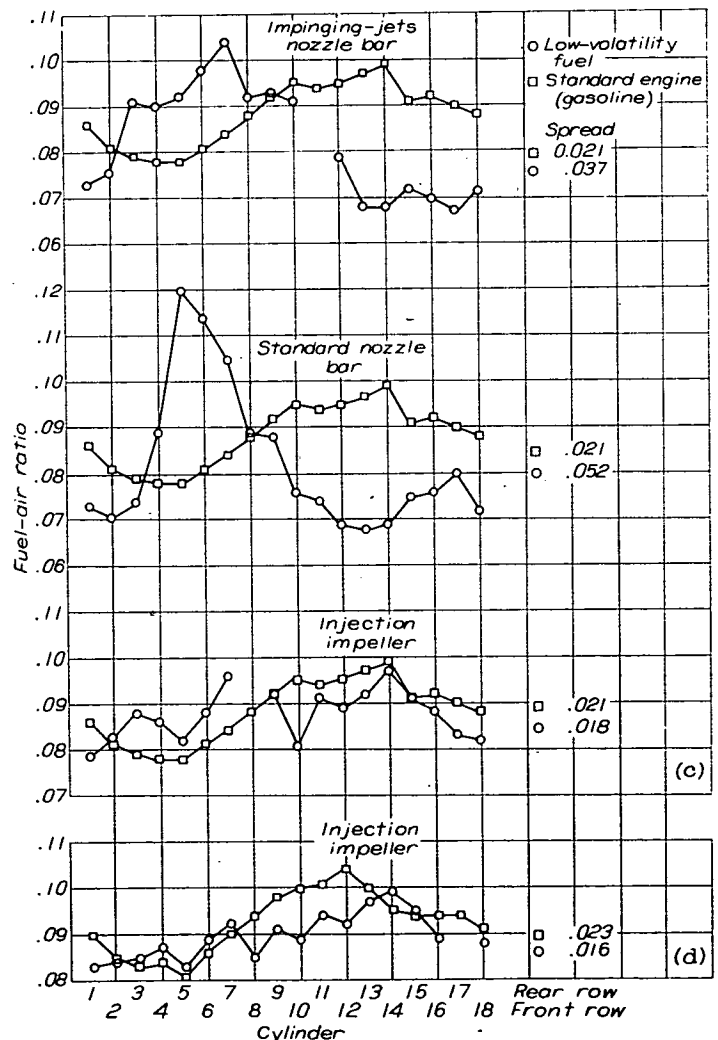
(a) Engine speed, 2000 rpm; engine power, 1200 brake horsepower; approximate average fuel-air ratio, 0.070.
(b) Engine speed, 2100 rpm; engine power, 1400 brake horsepower; approximate average fuel-air ratio, 0.082.

FIGURE IX-14.—Comparison of mixture distribution obtained with various methods of fuel injection. Combustion-air temperature, 100° F. (Fig. 6 of reference 5.)

In order to evaluate the three methods of injecting low-volatility fuel, a standard mixture-distribution pattern was selected. This pattern was produced at various powers by injection of gasoline through the standard nozzle bar. Comparisons of mixture-distribution patterns for the three methods of injecting low-volatility fuel with the pattern for the standard gasoline test are shown in figure IX-14. The spreads in fuel-air ratio obtained in the standard engine were 0.017, 0.024, 0.021, and 0.023 at 1200, 1400, 1600, and 1800 brake horsepower, respectively.

At 1200 brake horsepower (fig. IX-14(a)) the injection impeller gave a distribution pattern similar to that of the standard engine; the over-all spread in fuel-air ratio was slightly less than that of the standard engine. The other methods gave patterns somewhat different from those of the standard engine in that the over-all spreads in fuel-air ratio were greater.

At 1400 brake horsepower (fig. IX-14(b)) the injection impeller again gave better distribution with low-volatility fuel than the standard engine did with gasoline; however, the impinging-jets bar was worse than the standard engine.



(c) Engine speed, 2200 rpm; engine power, 1600 brake horsepower; approximate average fuel-air ratio, 0.087.
(d) Engine speed, 2300 rpm; engine power, 1800 brake horsepower; approximate average fuel-air ratio, 0.091.

FIGURE IX-14.—Concluded. Comparison of mixture distribution obtained with various methods of fuel injection. Combustion-air temperature, 100° F. (Fig. 6 of reference 5.)

Similar results were obtained at brake horsepowers of 1600 and 1800 (figs. IX-14(c) and IX-14(d)). Complete data were not obtained at 1800 brake horsepower inasmuch as this power was unattainable with either the standard nozzle bar or the impinging-jets bar.

It can be concluded from the foregoing data that the mixture-distribution patterns produced by the injection impeller with low-volatility fuel are better than those found with a standard nozzle bar and gasoline as a fuel.

EFFECTS OF FUEL VOLATILITY ON ENGINE PERFORMANCE

The effect of fuel volatility on engine performance was investigated by White and Engelman (reference 8). Four fuels were tested over a range of inlet-air temperatures between 46° F and 72° F. In general the results of this study were inconclusive; however, the following summary is offered.

The fuel having the lowest 90-percent point (A. S. T. M. Distillation Procedure D86-40) gave the best power and fuel economy. Differences in performance among the other fuels were relatively slight and could not be attributed to differences in volatility. The 90-percent points of the four fuels were 255°, 270°, 295°, and 306° F. The most volatile fuels were found to give more uniform mixture distribution.

FUEL VAPOR LOSS FROM AIRCRAFT FUEL TANKS

The development of high-altitude, long-range aircraft has resulted in considerable concern over the loss of fuel vapor through fuel tank vents during flight. The loss of fuel represents an increase in the fuel consumed during flight and thus causes a reduction in cruising range and load-carrying capacity.

The variables affecting fuel vapor loss were evaluated in a series of simulated-flight tests (reference 9). A small fuel tank on a bench-test installation (fig. IX-15) was employed to facilitate instrumentation and handling of the equipment during the tests. A similar apparatus was installed in a twin-engine airplane in order to correlate simulated-flight data with actual-flight data. (See fig. IX-16.) Details of the apparatus may be found in reference 9.

Effect of rate of climb on fuel vapor loss.—The fuel in the tank (fig. IX-15) was subjected to simulated flights at rates of climb of 1000, 2000, and 4000 feet per minute to an altitude of 40,000 feet; the results are shown in figure IX-17. These data are replotted in figure IX-18 with fuel vapor loss as a function of altitude. At any given altitude the loss increased only slightly with increased rate of climb for the rates tested; however, the losses due to foaming at the higher rates of climb when the fuel tank is filled close to capacity were not investigated.

Effect of altitude on fuel vapor loss.—The variation of fuel vapor loss with altitude in figure IX-18 is linear above some critical altitude (the theoretical altitude at which fuel vapor loss begins). For this linear portion of the curve the following formula was derived (reference 9):

$$L = \frac{Z - Z_c}{1.9}$$

where

L fuel vapor loss, percent

Z altitude, in 1000 feet

Z_c critical altitude, in 1000 feet (intersection of linear portion of loss-against-altitude curve with base line, fig. IX-18)

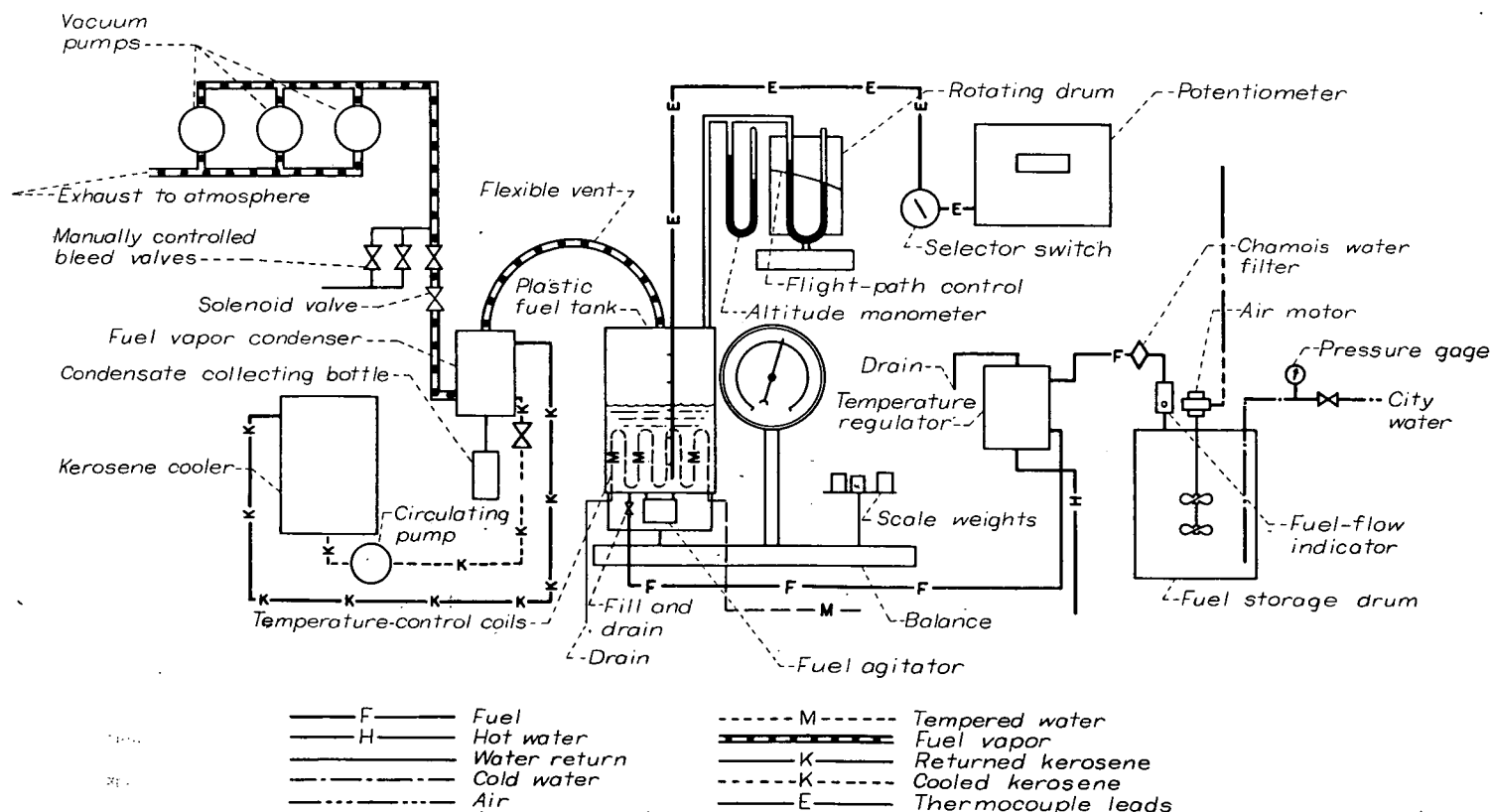


FIGURE IX-15.—Bench-test installation for determination of fuel vapor loss under simulated flight conditions. (Fig. 1 of reference 9.)

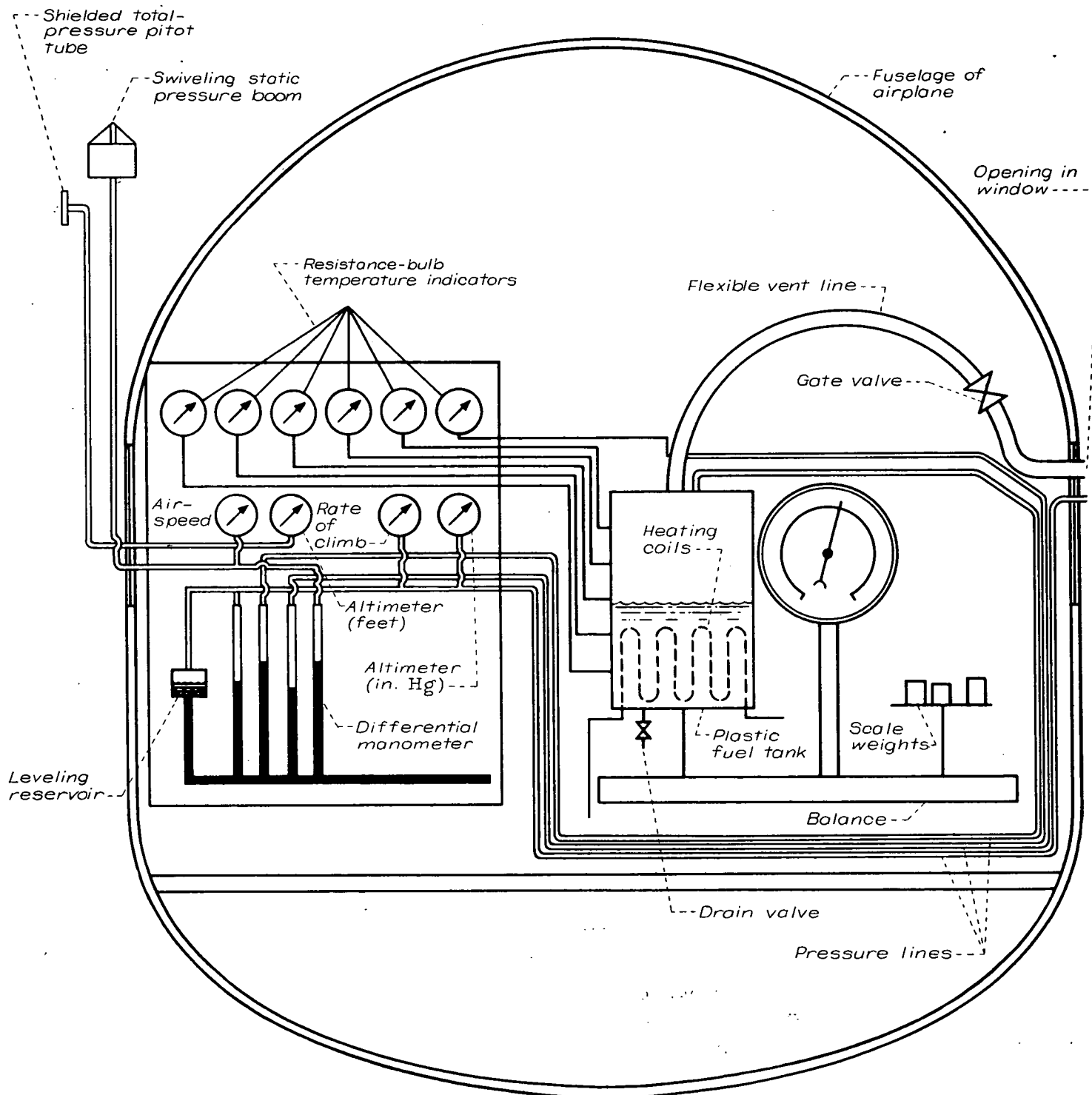


FIGURE IX-16. Flight-test installation for determination of fuel vapor loss. (Fig. 2 of reference 9.)

Effects of initial fuel temperature on fuel vapor loss.—

The effect of initial fuel temperature on fuel vapor loss is shown in figure IX-19. These results were obtained from simulated flight tests in which the condition of climb to 30,000 feet was investigated for each of the initial fuel temperatures shown on the figure. These data have been replotted in figure IX-20 to illustrate the variation of fuel vapor loss at the end of climb with initial fuel temperature. Above a fuel temperature of about 70° F the loss varies linearly with fuel temperature. It was suggested in reference 9 that the fuel vapor loss during climb to an altitude of

30,000 feet at initial fuel temperatures above 70° F could possibly be predicted from the following equation:

$$L = K(T - T_c)$$

where

L fuel vapor loss, percent

T initial fuel temperature, °F

T_c temperature above which loss varies linearly with temperature (intersection of linear portion of curve in fig. IX-20 with abscissa), °F.

K constant (0.18 percent per °F, from fig. IX-20)

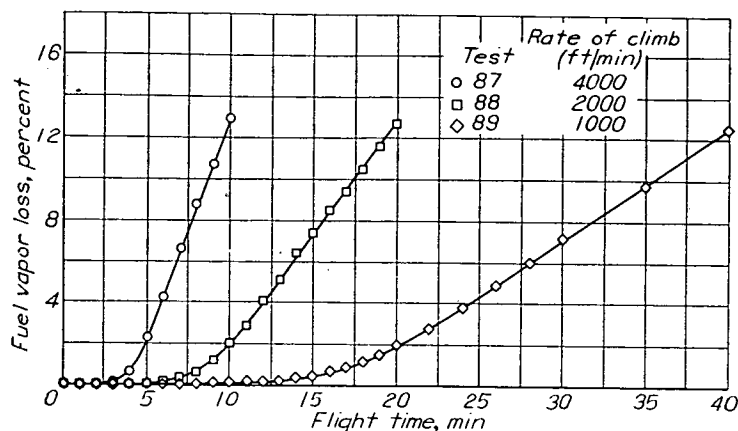


FIGURE IX-17.—Fuel vapor loss during simulated flights to 40,000 feet at various rates of climb. (Fig. 7 of reference 9.)

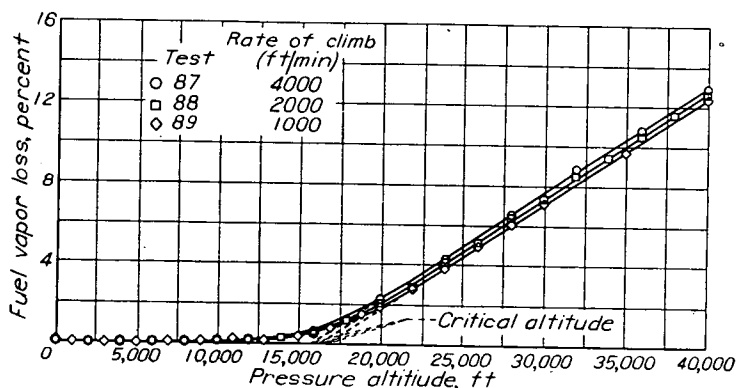


FIGURE IX-18.—Variation of fuel vapor loss with altitude during simulated flights at various rates of climb. (Fig. 8 of reference 9.)

Effect of fuel weathering on fuel vapor loss.—Several successive standard simulated flights were made on the same tank of fuel in order to determine the effect of weathering on vapor loss. (See fig. IX-21.) In this particular test the total volume loss of fuel resulting from the three simulated flights was about 20 percent. The effects of such losses on fuel properties are discussed later in this chapter.

Effects of fuel agitation on fuel vapor loss.—An attempt was made in reference 9 to determine the effects of agitation on vapor loss. The effects of low-amplitude vibrations were investigated by vibrating the tank vertically during a simulated flight. The tank was vibrated at frequencies of 168 and 120 cycles per second at amplitudes of 0.0009 and 0.0018 inch, respectively. An air-operated vibrator attached to the tank was used for this purpose. This type of vibration had no significant effect on fuel vapor loss.

The effect of turbulence on vapor loss was investigated (reference 9) by use of a three-bladed propeller installed in the tank. This mechanism simulated the turbulence arising from use of submerged fuel booster pumps. Two series of tests were made: one in which the fuel was thrust downward, and another in which the fuel was thrust upward. The results are shown in figure IX-22. For the range of speeds investigated the fuel vapor loss increased with speed irrespective of the direction of thrust.

A third method of agitation was investigated in which the fuel tank was oscillated to simulate sloshing of the fuel during flight. The tank was rocked through an angle of 5° at rates

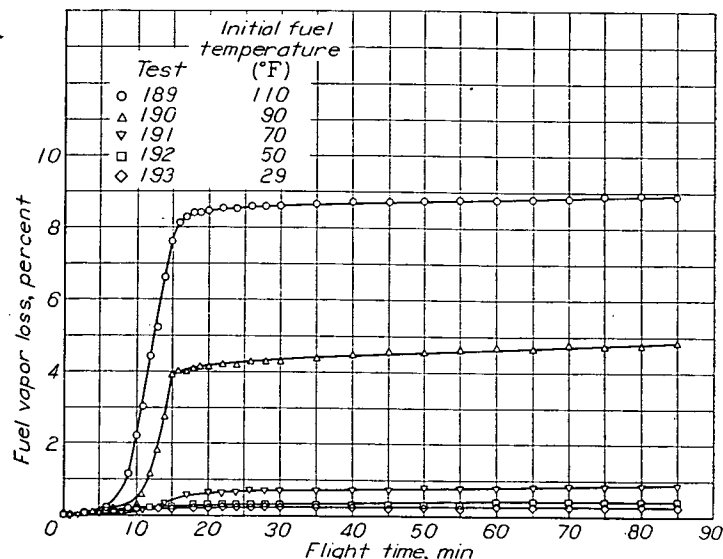


FIGURE IX-19.—Variation of fuel vapor loss with initial fuel temperature, during simulated flight. (Fig. 11 of reference 9.)

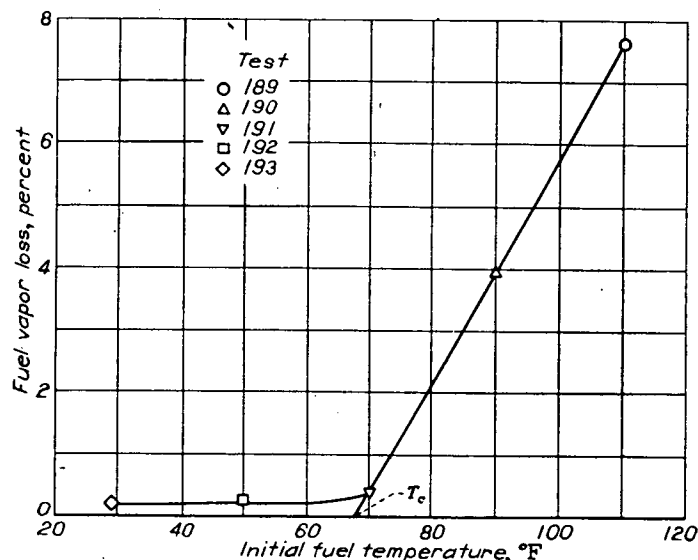


FIGURE IX-20.—Variation of fuel vapor loss at end of climb period with initial fuel temperature during simulated flight. (Fig. 12 of reference 9.)

of 40 and 60 cycles per minute. Fuel vapor loss was increased (fig. IX-23) by this method of agitation.

Flight correlation.—A comparison was made between the flight and bench-test apparatuses used for determination of fuel vapor loss. (See reference 9.) The flight paths followed in these tests are shown in figure IX-24. In both cases the initial fuel temperature was 108° F. The fuel vapor loss resulting from each of the two methods is shown in figure IX-25. The difference in vapor loss measured by the two methods was about 0.06 percent.

EFFECT OF FUEL VAPOR LOSS ON INSPECTION PROPERTIES OF AVIATION FUELS

The effects of fuel vapor loss on the properties of two typical service fuels (AN-F-28 (28-R) and AN-F-33 (33-R)) were investigated in reference 10. These fuels were weathered in a simulated-altitude apparatus shown diagrammatically in figure IX-26. This apparatus represents an improvement over the apparatus described in the preceding section of this chapter. (See fig. IX-15.)

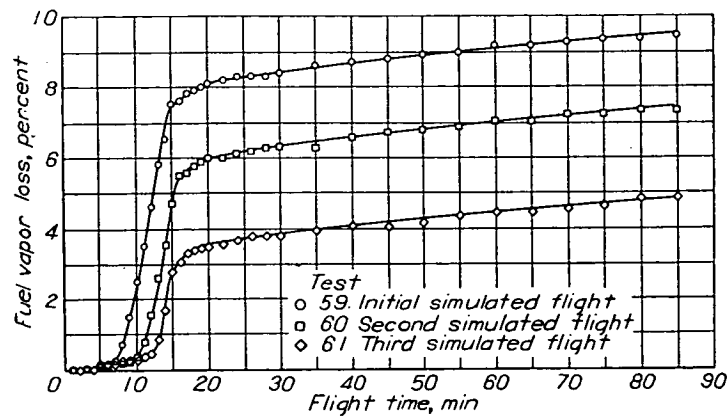


FIGURE IX-21.—Fuel vapor loss during successive standard simulated flights with given fuel sample. (Fig. 13 of reference 9.)

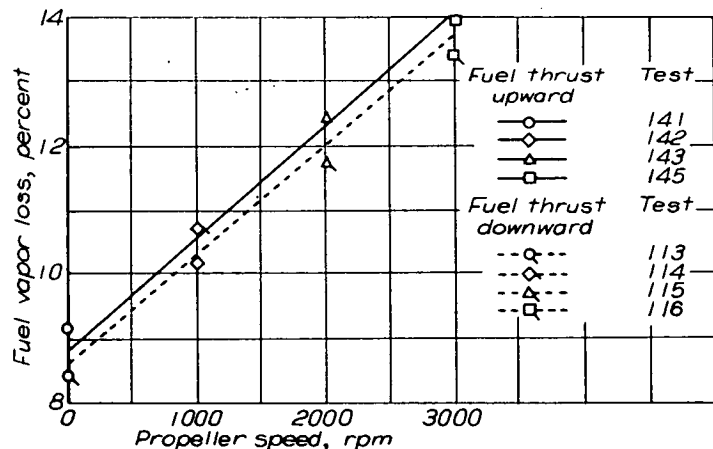


FIGURE IX-22.—Fuel vapor loss at end of standard simulated flight with induced fuel turbulence produced by rotating propeller with blade angle at 30°. (Fig. 17 of reference 9.)

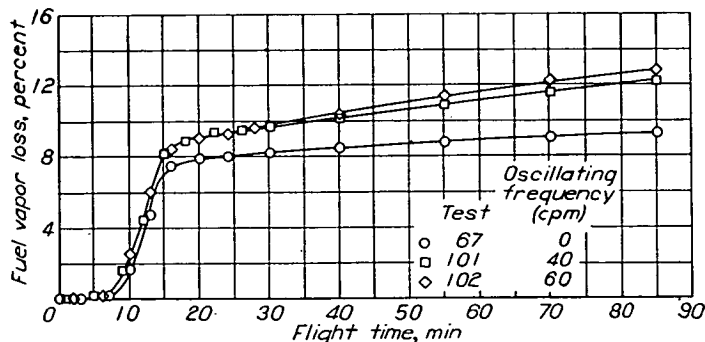


FIGURE IX-23.—Fuel vapor loss during standard simulated flight with fuel tank oscillated through angle of 5°. (Fig. 18 of reference 9.)

The fuels were weathered during a simulated flight consisting of climb at a rate of 1000 feet per minute to an altitude of 30,000 feet. The altitude of 30,000 feet was maintained for approximately 10 minutes after the end of the climb. Because fuel vapor loss is dependent on the temperature (fig. IX-19) of the fuel in an airplane tank at the time of take-off, two initial fuel temperatures (90° and 130° F) were used in the weathering tests of each fuel.

The data obtained from the weathering tests indicated that with 28-R about 3.6 percent (by weight) of the fuel was lost during a simulated flight in which the initial fuel temper-

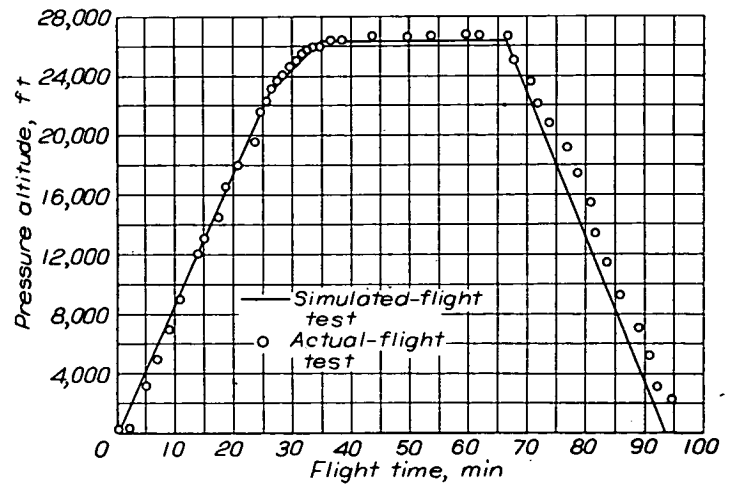


FIGURE IX-24.—Flight paths followed for both simulated and actual flight tests. (Fig. 19 of reference 9.)

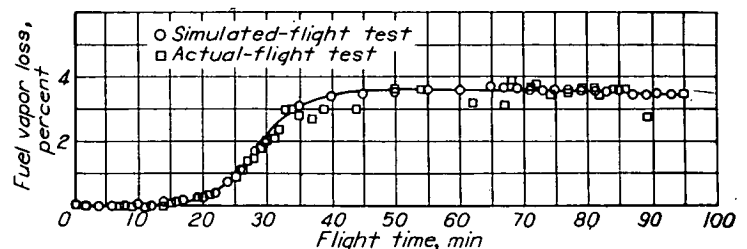


FIGURE IX-25.—Fuel vapor loss during simulated and actual flights. (Fig. 20 of reference 9.) Initial fuel temperature, 108° F.

ature was 90° F. For an initial temperature of 130° F the loss was about 12.8 percent. With 33-R the losses were about 3.5 and 14.3 percent at temperatures of 90° and 130° F, respectively.

Inspection data for both weathered and unweathered fuel samples are shown in table IX-3. The data in this table show that, as a result of the weathering loss, the distillation temperatures were increased and the Reid vapor pressures decreased. The greatest increase in distillation temperature occurred in the low temperature range. Specific gravities, aromatic concentrations, and tetraethyl lead concentrations were increased.

If the data for the weathered samples are compared with the specification limits, it is seen that the samples of 28-R and 33-R fuels, weathered from an initial temperature of 90° F, meet the requirements with the exception of lead concentrations. The 50-percent-evaporated temperature for 33-R is about 3° higher than permitted, but this difference is within the precision of the A. S. T. M. distillation procedure.

For an initial fuel temperature of 130° F the weathered sample of 28-R still meets the requirements with the exception of tetraethyl lead concentration, whereas 33-R is unacceptable because of the high 50-percent-evaporated temperature as well as the high tetraethyl lead concentration. Both weathered samples have a Reid vapor pressure of 4.6 pounds per square inch, which is a lower value than that of most aviation fuels. Under current aviation-fuel specifications, a low Reid vapor pressure is permissible as long as the low-end distillation temperatures are within the specified limits.

TABLE IX-3.—INSPECTION DATA FOR WEATHERED AND UNWEATHERED SAMPLES OF 28-R AND 33-R FUELS *

	28-R				33-R			
	AN-F-28 specifications	Unweathered	Weathered		AN-F-33 specifications	Unweathered	Weathered	
			^b 90° F	^b 130° F			^b 90° F	^b 130° F
Tetraethyl lead, ml/gal.	4.6 (max.)	4.61	4.81	5.49	4.6 (max.)	4.53	4.72	5.29
Specific gravity 60°/60° F.	-----	0.725	0.729	0.739	-----	0.708	0.710	0.718
Reid vapor pressure, lb/sq in.	7.0 (max.)	6.6	6.0	4.6	7.0 (max.)	6.6	5.9	4.6
Aromatics, percent by volume.	-----	15.1	15.3	17.7	-----	7.8	7.7	9.3
A. S. T. M. distillation								
Percentage evaporated	Temperature, ° F							
0.	167 (max.)	109	108	117	167 (max.)	103	104	116
10.	167 (min.)	137	142	160	167 (min.)	134	140	163
40.	221 (max.)	194	200	213	221 (max.)	196	204	220
50.	221 (min.)	213	217	225	221 (min.)	219	224	232
90.	284 (max.)	274	276	283	275 (max.)	272	274	279
End point.	356 (max.)	322	326	330	356 (max.)	344	349	357
Sum of 10- and 50-percent points.	307 (min.)	350	359	385	307 (min.)	353	364	395
Residue, percent.	1.5 (max.)	0.6	0.6	0.4	1.5 (max.)	0.9	0.9	0.8
Loss, percent.	1.5 (max.)	1.4	1.4	0.8	1.5 (max.)	1.1	1.1	0.7

* Table I of reference 10.

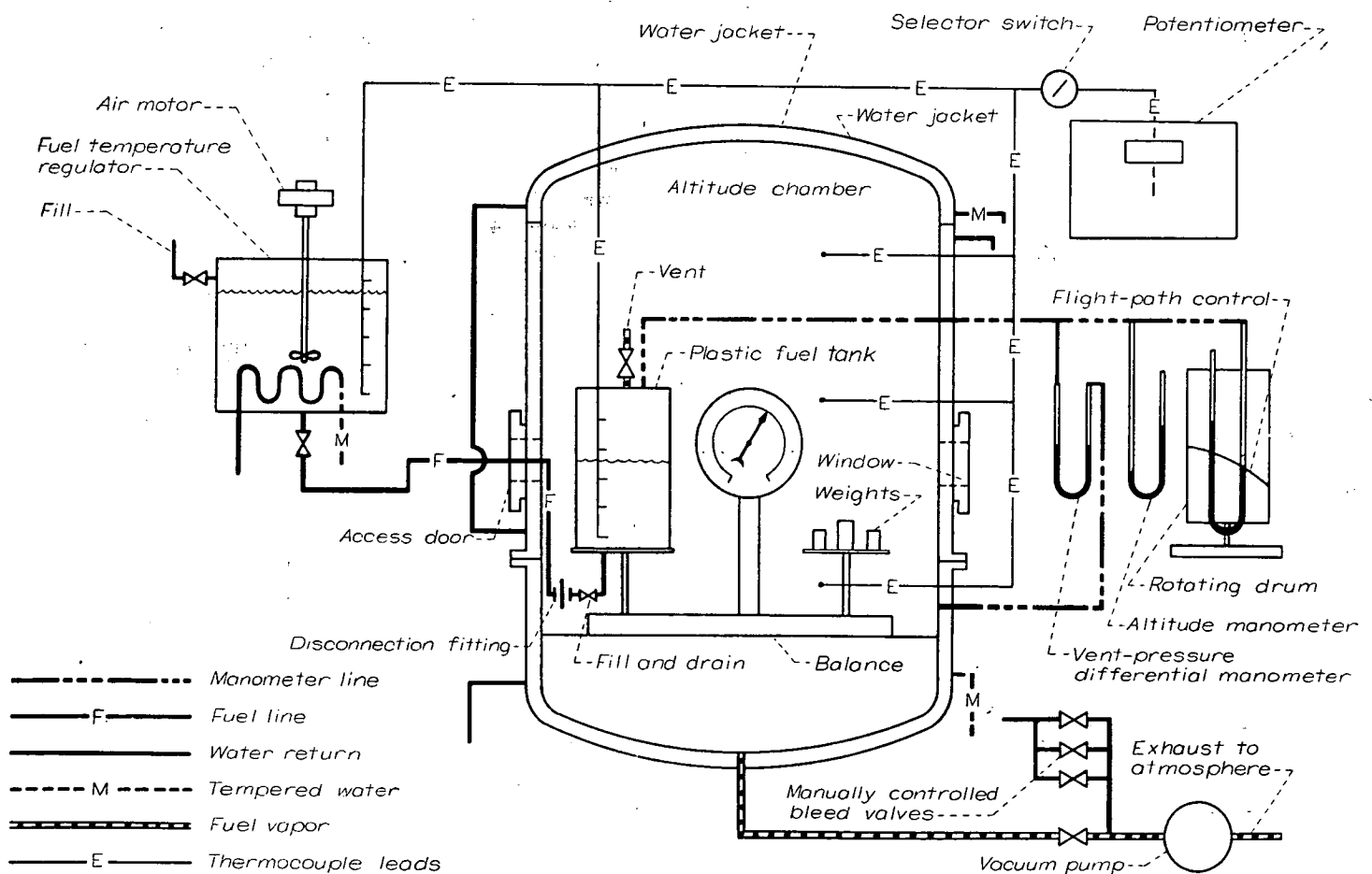
^b Temperature of fuel at start of simulated flight.

FIGURE IX-26.—Simulated-altitude bench-test installation for determination of fuel vapor loss. (Fig. 1 of reference 10.)

In an earlier investigation (reference 11) an equation was derived to permit estimation of fuel vapor loss by change in specific gravity.

$$L = K(A - A_0)$$

where

- L percentage fuel loss
 A final specific gravity of fuel at 20° C (68° F)
 A_0 initial specific gravity of fuel at 20° C (68° F)
 K constant, characteristic of each fuel

The values of the constant K were evaluated (reference 11) for the six fuels used in this study and were plotted against the initial specific gravity (A_0) in figure IX-27.

EFFECT OF FUEL VAPOR LOSS ON KNOCK-LIMITED PERFORMANCE

The A. S. T. M. Aviation (lean) and A. S. T. M. Supercharge (rich) antiknock ratings for 28-R and 33-R are shown in the following table (reference 10). Two rows of ratings are given for each fuel. The first row is milliliters of tetraethyl lead per gallon in S reference fuel and the second row is performance number. The A. S. T. M. Aviation ratings determined for unweathered 28-R and 33-R are higher than the nominal ratings for these fuels.

Fuel	Condition	A. S. T. M. Aviation rating	A. S. T. M. Supercharge rating
28-R	Nominal rating.....	0	1.28
28-R	Unweathered.....	100	130
28-R	Weathered (*90° F).....	0.08	1.31
28-R	Weathered (*130° F).....	103	130
33-R	Nominal rating.....	0.10	1.31
33-R	Unweathered.....	104	130
33-R	Weathered (*90° F).....	0.07	1.52
33-R	Weathered (*130° F).....	103	133
33-R	Nominal rating.....	0.47	2.78
33-R	Unweathered.....	115	145
33-R	Weathered (*90° F).....	0.68	2.68
33-R	Weathered (*130° F).....	120	144
33-R	Weathered (*90° F).....	0.75	2.67
33-R	Weathered (*130° F).....	121	144
33-R	Weathered (*130° F).....	0.75	2.92
33-R	Weathered (*130° F).....	121	146

*Temperature of fuel at start of simulated flight.

The data indicate that the loss of fuel vapor resulting from weathering has little or no effect on the ratings of the two fuels. If the changes in ratings can be assumed to be significant, the A. S. T. M. Aviation and A. S. T. M. Supercharge performance numbers of both fuels increase slightly. These comparisons should be valid inasmuch as the data were obtained on the same operating day.

REFERENCES

- Marble, Frank E., Butze, Helmut F., and Hickel, Robert O.: Study of the Mixture Distribution of a Double-Row Radial Aircraft Engine. NACA ARR E5105, 1945.
- Gerrish, Harold C., and Meem, J. Lawrence, Jr.: The Measurement of Fuel-Air Ratio by Analysis of the Oxidized Exhaust Gas. NACA Rep. 757, 1943.

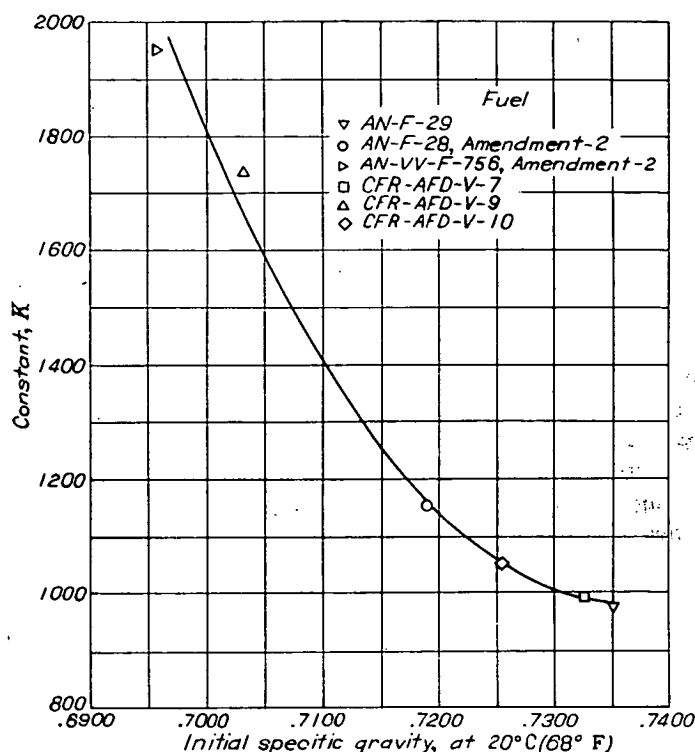


FIGURE IX-27.—Variation of constant K with initial specific gravity of fuel. (Fig. 6 of reference 11.)

- Cook, Harvey A., and Olson, Walter T.: Small-Orifice Tubes for Minimizing Dilution in Exhaust-Gas Samples. NACA ARR, Feb. 1943.
- Marble, Frank E., Ritter, William K., and Miller, Mahlon A.: Effect of the NACA Injection Impeller on the Mixture Distribution of a Double-Row Radial Aircraft Engine. NACA Rep. 821, 1945. (Formerly TN 1069.)
- Michel, Donald J., Hickel, Robert O., and Voit, Charles H.: Performance of a Double-Row Radial Aircraft Engine with Three Methods of Safety-Fuel Injection. NACA TN 1413, 1947.
- Schey, Oscar W., and Young, Alfred W.: Performance of a Fuel-Injection Spark-Ignition Engine Using a Hydrogenated Safety Fuel. NACA Rep. 471, 1933.
- Schey, Oscar W.: Performance of Aircraft Spark-Ignition Engines With Fuel Injection. SAE Jour., vol. 46, no. 4, April, 1940, pp. 166-176.
- White, H. Jack, and Engelman, Helmut M.: Effect of Fuel Volatility on Performance of a Wright R-2600-8 Engine as Influenced by Mixture Distribution. NACA ARR E4105, 1944.
- Stone, Charles S., Baker, Sol, and Black, Dugald O.: Flight Variables Affecting Fuel-Vapor Loss from a Fuel Tank. NACA MR E4L28, 1944.
- Barnett, Henry C., and Marsh, Edred T.: Effects of Fuel-Vapor Loss on Knock-Limited Performance and Inspection Properties of Aviation Fuels. NACA RB E6C01, 1946.
- Stone, Charles S., and Kramer, Walter E.: Fuel-Vaporization Loss as Determined by the Change in the Specific Gravity of the Fuel in an Aircraft Fuel Tank. NACA RB E5E19, 1945.

CHAPTER X

INTERNAL COOLING

Over a period of years, improvements in fuel performance have made it necessary to improve engine-cooling facilities in order to take full advantage of power potentialities of a given fuel. Cooling studies (reference 1) indicate that more and more heat must be disposed of through the engine walls as the specific power output of the engine is increased. If adequate cooling is assumed to be solely dependent on heat transfer through the cylinder walls, the search for better engine cooling is a continuing process accompanying the ever-increasing antiknock quality of conventional fuels. Although these comments pertain primarily to air-cooled engines, the same might be said of liquid-cooled engines in regard to radiator size and cooling drag.

Complete dependency on heat transfer through cylinder walls for adequate cooling can, however, be avoided by a method commonly called *internal cooling*. Internal cooling may be defined as the injection of an auxiliary liquid into the fuel-air charge at some point before the engine intake port. A desirable liquid for internal cooling should have a high latent heat of vaporization, since the more heat extracted from the charge fuel-air mixture, the greater the cooling attained. The reduction in temperature of the mixture achieved by the use of an internal coolant will result in lower cylinder temperatures and will extend the knock-free performance range. In addition to the advantage of increased permissible power, two other goals in the use of internal coolants are savings in fuel and savings in total liquid consumption.

Each of these objectives may be sought in itself by disregarding changes in the other two or it may be sought in combination with one or both of the others in an effort to balance advantages against disadvantages. The most outstanding advantage of internal cooling is, however, the increase in knock-limited or cooling-limited power that it makes possible.

Where gasoline shortages exist because of out-of-the-way destinations of transport airplanes or because of air routes requiring transportation by air of the gasoline used, the possibility of savings in gasoline is particularly pertinent. Such savings may result from the use of internal cooling rather than fuel enrichment to suppress knock at high

power output. In this case water would be the most efficient internal coolant if freezing temperatures were not encountered in the internal-coolant system. Temperatures below freezing would necessitate either the addition of a freezing-point depressant or the use of a lagged or heated water system. Special cases and requirements must obviously determine which of these methods is most advantageous.

The material that appears in the succeeding portions of this chapter considers only the case of internal cooling wherein the coolant is injected at some point in the intake-air system of the engine. Water injection was treated briefly in chapter I, however, where it was found that knock could be suppressed in an engine by injection of water directly into the combustion end zone. This method was suggested as a means of economizing on the quantity of coolant that might be required to reduce knock.

EFFECT OF INTERNAL COOLANTS ON ENGINE PERFORMANCE

Several investigations (references 2 to 4) have been conducted by the NACA in which the effectiveness of internal cooling with respect to knock-limited and temperature-limited performance has been illustrated. In one of these investigations (reference 4) a V-type 12-cylinder liquid-cooled aircraft engine was used as the test engine.

Water was continuously injected through 12 spray bars inserted into the intake manifolds through holes drilled in the top of the manifolds about 1 inch back from the faces of the manifold mounting flanges, as shown in figure X-1. The spray bars (fig. X-2) were of $\frac{5}{32}$ -inch-diameter, stainless steel tubing about $2\frac{1}{2}$ inches long with six holes, each 0.016 inch in diameter, arranged in two rows of three holes each, to spray water directly into each intake port. Water was applied to the spray bars by individual lines from a tank, which was kept under pressure with compressed air.

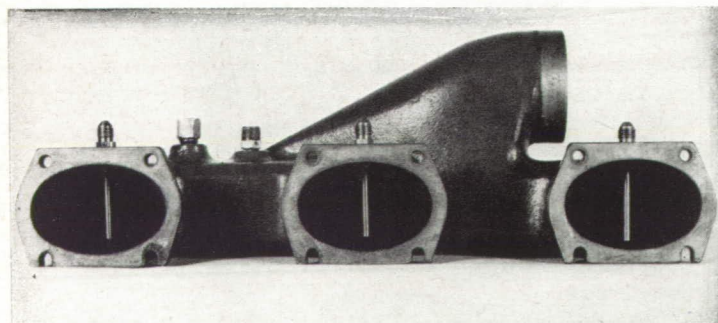


FIGURE X-1.—Intake manifold with water spray bars in position. (Fig. 1 of reference 4.)

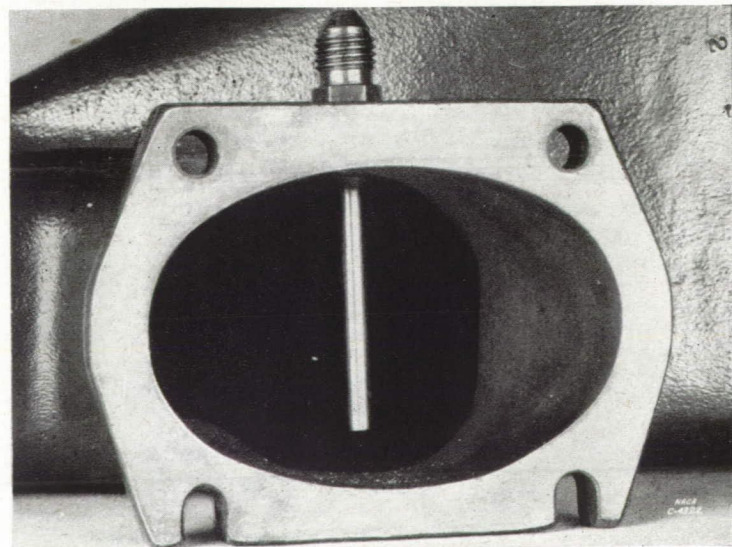
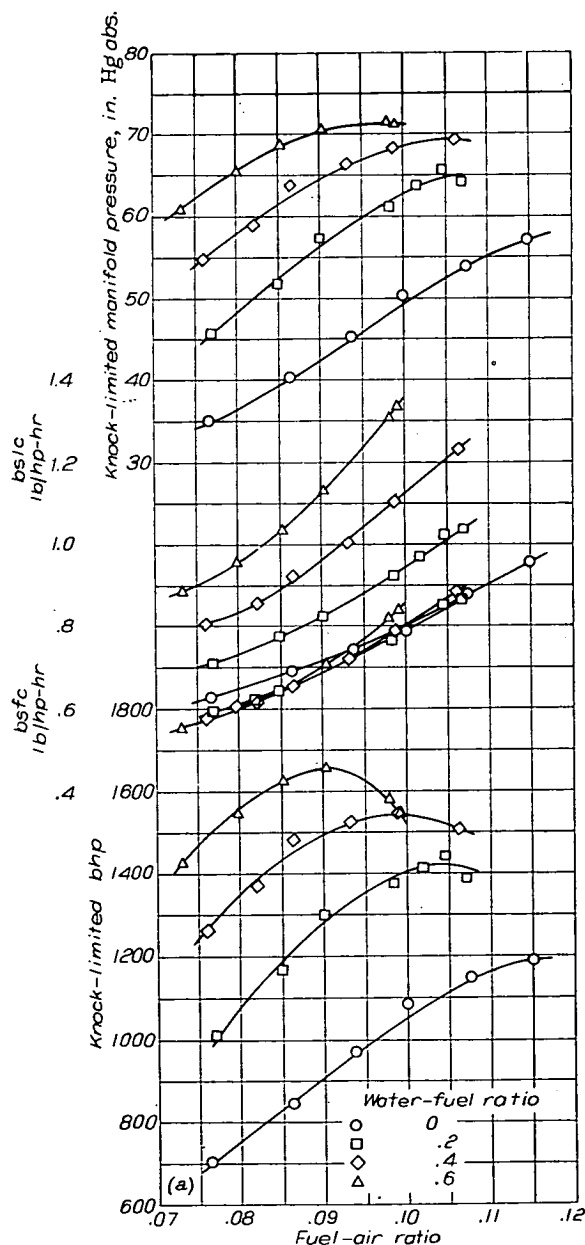


FIGURE X-2.—Closeup of water spray bar inserted in intake manifold. (Fig. 2 of reference 4.)

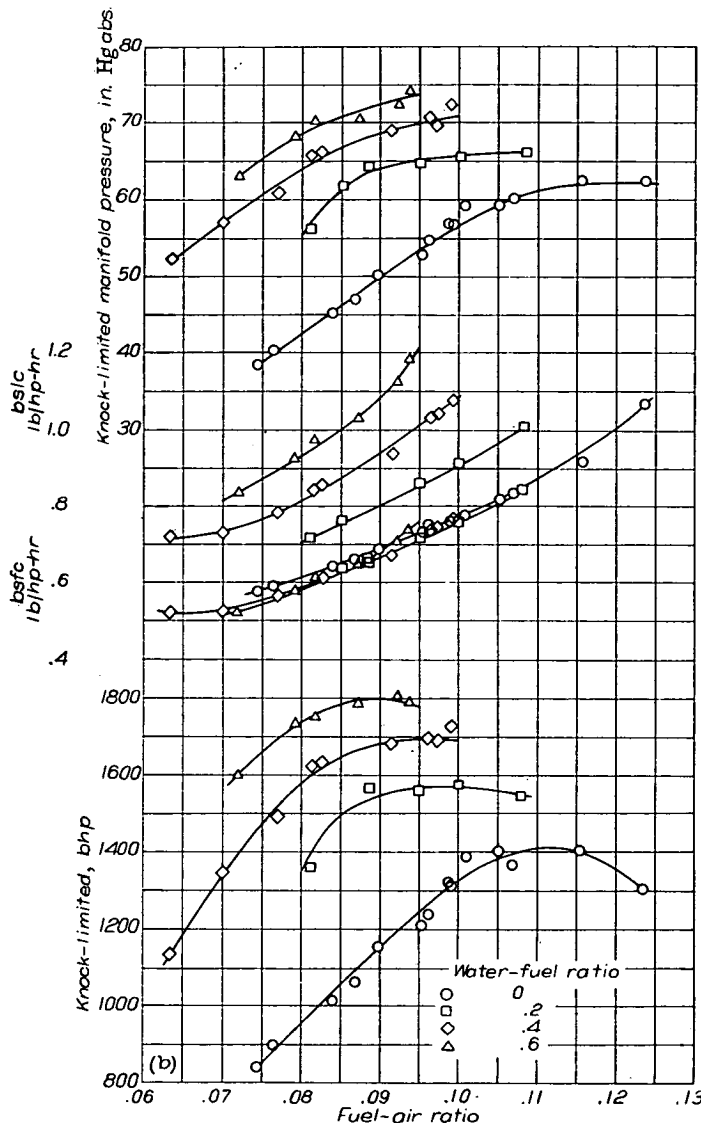
With this injection system, knock-limited performance data were obtained at carburetor-air temperatures of 158°, 101°, and 50° F (fig. X-3). In figure X-3 (a) the peaks of the knock-limited brake horsepower curves occur at successively leaner mixtures as the water-fuel ratio is increased. At a water-fuel ratio of 0.6, a rapid decrease in knock-limited power was found as the fuel-air ratio was increased beyond about 0.092. Similar results were found at the other carburetor-air temperatures; however, at a carburetor-air temperature of 50° F (fig. X-3 (c)) the sharp decline in knock-limited brake horsepower at a water-fuel ratio of 0.6 occurred at fuel-air ratios greater than 0.08.

At the three carburetor-air temperatures investigated, the brake specific fuel consumption was lower with water injection than without at fuel-air ratios leaner than 0.092. This



(a) Carburetor-air temperature, 158° F. (Fig. 4 of reference 4.)

FIGURE X-3.—Knock-limited performance with water injection. V-type 12-cylinder liquid-cooled engine; fuel, AN-F-28, Amendment-2; engine speed, 3000 rpm.



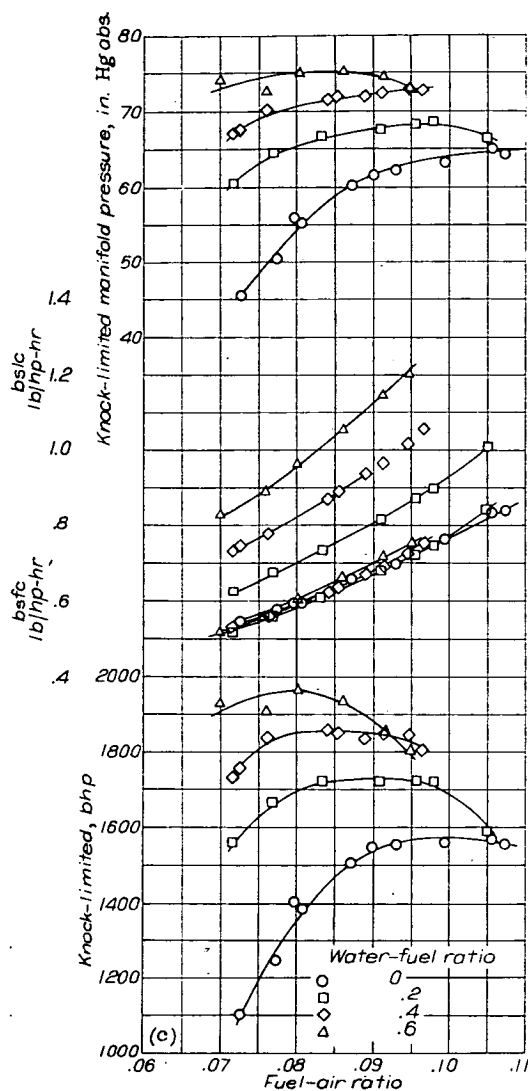
(b) Carburetor-air temperature, 101° F. (Fig. 5 of reference 4.)

FIGURE X-3.—Continued. Knock-limited performance with water injection. V-type 12-cylinder liquid-cooled engine; fuel, AN-F-28, Amendment-2; engine speed, 3000 rpm.

reduction is caused partly by increased mechanical efficiency of the engine at the high power outputs attainable with water injection.

Cross plots of the data in figure X-3 are shown in figure X-4 for two fuel-air ratios. With the exception of the data at 0.095 fuel-air ratio and 50° F carburetor-air temperature, increases of water-fuel ratio resulted in increases of knock-limited brake horsepower. For the excepted data, water-fuel ratios greater than 0.45 resulted in a decrease of knock-limited power.

The effect of internal cooling on cylinder-head temperature was also observed in reference 4. The average cylinder-head temperatures for the engine are shown in figure X-5 as a function of water-fuel ratio and knock-limited power. It is apparent in this figure that as the water-fuel ratio increases, the power increases continuously; however, the cylinder-head temperatures pass through a maximum. The water-fuel ratio at which this maximum occurs increases as the carburetor-air temperature increases.



(c) Carburetor-air temperature, 50° F. (Fig. 6 of reference 4.)

FIGURE X-3.—Concluded. Knock-limited performance with water injection. V-type 12-cylinder liquid-cooled engine; fuel, AN-F-28, Amendment-2; engine speed, 3000 rpm.

Effect of internal cooling on fuel consumption.—Under high-power cruise operation, over-all fuel enrichment may be necessary in order to prevent an increase in cylinder temperatures beyond the specified maximum. This practice obviously results in higher fuel consumption than would ordinarily be desired in the interest of range considerations. However, the same end result can be achieved by adding additional fuel only to the hottest cylinders.

This fact is substantiated by results obtained in an unpublished NACA investigation conducted in an air-cooled engine. The results indicated that lower brake specific fuel consumptions could be obtained at high cruising power, with no increase in maximum rear-spark-plug-gasket temperatures, by injecting additional fuel to the hottest cylinders and operating the engine in automatic-lean mixture setting instead of the automatic-rich mixture setting usually recommended. The fuel savings varied between 6 and 10 percent for two engines tested in flight and one engine tested in an altitude wind tunnel. It was found, however, that little or no fuel saving

is realized when additional fuel is supplied to the hot cylinders unless the temperature spread among the cylinders before enrichment is very large.

Another investigation conducted in an air-cooled engine (reference 5) showed that further gains in fuel economy could be achieved by using water instead of fuel to cool the hottest cylinders. In this particular study, it was found that the use of water instead of excess fuel to maintain engine temperature limits at powers normally requiring a fuel-air ratio of about 0.09 resulted in a decrease of approximately

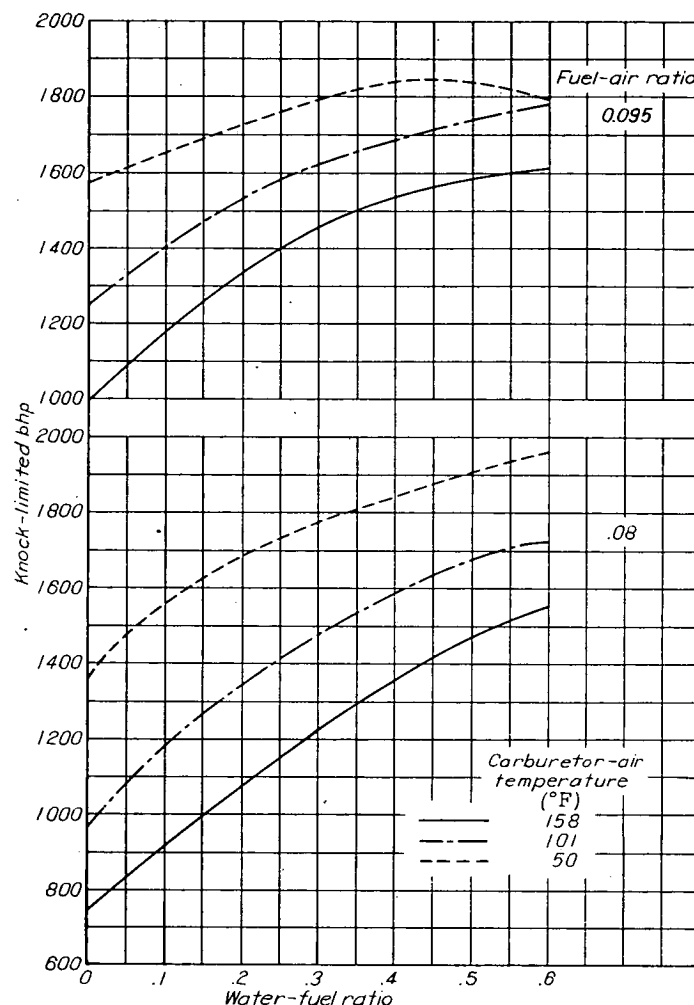


FIGURE X-4.—Variation of knock-limited brake horsepower with water-fuel ratio for three carburetor-air temperatures and two fuel-air ratios. V-type 12-cylinder liquid-cooled engine; fuel, AN-F-28, Amendment-2; engine speed, 3000 rpm. Cross plot from figure X-3. (Fig. 14 of reference 4.)

26 percent in brake specific fuel consumption with an increase of about 3 percent in brake specific liquid consumption.

It is obvious from the foregoing discussion that water injection in aircraft engines permits temperature-limited cruising powers to be reached at reduced engine speeds and increased brake mean effective pressures with fuel-air mixtures very near that for maximum economy. Furthermore, significant improvements in fuel consumption can be attained without appreciable increases in over-all liquid consumption.

These findings have been further substantiated by a study reported in reference 6 in which it was found that for typical air-cooled aircraft engines operating under cruising conditions in which overheating is ordinarily prevented by enriching the fuel-air mixture to the entire engine, the fuel consumption may be reduced 7 to 37 percent by adding water or fuel to only the overheated cylinders.

Effect of internal cooling on spark-advance requirements.—Internal cooling can also be utilized to take advantage of gains that may be achieved by retarding the spark timing of an engine. This fact was demonstrated in an investigation (reference 7) conducted in a liquid-cooled single-cylinder test engine. Two positions were used for the injection of

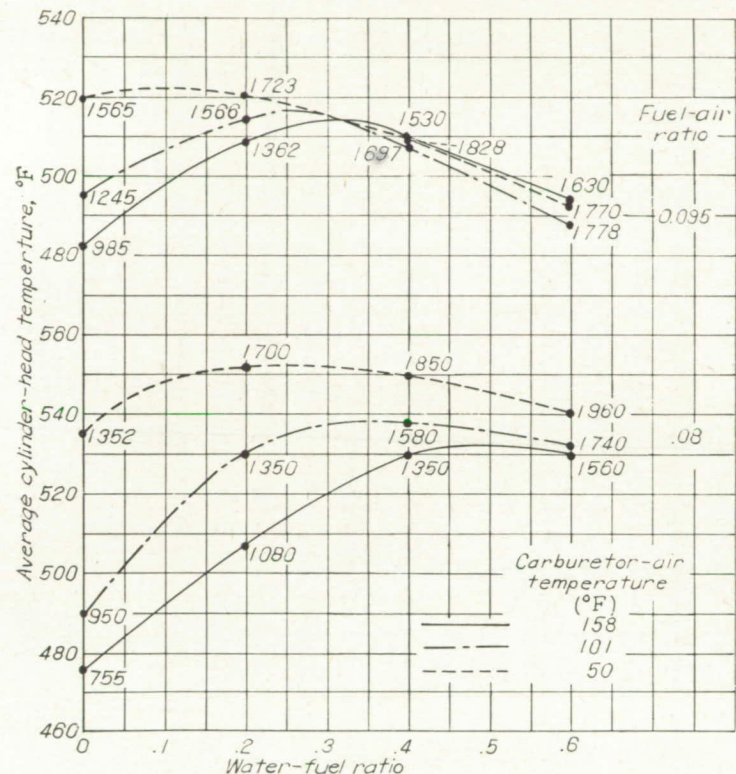


FIGURE X-5.—Variation of average cylinder-head temperature with water-fuel ratio at knock-limited power for three carburetor-air temperatures. V-type 12-cylinder liquid-cooled engine; fuel, AN-F-28, Amendment-2; engine speed, 3000 rpm. (Fig. 13 of reference 4.) Numbers on curves are values of knock-limited brake horsepower.

internal coolant: position A before the vaporization tank and position B at the intake elbow. In both positions the coolant was discharged downstream. The positions of injection are illustrated in figure X-6.

The effect of internal-coolant—fuel ratio on spark advance for peak power is shown in figure X-7. For both types of injection, the spark advance required for peak power was greater when the coolant was injected before the vaporization tank than at the injection elbow; however, the difference was small when a 50 : 50 mixture of water and ethyl alcohol was used as the coolant. This result indicates that the water, when injected before the vaporization tank and allowed to mix thoroughly with the fuel-air mixture, slowed the flame speed more than when injected at the intake elbow.

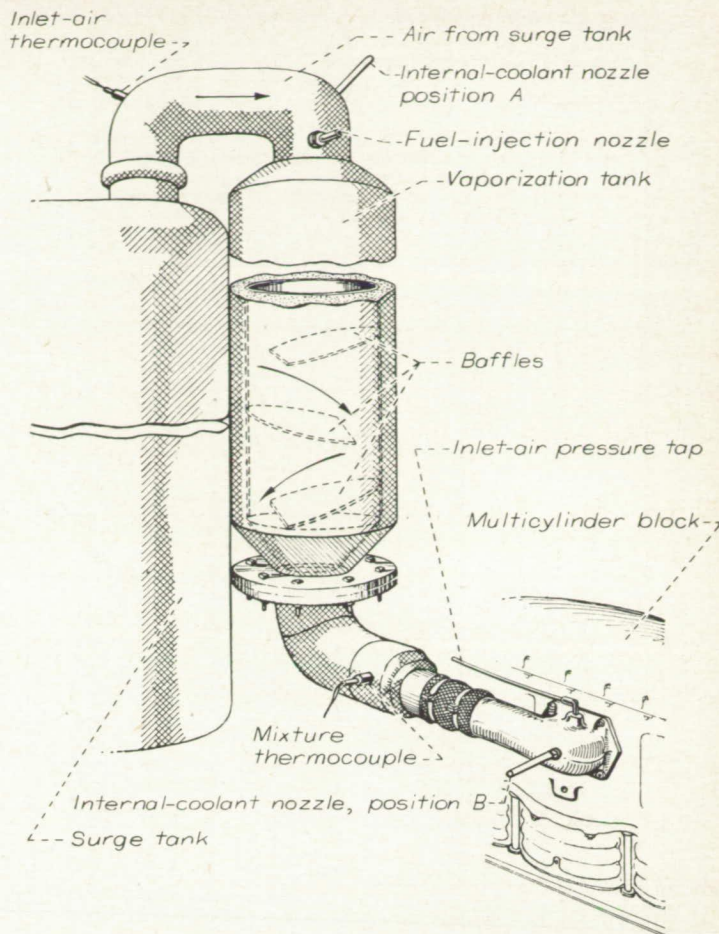


FIGURE X-6.—Induction system used with multicylinder-block adaptation to CUE crank-case showing two positions of internal-coolant nozzle. (Fig. 2 of reference 7).

The flame speed was slowed to about the same degree regardless of injection position for the 50 : 50 mixture of water and ethyl alcohol.

At the same internal-coolant—fuel ratios (fig. X-7) and for the same coolant injector location, the water—ethyl-alcohol mixture retarded burning less than the injection of pure water. This difference can be attributed to the fact that the alcohol is a fuel and as such is contributing to the combustion process.

In figure X-8 is illustrated the effect of internal-coolant—fuel ratio on the mixture temperatures corresponding to the data of figure X-7. The data for figure X-8 represent the case in which the coolants were injected ahead of the vaporization tank. The mixture temperature decreased until the fuel-air mixture became saturated and for further additions of coolant beyond this point the mixture temperature remained constant. In a similar manner, the power at peak-power spark advance increased because of the increased charge weight inducted into the cylinder until the internal-coolant—fuel ratio for saturation was reached, at which point the power leveled off. (See fig. X-9.) Beyond the point of complete saturation the additional cooling of the mixture must occur after the intake valves close, which makes it impossible to increase engine power through an effect on air flow. Further additions of coolant caused a

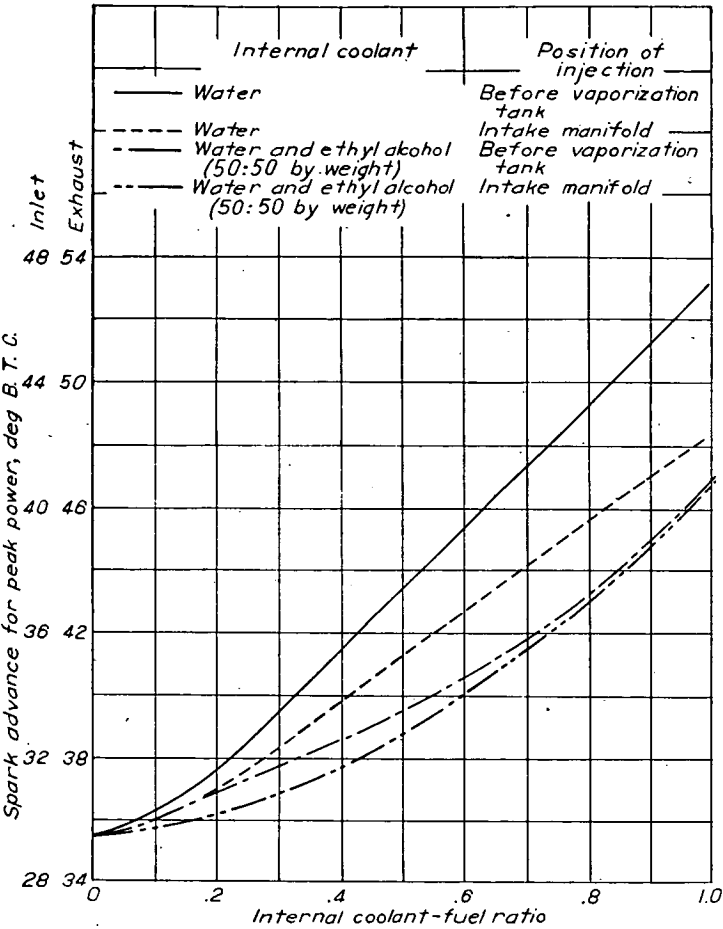


FIGURE X-7.—Effect of internal coolant-fuel ratio on spark advance for peak power for two internal coolants and two positions of injection. Single-cylinder adaptation of multicylinder engine to CUE crankcase; compression ratio, 6.65; engine speed, 3000 rpm; fuel-air ratio 0.08; inlet-oil temperature, 185° F; outlet-coolant temperature, 250° F; inlet-air temperature, 250° F; inlet-air pressure, 50 inches of mercury absolute. (Fig. 11 of reference 7.)

decrease in the power obtainable at peak-power spark advance because some heat of vaporization was extracted from the air during the compression stroke; decreased cycle efficiency resulted. The power increase at peak-power spark advance was small when coolant was injected at the intake elbow (fig. X-9) because so little time was available for charge cooling before the intake valves closed.

The percentage loss in power at various values of spark advance over that obtained by using peak-power spark advance is shown in figure X-10 for each internal-coolant-fuel ratio investigated. The data in this figure indicate that operation with normal spark timing at a given coolant-fuel ratio and position of injection results in a power loss approximately twice as high for water as for the mixture of water and ethyl alcohol.

USE OF INTERNAL COOLING FOR INCREASED TAKE-OFF POWER

In 1944 an analysis was made by the NACA to evaluate the use of internal cooling as a means of increasing take-off power. This study (reference 8) was made for four airplanes to determine the effects of a 25-percent increase in take-off power on the take-off load of the airplane. The operating

characteristics of the airplanes considered are presented in the following table:

Airplane	Normal take-off horsepower	Normal fuel capacity		Gross weight of airplane (lb)
		(gal)	(lb)	
Heavy bomber.....	4800	1433	8598	41,000
Pursuit.....	2000	210	1260	11,870
Torpedo bomber.....	1700	301	1806	15,364
Shipboard fighter.....	2000	344	2064	12,577

For these airplanes the estimated increase in take-off load for a 25-percent increase in take-off power is as follows:

Airplane	Load increase (percent)	Load increase (lb)	Usable load increase (lb)	Usable load (gal of gasoline)	Percentage increase (gal of gasoline)
Heavy bomber.....	11.5	4710	3631	427	30
Pursuit.....	12.0	1430	1097	129	65
Torpedo bomber.....	10.5	1610	1300	153	51
Shipboard fighter.....	10.5	1320	1037	122	36

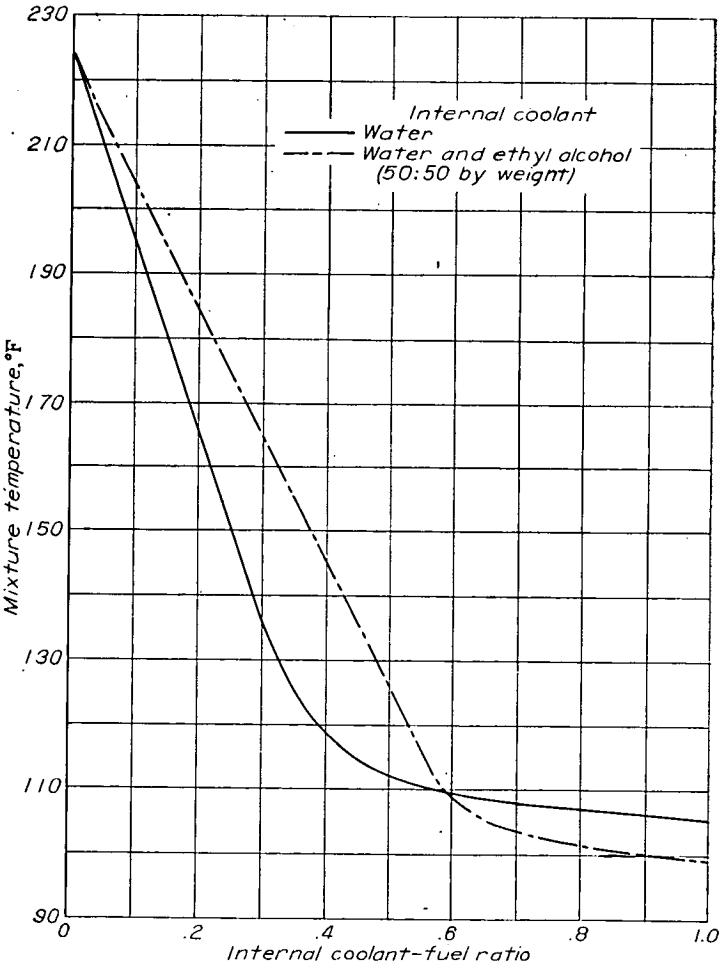


FIGURE X-8.—Effect of internal coolant-fuel ratio on mixture temperature for two internal coolants injected before the vaporization tank. Single-cylinder adaptation of multicylinder engine to CUE crankcase; compression ratio, 6.65; engine speed, 3000 rpm; fuel-air ratio, 0.03; inlet-oil temperature, 185° F; outlet-coolant temperature, 250° F; inlet-air temperature, 250° F; inlet-air pressure, 50 inches of mercury absolute. (Fig. 12 of reference 7.)

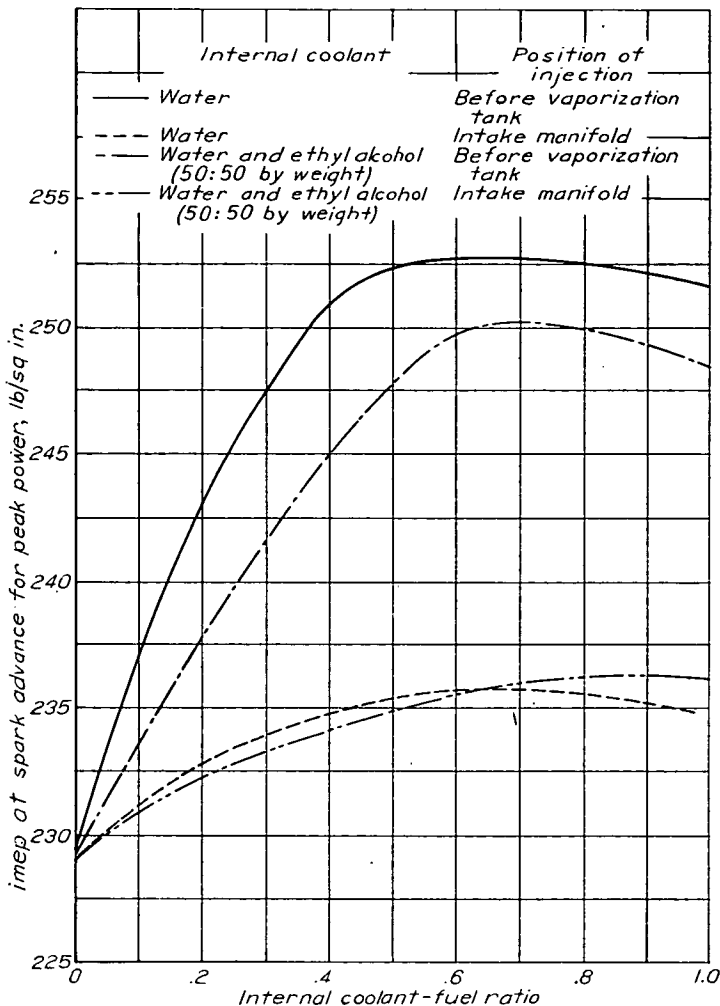


FIGURE X-9.—Effect of internal coolant-fuel ratio on engine power at spark advance for peak power for two internal coolants and two positions of injection. Single-cylinder adaptation of multicylinder engine to CUE crankcase; compression ratio, 6.65; engine speed, 3000 rpm; fuel-air ratio, 0.08; inlet-oil temperature, 185° F; outlet-coolant temperature, 250° F; inlet-air temperature, 250° F; inlet-air pressure, 50 inches of mercury absolute. (Fig. 13 of reference 7.)

The data presented in the foregoing table indicate that marked increases in the usable load, or in this usable load translated into gallons of gasoline, may be achieved through use of internal cooling for 25-percent increased take-off power. These numerical estimates are necessarily dependent upon the assumed values for many factors. For example, these particular calculations include the following assumptions:

- (1) that 0.78 pound of coolant per pound of fuel is required to produce a 25-percent increase in take-off power. This quantity of coolant is about 45 percent higher than is indicated by experimental data in order to provide a factor of safety in the calculations.
- (2) that sufficient coolant is provided for 5-minute operation
- (3) that the increase in propeller weight for the additional power output is about 100 pounds
- (4) that the coolant system exclusive of the tank weight is about 75 pounds.

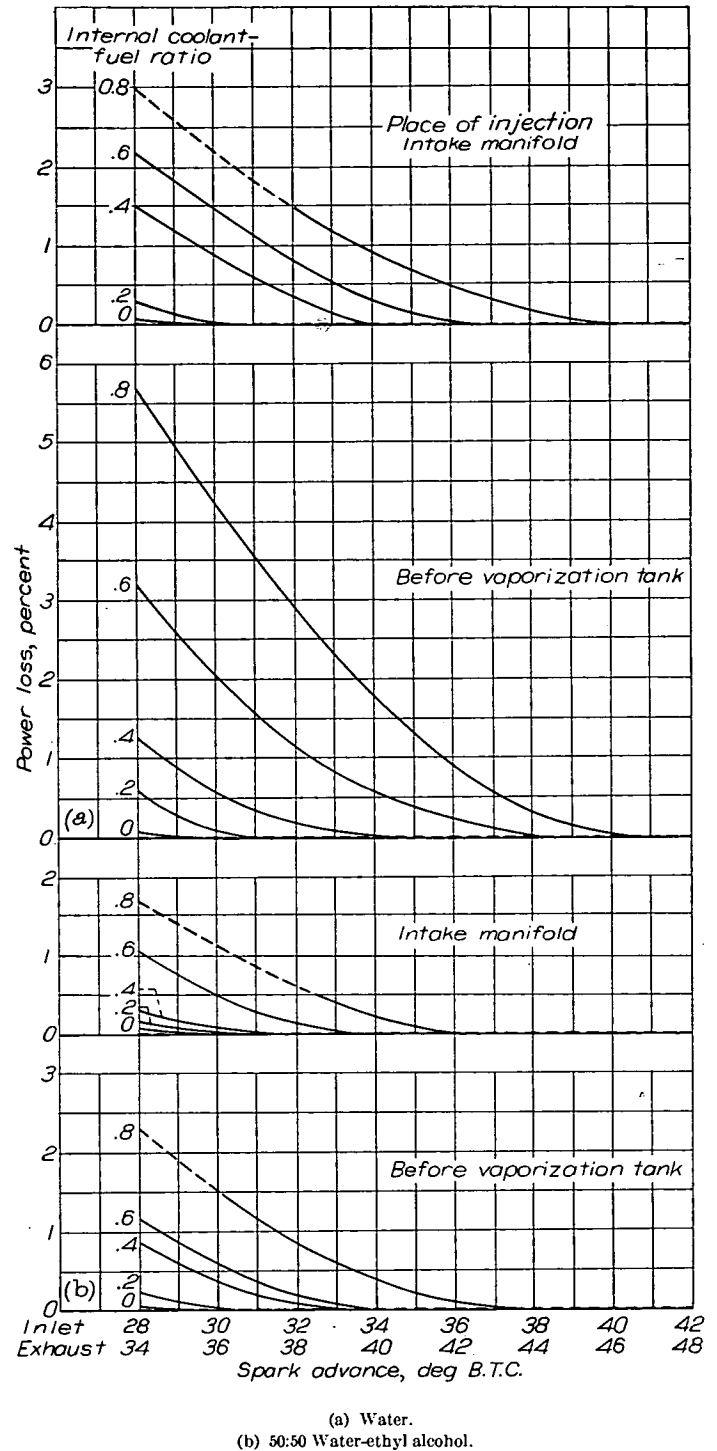
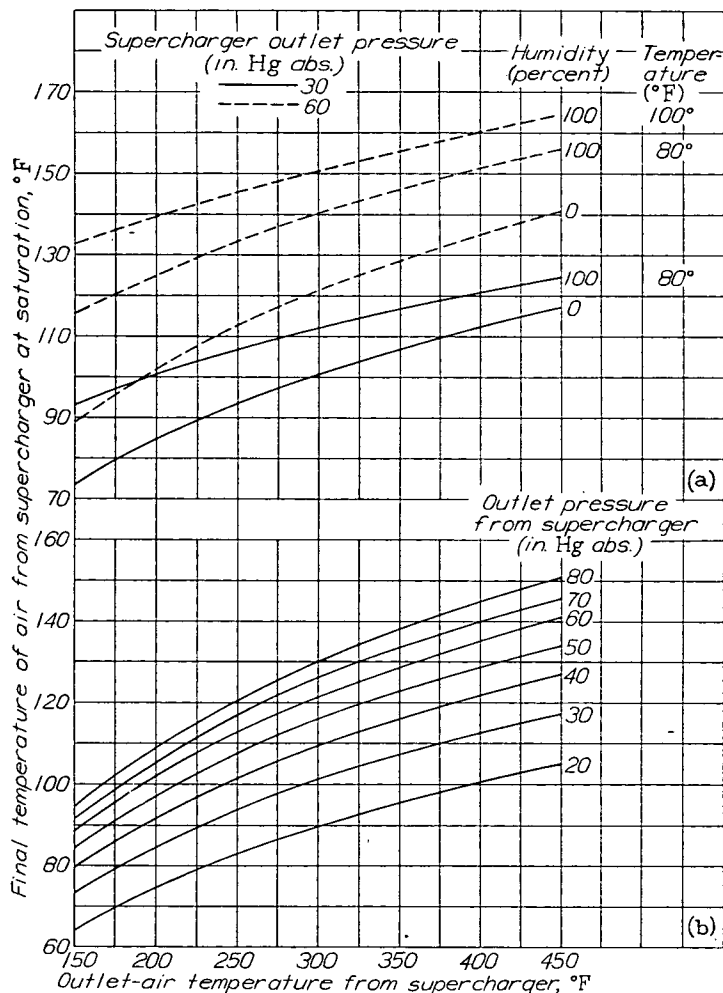


FIGURE X-10.—Loss in power incurred by operating without peak-power spark advance for several internal coolant-fuel ratios and for two positions of injection. Single-cylinder adaptation of multicylinder engine to CUE crankcase; compression ratio, 6.65; engine speed, 3000 rpm; fuel-air ratio, 0.08; inlet-oil temperature, 185° F; outlet-coolant temperature, 250° F; inlet-air temperature, 250° F; inlet-air pressure, 50 inches of mercury absolute. (Fig. 14 of reference 9.)

Other details of the estimates may be found in reference 8. In addition to the gains in usable load, certain gains may be realized in rate of climb and take-off run. The increase of 25 percent in take-off power indicated an increase in rate

of climb from 3100 to 4100 feet per minute up to an altitude of 12,000 feet for the pursuit-type airplane and 1600 to 2200 feet per minute for the torpedo bomber. The estimated take-off distances for the four airplanes are as follows:

Airplane	Normal take-off distance (ft)	Take-off distance 25-percent increase in take-off power (ft)
Heavy bomber.....	1700	1300
Pursuit.....	1300	1000
Torpedo bomber.....	340	200
Shipboard fighter.....	350	250



(a) Air at different humidities into supercharger.
(b) Dry air into supercharger.

FIGURE X-11.—Relation between outlet-air temperature from supercharger and final temperature of air when sufficient cooling water is induced for saturation of air. Water induced as a liquid at 60° F. (Fig. 7 of reference 9.)

PRACTICAL ASPECTS OF INTAKE-AIR COOLING BY WATER INJECTION

Inasmuch as the application of internal cooling to aircraft would necessarily require the installation of additional equipment, consideration has been given to means by which internal cooling might be applied without significant increase in aircraft weight. Calculations have been made (reference 9) to determine the extent to which the engine intake air is cooled by water injection. The results of these calculations,

presented in figure X-11, indicate that the degree of cooling achieved by water injection is sufficient to permit the elimination of normal intercooling and after-cooling in the supercharger. Consequently, the weight of equipment necessary for internal cooling might be offset by the elimination of the supercharger intercooler or aftercooler.

Further calculations reported in reference 9 indicate that the water to be used as internal coolant can be obtained by recovery from the exhaust gas. Sufficient water can be recovered from 50 percent of the exhaust gas to provide an inducted water-fuel ratio of 0.5.

EVALUATION OF VARIOUS LIQUIDS AS INTERNAL COOLANTS

As part of the problem of applying internal cooling to aircraft, extensive investigations have been conducted to evaluate the performance of various liquids as internal coolants. One such investigation, reported in reference 10, compares the performance of (1) water, (2) methyl alcohol and water, (3) ammonia, methyl alcohol, and water, (4) monomethylamine and water, (5) dimethylamine and water, and (6) trimethylamine and water. These studies were made on a high-speed supercharged CFR engine which is described in detail in reference 10. The internal coolant was continuously injected at room temperature into the injection elbow just above the fuel injection nozzle and parallel to the air flow.

The results of the investigation summarized in table X-1 indicate that when water is used as an internal coolant the greatest improvement in knock-limited performance of AN-F-28 fuel is achieved at lean fuel-air ratios. The mixture of methyl alcohol and water is a slightly better coolant than water at lean mixtures but considerably better at rich mixtures. As pointed out earlier in this chapter, the methyl alcohol is a fuel and as such contributes to the combustion process as well as the cooling. Of the coolants listed in table X-1, mixtures of water with monomethylamine or dimethylamine showed the greatest improvements in the knock-limited performance of AN-F-28; the increases in

TABLE X-1.—IMPROVEMENT IN KNOCK-LIMITED ENGINE PERFORMANCE OF AN-F-28 FUEL ACHIEVED BY INTERNAL COOLING

[CFR engine; compression ratio, 7.0; engine speed, 2500 rpm; inlet-air temperature, 250° F; inlet-coolant temperature, 250° F; spark advance, 30° B. T. C.]

Internal coolant (0.5 lb/lb fuel)	Relative power ratio imep (fuel + internal coolant) = ————— imep (fuel alone)			
	Fuel-air ratio			
	0.05	0.06	0.08	0.09
None.....	1.00	1.00	1.00	1.00
Water	1.14	1.52	1.41	1.28
Methylalcohol and water (70:30 by volume)	1.51	1.59	1.80	1.75
Monomethylamine and water (32:68 by weight)	1.98	• 1.81	• 1.86	• 1.83
Dimethylamine and water (26:74 by weight)	1.82	1.78	1.98	

• Afterfiring encountered.

knock-limited indicated mean effective pressure ranged between 78 and 98 percent for different fuel-air ratios. The use of trimethylamine-water solution as an internal coolant lowered the knock-limited performance of AN-F-28.

The addition of anhydrous ammonia to the solution of methyl alcohol and water before injection (reference 10) reduced the knock-inhibiting effects of the solution of methyl alcohol and water and promoted afterfiring. Afterfiring also occurred when monomethylamine-water solution was used as an internal coolant (table X-1).

The effect of inlet-air temperature on the knock-limited performance of the internal coolants discussed previously is also reported in reference 10. A summary of these data (table X-2) shows that the addition of water reduced the temperature sensitivity of AN-F-28 fuel. The other coolants appeared to increase the sensitivity at lean fuel-air ratios but decreased the sensitivity at rich fuel-air ratios.

Perhaps the most impressive example of the advantages found for the internal coolants evaluated in reference 10 is the knock-limited performance of monomethylamine-water mixture at an inlet-air temperature of 150° F. This particular internal coolant, when added to AN-F-28 fuel, allowed a knock-limited power of 1.96 horsepower per cubic inch of cylinder displacement (imep of 620 lb/sq in.) at a fuel-air ratio of 0.049. The corresponding indicated specific fuel and liquid consumptions were 0.37 and 0.55 pound per horsepower-hour, respectively.

The investigations reported in reference 10 were conducted in an engine installation in which the range of operation was limited to a fuel flow of 30 pounds per hour and an inlet-air pressure of 150 inches of mercury absolute. Because of the continued interest in monomethylamine and dimethylamine as internal coolants, the installation was later revised to extend the range of operation to a fuel flow of 80 pounds per hour and an inlet-air pressure of 225 inches of mercury absolute. Following these revisions the tests with monomethylamine and dimethylamine were resumed; however, two engine cylinders were cracked when the knock-limited performance with the internal coolants reached a level of 700

TABLE X-2.—EFFECT OF INLET-AIR TEMPERATURE ON KNOCK-LIMITED ENGINE PERFORMANCE OF AN-F-28 FUEL USED IN CONJUNCTION WITH INTERNAL COOLANTS

[CFR engine; compression ratio, 7.0; engine speed, 2500 rpm; inlet-coolant temperature, 250° F; spark advance, 30° B. T. C.]

Internal coolant (0.5 lb/lb fuel)	Imep at inlet-air temperature of 150° F			
	Imep at inlet-air temperature of 250° F			
	Fuel-air ratio			
	0.05	0.06	0.08	0.09
None.....	1.24	1.46	1.32	1.20
Water.....	----	1.23	1.11	1.09
Methyl alcohol and water (70:30 by volume).....	1.46	1.57	1.23	1.13
Monomethylamine and water (32:68 by weight).....	1.43	-----	-----	-----
Dimethylamine and water (26:74 by weight).....	1.42	* 1.51	-----	-----

* Afterfiring encountered at inlet-air temperature of 150° F.

TABLE X-3.—KNOCK-LIMITED RELATIVE POWERS RESULTING FROM THE USE OF INTERNAL COOLANTS WITH AN-F-28 (AMENDMENT 2) FUEL

[CFR engine; compression ratio, 7.0; engine speed, 2500 rpm; inlet-air temperature, 250° F; coolant temperature, 250° F; spark advance, 30° B. T. C.]

Internal coolant	Weight of coolant per pound of fuel (lb)	Imep (fuel+internal coolant) Imep (fuel alone)					
		Fuel-air ratio *					
		0.05	0.05	0.07	0.08	0.09	0.10
None	-----	1.00	1.00	1.00	1.00	1.00	1.00
Water.....	0.25 .50	--	1.25 1.48	1.21 1.48	1.21 1.37	1.16 1.24	1.08 1.13
Monomethylamine and water (32:68 by weight)	0.25 .50	1.78 2.22	1.46 2.16	1.66 2.18	1.75 2.17	1.70 2.14	1.60 2.18
Dimethylamine and water (27:73 by weight)	0.25 .50 .75	1.72 1.62 1.89	1.42 1.61 1.97	1.51 2.02 2.00	1.59 2.16 2.83	1.57 2.24 3.32	1.55 2.33 ---

* Any contribution of the amines to the energy of combustion was entirely neglected in computing fuel flows.

pounds per square inch. In order to resume the investigation, a specially designed CFR cylinder was obtained to permit studies at the exceptionally high powers attained with monomethylamine and dimethylamine. The investigation was again resumed and virtually completed before the next engine failure occurred.

In the extended study of the two internal coolants (reference 11) tests were conducted over a wider range of fuel-air ratio and at higher ratios of coolant to fuel. The results are summarized in table X-3. It is seen in this table that the amines are considerably better than water as internal coolants. Of the two amines examined, monomethylamine is superior to dimethylamine at coolant-fuel ratios of 0.25; however, at a coolant-fuel ratio of 0.50, monomethylamine is better than dimethylamine at lean fuel-air ratios but slightly poorer at rich fuel-air ratios. It is of interest to note that the injection of 0.75 pound of dimethylamine-water solution per pound of fuel permitted the attainment of a knock-limited indicated mean effective pressure of 967 pounds per square inch, corresponding to 3.05 indicated horsepower per cubic-inch displacement, at a fuel-air ratio of 0.092. Failure of a cylinder stud terminated the tests at this point; however, after an overhaul the tests were again resumed and the cylinder wall failed at an indicated mean effective pressure of 895 pounds per square inch during a test with dimethylamine-water solution at a coolant-fuel ratio of 0.75.

During the investigation reported in reference 11 a run was made to determine the influence of exhaust back pressure on the knock-limited performance when dimethylamine-water solution was used as an internal coolant. The results indicated that increases in exhaust back pressure had little or no effect on the performance at fuel-air ratios leaner than 0.095. At richer fuel-air ratios serious decreases in power output were encountered and engine operation was quite rough.

An examination of the data in reference 11 indicated that for certain power levels in the rich-mixture range the following combinations of internal coolants resulted in the lowest indicated specific liquid consumptions:

Imep range (lb/sq in.)	Corresponding minimum isle range (lb/hp-hr)	Internal coolant	Internal coolant- fuel ratio
Below 220	0.44 to 0.53	None	
220 to 370	0.53 to 0.63	Monomethylamine solution	0.25
370 to 440	0.63 to 0.71	Dimethylamine solution	.50
440 to 960	0.71 to 1.10	Dimethylamine solution	.75

The success of aliphatic amines as internal coolants, reported in references 10 and 11, led to further studies with other amines reported in reference 12. The additional amines evaluated were ethylenediamine, diethylamine, triethylamine, and butylamine. The results of the investigation (reference 12) are summarized in table X-4.

It is apparent in table X-4 that of all the amines tested monomethylamine and dimethylamine still offer the greatest possibilities as internal coolants, although ethylenediamine does permit higher knock-free power at lean fuel-air ratios.

In the investigation (reference 12) an effort was made to reach the highest possible knock-limited indicated mean effective pressure with dimethylamine. At a fuel-air ratio of 0.093 and a coolant-fuel ratio of 0.75, a knock-limited indicated mean effective pressure of 1024 pounds per square inch was attained. No engine failure occurred but this test was limited by the available intake-air supply.

TABLE X-4.—RELATIVE KNOCK-LIMITED POWERS RESULTING FROM USE OF INTERNAL-COOLANT ADDITIVES IN WATER AT COOLANT-FUEL RATIO OF 0.50

[CFR engine; fuel AN-F-28, Amendment-2; compression ratio, 7.0; engine speed, 2500 rpm; inlet-air temperature, 250° F; spark advance, 30° B. T. C.; jacket temperature, 250° F]

Internal-coolant additive	Additive in coolant solution (percent by weight)	Imep (fuel+water+additive) Imep (fuel+water alone)					
		Fuel-air ratio *					
		0.05	0.06	0.07	0.08	0.09	0.10
None	0	1.00	1.00	1.00	1.00	1.00	1.00
Monomethylamine	32	1.64	1.32	1.32	1.42	1.91	2.42
Dimethylamine	26	1.54	1.41	1.46	1.82	2.08	2.58
Ethylenediamine	25	1.81	1.61	1.38	1.44	1.45	1.42
Diethylamine	25	1.09	0.86	0.77	0.92	0.96	1.07
Triethylamine	25		0.92	0.71	0.80	0.84	0.83
Butylamine	25	1.02	0.82	0.90	0.93	0.93	0.96

* The amines were not considered as fuels and their heats of combustion were neglected in computing the fuel-air ratios.

In a later investigation (reference 13) the following compounds were evaluated as internal coolants; however, none were found to be as effective as monomethylamine, dimethylamine, and ethylenediamine:

Alkyl amines:

Isopropylamine
Isobutylamine
tert-Butylamine
Monoamylamine

Alkanolamines:

Ethanolamine
Diethanolamine
2-Amino-2-methyl-1-propanol

Amides:

Formamide

Amides—Continued

N-Ethylformamide
N-Ethylacetamide
N-Ethylpropionamide
N,N-Dimethylformamide
N,N-Diethylacetamide

Heterocyclic compounds:

2,2-Dimethylethylenimine
Morpholine
Pyridine
2-Methylpyridine
3-Methylpyridine
4-Methylpyridine
2,6-Dimethylpyridine
2-Vinylpyridine

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, May 15, 1951.

REFERENCES

1. Pinkel, Benjamin, and Ellerbrock, Herman H., Jr.: Correlation of Cooling Data from an Air-Cooled Cylinder and Several Multicylinder Engines. NACA Rep. 683, 1940.
2. Rothrock, Addison M., Krsek, Alois, Jr., and Jones, Anthony W.: The Induction of Water to the Inlet Air as a Means of Internal Cooling in Aircraft-Engine Cylinders. NACA Rep. 756, 1943. (Formerly NACA ARR, Aug. 1942.)
3. Wear, Jerrold D., Held, Louis F., and Slough, James W.: Some Effects of Internal Coolants on Knock Limited and Temperature-Limited Power as Determined in a Single-Cylinder Aircraft Test Engine. NACA ARR E4H31, 1944.
4. Harries, Myron L., Nelson, R. Lee, and Berguson, Howard E.: Effect of Water Injection on the Knock-Limited Performance of an Allison V-1710-99 Engine. NACA MR, Army Air Forces, Sept. 9, 1944.
5. Engelman, Helmuth W., and White, H. Jack: Use of Water Injection to Decrease Gasoline Consumption in an Aircraft Engine Cruising at High Power. NACA RB E4H12, 1944.
6. Biermann, Arnold E., Miller, George R., and Henneberry, Hugh M.: Economy of Internally Cooling Only the Overheated Cylinders of Aircraft Engines. NACA MR E5G14, Army Air Forces, July 14, 1945.
7. Pfender, John F., Dudugjian, Carl, and Lietzke, A. F.: Effect of Engine-Operating Variables and Internal Coolants on Spark-Advance Requirements of an Allison V-1710 Cylinder. NACA MR E5E18, Army Air Forces, May 18, 1945.
8. Rothrock, Addison M.: Use of Internal Coolant as a Means of Permitting Increase in Engine Take-Off Power. NACA RB 4A25, 1944.
9. Rothrock, Addison M.: Calculations of Intake-Air Cooling Resulting from Water Injection and of Water Recovery from Exhaust Gas. NACA RB E4H26, 1944.
10. Bellman, Donald R., and Evvard, John C.: Knock-Limited Performance of Several Internal Coolants. NACA Rep. 812, 1945. (Formerly NACA ACR 4B08.)
11. Bellman, Donald R., Moeckel, W. E., and Evvard, John C.: Knock-Limited Power Outputs from a CFR Engine Using Internal Coolants. I—Monomethylamine and Dimethylamine. NACA ARR E4L21, 1944.
12. Bellman, Donald R., Moeckel, W. E., and Evvard, John C.: Knock-Limited Power Outputs from a CFR Engine Using Internal Coolants. II—Six Aliphatic Amines. NACA ACR E5H31, 1945.
13. Imming, Harry S., and Bellman, Donald R.: Knock-Limited Power Outputs from a CFR Engine Using Internal Coolants. III—Four Alkyl Amines, Three Alkanolamines, Six Amides, and Eight Heterocyclic Compounds. NACA RM E6L05a, 1947.

APPENDIX A

ADDITIONAL DATA ON PERFORMANCE OF VARIOUS FUELS

TABLE A-1.—A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE PERFORMANCE NUMBERS OF LEADED AND UNLEADED BLENDS WITH ISOCTANE AND WITH MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE *

(a) Paraffins and olefins.

Paraffins and olefins	Formula	Performance number									
		A. S. T. M. Aviation method						A. S. T. M. Supercharge method ($F/A=0.11$)			
		Unleaded		4 ml TEL/gal		4 ml TEL/gal					
		Volume percent paraffin or olefin in blend with isooctane				Volume percent paraffin or olefin in blend with mixed base fuel ^b					
		10	20	10	20	10	25	50	10	25	50
Paraffins											
2-Methylbutane ^c	C ₅ H ₁₂	-----	-----	-----	-----	-----	128	-----	115	121	130
2,2-Dimethylbutane ^c	C ₆ H ₁₄	-----	-----	-----	-----	-----	129	-----	117	122	129
2,3-Dimethylbutane ^c	C ₆ H ₁₄	-----	-----	-----	-----	-----	129	-----	117	130	147
2,2,3-Trimethylbutane ^c	C ₇ H ₁₆	101	104	151	151	124	135	142	127	146	200
2,3-Dimethylpentane ^c	C ₇ H ₁₆	-----	88	-----	145	-----	118	-----	-----	114	-----
2,2,3-Trimethylpentane ^c	C ₈ H ₁₈	-----	-----	-----	-----	-----	130	-----	123	141	174
2,3,3-Trimethylpentane ^c	C ₈ H ₁₈	-----	-----	-----	-----	-----	124	-----	127	138	166
2,3,4-Trimethylpentane ^c	C ₈ H ₁₈	-----	-----	-----	-----	-----	122	-----	118	132	147
2,2,3,3-Tetramethylpentane ^c	C ₉ H ₂₀	-----	84	-----	128	-----	107	-----	127	156	>230
2,2,3,4-Tetramethylpentane ^c	C ₉ H ₂₀	96	96	145	133	120	118	111	125	141	175
2,2,4,4-Tetramethylpentane ^c	C ₉ H ₂₀	-----	-----	-----	-----	-----	118	-----	111	110	108
2,3,3,4-Tetramethylpentane ^c	C ₉ H ₂₀	93	93	137	131	117	110	106	125	143	192
2,4-Dimethyl-3-ethylpentane ^c	C ₉ H ₂₀	-----	86	-----	140	-----	115	-----	-----	127	-----
Olefins											
2,3-Dimethyl-2-pentene.....	C ₇ H ₁₄	-----	78	-----	108	-----	100	-----	-----	117	-----
2,3,4-Trimethyl-2-pentene.....	C ₈ H ₁₆	80	77	127	113	113	101	77	112	104	77
2,4,4-Trimethyl-1-pentene ^c	C ₈ H ₁₆	-----	-----	-----	-----	-----	105	-----	131	146	159
2,4,4-Trimethyl-2-pentene ^c	C ₈ H ₁₆	-----	-----	-----	-----	-----	108	-----	115	119	103
3,4,4-Trimethyl-2-pentene ^c	C ₈ H ₁₆	96	92	133	120	114	106	88	118	108	93

(b) Aromatics.

Aromatic	Formula	Performance number									
		A. S. T. M. Aviation method						A. S. T. M. Supercharge method ($F/A=0.11$)			
		Unleaded		4 ml TEL/gal		4 ml TEL/gal					
		Volume percent aromatic in blend with isooctane				Volume percent aromatic in blend with mixed base fuel ^b					
		10	20	10	20	10	25	50	10	25	50
Benzene	C ₆ H ₆	96	92	-----	-----	121	116	97±6	125	140	175
Methylbenzene	C ₇ H ₈	-----	94	-----	139	121	117	113	128	152	>290
Ethylbenzene	C ₈ H ₁₀	97	95	-----	138	120	119	106	130	155	>260
1,2-Dimethylbenzene		93	86	136	118	113	105	84	104	101	112
1,3-Dimethylbenzene		100	98	150	-----	125	123	122	131	166	297
1,4-Dimethylbenzene		98	99	148	138	125	122	-----	130	164	>300
<i>n</i> -Propylbenzene	C ₉ H ₁₂	96	93	146	-----	126	122	118	127	152	205
Isopropylbenzene		98	-----	-----	-----	122	122	-----	133	153	284
1-Methyl-2-ethylbenzene		-----	86	-----	124	-----	107	-----	-----	124	-----
1-Methyl-3-ethylbenzene		-----	95	-----	142	-----	124	-----	-----	168	-----
1-Methyl-4-ethylbenzene		96	88	-----	-----	122	120	-----	134	160	214
1,2,3-Trimethylbenzene		-----	85	-----	115	-----	105	-----	-----	104	-----
1,2,4-Trimethylbenzene		91	87	141	121	113	101	97	107	113	147
1,3,5-Trimethylbenzene		-----	-----	150	-----	123	127	-----	137	168	>300
<i>n</i> -Butylbenzene	C ₁₀ H ₁₄	96	89	-----	-----	118	118	111	123	135	156
Isobutylbenzene		97	93	-----	-----	120	119	116	125	144	174
<i>sec</i> -Butylbenzene		96	93	146	138	123	122	112	125	147	177
<i>tert</i> -Butylbenzene		98	99	151	142	125	127	126	135	162	287
1-Methyl-4-isopropylbenzene		98	95	-----	-----	126	123	113	130	158	223
1,2-Diethylbenzene		-----	84	-----	125	-----	107	-----	-----	124	-----
1,3-Diethylbenzene		96	92	145	-----	-----	123	-----	136	165	226
1,4-Diethylbenzene		-----	94	-----	136	-----	119	-----	-----	163	-----
1,3-Dimethyl-5-ethylbenzene		-----	95	-----	140	-----	124	-----	-----	171	-----
1-Methyl-3- <i>tert</i> -butylbenzene	C ₁₁ H ₁₆	-----	93	-----	141	-----	125	-----	-----	169	-----
1-Methyl-4- <i>tert</i> -butylbenzene		-----	97	-----	142	-----	124	-----	-----	176	-----
1-Methyl-3,5-diethylbenzene		-----	95	-----	140	-----	126	-----	-----	171	-----
1,3,5-Triethylbenzene	C ₁₂ H ₁₈	-----	93	-----	140	-----	122	-----	-----	170	-----

* Performance numbers greater than 161 were determined as follows:

$$\text{performance number} = 161 \frac{\text{imep of blend}}{\text{imep of isooctane} + 6 \text{ ml TEL/gal}}$$

^b A. S. T. M. Aviation and A. S. T. M. Supercharge performance numbers of mixed base fuel, 120 and 112, respectively.

^c A. S. T. M. Supercharge data for compound determined at an industrial laboratory; A. S. T. M. Aviation data determined at NACA Lewis laboratory.

TABLE A-1.—A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE PERFORMANCE NUMBERS OF LEADED AND UNLEADED BLENDS WITH ISOCTANE AND WITH MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE ^a—Concluded

(c) Ethers.

Ether	Formula	Performance number									
		A. S. T. M. Aviation method								A. S. T. M. Supercharge method (<i>F/A</i> =0.11)	
		Unleaded		4 ml TEL/gal		4 ml TEL/gal					
		Volume percent aromatics in blend with isooctane				Volume percent ether in blend with mixed base fuel ^b					
		10	20	10	20	10	25	50	10	25	50
Methyl <i>tert</i> -butyl ether.....	C ₈ H ₁₈ O	100	102	149	c 153	134	143	150	137	175	250
Ethyl <i>tert</i> -butyl ether.....	C ₈ H ₁₈ O	100	104	157	161	140	144	150	132	150	185
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O	103	104	160	161	137	149	160	126	150	185
Methyl phenyl ether (anisole).....	C ₇ H ₈ O	93	90	141	121	118	107	94	125	142	137
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O	99	97	140	120	120	111	100	128	146	137
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole).....	C ₈ H ₁₀ O	99	99	144	133	120	112	100	133	145	136
<i>m</i> -Methylanisole.....	C ₈ H ₁₀ O	-----	-----	-----	-----	-----	110	-----	-----	147	-----
<i>o</i> -Methylanisole.....	C ₈ H ₁₀ O	-----	-----	-----	-----	-----	82	-----	-----	94	-----
<i>p</i> - <i>tert</i> -Butylanisole.....	C ₁₀ H ₁₈ O	-----	-----	-----	-----	-----	110	-----	-----	147	-----
<i>n</i> -Propyl phenyl ether.....	C ₉ H ₁₂ O	-----	-----	-----	-----	-----	110	-----	-----	149	-----
Isopropyl phenyl ether.....	C ₉ H ₁₂ O	-----	-----	-----	-----	-----	110	-----	-----	150	-----
<i>tert</i> -Butyl phenyl ether.....	C ₁₀ H ₁₄ O	-----	-----	-----	-----	-----	107	-----	-----	137	-----
Methyl benzyl ether.....	C ₈ H ₁₀ O	-----	-----	-----	-----	-----	104	-----	-----	111	-----
Isopropyl benzyl ether.....	C ₁₀ H ₁₄ O	-----	-----	-----	-----	-----	119	-----	-----	140	-----
Phenyl methylal ether.....	C ₁₀ H ₁₂ O	-----	-----	-----	-----	-----	48	-----	-----	64	-----
Methyl methylal ether.....	C ₈ H ₁₀ O	-----	-----	-----	-----	-----	93	-----	-----	92	-----
Isopropyl methylal ether.....	C ₇ H ₁₀ O	-----	-----	-----	-----	-----	106	-----	-----	102	-----
<i>tert</i> -Butyl methylal ether.....	C ₈ H ₁₀ O	-----	-----	-----	-----	-----	106	-----	-----	109	-----
Dimethylal ether.....	C ₈ H ₁₀ O	-----	-----	-----	-----	-----	77	-----	-----	90	-----
Methyl cyclopropyl ether.....	C ₈ H ₁₀ O	-----	-----	-----	-----	-----	76	-----	-----	94	-----
Methyl cyclopentyl ether.....	C ₈ H ₁₂ O	-----	-----	-----	-----	-----	80	-----	-----	83	-----
Methyl cyclohexyl ether.....	C ₇ H ₁₄ O	-----	-----	-----	-----	-----	68	-----	-----	79	-----
Propylene oxide.....	C ₃ H ₆ O	-----	-----	-----	-----	-----	100	-----	-----	131	-----

^a Performance numbers greater than 161 were determined as follows:

$$\text{performance number} = 161 \frac{\text{imep of blend}}{\text{imep of isooctane} + 6 \text{ ml TEL/gal}}$$

^b A. S. T. M. Aviation and A. S. T. M. Supercharge performance numbers of mixed base fuel, 120 and 112, respectively.^c Approximate value.TABLE A-2.—A. S. T. M. SUPERCHARGE KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF BLENDS WITH MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE + 4 ML TEL PER GALLON

[Standard conditions]

(a) Paraffins and olefins.

Paraffins and olefins	Formula	Imep ratio *														
		Volume percent added paraffin or olefin in blend with mixed base fuel														
		10					25					50				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Paraffins																
2,2,3-Trimethylbutane..... 2,3-Dimethylpentane.....	C ₇ H ₁₆	1.02	1.05	1.08	1.09	1.11	1.12 .96	1.21 .96	1.26 .98	1.29 1.00	1.30 1.00	1.49	1.51	1.66	1.73	1.74
2,2,3,3-Tetramethylpentane..... 2,2,3,4-Tetramethylpentane..... 2,3,3,4-Tetramethylpentane..... 2,4-Dimethyl-3-ethylpentane.....	C ₉ H ₂₀	--- 0.90 .95 ---	--- 0.93 .94 ---	--- 1.04 1.03 ---	--- 1.08 1.09 ---	--- 1.10 1.11 ---	--- 0.83 .85 .87 1.05	--- 0.83 .88 .84 1.05	--- 1.06 1.03 1.06 1.07	--- 1.31 1.19 1.23 1.09	--- 1.39 1.24 1.28 1.10	--- --- .75 ---	--- --- 0.75 .70 ---	--- --- 0.88 1.06 ---	--- --- 1.46 1.50 ---	--- --- 1.59 1.76 ---
Olefins																
2,3-Dimethyl-2-pentene..... 2,3,4-Trimethyl-2-pentene..... 3,4,4-Trimethyl-2-pentene.....	C ₇ H ₁₄ C ₈ H ₁₆	--- 0.84 .87	--- 0.82 .87	--- 0.92 1.00	--- 1.00 1.02	--- 1.00 1.03	0.75 0.69 .75	0.79 0.67 .73	0.88 0.72 .79	0.95 0.86 .91	1.02 0.93 .97	--- 0.59 .66	--- 0.50 .59	--- 0.48 .59	--- 0.58 .72	--- 0.67 .84

^a Imep ratio = $\frac{\text{knock-limited imep of blend with 4 ml TEL/gal}}{\text{knock-limited imep of mixed base fuel with 4 ml TEL/gal}}$

TABLE A-2.—A. S. T. M. SUPERCHARGE KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF BLENDS WITH MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE+4 ML TEL PER GALLON—Concluded

[Standard conditions]

(b) Aromatics.

Aromatic	Formula	Imep ratio *														
		Volume percent aromatic in blend with mixed base fuel														
		10					25					50				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Benzene.....	C ₆ H ₆	1.03	1.01	1.04	1.05	1.07	1.06	1.10	1.12	1.15	1.19	0.74	0.73	1.23	1.39	1.51
Methylbenzene.....	C ₇ H ₈	1.01	1.05	1.08	1.10	1.13	1.00	1.04	1.18	1.29	1.37	0.84	0.99	1.27	2.43	----
Ethylbenzene.....	C ₈ H ₁₀	1.06	1.07	1.11	1.12	1.14	1.02	1.08	1.21	1.29	1.37	0.99	1.07	1.44	1.71	----
1,2-Dimethylbenzene.....		.95	.94	.95	.92	.91	.76	.80	.89	.91	.68	.73	.87	.95	1.00	
1,3-Dimethylbenzene.....		.93	1.02	1.12	1.15	1.16	.82	.98	1.33	1.45	1.51	.80	.92	1.72	2.36	2.70
1,4-Dimethylbenzene.....		1.04	1.07	1.10	1.10	1.11	1.01	1.09	1.28	1.39	1.50	.94	1.03	1.57	2.83	----
<i>n</i> -Propylbenzene.....	C ₉ H ₁₂	0.94	0.99	1.08	1.10	1.11	0.94	1.08	1.24	1.30	1.38	0.73	0.78	1.18	1.37	1.86
Isopropylbenzene.....		1.14	1.13	1.13	1.11	1.12	1.05	1.07	1.14	1.23	1.30	.93	1.02	1.41	1.92	2.58
1-Methyl-2-ethylbenzene.....		----	----	----	----	----	.86	.91	1.03	1.08	1.09	----	----	----	----	----
1-Methyl-3-ethylbenzene.....		----	----	----	----	----	1.11	1.14	1.28	1.40	1.47	----	----	----	----	----
1-Methyl-4-ethylbenzene.....		1.01	1.04	1.11	1.15	1.15	.98	1.01	1.21	1.34	1.43	.93	.98	1.27	1.51	1.93
1,2,3-Trimethylbenzene.....		----	----	----	----	----	.82	.85	.90	.93	.94	----	----	----	----	----
1,2,4-Trimethylbenzene.....		.92	.95	.98	.98	.97	.81	.85	1.02	1.03	1.02	.81	.83	1.04	1.21	1.34
1,3,5-Trimethylbenzene.....		1.09	1.10	1.13	1.16	1.18	.94	.95	1.26	1.39	1.49	1.02	1.04	1.61	2.86	----
<i>n</i> -Butylbenzene.....	C ₁₀ H ₁₄	0.95	1.00	1.04	1.07	1.08	0.87	0.96	1.12	1.16	1.18	0.77	0.91	1.23	1.34	1.42
Isobutylbenzene.....		.95	1.04	1.09	1.09	1.11	.81	.97	1.20	1.25	1.30	.90	1.02	1.28	1.44	1.59
<i>sec</i> -Butylbenzene.....		1.01	.99	1.03	1.07	1.09	1.03	1.01	1.14	1.25	1.31	.98	.97	1.17	1.43	1.63
<i>tert</i> -Butylbenzene.....		1.06	1.06	1.10	1.14	1.15	1.12	1.17	1.31	1.42	1.47	1.06	1.13	1.46	2.28	2.58
1-Methyl-4-isopropylbenzene.....		1.01	1.06	1.12	1.15	1.15	.91	1.03	1.23	1.34	1.43	.77	.87	1.35	1.84	2.03
1,2-Diethylbenzene.....		----	----	----	----	----	.93	.96	1.01	1.04	1.07	----	----	----	----	----
1,3-Diethylbenzene.....		1.04	1.09	1.18	1.18	1.19	.99	1.07	1.28	1.42	1.51	.82	.86	1.23	1.81	2.09
1,4-Diethylbenzene.....		----	----	----	----	----	1.10	1.13	1.22	1.34	1.45	----	----	----	----	----
1,3-Dimethyl-5-ethylbenzene.....		----	----	----	----	----	1.07	1.11	1.28	1.41	1.49	----	----	----	----	----
1-Methyl-3- <i>tert</i> -butylbenzene.....	C ₁₁ H ₁₆	----	----	----	----	----	1.10	1.15	1.28	1.40	1.49	----	----	----	----	----
1-Methyl-4- <i>tert</i> -butylbenzene.....		----	----	----	----	----	1.11	1.14	1.28	1.43	1.53	----	----	----	----	----
1-Methyl-3, 5-diethylbenzene.....		----	----	----	----	----	1.10	1.15	1.27	1.39	1.50	----	----	----	----	----
1,3,5-Triethylbenzene.....	C ₁₂ H ₁₈	----	----	----	----	----	1.07	1.11	1.25	1.40	1.51	----	----	----	----	----

(c) Ethers.

Ether	Formula	Imep ratio *														
		Volume percent ether in blend with mixed base fuel														
		10					25					50				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Methyl <i>tert</i> -butyl ether.....	C ₅ H ₁₂ O	1.07	1.01	1.11	1.18	1.21	1.34	1.13	1.39	1.52	1.59	2.22	1.44	1.85	2.08	2.34
Ethyl <i>tert</i> -butyl ether.....	C ₆ H ₁₄ O	1.13	1.12	1.14	1.16	1.17	1.37	1.15	1.25	1.34	1.37	1.97	1.25	1.48	1.56	1.70
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O	1.13	1.07	1.09	1.10	1.11	1.34	1.26	1.24	1.29	1.33	2.5 b	1.45	1.45	1.53	1.66
Methyl phenyl ether (anisole).....	C ₇ H ₈ O	.94	.93	1.03	1.07	1.10	.87	.72	.96	1.13	1.25	.89	.63	.55	.92	1.21
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O	.97	.94	1.05	1.11	1.12	.99	.81	1.05	1.21	1.30	.95	.65	.64	.96	1.21
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole).....	C ₈ H ₁₀ O	.99	.90	1.05	1.13	1.16	.91	.74	.98	1.15	1.29	.94	.65	.56	.98	1.19
<i>m</i> -Methylanisole.....	C ₈ H ₁₀ O	----	----	----	----	----	.97	.90	1.07	1.21	1.31	----	----	----	----	----
<i>o</i> -Methylanisole.....	C ₈ H ₁₀ O	----	----	----	----	----	.60	.52	.57	.75	.85	----	----	----	----	----
<i>p</i> - <i>tert</i> -Butylanisole.....	C ₁₁ H ₁₆ O	----	----	----	----	----	.95	.91	.94	1.10	1.32	----	----	----	----	----
<i>n</i> -Propyl phenyl ether.....	C ₉ H ₁₂ O	----	----	----	----	----	1.06	1.00	1.17	1.27	1.35	----	----	----	----	----
Isopropyl phenyl ether.....	C ₉ H ₁₂ O	----	----	----	----	----	.99	.90	1.05	1.26	1.35	----	----	----	----	----
<i>tert</i> -Butyl phenyl ether.....	C ₁₀ H ₁₄ O	----	----	----	----	----	.88	.81	.92	1.06	1.21	----	----	----	----	----
Methyl benzyl ether.....	C ₈ H ₁₀ O	----	----	----	----	----	.83	.89	.95	.96	.99	----	----	----	----	----
Isopropyl benzyl ether.....	C ₁₀ H ₁₄ O	----	----	----	----	----	.95	.96	1.10	1.20	1.24	----	----	----	----	----
Phenyl methallyl ether.....	C ₁₀ H ₁₂ O	----	----	----	----	----	.48	.44	.38	.45	.54	----	----	----	----	----
Methyl methallyl ether.....	C ₈ H ₁₀ O	----	----	----	----	----	.64	.66	.79	.81	.83	----	----	----	----	----
Isopropyl methallyl ether.....	C ₉ H ₁₂ O	----	----	----	----	----	.77	.71	.85	.90	.92	----	----	----	----	----
<i>tert</i> -Butyl methallyl ether.....	C ₉ H ₁₂ O	----	----	----	----	----	.68	.66	.81	.90	.97	----	----	----	----	----
Dimethallyl ether.....	C ₈ H ₁₄ O	----	----	----	----	----	.64	.58	.68	.78	.81	----	----	----	----	----
Methyl cyclopropyl ether.....	C ₆ H ₈ O	----	----	----	----	----	.61	.57	.72	.82	.85	----	----	----	----	----
Methyl cyclopentyl ether.....	C ₈ H ₁₂ O	----	----	----	----	----	.67	.61	.69	.75	.75	----	----	----	----	----
Methyl cyclohexyl ether.....	C ₇ H ₁₀ O	----	----	----	----	----	.63	.65	.61	.67	.70	----	----	----	----	----
Propylene oxide.....	C ₃ H ₆ O	----	----	----	----	----	.94	.87	1.01	1.12	1.15	----	----	----	----	----

* Imep ratio = $\frac{\text{knock-limited imep of blend with 4 ml TEL/gal}}{\text{knock-limited imep of mixed base fuel with 4 ml TEL/gal}}$

b Approximate value.

TABLE A-3.—17.6 ENGINE KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF UNLEADED BLENDS WITH ISOCTANE

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(a) Paraffins and olefins.

Paraffins and olefins	Formula	Imep ratio *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent added paraffin or olefin in blend with isooctane														
		10					25					50				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Paraffins																
2,2,3-Trimethylbutane.....	C ₇ H ₁₆	1.02	1.02	1.03	1.09	1.10	1.08	1.07	1.09	1.17	1.20	1.16	1.15	1.21	1.20	1.20
2,3-Dimethylpentane.....		----	----	----	----	----	.95	.96	.94	.94	.94	.93	.93	.95	.94	.93
2,2,3,3-Tetramethylpentane.....	C ₉ H ₂₀	----	----	----	----	----	1.12	1.14	1.20	1.32	1.31	1.23	1.22	1.27	1.27	1.27
2,2,3,4-Tetramethylpentane.....		.96	.96	1.01	1.11	1.13	.91	.91	1.00	1.15	1.22	1.09	1.10	1.15	1.21	1.22
2,3,3,4-Tetramethylpentane.....		.97	.98	1.03	1.06	1.13	.95	.96	1.07	1.16	1.27	1.17	1.19	1.22	1.27	1.27
2,4-Dimethyl-3-ethylpentane.....		----	----	----	----	----	.99	1.00	1.00	1.01	1.00	1.03	1.02	1.02	.98	.97
Olefins																
2,3-Dimethyl-2-pentene.....	C ₇ H ₁₄	----	----	----	----	----	0.99	0.99	1.01*	1.09	1.15	1.14	1.12	1.17	1.23	1.27
2,3,4-Trimethyl-2-pentene.....	C ₈ H ₁₆	0.90	0.90	0.89	0.91	0.96	0.79	0.79	0.78	0.82	0.89	0.95	0.95	0.96	0.99	1.02
3,4,4-Trimethyl-2-pentene.....		.96	.98	.99	1.00	1.05	.93	.94	.94	.98	1.06	1.05	1.04	1.08	1.16	1.20

(b) Aromatics.

Aromatic	Formula	Imep ratio *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent aromatic in blend with isooctane														
		10					25					50				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Benzene.....	C ₆ H ₆	1.00	1.01	1.02	1.03	1.04	1.00	1.01	1.02	1.10	1.17	1.08	1.06	1.08	1.12	1.15
Methylbenzene.....	C ₇ H ₈	1.04	1.02	1.05	1.08	1.11	1.04	1.02	1.11	1.21	1.26	1.05	1.07	1.18	1.21	1.24
Ethylbenzene.....	C ₈ H ₁₀	1.02	1.02	1.04	1.11	1.16	1.02	1.00	1.07	1.21	1.31	1.19	1.21	1.28	1.35	1.36
1,2-Dimethylbenzene.....		.95	.95	.93	.97	1.00	.90	.89	.87	.93	.99	.97	.98	1.00	1.02	1.06
1,3-Dimethylbenzene.....		.97	.98	1.03	1.07	1.08	.93	.94	1.08	1.14	1.18	1.14	1.15	1.22	1.35	1.40
1,4-Dimethylbenzene.....		1.01	1.01	1.10	1.12	1.14	1.04	1.04	1.16	1.28	1.38	1.15	1.20	1.29	1.34	1.40
n-Propylbenzene.....	C ₉ H ₁₂	0.99	0.99	1.00	1.10	1.16	0.99	0.99	1.02	1.15	1.23	1.14	1.14	1.19	1.30	1.32
Isopropylbenzene.....		1.01	1.00	1.09	1.14	1.16	1.05	.98	1.07	1.20	1.29	1.15	1.20	1.29	1.32	1.34
1-Methyl-2-ethylbenzene.....		---	---	---	---	---	1.03	1.01	1.03	1.07	1.10	1.03	1.04	1.09	1.08	1.10
1-Methyl-3-ethylbenzene.....		---	---	---	---	---	1.11	1.11	1.21	1.29	1.33	1.28	1.31	1.39	1.41	1.43
1-Methyl-4-ethylbenzene.....		1.00	1.00	1.05	1.08	1.09	1.00	1.00	1.11	1.14	1.18	1.01	1.03	1.11	1.17	1.21
1,2,3-Trimethylbenzene.....		---	---	---	---	---	.97	.97	.99	1.00	1.02	1.04	1.03	1.05	1.07	1.09
1,2,4-Trimethylbenzene.....		.91	.93	.95	.99	1.01	.87	.86	.91	.96	1.05	.96	.95	1.02	1.07	1.10
1,3,5-Trimethylbenzene.....		.96	.98	1.03	1.15	1.21	.91	.95	1.08	1.27	1.42	1.15	1.16	1.31	1.48	1.51
n-Butylbenzene.....	C ₁₀ H ₁₄	0.95	0.98	0.97	0.99	1.02	0.93	0.96	0.96	0.99	1.04	1.04	1.05	1.08	1.10	1.09
Isobutylbenzene.....		.98	.98	.99	1.02	1.07	.97	.94	.96	1.03	1.11	1.10	1.11	1.15	1.19	1.20
sec-Butylbenzene.....		.98	1.00	1.05	1.08	1.10	.98	.99	1.08	1.16	1.26	1.14	1.12	1.19	1.23	1.23
tert-Butylbenzene.....		.98	1.00	1.05	1.12	1.19	.95	.95	1.08	1.15	1.27	1.24	1.23	1.32	1.41	1.43
1-Methyl-4-isopropylbenzene.....		.98	.99	1.04	1.07	1.12	.94	.95	1.02	1.10	1.20	1.14	1.14	1.21	1.33	1.37
1,2-Diethylbenzene.....		---	---	---	---	---	1.03	1.01	1.03	1.08	1.11	1.04	1.06	1.11	1.11	1.12
1,3-Diethylbenzene.....		1.01	1.01	1.05	1.12	1.19	1.05	1.04	1.11	1.24	1.38	1.27	1.27	1.39	1.46	1.49
1,4-Diethylbenzene.....		---	---	---	---	---	1.11	1.12	1.23	1.34	1.41	1.32	1.33	1.41	1.44	1.41
1,3-Dimethyl-5-ethylbenzene.....		---	---	---	---	---	1.10	1.07	1.17	1.24	1.29	1.24	1.26	1.31	1.35	1.32
1-Methyl-3-tert-butylbenzene.....	C ₁₁ H ₁₆	---	---	---	---	---	1.16	1.16	1.26	1.36	1.37	1.30	1.33	1.37	1.39	1.38
1-Methyl-4-tert-butylbenzene.....		---	---	---	---	---	1.12	1.14	1.22	1.31	1.37	1.31	1.32	1.37	1.40	1.38
1-Methyl-3,5-diethylbenzene.....		---	---	---	---	---	1.10	1.11	1.23	1.36	1.44	1.30	1.36	1.48	1.51	1.51
1,3,5-Triethylbenzene.....	C ₁₂ H ₁₈	---	---	---	---	---	1.09	1.09	1.19	1.31	1.42	1.35	1.37	1.46	1.50	1.48

* Imep ratio = $\frac{\text{knock-limited imep of blend}}{\text{knock-limited imep of isooctane}}$

TABLE A-3.—17.6 ENGINE KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF UNLEADED BLENDS WITH ISOCTANE—Concluded

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]
(c) Ethers.

Ether	Formula	Imep ratio *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent ether in blend with isoctane														
		10					20					20				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Methyl <i>tert</i> -butyl ether.....	C ₈ H ₁₈ O	1.06	1.07	1.10	1.16	1.16	1.15	1.14	1.20	1.34	1.38	1.26	1.28	1.35	1.37	1.35
Ethyl <i>tert</i> -butyl ether.....	C ₈ H ₁₈ O	1.14	1.15	1.19	1.20	1.21	1.24	1.25	1.28	1.37	1.45	1.33	1.32	1.31	1.35	1.37
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O	1.11	1.11	1.12	1.15	1.14	1.19	1.19	1.21	1.27	1.30	1.29	1.29	1.27	1.27	1.26
Methyl phenyl ether (anisole).....	C ₇ H ₈ O	1.06	1.05	1.07	1.10	1.15	1.13	1.10	1.09	1.20	1.32	1.28	1.27	1.26	1.31	1.34
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O	1.12	1.11	1.11	1.16	1.18	1.27	1.24	1.17	1.31	1.40	1.40	1.37	1.33	1.41	1.41
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole)---	C ₈ H ₁₀ O	1.12	1.09	1.12	1.18	1.21	1.20	1.14	1.17	1.28	1.38	1.29	1.29	1.34	1.42	1.44

$$* \text{ Imep ratio} = \frac{\text{knock-limited imep of blend}}{\text{knock-limited imep of isoctane}}$$

TABLE A-4.—17.6 ENGINE KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF BLENDS WITH ISOCTANE+4 ML TEL PER GALLON

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]
(a) Paraffins and olefins.

Paraffins and olefins	Formula	Imep ratio *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent added paraffin or olefin in blend with isoctane														
		10					20					20				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Paraffins																
2,2,3-Trimethylbutane.....	C ₇ H ₁₆	1.05	1.06	1.09	1.10	1.10	1.13	1.16	1.22	1.23	1.24	1.20	1.18	1.22	1.20	1.19
2,3-Dimethylpentane.....		----	----	----	----	----	.96	.97	.98	.96	.95	.96	.97	.97	.96	.95
2,2,3,3-Tetramethylpentane.....	C ₈ H ₂₀	----	----	----	----	----	1.19	1.18	1.31	1.34	1.32	1.23	1.25	1.28	1.29	1.29
2,2,3,4-Tetramethylpentane.....		1.02	1.03	1.04	1.09	1.11	1.04	1.06	1.10	1.19	1.23	1.17	1.18	1.21	1.21	1.19
2,3,3,4-Tetramethylpentane.....		1.05	1.05	1.09	1.14	1.17	1.10	1.10	1.15	1.24	1.27	1.17	1.18	1.20	1.20	1.19
2,4-Dimethyl-3-ethylpentane.....		----	----	----	----	----	1.01	1.01	1.02	1.01	.99	1.00	1.01	1.01	1.01	1.00
Olefins																
2,3-Dimethyl-2-pentene.....	C ₇ H ₁₄	----	----	----	---	----	0.95	0.95	1.00	1.13	1.16	1.14	1.14	1.19	1.24	1.23
2,3,4-Trimethyl-2-pentene.....	C ₈ H ₁₆	0.91	0.91	0.89	0.94	0.97	0.81	0.81	0.78	0.85	0.92	0.99	1.00	1.00	1.05	1.09
3,4,4-Trimethyl-2-pentene.....		.96	.96	.97	1.01	1.05	.93	.92	.92	.99	1.07	1.10	1.10	1.10	1.15	1.18

$$* \text{ Imep ratio} = \frac{\text{knock-limited imep of blend with 4 ml TEL/gal}}{\text{knock-limited imep of isoctane with 4 ml TEL/gal}}$$

TABLE A-4.—17.6 ENGINE KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF BLENDS WITH ISOCTANE+4 ML TEL PER GALLON—Continued

Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(b) Aromatics.

Aromatic	Formula	Imep ratio *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent aromatic in blend with isooctane														
		10					20					20				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Benzene.....	C ₆ H ₆	1.00	1.02	1.06	1.08	1.09	0.99	1.04	1.10	1.14	1.17	1.11	1.11	1.17	1.21	1.23
Methylbenzene.....	C ₇ H ₈	1.01	1.03	1.04	1.05	1.10	1.05	1.07	1.11	1.19	1.26	1.13	1.17	1.23	1.29	1.31
Ethylbenzene.....	C ₈ H ₁₀	1.07	1.07	1.08	1.10	1.13	1.14	1.17	1.22	1.31	1.36	1.29	1.33	1.36	1.37	1.38
1,2-Dimethylbenzene.....		.89	.91	.88	.87	.86	.77	.82	.84	.83	.82	.81	.82	.84	.86	.87
1,3-Dimethylbenzene.....		1.07	1.09	1.13	1.19	1.23	1.12	1.16	1.29	1.41	1.46	1.28	1.29	1.44	1.48	1.51
1,4-Dimethylbenzene.....		1.04	1.08	1.12	1.14	1.16	1.05	1.13	1.23	1.34	1.41	1.23	1.28	1.42	1.47	1.48
n-Propylbenzene.....	C ₉ H ₁₂	1.05	1.05	1.09	1.14	1.16	1.11	1.13	1.21	1.29	1.33	1.34	1.38	1.42	1.36	1.31
Isopropylbenzene.....		1.04	1.10	1.12	1.16	1.20	1.10	1.12	1.19	1.31	1.38	1.33	1.35	1.35	1.38	1.41
1-Methyl-2-ethylbenzene.....		-----	-----	-----	-----	-----	.96	1.00	1.03	1.01	.98	1.00	1.00	1.01	1.00	.99
1-Methyl-3-ethylbenzene.....		-----	-----	-----	-----	-----	1.26	1.32	1.39	1.46	1.46	1.42	1.46	1.52	1.48	1.47
1-Methyl-4-ethylbenzene.....		1.03	1.04	1.14	1.17	1.20	1.11	1.12	1.21	1.33	1.41	1.34	1.35	1.45	1.42	1.41
1,2,3-Trimethylbenzene.....		-----	-----	-----	-----	-----	.83	.87	.90	.89	.91	.87	.88	.90	.93	.94
1,2,4-Trimethylbenzene.....		.91	.94	.97	.95	.94	.85	.88	.95	.94	.94	.88	.90	.94	.94	.95
1,3,5-Trimethylbenzene.....		1.00	1.03	1.12	1.16	1.17	1.02	1.07	1.24	1.40	1.47	1.26	1.29	1.49	1.55	1.55
n-Butylbenzene.....		C ₁₀ H ₁₄	1.01	1.03	1.04	1.03	1.06	1.02	1.04	1.07	1.10	1.11	1.10	1.11	1.12	1.14
Isobutylbenzene.....	1.09		1.08	1.09	1.14	1.15	1.14	1.15	1.18	1.25	1.29	1.20	1.22	1.23	1.25	1.24
sec-Butylbenzene.....	1.09		1.08	1.08	1.14	1.17	1.16	1.13	1.20	1.29	1.35	1.26	1.28	1.30	1.31	1.31
tert-Butylbenzene.....	1.12		1.12	1.12	1.16	1.18	1.32	1.30	1.31	1.36	1.41	1.37	1.36	1.39	1.41	1.42
1-Methyl-4-isopropylbenzene.....	1.08		1.08	1.13	1.17	1.19	1.12	1.13	1.22	1.39	1.45	1.30	1.32	1.42	1.42	1.43
1,2-Diethylbenzene.....	-----		-----	-----	-----	-----	1.00	1.03	1.07	1.11	1.10	1.11	1.10	1.11	1.10	1.08
1,3-Diethylbenzene.....	1.08		1.11	1.18	1.22	1.25	1.17	1.25	1.32	1.48	1.54	1.45	1.47	1.53	1.53	1.53
1,4-Diethylbenzene.....	-----		-----	-----	-----	-----	1.25	1.27	1.32	1.42	1.45	1.40	1.47	1.53	1.49	1.47
1,3-Dimethyl-5-ethylbenzene.....	-----		-----	-----	-----	-----	1.17	1.24	1.35	1.38	1.38	1.39	1.46	1.53	1.46	1.43
1-Methyl-3-tert-butylbenzene.....	C ₁₁ H ₁₆		-----	-----	-----	-----	-----	1.22	1.24	1.32	1.41	1.42	1.36	1.40	1.45	1.47
1-Methyl-4-tert-butylbenzene.....		-----	-----	-----	-----	-----	1.23	1.25	1.32	1.36	1.36	1.37	1.40	1.44	1.45	1.45
1-Methyl-3,5-diethylbenzene.....		-----	-----	-----	-----	-----	1.22	1.25	1.36	1.44	1.50	1.47	1.54	1.59	1.56	1.54
1,3,5-Triethylbenzene.....	C ₁₂ H ₁₈	-----	-----	-----	-----	-----	1.20	1.24	1.36	1.49	1.51	1.49	1.58	1.59	1.53	1.53

* Imep ratio = $\frac{\text{knock-limited imep of blend with 4 ml TEL/gal}}{\text{knock-limited imep of isooctane with 4 ml TEL/gal}}$

TABLE A-4.—17.6 ENGINE KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF BLENDS WITH ISOCTANE+4 ML TEL PER GALLON—Concluded

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(c) Ethers.

Ether	Formula	Imep ratio *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent ether in blend with isooctane														
		10					20					20				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Methyl <i>tert</i> -butyl ether.....	C ₅ H ₁₂ O	1.14	1.15	1.19	1.21	1.22	1.26	1.26	1.32	1.42	1.43	1.41	1.43	1.45	1.47	1.46
Ethyl <i>tert</i> -butyl ether.....	C ₆ H ₁₄ O	1.15	1.13	1.12	1.15	1.15	1.41	1.39	1.37	1.37	1.36	1.41	1.42	1.40	1.37	1.33
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O	1.13	1.11	1.13	1.15	1.15	1.32	1.27	1.24	1.27	1.27	1.33	1.35	1.31	1.27	1.24
Methyl phenyl ether (anisole).....	C ₇ H ₈ O	1.07	1.07	1.07	1.13	1.16	1.16	1.14	1.11	1.26	1.37	1.34	1.33	1.33	1.39	1.40
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O	1.11	1.09	1.15	1.18	1.18	1.28	1.25	1.29	1.39	1.45	1.44	1.43	1.43	1.44	1.45
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole)...	C ₈ H ₁₀ O	1.12	1.12	1.16	1.26	1.29	1.16	1.17	1.25	1.36	1.43	1.43	1.41	1.40	1.45	1.48

* Imep ratio = $\frac{\text{knock-limited imep of blend with 4 ml TEL/gal}}{\text{knock-limited imep of isooctane with 4 ml TEL/gal}}$

TABLE A-5.—KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF BLENDS WITH MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE+4 ML TEL PER GALLON

(a) Paraffins and olefins; 25 volume percent blends.

[For 17.6 engine: compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C. For full-scale single-cylinder at simulated take-off: compression ratio, 7.3; engine speed, 2500 rpm; inlet-air temperature, 250° F; spark advance, 30° B. T. C.; cooling-air flow such that rear-spark-plug-bushing temperature equals 365° F at 140 bme and 0.10 fuel-air ratio. For full-scale single-cylinder engine at simulated cruise: same conditions as for take-off except for engine speed, 2000 rpm; inlet-air temperature, 210° F.]

Paraffins and olefins	Formula	Imep ratio *																			
		17.6 engine										Full-scale single-cylinder engine									
		Inlet-air temperature, 250° F					Inlet-air temperature, 100° F					Simulated take-off conditions					Simulated cruise conditions				
		Fuel-air ratio																			
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Paraffins																					
2-Methylbutane.....	C ₅ H ₁₂	----	----	----	----	----	----	----	----	----	1.07	1.05	1.14	1.10	1.09	1.10	1.12	1.14	1.11	1.07	
2,2-Dimethylbutane.....	C ₆ H ₁₄	----	----	----	----	----	----	----	----	----	1.13	1.10	1.08	1.07	1.05	1.10	1.13	1.10	1.07	1.07	
2,3-Dimethylbutane.....		----	----	----	----	----	----	----	----	----	1.27	1.25	1.19	1.12	1.10	1.27	1.26	1.19	1.13	1.08	
2,2,3-Trimethylbutane.....	C ₇ H ₁₆	1.24	1.24	1.32	1.30	1.28	1.28	1.29	1.31	1.31	1.28	1.39	1.31	1.34	1.28	1.27	1.35	1.31	1.26	1.30	1.25
2,3-Dimethylpentane.....		1.03	1.03	1.05	1.04	1.02	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
2,2,3-Trimethylpentane.....	C ₈ H ₁₈	----	----	----	----	----	----	----	----	----	----	1.26	1.22	1.27	1.25	1.23	1.27	1.26	1.26	1.28	1.24
2,3,3-Trimethylpentane.....		----	----	----	----	----	----	----	----	----	----	1.26	1.21	1.22	1.21	1.20	1.19	1.20	1.18	1.22	1.21
2,3,4-Trimethylpentane.....		----	----	----	----	----	----	----	----	----	----	1.15	1.06	1.16	1.17	1.15	1.17	1.16	1.17	1.18	1.14
2,2,3,3-Tetramethylpentane.....	C ₉ H ₂₀	1.26	1.26	1.35	1.39	1.37	----	----	----	----	----	----	----	----	----	----	1.22	1.20	1.26	1.29	1.22
2,2,3,4-Tetramethylpentane.....		1.12	1.10	1.16	1.23	1.26	1.20	1.21	1.24	1.24	1.24	----	----	----	----	----	1.22	1.20	1.26	1.29	1.22
2,2,4,4-Tetramethylpentane.....		1.13	1.11	1.19	1.29	1.30	1.22	1.22	1.27	1.30	1.28	1.14	1.06	0.99	0.99	1.02	1.05	1.03	.98	1.00	1.00
2,3,3,4-Tetramethylpentane.....		1.04	1.06	1.08	1.08	1.07	----	----	----	----	----	----	----	----	----	----	1.21	1.15	1.21	1.27	1.28
2,4-Dimethyl-3-ethylpentane.....																					
Olefins																					
2,3-Dimethyl-2-pentene.....	C ₇ H ₁₄	1.03	1.07	1.12	1.16	1.20	1.12	1.11	1.11	1.14	1.16	----	----	----	----	----	0.95	0.96	1.05	1.00	1.04
2,3,4-Trimethyl-2-pentene.....	C ₈ H ₁₆	.94	.94	.94	.99	1.05	1.12	1.11	1.11	1.14	1.16	----	----	----	----	----	0.95	0.96	1.05	1.00	1.04
2,4,4-Trimethyl-1-pentene.....		----	----	----	----	----	----	----	----	----	----	1.04	1.01	1.14	1.32	1.37	1.26	1.23	1.31	1.35	1.33
2,4,4-Trimethyl-2-pentene.....		----	----	----	----	----	----	----	----	----	----	1.06	1.01	1.08	1.12	1.21	1.25	1.20	1.17	1.12	1.15
3,4,4-Trimethyl-2-pentene.....		.99	1.01	1.04	1.10	1.13	1.25	1.22	1.20	1.23	1.25	----	----	----	----	----	1.18	1.12	1.17	1.13	1.12

(b) Aromatics

Aromatic	Formula	Imep ratio *																			
		17.6 engine										Full-scale single-cylinder engine									
		Inlet-air temperature, 250° F					Inlet-air temperature, 100° F					Simulated take-off conditions					Simulated cruise conditions				
		Fuel-air ratio																			
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Benzene.....	C ₆ H ₆	1.04	1.08	1.15	1.22	1.28	1.18	1.21	1.23	1.27	1.28	0.97	0.93	1.14	1.19	1.20	1.20	1.19	1.17	1.23	1.25
Methylbenzene.....	C ₇ H ₈	----	----	----	----	----	----	----	----	----	----	1.10	1.12	1.41	1.40	1.38	1.28	1.30	1.41	1.49	1.42
Ethylbenzene.....	C ₈ H ₁₀	----	----	----	----	----	----	----	----	----	----	1.12	1.22	1.33	1.36	1.34	1.35	1.39	1.45	1.39	1.35
1,2-Dimethylbenzene.....		0.81	0.84	0.89	0.91	0.92	0.89	0.90	0.91	0.92	0.94	.77	.71	.98	.99	.99	.77	.82	.98	.99	.98
1,3-Dimethylbenzene.....		1.21	1.22	1.46	1.53	1.54	1.34	1.38	1.51	1.56	1.57	1.19	1.21	1.55	1.64	1.64	1.23	1.31	1.68	1.62	1.58
1,4-Dimethylbenzene.....		----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
<i>n</i> -Propylbenzene.....	C ₉ H ₁₂	1.22	1.24	1.31	1.31	1.34	1.34	1.36	1.39	1.38	1.40	----	----	----	----	----	1.47	1.45	1.48	1.40	1.38
Isopropylbenzene.....		1.25	1.27	1.40	1.45	1.48	1.45	1.47	1.50	1.51	1.52	1.25	1.27	1.39	1.47	1.43	1.40	1.36	1.52	1.48	1.41
1-Methyl-2-ethylbenzene.....		1.02	1.06	1.09	1.06	1.05	----	----	----	----	----	1.25	1.27	1.39	1.47	1.43	1.40	1.36	1.52	1.48	1.41
1-Methyl-3-ethylbenzene.....		1.34	1.43	1.54	1.56	1.57	----	----	----	----	----	1.25	1.27	1.39	1.47	1.43	1.40	1.36	1.52	1.48	1.41
1-Methyl-4-ethylbenzene.....		1.05	1.09	1.25	1.41	1.48	1.33	1.33	1.59	1.59	1.58	1.19	1.18	1.59	1.62	1.58	1.53	1.52	1.65	1.66	1.56
1,2,3-Trimethylbenzene.....		.94	.94	.99	.98	.99	1.33	1.33	1.59	1.59	1.58	1.19	1.18	1.59	1.62	1.58	1.53	1.52	1.65	1.66	1.56
1,2,4-Trimethylbenzene.....		.86	.92	1.01	1.03	1.05	.96	.99	1.04	1.04	1.06	.88	.87	1.11	1.10	1.10	.74	.87	1.05	1.08	1.07
1,3,5-Trimethylbenzene.....		1.13	1.22	1.45	1.58	1.66	----	----	----	----	----	1.19	1.25	1.63	1.72	1.72	1.34	1.39	1.54	1.62	1.64
<i>n</i> -Butylbenzene.....	C ₁₀ H ₁₄	1.03	1.04	1.12	1.16	1.18	1.13	1.14	1.17	1.18	1.19	----	----	----	----	----	1.11	1.10	1.20	1.22	1.19
Isobutylbenzene.....		1.09	1.13	1.18	1.23	1.26	1.27	1.27	1.29	1.27	1.25	1.16	1.11	1.41	1.35	1.38	1.35	1.31	1.35	1.34	1.31
<i>sec</i> -Butylbenzene.....		1.22	1.22	1.27	1.33	1.41	1.31	1.32	1.36	1.40	1.40	1.16	1.11	1.41	1.35	1.38	1.34	1.34	1.37	1.38	1.35
<i>tert</i> -Butylbenzene.....		1.24	1.22	1.34	1.41	1.44	1.46	1.52	1.54	1.54	1.54	1.24	1.30	1.50	1.53	1.52	1.49	1.49	1.57	1.55	1.49
1-Methyl-4-isopropylbenzene.....		1.23	1.23	1.36	1.44	1.49	1.41	1.44	1.51	1.53	1.53	1.24	1.30	1.50	1.53	1.52	1.51	1.51	1.60	1.52	1.52
1,2-Diethylbenzene.....		1.07	1.11	1.14	1.14	1.13	----	----	----	----	----	1.26	1.41	1.71	1.62	1.54	1.67	1.71	1.78	1.63	1.55
1,3-Diethylbenzene.....		1.25	1.26	1.43	1.48	1.51	1.47	1.48	1.57	1.60	1.60	1.26	1.41	1.71	1.62	1.54	1.67	1.71	1.78	1.63	1.55
1,4-Diethylbenzene.....		1.43	1.50	1.57	1.55	1.52	----	----	----	----	----	1.26	1.41	1.71	1.62	1.54	1.67	1.71	1.78	1.63	1.55
1,3-Dimethyl-5-ethylbenzene.....		1.36	1.45	1.49	1.49	1.47	----	----	----	----	----	1.26	1.41	1.71	1.62	1.54	1.67	1.71	1.78	1.63	1.55
1-Methyl-3- <i>tert</i> -butylbenzene.....	C ₁₁ H ₁₆	1.40	1.44	1.52	1.55	1.50	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
1-Methyl-4- <i>tert</i> -butylbenzene.....		1.44	1.49	1.57	1.59	1.57	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
1-Methyl-3,5-diethylbenzene.....		1.44	1.50	1.65	1.71	1.69	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----
1,3,5-Triethylbenzene.....	C ₁₂ H ₁₈	1.44	1.51	1.62	1.63	1.60	----	----	----	----	----	----	----	----	----	----	----	----	----	----	----

* Imep ratio = $\frac{\text{knock-limited imep of blend with 4 ml TEL/gal}}{\text{knock-limited imep of mixed base fuel with 4 ml TEL/gal}}$

TABLE A-5.—KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF BLENDS WITH MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE+4 ML TEL PER GALLON—Concluded

(c) Ethers.

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

Ether	Formula	Volume percent ether in blend	17.6 Engine imep ratio *									
			Inlet-air temperature (°F)									
			100					250				
			Fuel-air ratio									
			0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Methyl <i>tert</i> -butyl ether.....	C ₅ H ₁₂ O	10	1.18	1.17	1.17	1.18	1.16	1.14	1.14	1.15	1.15	1.13
Ethyl <i>tert</i> -butyl ether.....	C ₆ H ₁₄ O		1.11	1.11	1.10	1.09	1.06	1.11	1.12	1.13	1.13	1.13
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O		1.11	1.11	1.11	1.10	1.11	1.13	1.13	1.10	1.10	1.11
Methyl phenyl ether (anisole).....	C ₇ H ₈ O		1.18	1.20	1.17	1.18	1.18	1.11	1.11	1.12	1.15	1.16
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O		1.17	1.16	1.16	1.17	1.17	1.15	1.14	1.11	1.12	1.15
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole).....	C ₈ H ₁₀ O		1.18	1.15	1.18	1.19	1.20	1.15	1.15	1.19	1.19	1.20
Methyl <i>tert</i> -butyl ether.....	C ₅ H ₁₂ O	25	1.51	1.50	1.50	1.53	1.51	1.45	1.43	1.49	1.52	1.51
Ethyl <i>tert</i> -butyl ether.....	C ₆ H ₁₄ O		1.44	1.43	1.41	1.36	1.32	1.47	1.45	1.38	1.39	1.39
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O		1.36	1.37	1.38	1.35	1.33	1.33	1.33	1.31	1.31	1.31
Methyl phenyl ether (anisole).....	C ₇ H ₈ O		1.46	1.48	1.45	1.50	1.54	1.31	1.29	1.27	1.36	1.41
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O		1.57	1.55	1.54	1.55	1.56	1.39	1.34	1.30	1.37	1.43
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole).....	C ₈ H ₁₀ O		1.55	1.50	1.52	1.59	1.65	1.38	1.33	1.37	1.44	1.52
Methyl <i>tert</i> -butyl ether.....	C ₅ H ₁₂ O	50	2.28	2.31	2.56	2.59	2.40	2.07	1.84	2.28	2.46	2.44
Ethyl <i>tert</i> -butyl ether.....	C ₆ H ₁₄ O		2.27	2.35	2.28	2.14	2.02	1.99	1.88	1.76	1.90	2.01
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O		2.14	2.16	2.10	2.00	1.98	2.00	1.98	1.88	1.90	1.95
Methyl phenyl ether (anisole).....	C ₇ H ₈ O		-----	-----	-----	2.63	2.72	1.71	1.50	1.41	1.74	2.07
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O		-----	2.80	2.61	2.63	2.72	1.77	1.64	1.52	1.99	2.22
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole).....	C ₈ H ₁₀ O		-----	2.71	2.42	2.74	2.96	1.43	1.28	1.24	1.40	1.65

* Imep ratio = $\frac{\text{knock-limited imep of blend with 4 ml TEL/gal}}{\text{knock-limited imep of mixed base fuel with 4 ml TEL/gal}}$

TABLE A-6.—17.6 ENGINE TEMPERATURE SENSITIVITY OF BLENDS RELATIVE TO ISOCTANE AND MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE+4 ML TEL PER GALLON

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(a) Paraffins and olefins.

Paraffins and olefins	Formula	Relative temperature sensitivity *														
		20 volume percent added paraffin or olefin in blend with isooctane										25 volume percent added paraffin or olefin in blend with mixed base fuel				
		Unleaded					4 ml TEL/gal									
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Paraffins																
2, 2, 3-Trimethylbutane..... 2, 3-Dimethylpentane.....	C ₇ H ₁₆	1.05 1.00	1.05 .95	1.10 1.00	1.05 1.00	1.00 1.00	1.05 1.00	1.00 1.00	1.00 1.00	1.00 1.00	0.95 1.00	1.05 ----	1.05 ----	1.00 ----	1.00 ----	1.00 ----
2, 2, 3, 3-Tetramethylpentane..... 2, 2, 3, 4-Tetramethylpentane..... 2, 3, 3, 4-Tetramethylpentane..... 2, 4-Dimethyl-3-ethylpentane.....	C ₈ H ₂₀	1.10 1.20 1.25 1.05	1.05 1.20 1.25 1.00	1.05 1.15 1.15 1.00	0.95 1.05 1.10 1.00	0.95 1.00 1.00 .95	1.05 1.05 1.05 1.00	1.05 1.10 1.05 1.00	1.00 1.10 1.05 1.00	0.95 1.00 .95 1.00	0.95 .95 .95 1.00	---- 1.05 1.10 ----	---- 1.10 1.10 ----	---- 1.05 1.05 ----	---- 1.00 1.00 ----	---- 1.00 1.00 ----
Olefins																
2, 3-Dimethyl-2-pentene.....	C ₇ H ₁₄	1.15	1.15	1.15	1.15	1.10	1.20	1.20	1.20	1.10	1.05	----	----	----	----	----
2, 3, 4-Trimethyl-2-pentene..... 3, 4, 4-Trimethyl-2-pentene.....	C ₈ H ₁₆	1.20 1.15	1.20 1.10	1.25 1.15	1.20 1.20	1.15 1.15	1.20 1.20	1.25 1.20	1.30 1.20	1.25 1.15	1.20 1.10	1.20 1.25	1.20 1.20	1.20 1.15	1.15 1.10	1.10 1.10

* Relative temperature sensitivity = $\frac{\text{imep ratio of blend (inlet-air temperature, 100° F)}}{\text{imep ratio of blend (inlet-air temperature, 250° F)}}$

TABLE A-6.—17.6 ENGINE TEMPERATURE SENSITIVITY OF BLENDS RELATIVE TO ISOCTANE AND MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE+4 ML TEL PER GALLON—Continued

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(b) Aromatics.

Aromatic	Formula	Relative temperature sensitivity *														
		20 volume percent aromatic in blend with isooctane										25 volume percent aromatic in blend with mixed base fuel				
		Unleaded					4 ml TEL/gal									
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Benzene.....	C ₆ H ₆	1.10	1.05	1.05	1.00	1.00	1.10	1.05	1.05	1.05	1.05	1.15	1.10	1.05	1.05	1.00
Methylbenzene.....	C ₇ H ₈	1.00	1.05	1.05	1.00	1.00	1.10	1.10	1.10	1.10	1.05	----	----	----	----	----
Ethylbenzene.....	C ₈ H ₁₀	1.15	1.20	1.20	1.10	1.05	1.15	1.15	1.10	1.05	1.00	----	----	----	----	----
1, 2-Dimethylbenzene.....		1.10	1.10	1.15	1.10	1.05	1.05	1.00	1.00	1.05	1.05	1.10	1.05	1.00	1.00	1.00
1, 3-Dimethylbenzene.....		1.25	1.20	1.15	1.20	1.20	1.15	1.10	1.10	1.05	1.05	1.10	1.15	1.05	1.00	1.00
1, 4-Dimethylbenzene.....		1.10	1.15	1.10	1.05	1.00	1.15	1.15	1.15	1.10	1.05	1.10	1.15	1.05	1.00	1.00
n-Propylbenzene.....	C ₉ H ₁₂	1.15	1.15	1.15	1.15	1.05	1.20	1.20	1.15	1.05	1.00	1.10	1.10	1.05	1.05	1.05
Isopropylbenzene.....		1.10	1.20	1.20	1.10	1.05	1.20	1.20	1.15	1.05	1.00	----	----	----	----	----
1-Methyl-2-ethylbenzene.....		1.00	1.05	1.05	1.00	1.00	1.05	1.00	1.00	1.00	1.00	----	----	----	----	----
1-Methyl-3-ethylbenzene.....		1.15	1.20	1.15	1.10	1.05	1.15	1.10	1.10	1.00	1.00	----	----	----	----	----
1-Methyl-4-ethylbenzene.....		1.00	1.05	1.00	1.05	1.05	1.20	1.20	1.20	1.05	1.00	1.25	1.20	1.25	1.15	1.05
1, 2, 3-Trimethylbenzene.....		1.05	1.05	1.05	1.05	1.05	1.05	1.00	1.00	1.05	1.05	----	----	----	----	----
1, 2, 4-Trimethylbenzene.....		1.10	1.10	1.10	1.10	1.05	1.05	1.00	1.00	1.00	1.00	1.10	1.10	1.05	1.00	1.00
1, 3, 5-Trimethylbenzene.....		1.25	1.20	1.20	1.15	1.05	1.25	1.20	1.20	1.10	1.05	----	----	----	----	----
n-Butylbenzene.....	C ₁₀ H ₁₄	1.10	1.10	1.10	1.10	1.05	1.10	1.05	1.05	1.05	1.05	1.10	1.10	1.05	1.00	1.00
Isobutylbenzene.....		1.15	1.20	1.20	1.15	1.10	1.05	1.05	1.05	1.00	.95	1.15	1.10	1.10	1.05	1.00
sec-Butylbenzene.....		1.15	1.15	1.10	1.05	1.00	1.10	1.15	1.10	1.00	.95	1.05	1.10	1.05	1.05	1.00
tert-Butylbenzene.....		1.30	1.30	1.20	1.25	1.15	1.05	1.05	1.05	1.05	1.00	1.20	1.25	1.15	1.10	1.05
1-Methyl-4-isopropylbenzene.....		1.20	1.20	1.20	1.20	1.15	1.15	1.15	1.15	1.00	1.00	1.15	1.15	1.10	1.05	1.05
1, 2-Diethylbenzene.....		1.00	1.05	1.10	1.00	1.00	1.10	1.05	1.05	1.00	1.00	----	----	----	----	----
1, 3-Diethylbenzene.....		1.20	1.20	1.25	1.20	1.10	1.25	1.20	1.15	1.05	1.00	1.20	1.15	1.10	1.10	1.05
1, 4-Diethylbenzene.....		1.20	1.20	1.15	1.10	1.00	1.10	1.15	1.15	1.05	1.00	----	----	----	----	----
1, 3-Dimethyl-5-ethylbenzene.....		1.15	1.15	1.15	1.10	1.05	1.20	1.20	1.15	1.05	1.05	----	----	----	----	----
1-Methyl-3-tert-butylbenzene.....	C ₁₁ H ₁₆	1.10	1.15	1.10	1.00	1.00	1.10	1.15	1.10	1.05	1.00	----	----	----	----	----
1-Methyl-4-tert-butylbenzene.....		1.15	1.15	1.10	1.05	1.00	1.10	1.10	1.10	1.05	1.05	----	----	----	----	----
1-Methyl-3, 5-diethylbenzene.....		1.20	1.20	1.20	1.10	1.05	1.20	1.25	1.10	1.10	1.00	----	----	----	----	----
1, 3, 5-Triethylbenzene.....	C ₁₂ H ₁₈	1.25	1.25	1.25	1.15	1.05	1.25	1.25	1.10	1.00	1.00	----	----	----	----	----

* Relative temperature sensitivity = $\frac{\text{imep ratio of blend (inlet-air temperature, 100° F)}}{\text{imep ratio of blend (inlet-air temperature, 250° F)}}$

TABLE A-6.—17.6 ENGINE TEMPERATURE SENSITIVITY OF BLENDS RELATIVE TO ISOCTANE AND MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE+4 ML TEL PER GALLON—Concluded

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(c) Ethers.

Ether	Formula	Relative temperature sensitivity *																			
		Ether in blend with mixed base fuel, percent by volume																			
		20					10					25					50				
		Unleaded					4 ml TEL/gal														
		Fuel-air ratio																			
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Methyl <i>tert</i> -butyl ether.....	C ₅ H ₁₂ O	1.10	1.10	1.15	1.05	1.00	1.10	1.15	1.10	1.05	1.00	1.05	1.05	1.00	1.00	1.05	1.05	1.00	1.00	1.00	
Ethyl <i>tert</i> -butyl ether.....	C ₆ H ₁₄ O	1.05	1.05	1.00	1.00	.95	1.00	1.00	1.00	1.00	1.00	1.00	.95	.95	.95	1.00	1.00	1.00	1.00	1.00	
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O	1.10	1.10	1.05	1.00	.95	1.00	1.05	1.05	1.00	.95	1.00	1.00	1.00	1.00	1.00	1.05	1.05	1.05	1.00	
Methyl phenylether (anisole)...	C ₇ H ₈ O	1.15	1.15	1.15	1.10	1.00	1.15	1.15	1.20	1.10	1.05	1.05	1.10	1.05	1.05	1.00	1.10	1.15	1.15	1.10	
Ethyl phenyl ether (phenetole).	C ₈ H ₁₀ O	1.10	1.10	1.15	1.10	1.00	1.15	1.15	1.10	1.05	1.00	1.00	1.00	1.05	1.05	1.00	1.15	1.15	1.20	1.15	
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole)	C ₈ H ₁₀ O	1.05	1.15	1.15	1.10	1.05	1.25	1.20	1.10	1.05	1.05	1.00	1.00	1.00	1.00	1.00	1.10	1.15	1.10	1.10	

* Relative temperature sensitivity = $\frac{\text{imep ratio of blend (inlet-air temperature, 100° F)}}{\text{imep ratio of blend (inlet-air temperature, 250° F)}}$

TABLE A-7.—17.6 ENGINE LEAD SUSCEPTIBILITY OF BLENDS RELATIVE TO ISOCTANE

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(a) Paraffins and olefins.

Paraffins and olefins	Formula	Relative lead susceptibility *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent paraffin or olefin in blend with isooctane														
		10					20					20				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Paraffins																
2, 2, 3-Trimethylbutane..... 2, 3-Dimethylpentane.....	C ₇ H ₁₆	1.05 ---	1.05 ---	1.05 ---	1.00 ---	1.00 ---	1.05 1.00	1.10 1.00	1.10 1.05	1.05 1.05	1.05 1.00	1.05 1.05	1.05 1.05	1.00 1.00	1.00 1.00	1.00 1.00
2, 2, 3, 3-Tetramethylpentane..... 2, 2, 3, 4-Tetramethylpentane..... 2, 3, 3, 4-Tetramethylpentane..... 2, 4-Dimethyl-3-ethylpentane.....	C ₉ H ₂₀	--- 1.05 1.10 ---	--- 1.05 1.05 ---	--- 1.05 1.05 ---	--- 1.00 1.10 ---	--- 1.00 1.05 ---	1.05 1.15 1.15 1.05	1.05 1.15 1.15 1.00	1.10 1.10 1.05 1.00	1.00 1.05 1.05 1.00	1.00 1.00 1.00 1.00	1.00 1.05 1.00 .95	1.00 1.05 1.00 1.00	1.00 1.05 1.00 .95	1.00 1.00 1.00 1.00	1.00 1.00 .95 1.05
Olefins																
2, 3-Dimethyl-2-pentene.....	C ₇ H ₁₄	---	---	---	---	---	0.95	0.95	1.00	1.05	1.00	1.00	1.00	1.00	1.00	0.95
2, 3, 4-Trimethyl-2-pentene..... 3, 4, 4-Trimethyl-2-pentene.....	C ₈ H ₁₆	1.00 1.00	1.00 1.00	1.00 1.00	1.05 1.00	1.00 1.00	1.05 1.00	1.05 1.00	1.00 1.00	1.05 1.00	1.05 1.00	1.05 1.05	1.05 1.05	1.05 1.00	1.05 1.00	1.05 1.00

* Relative lead susceptibility = $\frac{\text{imep ratio of blend with 4 ml TEL/gal}}{\text{imep ratio of blend with 0 ml TEL/gal}}$

TABLE A-7.—17.6 ENGINE LEAD SUSCEPTIBILITY OF BLENDS RELATIVE TO ISOCTANE—Continued

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(b) Aromatics.

Aromatic	Formula	Relative lead susceptibility *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent aromatic in blend with isooctane														
		10					20					20				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Benzene.....	C ₆ H ₆	1.00	1.00	1.05	1.05	1.05	1.00	1.05	1.10	1.05	1.00	1.05	1.05	1.10	1.10	1.05
Methylbenzene.....	C ₇ H ₈	0.95	1.00	1.00	0.95	1.00	1.00	1.05	1.00	1.00	1.00	1.10	1.10	1.05	1.05	1.05
Ethylbenzene.....	C ₈ H ₁₀	1.05	1.05	1.05	1.00	0.95	1.10	1.15	1.15	1.10	1.05	1.10	1.10	1.05	1.00	1.00
1, 2-Dimethylbenzene.....		.95	.95	.95	.90	.85	.85	.90	.95	.90	.85	.85	.85	.85	.85	.80
1, 3-Dimethylbenzene.....		1.10	1.10	1.10	1.10	1.15	1.20	1.25	1.20	1.25	1.25	1.10	1.10	1.20	1.10	1.10
1, 4-Dimethylbenzene.....		1.05	1.05	1.00	1.00	1.00	1.00	1.10	1.05	1.05	1.00	1.05	1.05	1.10	1.10	1.05
n-Propylbenzene.....	C ₉ H ₁₂	1.05	1.05	1.10	1.05	1.00	1.10	1.15	1.20	1.10	1.10	1.20	1.20	1.20	1.05	1.00
Isopropylbenzene.....		1.05	1.10	1.05	1.00	1.05	1.05	1.15	1.10	1.10	1.05	1.15	1.15	1.05	1.05	1.05
1-Methyl-2-ethylbenzene.....		---	---	---	---	---	.95	1.00	1.00	.95	.90	.95	.95	.90	.90	.90
1-Methyl-3-ethylbenzene.....		---	---	---	---	---	1.15	1.20	1.15	1.15	1.10	1.10	1.10	1.10	1.05	1.05
1-Methyl-4-ethylbenzene.....		1.05	1.05	1.10	1.10	1.10	1.10	1.10	1.10	1.15	1.20	1.35	1.30	1.30	1.20	1.15
1, 2, 3-Trimethylbenzene.....		---	---	---	---	---	.85	.90	.90	.90	.90	.85	.85	.85	.85	.85
1, 2, 4-Trimethylbenzene.....		1.00	1.00	1.00	.95	.95	1.00	1.00	1.05	1.00	.90	.90	.95	.90	.90	.85
1, 3, 5-Trimethylbenzene.....		1.05	1.05	1.10	1.00	.95	1.10	1.15	1.15	1.10	1.05	1.10	1.10	1.15	1.05	1.05
n-Butylbenzene.....	C ₁₀ H ₁₄	1.05	1.05	1.05	1.05	1.05	1.10	1.10	1.10	1.10	1.05	1.05	1.05	1.05	1.05	1.05
Isobutylbenzene.....		1.10	1.10	1.10	1.10	1.05	1.20	1.20	1.25	1.20	1.15	1.10	1.10	1.10	1.05	1.05
sec-Butylbenzene.....		1.10	1.10	1.05	1.05	1.05	1.20	1.15	1.10	1.10	1.05	1.10	1.15	1.10	1.05	1.05
tert-Butylbenzene.....		1.15	1.10	1.05	1.05	1.00	1.40	1.35	1.20	1.20	1.10	1.10	1.10	1.05	1.00	1.00
1-Methyl-4-isopropylbenzene.....		1.10	1.10	1.10	1.10	1.05	1.20	1.20	1.20	1.25	1.20	1.15	1.15	1.15	1.05	1.05
1, 2-Diethylbenzene.....		---	---	---	---	---	.95	1.00	1.05	1.00	1.00	1.05	1.05	1.00	1.00	.95
1, 3-Diethylbenzene.....		1.05	1.10	1.10	1.10	1.05	1.10	1.20	1.20	1.20	1.10	1.15	1.15	1.10	1.05	1.05
1, 4-Diethylbenzene.....		---	---	---	---	---	1.15	1.15	1.10	1.05	1.05	1.05	1.10	1.10	1.05	1.05
1, 3-Dimethyl-5-ethylbenzene.....		---	---	---	---	---	1.05	1.15	1.15	1.10	1.05	1.10	1.15	1.15	1.10	1.10
1-Methyl-3-tert-butylbenzene.....	C ₁₁ H ₁₆	---	---	---	---	---	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05	1.05
1-Methyl-4-tert-butylbenzene.....		---	---	---	---	---	1.10	1.10	1.10	1.05	1.05	1.05	1.05	1.05	1.05	1.05
1-Methyl-3, 5-diethylbenzene.....		---	---	---	---	---	1.10	1.15	1.10	1.05	1.05	1.15	1.15	1.10	1.05	1.00
1, 3, 5-Triethylbenzene.....	C ₁₂ H ₁₈	---	---	---	---	---	1.10	1.15	1.15	1.15	1.05	1.10	1.15	1.05	1.00	1.05

* Relative lead susceptibility = $\frac{\text{imep ratio of blend with 4 ml TEL/gal}}{\text{imep ratio of blend with 0 mo TEL/gal}}$

TABLE A-7.—17.6 ENGINE LEAD SUSCEPTIBILITY OF BLENDS RELATIVE TO ISOCTANE—Concluded

[Compression ratio, 7.0; engine speed, 1800 rpm; coolant temperature, 212° F; spark advance, 30° B. T. C.]

(c) Ethers.

Ether	Formula	Relative lead susceptibility *														
		Inlet-air temperature, 250° F										Inlet-air temperature, 100° F				
		Volume percent ether in blend with isooctane														
		10					20					20				
		Fuel-air ratio														
		0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11	0.065	0.07	0.085	0.10	0.11
Methyl <i>tert</i> -butyl ether.....	C ₅ H ₁₂ O	1.10	1.05	1.10	1.05	1.05	1.10	1.10	1.10	1.05	1.05	1.15	1.10	1.10	1.05	1.10
Ethyl <i>tert</i> -butyl ether.....	C ₈ H ₁₆ O	1.00	1.00	.95	.95	.95	1.15	1.10	1.05	1.00	.95	1.05	1.05	1.05	1.00	.95
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₆ O	1.00	1.00	1.00	1.00	1.00	1.10	1.05	1.00	1.00	1.00	1.05	1.05	1.05	1.00	1.00
Methyl phenyl ether (anisole).....	C ₇ H ₈ O	1.00	1.00	1.00	1.00	1.00	1.05	1.05	1.00	1.05	1.05	1.05	1.05	1.05	1.05	1.05
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O	1.00	1.00	1.05	1.00	1.00	1.00	1.00	1.10	1.05	1.05	1.05	1.05	1.10	1.00	1.00
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole).....	C ₈ H ₁₀ O	1.00	1.05	1.05	1.05	1.05	.95	1.05	1.05	1.05	1.05	1.10	1.10	1.05	1.00	1.05

* Relative lead susceptibility = $\frac{\text{imep ratio of blend with 4 ml TEL/gal}}{\text{imep ratio of blend with 0 ml TEL/gal}}$

TABLE A-8.—FULL-SCALE SINGLE-CYLINDER ENGINE KNOCK-LIMITED INDICATED MEAN EFFECTIVE PRESSURE RATIOS OF ETHER BLENDS WITH MIXED BASE FUEL CONSISTING OF 87.5 PERCENT (BY VOLUME) ISOCTANE AND 12.5 PERCENT *n*-HEPTANE+4 ML TEL PER GALLON

[Full-scale cruise conditions; compression ratio, 7.3; engine speed, 1800 rpm; inlet-air temperature, 210° F; spark advance, 30° B. T. C.; cooling-air flow such that rear-spark-plug-bushing temperature equals 365° F at 140 bmep and 0.10 fuel-air ratio]

Ether	Formula	Imep ratio *				
		10 volume percent ether in blend with mixed base fuel				
		Fuel-air ratio				
		0.065	0.07	0.085	0.10	0.11
Methyl <i>tert</i> -butyl ether.....	C ₅ H ₁₂ O	1.27	1.21	1.24	1.25	1.19
Ethyl <i>tert</i> -butyl ether.....	C ₈ H ₁₆ O	1.19	1.16	1.19	1.16	1.11
Isopropyl <i>tert</i> -butyl ether.....	C ₇ H ₁₄ O	1.17	1.16	1.15	1.15	1.14
Methyl phenyl ether (anisole).....	C ₇ H ₈ O	1.11	1.08	1.08	1.11	1.11
Ethyl phenyl ether (phenetole).....	C ₈ H ₁₀ O	1.13	1.11	1.11	1.15	1.12
Methyl <i>p</i> -tolyl ether (<i>p</i> -methylanisole).....	C ₈ H ₁₀ O	1.20	1.16	1.15	1.17	1.19

* Imep ratio = $\frac{\text{knock-limited imep of blend with 4 ml TEL/gal}}{\text{knock-limited imep of mixed base fuel with 4 ml TEL/gal}}$

TABLE A-9.—INSPECTION DATA FOR PURE FUEL STOCKS AND FUEL BLENDS

The following abbreviations are used throughout the table: VBS for virgin base stock; alkylate for aviation alkylate; one-pass stock for one-pass catalytic stock; and MTB ether for methyl tert-butyl ether.]

Fuel	Fuel composition	A. S. T. M. distillation date										Specific gravity at 60° F	Copper dish gum (mg/100 ml)	Reid vapor pressure (lb/sq in.)	Aromatics* (percent by volume)	Freezing point (°F)	Hydrogen-carbon ratio	Net heat of combustion (Btu/lb)	Refractive index n_D at 68° F	Bromine number (cc/g)	Olefins ^b (percent by weight)
	Components	TEL concentration (ml/gal)	Initial boiling point (°F)	10% point (°F)	40% point (°F)	50% point (°F)	90% point (°F)	Final boiling point (°F)	Sum of 10 and 50% points (°F)	Residue (percent)	Loss (percent)										
A-202	VBS.....	0	114	145	172	180	229	298	325	0.4	1.1	0.713	None	5.9	7.3	Below -76	0.175	19,000	1.3988	2.8	1.7
A-118	50% alkylate+50% VBS.....	4.04	112	150	188	200	254	336	350	0.7	1.1	0.707	-----	5.9	-----	-----	-----	-----	1.3957	-----	-----
A-203	Alkylate.....	0	113	160	215	226	272	364	366	.5	1.0	.698	2	4.7	None	Below -76	0.189	19,100	1.3928	0.2	0.1
A-132	30% one-pass stock+70% VBS.....	4.08	112	140	170	182	251	319	322	0.5	1.3	0.723	-----	6.0	-----	-----	-----	-----	1.4043	-----	-----
A-116	50% one-pass stock+50% VBS.....	4.06	103	134	170	184	269	321	318	.5	1.0	.727	-----	7.1	-----	-----	-----	-----	1.4080	-----	-----
A-119	80% one-pass stock+20% VBS.....	4.05	116	130	171	190	285	326	320	.4	1.4	.735	-----	7.4	-----	-----	-----	-----	1.4141	-----	-----
A-122	30% one-pass stock+70% alkylate.....	3.96	110	147	202	218	272	345	365	0.6	1.2	0.718	-----	5.8	-----	-----	-----	-----	1.4005	-----	-----
A-117	50% one-pass stock+50% alkylate.....	4.00	105	140	194	213	280	341	353	.5	1.0	.721	-----	6.5	-----	-----	-----	-----	1.4051	-----	-----
A-121	80% one-pass stock+20% alkylate.....	4.04	106	131	181	204	288	334	335	.7	1.3	.732	-----	7.2	-----	-----	-----	-----	1.4123	-----	-----
A-204	One-pass stock.....	0	105	128	170	193	287	332	321	.5	1.0	.739	-----	7.9	24.8	Below -76	0.155	18,700	1.4177	30.4	17.6
A-136	20% triptane+80% VBS.....	4.10	124	150	172	178	218	299	328	0.7	0.8	0.712	-----	6.0	-----	Below -76	-----	-----	1.3969	-----	-----
A-137	40% triptane+60% VBS.....	4.05	128	156	172	177	205	287	333	.6	.6	.707	-----	5.3	-----	Below -76	-----	-----	1.3953	-----	-----
A-138	60% triptane+40% VBS.....	4.08	142	162	174	176	194	274	338	.7	.5	.704	-----	4.5	-----	Below -76	-----	-----	1.3938	-----	-----
A-272	20% triptane+80% alkylate.....	(d)	122	164	202	210	259	346	374	0.9	0.6	0.701	-----	4.4	-----	Below -76	-----	-----	-----	-----	-----
A-273	40% triptane+60% alkylate.....	(d)	130	167	192	198	251	335	365	.7	.8	.701	-----	4.1	-----	Below -76	-----	-----	-----	-----	-----
A-274	60% triptane+40% alkylate.....	(d)	144	169	183	188	230	326	357	.8	.7	.700	-----	4.0	-----	Below -76	-----	-----	-----	-----	-----
A-275	80% triptane+20% alkylate.....	(d)	158	172	178	180	202	296	352	.8	.2	.699	-----	3.6	-----	Below -76	-----	-----	-----	-----	-----
A-276	20% triptane+80% one-pass stock.....	(d)	107	136	172	186	281	321	322	0.4	1.4	0.734	-----	6.7	-----	Below -76	-----	-----	-----	-----	-----
A-277	40% triptane+60% one-pass stock.....	(d)	120	147	174	183	275	320	330	.9	.9	.725	-----	5.9	-----	Below -76	-----	-----	-----	-----	-----
A-278	60% triptane+40% one-pass stock.....	(d)	128	156	174	179	234	316	335	.4	.6	.716	-----	5.1	-----	Below -76	-----	-----	-----	-----	-----
A-279	80% triptane+20% one-pass stock.....	(d)	141	166	177	179	196	304	345	.5	.5	.710	-----	4.2	-----	Below -76	-----	-----	-----	-----	-----
A-206	Triptane.....	0	172	174	174	174	175	180	348	.4	.8	.694	1	3.0	-----	Below -18	0.191	19,200	1.3896	-----	-----
A-397	20% diisopropyl+80% VBS.....	(d)	120	143	160	168	224	287	311	0.9	0.8	0.705	-----	6.6	-----	-----	-----	-----	-----	-----	-----
A-398	40% diisopropyl+60% VBS.....	(d)	123	138	151	156	214	285	294	.7	1.5	.695	-----	5.8	-----	-----	-----	-----	-----	-----	-----
A-399	60% diisopropyl+40% VBS.....	(d)	128	138	146	148	197	258	286	.9	1.1	.686	-----	5.8	-----	-----	-----	-----	-----	-----	-----
A-400	80% diisopropyl+20% VBS.....	(d)	130	136	140	142	165	254	278	.7	1.0	.677	-----	7.0	-----	-----	-----	-----	-----	-----	-----
A-405	20% diisopropyl+80% alkylate.....	(d)	120	151	187	202	263	344	353	1.1	0.7	0.695	-----	5.5	-----	-----	-----	-----	-----	-----	-----
A-406	40% diisopropyl+60% alkylate.....	(d)	120	140	155	162	252	330	302	1.2	.3	.687	-----	6.3	-----	-----	-----	-----	-----	-----	-----
A-407	60% diisopropyl+40% alkylate.....	(d)	127	140	156	156	246	322	296	1.1	.6	.681	-----	6.4	-----	-----	-----	-----	-----	-----	-----
A-408	80% diisopropyl+20% alkylate.....	(d)	130	136	141	143	208	295	279	1.0	.6	.675	-----	6.8	-----	-----	-----	-----	-----	-----	-----
A-401	20% diisopropyl+80% one-pass stock.....	(d)	112	132	157	172	287	322	304	0.5	1.5	0.728	-----	6.9	-----	-----	-----	-----	-----	-----	-----
A-402	40% diisopropyl+60% one-pass stock.....	(d)	117	133	150	157	277	321	290	.8	.9	.713	-----	7.1	-----	-----	-----	-----	-----	-----	-----
A-403	60% diisopropyl+40% one-pass stock.....	(d)	121	133	143	147	257	315	280	.4	1.2	.698	-----	7.2	-----	-----	-----	-----	-----	-----	-----
A-404	80% diisopropyl+20% one-pass stock.....	(d)	127	134	138	140	169	305	274	.7	.7	.683	-----	7.3	-----	-----	-----	-----	-----	-----	-----
A-433	Diisopropyl.....	0	132	134	135	135	135	136	269	None	1.2	.666	None	7.4	-----	Below -76	0.196	19,100	-----	-----	-----
A-411	20% neohexane+80% VBS.....	(d)	118	138	157	165	227	289	303	0.9	0.9	0.705	-----	7.0	-----	-----	-----	-----	-----	-----	-----
A-412	40% neohexane+60% VBS.....	(d)	118	133	145	151	219	285	284	.8	.9	.694	-----	7.5	-----	-----	-----	-----	-----	-----	-----
A-413	60% neohexane+40% VBS.....	(d)	113	127	135	138	201	266	265	.6	1.3	.683	-----	8.1	-----	-----	-----	-----	-----	-----	-----
A-414	80% neohexane+20% VBS.....	(d)	118	128	126	128	161	241	251	.9	1.1	.672	-----	9.1	-----	-----	-----	-----	-----	-----	-----
A-415	20% neohexane+80% alkylate.....	(d)	116	146	183	203	265	345	349	1.1	0.8	0.693	-----	5.9	-----	-----	-----	-----	-----	-----	-----
A-416	40% neohexane+60% alkylate.....	(d)	117	134	157	169	257	332	303	1.2	.8	.686	-----	6.0	-----	-----	-----	-----	-----	-----	-----
A-417	60% neohexane+40% alkylate.....	(d)	116	125	136	142	247	322	267	1.1	1.1	.677	-----	7.9	-----	-----	-----	-----	-----	-----	-----
A-418	80% neohexane+20% alkylate.....	(d)	116	122	127	129	220	290	251	.9	1.4	.670	-----	8.8	-----	-----	-----	-----	-----	-----	-----
A-420	20% neohexane+80% one-pass stock.....	(d)	107	127	151	167	283	323	294	0.6	1.3	0.728	-----	7.9	-----	-----	-----	-----	-----	-----	-----
A-421	40% neohexane+60% one-pass stock.....	(d)	110	125	140	148	275	318	273	.5	1.4	.712	-----	8.3	-----	-----	-----	-----	-----	-----	-----
A-422	60% neohexane+40% one-pass stock.....	(d)	112	121	130	134	259	313	255	.5	1.2	.696	-----	8.5	-----	-----	-----	-----	-----	-----	-----
A-423	80% neohexane+20% one-pass stock.....	(d)	116	120	125	126	181	302	246	.6	1.6	.678	-----	9.1	-----	-----	-----	-----	-----	-----	-----
A-500	Neohexane.....	0	119	120	120	120	121	123	240	None	1.2	.658	1	9.7	-----	-----	0.197	19,100	-----	-----	-----
A-123	20% isopentane+80% VBS.....	4.06	102	122	148	160	224	297	282	0.7	1.3	0.702	-----	9.3	-----	-----	-----	-----	1.3907	-----	-----
A-124	40% isopentane+60% VBS.....	4.06	96	108	124	134	216	291	242	.7	2.8	.679	-----	11.8	-----	-----	-----	-----	1.3822	-----	-----
A-134	60% isopentane+40% VBS.....	4.16	90	98	106	110	200	266	208	.8	2.4	.664	-----	14.4	-----	-----	-----	-----	1.3732	-----	-----
A-375	20% isopentane+80% alkylate.....	(d)	104	127	175	208	262	344	335	0.9	2.0	0.687	-----	8.4	-----	-----	-----	-----	-----	-----	-----
A-376	40% isopentane+60% alkylate.....	(d)	97	110	134	154	256	335	264	.7	1.4	.673	-----	11.6	-----	-----	-----	-----	-----	-----	-----
A-388	20% isopentane+80% one-pass stock.....	(d)	102	118	146	167	288	322	285	0.4	2.4	0.720	-----	9.9	-----	-----	-----	-----	-----	-----	-----
A-389	40% isopentane+60% one-pass stock.....	(d)	96	106	121	129	278	318	235	.6	2.1	.699	-----	12.5	-----	-----	-----	-----	-----	-----	-----
A-209	Isopentane.....	0	80	82	85	86	90	92	168	None	3.5	.624	None	19.6	-----	Below -76	0.202	19,100	1.3544	-----	-----
A-139	20% hot-acid octane+80% VBS.....	4.10	118	154	184	193	232	298	347	0.5	1.0	0.715	5	5.8	-----	-----	-----	-----	1.3992	-----	-----
A-140	40% hot-acid octane+60% VBS.....	4.04	123	166	198	206	234	290	372	.6	1.2	.716	6	5.0	-----	-----	-----	-----	1.3997	-----	-----
A-141	60% hot-acid octane+40% VBS.....	4.02	122	176	209	216	234	272	392	.7	.5	.715	6	4.3	-----	-----	-----	-----	1.4002	-----	-----
A-367	20% hot-acid octane+80% alkylate.....	(d)	112	169	218	225	259	341	394	1.2	0.6	0.703	-----	4.4	-----	-----	-----	-----	-----	-----	-----
A-368	40% hot-acid octane+60% alkylate.....	(d)	124	184	223	227	253	342	411	.8	1.2	.707	-----	3.8	-----	-----	-----	-----	-----	-----	-----
A-369	60% hot-acid octane+40% alkylate.....	(d)	132	193	224	226	243	315	419	.8	1.5	.710	-----	3.1	-----	-----	-----	-----	-----	-----	-----
A-370	80% hot-acid octane+20% alkylate.....	(d)	144	208	224	225	239	279	433	.8	1.4	.714	-----	2.9	-----	-----	-----	-----	-----	-----	-----

See footnotes at end of table.

TABLE A-9.—INSPECTION DATA FOR PURE FUEL STOCKS AND FUEL BLENDS—Continued

Fuel	Fuel composition	A. S. T. M. distillation date										Specific gravity at 60° F	Copper dish gum (mg/100 ml)	Reid vapor pressure (lb/sq in.)	Aromatics * (percent by volume)	Freezing point (°F)	Hydrogen-carbon ratio	Net heat of combustion (Btu/lb)	Refractive index n_D at 68° F	Bromine number (cc/g)	Olefins ^b (percent by weight)
	Components	TEL concentration (ml/gal)	Initial boiling point (°F)	10% point (°F)	40% point (°F)	50% point (°F)	90% point (°F)	Final boiling point (°F)	Sum of 10 and 50% points (°F)	Residue (percent)	Loss (percent)										
A-371	20% hot-acid octane+80% one-pass stock	(d)	114	142	189	208	282	322	350	0.4	1.4	0.737	-----	6.5	-----	-----	-----	-----	-----	-----	
A-372	40% hot-acid octane+60% one-pass stock	(d)	124	155	206	218	269	321	373	.6	1.4	.732	-----	5.5	-----	-----	-----	-----	-----	-----	
A-373	60% hot-acid octane+40% one-pass stock	(d)	115	165	215	222	250	310	387	.8	1.1	.727	-----	4.5	-----	-----	-----	-----	-----	-----	
A-374	80% hot-acid octane+20% one-pass stock	(d)	129	195	222	224	240	294	419	.7	.5	.722	-----	3.6	-----	-----	-----	-----	-----	-----	
A-205	Hot-acid octane	0	174	216	224	224	230	257	440	.6	.9	.715	6	2.7	None	Below -76	0.188	19,200	1.4009	2.6	1.7
A-148	20% mixed xylenes+80% VBS	4.07	116	152	191	203	272	296	355	0.4	0.6	0.746	-----	5.3	26.4	Below -76	-----	-----	1.4180	-----	-----
A-444	40% mixed xylenes+60% VBS	(d)	122	167	215	232	277	288	399	.3	.5	.775	-----	4.2	-----	-----	-----	-----	-----	-----	
A-445	60% mixed xylenes+40% VBS	(d)	128	196	252	264	277	287	460	.3	.4	.805	-----	3.0	-----	-----	-----	-----	-----	-----	
A-446	80% mixed xylenes+20% VBS	(d)	140	220	266	270	278	283	490	.4	.6	.824	-----	2.2	-----	-----	-----	-----	-----	-----	
A-149	20% mixed xylenes+80% alkylate	3.96	121	176	234	242	276	342	418	0.8	0.7	0.731	1	3.9	20.5	Below -76	-----	-----	1.4132	-----	-----
A-262	40% mixed xylenes+60% alkylate	(d)	124	191	249	255	281	320	446	.5	1.3	.768	-----	3.0	-----	-----	-----	-----	-----	-----	
A-263	60% mixed xylenes+40% alkylate	(d)	130	217	262	266	280	306	483	.5	.5	.799	-----	2.2	-----	-----	-----	-----	-----	-----	
A-264	80% mixed xylenes+20% alkylate	(d)	136	238	268	271	279	295	509	.8	.8	.821	-----	1.9	-----	-----	-----	-----	-----	-----	
A-150	20% mixed xylenes+80% one-pass stock	4.10	106	146	200	230	285	315	376	0.3	0.9	0.767	-----	6.5	42.3	Below -76	-----	-----	1.4333	-----	-----
A-266	40% mixed xylenes+60% one-pass stock	(d)	115	159	244	262	284	311	421	.8	.8	.793	-----	5.2	-----	-----	-----	-----	-----	-----	
A-267	60% mixed xylenes+40% one-pass stock	(d)	115	154	251	264	282	303	418	.1	1.1	.800	-----	4.4	-----	-----	-----	-----	-----	-----	
A-268	80% mixed xylenes+20% one-pass stock	(d)	126	222	273	275	280	294	497	.2	.8	.838	-----	2.4	-----	-----	-----	-----	-----	-----	
A-237	Mixed xylenes	0	273	275	276	276	277	278	551	.2	None	.867	2	.5	-----	Below -40	0.106	17,600	1.4959	-----	-----
A-163	20% cumene+80% VBS	4.12	116	152	188	202	294	321	354	0.5	0.7	0.737	-----	5.3	25.6	Below -76	-----	-----	1.4169	-----	-----
A-244	40% cumene+60% VBS	(d)	123	161	218	238	300	330	399	.4	.9	.775	-----	4.4	43.0	Below -76	-----	-----	1.4353	-----	-----
A-246	60% cumene+40% VBS	(d)	126	180	265	284	302	335	464	.6	.6	.804	-----	3.2	62.0	Below -76	-----	-----	1.4538	-----	-----
A-247	80% cumene+20% VBS	(d)	130	228	296	298	306	339	526	.5	.5	.832	-----	2.0	-----	Below -76	-----	-----	-----	-----	-----
A-164	20% cumene+80% alkylate	3.99	116	170	234	244	300	342	414	0.7	0.5	0.732	3	4.0	19.7	Below -76	-----	-----	1.4122	-----	-----
A-249	40% cumene+60% alkylate	(d)	118	188	257	267	302	340	455	.4	.8	.767	-----	3.2	39.0	Below -76	-----	-----	1.4320	-----	-----
A-250	60% cumene+40% alkylate	(d)	124	206	271	280	304	339	486	.5	.5	.787	-----	2.6	-----	Below -76	-----	-----	-----	-----	-----
A-251	80% cumene+20% alkylate	(d)	140	262	293	296	304	342	558	.5	.5	.831	-----	1.4	-----	Below -76	-----	-----	-----	-----	-----
A-165	20% cumene+80% one-pass stock	3.91	106	136	203	234	299	325	370	0.6	0.4	0.769	-----	6.4	40.0	Below -76	-----	-----	1.4325	-----	-----
A-253	40% cumene+60% one-pass stock	(d)	110	144	240	268	301	330	412	1.0	1.0	.787	-----	5.3	-----	Below -76	-----	-----	-----	-----	-----
A-254	60% cumene+40% one-pass stock	(d)	120	174	285	289	303	336	463	.4	1.1	.815	-----	3.8	-----	Below -76	-----	-----	-----	-----	-----
A-255	80% cumene+20% one-pass stock	(d)	125	216	292	295	303	340	511	1.0	1.0	.836	-----	2.4	-----	Below -76	-----	-----	-----	-----	-----
A-238	Cumene	0	293	298	300	300	305	349	598	.5	.5	.863	3	.3	-----	Below -76	0.113	17,800	1.4903	-----	-----
A-170	20% benzene+80% VBS	4.11	118	148	167	174	218	299	322	0.5	0.5	0.748	-----	6.0	26.0	-----	-----	-----	1.4175	-----	-----
A-342	40% benzene+60% VBS	(d)	126	154	168	173	204	286	327	.5	1.0	.772	-----	5.6	-----	-----	-----	-----	-----	-----	-----
A-343	60% benzene+40% VBS	(d)	144	164	171	174	185	266	338	.4	1.1	.823	-----	4.6	-----	-----	-----	-----	-----	-----	-----
A-344	80% benzene+20% VBS	(d)	148	168	172	174	179	238	342	.4	.6	.844	-----	4.2	-----	-----	-----	-----	-----	-----	-----
A-168	20% benzene+80% alkylate	3.92	120	156	190	200	258	345	356	0.7	0.3	0.734	4	4.8	19.5	-----	-----	-----	1.4124	-----	-----
A-359	40% benzene+60% alkylate	(d)	126	162	180	186	245	333	348	.8	1.2	.770	-----	4.7	-----	-----	-----	-----	-----	-----	-----
A-360	60% benzene+40% alkylate	(d)	143	166	177	179	219	324	345	.6	.9	.803	-----	4.4	-----	-----	-----	-----	-----	-----	-----
A-361	80% benzene+20% alkylate	(d)	154	170	175	176	185	293	346	.6	.6	.841	-----	3.7	-----	-----	-----	-----	-----	-----	-----
A-169	20% benzene+80% one-pass stock	4.08	107	134	170	182	282	321	316	0.3	0.9	0.768	-----	7.0	40.0	-----	-----	-----	1.4333	-----	-----
A-363	40% benzene+60% one-pass stock	(d)	117	147	171	179	273	318	326	.5	1.0	.822	-----	6.0	-----	-----	-----	-----	-----	-----	-----
A-364	60% benzene+40% one-pass stock	(d)	122	153	173	177	227	314	330	.4	1.1	.841	-----	5.2	-----	-----	-----	-----	-----	-----	-----
A-365	80% benzene+20% one-pass stock	(d)	143	164	173	175	191	316	339	.4	.8	.850	-----	4.4	-----	-----	-----	-----	-----	-----	-----
A-210	Benzene	0	170	173	173	173	174	175	346	.2	.8	.883	1	3.5	-----	45	0.084	17,400	1.4989	-----	-----
A-145	20% toluene+80% VBS	4.10	122	157	186	194	230	294	351	0.6	0.9	0.746	-----	5.3	26.2	Below -76	-----	-----	1.4178	-----	-----
A-322	40% toluene+60% VBS	(d)	126	163	193	201	231	282	364	.4	.6	.760	-----	5.1	-----	-----	-----	-----	-----	-----	-----
A-323	60% toluene+40% VBS	(d)	135	179	208	215	230	256	394	.5	1.0	.792	-----	4.0	-----	-----	-----	-----	-----	-----	-----
A-324	80% toluene+20% VBS	(d)	148	203	222	224	230	252	427	.1	.9	.823	-----	2.6	-----	-----	-----	-----	-----	-----	-----
A-146	20% toluene+80% alkylate	3.96	118	172	216	223	254	349	395	1.0	0.5	0.733	-----	4.0	19.8	Below -76	-----	-----	1.4126	-----	-----
A-326	40% toluene+60% alkylate	(d)	119	185	221	224	241	364	409	.8	.7	.766	-----	3.3	-----	-----	-----	-----	-----	-----	-----
A-327	60% toluene+40% alkylate	(d)	131	203	223	225	233	311	428	1.1	.1	.800	-----	2.6	-----	-----	-----	-----	-----	-----	-----
A-328	80% toluene+20% alkylate	(d)	155	216	226	227	231	263	443	1.1	.1	.832	-----	2.0	-----	-----	-----	-----	-----	-----	-----
A-147	20% toluene+80% one-pass stock	4.05	106	140	192	213	274	320	353	0.6	0.9	0.766	-----	6.6	44.0	Below -76	-----	-----	1.4333	-----	-----
A-332	40% toluene+60% one-pass stock	(d)	119	158	212	224	259	322	382	.3	1.2	.793	-----	5.1	-----	-----	-----	-----	-----	-----	-----
A-333	60% toluene+40% one-pass stock	(d)	119	180	223	228	246	313	408	.5	.7	.817	-----	3.9	-----	-----	-----	-----	-----	-----	-----
A-334	80% toluene+2ne																				

TABLE A-10.—PERFORMANCE RATINGS OBTAINED IN A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE ENGINES

[Three rows of values are given for each fuel: The first row is imep, lb/sq in.; the second row for A. S. T. M. Aviation ratings is octane number or tetraethyl lead in S reference fuel, ml/gal; the second row for A. S. T. M. Supercharge ratings is S reference fuel in M reference fuel, percent, or tetraethyl lead in S reference fuel, ml/gal; the third row is performance number. The following abbreviations are used throughout the table: VBS for virgin base stock; alkylate for aviation alkylate; one-pass stock for one-pass catalytic stock; MTB ether for methyl *tert*-butyl ether.]

Fuel	Fuel composition* (by volume)	A. S. T. M. Aviation ratings	A. S. T. M. Supercharge ratings ^b					
			Fuel-air ratio					
			0.065	0.070	0.085	0.095	0.100	0.110
A-355	VBS.....	90.7 75	73 96.6 91	83 99.8 99	122 0.08 103	137 99.8 99	141 99.0 97	143 97.8 94
A-118	50% alkylate+50% VBS.....	98.8 96	86 0.10 104	99 0.19 107	143 0.34 111	159 0.33 111	162 0.29 110	165 0.24 109
A-356	Alkylate.....	0.64 119	104 0.55 117	129 0.93 124	176 1.57 134	190 1.71 135	195 1.87 137	201 2.14 140
A-132	30% one-pass stock+70% VBS.....	90.6 75	72 93.8 84	71 90.0 78	116 100 100	130 98.0 94	136 97.5 94	145 97.7 94
A-116	50% one-pass stock+50% VBS.....	90.9 76	64 88.6 76	76 93.1 84	116 100 100	137 0.01 101	145 0.01 101	156 0.06 103
A-119	80% one-pass stock+20% VBS.....	92.7 79	67 90.6 78	76 93.1 84	114 99.2 97	142 0.09 104	154 0.16 106	165 0.24 109
A-122	30% one-pass stock+70% alkylate.....	0.15 106	82 100 100	103 0.26 110	152 0.45 114	172 0.58 117	178 0.75 121	182 0.83 123
A-117	50% one-pass stock+50% alkylate.....	100 100	76 96.3 91	91 0.06 103	143 0.34 111	167 0.44 114	176 0.58 117	186 1.17 129
A-121	80% one-pass stock+20% alkylate.....	96.3 88	72 93.8 84	79 95 84	123 0.09 104	149 0.19 107	160 0.26 110	177 0.48 115
A-410	One-pass stock.....	93.4 81	73 96.6 91	83 99.8 99	125 0.12 105	151 0.16 106	164 0.28 110	179 0.49 115
A-136	20% triptane+80% VBS.....	94.2 83	74 95.0 88	90 0.05 102	134 0.23 108	155 0.27 110	162 0.27 110	167 0.23 110
A-137	40% triptane+60% VBS.....	0.18 107	100 0.43 114	119 0.55 117	164 0.96 125	191 1.75 136	201 2.07 139	205 2.07 139
A-138	60% triptane+40% VBS.....	0.67 120	117 1.20 129	142 1.58 134	224 5.54 160	260 175	264 175	269 175
A-272	20% triptane+80% alkylate.....	1.08 127	90 0.19 106	126 0.88 123	185 2.13 140	213 3.17 148	225 3.79 152	237 4.57 156
A-273	40% triptane+60% alkylate.....	2.43 142	98 0.38 113	126 0.88 123	225 5.69 160	262 177	274 182	283 184
A-274	60% triptane+40% alkylate.....	2.70 145	111 0.90 124	159 2.76 145	275 195	316 213	326 216	334 218
A-275	80% triptane+20% alkylate.....	3.06 147	139 2.59 144	190 5.90 161	314	---	---	---

* Each fuel contains approximately 4 ml TEL/gal.

^b Based on fixed reference-fuel framework (ch. VIII, reference 13).

TABLE A-10.—PERFORMANCE RATINGS OBTAINED IN A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE ENGINES—Continued

Fuel	Fuel composition* (by volume)	A. S. T. M. Aviation ratings	A. S. T. M. Supercharge ratings ^b					
			Fuel-air ratio					
			0.065	0.070	0.085	0.095	0.100	0.110
A-276	20% triptane+80% one-pass stock. ^c	98.8 96	66 90.0 77	72 90.7 78	117 0.01 101	146 0.14 105	160 0.26 110	186 1.17 135
A-277	40% triptane+60% one-pass stock. ^c	0.08 103	81 99.4 99	89 0.05 101	139 0.29 111	176 0.88 124	195 1.77 136	231 3.86 152
A-278	60% triptane+40% one-pass stock. ^c	0.48 115	100 0.43 114	109 0.36 113	171 1.36 131	218 3.52 150	244 ---	291 190
A-279	80% triptane+20% one-pass stock. ^c	1.80 136	126 1.63 134	147 1.82 137	290 ---	361 ---	391 ---	---
A-271	Triptane ^c	3.30 149	204 191	262 ---	393 ---	---	---	---
A-397	20% diisopropyl+80% VBS.....	96.6 90	77 96.9 91	91 0.08 103	132 0.20 108	149 0.19 107	154 0.16 106	155 0.04 101
A-398	40% diisopropyl+60% VBS.....	0.09 103	81 99.4 98	96 0.16 106	143 0.44 112	167 0.44 114	175 0.50 116	180 0.67 120
A-399	60% diisopropyl+40% VBS.....	0.33 111	96 0.33 111	108 0.34 112	163 0.90 124	187 1.55 134	197 1.86 137	207 2.21 141
A-400	80% diisopropyl+20% VBS.....	1.17 128	111 1.10 127	141 1.56 134	202 2.58 144	226 4.14 153	236 5.07 158	250 163
A-405	20% diisopropyl+80% alkylate.....	0.90 124	125 1.78 136	146 1.78 136	192 3.49 144	210 4.29 146	217 3.21 148	222 3.24 148
A-406	40% diisopropyl+60% alkylate.....	1.45 132	138 2.47 143	158 2.67 144	206 3.49 150	227 4.29 154	234 4.80 156	240 5.00 157
A-407	60% diisopropyl+40% alkylate.....	1.40 132	132 1.91 138	154 2.29 141	212 3.87 152	240 ---	252 167	263 171
A-408	80% diisopropyl+20% alkylate.....	2.10 139	136 2.24 141	162 3.05 147	226 5.85 161	261 ---	275 182	292 190
A-401	20% diisopropyl+80% one-pass stock.	96.1 88	79 98.1 95	89 0.05 102	132 0.20 108	163 0.39 113	177 0.67 120	195 1.60 134
A-402	40% diisopropyl+60% one-pass stock.	0.06 102	84 0.05 102	99 0.20 108	150 0.43 114	177 0.95 125	189 1.48 133	209 2.34 142
A-403	60% diisopropyl+40% one-pass stock.	0.24 108	96 0.03 111	114 1.02 114	165 2.00 126	196 2.72 138	210 4.29 145	235 4.29 154
A-404	80% diisopropyl+20% one-pass stock.	0.68 120	120 1.34 131	143 1.65 135	197 2.90 146	229 4.57 155	245 162	272 177
A-393	Diisopropyl ^c	2.41 142	147 3.53 150	173 4.11 153	246 ---	289 195	304 ---	324 ---

* Each fuel contains approximately 4 ml TEL/gal.

^b Based on fixed reference-fuel framework (ch. VIII, reference 13).

^c Knock-limited performance of the engine with one-pass catalytic stock was low on day fuels were investigated.

^d Estimated value.

* Values for knock-limited imep were averaged from three curves.

TABLE A-10.—PERFORMANCE RATINGS OBTAINED IN A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE ENGINES—Continued

Fuel	Fuel composition ^a (by volume)	A. S. T. M. Aviation ratings	A. S. T. M. Supercharge ratings ^b					
			Fuel-air ratio					
			0.065	0.070	0.085	0.095	0.100	0.110
A-411	20% neohexane+80% VBS....	94.5 84	74 95.0 88	86 100 100	124 100 104	142 109 103	147 0.05 102	150 99.2 98
A-412	40% neohexane+60% VBS....	0.05 102	81 99.4 98	97 0.17 106	138 0.28 110	158 0.31 111	164 0.32 111	167 0.28 110
A-413	60% neohexane+40% VBS....	0.36 112	93 0.26 110	108 0.34 112	159 0.67 120	178 1.03 126	183 1.17 128	187 1.25 130
A-414	80% neohexane+20% VBS....	2.00 138	108 0.75 121	130 1.06 127	182 1.95 138	203 2.48 143	208 2.57 143	210 2.41 142
A-415	20% neohexane+80% alkylate.	1.10 127	112 0.95 125	130 1.06 127	172 1.41 132	193 1.85 137	199 1.95 138	202 1.91 138
A-416	40% neohexane+60% alkylate.	1.50 133	118 1.25 130	137 1.38 131	186 2.19 140	203 2.48 143	207 2.50 143	209 2.35 142
A-417	60% neohexane+40% alkylate.	2.57 143	124 1.53 133	146 1.78 136	195 2.78 145	212 3.10 147	215 3.07 147	216 2.83 145
A-418	80% neohexane+20% alkylate.	3.36 149	137 2.35 142	158 2.67 144	208 3.61 151	224 3.93 152	227 3.93 152	226 3.52 150
A-420	20% neohexane+80% one-pass stock.	96.6 90	79 98.1 95	95 0.14 105	146 0.38 113	171 0.50 116	180 0.92 124	190 1.38 131
A-421	40% neohexane+60% one-pass stock.	0.10 104	86 0.10 104	107 0.33 104	165 1.02 126	190 1.70 135	197 1.86 137	203 1.96 138
A-422	60% neohexane+40% one-pass stock.	0.33 111	108 0.75 121	138 1.43 132	192 2.58 143	210 2.97 146	215 3.07 147	217 2.90 146
A-423	80% neohexane+20% one-pass stock.	1.06 135	132 1.91 138	162 3.05 147	214 4.00 153	230 4.72 156	233 4.67 156	234 4.14 153
A-394	Neohexane ^c	6.00 161	159 4.76 156	187 5.58 160	230 4.00 163	240 5.87 162	242 5.87 161	243 5.43 159
A-123	20% isopentane+80% VBS....	94.4 83	72 93.8 84	87 0.02 101	127 0.14 106	141 0.07 103	146 0.03 101	149 98.9 99
A-124	40% isopentane+60% VBS....	99.1 97	80 98.8 96	99 0.20 108	139 0.29 110	151 0.21 108	155 0.18 107	159 0.12 105
A-134	60% isopentane+40% VBS....	0.23 108	87 0.12 105	112 0.41 114	153 0.46 114	168 0.46 114	172 0.45 114	174 0.42 113
A-375	20% isopentane+80% alkylate.	0.92 124	121 1.39 131	144 1.69 135	186 2.19 140	201 2.34 142	204 2.29 141	204 2.00 138
A-376	40% isopentane+60% alkylate.	0.99 125	121 1.39 131	144 1.69 135	191 2.52 143	203 2.48 143	205 2.36 142	204 2.00 138
A-388	20% isopentane+80% one-pass stock.	93.8 87	78 97.5 94	87 0.02 101	132 0.20 108	160 0.34 111	173 0.47 115	188 1.29 130
A-389	40% isopentane+60% one-pass stock.	100 100	85 0.07 103	97 0.17 107	140 0.30 111	168 0.46 115	180 0.92 124	192 1.47 133

^a Each fuel contains approximately 4 ml TEL/gal.^b Based on fixed reference-fuel framework (ch. VII, reference 13).^c Values for knock-limited imep were averaged from two curves.

TABLE A-10.—PERFORMANCE RATINGS OBTAINED IN A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE ENGINES—Continued

Fuel	Fuel composition ^a (by volume)	A. S. T. M. Aviation ratings	A. S. T. M. Supercharge ratings ^b					
			Fuel-air ratio					
			0.065	0.070	0.085	0.095	0.100	0.110
A-139	20% hot-acid octane+80% VBS.	94.3 83	70 92.5 83	83 98.0 94	128 0.15 106	147 0.16 106	151 0.11 105	154 0.02 101
A-140	40% hot-acid octane+60% VBS.	100 100	74 95.0 94	89 0.03 101	143 0.34 111	168 0.46 114	173 0.47 115	179 0.58 117
A-141	60% hot-acid octane+40% VBS.	0.18 107	84 0.05 102	106 0.31 111	165 1.02 126	191 1.75 136	198 1.91 138	207 2.21 140
A-367	20% hot-acid octane+80% alkylate.	0.82 123	121 1.39 131	142 1.60 134	188 2.32 141	205 2.62 144	210 2.72 145	215 2.76 145
A-368	40% hot-acid octane+60% alkylate.	0.72 121	125 1.58 134	148 1.87 137	200 3.10 147	219 3.59 150	226 3.86 152	235 4.29 154
A-369	60% hot-acid octane+40% alkylate.	0.88 124	129 1.77 136	154 2.29 141	216 4.31 154	240 4.62 162	248 4.64 164	257 4.68 168
A-370	80% hot-acid octane+20% alkylate.	0.72 121	129 1.77 136	154 2.29 141	238 2.29 169	269 182 182	276 183 182	280 182 182
A-371	20% hot-acid octane+80% one-pass stock.	95.1 86	80 98.8 95	90 0.06 102	138 0.28 110	170 0.49 115	185 1.30 130	206 2.14 140
A-372	40% hot-acid octane+60% one-pass stock.	100 100	88 0.14 105	97 0.17 107	154 0.48 115	192 1.80 136	208 2.57 143	229 3.73 151
A-373	60% hot-acid octane+40% one-pass stock.	0.18 107	90 0.19 107	101 0.23 108	164 0.96 125	203 2.48 143	220 3.43 149	245 5.71 160
A-374	80% hot-acid octane+20% one-pass stock.	0.45 115	99 0.41 114	115 0.45 115	187 2.26 141	224 3.93 152	240 5.60 160	268 175 175
A-330	Hot-acid octane ^c	1.08 127	131 1.86 137	159 2.76 145	250 178 178	289 195 195	304 195 195	317 195 195
A-257	20% mixed xylenes+80% VBS.	92.6 79	68 91.3 79	75 94.7 86	114 99.2 97	132 98.7 96	138 98.1 94	148 98.6 96
A-258	40% mixed xylenes+60% VBS.	95.5 86	69 91.9 80	78 94.7 86	117 0.03 101	147 0.16 106	160 0.26 110	182 0.83 123
A-259	60% mixed xylenes+40% VBS.	95.2 86	74 95.0 87	85 99.3 99	146 0.38 113	194 1.90 137	216 3.14 148	253 165 165
A-260	80% mixed xylenes+20% VBS.	0.04 101	84 0.05 102	95 0.14 105	214 4.00 153	240 4.00 153	240 4.00 153	240 4.00 153
A-261	20% mixed xylenes+80% alkylate.	0.52 116	88 0.14 105	101 0.23 108	158 0.65 119	194 1.90 137	208 2.57 143	227 3.59 150
A-262	40% mixed xylenes+60% alkylate.	0.27 110	82 100 100	95 0.14 105	153 0.46 115	206 2.69 144	252 167 167	287 187 187
A-263	60% mixed xylenes+40% alkylate.	0.14 105	85 0.07 103	98 0.19 107	181 1.89 137	274 185 185	274 185 185	274 185 185
A-264	80% mixed xylenes+20% alkylate.	0.27 110	87 0.12 105	103 0.27 110	260 185 185	336 185 185	370 185 185	370 185 185

^a Each fuel contains approximately 4 ml TEL/gal.^b Based on fixed reference-fuel framework (ch. VIII, reference 13).^c Values for knock-limited imep were averaged from two curves

TABLE A-10.—PERFORMANCE RATINGS OBTAINED IN A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE ENGINES—Continued

Fuel	Fuel composition* (by volume)	A. S. T. M. Aviation ratings	A. S. T. M. Supercharge ratings ^b					
			Fuel-air ratio					
			0.065	0.070	0.085	0.095	0.100	0.110
A-265	20% mixed xylenes+80% one-pass stock ^c	94.7 84	71 93.1 83	74 92.0 81	111 97.9 94	138 0.03 101	151 0.11 105	178 0.50 116
A-266	40% mixed xylenes+60% one-pass stock ^c	97.5 92	80 98.8 95	86 100 100	133 0.21 108	172 0.58 118	196 1.81 136	246 5.86 161
A-267	60% mixed xylenes+40% one-pass stock ^c	98.8 96	95 0.31 111	100 0.22 108	184 2.06 139	251 — 169	282 — 187	339 — —
A-268	80% mixed xylenes+20% one-pass stock ^c	0.16 106	102 0.48 115	106 0.31 111	351 — —	— — —	— — —	— — —
A-256	Mixed xylenes ^c	0.92 124	105 0.60 118	122 0.69 120	— — —	— — —	— — —	— — —
A-245	20% cumene+80% VBS	92.4 78	67 90.6 78	72 90.7 78	98 92.5 82	123 95.7 88	134 96.9 91	154 0.02 101
A-244	40% cumene+60% VBS	92.7 79	67 90.6 78	70 89.3 76	95 91.3 80	117 93.7 84	130 95.6 88	160 0.14 105
A-246	60% cumene+40% VBS	94.2 83	67 90.6 78	72 90.7 78	94 90.8 78	118 94.0 85	132 96.3 91	174 0.42 114
A-247	80% cumene+20% VBS	96.0 88	77 96.9 91	76 93.3 84	90 89.2 76	120 94.7 86	151 0.11 105	— — —
A-248	20% cumene+80% alkylate	0.32 111	98 0.38 113	102 0.25 109	143 0.34 111	172 0.58 117	187 1.39 131	215 2.76 145
A-249	40% cumene+60% alkylate	0.11 105	71 93.1 84	76 93.3 84	113 98.8 95	148 0.17 106	171 0.44 114	233 4.00 153
A-250	60% cumene+40% alkylate	0.03 101	77 96.9 91	76 93.3 84	94 90.8 78	124 96.0 90	149 0.08 103	277 — 180
A-251	80% cumene+20% alkylate	97.7 93	74 95.0 87	73 91.3 80	86 87.5 73	114 92.7 82	160 0.26 110	— — —
A-252	20% cumene+80% one-pass stock	93.0 80	70 92.5 82	69 88.6 75	91 89.6 76	120 94.3 85	137 97.8 93	175 0.44 114
A-253	40% cumene+60% one-pass stock	93.6 82	70 92.5 82	67 87.4 74	75 82.9 67	95 86.3 72	112 90.0 77	168 0.30 111
A-254	60% cumene+40% one-pass stock	93.0 80	66 90.0 78	62 84.0 68	65 78.8 62	81 81.3 65	94 84.4 69	153 100 100
A-255	80% cumene+20% one-pass stock	95.0 85	66 90.0 78	63 84.8 69	70 80.8 64	98 87.3 74	141 99.1 98	— — —
A-240	Cumene ^c	95.0 85	77 96.9 91	75 92.7 83	87 87.9 74	122 95.3 88	— — —	— — —

* Each fuel contains approximately 4 ml TEL/gal.

^b Based on fixed reference-fuel framework (ch. VIII, reference 13).^c Knock-limited performance of engine with one-pass catalytic stock was low on day fuels were investigated.

• Values for knock-limited imep were averaged from three curves.

TABLE A-10.—PERFORMANCE RATINGS OBTAINED IN A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE ENGINES—Continued

Fuel	Fuel composition* (by volume)	A. S. T. M. Aviation ratings	A. S. T. M. Supercharge ratings ^b					
			Fuel-air ratio					
			0.065	0.070	0.085	0.095	0.100	0.110
A-341	20% benzene+80% VBS	91.4 76	78 97.5 93	85 99.4 97	134 0.22 108	155 0.27 110	162 0.29 110	168 0.30 111
A-342	40% benzene+60% VBS	92.4 78	81 99.4 97	89 0.05 102	146 0.37 112	178 1.03 126	190 1.63 134	208 2.28 141
A-343	60% benzene+40% VBS	94.2 83	78 97.5 93	85 99.4 97	244 — 173	346 — —	384 — —	— — —
A-344	80% benzene+20% VBS	96.2 88	98 0.38 113	115 0.45 115	— — —	— — —	— — —	— — —
A-358	20% benzene+80% alkylate	0.43 114	117 1.20 129	127 0.92 124	182 1.95 137	212 3.10 147	222 3.57 150	234 4.14 153
A-359	40% benzene+60% alkylate	0.12 105	102 0.48 115	112 1.95 114	182 4.72 156	230 — 168	253 — 192	295 — —
A-360	60% benzene+40% alkylate	100 100	102 0.48 115	110 0.38 113	336 — —	— — —	— — —	— — —
A-361	80% benzene+20% alkylate	98.3 94	119 1.30 130	178 4.63 156	— — —	— — —	— — —	— — —
A-362	20% benzene+80% one-pass stock	93.8 82	77 96.9 91	86 100 100	142 0.33 111	172 0.58 118	184 1.25 130	203 1.96 138
A-363	40% benzene+60% one-pass stock	92.0 78	82 100 100	79 95.3 88	160 0.73 121	213 3.17 148	238 5.33 159	264 — 172
A-364	60% benzene+40% one-pass stock	91.5 77	68 91.3 80	72 90.7 78	191 2.52 143	254 — 172	280 — 186	328 — —
A-365	80% benzene+20% one-pass stock	93.0 80	94 0.29 110	93 0.11 105	— — —	— — —	— — —	— — —
A-340	Benzene ^c	87 68	199 186	— 196	— —	— —	— —	— —
A-321	20% toluene+80% VBS	93.7 82	85 0.07 103	96 0.16 106	137 0.26 110	156 0.29 110	164 0.32 111	172 0.27 110
A-322	40% toluene+60% VBS	95.1 85	92 0.24 109	96 0.16 106	175 1.57 134	228 4.43 155	245 — 162	266 — 173
A-323	60% toluene+40% VBS	97.0 91	88 0.14 105	95 0.14 105	204 3.36 149	303 — —	346 — —	425 — —
A-324	80% toluene+20% VBS	98.8 96	101 0.45 115	113 0.42 114	340 — —	— — —	— — —	— — —

* Each fuel contains approximately 4 ml TEL/gal.

^b Based on fixed reference-fuel framework (ch. VIII, reference 13).^c Values for knock-limited imep were averaged from two curves.

TABLE A-10.—PERFORMANCE RATINGS OBTAINED IN
A. S. T. M. AVIATION AND A. S. T. M. SUPERCHARGE
ENGINES—Concluded

Fuel	Fuel composition* (by volume)	A. S. T. M. Aviation rat- ings	A. S. T. M. Supercharge ratings ^b					
			Fuel-air ratio					
			0.065	0.070	0.085	0.095	0.100	0.110
A-325	20% toluene+80% alkylate....	0.48 115	121 1.39 131	139 1.47 132	191 2.52 143	221 3.73 151	232 4.53 155	249 162
A-326	40% toluene+60% alkylate....	0.54 116	108 0.75 121	128 0.97 125	223 5.38 159	275 186	308	348
A-327	60% toluene+40% alkylate....	0.25 109	100 0.43 114	105 0.30 111	300			
A-328	80% toluene+20% alkylate....	0.16 106	108 0.75 121	116 0.47 115				
A-331	20% toluene+80% one-pass stock.	95.1 85	80 98.8 95	90 0.06 103	137 0.26 110	169 0.47 115	184 1.25 130	212 2.55 143
A-332	40% toluene+60% one-pass stock.	95.3 86	85 0.07 103	92 0.09 103	151 0.44 114	202 2.41 142	224 3.72 151	262 171
A-333	60% toluene+40% one-pass stock.	97.4 91	91 0.21 108	95 0.14 105	178 1.73 135	270 182	319	
A-334	80% toluene+20% one-pass stock.	0.10 104	102 0.48 115	106 0.31 111				
A-320	Toluene ^c	0.57 118	134 2.00 138	140 1.51 133				
A-336	20% MTB ether+80% VBS....	98.8 96	95 0.31 111	101 0.23 108	144 0.30 111	170 0.49 115	179 0.83 121	187 1.25 130
A-337	40% MTB ether+60% VBS....	0.12 105	112 0.95 125	113 0.42 114	165 1.02 126	204 2.55 143	223 3.64 151	253 165
A-338	60% MTB ether+40% VBS....	0.92 124	192 180	163 3.14 148	228 162	281 190	307	355
A-339	80% MTB ether+20% VBS....	2.61 144		239	309	379		
A-347	20% MTB ether+80% al- kylate.	1.68 135	143 3.06 146	155 2.38 142	230 163	258 174	268 178	281 183
A-348	40% MTB ether+60% al- kylate.	2.30 141	166 5.43 159	174 4.21 154	258 183	312	338	377
A-349	60% MTB ether+40% al- kylate.	2.50 143	258	229 193	327	406	442	
A-350	80% MTB ether+20% al- kylate.	6.0 161	307	271	374			
A-351	20% MTB ether+80% one- pass stock.	96.1 88	87 0.12 105	91 0.08 103	144 0.35 112	179 1.10 127	194 1.72 135	218 2.97 146
A-352	40% MTB ether+60% one- pass stock.	0.14 105	113 1.00 126	112 0.41 114	163 0.85 123	204 2.55 143	225 3.79 152	269 175
A-353	60% MTB ether+40% one- pass stock.	0.47 115	185 173	160 2.86 146	230 163	289 195	319	376
A-354	80% MTB ether+20% one- pass stock.	1.00 126	269	237 200	301	370		
A-335	MTB ether ^c	>6.00 >161		330	406			

* Each fuel contains approximately 4 ml TEL/gal.

^b Based on fixed reference-fuel framework (ch. VIII, reference 13).

^c Values for knock-limited imep were averaged from two curves.

APPENDIX B

ADDITIONAL BLENDING CHARTS FOR TERNARY FUEL BLENDS¹

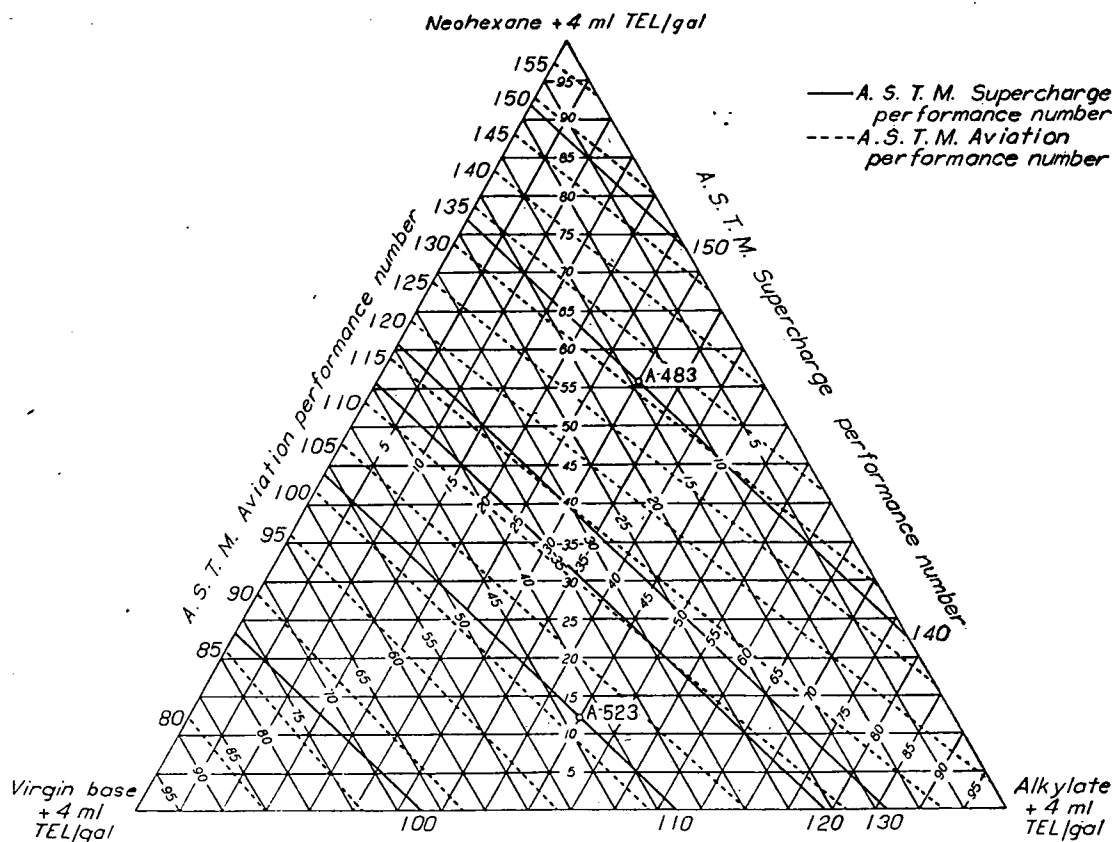
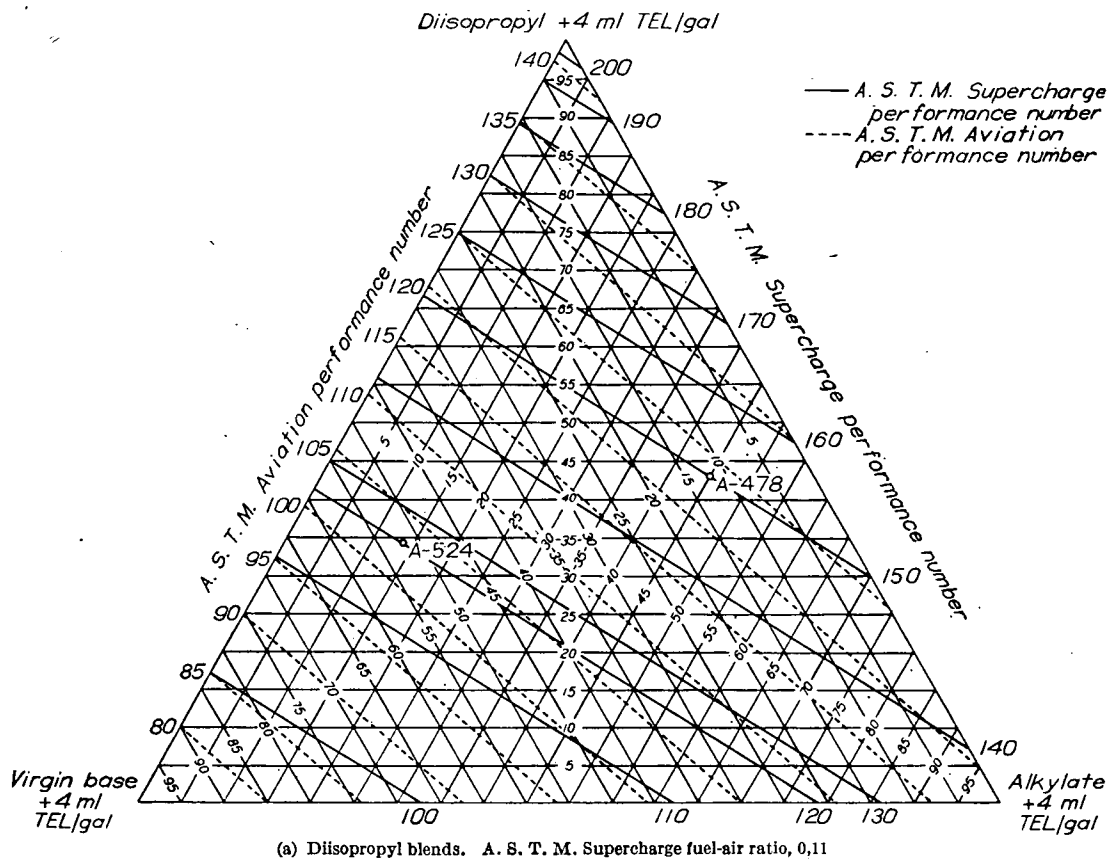


FIGURE B-1.—Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and virgin base stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods.

¹ The charts of this appendix were reproduced from NACA Report 904 entitled "Estimation of F-3 and F-4 Knock-Limited Performance Ratings for Ternary and Quaternary Blends Containing Triptane or Other High-Antiknock Aviation-Fuel Blending Agents" by Henry C. Barnett.

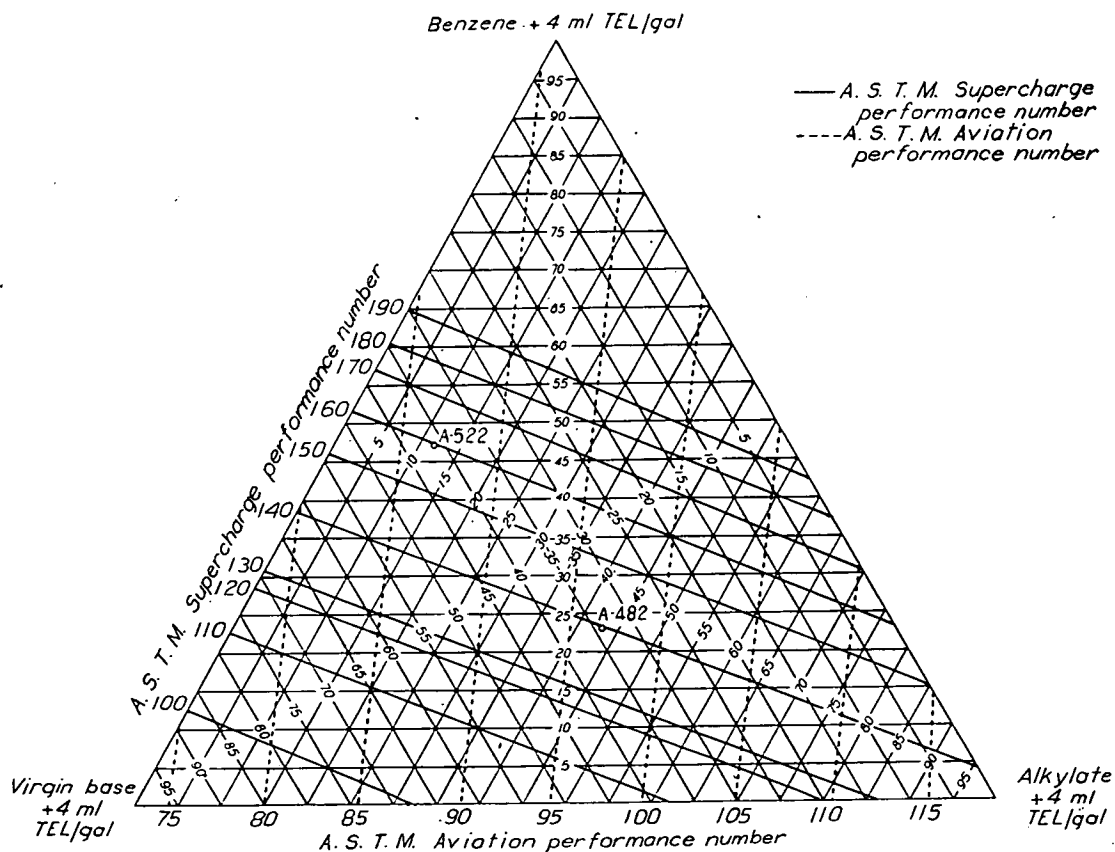
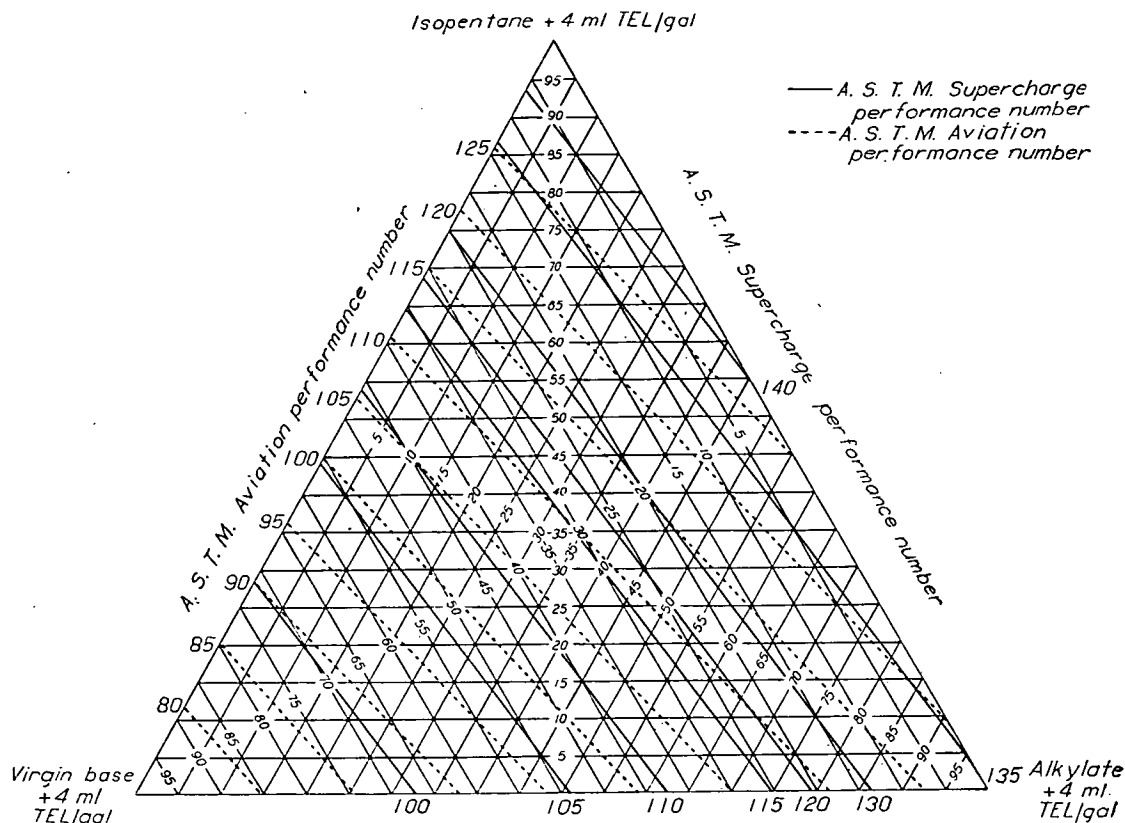
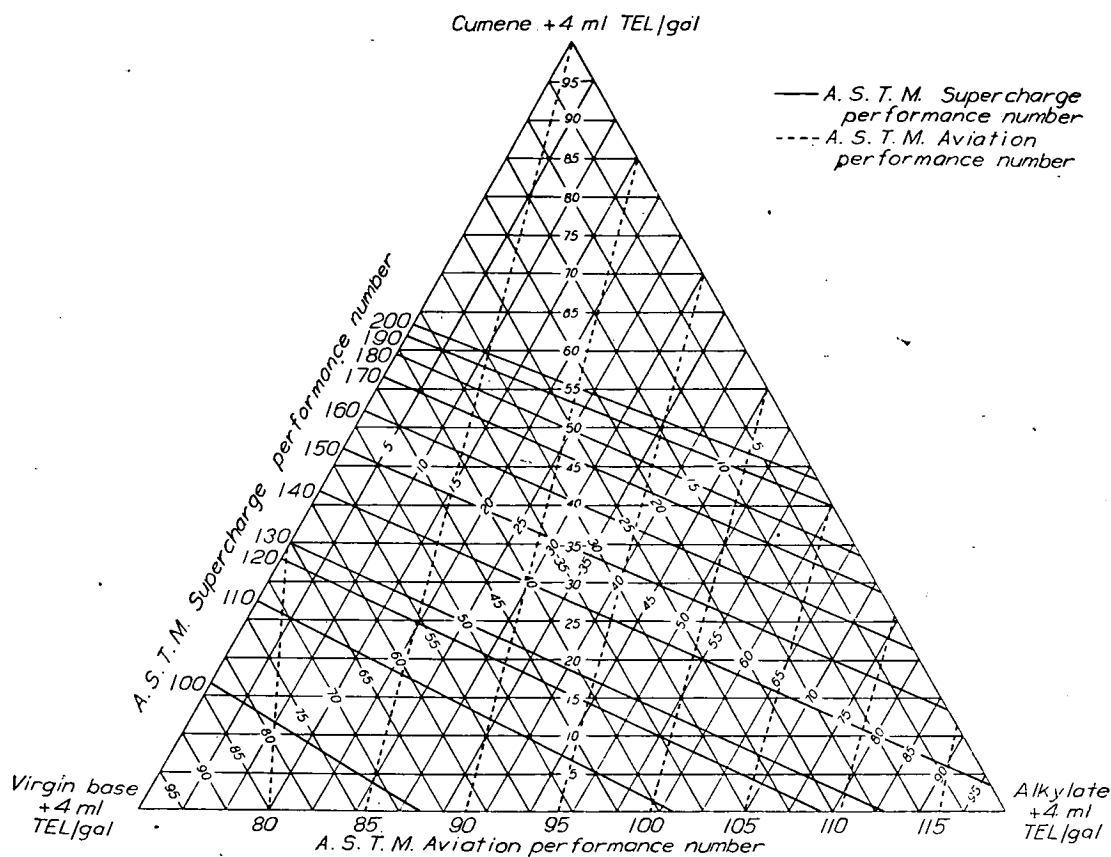
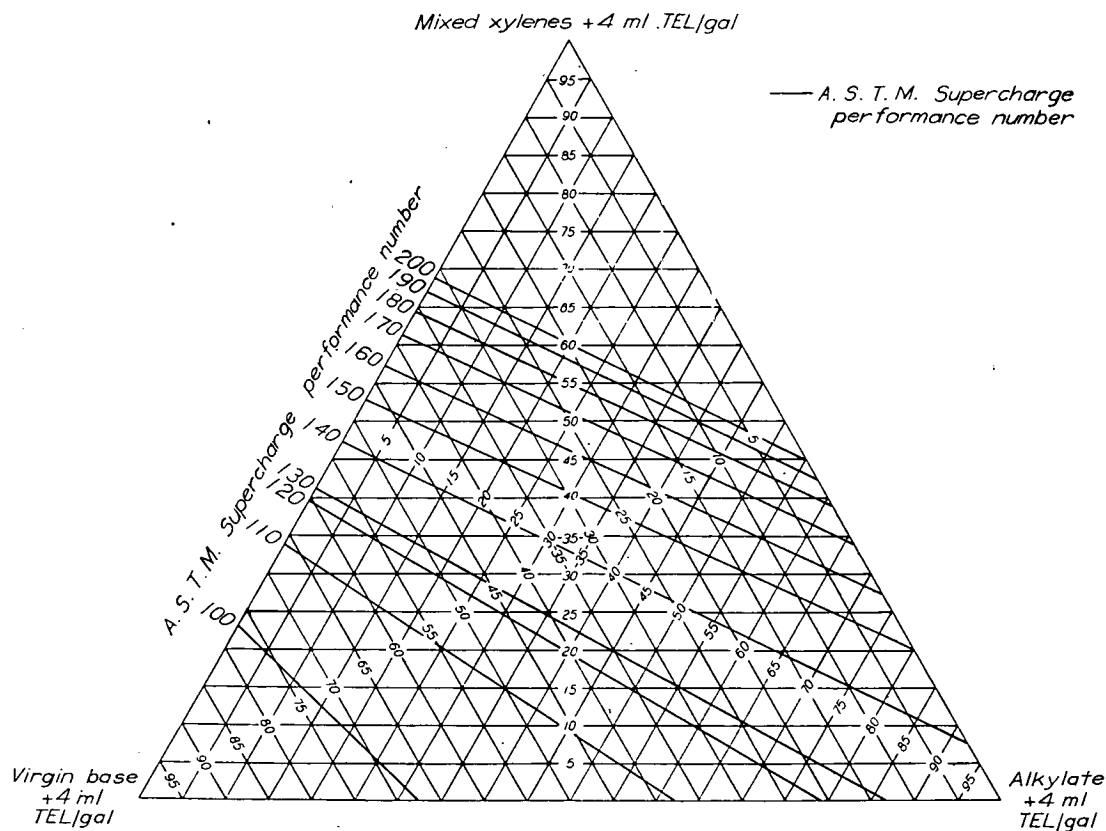


FIGURE B-1.—Continued. Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and virgin base stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods.



(e) Cumene blends. A. S. T. M. Supercharge ratings at fuel-air ratio for peak power.



(f) Mixed xylene blends. A. S. T. M. Supercharge fuel-air ratio, 0.11.

FIGURE B-1.—Continued. Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and virgin base stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods.

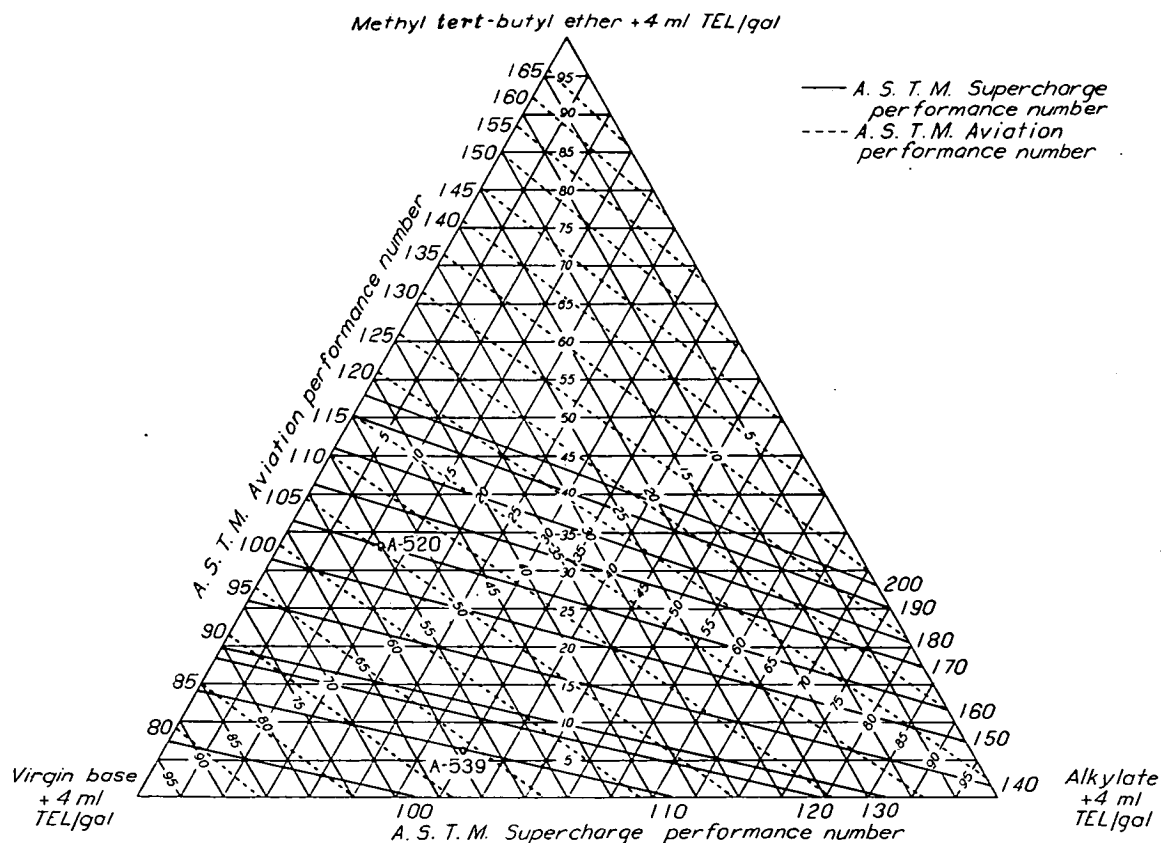
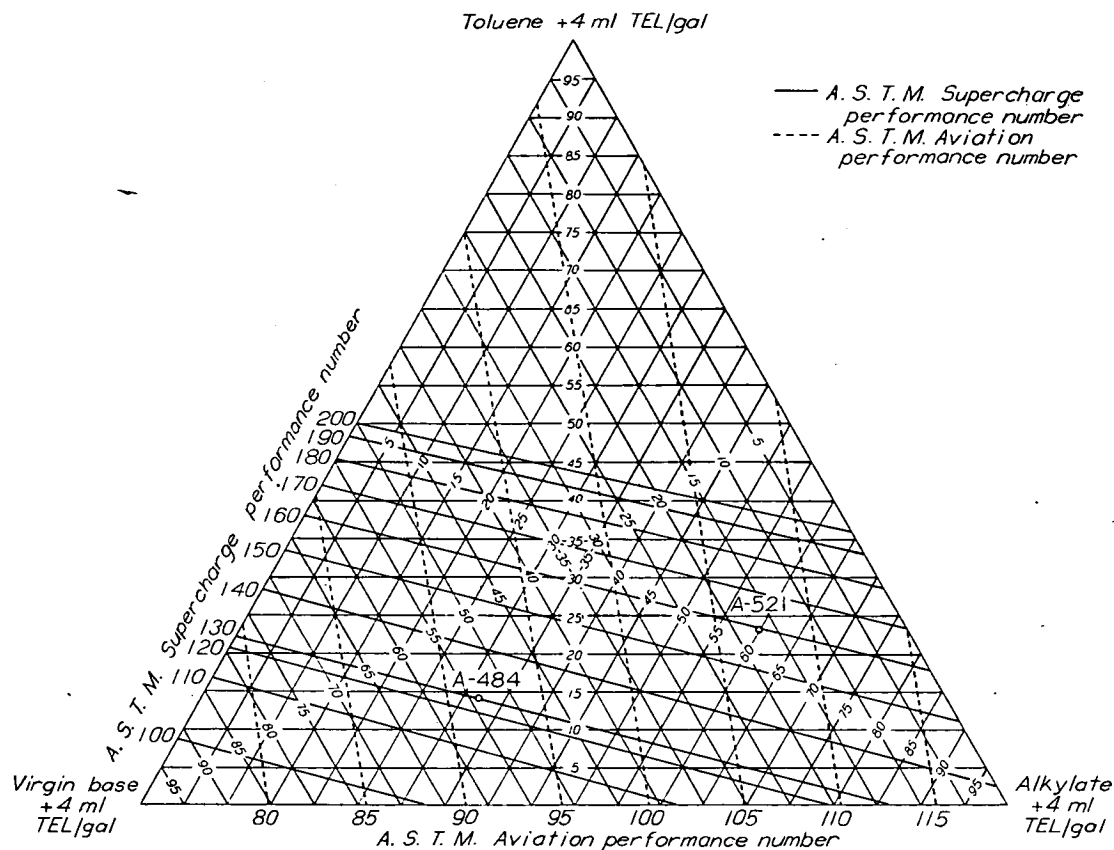


FIGURE B-1.—Concluded. Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and virgin base stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods.

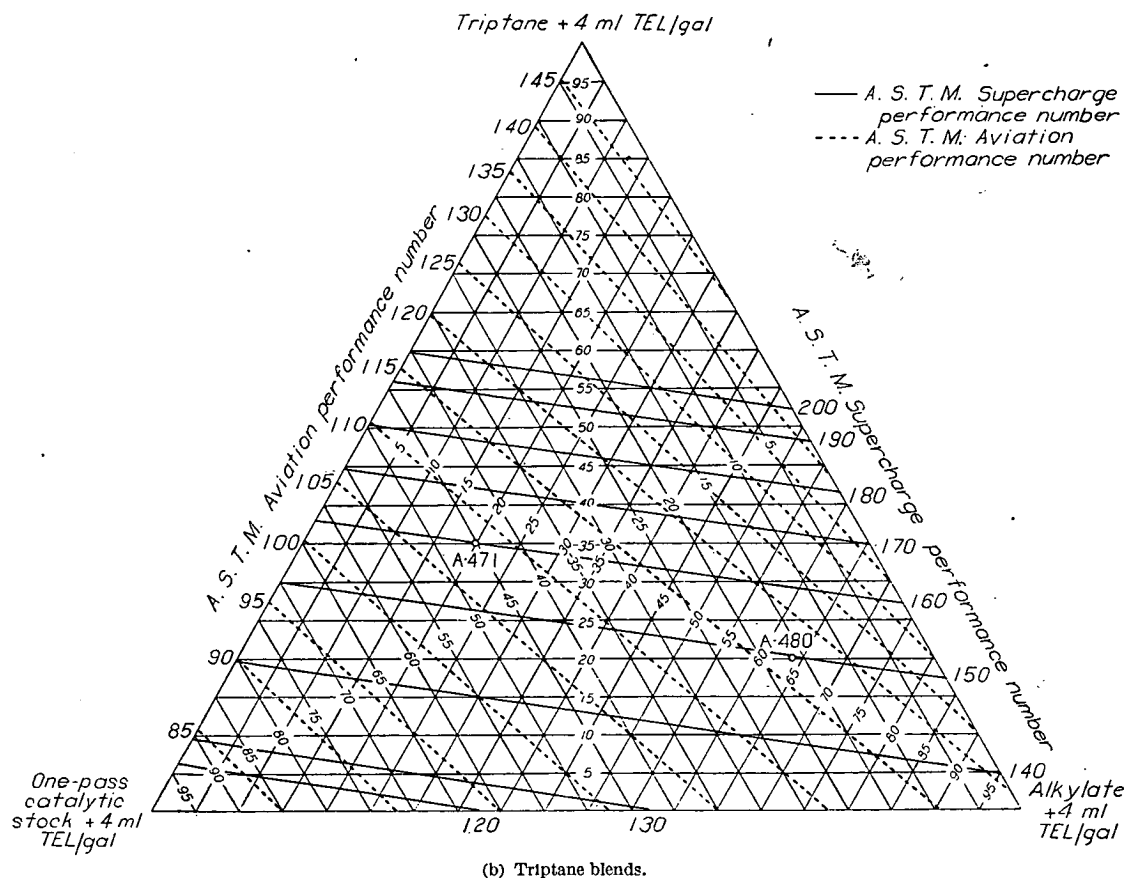
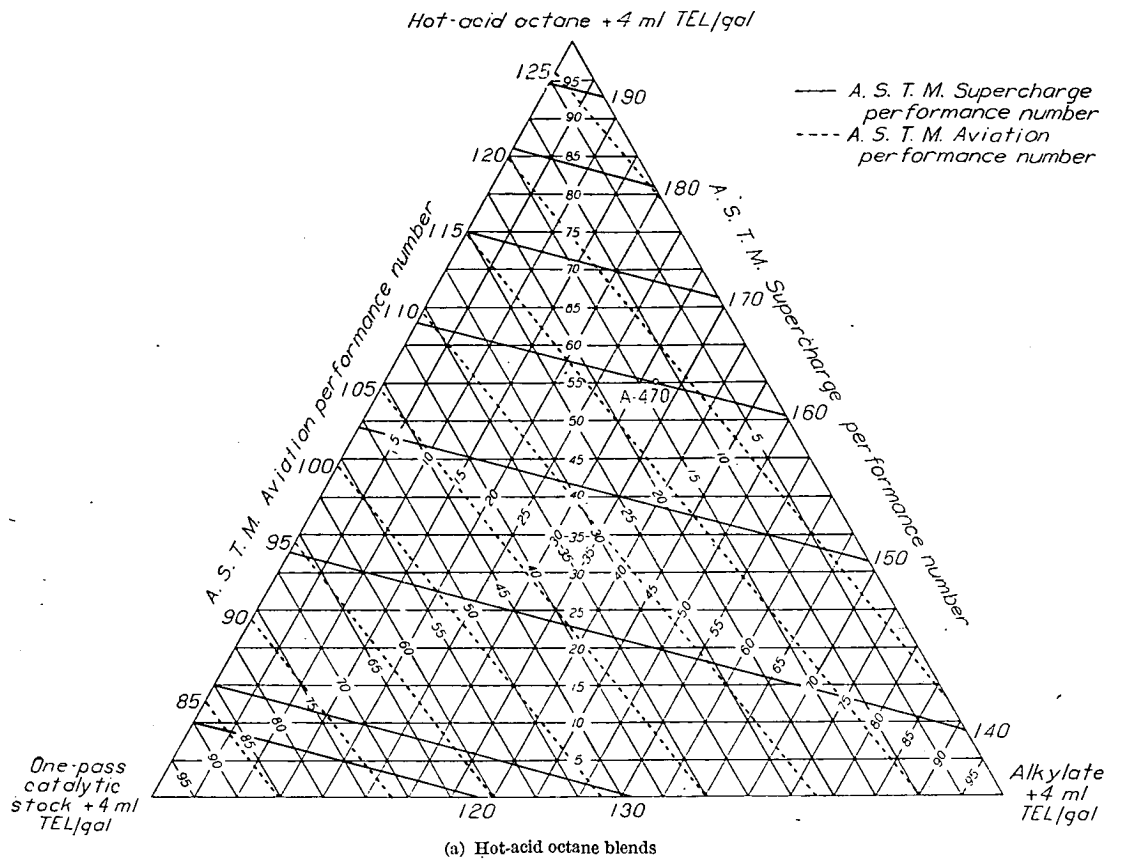


FIGURE B-2.—Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and one-pass catalytic stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. A. S. T. M. Supercharge fuel-air ratio, 0.11.

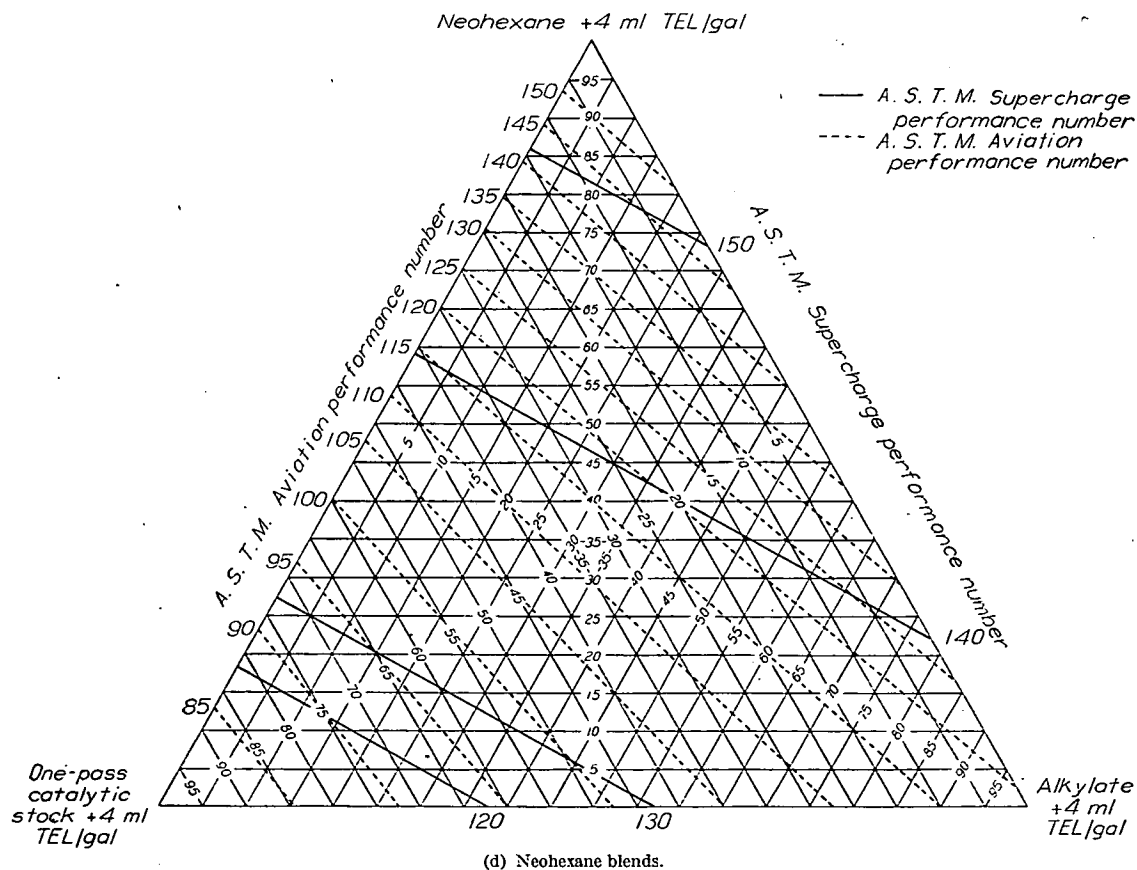
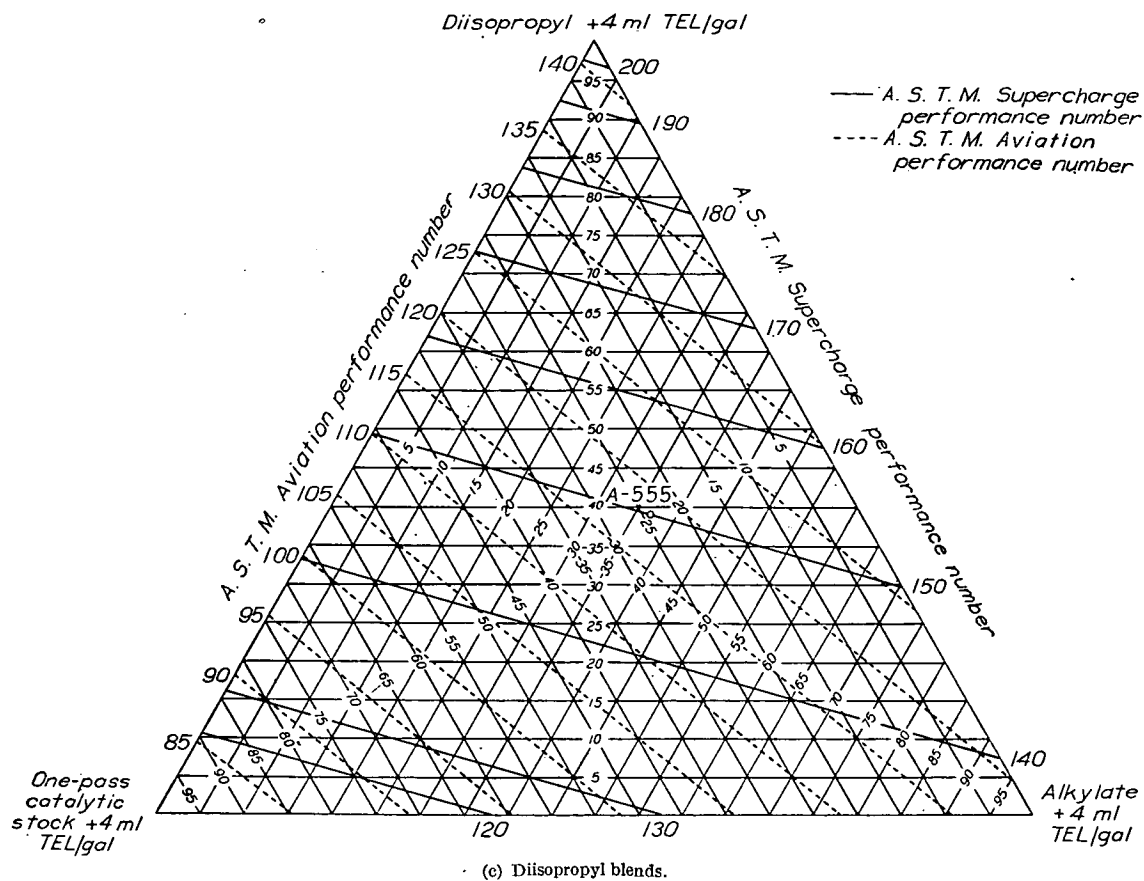


FIGURE B-2.—Continued. Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and one-pass catalytic stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. A. S. T. M. Supercharge fuel-air ratio, 0.11.

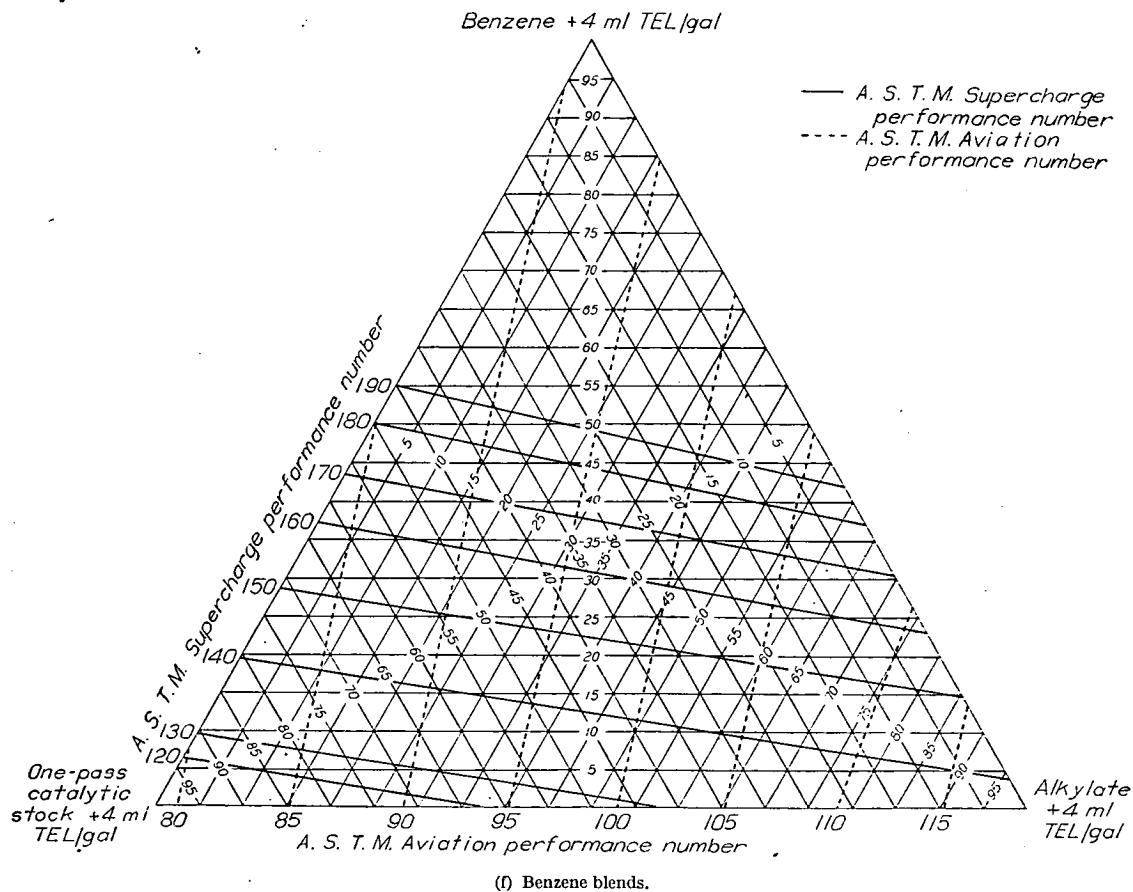
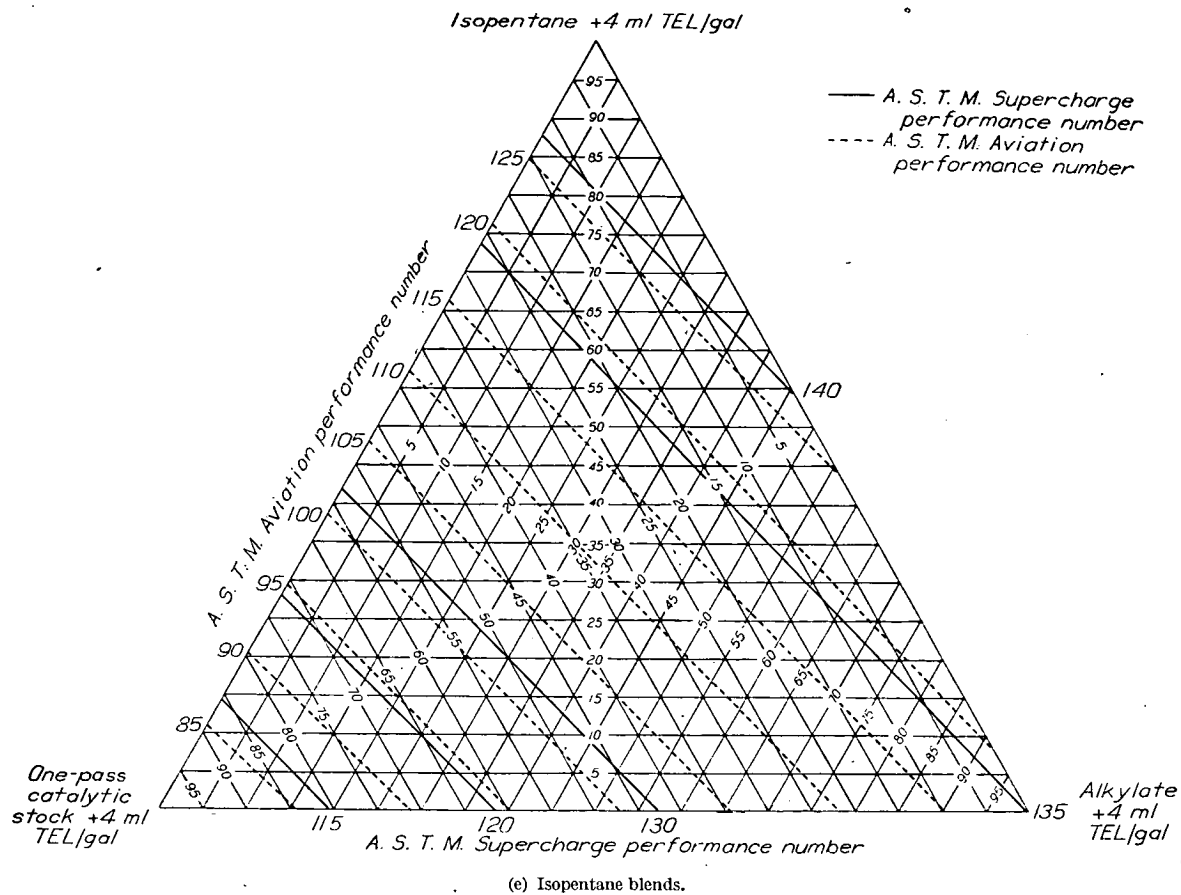


FIGURE B-2.—Continued. Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and one-pass catalytic stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. A. S. T. M. Supercharge fuel-air ratio, 0.11.

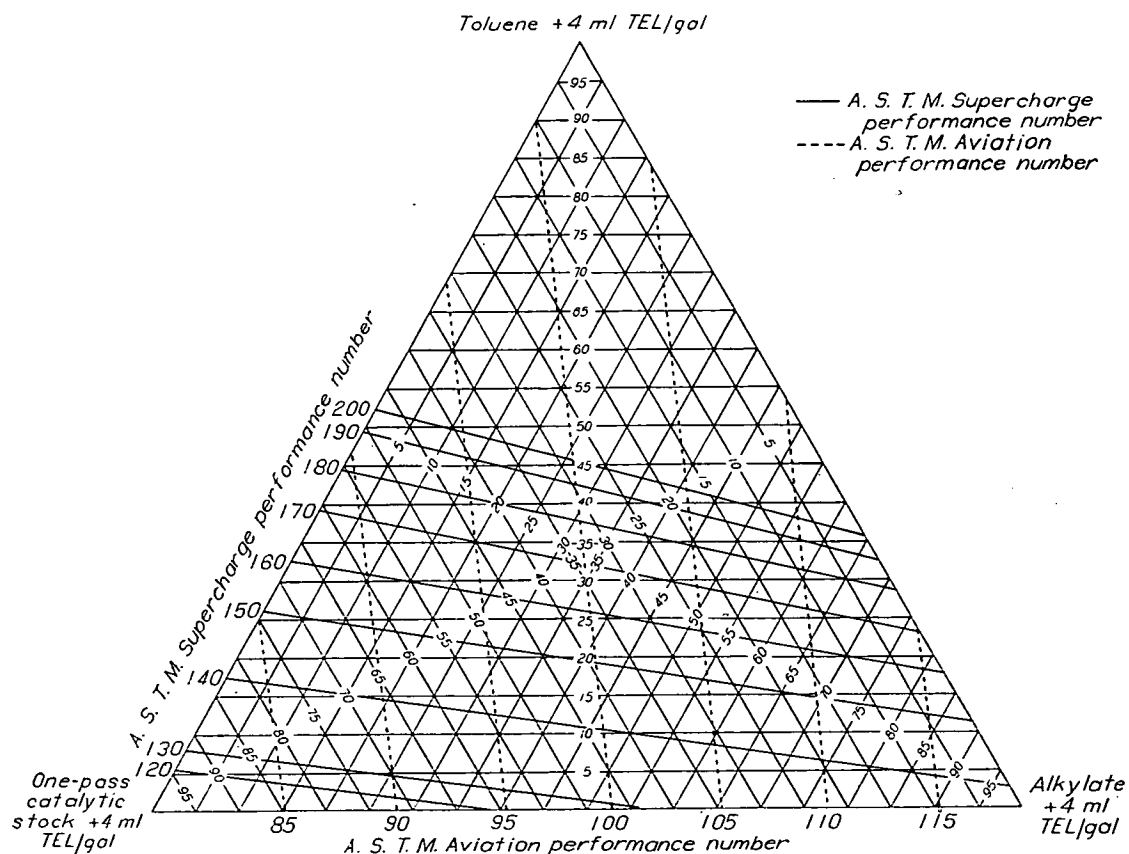
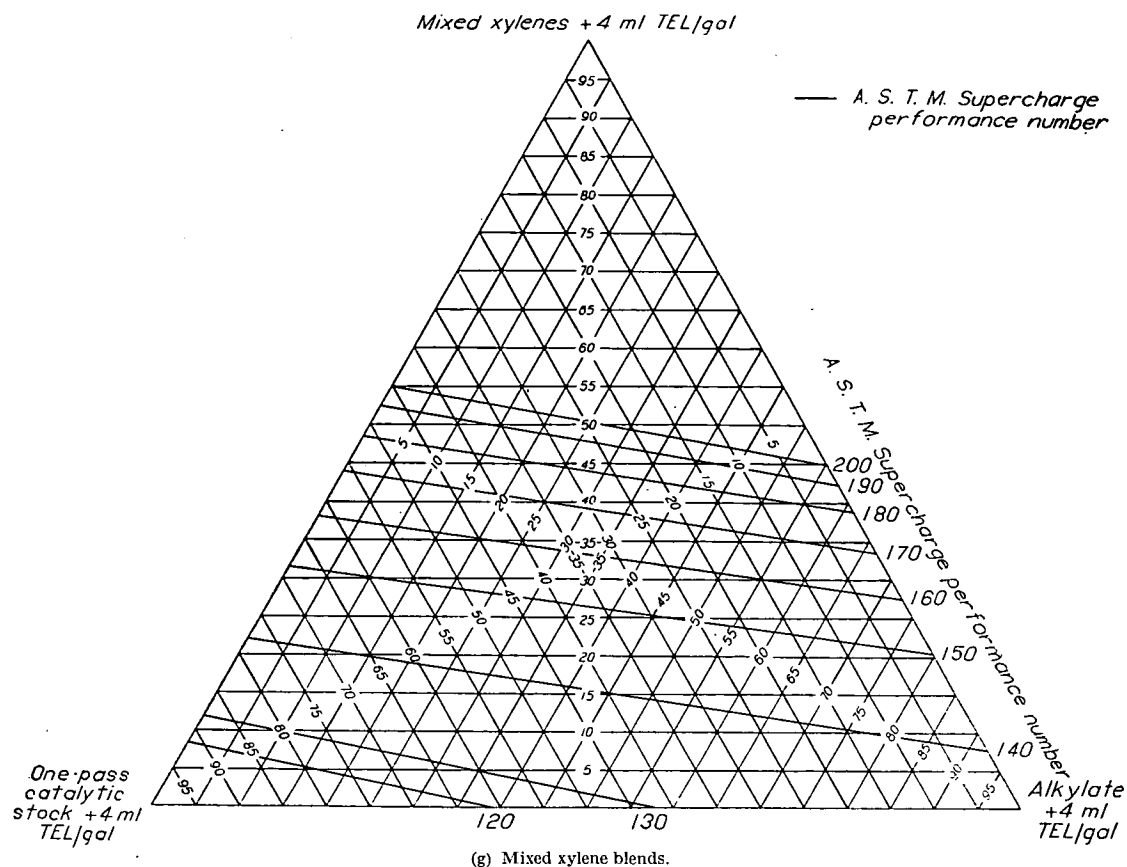


FIGURE B-2.—Continued. Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and one-pass catalytic stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. A. S. T. M. Supercharge fuel-air ratio, 0.11.

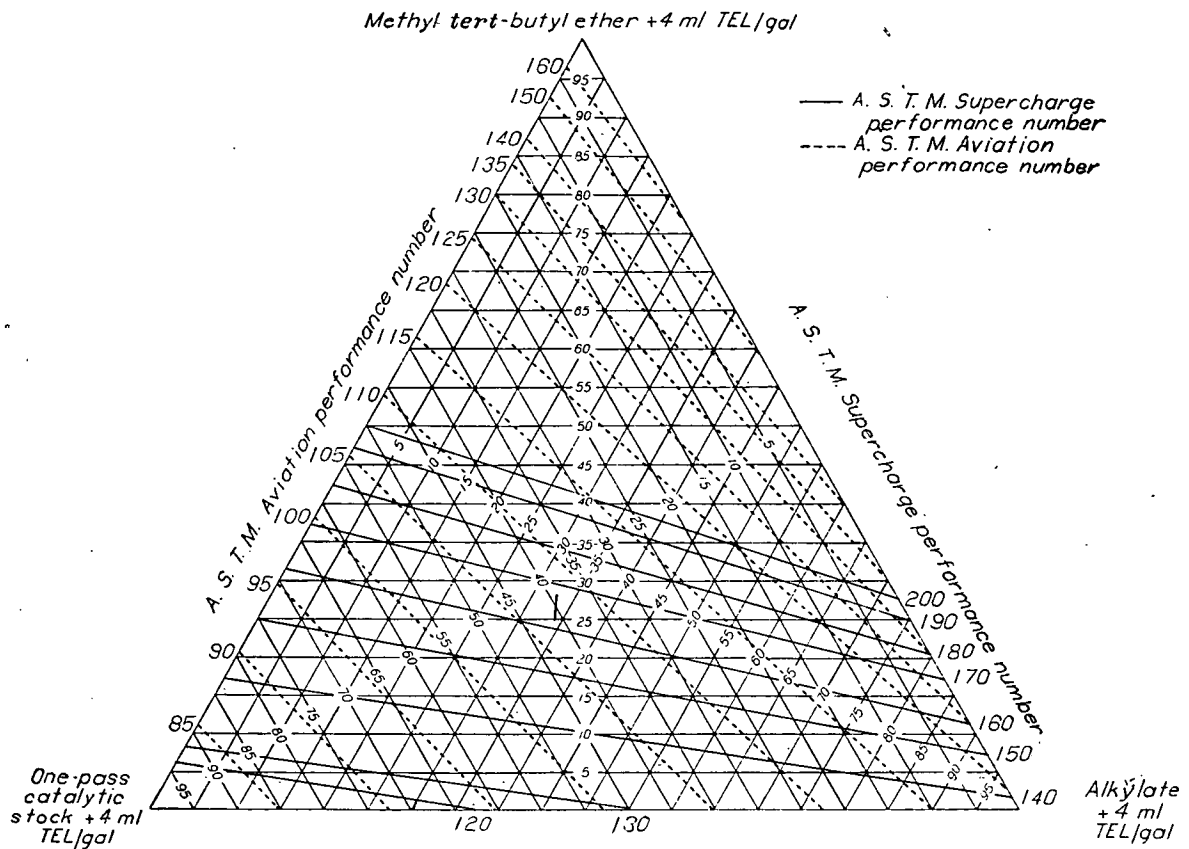


FIGURE B-2.—Concluded. Blending charts for ternary blends containing high-antiknock blending agents, aviation alkylate, and one-pass catalytic stock by A. S. T. M. Aviation and A. S. T. M. Supercharge methods. A. S. T. M. Supercharge fuel-air ratio, 0.11.

REPORT 1027

BUCKLING OF THIN-WALLED CYLINDER UNDER AXIAL COMPRESSION AND INTERNAL PRESSURE¹

By HSU LO, HAROLD CRATE, and EDWARD B. SCHWARTZ

SUMMARY

An investigation was made of a thin-walled cylinder under axial compression and various internal pressures to study the effect of the internal pressure on the compressive buckling stress of the cylinder. A theoretical analysis based on a large-deflection theory was also made. The theoretically predicted increase of compressive buckling stress due to internal pressure agrees fairly well with the experimental results.

INTRODUCTION

The buckling of thin-walled cylinders under axial compression and lateral pressure has been investigated by Flügge (reference 1) who found that the effect of the internal pressure on the buckling load is negligible. Flügge's conclusion is in contradiction to the results of a series of tests, made at the Langley Aeronautical Laboratory of the NACA, of two curved panels under axial compression and various lateral pressures. These test results, reported in reference 2, showed an appreciable strengthening effect of the lateral pressure on the buckling load of the curved panels. The apparent discrepancy between these experimental results and the prediction by Flügge's theory made it desirable to investigate this problem further. Consequently, additional tests were made of a cylinder under axial compression and various internal pressures for which results are presented herein. A theoretical analysis of this problem is also presented which differs from that of Flügge in that the present analysis is based on large-, rather than small-, deflection theory.

APPARATUS AND PROCEDURES

Test specimen.—The specimen used for the tests was a cylinder, 32 inches long with a 15-inch inside radius, made of 24S-T aluminum alloy sheet of 0.0249-inch average thickness. It was closely riveted around two heavy steel rings, one at each end. The butt joint of the two longitudinal edges was covered both inside and outside by straps, 0.032 inch thick and 1½ inches wide, along the total length of the cylinder. (See fig. 1.)

The two heavy steel rings were made of ½- by 4-inch steel bar stock rolled to the diameter of the cylinder. Two ¾- by 2-inch spreader bars were used to reinforce the ring as shown in figure 1. A ring with a flange, machined flat, was fastened to the ½- by 4-inch steel ring to provide an even bearing surface on which a steel cover plate was fitted.

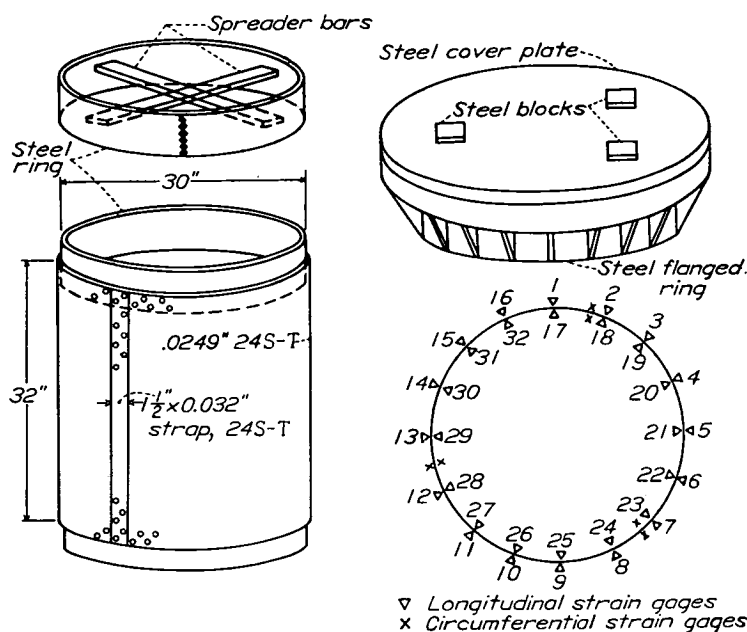


FIGURE 1.—Test specimen and strain-gage positions.

Three steel blocks were placed on top of the plate. The applied compressive load was transmitted from the machine head through the three steel blocks to the cover plate. The joint between the cylinder and the cover plates was sealed.

Equally spaced along the inside circumference of the cylinder at midlength were 16 strain gages, and directly opposite to them on the outside were 16 more gages. These gages were placed to measure strains in the longitudinal direction. Six more gages, three inside and three outside, were placed to measure the circumferential strains.

Test procedures.—The specimen was subjected to compressive load in the 1,200,000-pound universal testing machine of the Langley Structures Research Laboratory. Compressed air was used to produce internal pressure, which could be maintained at any desired constant value. The pressure was measured by a manometer. The strains were recorded by standard electric strain-gage equipment and the end-shortening was measured by dial gages.

The cylinder was preloaded and the strain-gage readings were taken. The three steel blocks were so adjusted that all longitudinal strain-gage readings around the circumference of the cylinder were equal.

¹ Supersedes NACA TN 2021, "Buckling of Thin-Walled Cylinder under Axial Compression and Internal Pressure" by Hsu Lo, Harold Crate, and Edward B. Schwartz, 1950.

The compressed air was then let into the cylinder until the desired internal pressure was reached. The axial compressive load was increased in increments until buckling was observed. At each load increment, all gage readings were recorded. The load was then decreased until the buckles disappeared and increased a second time to check the reading obtained the first time. During all these steps the internal pressure was maintained constant.

The axial load was then reduced and the internal pressure was changed to another value. For each value of internal pressure the same procedure was repeated.

EXPERIMENTAL RESULTS

A typical experimental result is shown in figures 2(a) and 2(b) for the case in which the internal pressure was $1\frac{1}{2}$ psi. In figure 2(a) the compressive load is plotted against the strain-gage readings for four different pairs of gages, within the range where the load-strain relation is linear. In figure 2(b) the strain-gage reading is plotted for all strain gages at three compressive loadings close to the buckling load. Figure 2(b) indicates that buckling occurs at a compressive load of 12,700 pounds between strain gages 22 and 23. (Note the intersection of the curves at two consecutive loadings.) A buckle at this location was observed during the test. The compressive load at which this phenomenon occurs is considered the buckling load.

Since the buckling occurs locally and not simultaneously at all the gages, the local buckling strain is obtained by dividing the buckling load by the slope of the linear portion of the load-strain curve corresponding to the gage at which the buckling occurs. The corresponding stress is the buckling stress. The buckling stresses for various internal pressures were determined in this same way.

The results are tabulated in table 1 and plotted in figure 3 in terms of the two nondimensional parameters

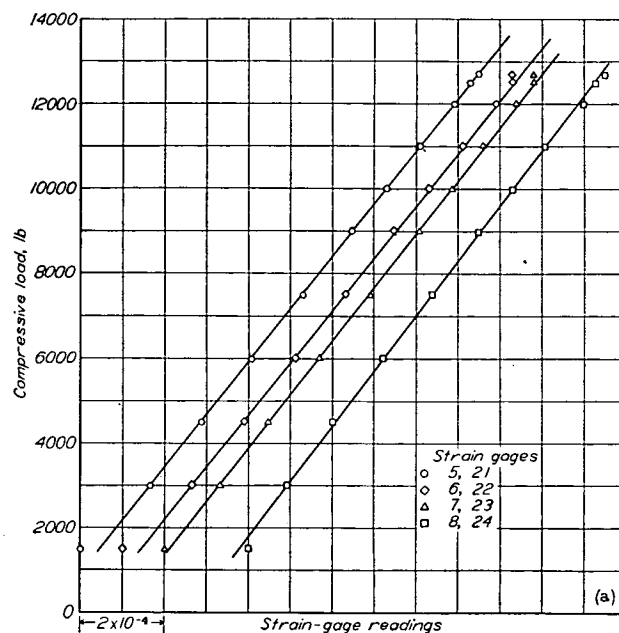
$$\bar{\sigma}_{cr} = \frac{\sigma_{u_{cr}}}{E} \frac{R}{t}$$

$$\bar{p} = \frac{p}{E} \left(\frac{R}{t} \right)^2$$

where $\sigma_{u_{cr}}$ is the buckling stress, p is the internal pressure, R is the radius of the cylinder, t is the wall thickness, and E is Young's modulus. Except for the first test corresponding to $\bar{p}=0.1028$ in which the cylinder had undergone no previous buckling, all the tests were carried out on the cylinder with possible permanent set.

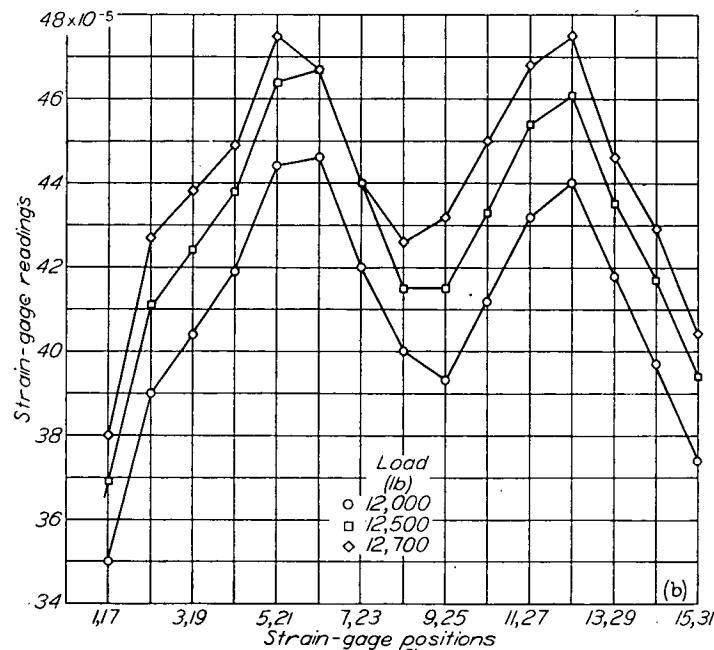
THEORETICAL RESULTS

A theoretical analysis for calculating the buckling stress of a cylindrical shell under axial compression and internal pressure was obtained by a "large-deflection" theory for



(a) Linear part of load-strain curve for four typical pairs of strain gages.

FIGURE 2.—Typical experimental result. Internal pressure, $1\frac{1}{2}$ psi.



(b) Close to buckling load.

FIGURE 2.—Concluded.

which details are given in the appendix. The large-deflection theory was first advanced by Von Kármán and Tsien (reference 3) in the study of buckling stress of cylindrical shells under axial compression (but without internal pressure). This theory was subsequently improved by Leggett and Jones (reference 4). In reference 3 the buckling stress was shown to depend on whether the load was applied by a

TABLE 1

BUCKLING STRESSES FOR VARIOUS INTERNAL PRESSURES

Experimental			Theoretical		
$\bar{p}$	$\bar{\sigma}$	$\Delta\bar{\sigma}_{cr}$	$\bar{p}$	$\bar{\sigma}$	$\Delta\bar{\sigma}_{cr}$
0	0.1936	0	0	0.376	0
.01715	.252	.058	.02	.444	.068
.03425	.277	.083	.04	.480	.104
.0514	.309	.116	.06	.506	.130
.0685	.350	.156	.08	.528	.152
.0856	.363	.170	.10	.547	.171
.1028	.407	.213	.12	.565	.189
			.14	.581	.205

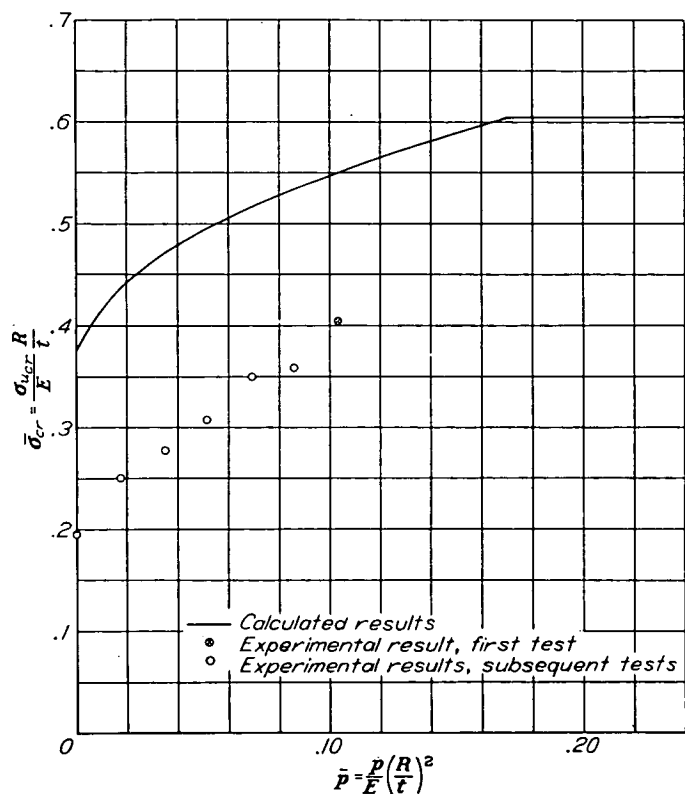


FIGURE 3.—Comparison of theoretical and experimental results of the buckling stress at various internal pressures.

rigid loading machine or by a dead-weight machine. In the present analysis, the loading machine is assumed to be rigid.

The existing procedures for computation of the buckling stress by large-deflection theory involve the solution of four simultaneous nonlinear equations for each pressure loading. The numerical work is quite lengthy. The method used in the present study introduces a fifth equation which governs the conditions at which the buckling occurs. The fifth equation is based on consideration of conservation of energy, which is an extension of Tsien's buckling criterion given in reference 5. Although a solution of five simultaneous equations is now necessary, the numerical work is actually

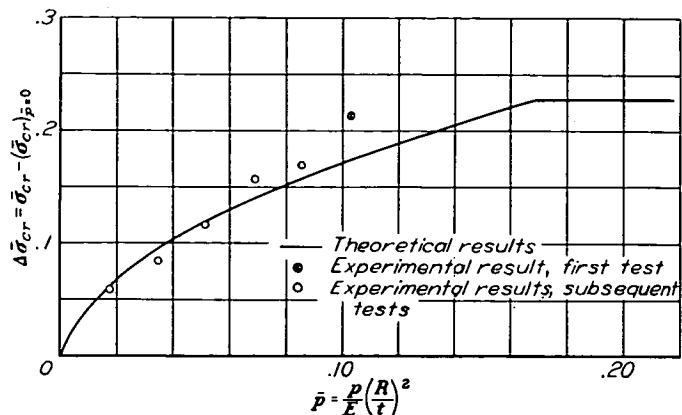


FIGURE 4.—Theoretical and experimental results showing the increment of buckling stress due to internal pressure.

reduced to a small fraction of that required if the existing procedures were used. This reduction in labor is made possible through a proper choice of the parameters in the equations and the process of the computations. The results calculated by the present method are presented in table 1 and are represented by the solid-line curve in figure 3. The curve is cut off at a value of $\bar{\sigma}_{cr}=0.605$, corresponding to $\bar{p}\approx 0.169$. This constant value of $\bar{\sigma}_{cr}=0.605$ for $\bar{p}>0.169$ is the same as that obtained by the classical theory.

DISCUSSION AND CONCLUSIONS

From the theoretical and experimental results shown in figure 3, the internal pressure is seen to have an appreciable strengthening effect on the cylinder. Although the two curves obtained from theoretical and experimental results do not coincide, both show the same trends as regards the effect of internal pressure on the buckling stresses. If the increment of the buckling stress $\Delta\bar{\sigma}_{cr}$ due to the presence of internal pressure (that is, the difference between the buckling stress with the pressure $\bar{\sigma}_{cr}$ and that without the pressure $(\bar{\sigma}_{cr})_{\bar{p}=0}$) is plotted against the internal pressure, as shown in figure 4, a good agreement is obtained between the theoretical and experimental results. These data indicate that, although further improvement of the theory is necessary for the determination of the magnitude of the compressive buckling stress, the theory gives a fairly good prediction of the increase of buckling compressive stress that may be expected as a result of internal pressure. The discrepancy between the theoretical curve and the experimental curve of figure 3 is believed to be caused by such factors as manufacturing imperfections in the specimens, material irregularities, and energy absorbed by the loading machine, which have not been included in the theory.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., October 12, 1949.

APPENDIX

THEORETICAL ANALYSIS OF BUCKLING LOAD OF CYLINDRICAL SHELLS UNDER AXIAL COMPRESSION AND INTERNAL PRESSURE BY LARGE-DEFLECTION THEORY

BACKGROUND OF THEORY

The use of large-deflection theory for shells under axial compression was first advanced by Von Kármán and Tsien (reference 3) in an attempt to explain the discrepancies between the buckling loads predicted by classical theory and those obtained from experimental results. (See, for instance, reference 6.) The results of reference 3 indicated that cylindrical shells can be maintained in equilibrium in the buckled state by a compressive load considerably lower than that predicted by classical theory. A plausible explanation of this result is that, before the classical buckling load is reached during a test, the cylindrical shell "jumps" from an equilibrium unbuckled state to an equilibrium buckled state. The physical phenomena of the jump were further examined in reference 5 by Tsien.

The treatment of Von Kármán and Tsien in reference 3 was left incomplete, however, in that the equilibrium positions at the buckled state were determined by differentiating the total potential energy with respect to some but not all of the physical parameters involved. The resulting equations gave a relation between the average compressive stress σ and the end-shortening ϵ in terms of the remaining parameters. A set of curves of σ against ϵ were thus obtained for various combinations of the remaining parameters.

Improvement of the theory of Von Kármán and Tsien was made by Leggett and Jones (reference 4), who took the derivatives of the energy with respect to all the parameters and thus obtained a single curve between σ and ϵ , representing all equilibrium positions of the cylindrical shell in the buckled state. The same result was obtained by Michielsen (reference 7) in a similar process. Such a curve is shown by BC of figure 5.

Theoretically, when the cylinder is compressed, the relation between σ and ϵ follows the straight line ODA which represents the unbuckled state and will reach the point A if everything is perfect; the cylinder then buckles and the relationship follows the curve ABC which represents the buckled state. Before point A is reached, however, some external disturbance may possibly cause the cylinder to jump from the unbuckled state represented by the point D to the buckled state represented by the point E. The positions of D and E on the respective curves depend on the actual physical conditions of the jump.

If the physical condition which governs the jump is known or defined, the buckling stress corresponding to the point D can be obtained directly without going through the labor of finding the curve ABC. This procedure can greatly reduce the amount of numerical work.

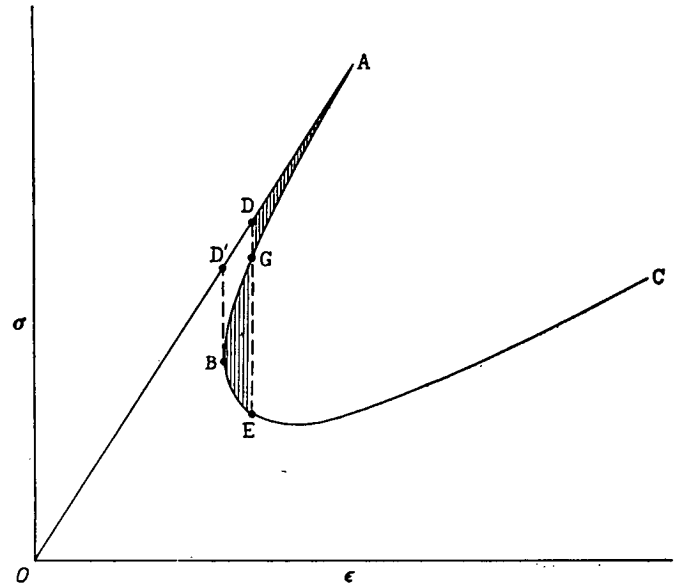


FIGURE 5.—Relation between the average compressive stress σ and the end-shortening ϵ .

In reference 5, Tsien introduced a criterion which governs the jump DE for the condition of loading obtained in a rigid testing machine; namely, that the strain energy remains the same before and after the jump and that the jump occurs at constant end-shortening. According to this criterion the line DE must be vertical and must cut the curve ABC in such a way that the two shaded areas ADG and GBE are equal. In fact, the area ADG represents the additional energy that is needed to assist the cylinder in jumping from a condition represented by D to that represented by G and the area GBE represents the energy that is given up by the cylinder when it arrives at the lower energy level, point E. The energy represented by the area ADG is very small, and therefore a slight disturbance from the surrounding air might assist the cylinder to jump from the unbuckled state to the buckled state at a compressive stress well below the classical buckling stress corresponding to point A.

Since the external disturbance is required to assist the cylinder to jump from the state corresponding to D to that corresponding to G, a slightly larger external disturbance can well cause the cylinder to make the transition from the state represented by D' to that represented by B, except that in the case in which the cylinder jumps from D' to B the cylinder absorbs the energy of the external disturbance and does not re-emit it. The buckling stress can be then as low as point D'. This fact was pointed out by Tsien in reference 8.

In addition to the two criterions just mentioned, there are still others that might be used. In view of the fact that the choice of the buckling criterion is a much less important factor in the determination of buckling stress than are such other factors as, for example, the initial imperfections, Tsien's criterion of reference 5, as represented by the line DE, is as reasonable as any other, and the choice of this criterion greatly simplifies the numerical work.

Tsien's criterion of reference 5 cannot be applied directly to the present analysis, however, because with the presence of the internal pressure the strain energy is no longer the same before and after the jump. In addition, the criterion is applied herein in quite a different manner from that of reference 5. In reference 5, a series of values of wave number n and aspect ratio β were chosen and the criterion was applied to each pair of values of n and β ; the pair of values of n and β which gave the minimum value of buckling load was considered to correspond to the buckling condition. In the present analysis, since the variation of σ with ϵ can be plotted only as a single curve, this criterion need be applied only once for each internal pressure. The results correspond to the minimum-potential-energy condition.

In the derivation of the present analysis, the basic equations in reference 3 are first extended to include the effect of internal pressure, Tsien's criterion governing the jump for rigid machine loading (reference 5) is modified, and the buckling stress is finally obtained.

SYMBOLS

A list of symbols follows. Most of the symbols used in the present report are the same as those in reference 3; exceptions are the use of μ for Poisson's ratio, λ for wave length, and β for aspect ratio of the buckled waves.

λ_a	half wave length in longitudinal direction
λ_b	half wave length in circumferential direction
f_0, f_1, f_2	parameters used in deflection function
m	number of waves in longitudinal direction within length equal to circumference of cylinder
n	number of waves in circumference
p	internal pressure
t	thickness of cylinder wall
x, y	coordinates measured in longitudinal and circumferential directions, respectively
u	component of displacement of a point on median surface of shell in x -direction
w	component of displacement of a point on median surface of shell in radial direction

α	measure of average circumferential stress per wave length in longitudinal direction
ϵ	end-shortening of cylinder
σ	average compressive stress
$\beta = \frac{m}{n}$	aspect ratio of buckled waves
μ	Poisson's ratio
ϕ	total potential energy
E	Young's modulus
ψ	strain energy
$B_1, B_2, \dots B_5$ $B_1', B_2', \dots B_5'$	certain functions of β
$D_1, D_2, \dots D_5$	certain functions of ρ and β
$(D)_\rho = \frac{\partial D}{\partial \rho}$	(D represents the functions $D_1, D_2, \dots D_5$)
$(D)_\beta = \beta \frac{\partial D}{\partial \beta}$	(D represents the functions $D_1, D_2, \dots D_5$)
R	radius of cylinder
W_1	elastic extensional energy
W_2	bending energy
W_3	work done by applied compressive load
W_p	work done by internal pressure

Nondimensional parameters:

$$\rho = \frac{f_2}{f_1}$$

$$\zeta = f_1 \frac{R}{t}$$

$$\eta = n^2 \frac{t}{R}$$

$$\bar{\sigma} = \frac{\sigma}{E} \frac{R}{t}$$

$$\bar{p} = \frac{p}{t} \left(\frac{R}{t} \right)^2$$

$$\bar{W} = \frac{W}{\frac{1}{2} E t \lambda_a \lambda_b}$$

($\bar{W}_1, \bar{W}_2, \bar{W}_3$, and $\bar{W}_p$ are defined in the same manner.)

$$\bar{\phi} = \frac{\phi}{\frac{1}{2} E t \lambda_a \lambda_b}$$

$$\bar{\psi} = \frac{\psi}{\frac{1}{2} E t \lambda_a \lambda_b}$$

$$\bar{\epsilon} = \epsilon \frac{R}{t}$$

$$\bar{\sigma}' = \left(\bar{\sigma} - \frac{1}{\beta^2} \bar{p} \right) \eta D_1$$

Subscripts:

o

pertaining to buckling condition

u

unbuckled state just prior to buckling

cr

buckling condition

DERIVATION OF BASIC EQUATIONS

Three basic equations are derived in the following analysis to include the effect of internal pressure. They are the expression for the total potential energy, the expression for the strain energy, and the relation between the end-shortening and the average compressive stress.

In order to calculate the total potential energy, the work done by the internal pressure should be included in addition to the energies W_1 , W_2 , and W_3 , which are given by equations (25), (26), and (27) of reference 3 as:

$$\frac{W_1}{\frac{1}{2}Et\lambda_a\lambda_b} = 4 \left[(1-\mu^2) \left(\frac{\sigma}{E} \right)^2 + n^4 \left(\frac{3}{64} f_1^2 + \frac{1}{16} f_1 f_2 + \frac{1}{16} f_2^2 \right)^2 + \left(f_0 + \frac{1}{4} f_1 \right)^2 - 2n^2 \left(\frac{3}{64} f_1^2 + \frac{1}{16} f_1 f_2 + \frac{1}{16} f_2^2 \right) \left(f_0 + \frac{1}{4} f_1 \right) \right] + \left[\frac{A^2}{8} + \frac{B^2\beta^4}{8} + \frac{\beta^4 C^2}{(1+\beta^2)^2} + \frac{\beta^4 D^2}{(1+9\beta^2)^2} + \frac{\beta^4 G^2}{(9+\beta^2)^2} + \frac{\beta^4 H^2}{16(1+\beta^2)^2} \right] \quad (1)$$

$$\frac{W_2}{\frac{1}{2}Et\lambda_a\lambda_b} = \frac{1}{6(1-\mu^2)} \left(\frac{t}{R} \right)^2 n^4 \left\{ f_1^2 \left[\frac{1}{8} (1+\beta^2)^2 + \frac{1}{4} (1+\beta^4) \right] + (1+\beta^4) f_1 f_2 + (1+\beta^4) f_2^2 \right\} \quad (2)$$

$$\frac{W_3}{\frac{1}{2}Et\lambda_a\lambda_b} = \left[2(1-\mu^2) \left(\frac{\sigma}{E} \right)^2 + n^2 \frac{\sigma}{E} (\mu + \beta^2) \left(\frac{3}{32} f_1^2 + \frac{1}{8} f_1 f_2 + \frac{1}{8} f_2^2 \right) - 2\mu \frac{\sigma}{E} \left(f_0 + \frac{1}{4} f_1 \right) \right] \quad (3)$$

where a , b , μ , and ν have been changed to λ_a , λ_b , β , and μ , respectively, to agree with the notation of the present report and where

$$A = \frac{1}{8} f_1^2 n^2 - \left(\frac{1}{2} f_1 + f_2 \right)$$

$$B = \frac{1}{8} f_1^2 n^2$$

$$C = \frac{1}{2} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right) - \frac{1}{2} f_1$$

$$D = \frac{1}{4} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right)$$

$$G = \frac{1}{4} f_1 n^2 \left(\frac{1}{2} f_1 + f_2 \right)$$

and

$$H = n^2 \left(\frac{1}{2} f_1 + f_2 \right)^2$$

The work done by the internal pressure is, for a complete wave panel,

$$W_p = -4 \int_0^{\lambda_a} \int_0^{\lambda_b} w p \, dx \, dy$$

The negative sign is introduced because the radial deflection w is considered positive inward.

If the same deflection function given in equation (16) of reference 3 is used, that is,

$$\frac{w}{R} = \left(f_0 + \frac{f_1}{4} \right) + \frac{f_1}{2} \left(\cos \frac{mx}{R} \cos \frac{ny}{R} + \frac{1}{4} \cos \frac{2mx}{R} + \frac{1}{4} \cos \frac{2ny}{R} \right) + \frac{f_2}{4} \left(\cos \frac{2mx}{R} + \cos \frac{2ny}{R} \right) \quad (4)$$

the work done by the internal pressure becomes

$$W_p = -4pR\lambda_a\lambda_b \left(f_0 + \frac{f_1}{4} \right). \quad (5)$$

If the total potential energy

$$\phi = W_1 + W_2 - W_3 - W_p$$

is differentiated with respect to f_0 and the derivative is set equal to zero, the following expression is obtained:

$$f_0 + \frac{f_1}{4} = \frac{1}{16} n^2 \left(\frac{3}{4} f_1^2 + f_1 f_2 + f_2^2 \right) - \mu \frac{\sigma}{E} - \frac{pR}{Et} \quad (6)$$

Substitute this expression into equations (1), (2), (3), and (5) for W_1 , W_2 , W_3 , and W_p , respectively, and the following equations are obtained:

$$\begin{aligned} \frac{W_1}{\frac{1}{2}Et\lambda_a\lambda_b} = & 4 \left[\left(\frac{\sigma}{E} \right)^2 + 2\mu \left(\frac{\sigma}{E} \right) \left(\frac{pR}{Et} \right) + \left(\frac{pR}{Et} \right)^2 \right] + \\ & n^4 \left(B_1 f_1^4 + B_2 f_1^3 f_2 + B_3 f_1^2 f_2^2 + B_4 f_1 f_2^3 + \frac{B_4}{2} f_2^4 \right) - \\ & n^2 \left[\left(2B_4 + \frac{1}{64} \right) f_1^3 + \left(4B_4 + \frac{1}{32} \right) f_1^2 f_2 \right] + \\ & \left[\left(2B_4 + \frac{1}{32} \right) f_1^2 + \frac{1}{8} f_1 f_2 + \frac{1}{8} f_2^2 \right] \end{aligned} \quad (7)$$

$$\frac{W_2}{\frac{1}{2}Et\lambda_a\lambda_b} = \left(\frac{t}{R} \right)^2 (B_5 f_1^2 + B_6 f_1 f_2 + B_6 f_2^2) n^4 \quad (8)$$

$$\begin{aligned} \frac{W_3}{\frac{1}{2}Et\lambda_a\lambda_b} = & 4 \left[2 \left(\frac{\sigma}{E} \right)^2 + 2\mu \left(\frac{\sigma}{E} \right) \left(\frac{pR}{Et} \right) + \right. \\ & \left. \frac{1}{8} \left(\frac{\sigma}{E} \right) n^2 \beta^2 \left(\frac{3}{4} f_1^2 + f_1 f_2 + f_2^2 \right) \right] \end{aligned} \quad (9)$$

$$\frac{W_p}{\frac{1}{2}Et\lambda_a\lambda_b} = 8 \left[\left(\frac{pR}{Et} \right)^2 + 8\mu \left(\frac{\sigma}{E} \right) \left(\frac{pR}{Et} \right) - n^2 \left(\frac{pR}{Et} \right) \frac{1}{16} \left(\frac{3}{4} f_1^2 + f_1 f_2 + f_2^2 \right) \right] \quad (10)$$

where

$$B_1 = \frac{1}{64} \left[\frac{1+\beta^4}{8} + \frac{17}{4} \frac{\beta^4}{(1+\beta^2)^2} + \frac{\beta^4}{(1+9\beta^2)^2} + \frac{\beta^4}{(9+\beta^2)^2} \right]$$

$$B_2 = \frac{1}{16} \left[\frac{9}{2} \frac{\beta^4}{(1+\beta^2)^2} + \frac{\beta^4}{(1+9\beta^2)^2} + \frac{\beta^4}{(9+\beta^2)^2} \right]$$

$$B_3 = \frac{1}{16} \left[\frac{11}{2} \frac{\beta^4}{(1+\beta^2)^2} + \frac{\beta^4}{(1+9\beta^2)^2} + \frac{\beta^4}{(9+\beta^2)^2} \right]$$

$$B_4 = \frac{1}{8} \frac{\beta^4}{(1+\beta^2)^2}$$

$$B_5 = \frac{1}{6(1-\mu^2)} \left[\frac{1}{8} (1+\beta^2)^2 + \frac{1}{4} (1+\beta^4) \right]$$

$$B_6 = \frac{1}{6(1-\mu^2)} (1+\beta^4)$$

Equations (7) to (10) may be expressed in terms of the nondimensional parameters $\bar{\sigma}$, $\bar{p}$, ρ , η , ζ , and $\bar{W}$ as

$$\bar{W}_1 = 4(\bar{\sigma}^2 + 2\mu\bar{\sigma}\bar{p} + \bar{p}^2) + \eta^2\zeta^4 D_2 + \eta\zeta^3(-D_3) + \zeta^2 D_4 \quad (11)$$

$$\bar{W}_2 = \eta^2\zeta^2 D_5 \quad (12)$$

$$\bar{W}_3 = 8\bar{\sigma}^2 + 8\mu\bar{\sigma}\bar{p} + \bar{\sigma}\eta\zeta^2 D_1 \quad (13)$$

$$\bar{W}_p = 8\mu\bar{\sigma}\bar{p} + 8\bar{p}^2 - \frac{1}{\beta^2} \bar{p}\eta\zeta^2 D_1 \quad (14)$$

where

$$D_1 = \frac{1}{2} \beta^2 \left(\frac{3}{4} + \rho + \rho^2 \right)$$

$$D_2 = B_1 + B_2\rho + B_3\rho^2 + B_4\rho^3 + \frac{1}{2} B_4\rho^4$$

$$D_3 = \left(2B_4 + \frac{1}{64} \right) (1+2\rho)$$

$$D_4 = \left(2B_4 + \frac{1}{32} \right) + \frac{1}{8} (\rho + \rho^2)$$

$$D_5 = B_5 + B_6(\rho + \rho^2)$$

The nondimensional total-potential-energy parameter $\bar{\phi}$ is

$$\begin{aligned} \bar{\phi} &= \bar{W}_1 + \bar{W}_2 - \bar{W}_3 - \bar{W}_p \\ &= -4(\bar{\sigma}^2 + 2\mu\bar{\sigma}\bar{p} + \bar{p}^2) - \left(\bar{\sigma} - \frac{1}{\beta^2} \bar{p} \right) \eta\zeta^2 D_1 + \\ &\quad \eta^2\zeta^4 D_2 - \eta\zeta^3 D_3 + \zeta^2 D_4 + \eta^2\zeta^2 D_5 \end{aligned} \quad (15)$$

The nondimensional strain-energy parameter $\bar{\psi}$ is

$$\begin{aligned} \bar{\psi} &= \bar{W}_1 + \bar{W}_2 \\ &= 4(\bar{\sigma}^2 + 2\mu\bar{\sigma}\bar{p} + \bar{p}^2) + \eta^2\zeta^4 D_2 - \\ &\quad \eta\zeta^3 D_3 + \zeta^2 D_4 + \eta^2\zeta^2 D_5 \end{aligned} \quad (16)$$

The relation between the end-shortening ϵ and the average compressive stress can be determined from equation (23) of reference 3 by integrating; thus,

$$\begin{aligned} \epsilon &= -\frac{1}{\lambda_a} \int_0^{\lambda_a} \frac{\partial u}{\partial x} dx \\ &= \left(\frac{\sigma}{E} + \mu \frac{\alpha}{E} \right) + \frac{1}{16} n^2 \beta^2 \left(\frac{3}{4} f_1^2 + f_2^2 + f_1 f_2 \right) \end{aligned}$$

where α/E , as determined from equation (24) of reference 3, together with equation (6) herein is

$$\frac{\alpha}{E} = \frac{pR}{Et}$$

Therefore, the relation between $\bar{\epsilon}$ and $\bar{\sigma}$ in nondimensional form becomes

$$\begin{aligned} \bar{\epsilon} &= \epsilon \frac{R}{t} \\ &= \bar{\sigma} + \mu\bar{p} + \frac{1}{8} \eta\zeta^2 D_1 \end{aligned} \quad (17)$$

Equations (15), (16), and (17) are for cylinders in the buckled state. For cylinders in the unbuckled state, the corresponding equations are

$$\bar{\phi}_u = -4(\bar{\sigma}_u^2 + 2\mu\bar{\sigma}_u\bar{p} + \bar{p}^2) \quad (18)$$

$$\bar{\psi}_u = 4(\bar{\sigma}_u^2 + 2\mu\bar{\sigma}_u\bar{p} + \bar{p}^2) \quad (19)$$

$$\bar{\epsilon}_u = \bar{\sigma}_u + \mu\bar{p} \quad (20)$$

EQUILIBRIUM POSITIONS OF CYLINDERS IN BUCKLED STATE

The equilibrium positions of cylinders in the buckled state can be obtained by differentiating the total potential energy of equation (15) with respect to each of the parameters η , ζ , ρ , and β and by setting the derivatives equal to zero. Four simultaneous, nonlinear equations are thus obtained:

$$\left. \begin{aligned} \frac{\partial \bar{\phi}}{\partial \eta} &= 0 = - \left[\left(\bar{\sigma} - \frac{1}{\beta^2} \bar{p} \right) \eta D_1 - 2(\eta\zeta)^2 D_2 + (\eta\zeta) D_3 - 2\eta^2 D_5 \right] \left(\frac{\zeta^2}{\eta} \right) \\ \frac{\partial \bar{\phi}}{\partial \zeta} &= 0 = - \left[\left(\bar{\sigma} - \frac{1}{\beta^2} \bar{p} \right) \eta D_1 - 2(\eta\zeta)^2 D_2 + 1.5(\eta\zeta) D_3 - \right. \\ &\quad \left. D_4 - \eta^2 D_5 \right] (2\zeta) \\ \frac{\partial \bar{\phi}}{\partial \rho} &= 0 = - \left[\left(\bar{\sigma} - \frac{1}{\beta^2} \bar{p} \right) \eta (D_1)_\rho - (\eta\zeta)^2 (D_2)_\rho + (\eta\zeta) (D_3)_\rho - \right. \\ &\quad \left. (D_4)_\rho - \eta^2 (D_5)_\rho \right] \zeta^2 \\ \frac{\partial \bar{\phi}}{\partial \beta} &= 0 = - \left[\left(\bar{\sigma} - \frac{1}{\beta^2} \bar{p} \right) \eta (D_1)_\beta - (\eta\zeta)^2 (D_2)_\beta + (\eta\zeta) (D_3)_\beta - \right. \\ &\quad \left. (D_4)_\beta - \eta^2 (D_5)_\beta + \frac{2}{\beta^2} \bar{p} \eta D_1 \right] \frac{\zeta^2}{\beta} \end{aligned} \right\} \quad (21)$$

where

$$(D_1)_\rho = \frac{1}{2} \beta^2 (1 + 2\rho)$$

$$(D_2)_\rho = B_2 + 2B_3\rho + 3B_4\rho^2 + 2B_4\rho^3$$

$$(D_3)_\rho = 2 \left(B_4 + \frac{1}{64} \right)$$

$$(D_4)_\rho = \frac{1}{8} (1 + 2\rho)$$

$$(D_5)_\rho = B_6 (1 + 2\rho)$$

$$(D_1)_\beta = \beta^2 \left(\frac{3}{4} + \rho + \rho^2 \right) = 2D_1$$

$$(D_2)_\beta = B_1' + B_2'\rho + B_3'\rho^2 + B_4'\rho^3 + \frac{1}{2} B_4'\rho^4$$

$$(D_3)_\beta = 2B_4' (1 + 2\rho)$$

$$(D_4)_\beta = 2B_4'$$

$$(D_5)_\beta = B_5' + B_6'(\rho + \rho^2)$$

and

$$B_1' = \frac{1}{16} \left[\frac{\beta^4}{8} + \frac{17}{4} \frac{\beta^4}{(1+\beta^2)^3} + \frac{\beta^4}{(1+9\beta^2)^3} + 9 \frac{\beta^4}{(9+\beta^2)^3} \right]$$

$$B_2' = \frac{1}{4} \left[\frac{9}{2} \frac{\beta^4}{(1+\beta^2)^3} + \frac{\beta^4}{(1+9\beta^2)^3} + 9 \frac{\beta^4}{(9+\beta^2)^3} \right]$$

$$B_3' = \frac{1}{4} \left[\frac{11}{2} \frac{\beta^4}{(1+\beta^2)^3} + \frac{\beta^4}{(1+9\beta^2)^3} + 9 \frac{\beta^4}{(9+\beta^2)^3} \right]$$

$$B_4' = \frac{1}{2} \frac{\beta^4}{(1+\beta^2)^3}$$

$$B_5' = \frac{1}{6(1-\mu^2)} \frac{1}{2} \beta^2 (1 + 3\beta^2)$$

$$B_6' = \frac{4}{6(1-\mu^2)} \beta^4$$

Let

$$\bar{\sigma}' = \left(\bar{\sigma} - \frac{1}{2} \bar{p} \right) \eta D_1$$

The four simultaneous equations (21) become

$$\bar{\sigma}' = (\eta \zeta)^2 (2D_2) - (\eta \zeta) D_3 + 2\eta^2 D_5 \quad (22a)$$

$$\bar{\sigma}' = (\eta \zeta)^2 (2D_2) - (\eta \zeta) (1.5D_3) + D_4 + \eta^2 D_5 \quad (22b)$$

$$\bar{\sigma}' = [(\eta \zeta)^2 (D_2)_\rho - (\eta \zeta) (D_3)_\rho + (D_4)_\rho + \eta^2 (D_5)_\rho] \frac{D_1}{(D_1)_\rho} \quad (22c)$$

$$\bar{\sigma}' = [(\eta \zeta)^2 (D_2)_\beta - (\eta \zeta) (D_3)_\beta + (D_4)_\beta + \eta^2 (D_5)_\beta] \frac{1}{2} - \frac{\bar{p}}{\beta^2} \eta D_1 \quad (22d)$$

Theoretically these four simultaneous equations can be solved for η , ζ , ρ , and β in terms of $\bar{\sigma}$ for a given pressure. If they are substituted into equation (17), a relation between end-shortening $\bar{\epsilon}$ and the compressive stress $\bar{\sigma}$ is obtained which represents all equilibrium positions at the buckled state. In fact, this solution is essentially that obtained by Leggett and Jones (reference 4) and Michielsen (reference 7) for cylinders with axial compression but no internal pressure.

Practically, however, the solution of the four simultaneous equations (22) requires a long and tedious numerical process. If only the buckling stress is required, calculation of only one point on the curve of $\bar{\sigma}$ against $\bar{\epsilon}$ rather than the whole curve is necessary. This solution can be obtained by the introduction of one more equation which governs the condition at buckling.

BUCKLING CRITERION

In reference 5, Tsien gave the following criterion which governs the condition at buckling: That the strain-energy of the buckled cylinder is the same as the strain energy of the unbuckled cylinder when the cylinder is tested in the rigid testing machine so that the end-shortening does not change during buckling. This criterion is apparently established from considerations of conservation of energy. Although other physical criteria can be used (for instance, see reference 8), the criterion of reference 5 was chosen and extended to include the case for which the internal pressure is present. The choice of this criterion simplifies the numerical work.

When internal pressure is present in the cylinder, work is done by the pressure during buckling. The strain energy in the buckled state is no longer equal to that in the unbuckled state, but

$$\bar{\psi} = \bar{\psi}_u + \Delta \bar{W}_p \quad (23)$$

where $\Delta \bar{W}_p$ is the work done by the pressure during buckling, or

$$\begin{aligned} \Delta \bar{W}_p &= \frac{1}{\frac{1}{2} E t \lambda_a \lambda_b} \left(\frac{R}{t} \right)^2 \left[- \int_0^{\lambda_a} \int_0^{\lambda_b} p(w - w_u) dx dy \right] \\ &= \bar{W}_p - (\bar{W}_p)_u \end{aligned} \quad (24)$$

Equation (14) can be rearranged as follows:

$$\bar{W}_p = 8(\bar{p}^2 + \mu \bar{\sigma} \bar{p}) - \frac{1}{\beta^2} \bar{p} \eta \zeta^2 D_1 \quad (25)$$

Therefore, for the unbuckled state, the last term is eliminated and

$$(\bar{W}_p)_u = 8(\bar{p}^2 + \mu \bar{\sigma}_u \bar{p}) \quad (26)$$

Then, from equations (24), (25), and (26)

$$\Delta \bar{W}_p = 8\mu \bar{p} (\bar{\sigma} - \bar{\sigma}_u) - \frac{1}{\beta^2} \bar{p} \eta \zeta^2 D_1 \quad (27)$$

The buckling criterion becomes (equations (23) and (27))

$$\bar{\psi} = \bar{\psi}_u + 8\mu\bar{p}(\bar{\sigma} - \bar{\sigma}_u) - \frac{1}{\beta^2} \bar{p}\eta\zeta^2 D_1$$

or, from equations (16) and (19),

$$4(\bar{\sigma}^2 + 2\mu\bar{\sigma}\bar{p} + \bar{p}^2) + \eta^2\zeta^4 D_2 - \eta\zeta^3 D_3 + \zeta^2 D_4 + \eta^2\zeta^2 D_5 \\ = 4(\bar{\sigma}_u^2 + 2\mu\bar{\sigma}_u\bar{p} + \bar{p}^2) + 8(\bar{\sigma} - \bar{\sigma}_u)\mu\bar{p} - \frac{1}{\beta^2} \bar{p}\eta\zeta^2 D_1 \quad (28)$$

Since the end-shortening remains unchanged during buckling, that is, $\bar{\epsilon} = \bar{\epsilon}_u$, the following relation is obtained from equations (17) and (20):

$$\bar{\sigma}_u = \bar{\sigma} + \frac{1}{8} \eta\zeta^2 D_1 \quad (29)$$

If this relation is substituted in equation (28) and if the relation $\bar{\sigma}' = \left(\bar{\sigma} - \frac{1}{\beta^2} p\right) \bar{\eta} D_1$ is used, the buckling criterion becomes

$$\bar{\sigma}' = (\eta\zeta)^2 \left(D_2 - \frac{1}{16} D_1^2\right) - (\eta\zeta) D_3 + D_4 + \eta^2 D_5 \quad (30)$$

The solution of the five equations (22a), (22b), (22c), (22d), and (30) gives the buckling stress for a given internal pressure. The following section presents a very simple method for the solution of these five simultaneous equations.

METHOD OF SOLUTION

From equations (22b) and (30) and equations (22a) and (22c), the following equations are obtained:

$$\eta\zeta = \frac{D_3}{2 \left(D_2 + \frac{1}{16} D_1^2\right)} \quad (31)$$

$$\eta^2 = \frac{D_4 - \frac{1}{2} D_3(\eta\zeta)}{D_5} \quad (32)$$

For a preassigned value of β , assume various values of ρ and compute $\eta\zeta$ and η^2 from equations (31) and (32). Substitute these values in equations (22a) and (22c) to obtain $(\bar{\sigma}')_a$ and $(\bar{\sigma}')_c$, respectively. Plot both $(\bar{\sigma}')_a$ and $(\bar{\sigma}')_c$ against ρ .

The intersection of these two curves determines a pair of values $\bar{\sigma}'$ and ρ which are called $\bar{\sigma}'_o$ and ρ_o . The corresponding values of $(\eta\zeta)_o$ and $(\eta^2)_o$ are computed and substituted in equation (22d) from which the pressure p can be calculated. For each assigned value of β , there are obtained corresponding values of $\bar{\sigma}'_o$ and $\bar{p}$. A curve of $\bar{\sigma}'_o$ against $\bar{p}$ can thus be determined. If the following relations are used

$$\bar{\sigma}'_o = (\eta D_1)_o \left(\bar{\sigma}_o - \frac{1}{\beta^2} \bar{p}\right)$$

$$\bar{\sigma}_{cr} = (\bar{\sigma}_u)_o = \bar{\sigma}_o + \frac{1}{8} (\eta\zeta^2 D_1)_o$$

the relation between $\bar{\sigma}_{cr}$ and $\bar{p}$ is obtained as shown in figure 3.

CUT-OFF BUCKLING STRESS

When equation (31) is derived from equations (22b) and (30), a factor $(\eta\zeta) = 0$ is also obtained. If this relation is used instead of equation (31), it can be shown that the buckling stress $\bar{\sigma}_{cr}$ can never exceed the classical buckling stress 0.605 which is independent of pressure.

REFERENCES

1. Flügge, W.: Die Stabilität der Kreiszylinderschale. Ing.-Archiv, Bd. III, Heft 5, Dec. 1932, pp. 463-506.
2. Rafel, Norman: Effect of Normal Pressure on the Critical Compressive Stress of Curved Sheet. NACA RB, Nov. 1942.
3. Von Kármán, Theodore, and Tsien, Hsue-Shen: The Buckling of Thin Cylindrical Shells under Axial Compression. Jour. Aero. Sci., vol. 8, no. 8, June 1941, pp. 303-312.
4. Leggett, D. M. A., and Jones, R. P. N.: The Behavior of a Cylindrical Shell under Axial Compression When the Buckling Load Has Been Exceeded. R. & M. No. 2190, British A.R.C., 1942.
5. Tsien, Hsue-Shen: A Theory for the Buckling of Thin Shells. Jour. Aero. Sci., vol. 9, no. 10, Aug. 1942, pp. 373-384.
6. Lundquist, Eugene E.: Strength Tests of Thin-Walled Duralumin Cylinders in Compression. NACA Rep. 473, 1933.
7. Michielsen, Herman F.: The Behavior of Thin Cylindrical Shells after Buckling under Axial Compression. Jour. Aero. Sci., vol. 15, no. 12, Dec. 1948, pp. 738-744.
8. Tsien, Hsue-Shen: Lower Buckling Load in the Non-Linear Buckling Theory for Thin Shells. Quarterly Appl. Math., vol. V, no. 2, July 1947, pp. 236-237.

REPORT 1028

EFFECT OF ASPECT RATIO ON THE AIR FORCES AND MOMENTS OF HARMONICALLY OSCILLATING THIN RECTANGULAR WINGS IN SUPERSONIC POTENTIAL FLOW¹

By CHARLES E. WATKINS

SUMMARY

This report treats the effect of aspect ratio on the air forces and moments of an oscillating flat rectangular wing in supersonic potential flow. The linearized velocity potential for the wing undergoing sinusoidal torsional oscillations simultaneously with sinusoidal vertical translations is derived in the form of a power series in terms of a frequency parameter. The series development is such that the differential equation for the velocity potential is satisfied to the required power of the frequency parameter considered and the linear boundary conditions are satisfied exactly. The method of solution can be utilized for other plan forms—that is, plan forms for which certain steady-state solutions are known.

Simple, closed expressions that include the reduced frequency to the third power, which is sufficient for application to a large class of practicable problems, are given for the velocity potential, the components of total force and moment coefficients, and the components of chordwise section force and moment coefficients. The components of total force and moment coefficients indicate the over-all effect of aspect ratio on these quantities; however, the components of chordwise coefficients yield more information because they account for the spanwise distribution of aerodynamic loading of a rectangular wing and may therefore be more useful for flutter calculations. It is found that the components of force and moment coefficients for a small-aspect-ratio wing may deviate considerably from those of an infinite-aspect-ratio wing. Thickness effects which may alter some of the conclusions are not taken into account in the analysis. Results of some selected calculations are presented in several figures and discussed.

INTRODUCTION

The effect of aspect ratio on the single-degree torsional instability of a finite rectangular wing oscillating in a supersonic stream was treated in reference 1 by expanding, in powers of the frequency of oscillation, the linearized velocity potential developed in reference 2. Since only slow oscillations were considered pertinent to single-degree torsional instability, terms in the expansion involving the frequency of oscillation to powers higher than the first were not considered.

In the present report the expanded linearized velocity potential is used to study the effect of aspect ratio on the air

forces and moments of an oscillating, thin, flat, finite, rectangular wing when higher powers of the frequency of oscillation are taken into account. The motions considered are sinusoidal torsional oscillations about a spanwise axis taken simultaneously with sinusoidal vertical translations of this axis. The velocity potential is developed by use of sources and doublets, so as to include all powers of the frequency of oscillations up to any desired power. Simple, closed expressions are given for the velocity potential, components of the total force and moment coefficients, and components of the chordwise section force and moment coefficients involving powers of the frequency up to and including the third power. Extension of the results to include higher powers of the frequency is straightforward.

A recent publication, reference 3, that became available after this investigation was completed, is partly devoted to the treatment of a rectangular wing undergoing the same types of harmonic motions as those considered herein. The velocity potential is determined in the form of a double integral, by application of the Fourier transform to the boundary-value problem for this potential, and expressions for forces and moments are given in terms of this double integral. The reduction of the integral expressions of reference 3 to forms desirable for flutter calculations—that is, chordwise section forces and moments—is not given.

SYMBOLS

ϕ	disturbance-velocity potential
x, y, z	rectangular coordinates attached to wing moving in negative x -direction
ξ, η	rectangular coordinates used to represent space location of sources or doublets in xy -plane
Z_m	function defining mean ordinates of any chordwise section of wing such as $y=y_1$ as shown in figure 1
$w(x, y_1, t)$	vertical velocity at surface of wing along chordwise section at $y=y_1$
x_0	abscissa of axis of rotation of wing (elastic axis) as shown in figure 1
t	time
h	vertical displacement of axis of rotation
h_0	amplitude of vertical displacement of axis of rotation, positive downward

¹ Supersedes NACA TN 2064, "Effect of Aspect Ratio on the Air Forces and Moments of Harmonically Oscillating Thin Rectangular Wings in Supersonic Potential Flow" by Charles E. Watkins, 1950.

α	angle of attack
α_0	amplitude of angular displacement about axis of rotation, positive leading edge up
$\dot{h}, \dot{\alpha}$	time derivative of h and α , respectively
V	velocity of main stream
c	velocity of sound
M	free-stream Mach number (V/c)
$\beta = \sqrt{M^2 - 1}$	
$\tau_1, \tau_2, \eta_1, \eta_2$	functions defined with equation (7)
$W(\xi, \eta)$	function used to represent space variation of source and doublet strengths
$w(t)$	function used to represent time variation of source and doublet strengths
ω	frequency of oscillation
$\bar{\omega} = \frac{M^2 \omega}{V \beta^2}$	
k	reduced frequency ($\omega b/V$)
$R = \beta \sqrt{(\eta - \eta_1)(\eta_2 - \eta)}$	
a_{nm}	represents functions of $\bar{\omega}$, x , and M , defined in equations (15)
f_1, f_2, f_3	represent functions of x , x_0 , and $\bar{\omega}$, defined in equations (19)
D_n	function used to denote doublet distributions (see equation (22))
F_n	function defined in equation (28)
G_n	function defined in equation (29)
ρ	density
Δp	local pressure difference measured positive downward, defined in equation (31)
b	half-chord
s	half-span
A	aspect ratio (s/b)
$\bar{P}$	total force acting on wing defined in equation (32)
$\bar{L}_1, \bar{L}_2, \bar{L}_3, \bar{L}_4$	components of total force coefficients, defined in equations (35)
$\bar{M}_\alpha$	total moment acting on wing, defined in equation (36)
$\bar{M}_1, \bar{M}_2, \bar{M}_3, \bar{M}_4$	components of total moment coefficients, defined in equations (38)
P	section force (total force at any spanwise station), defined in equation (39)
L_1, L_2, L_3, L_4	components of section force coefficients, defined in equations (41) and (42)
M_α	section moment (total moment at any spanwise station), defined in equation (40)
M_1, M_2, M_3, M_4	components of section moment coefficients, defined in equations (43) and (44)
$\bar{F}_n, \bar{G}_n$	functions related to F_n and G_n , defined in appendix

ANALYSIS

BOUNDARY-VALUE PROBLEMS FOR VELOCITY POTENTIALS

Consider a thin flat rectangular wing moving at a constant supersonic speed in a chordwise direction normal to its leading edge as shown in figure 1. The boundary-value problems for the velocity potential for such a wing may be conveniently classified into two types associated with the nature of the flow over different portions of the wing. On the portion of the wing between the Mach cones emanating from the foremost point of each tip (region N in fig. 1 (a)) there is no interaction between the flow on the upper and lower surfaces of the wing. The type of boundary-value problem

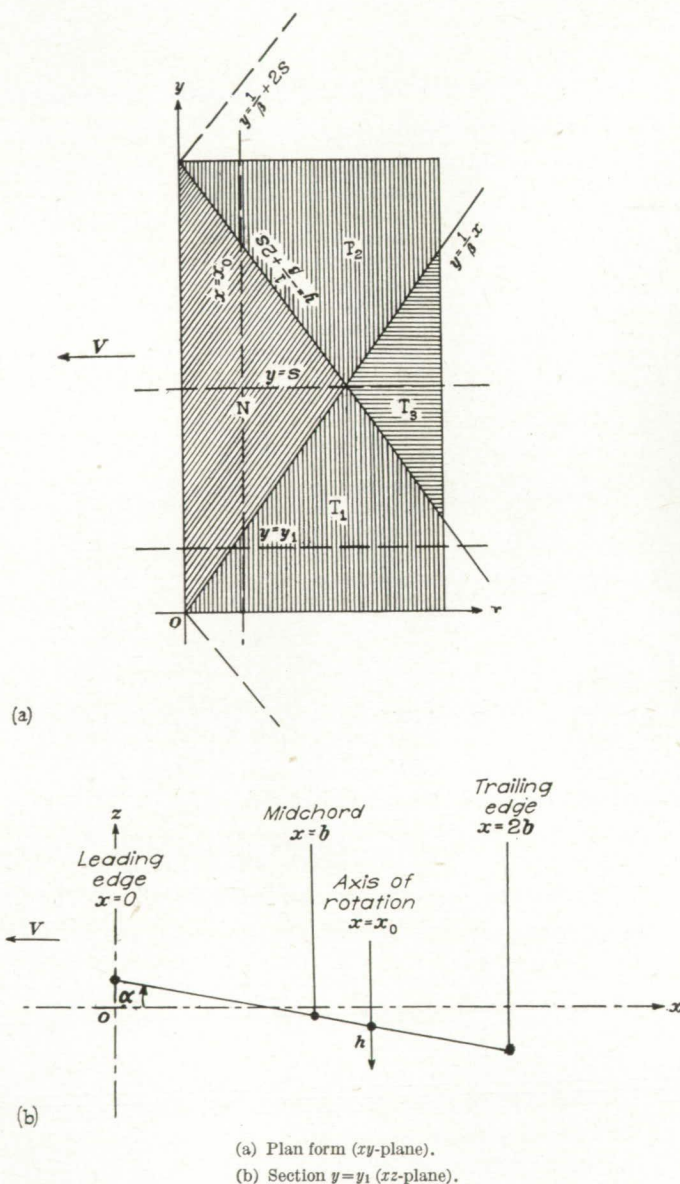


FIGURE 1.—Sketch illustrating chosen coordinate system and the two degrees of freedom α and h .

for this portion of the wing is referred to herein as "purely supersonic" and the velocity potential for region N is denoted by ϕ_N . On portions of the wing within the Mach cones emanating from the foremost point of each tip (regions T₁, T₂, and T₃ in fig. 1 (a)), there is interaction between the flow on the upper and lower surfaces of the wing. The type of boundary-value problem for these portions of the wing is referred to as "mixed supersonic" and the velocity potentials for these regions are designated by ϕ_{T_1} , ϕ_{T_2} , and ϕ_{T_3} , respectively. The complete velocity potential at a point may then be expressed as ϕ_N , ϕ_{T_1} , ϕ_{T_2} , or ϕ_{T_3} according to the region that contains the point.

As customary in linear theory, as applied to thin flat surfaces, the boundary conditions are to be ultimately satisfied by the velocity potentials at the projection of the wing onto a plane (the xy -plane) with respect to which all deflections are considered small and which lies parallel to the free-stream direction. Thickness effects are not taken into account; hence, the velocity potentials are associated only with conditions that yield lift and are consequently anti-symmetrical with respect to the plane of the projected wing. It is therefore necessary to consider the potentials at only one surface, upper or lower, of the projected wing. The upper surface is chosen for this analysis.

The differential equation for the propagation of small disturbances that must be satisfied by the velocity potentials is (when referred to a rectangular coordinate system x, y, z with the xy -plane coincident with the reference plane and moving uniformly in the negative x -direction, fig. 1)

$$\frac{1}{c^2} \left(\frac{\partial}{\partial t} + V \frac{\partial}{\partial x} \right)^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \quad (1)$$

The boundary conditions that must be satisfied by the velocity potential are: (a) In regions T₁, T₂, T₃, and N the flow must be tangent to the surface of the wing or

$$\left(\frac{\partial \phi}{\partial z} \right)_{z \rightarrow 0} = w(x, y, t) = V \frac{\partial Z_m}{\partial x} + \frac{\partial Z_m}{\partial t} \quad (2)$$

where Z_m is the vertical displacement of the ordinates of the surface of any chordwise section of the wing (see fig. 1 (b)). (b) In regions T₁ and T₂ the pressure must fall to zero along the wing tips and remain zero in the portion of the Mach cones emanating from the foremost points of the wing tips not occupied by the wing. (Another condition, that the potential must be zero ahead of the wing and in the region off the wing adjacent to the Mach cones emanating from the foremost points of the tips, is automatically satisfied by the type of source and doublet synthesis employed in the solutions.)

For the particular case of a wing independently performing small sinusoidal torsional oscillations of amplitude $|\alpha_0|$ and frequency ω about some spanwise axis x_0 and small sinusoidal vertical translations of amplitude $|h_0|$ and frequency ω , the equation of Z_m is

$$Z_m = e^{i\omega t} [\alpha_0(x - x_0) + h_0] = \alpha(x - x_0) + h \quad (3)$$

Substituting this expression for Z_m into equation (2) gives

$$w(x, y, t) = V\alpha + \dot{\alpha}(x - x_0) + \dot{h} \quad (4)$$

The velocity potential may thus be expressed as the sum of separate effects due to position and motion of the wing associated with the individual terms in equation (4) as

$$\phi = \phi_a + \phi_\alpha + \phi_h \quad (5)$$

DERIVATION OF ϕ_N

The boundary-value problem in the purely supersonic region (fig. 2 (a)) is the same as that for the two-dimensional wing treated in reference 4. This problem is there shown to be satisfied by a distribution of sources referred to, in this case, as moving sources because of the uniform motion; that is,

$$\phi_N(x, y, z, t) = -\frac{1}{2\beta\pi} \int_0^{x-\beta z} \int_{\eta_1}^{\eta_2} W(\xi, \eta) \phi_1 d\eta d\xi \quad (6)$$

In equation (6), $W(\xi, \eta)$ represents the space variation of source strength and must be evaluated in accordance with the individual terms of equation (4), and ϕ_1 is the potential of a moving source situated at the point $(\xi, \eta, 0)$ that may be expressed as

$$\phi_1 = \frac{w(t - \tau_1) + w(t - \tau_2)}{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}} \quad (7)$$

where $w(t)$ is the time variation of source strength and the symbols with subscripts appearing in equation (7) are defined as

$$\tau_1 = \frac{M(x - \xi)}{c\beta^2} - \frac{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\beta c}$$

$$\tau_2 = \frac{M(x - \xi)}{c\beta^2} + \frac{\sqrt{(\eta - \eta_1)(\eta_2 - \eta)}}{\beta c}$$

$$\eta_1 = y - \frac{1}{\beta} \sqrt{(x - \xi)^2 - \beta^2 z^2}$$

$$\eta_2 = y + \frac{1}{\beta} \sqrt{(x - \xi)^2 - \beta^2 z^2}$$

The time variation of source strength $w(t)$ for harmonic oscillations may be written as

$$w(t) = e^{i\omega t} \quad (8)$$

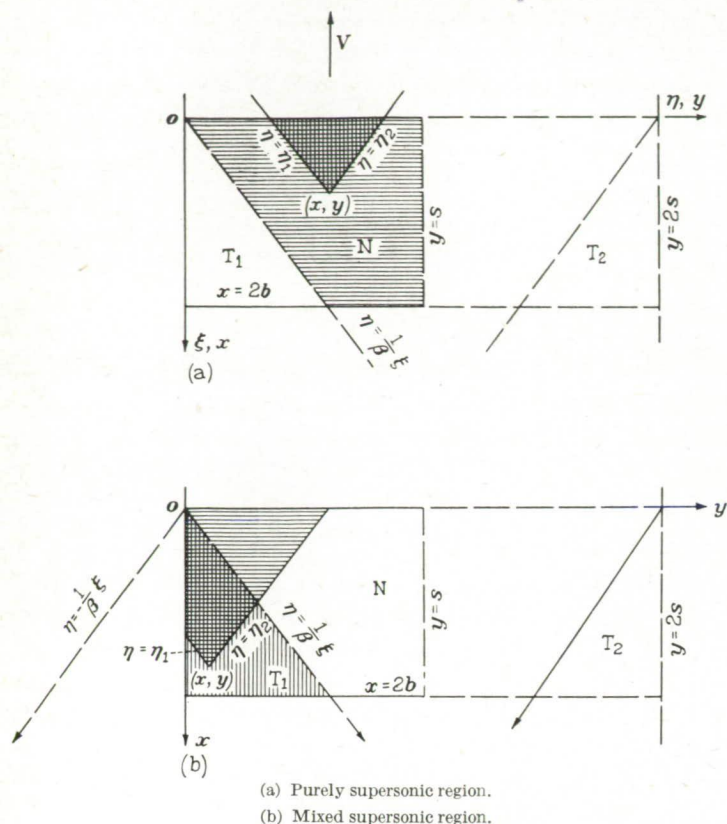


FIGURE 2.—Sketch illustrating areas of integration for purely supersonic and mixed supersonic regions of flow.

The numerator in equation (7) thus becomes

$$\begin{aligned} w(t-\tau_1) + w(t-\tau_2) &= e^{i\omega(t-\tau_1)} + e^{i\omega(t-\tau_2)} \\ &= 2e^{i\omega t} e^{-i\omega \frac{\tau_2+\tau_1}{2}} \cos \omega \frac{\tau_2-\tau_1}{2} \end{aligned} \quad (9)$$

Substituting equations (7) and (9) into equation (6) yields

$$\phi_N(x, y, z, t) = -\frac{e^{i\omega t}}{\pi} \int_0^{x-\beta z} \int_{\eta_1}^{\eta_2} \frac{W(\xi, \eta) e^{-i\bar{\omega}(x-\xi)} \cos\left(\frac{\bar{\omega}}{M} R\right) d\eta d\xi}{R} \quad (10)$$

where, for brevity,

$$\bar{\omega} = \frac{\omega M}{c\beta^2} = \frac{M^2 \omega}{V\beta^2}$$

and

$$R = \sqrt{(x-\xi)^2 - \beta^2(y-\eta)^2 - \beta^2 z^2} = \beta \sqrt{(\eta-\eta_1)(\eta_2-\eta)}$$

The values of $W(\xi, \eta)$ associated with the different terms of equation (4) are

For h

$$W(\xi, \eta) = \frac{iV\beta^2 \bar{\omega}}{M^2} h_0 \quad (11)$$

For $V\alpha$

$$W(\xi, \eta) = V\alpha_0 \quad (12)$$

For $\dot{\alpha}(x-x_0)$

$$W(\xi, \eta) = \frac{iV\beta^2 \bar{\omega}}{M^2} \alpha_0 (\xi - x_0) \quad (13)$$

If any of the values of $W(\xi, \eta)$ given in equations (11), (12), and (13) is put into equation (10), the integration with

respect to η can be readily performed and the remaining integral evaluated as a series of Bessel functions. (See, for example, reference 4.) However, in order to be consistent with and to lead naturally to a succeeding part of the analysis the integrand is expanded into a Maclaurin's series with respect to $\bar{\omega}$. The expansion yields

$$\begin{aligned} \phi_N(x, y, z, t) &= -\frac{e^{i\omega t}}{\pi} \int_0^{x-\beta z} \int_{\eta_1}^{\eta_2} W(\xi, \eta) \left[\left(a_{01} \frac{1}{R} + a_{11} \frac{\xi}{R} + \dots \right. \right. \\ &\quad \left. \left. + a_{n1} \frac{\xi^n}{R} + \dots \right) + (a_{02} R + a_{12} \xi R + \dots \right. \\ &\quad \left. + a_{n2} \xi^n R + \dots \right) + \dots + (a_{0m} R^{2m-3} + a_{1m} \xi R^{2m-3} \\ &\quad \left. + \dots + a_{nm} \xi^n R^{2m-3} + \dots \right) + \dots \Big] d\eta d\xi \end{aligned} \quad (14)$$

where the coefficients a_{nm} are functions of $\bar{\omega}$, x , and M ; those coefficients involving $\bar{\omega}$, up to and including the third power, are

$$\left. \begin{aligned} a_{01} &= 1 - i\bar{\omega}x - \frac{\bar{\omega}^2}{2}x^2 + \frac{i\bar{\omega}^3}{6}x^3 \\ a_{11} &= i\bar{\omega} + \bar{\omega}^2x - \frac{i\bar{\omega}^3}{2}x^2 \\ a_{21} &= -\frac{\bar{\omega}^2}{2} + \frac{i\bar{\omega}^3}{2}x \\ a_{31} &= -\frac{i\bar{\omega}^3}{6} \\ a_{02} &= -\frac{\bar{\omega}^2}{2M^2} + \frac{i\bar{\omega}^3}{2M^2}x \\ a_{12} &= -\frac{i\bar{\omega}^3}{2M^2} \end{aligned} \right\} \quad (15)$$

Observe the following identity that is valid regardless of the highest power of $\bar{\omega}$ considered and that will be of subsequent use, namely

$$a_{01} + xa_{11} + \dots + x^n a_{n1} \equiv 1 \quad (16)$$

It will be noted in equation (14) that the potential of a moving source when expanded in terms of the frequency appears as a series of terms similar to steady-state source potentials plus series of terms involving various powers of R . By grouping the terms in equation (14) with respect to powers of ξ , the following form of the source potential convenient for later use is obtained:

$$\begin{aligned} \phi_N(x, y, z, t) &= -\frac{e^{i\omega t}}{\pi} \int_0^{x-\beta z} \int_{\eta_1}^{\eta_2} W(\xi, \eta) \left[\left(a_{01} \frac{1}{R} + a_{02} R + \dots \right. \right. \\ &\quad \left. \left. + a_{0m} R^{2m-3} + \dots \right) + \xi \left(a_{11} \frac{1}{R} + a_{12} R + \dots \right. \right. \\ &\quad \left. \left. + a_{1m} R^{2m-3} + \dots \right) + \dots + \xi^n \left(a_{n1} \frac{1}{R} + a_{n2} R \right. \right. \\ &\quad \left. \left. + \dots + a_{nm} R^{2m-3} + \dots \right) + \dots \right] d\eta d\xi \end{aligned} \quad (17)$$

With the terms of the series grouped in this manner, in view of the fact that the differential equation (1) is independent

of ξ , it is apparent that the coefficient of each power of ξ in equation (17) is a solution to the differential equation.

If the values of $W(\xi, \eta)$ in equations (11), (12), and (13) are put into either equation (14) or equation (17), the integrations of each term can be easily carried out in closed form. Moreover it can readily be shown that, when all the terms involving $\bar{\omega}$ up to a given power are taken into account, the differential equation (1) is satisfied to the highest power of $\bar{\omega}$ considered. The boundary condition of tangential flow as expressed in equation (4) is satisfied exactly and does not depend on the order of $\bar{\omega}$ considered.

Putting the values of $W(\xi, \eta)$ in equations (11), (12), and (13) successively into either equation (14) or equation (17), carrying out the indicated integration, and setting $z=0$ yields for the velocity potential, to the third power of $\bar{\omega}$ at the upper surface of the wing, in the purely supersonic region:

$$\phi_N = -\frac{1}{\beta}(\bar{h}f_1 + V\alpha f_2 + \dot{\alpha}f_3) \quad (18)$$

where

$$\left. \begin{aligned} f_1 &= x - \frac{i\bar{\omega}}{2}x^2 - \frac{\bar{\omega}^2}{12M^2}(2\beta^2 + 3)x^3 \\ f_2 &= x - \frac{i\bar{\omega}}{2}x^2 - \frac{\bar{\omega}^2}{12M^2}(2\beta^2 + 3)x^3 + \frac{i\bar{\omega}^3}{48M^2}(2\beta^2 + 5)x^4 \\ f_3 &= \frac{x}{2}(x - 2x_0) - \frac{i\bar{\omega}x^2}{6}(x - 3x_0) - \frac{\bar{\omega}^2}{48M^2}(2\beta^2 + 3)x^3(x - 4x_0) \end{aligned} \right\} \quad (19)$$

DERIVATION OF ϕ_{T_1} , ϕ_{T_2} , AND ϕ_{T_3} [†]

In order to satisfy the boundary-value problem in regions of mixed supersonic flow it is convenient to start with the potential of a moving doublet. Then, for a given order of the frequency of oscillation, this potential, as will be shown in the following analysis, can be modified so that when integrated over the appropriate region the results will satisfy, as in the purely supersonic case, the differential equation to the given order of the frequency and will satisfy the condition of tangential flow exactly. The potential of the type of doublet required may be obtained by partial differentiation of the potential of a moving source (see integrand of equation (17)) with respect to the direction normal to the plane of the wing, namely

$$\begin{aligned} \phi_2 &= \frac{e^{i\omega t}}{\pi} \frac{\partial}{\partial z} \left[\left(a_{01} \frac{1}{R} + a_{02}R + \dots + a_{0m}R^{2m-3} + \dots \right) + \right. \\ &\quad \xi \left(a_{11} \frac{1}{R} + a_{12}R + \dots + a_{1m}R^{2m-3} + \dots \right) + \\ &\quad \left. \dots + \xi^n \left(a_{n1} \frac{1}{R} + a_{n2}R + \dots + a_{nm}R^{2m-3} + \dots \right) + \dots \right] \quad (20) \end{aligned}$$

Examination of equation (20), like equation (17), shows that the coefficient of each power of ξ satisfies the differential equation and, since the differential equation is linear, it is permissible in synthesizing the solution to the boundary-

value problem to weight these coefficients separately. Furthermore the coefficient of each power of ξ consists of a term that has the form of a steady-state doublet potential, namely

$$a_{n1} \frac{\partial}{\partial z} \left(\frac{1}{R} \right) \quad (21)$$

plus a series of other terms involving various powers of R . In the following analysis, attention is first directed to the treatment of the first term of the coefficient ξ^n , (expression (21)). The other terms will be treated subsequently.

Expression (21) has the form that in steady flow is convenient for treating the (antisymmetric) problem of satisfying the condition of tangential flow for a distribution of normal velocity prescribed, at the wing surface, independent of y but proportional to x^n ; that is, a weight or distribution function $D_n(\xi, \eta)$ can be determined so that

$$\lim_{z \rightarrow 0} \frac{\partial^2}{\partial z^2} \int_r \int D_n(\xi, \eta) \frac{\xi^n}{R} d\xi d\eta = \pi x^n \quad n=(0, 1, 2, \dots) \quad (22)$$

where the region of integration r is the portion of the wing situated in the fore cone emanating from the field point (x, y, z) (shown in fig. 2(b) for the rectangular wing with $z=0$).

The distribution function D_n for rectangular wings may be easily determined when D_0 , the distribution function for this wing at constant angle of attack in steady supersonic flow, is known. The expression for D_0 is derived in reference 1 and found to be

$$D_0 = \frac{2}{\beta} \left[\sqrt{\beta\eta(\xi - \beta\eta)} + \xi \sin^{-1} \sqrt{\frac{\beta\eta}{\xi}} \right] \quad (23)$$

From this expression and equation (22) it follows by direct substitution and reduction that

$$D_n = n! \int_0^\xi \int_0^\xi \dots \int_0^\xi D_0(\xi, \eta) (d\xi)^n = n \int_0^\xi (\xi - \lambda)^{n-1} D_0(\lambda, \eta) d\lambda \quad (24)$$

With D_n known so that equation (22) is satisfied it may be demonstrated, with use of identity (16), that these "doublet-type" terms alone satisfy the condition of tangential flow. For example, let it be required to satisfy this condition for vertical translations; then,

$$\begin{aligned} w(x, t) &= \frac{h}{\pi} \frac{\partial^2}{\partial z^2} \int_r \int \sum_{m=0}^n \left[a_{m1} D_m(\xi, \eta) \frac{\xi^m}{R} \right] d\xi d\eta \\ &= h(a_{01} + x a_{11} + \dots + x^n a_{n1}) = h \end{aligned}$$

The next step in the analysis is to consider terms of the type

$$\frac{\partial}{\partial z} (a_{nm} R^{2m-3}) \quad m > 1 \quad (25)$$

appearing in the coefficients of ξ^n (equation (20)). It is to be noted that, when the power of $\bar{\omega}$ to which the potential is to be derived is chosen, the number of terms in equation (20) that are to be treated is determined by the expressions a_{mn} that contain $\bar{\omega}$ to the chosen order (see equations (15) for $\bar{\omega}$ to the third power).

[†]Although the derivation of these potentials in NACA TN 2064 led to correct results, the procedure followed therein is based on erroneous arguments. The present procedure is correct and general.

If the distribution function D_n (equation (24)) is introduced into equation (20) and the resulting equation is integrated over the appropriate region of the rectangle (fig. 2(b)), it is found that in the limit as $z \rightarrow 0$ terms of the type given in expression (25) do not contribute to the resulting potential, but they do contribute to the resulting vertical velocity. Therefore, since it has been shown that the doublet-type terms, taken alone, satisfy the condition of tangential flow, the distribution of vertical velocity would now contain extraneous terms that need to be canceled. This canceling may be achieved by what is essentially an iterative process: the functions D_n (equation (24)) are used with terms in equation (20) that have the form given in expression (25) to calculate the velocity that is to be canceled; then, terms involving $1/R$ that may be individually weighted and which, of course, must satisfy the differential equation to the order of $\bar{\omega}$ to which the velocity potential is to be derived are added to equation (20); finally, by making use of equation (22), weight or distribution functions for these additional terms may be determined so as to cancel the extraneous velocity.

This process is illustrated for ω to the third power as follows:

$$\begin{aligned} \phi_{r_1} = & -\lim_{z \rightarrow 0} \frac{e^{i\omega t}}{\pi} \frac{\partial}{\partial z} \int_r \int W(\xi, \eta) \left[D_0 \left(a_{01} \frac{1}{R} + a_{02} R \right) + \right. \\ & D_1 \xi \left(a_{11} \frac{1}{R} + a_{12} R \right) + D_2 a_{21} \frac{\xi^2}{R} + D_3 a_{31} \frac{\xi^3}{R} + \\ & \left. a_{02} (\xi D_1 - D_2) \frac{\xi^2}{R} + a_{12} \left(\frac{\xi D_2 - D_3}{2} \right) \frac{\xi^2}{R} \right] d\xi d\eta \quad (26) \end{aligned}$$

where it is noted in the integrand that two extra terms have been added to equation (20) to accomplish the desired canceling. Substituting the values of $W(\xi, \eta)$ defined in equations (11), (12), and (13) into equation (26) gives the expression for the velocity potential at the upper surface of the wing

$$\begin{aligned} \phi_{r_1} = & -\frac{1}{\beta\pi} \left(h \left[2F_1 - \left(2i\bar{\omega} + \frac{\bar{\omega}^2 x}{M^2} \right) F_2 - \frac{\beta^2 \bar{\omega}^2}{M^2} F_3 \right] + \right. \\ & V\alpha \left[2F_1 - \left(2i\bar{\omega} + \frac{\bar{\omega}^2}{M^2} x - \frac{i\bar{\omega}^3}{2M^2} x^2 \right) F_2 - \right. \\ & \left. \frac{\beta^2 \bar{\omega}^2}{M^2} F_3 + \frac{i\bar{\omega}^3}{6M^2} (2\beta^2 - 1) F_4 \right] + \alpha \left\{ 2(x - x_0) F_1 - \right. \\ & \left[2 + 2i\bar{\omega}(x - x_0) + \frac{\bar{\omega}^2}{2M^2} (x^2 - 2xx_0) \right] F_2 + \\ & \left. \left[2i\bar{\omega} - \frac{\beta^2 \bar{\omega}^2}{M^2} (x - x_0) \right] F_3 + \frac{\bar{\omega}^2}{2M^2} (2\beta^2 + 1) F_4 \right\} \quad (27) \end{aligned}$$

where the terms are grouped conveniently by the definition of F_n in the following integral:

$$F_n = \int_0^x x^{n-1} \sin^{-1} \sqrt{\frac{\beta y}{x}} dx \quad (n=1, 2, 3, 4) \quad (28)$$

(The functions F_n , given in equation (28), and certain related functions are of particular importance in the remainder of this development. Integrated values of this function for the first few values of n and expressions for related functions needed later in this analysis are given in the appendix.)

Examination of equation (27) shows that along the Mach line $x = \beta y$, separating region T_1 from region N , the expression ϕ_{r_1} reduces to the expression for ϕ_N given in equation (18).

The corresponding potentials for regions T_2 and T_3 can now be obtained. The potential ϕ_{r_2} is obtained from equation (27) by merely substituting $2s - y$ for y in equation (28) so that

$$G_n = \int_0^x x^{n-1} \sin^{-1} \sqrt{\frac{\beta(2s-y)}{x}} dx \quad (n=1, 2, 3, 4) \quad (29)$$

The potential in region T_3 (that is for $1 \leq A\beta < 2$) is a simple superposition of the potentials for regions N , T_1 , and T_2 , as in the steady case (see, for example, reference 5), and may be written as

$$\phi_{r_3} = \phi_{r_1} + \phi_{r_2} - \phi_N \quad (30)$$

FORCES AND MOMENTS

Two types of force and moment coefficients are derived. First, in order to gain some insight into the over-all effect of aspect ratio on the forces and moments, expressions for total force and moment coefficients are derived. Then, in order to present expressions that are more suitable for use in flutter calculations, expressions for section force and moment coefficients for any station along the span are derived.

Total forces and moments.—The local pressure difference between the upper and lower surfaces on the wing may be written

$$\Delta p = -2\rho \left(\frac{\partial \phi}{\partial t} + V \frac{\partial \phi}{\partial x} \right) \quad (31)$$

In order to derive expressions for total forces and total moments, it is only necessary to consider the velocity potential in two regions—either regions N and T_1 or regions N and T_2 . Therefore the expression for the total force, positive downward, on the wing may be written as

$$\bar{P} = -2 \int_N \int \Delta p_N dy dx - 2 \int_{T_1} \int \Delta p_{r_1} dy dx \quad (32)$$

where Δp_N is to be calculated from equation (18) and the integrations in the first term are to be extended over the shaded portion of region N shown in figure 2 (a), and where Δp_{r_1} is to be calculated from equation (27) and the integrations in the second term are to be extended over region T_1 . (The integrations in the first term are simple and may be performed by inspection. Those in the second term may be readily performed by making use of the relations given in the appendix.)

After the indicated integrations have been performed and all position coordinates involved have been referred to the chord $2b$ (but the original coordinate symbols maintained), the results can be written as

$$\bar{P} = -8\rho b^2 V^2 k^2 A e^{i\omega t} \left[\frac{h_0}{b} (\bar{L}_1 + i\bar{L}_2) + \alpha_0 (\bar{L}_3 + i\bar{L}_4) \right] \quad (33)$$

where the reduced frequency k is related to ω and $\bar{\omega}$ by the relations

$$k = \frac{b\omega}{V} = \frac{b\beta^2}{M^2} \bar{\omega} \quad (34)$$

and where

$$\bar{L}_1 = \frac{1}{3\beta^4} \left(3\beta - \frac{2+\beta^2}{A} \right) \quad (35a)$$

$$\bar{L}_2 = \frac{1}{\beta k} - \frac{M^2 k}{\beta^5} - \frac{1}{2A} \left[\frac{1}{\beta^2 k} - \frac{M^2 k}{3\beta^6} (4+\beta^2) \right] \quad (35b)$$

$$\bar{L}_3 = \frac{1}{\beta k^2} - \frac{1}{3\beta^5} (3+\beta^2+6\beta^2 x_0) - \frac{1}{2A} \left\{ \frac{1}{\beta^2 k^2} - \frac{1}{3\beta^6} [(3\beta^2+4)+4\beta^2 x_0(2+\beta^2)] \right\} \quad (35c)$$

$$\bar{L}_4 = \frac{1}{\beta^3 k} (\beta^2 - 1 - 2\beta^2 x_0) + \frac{M^2 k}{6\beta^7} (5+\beta^2+12\beta^2 x_0) + \frac{1}{3A} \left[\frac{1}{\beta^4 k} (2+3\beta^2 x_0) - \frac{M^2 k}{5\beta^8} (8+4\beta^2+20\beta^2 x_0+5\beta^4 x_0) \right] \quad (35d)$$

The quantities $\bar{L}_i$ ($i=1, 2, 3, 4$) are the in-phase and out-of-phase components of the total force coefficients, $\bar{L}_1$ and $\bar{L}_3$ being the in-phase and $\bar{L}_2$ and $\bar{L}_4$ being the out-of-phase components. It will be noted that $\bar{L}_1$ and $\bar{L}_2$ are associated only with vertical translations of the wing and are independent of axis-of-rotation location x_0 . The components $\bar{L}_3$ and $\bar{L}_4$ are associated with angular position and rotation of the wing about any axis $x=x_0$ and depend partly on the location of x_0 .

The total moment, positive clockwise, on the wing about the arbitrary axis of rotation $x=x_0$ is

$$\bar{M}_a = -2 \int_N \int (x-x_0) \Delta p_N dy dx - 2 \int_{T_1} \int (x-x_0) \Delta p_{T_1} dy dx \quad (36)$$

If steps similar to those required to obtain equation (33) are performed, there is obtained

$$\bar{M}_a = -8\rho b^3 V^2 k^2 A e^{i\omega t} \left[\frac{h_0}{b} (\bar{M}_1 + i\bar{M}_2) + \alpha_0 (\bar{M}_3 + i\bar{M}_4) \right] \quad (37)$$

where

$$\bar{M}_1 = \frac{2}{3\beta^3} (2-3x_0) - \frac{1}{6A\beta^4} [3\beta^2+1-4x_0(2+\beta^2)] \quad (38a)$$

$$\bar{M}_2 = \frac{1}{\beta k} (1-2x_0) - \frac{M^2 k}{2\beta^5} (3-4x_0) - \frac{1}{3A} \left\{ \frac{1}{\beta^2 k} (2-3x_0) - \frac{M^2 k}{5\beta^6} [4(\beta^2+4)(4-5x_0)] \right\} \quad (38b)$$

$$\bar{M}_3 = \frac{1}{\beta k^2} (1-2x_0) - \frac{1}{2\beta^5} [3+\beta^2+4x_0(\beta^2-1)-8\beta^2 x_0^2] - \frac{1}{3A} \left\{ \frac{1}{\beta^2 k^2} (2-3x_0) - \frac{1}{5\beta^6} [4(4+3\beta^2)+5x_0(3\beta^4+3\beta^2-4)-20\beta^2 x_0^2(\beta^2+2)] \right\} \quad (38c)$$

$$\bar{M}_4 = \frac{2}{3\beta^3 k} [2(\beta^2-1)-3x_0(2\beta^2-1)+6\beta^2 x_0^2] + \frac{M^2 k}{15\beta^7} (20+4\beta^2-25x_0+40\beta^2 x_0-60\beta^2 x_0^2) + \frac{1}{3A} \left\{ \frac{1}{\beta^4 k} [3+4x_0(\beta^2-1)-6\beta^2 x_0^2] - \frac{M^2 k}{15\beta^8} [20(2+\beta^2)-24x_0(2-3\beta^2-\beta^4)-30\beta^2 x_0^2(4+\beta^2)] \right\} \quad (38d)$$

The quantities $\bar{M}_1$ and $\bar{M}_2$ are, respectively, the in-phase and out-of-phase components of total moment coefficients about the axis $x=x_0$ associated with vertical translations of the wing; $\bar{M}_3$ and $\bar{M}_4$ are the corresponding components due to angular position and rotation of the wing about $x=x_0$.

It is of interest to note in equations (35) and (38) that the components $\bar{L}_1$ and $\bar{M}_1$ do not involve the reduced frequency k . The effect of frequency on these two components comes from terms involving the frequency to the fourth and higher powers; but for values of k thought likely to be encountered in supersonic flutter ($k < 0.1$), the contribution of these higher-power terms to any of the components in equations (35) and (38) is, for the most part, negligible.

Section forces and moments.—The section forces and moments at any spanwise station are derived by integrating the pressure difference along the chord for the forces and the pressure difference multiplied by a moment arm for the moments. Since the distribution over the entire wing is symmetrical with regard to the midspan section, it is only necessary to derive expressions for the forces and moments at any station of the half-span adjacent to the origin. (See figs. 1, 2, and 3.)

Under the restrictions previously stipulated, two cases that can arise are considered (see fig. 3): (1) the Mach lines from the tips do not intersect on the wing (or $A\beta > 2$), and (2) the Mach lines intersect on the wing but the Mach line from one tip does not intersect the opposite tip ahead of the trailing edge (or $1 \leq A\beta \leq 2$). Only the final forms of the section force and moment equations are given. These forms are easily calculated by deriving the pressure difference for the different regions from the appropriate velocity potential, making use of figure 3 to determine the limits of

integration for the regions involved, and using the relations given in the appendix to carry out the more troublesome integrations. The integrated expression for any region can then be reduced to the forms

$$P = -4\rho b V^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (L_1 + iL_2) + \alpha_0 (L_3 + iL_4) \right] \quad (39)$$

and

$$M_a = -4\rho b^2 V^2 k^2 e^{i\omega t} \left[\frac{h_0}{b} (M_1 + iM_2) + \alpha_0 (M_3 + iM_4) \right] \quad (40)$$

where the position coordinates are referred to the chord length $2b$.

The components of force and moment coefficients for the half-span adjacent to the origin are as follows:

Case 1 (see fig. 3(a)): For any section between the tip and the point where the Mach line intersects the trailing edge, or where $0 < y < \frac{1}{\beta}$, the components of section force coefficients are

$$L_1 = -\frac{4}{\beta\pi} \left(\bar{F}_1 - \frac{1+2\beta^2}{\beta^2} \bar{F}_2 \right) \quad (41a)$$

$$L_2 = \frac{4}{\beta\pi} \left\{ \frac{1}{2k} \bar{F}_1 + \frac{M^2 k}{\beta^4} [(2\beta^2 - 1)\bar{F}_2 - 3\beta^2 \bar{F}_3] \right\} \quad (41b)$$

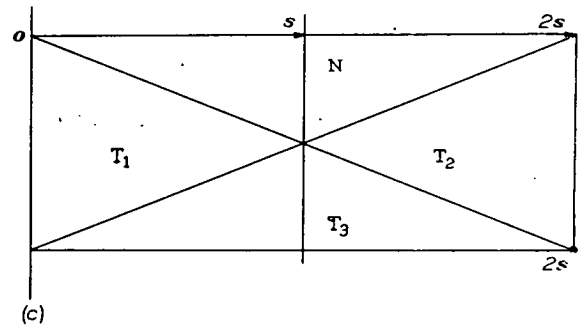
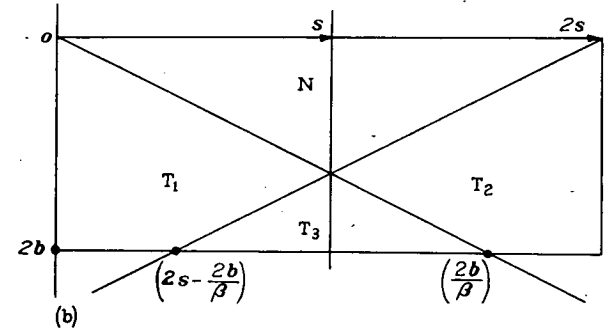
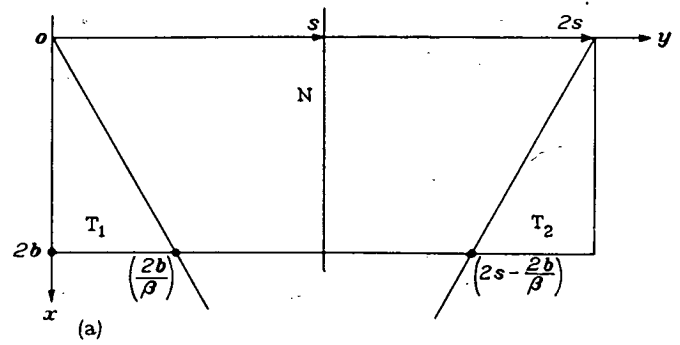
$$L_3 = \frac{4}{\beta\pi} \left[\frac{1}{2k^2} \bar{F}_1 - (1 - 2x_0) \bar{F}_1 + \frac{6\beta^4 + 3\beta^2 - 1}{\beta^4} \bar{F}_2 - \frac{2(2\beta^2 + 1)x_0}{\beta^2} \bar{F}_2 - \frac{6\beta^2 + 5}{\beta^2} \bar{F}_3 \right] \quad (41c)$$

$$L_4 = \frac{4}{\beta\pi} \left\{ \frac{1}{k} \left[(2 - x_0) \bar{F}_1 - \frac{3\beta^2 + 1}{\beta^2} \bar{F}_2 \right] + \frac{M^2 k}{3\beta^6} [3(1 - \beta^2 + 2\beta^4 + 2\beta^2 x_0 - 4\beta^4 x_0) \bar{F}_2 - 6\beta^4 (4 - 3x_0) \bar{F}_3 - (1 - 7\beta^2 - 20\beta^4) \bar{F}_4] \right\} \quad (41d)$$

where $\bar{F}_n$ ($n=1, 2, 3, 4$), given in the appendix, is obtained from F_n (equation (28)) when $x=2b$. For any section between the point where the Mach line intersects the trailing edge and the midspan, or where $\frac{1}{\beta} \leq y \leq \frac{A}{2}$, the components of section forces are:

$$\left. \begin{aligned} L_1 &= \frac{1}{\beta^3} \\ L_2 &= \frac{1}{\beta k} - \frac{M^2 k}{\beta^5} \\ L_3 &= \frac{1}{\beta k^2} - \frac{1}{3\beta^5} [(3 + \beta^2) + 6\beta^2 x_0] \\ L_4 &= \frac{1}{\beta^3 k} [(\beta^2 - 1) - 2\beta^2 x_0] + \frac{M^2 k}{6\beta^7} (5 + \beta^2 + 12\beta^2 x_0) \end{aligned} \right\} \quad (42)$$

As a check on the results in equations (41) and (42) the expressions in equations (41) reduce to those in equations (42) when $y = \frac{1}{\beta}$.



(a) Mach lines from tips do not intersect on wing.

(b) Mach lines from tips intersect on wing but Mach line from one tip does not intersect opposite tip.

(c) Mach lines from tips intersect on wing and Mach line from one tip intersects opposite tip at trailing edge.

FIGURE 3.—Sketch illustrating different Mach line locations accounted for in analysis.

The components of section moment coefficients for case 1 are as follows:

For $0 < y < \frac{1}{\beta}$,

$$M_1 = -\frac{4}{\beta\pi} \left[(1 - 2x_0) \bar{F}_1 + 2x_0 \frac{2\beta^2 + 1}{\beta^2} \bar{F}_2 - \frac{3\beta^2 + 2}{\beta^2} \bar{F}_3 \right] \quad (43a)$$

$$M_2 = \frac{4}{\beta\pi} \left\{ \frac{1}{k} (\bar{F}_2 - x_0 \bar{F}_1) + \frac{M^2 k}{\beta^2} \left[\frac{(2\beta^2 - 1)(1 - 2x_0)}{\beta^2} \bar{F}_2 + 6x_0 \bar{F}_3 - \frac{4\beta^2 + 1}{\beta^2} \bar{F}_4 \right] \right\} \quad (43b)$$

$$M_3 = \frac{4}{\beta\pi} \left[\frac{1}{k^2} (\bar{F}_2 - x_0 \bar{F}_1) - \frac{4(1-3x_0+3x_0^2)}{3} \bar{F}_1 + \frac{(6\beta^4+3\beta^2-1)(1-2x_0)+4\beta^2x_0^2(1+2\beta^2)}{\beta^4} \bar{F}_2 + \frac{6(1+\beta^2)}{\beta^2} x_0 \bar{F}_3 - \frac{20\beta^4+21\beta^2+3}{3\beta^4} \bar{F}_4 \right] \quad (43c)$$

$$M_4 = \frac{4}{\beta\pi} \left(\frac{2}{k} \left[(1-x_0)^2 \bar{F}_1 + \frac{2\beta^2+1}{\beta^2} x_0 \bar{F}_2 - \frac{2\beta^2+1}{\beta^2} \bar{F}_2 \right] + \frac{4M^2k}{\beta^2} \left\{ \frac{1}{6\beta^4} [(4\beta^4-2\beta^2+1)(1-3x_0+3x_0^2)+1-3x_0^2] \bar{F}_2 - (2-4x_0+3x_0^2) \bar{F}_3 - \frac{x_0}{6\beta^4} (8\beta^4+4\beta^2-1) \bar{F}_4 + \frac{1}{3\beta^2} (5\beta^2+3) \bar{F}_5 \right\} \right) \quad (43d)$$

and for $\frac{1}{\beta} \leq y \leq \frac{A}{2}$,

$$\left. \begin{aligned} M_1 &= \frac{2}{3\beta^3} (2-3x_0) \\ M_2 &= \frac{1}{\beta k} (1-2x_0) - \frac{M^2k}{2\beta^5} (3-4x_0) \\ M_3 &= \frac{1}{\beta k^2} (1-2x_0) - \frac{1}{2\beta^5} (3+\beta^2-4x_0+4\beta^2x_0-8\beta^2x_0^2) \\ M_4 &= \frac{2}{3\beta^3k} [2(\beta^2-1)-3x_0(2\beta^2-1)+6\beta^2x_0^2] + \frac{M^2k}{15\beta^7} [4(5+\beta^2)+5x_0(8\beta^2-5)-60\beta^2x_0^2] \end{aligned} \right\} \quad (44)$$

The expressions in equations (43) reduce to those in equations (44) when $y = \frac{1}{\beta}$. The expressions in equations (42) and (44) correspond to the more exact two-dimensional components of force and moment coefficients derived in reference 4. For values of $k < 0.1$ these expressions yield, for the most part, values that are in good agreement with those that may be calculated from the tables in reference 4.

Case 2 (see fig. 3 (b)): For any section between the tip at $y=0$ and the point where the Mach line from the tip at $y=2s$ intersects the trailing edge (or where $0 < y \leq A - \frac{1}{\beta}$), the components of section force coefficients are given by equations (41) and the components of section moment coefficients, by equation (43). For any section between the point where the Mach line from the tip at $y=2s$ intersects the trailing edge and the midspan (or where $A - \frac{1}{\beta} < y \leq \frac{A}{2}$), the components of section force coefficients are

$$L_1 = - \left\{ \frac{4}{\beta\pi} \left[(\bar{F}_1 + \bar{G}_1) - \frac{1+2\beta^2}{\beta^2} (\bar{F}_2 + \bar{G}_2) \right] + \frac{1}{\beta^3} \right\} \quad (45a)$$

$$L_2 = \frac{4}{\beta\pi} \left\{ \frac{1}{2k} (\bar{F}_1 + \bar{G}_1) + \frac{M^2k}{\beta^4} [(2\beta^2-1)(\bar{F}_2 + \bar{G}_2) - 3\beta^2(\bar{F}_3 + \bar{G}_3)] - \left(\frac{1}{\beta k} - \frac{M^2k}{\beta^5} \right) \right\} \quad (45b)$$

$$L_3 = \frac{4}{\beta\pi} \left[\frac{1}{2k^2} (\bar{F}_1 + \bar{G}_1) - 2(1-2x_0)(\bar{F}_1 + \bar{G}_1) + \frac{6\beta^4+3\beta^2-1}{\beta^4} (\bar{F}_2 + \bar{G}_2) - \frac{2(2\beta^2+1)x_0}{\beta^2} (\bar{F}_2 + \bar{G}_2) - \frac{6\beta^2+5}{\beta^2} (\bar{F}_3 + \bar{G}_3) \right] - \left\{ \frac{1}{\beta k^2} - \frac{1}{3\beta^5} [(3+\beta^2)+6\beta^2x_0] \right\} \quad (45c)$$

$$L_4 = \frac{4}{\beta\pi} \left\{ \frac{1}{k} \left[(2-x_0)(\bar{F}_1 + \bar{G}_1) - \frac{3\beta^2+1}{\beta^2} (\bar{F}_2 + \bar{G}_2) \right] + \frac{M^2k}{3\beta^6} [3(1-\beta^2+2\beta^4+2\beta^2x_0-4\beta^4x_0)(\bar{F}_2 + \bar{G}_2) - 6\beta^4(4-3x_0)(\bar{F}_3 + \bar{G}_3) - (1-7\beta^2-20\beta^4)(\bar{F}_4 + \bar{G}_4)] \right\} - \left\{ \frac{1}{\beta^3k} [(\beta^2-1)-2\beta^2x_0] + \frac{M^2k}{6\beta^7} (5+\beta^2+12\beta^2x_0) \right\} \quad (45d)$$

where $\bar{G}_n$ ($n=1, 2, 3, 4$), given in the appendix, is obtained from G_n (equation (29)) when $x=2b$. The corresponding components of section moment coefficients are

$$M_1 = - \left\{ \frac{4}{\beta\pi} \left[(1-2x_0)(\bar{F}_1 + \bar{G}_1) + \frac{2x_0(2\beta^2+1)}{\beta^2} (\bar{F}_2 + \bar{G}_2) \right] - \frac{3\beta^2+1}{\beta^2} (\bar{F}_3 + \bar{G}_3) + \frac{2}{3\beta^3} (2-3x_0) \right\} \quad (46a)$$

$$M_2 = \frac{4}{\beta\pi} \left\{ \frac{1}{k} [(\bar{F}_2 + \bar{G}_2) - x_0(\bar{F}_1 + \bar{G}_1)] + \frac{M^2k}{\beta^2} \left[\frac{(2\beta^2-1)(1-2x_0)}{\beta^2} (\bar{F}_2 + \bar{G}_2) + 6x_0(\bar{F}_3 + \bar{G}_3) - \frac{4\beta^2+1}{\beta^2} (\bar{F}_4 + \bar{G}_4) \right] \right\} - \left[\frac{1}{\beta k} (1-2x_0) - \frac{M^2k}{2\beta^5} (3-4x_0) \right] \quad (46b)$$

$$M_3 = \frac{4}{\beta\pi} \left\{ \frac{1}{k^2} [(\bar{F}_2 + \bar{G}_2) - x_0(\bar{F}_1 + \bar{G}_1)] - 4 \frac{1-3x_0+3x_0^2}{3} (\bar{F}_1 + \bar{G}_1) + \frac{(6\beta^4+3\beta^2-1)(1-2x_0)+4\beta^2x_0^2(1+2\beta^2)}{\beta^4} (\bar{F}_2 + \bar{G}_2) + \frac{6x_0(1+\beta^2)}{\beta^2} (\bar{F}_3 + \bar{G}_3) - \frac{20\beta^4+21\beta^2+3}{3\beta^4} (\bar{F}_4 + \bar{G}_4) \right\} - \left\{ \frac{1}{\beta k^2} (1-2x_0) - \frac{1}{2\beta^5} [(3+\beta^2)-4x_0(1-\beta^2)-8\beta^2x_0^2] \right\} \quad (46c)$$

$$M_4 = \frac{4}{\beta\pi} \left\{ \frac{2}{k} \left[(1-x_0)(\bar{F}_1 + \bar{G}_1) + \frac{2\beta^2+1}{\beta^2} x_0(\bar{F}_2 + \bar{G}_2) - \frac{2\beta^2+1}{\beta^2} (\bar{F}_3 + \bar{G}_3) \right] + \frac{4M^2k}{\beta^2} \left\{ \frac{1}{6\beta^4} [(4\beta^4-2\beta^2+1)(1-3x_0+3x_0^2)+1-3x_0^2](\bar{F}_2 + \bar{G}_2) - (2-4x_0+3x_0^2)(\bar{F}_3 + \bar{G}_3) - \frac{x_0}{6\beta^4} (8\beta^4+4\beta^2-1)(\bar{F}_4 + \bar{G}_4) + \frac{1}{3\beta^2} (5\beta^2+3)(\bar{F}_5 + \bar{G}_5) \right\} \right\} - \left\{ \frac{2}{3\beta^3k} [2(\beta^2-1)-3x_0(2\beta^2-1)+6\beta^2x_0^2] + \frac{M^2k}{15\beta^7} [4(5+\beta^2)+5x_0(8\beta^2-5)-60\beta^2x_0^2] \right\} \quad (46d)$$

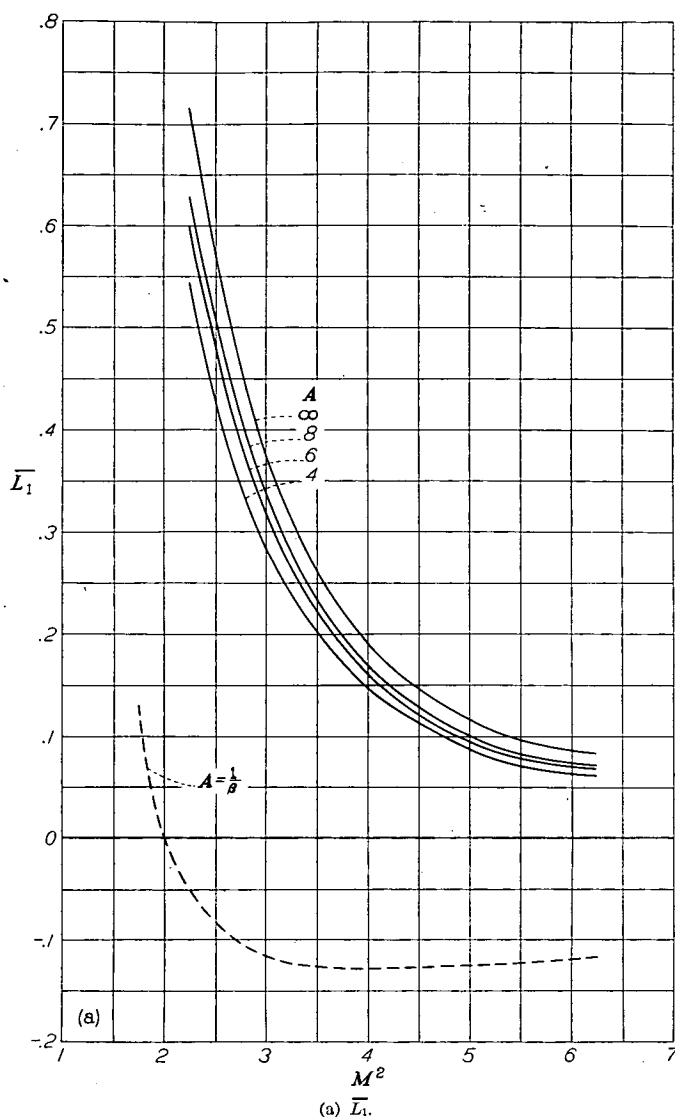


FIGURE 4.—Components of total force coefficients as functions of M^2 for $x_0=0.4$, $k=0.02$, and various values of A .

For the limiting condition of case 2—that is, when the Mach line from one tip intersects the opposite tip at the trailing edge, or $A=1/\beta$ (see fig. 3(c))—the components of section force coefficients are given by equations (45) and the corresponding components of moment coefficients, by equations (46).

SOME PARTICULAR CALCULATIONS AND DISCUSSIONS

From the expressions for total force and moment coefficients (equations (35) and (38), respectively) the over-all effect of aspect ratio on the magnitude of the forces and moments can be calculated for particular values of the parameters M , k , x_0 , and A . Examination of these equations shows that varying some of the parameters might cause some terms in the equations to vanish and to change sign. For example, if x_0 is continuously increased from some value less than $1/2$ to some value greater than $1/2$, the first terms in the expressions for $\bar{M}_2$ and $\bar{M}_3$ vanish at $x_0=1/2$ and

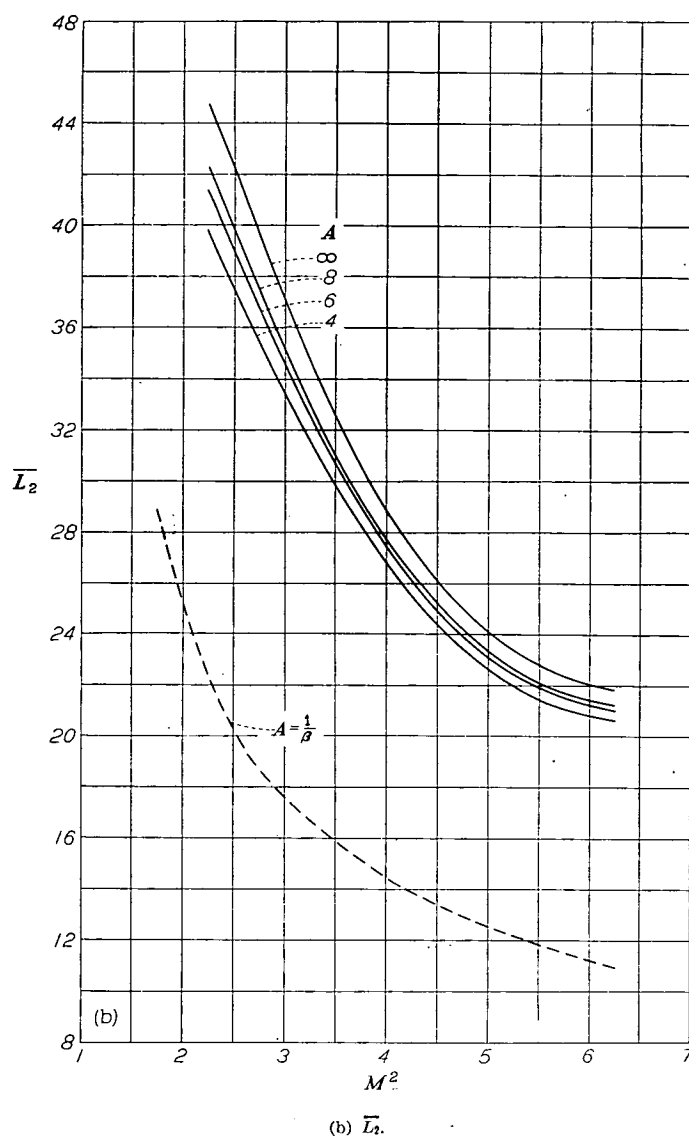


FIGURE 4.—Continued.

change sign when x_0 becomes greater than $1/2$. In particular, decreasing the aspect ratio decreases the components of force and moment coefficients $\bar{L}_1$, $\bar{L}_2$, $\bar{L}_3$, $\bar{M}_1$, $\bar{M}_2$, and $\bar{M}_3$ but increases the two important components $\bar{L}_4$ and $\bar{M}_4$.

Although the effect of aspect ratio may change considerably with only a small change in any one (or more) of the parameters M , k , and x_0 , some insight into what the over-all effect might be can be gained from calculations of all the components of total force and moment coefficients for various values of M and A and fixed values of the parameters k and x_0 . Results of such a set of calculations are presented in figures 4 to 7.

In figures 4 and 5 the components of total force and moment coefficients for various values of A and for $x_0=0.4$ and $k=0.02$ are plotted as functions of M^2 . The curves in these figures calculated for infinite aspect ratio correspond to the two-dimensional results of reference 4. The dashed curves represent calculations for aspect ratio and Mach number combinations that cause the Mach lines from one tip to intersect the opposite tip at the trailing edge so that

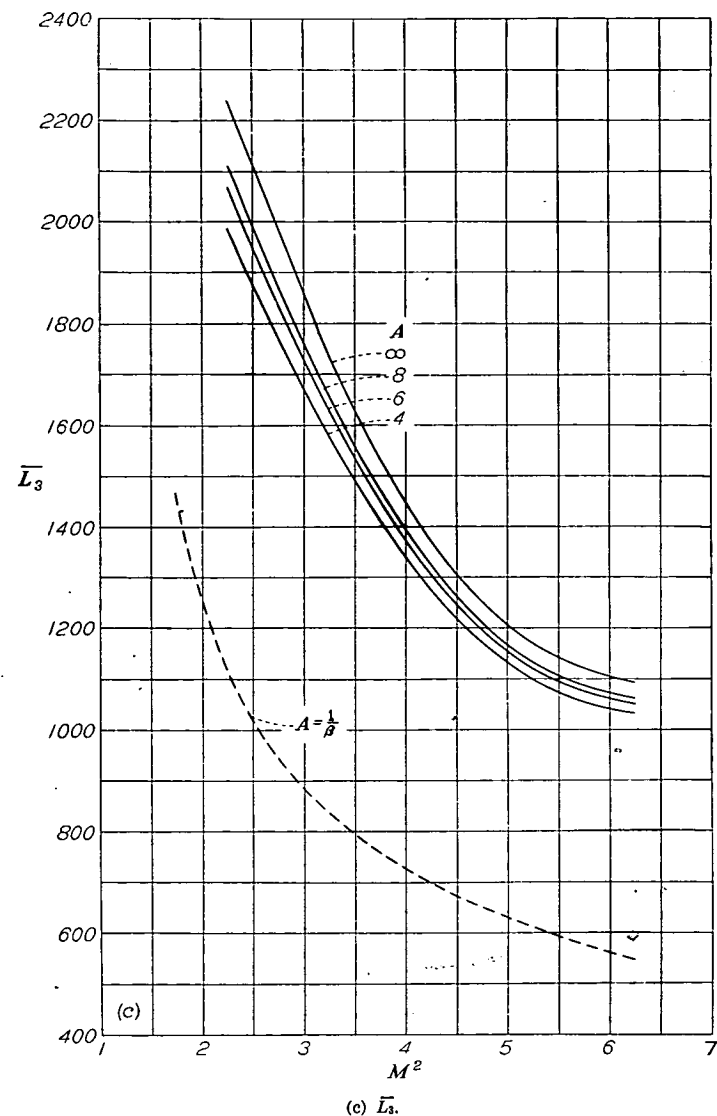


FIGURE 4.—Continued.

along the dashed curves the aspect ratio is not constant but varies with M^2 according to the previously given expression

$$A = \frac{1}{\beta} = \frac{1}{\sqrt{M^2 - 1}}$$

The difference, at any value of M^2 , between the dashed curves and the curves corresponding to infinite aspect ratio in figures 4 and 5 is, therefore, for the chosen values of k and x_0 , the maximum effect of aspect ratio on the components of total force and moment coefficients for a rectangular wing under the restrictions of the foregoing analysis. It will be noted in figures 4 and 5 that, when the aspect ratio is small, the deviation of the three-dimensional results from two-dimensional results may be quite large.

In figures 6 and 7 the components of the total force and moment coefficients are plotted as functions of aspect ratio for $x_0=0.4$, $k=0.02$, and some particular values of M . It will be noted in these figures that all the components of force and moment coefficients undergo rapid changes with

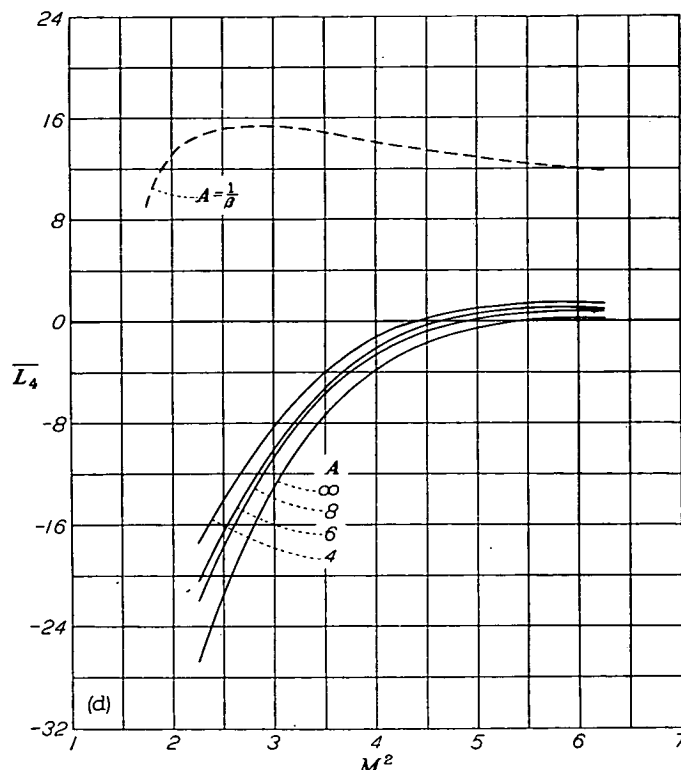


FIGURE 4.—Concluded.

respect to varying aspect ratio when A becomes less than 4 or 5. It may be remarked that the directions of the changes with respect to aspect ratio appear to be such that they would have favorable effects on the flutter characteristics of a wing.

The spanwise distribution of the components of section force and moment coefficients computed from equations (41) to (44) for $A=4$, $x_0=0.4$, $k=0.02$, and $M=2$ are plotted in figures 8 and 9. The portions of the curves in these figures corresponding to values of y in the range $\frac{1}{\beta} \leq y \leq A - \frac{1}{\beta}$ are the two-dimensional values, and the effect of aspect ratio may be noted in the tip regions, $0 \leq y \leq \frac{1}{\beta}$ and $A - \frac{1}{\beta} \leq y \leq A$, as deviations from these two-dimensional values.

In conclusion, it may be stated that, in regard to the effect of aspect ratio on supersonic flutter, an important item that has not been discussed herein but can be studied for any particular case with the aid of equations (33) and (36) is the change in center of pressure, associated with prescribed motions of the wing, with change in aspect ratio. An investigation to find the effect that thickness might have on the center-of-pressure location is also needed. An extension of the foregoing analysis to include the effect of an aileron as an additional degree of freedom would follow in a straightforward manner.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., January 6, 1950.

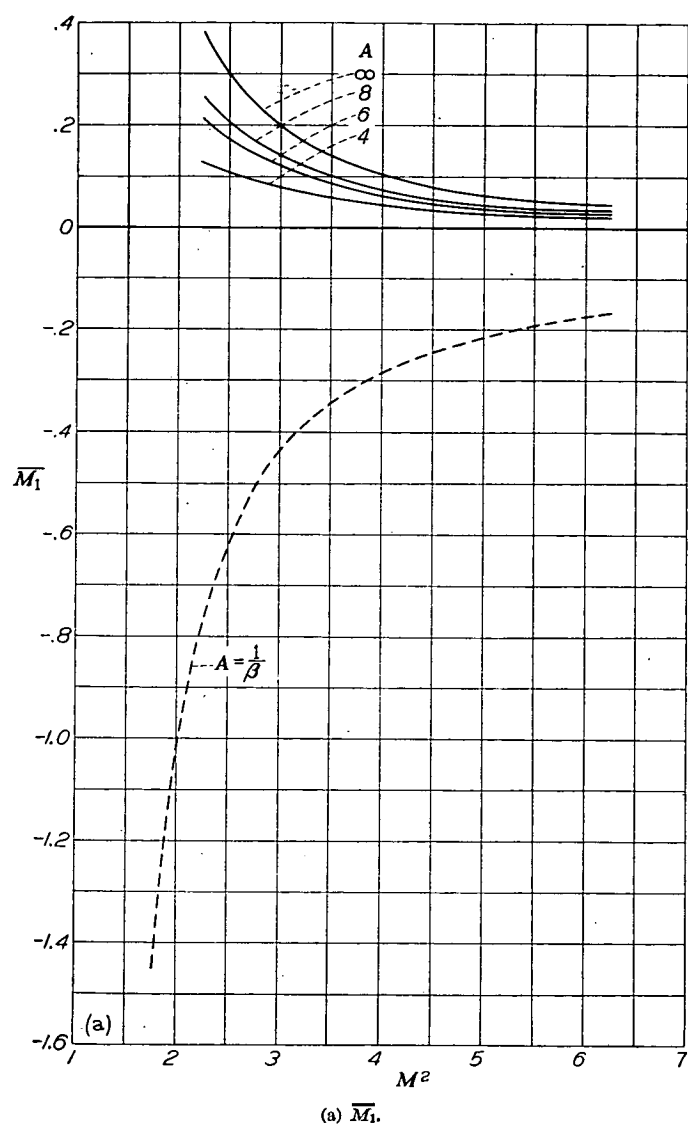


FIGURE 5.—Components of total moment coefficients as functions of M^2 for $x_0=0.4$, $k=0.02$, and various values of A .

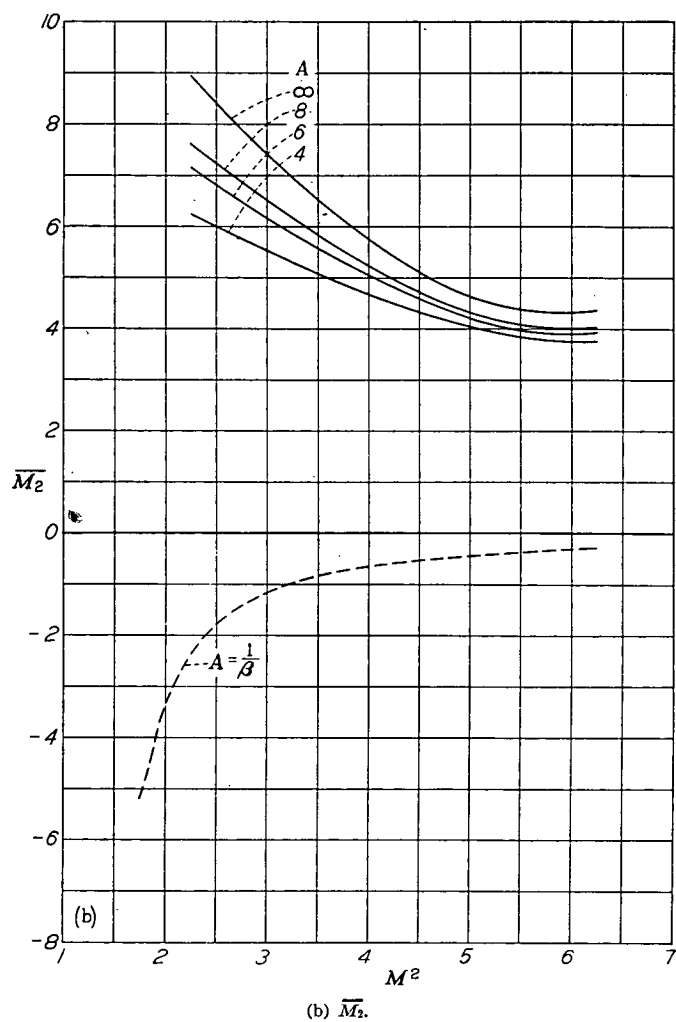


FIGURE 5.—Continued.

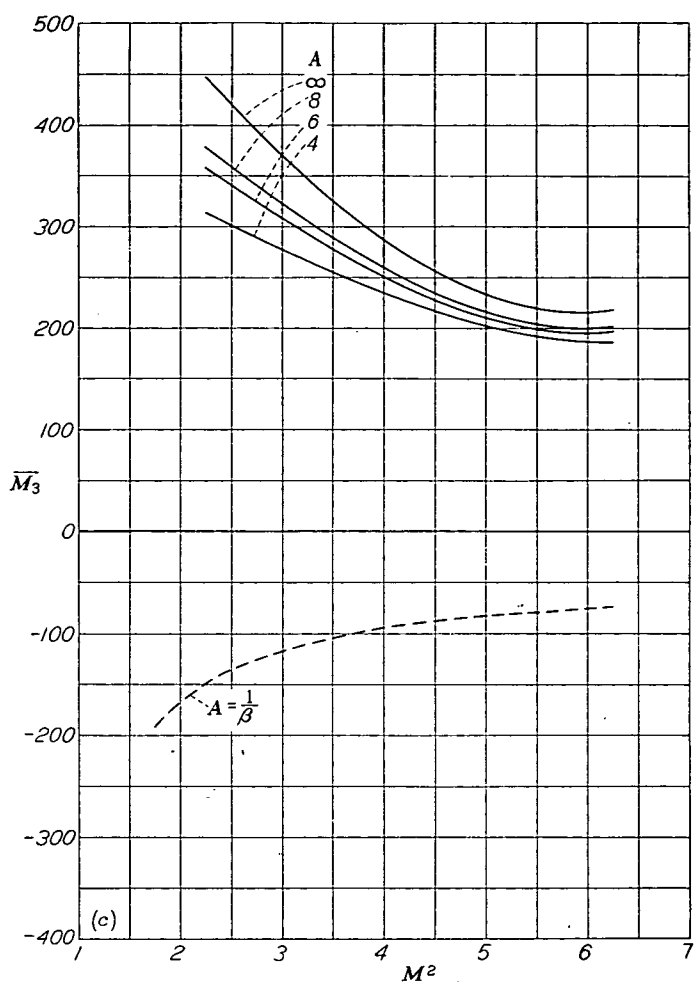
(c) $\overline{M}_3$.

FIGURE 5.—Continued.

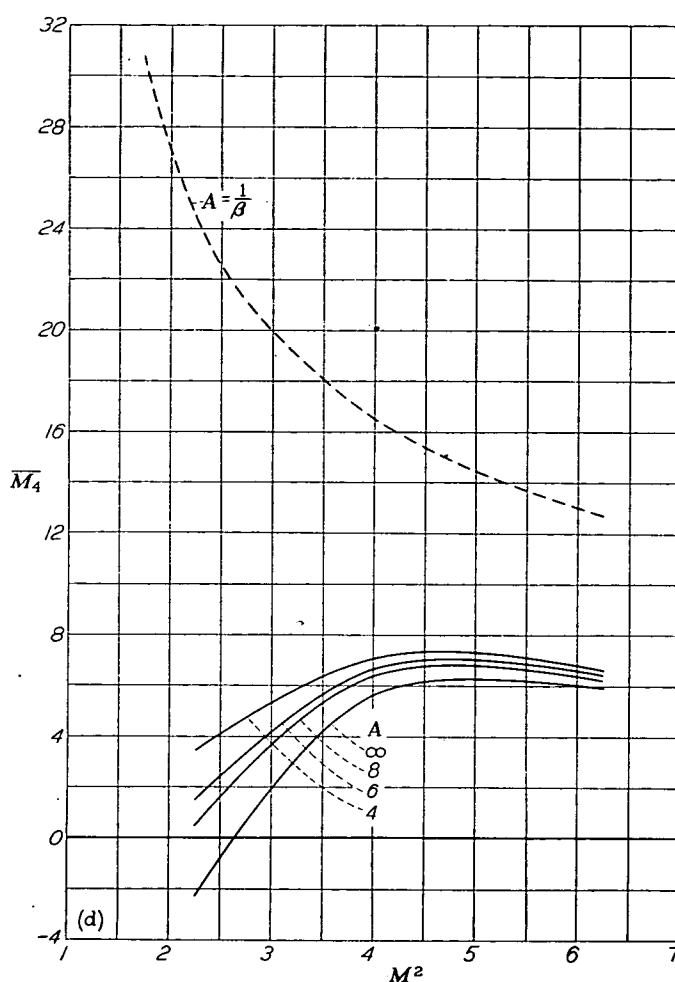
(d) $\overline{M}_4$.

FIGURE 5.—Concluded.

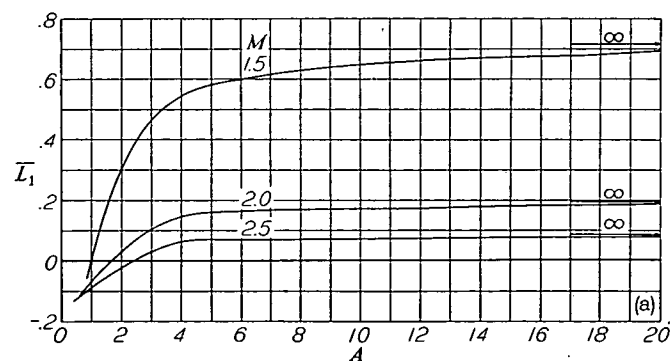
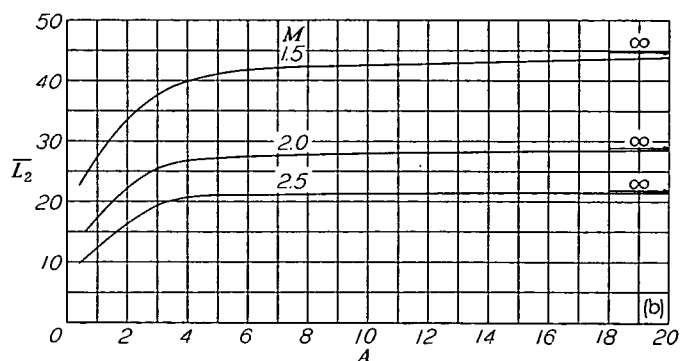
(a) $\overline{L}_1$.FIGURE 6.—Components of total force coefficients as functions of A for $x_0=0.4$, $k=0.02$, and various values of M .(b) $\overline{L}_2$.

FIGURE 6.—Continued.

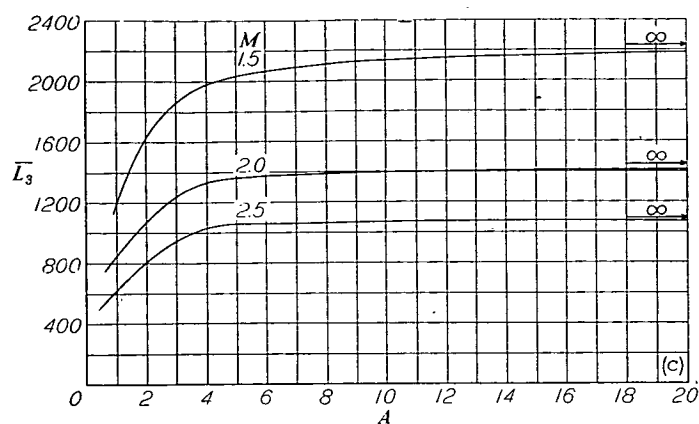
(c) $\bar{L}_3$.

FIGURE 6.—Continued.

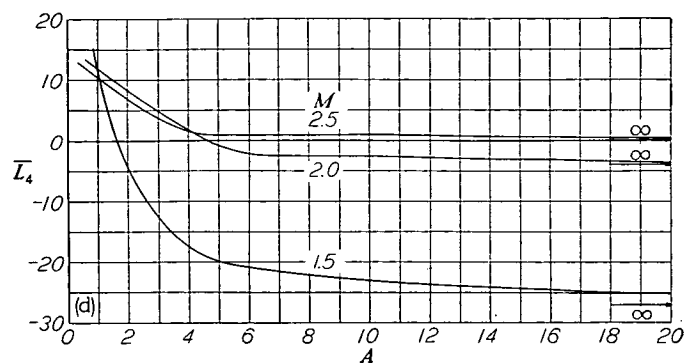
(d) $\bar{L}_4$.

FIGURE 6.—Concluded.

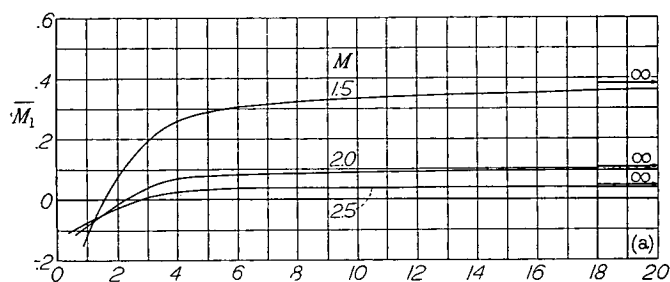
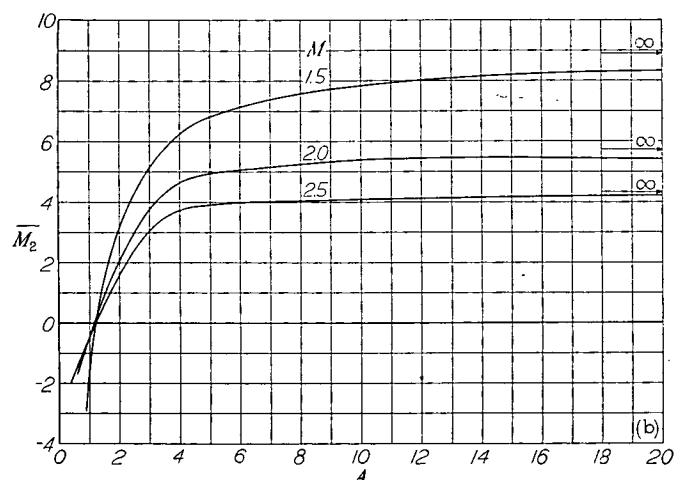
(a) $\bar{M}_1$.FIGURE 7.—Components of total moment coefficients as functions of A for $x_0=0.4$, $k=0.02$, and various values of M .(b) $\bar{M}_2$.

FIGURE 7.—Continued.

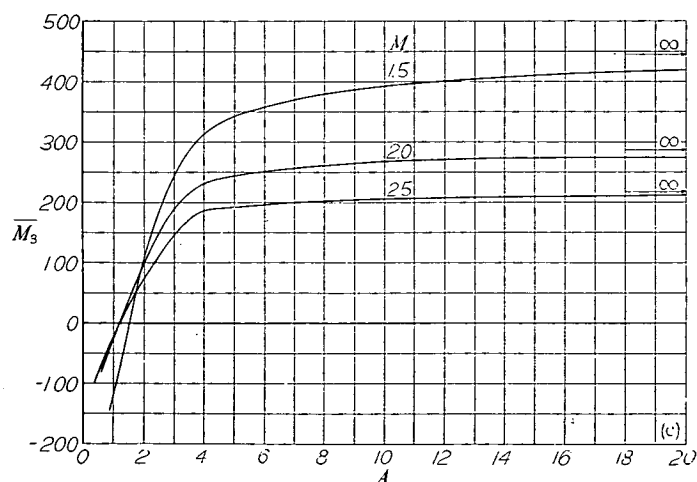
(c) $\bar{M}_3$.

FIGURE 7.—Continued.

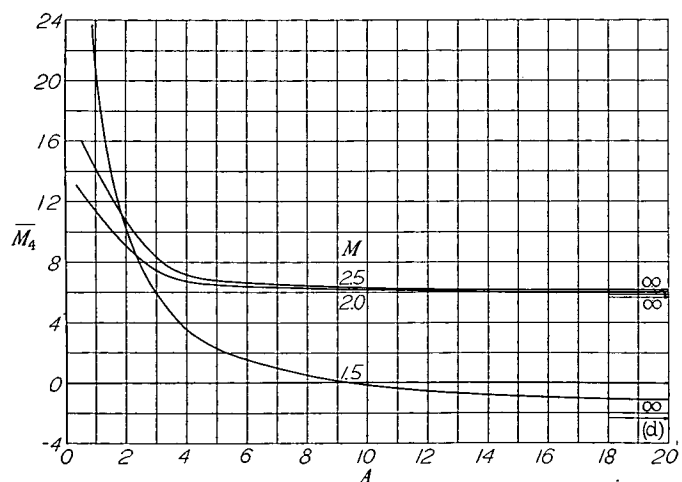
(d) $\bar{M}_4$.

FIGURE 7.—Concluded.

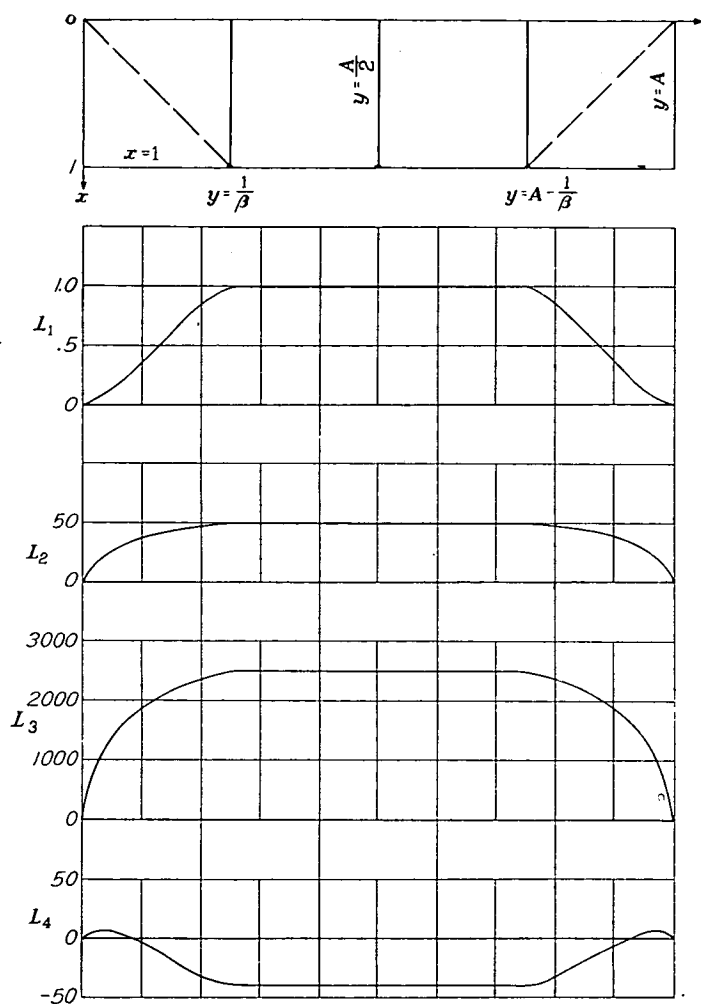


FIGURE 8.—Spanwise distribution of components of section force coefficients for $x_0=0.4$, $k=0.02$, $M=2$, and $A=4$.

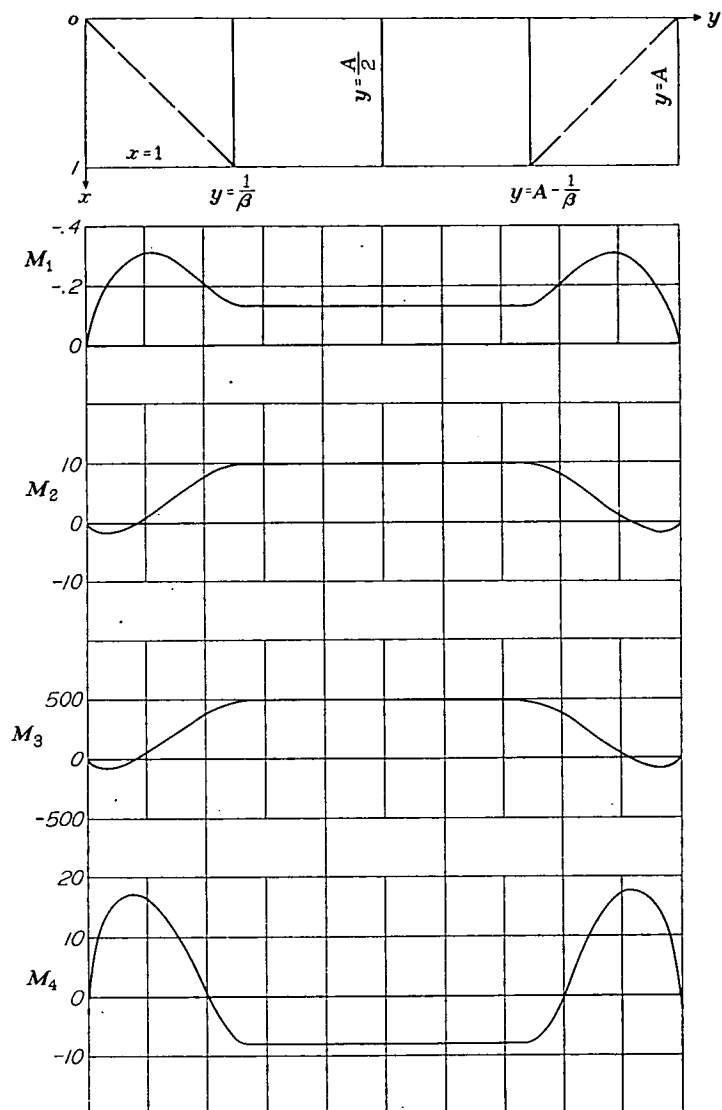


FIGURE 9.—Spanwise distribution of components of section moment coefficients for $x_0=0.4$, $k=0.02$, $M=2$, and $A=4$.

APPENDIX

SOME INTEGRATED VALUES OF $F_n, G_n, \bar{F}_n, \bar{G}_n$, AND OTHER RELATED FUNCTIONS

Values of F_n and G_n .—The values of the functions F_n (equation (28)) and G_n (equation (29)) for the first few values of n are as follows:

$$F_n = \int_0^x x^{n-1} \sin^{-1} \sqrt{\beta y/x} dx \quad (n=1, 2, 3, \dots)$$

$$F_1 = \sqrt{\beta y(x-\beta y)} + x \sin^{-1} \sqrt{\beta y/x}$$

$$F_2 = \frac{x+2\beta y}{6} \sqrt{\beta y(x-\beta y)} + \frac{x^2}{2} \sin^{-1} \sqrt{\beta y/x}$$

$$F_3 = \frac{3x^2+4\beta yx+8\beta^2 y^2}{45} \sqrt{\beta y(x-\beta y)} + \frac{x^3}{3} \sin^{-1} \sqrt{\beta y/x}$$

$$F_4 = \frac{5x^3+6\beta yx^2+8\beta^2 y^2x+16\beta^3 y^3}{140} \sqrt{\beta y(x-\beta y)} + \frac{x^4}{4} \sin^{-1} \sqrt{\beta y/x}$$

$$F_5 = \frac{35x^4+40\beta yx^3+48\beta^2 y^2x^2+64\beta^3 y^3x+128\beta^4 y^4}{1575} \sqrt{\beta y(x-\beta y)} + \frac{x^5}{5} \sin^{-1} \sqrt{\beta y/x}$$

$$G_n = \int_0^x x^{n-1} \sin^{-1} \sqrt{\frac{\beta(2s-y)}{x}} dx \quad (n=1, 2, 3, \dots)$$

$$G_1 = \sqrt{\beta(2s-y)[x-\beta(2s-y)]} + x \sin^{-1} \sqrt{\frac{\beta(2s-y)}{x}}$$

$$G_2 = \frac{x+2\beta(2s-y)}{6} \sqrt{\beta(2s-y)[x-\beta(2s-y)]} + \frac{x^2}{2} \sin^{-1} \sqrt{\frac{\beta(2s-y)}{x}}$$

$$G_3 = \frac{3x^2+4\beta x(2s-y)+8\beta^2(2s-y)^2}{45} \sqrt{\beta(2s-y)[x-\beta(2s-y)]} + \frac{x^3}{3} \sin^{-1} \sqrt{\frac{\beta(2s-y)}{x}}$$

$$G_4 = \frac{5x^3+6x^2\beta(2s-y)+8x\beta^2(2s-y)^2+16\beta^3(2s-y)^3}{140} \sqrt{\beta(2s-y)[x-\beta(2s-y)]} + \frac{x^4}{4} \sin^{-1} \sqrt{\frac{\beta(2s-y)}{x}}$$

$$G_5 = \frac{35x^4+40x^3\beta(2s-y)+48x^2\beta^2(2s-y)^2+64x\beta^3(2s-y)^3+128\beta^4(2s-y)^4}{1575} \sqrt{\beta(2s-y)[x-\beta(2s-y)]} + \frac{x^5}{5} \sin^{-1} \sqrt{\frac{\beta(2s-y)}{x}}$$

Values of $\bar{F}_n$ and $\bar{G}_n$.—The following expressions define the functions $\bar{F}_n$ and $\bar{G}_n$ appearing in equations (41), (43), (45), and (46) in the body of the report. In these expressions the variable y has been referred to the chord $2b$; that is $y/2b$ has been replaced by y and in the expressions for $\bar{G}_n$ the ratio s/b has been replaced by A :

$$\bar{F}_1 = \sqrt{\beta y(1-\beta y)} + \sin^{-1} \sqrt{\beta y}$$

$$\bar{F}_2 = \frac{1+2\beta y}{6} \sqrt{\beta y(1-\beta y)} + \frac{1}{2} \sin^{-1} \sqrt{\beta y}$$

$$\bar{F}_3 = \frac{3+4\beta y+8\beta^2 y^2}{45} \sqrt{\beta y(1-\beta y)} + \frac{1}{3} \sin^{-1} \sqrt{\beta y}$$

$$\bar{F}_4 = \frac{5+6\beta y+8\beta^2 y^2+16\beta^3 y^3}{140} \sqrt{\beta y(1-\beta y)} + \frac{1}{4} \sin^{-1} \sqrt{\beta y}$$

$$\bar{F}_5 = \frac{35+40\beta y+48\beta^2 y^2+64\beta^3 y^3+128\beta^4 y^4}{1575} \sqrt{\beta y(1-\beta y)} + \frac{1}{5} \sin^{-1} \sqrt{\beta y}$$

$$\bar{\bar{G}}_1 = \sqrt{\beta(A-y)[1-\beta(A-y)]} + \sin^{-1} \sqrt{\beta(A-y)}$$

$$\bar{\bar{G}}_2 = \frac{1+2\beta(A-y)}{6} \sqrt{\beta(A-y)[1-\beta(A-y)]} + \frac{1}{2} \sin^{-1} \sqrt{\beta(A-y)}$$

$$\bar{\bar{G}}_3 = \frac{3+4\beta(A-y)+8\beta^2(A-y)^2}{45} \sqrt{\beta(A-y)[1-\beta(A-y)]} + \frac{1}{3} \sin^{-1} \sqrt{\beta(A-y)}$$

$$\bar{\bar{G}}_4 = \frac{5+6\beta(A-y)+8\beta^2(A-y)^2+16\beta^3(A-y)^3}{140} \sqrt{\beta(A-y)[1-\beta(A-y)]} + \frac{1}{4} \sin^{-1} \sqrt{\beta(A-y)}$$

$$\bar{\bar{G}}_5 = \frac{35+40\beta(A-y)+48\beta^2(A-y)^2+64\beta^3(A-y)^3+128\beta^4(A-y)^4}{1575} \sqrt{\beta(A-y)[1-\beta(A-y)]} + \frac{1}{5} \sin^{-1} \sqrt{\beta(A-y)}$$

Some integral relations for F_n and G_n .—Some pertinent integral relations for F_n are as follows:

$$\int_0^x F_n dx = x F_n - F_{n+1}$$

$$\int_0^x x F_n dx = \frac{1}{2} (x^2 F_n - F_{n+2})$$

$$\int_0^x x^2 F_n dx = \frac{1}{3} (x^3 F_n - F_{n+3})$$

$$\int_0^x (x-x_0) F_n dx = \frac{x^2-2xx_0}{2} F_n + x_0 F_{n+1} - \frac{1}{2} F_{n+2}$$

$$\int_0^x (x-x_0)^2 F_n dx = \frac{x^3-3x_0x^2+3x_0^2x}{3} F_n - x_0^2 F_{n+1} + x_0 F_{n+2} - \frac{1}{3} F_{n+3}$$

Corresponding integral relations for G_n may be obtained from these relations by simply replacing F by G .

Integral relations for $\bar{\bar{F}}_n$.—Integral relations for $\bar{\bar{F}}_n$ that may be used in calculating total forces and moments from sectional forces and moments are as follows:

$$2b \int_0^{1/\beta} \bar{\bar{F}}_1 dy = \frac{3b\pi}{4\beta}$$

$$2b \int_0^{1/\beta} \bar{\bar{F}}_2 dy = \frac{b\pi}{3\beta}$$

$$2b \int_0^{1/\beta} \bar{\bar{F}}_3 dy = \frac{5b\pi}{24\beta}$$

$$2b \int_0^{1/\beta} \bar{\bar{F}}_4 dy = \frac{3b\pi}{20\beta}$$

$$2b \int_0^{1/\beta} \bar{\bar{F}}_5 dy = \frac{7b\pi}{60\beta}$$

REFERENCES

1. Watkins, Charles E.: Effect of Aspect Ratio on Undamped Torsional Oscillations of a Thin Rectangular Wing in Supersonic Flow. NACA TN 1895, 1949.
2. Garrick, I. E., and Rubinow, S. I.: Theoretical Study of Air Forces on an Oscillating or Steady Thin Wing in a Supersonic Main Stream. NACA Rep. 872, 1947.
3. Miles, John W.: The Oscillating Rectangular Airfoil at Supersonic Speeds. TM RRB-15, U. S. Naval Ordnance Test Station, Inyokern, Calif., June 1, 1949.
4. Garrick, I. E., and Rubinow, S. I.: Flutter and Oscillating Air-Force Calculations for an Airfoil in a Two-Dimensional Supersonic Flow. NACA Rep. 846, 1946.
5. Snow, R. M., and Bonney, E. A.: Aerodynamic Characteristics of Wings at Supersonic Speeds. Bumblebee Rep. No. 55, The Johns Hopkins Univ., Appl. Phys. Lab., March 1947.

REPORT 1029

COMPRESSIVE STRENGTH OF FLANGES¹

By ELBRIDGE Z. STOWELL

SUMMARY

The maximum compressive stress carried by a hinged flange is computed from a deformation theory of plasticity combined with the theory for finite deflections for this structure. The computed stresses agree well with those found experimentally. Empirical observation indicates that the results will also apply fairly well to the more commonly used flanges which are not hinged.

INTRODUCTION

Ordinarily the ability of columns and plates to carry additional load does not entirely cease when they buckle. If the load is increased sufficiently beyond the buckling load, they will ultimately refuse to carry more load, with subsequent permanent distortion. In the case of columns, the maximum load is not far above the buckling load (see reference 1); in the case of plates, there may be a considerable spread between the two loads.

The first essential requirement for the solution of the problem of maximum load is the existence of a finite-deflection theory for the behavior of the structure. Maximum load always occurs at some finite deflection or distortion beyond the buckling load. The problem of the load for a given distortion is thus nonlinear even without the introduction of plasticity. Few such solutions exist for post-buckling behavior of structures even in the elastic region.

The second essential requirement for computation of maximum load is the ability to describe the nonlinear behavior of the structure that results from plasticity of the material. Neither columns nor plates would ever possess a maximum load in compression, if the material of which the structure was made obeyed Hooke's law at all times, although they might be tremendously distorted. In such a structure it would always be possible to add still another increment of load, which would result in still another increment of distortion. The question of a maximum load must therefore be directly linked with the failure of the material to obey Hooke's law—that is, with the plasticity of the material and the nonlinear behavior of the structure which results from that plasticity.

For the calculation of the maximum load carried by a buckled structure, these two essential but difficult requirements must be met. This report treats the maximum compressive strength of a simple plate structure for which the effects of both types of nonlinearity can be found—that is, the compressed flange hinged along one side edge.

The maximum load carried by a long hinged flange is computed as follows: The strain distribution across the flange at any angle of twist is found from knowledge of nonlinearity due to finite deflection. This elastic strain distribution is assumed to persist into the plastic region. This strain distribution is transformed, with the aid of knowledge of nonlinearity due to plasticity, into a stress distribution by means of some appropriate stress-strain relation. The load carried by the flange at the particular twist is then obtained by integrating the stress distribution across the flange. The load is then investigated to see if it has a maximum value as the twist increases; the maximum load should correspond with the experimentally observed maximum load.

Experimental data on the behavior of hinged flanges have been obtained in the Langley Structures Research Division by the methods of reference 2. These data are used in the present report for comparison with theoretical relations.

The theoretical treatment of the behavior of a hinged flange commences in the next section with a discussion of the effects due to finite deflections. Details of the theoretical calculations are presented in two appendixes.

NONLINEAR BEHAVIOR DUE TO FINITE DEFLECTION

Theoretical strain relations.—A flange of length L , width b , and thickness t is shown in figure 1 together with the coordinate system. The flange is hinged along the line $z=0$ and has a free edge along the line $z=b$. Compression is applied longitudinally.

The load is applied uniformly at first. The theory of appendix A shows that, for strains below a certain critical strain ϵ_{cr} , the flange will shorten without twisting. The critical strain ϵ_{cr} at which twisting begins is shown to be

$$\epsilon_{cr} = \frac{(t/b)^2}{2(1+\mu)} + \frac{1}{3} \left(\frac{\pi t}{L} \right)^2 \quad (1)$$

where μ is Poisson's ratio.

As the load is increased beyond that required to start twisting, both the middle-surface strain and the stress distribution across the flange width become nonuniform, larger than the average at the hinge, less than the average at the free edge. The middle-surface strain at any point (x, z) of the flange is shown by the theory of appendix A to be

$$\epsilon_z = \frac{(t/b)^2}{2(1+\mu)} + \frac{m^2}{12} + \frac{5}{24} \frac{k^2 m^2}{1+k^2} \left(1 - 3 \frac{z^2}{b^2} \right) \text{sn}^2 \left(\frac{4Kx}{L} \right) \quad (2a)$$

¹ Supersedes NACA TN 2020, "Compressive Strength of Flanges" by Elbridge Z. Stowell, 1950.

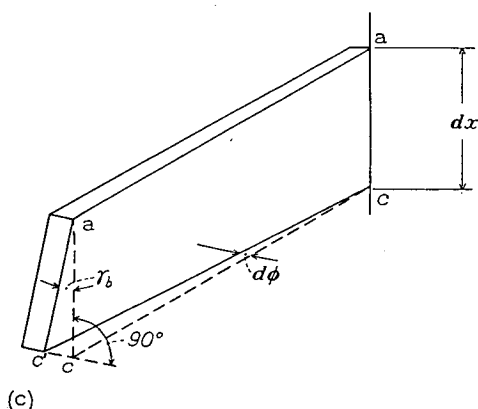
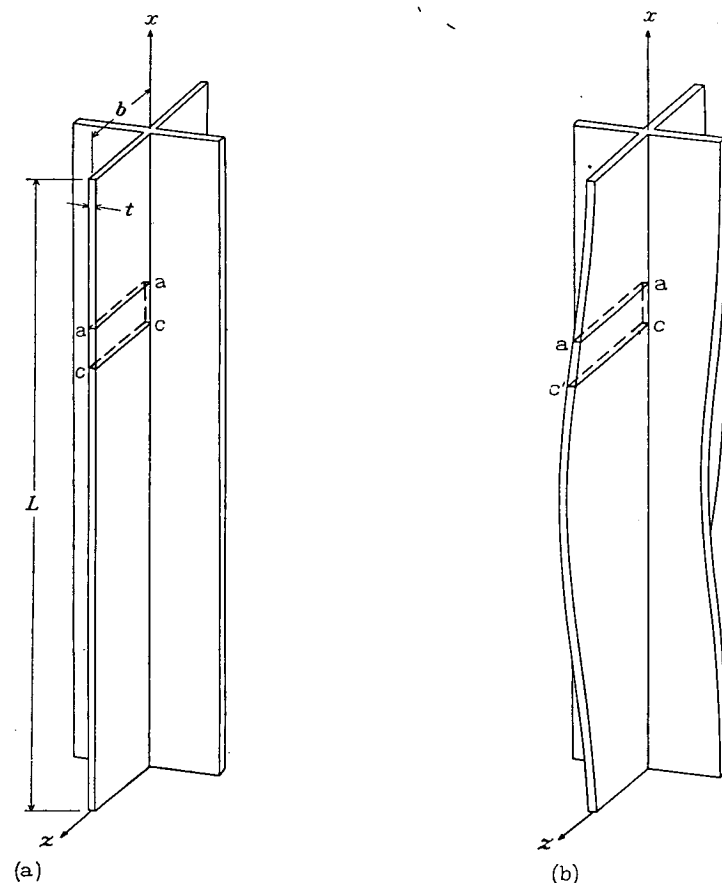
in which k^2 is a parameter lying between 0 and 1 which specifies the amount of twist,

$$K = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

is the complete elliptic integral of the first kind, and

$$m^2 = 12 \left[\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} \right] = K^2(1+k^2) \left(\frac{4t}{L} \right)^2$$

The average middle-surface strain ϵ_{av} in the elastic range is the average stress divided by the elastic modulus E .



(a) Without distortion. (b) With large distortion.
(c) Enlargement of section aac'c.

FIGURE 1.—Cruciform section, consisting of four identical flanges, before and after buckling. Coordinate system is shown on one flange.

Thus, if a value is assigned to k^2 (a certain amount of twist), both the quantities K and m^2 are determined; the strain at any point (x, z) may then be computed.

Equation (2a) may be simplified as shown in appendix A to the following expression which holds over the essentially straight part of the flange:

$$\epsilon_x = \epsilon_{av} + \frac{5}{4} (\epsilon_{av} - \epsilon_{cr}) \left(1 - 3 \frac{z^2}{b^2} \right) \quad (2b)$$

Theory also shows that over most of the flange length (except at the middle and extreme ends) the relation between the middle-surface strain at the hinge ϵ_h , the average middle-surface strain over the width of the flange ϵ_{av} , and the critical strain ϵ_{cr} is

$$\epsilon_{av} = \frac{4}{9} \epsilon_h + \frac{5}{9} \epsilon_{cr} \quad (3)$$

and the rotation ϕ_{max} at the middle of the flange is

$$\phi_{max} = \sqrt{5} \frac{t}{b} \cosh^{-1} \frac{1}{\sqrt{1 - k^2}} \quad (4a)$$

or approximately

$$\phi_{max} = 1.37 \frac{L}{b} \sqrt{\epsilon_{av} - \epsilon_{cr}} - 1.55 \frac{t}{b} \quad (4b)$$

Relations (1), (2), (3), and (4) are susceptible to experimental check, and the following section describes the results of experiments designed to test these relations.

Experimental check of strain relations.—The hinged flange shown in figure 1 was realized experimentally by the cruciform column shown under test in figure 2. The cruciform column has four identical flanges which, if equally loaded, will twist at the same time without restraint to each other; thus the condition of zero restraint against rotation is fulfilled. The columns were all sufficiently short to cause them to buckle by twisting rather than by Euler bending.

The tests included measurement of the stress-strain curve for the material from which the different groups of specimens were made, determination of the buckling and maximum load for each specimen, a study of the strain distribution across the flanges of two specimens, and a measurement of rotation of each specimen at the middle.

Results of the buckling-load measurements and their connection with the stress-strain curves for the specimens were given in reference 3 and are shown in figure 3 of this report where the buckling stress is plotted against the calculated elastic buckling strain. Because the experimental points follow along the stress-strain curve, the proper reduced modulus for pure twisting in the plastic range is concluded to be the secant modulus, which agrees with the theoretical value of reference 3.

The relation between the computed and experimental middle-surface strain distribution over the width of the flange for one specimen at the quarter height for a number of different loads is shown in figure 4. The highest average stresses exceeded the proportional limit of the material. The measured strains for the four flanges were averaged to give the points shown in the figure. These average strains

were somewhat larger than the ratio of average stress to E at the very highest loads where plasticity reduced the average effective modulus. From the experimentally observed average strain across the flange at each load and the critical strain at which buckling began, the corresponding theoretical strain distributions were computed from equation (2b) and are presented as the curves in figure 4. This calculated strain distribution agrees fairly well with that observed experimentally.

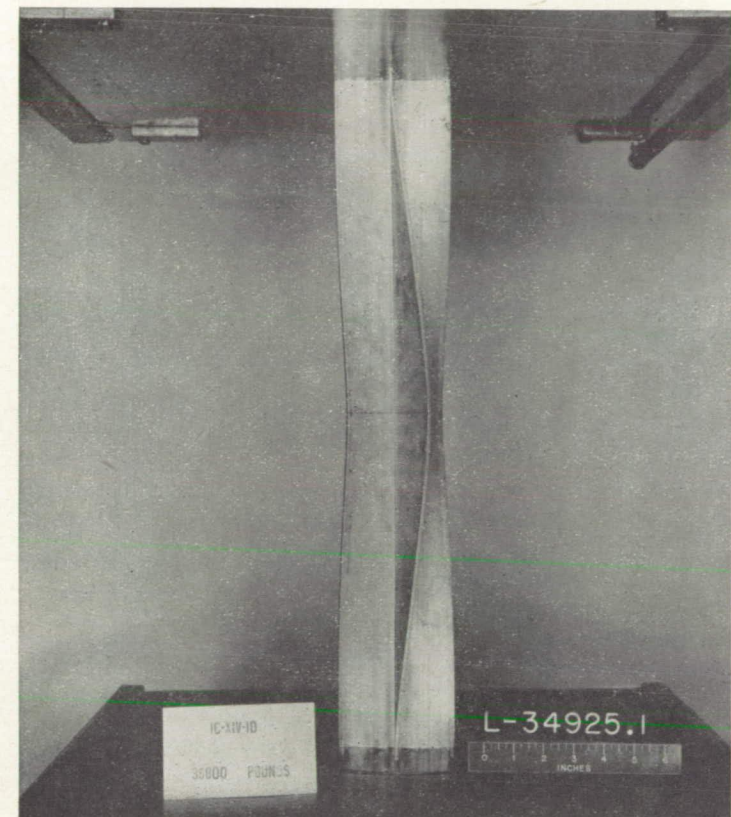


FIGURE 2.—Buckling of a cruciform section in compression.

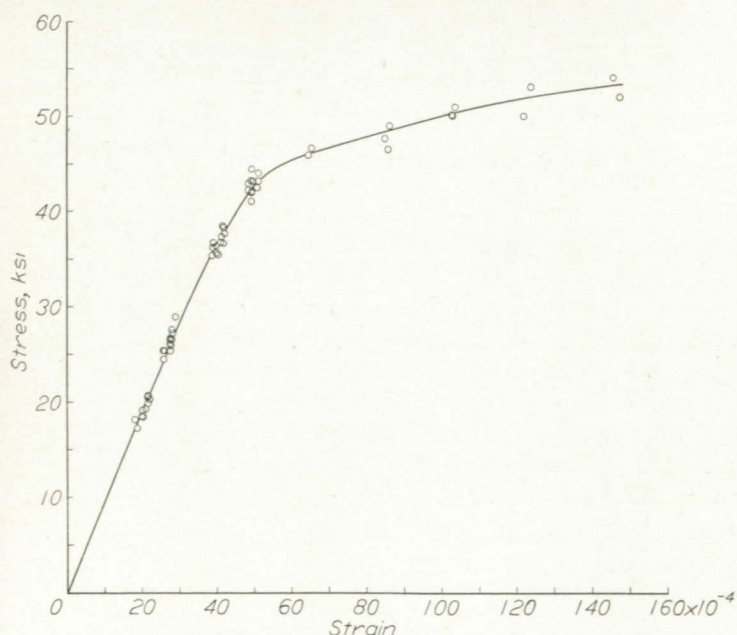


FIGURE 3.—Experimental values of the buckling stress for cruciform-section columns of 24S-T4 extruded aluminum alloy compared with the compressive stress-strain curve for that material.

The relation between average strain, corner strain, and critical strain given by equation (3) was investigated experimentally. From measurement of the strain in two opposite flanges of one buckled specimen, averages were taken to give mean values of ϵ_{av} and ϵ_h . The critical strain ϵ_{cr} was also accurately known. Figure 5 shows the theoretical relation of equation (3) compared with the averaged experimental points. The agreement is good. The strain ϵ_h ceases to be elastic at a value of 0.0025, so that both the curve and the points extend well into the plastic region. The persistence of the agreement between equation (3) and the experimental points up to the highest strains indicates that, even though equation (3) was derived on an elastic basis, it is a good approximation in the plastic region also.

Figure 6 compares the theoretical rotation of three cruciform specimens of widely different lengths with the measured rotations. The ordinate in figure 6 is the shortening δ/L , which is the hinge strain ϵ_h . Rotation was measured by a pointer attached to the flange and moving past a circular scale. Equation (4b) was used to compute the theoretical rotations. The agreement between theory and experiment is good in this case also.

NONLINEAR BEHAVIOR DUE TO PLASTICITY OF THE MATERIAL

The material of the flanges (24S-T aluminum alloy) is defined by the stress-strain curve of figure 3. The figure shows that above 25 ksi the material starts to depart from

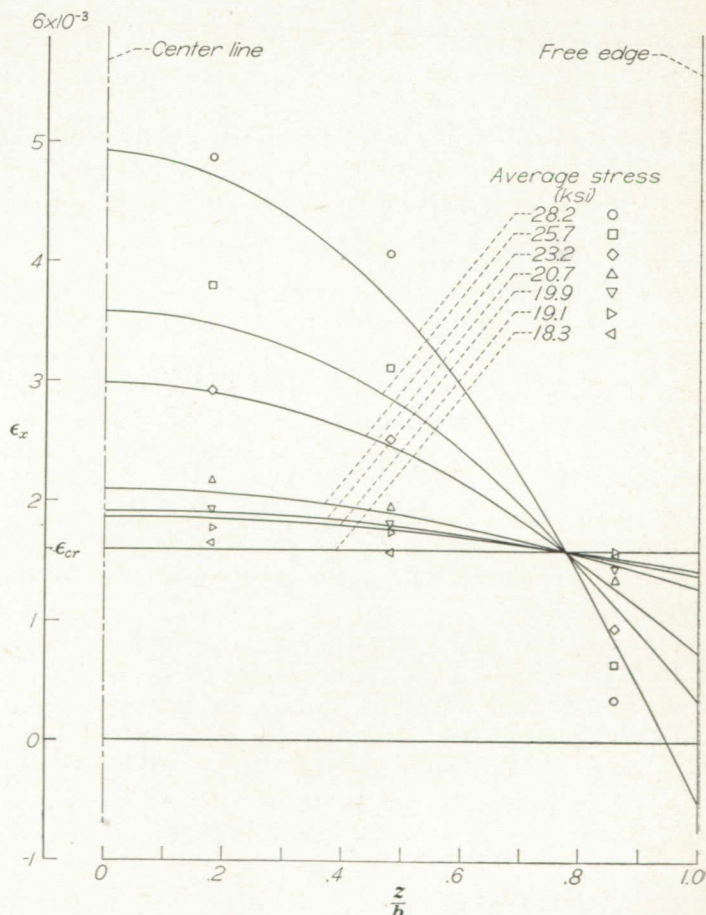


FIGURE 4.—Theoretical middle-surface strain distribution across a hinged flange at the quarter-length station along a cruciform-section column compared with experiment. (Experimental values are average for the four flanges; $\epsilon_{cr} = 0.0016$.)

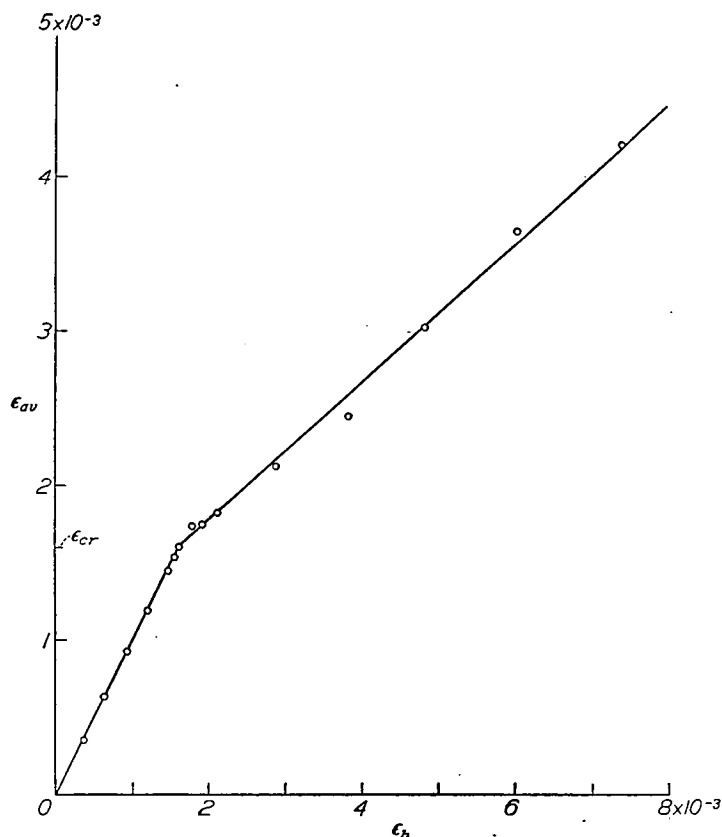


FIGURE 5.—Theoretical middle-surface strain relation between ϵ_{av} , ϵ_h , and ϵ_{cr} for a hinged flange compared with experiment. ($\epsilon_{av} = \frac{4}{9}\epsilon_h + \frac{5}{9}\epsilon_{cr}$ for $\epsilon_{av} > \epsilon_{cr}$.)

purely elastic behavior and becomes partly plastic. As a result of this plasticity, the flanges exhibit nonlinear behavior above about 25 ksi.

The most elementary consequence of the plastic nonlinear behavior is the substitution of E_{sec} for E in the formula for critical stress which, for a hinged flange, is (reference 3)

$$\sigma_{cr} = E_{sec} \epsilon_{cr} \quad (5)$$

Another consequence of the nonlinear behavior due to plasticity is the existence of a maximum load. Experimentally, as the load is increased more and more, the twist of the flange will increase until a value of load is reached at which the flange ceases to carry more load; this value is the maximum load. As was pointed out in the introduction, if the material of the flange obeyed Hooke's law strictly at all times, the rotation of the flange would increase indefinitely with increase in load. The existence of a maximum load is therefore directly attributable to plasticity of the material.

As the structure twists more and more beyond the buckling load, greater and greater shear strains are set up through the thickness of the flange. The shear strains are zero at the middle surface and have opposite signs at the faces. These shear strains will combine with the compressive strain already present to form a strain intensity; at a point where the compressive strain is ϵ_z and the shear strain is γ ,

the strain intensity is $\epsilon_i = \sqrt{\epsilon_z^2 + \frac{\gamma^2}{3}}$. (In order not to have to

consider variations of γ through the thickness, a mean value of γ^2 is used.) According to the deformation theory of

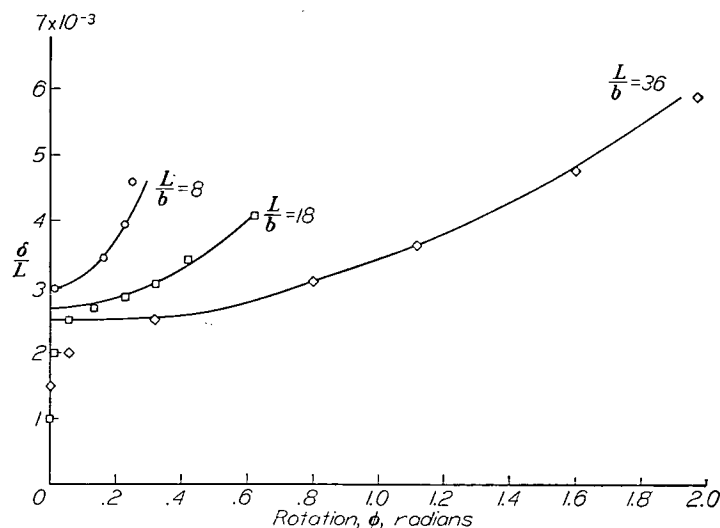


FIGURE 6.—Relation between the shortening δ/L and the rotation ϕ of a hinged flange compared with experiment. $\frac{b}{t} = 12$.

plasticity used herein (reference 3), the value of ϵ_i at any stage of deformation determines the reduced modulus of the material at that stage.

Since the maximum load always occurs at a finite rotation of the flange, the two effects of nonlinearity must be combined in order to account for the maximum load. Such a combination is effected in appendix B and the results are given in the following section.

MAXIMUM LOAD OF A FLANGE

It is shown in appendix B how the maximum load on a hinged flange may be computed from the dimensions of the flange and the stress-strain curve for the material.

The middle-surface strain distribution across the flange is given by equation (2a). In addition to these strains which arise directly from the compressive load, there are also shear strains in the flange due to its twist. These shear strains become as large as two-thirds of the compressive strains upon which they are superposed. Although, strictly speaking, the deformation theory of plasticity has only been shown to hold for simple loading (reference 4), its validity is also assumed herein for complex loading. The square of the compressive strains and the mean square of the shear strains were added in the proper manner to give a strain intensity. (The highly localized effects of bending at the middle and ends have been neglected.) From the compressive stress-strain curve for the material the value of the secant modulus E_{sec} was read at this strain intensity. For increasing strain intensities the compressive stress σ at any point across the width of the flange is then simply E_{sec} times the compressive strain at the point. Near the free edge the strain intensity decreases; in such a case, the elastic modulus E is used to compute the corresponding stress reduction. The average stress σ_{av} across the flange is then

$$\sigma_{av} = \frac{1}{b} \int_0^b \sigma dz \quad (6)$$

The value of σ_{av} is computed for a number of different twists until a maximum average stress σ_{max} is found.

Figure 7 shows the results plotted in a nondimensional form similar to that employed in reference 2. The parameters used have some theoretical justification and have the effect of making the information given by the plot largely independent of the material. The agreement between the computed curve and the experimental points for cruciform-section columns is satisfactory.

The fact that maximum loads may be computed in this case solely on the basis of deformation theory suggests that the theory is sufficiently accurate when the stress state changes from pure compression to combined compression and shear, for shear strains up to two-thirds of the largest compressive strains.

An interesting side light on this computation is revealed by the values of stress intensity at the supported edge when the load is a maximum. The stress intensity for eight widely different cruciforms is a constant, to about 1 percent, equal to about 47 ksi. (See table 1.) This value is close to the yield stress for the material (46 ksi).

When the flanges are present in actual structures, they are generally connected to other members which offer a certain elastic restraint against rotation along the supported edge. The question arises as to what effect this connection has upon the calculations based on the assumption of a hinge connection. The elastic restraint along the supported edge will have two major effects: The critical strain will be appreciably raised and the effective length L of the buckles will be appreciably shortened. A necessary consequence is that the rotation (which is proportional to L) is reduced

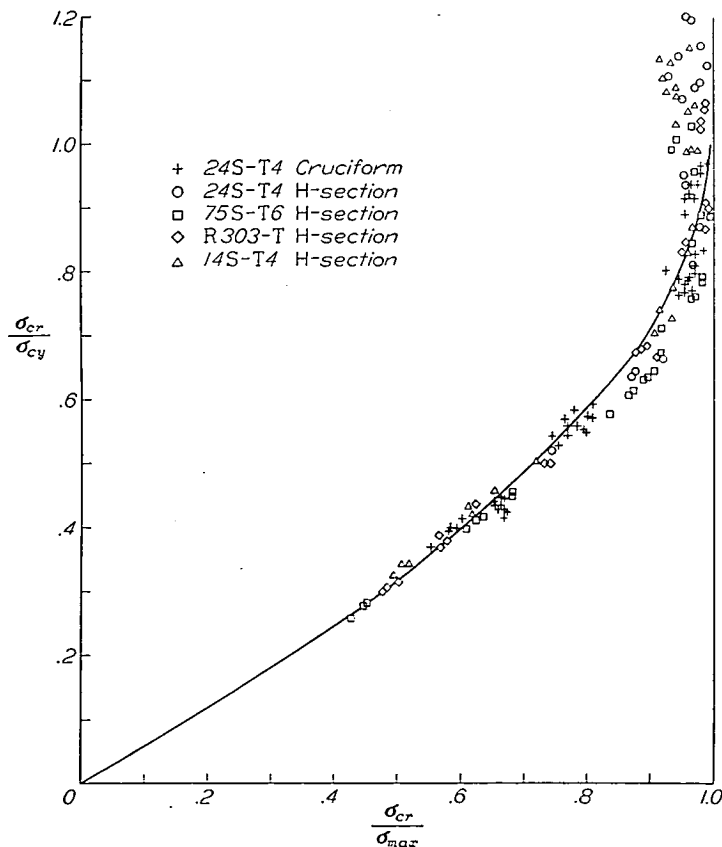


FIGURE 7.—Comparison of theoretical curve for the maximum strength of 24S-T4 aluminum alloy cruciforms with test results. Compressive yield stress σ_{cy} = 46 ksi. (Experimental values for H-sections of various aluminum alloys have been added for comparison with the theoretical curve.)

and, therefore, is more nearly of the shape of a circular sine along the length of the flange than it would be when a hinge is present along the joint. A third effect is the introduction of a slight curvature across the width of the flange. When the revised critical strain and the revised length are inserted into the formulas of appendix A, which were derived for a flange supported along a hinge, it is found that the rotation and the strain relations may still be accurately predicted for flanges with restraint along the supported edge. Such a result seems to indicate that the small amount of transverse curvature introduced by the restraint does not have an important effect on the formulas.

In view of the fact that the theory of appendix A applies fairly well to flanges with restrained edges, it might be expected that the maximum strength, also, might be given by the same theory. Experiment shows that such is the case; the values of maximum strength for H-sections are included in the experimental points shown in figure 7 and the points intermingle with the cruciform points such that one set cannot be distinguished from the other. The theory of this report may then be said to apply approximately to flanges with elastic restraint along one side edge as well as to flanges without elastic restraint.

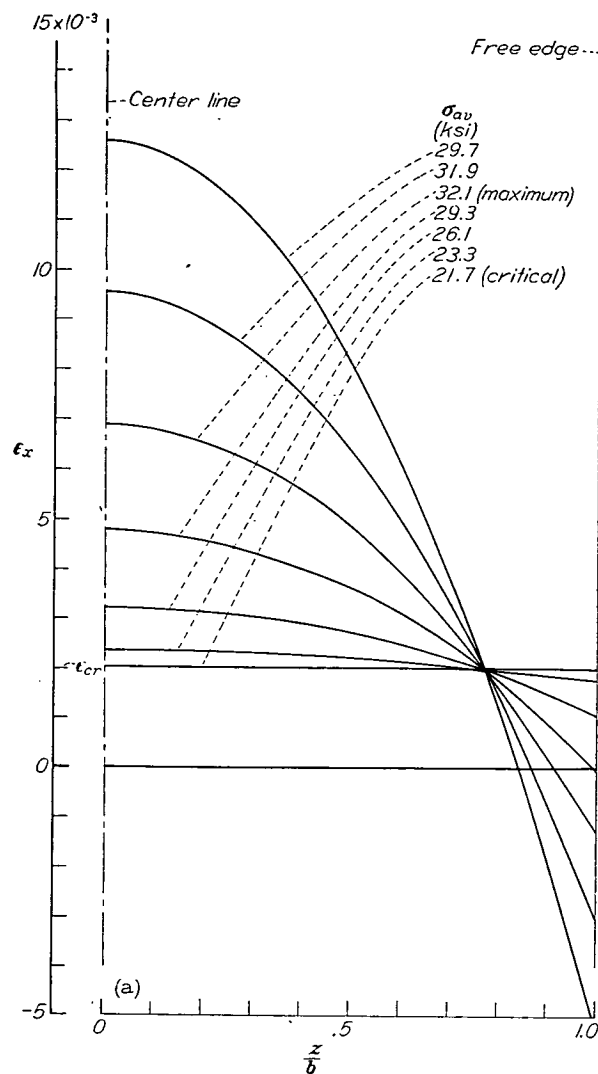
CAUSE OF MAXIMUM LOAD

Maximum load occurs when it is no longer possible for the stress, on the average, to grow with increasing strain. The natural tendency for the stress to grow is defeated by the decrease in effective modulus.

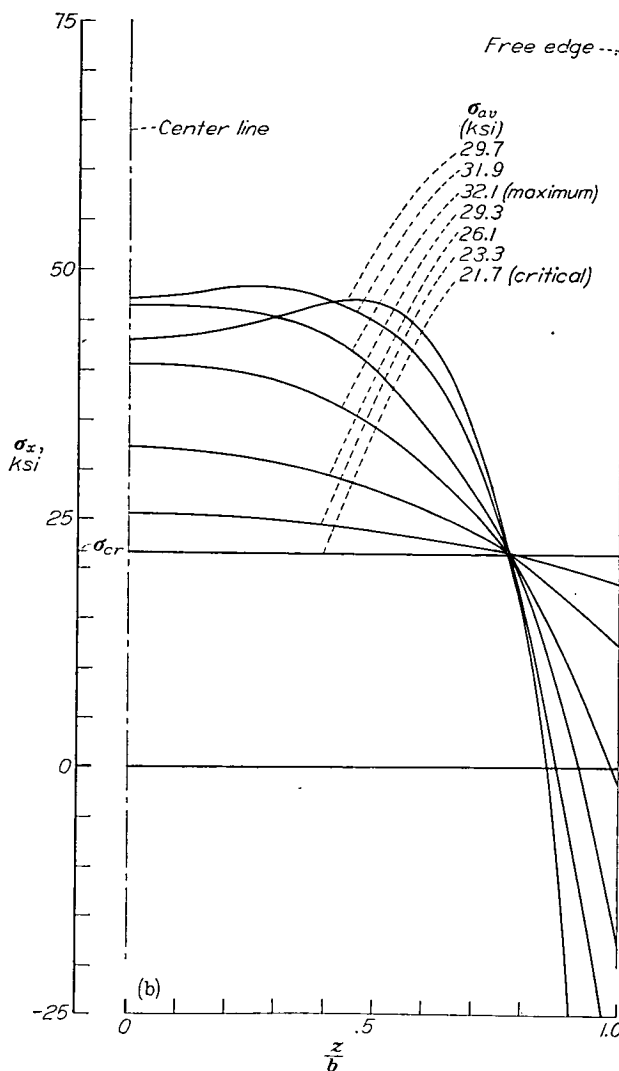
In order to illustrate this effect graphically, figures 8(a) and 8(b) have been prepared. These figures illustrate the calculated strain and stress distributions across a hinged flange of 24S-T4 aluminum alloy and of proportions $\frac{b}{t}=14$ and $\frac{L}{b}=12$. These distributions hold over the greater part of the flange where the bending is negligible. Up to the critical strain of 0.002 and the critical stress of 21.5 psi, the distributions are uniform. As the load is increased beyond the critical value, the distributions become more and more nonuniform as a result of twisting of the flange. With increasing load, the strain increases faster at the hinge than at the middle of the flange as shown in figure 8(a). For a time, the corresponding stress also increases faster at the hinge than at the middle of the flange, as shown in figure 8(b). Eventually, however, the strain intensity at the hinge (averaged over the thickness) becomes so large that the modulus is greatly reduced. When that occurs, the stress at the hinge line ceases to grow with increase in strain and even

TABLE 1.—SHOWING CONSTANCY OF STRESS INTENSITY AT HINGE LINE AT MAXIMUM LOAD

Specimen			At failure	
b/t	L/b	σ_{cr} (ksi)	σ_{max} (ksi)	σ_i (ksi)
8	12	45.9	45.7	46.6
9	18	40.6	40.0	45.9
10	4	44.8	44.0	47.5
10	10	37.6	38.0	47.5
11	10	33.4	36.2	47.1
12	4	37.3	39.2	46.6
13	10	25.8	31.5	48.2
14	12	21.7	31.3	48.2



(a) Strain distribution.



(b) Stress distribution.

FIGURE 8.—Theoretical middle-surface strain and stress distribution across a flange at the quarter-length station along a typical cruciform. $\frac{b}{t} = 14$; $\frac{L}{b} = 12$.

starts to decrease (see fig. 8(b)). The maximum area under the stress curve, and therefore the maximum load, occurs just as the hinge stress starts to recede.

CONCLUSIONS

A theoretical analysis of the compressive strength of flanges, based on a deformation theory of plasticity combined with the theory for finite deflections for this structure, and comparison with experimental data lead to the following conclusions:

1. The maximum load for a flange under compression and hinged along one edge may be accurately computed from the dimensions of the flange and the compressive stress-strain curve for the material.

2. Maximum load occurs when, because of the onset of plasticity, the effective modulus has been reduced to such a low value that it is no longer possible for the average stress to increase with increasing strain. Failure is not a local

phenomenon but is an integrated effect over the cross section of the flange.

3. For a wide variety of cruciform sections, the stress intensity (averaged over the thickness) along the hinge line at maximum load is a constant to about 1 percent. This value of stress intensity is very close to the yield stress for the material.

4. The fact that maximum loads may be computed in this case suggests that the deformation theory of plasticity is sufficiently accurate when the stress state changes from compression to combined compression and shear in the case when the shear strains are less than about two-thirds of the compressive strains.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., December 9, 1949.

APPENDIX A

FINITE DEFLECTION THEORY FOR A HINGED FLANGE UNDER COMPRESSION

ELLIPTIC-FUNCTION SOLUTION

The coordinate system and dimensions of the hinged flange (one-fourth of a cruciform-section column) are shown in figure 1 (a); the form of the distorted shape is shown in figure 1 (b). The fundamental hypothesis of the calculation is that at any section $x = \text{Constant}$ there is no curvature of the flange in the direction of z . The correctness of this hypothesis is amply borne out by tests on the flanges while under twist. With this assumption it becomes possible to avoid a formalized plate treatment of the problem.

For infinitesimal rotations, the differential equation of equilibrium for a column under the simultaneous action of a compressive stress σ and torque T has been shown by Wagner (reference 5) to be

$$(GJ - \sigma I_p) \frac{d\phi}{dx} - EC_{BT} \frac{d^3\phi}{dx^3} = T \quad (\text{A1})$$

where

$GJ \frac{d\phi}{dx}$ St. Venant component of internal resisting torque

$\sigma I_p \frac{d\phi}{dx}$ component of internal torque due to application of compressive force. (This component is not a resisting torque but aids the applied torque T in twisting the column; its sign is therefore negative.)

$-EC_{BT} \frac{d^3\phi}{dx^3}$ component of internal resisting torque due to bending of column as it twists

For the case in which the applied torque T is zero, such as for a compressed hinged flange, equation (A1) becomes

$$(GJ - \sigma I_p) \frac{d\phi}{dx} - EC_{BT} \frac{d^3\phi}{dx^3} = 0 \quad (\text{A2})$$

As previously mentioned, equations (A1) and (A2) are limited to infinitesimal rotations and thus cannot be used to determine the behavior of a column above the buckling load where rotations may become large.

In order to investigate the behavior of a compressed hinged flange above buckling, a theory which permits the calculation of the large deformations which may occur after buckling must be employed. The differential equation (A2) must therefore be amended to include the effects which appear at finite values of the rotation ϕ .

DERIVATION OF THE BASIC DIFFERENTIAL EQUATION FOR FINITE ROTATIONS

The effects of finite rotation involve the changes in the middle-surface strain that occur after buckling. As the plate twists, the longitudinal fibers will be inclined at a small angle to the hinge line as shown in figure 1 (b). As a result, the longitudinal fibers are stretched in varying amounts and the horizontal components of the forces along the fibers produce a torque which resists twisting of the plate. The resisting torque increases very rapidly with twisting of the plate, which thus becomes progressively stiffer. The rapid increase of stiffness with rotation provides the required mechanism for maintaining the rotation at a finite value.

Stretching of middle-surface fibers after buckling.—A short section of the plate as shown in figure 1 (c) will have the length ac before the plate buckles. After buckling, the length ac' will be greater than ac because ac' is inclined at an angle γ_b with the hinge line. (See fig. 1 (c).) Thus the strain at the free edge ϵ_b due to stretching for small values of γ_b is

$$-\epsilon_b = \frac{ac' - ac}{ac} = \sec \gamma_b - 1 \approx \frac{\gamma_b^2}{2} \quad (\text{A3})$$

(The strain ϵ_b is positive when compressive.)

If the line aa (fig. 1 (c)) has been rotated an angle ϕ from its original position, the free-edge fiber at c moves a distance $b(\phi + d\phi)$. The angle of inclination of the free-edge fiber is thus

$$\gamma_b = \frac{b(\phi + d\phi) - b\phi}{dx} = b \frac{d\phi}{dx} \quad (\text{A4})$$

If the point c is not at the free edge but at some interior position a distance z from the hinge line, it can similarly be shown that

$$-\epsilon_z = \frac{\gamma_z^2}{2} \quad (\text{A5})$$

$$\gamma_z = z \frac{d\phi}{dx} \quad (\text{A6})$$

From equations (A5) and (A6), the strain ϵ_z resulting from the stretching action can be given as

$$\epsilon_z = -\frac{z^2}{2} \left(\frac{d\phi}{dx} \right)^2 \quad (\text{A7})$$

Equation (A7) gives the difference between the hinge-line strain and the strain at any fiber due to the stretching action, for a given position along the width of the flange. It is this

difference which causes the middle-surface strain distribution after buckling to differ from the uniform strain distribution at the instant of buckling and which will now be considered.

Middle-surface strain distribution after buckling.—A compressive load P applied to the hinged flange will cause the ends to approach each other by a distance δ . The unit shortening e is δ/L . Equilibrium of the internal compressive forces with the applied force P requires that

$$P = tE \int_0^b (e + \epsilon_z) \cos \gamma_z dz \quad (\text{A8})$$

The angle γ_z is usually so small that $\cos \gamma_z$ may be taken as unity. Then, substituting the expression for ϵ_z from equation (A7) into equation (A8) yields

$$P = tE \int_0^b \left[e - \frac{z^2}{2} \left(\frac{d\phi}{dx} \right)^2 \right] dz = EA \left[e - \frac{b^2}{6} \left(\frac{d\phi}{dx} \right)^2 \right] \quad (\text{A9})$$

The unit shortening e is therefore

$$e = \frac{P}{AE} + \frac{b^2}{6} \left(\frac{d\phi}{dx} \right)^2 \quad (\text{A10})$$

The ratio P/AE is the average strain over the cross section. If P/AE is denoted by ϵ_{av} , equation (A10) becomes

$$e = \epsilon_{av} + \frac{\gamma_b^2}{6} \quad (\text{A11})$$

The longitudinal middle-surface strain ϵ_x at any fiber z in the cross section is therefore

$$\epsilon_x = e + \epsilon_z = \epsilon_{av} + \frac{\gamma_b^2}{6} - \frac{z^2}{2} \left(\frac{d\phi}{dx} \right)^2 \quad (\text{A12})$$

Moment due to axial stress after buckling.—The longitudinal strain ϵ_x does not have the direction of the hinge line but of the slightly inclined longitudinal fibers (the angle γ_z , equation (A6)). Consequently, $E\epsilon_x$ has components perpendicular to the hinge line which create the moment ΔM resulting from the applied compressive force.

The component of $E\epsilon_x$ perpendicular to the hinge line at any fiber z is $E\epsilon_x \sin \gamma_z$ and for small angles is approximately equal to $E\epsilon_x z \frac{d\phi}{dx}$. As this component has a lever arm z , the internal resisting moment ΔM is

$$\Delta M = -Et \int_0^b \left(\epsilon_x z \frac{d\phi}{dx} \right) z dz \quad (\text{A13})$$

Substituting the expression for ϵ_x from equation (A12) into equation (A13) results in the following relationship:

$$\Delta M = -\sigma I_p \frac{d\phi}{dx} + \frac{2}{15} E b^2 I_p \left(\frac{d\phi}{dx} \right)^3 \quad (\text{A14})$$

The term $\sigma I_p \frac{d\phi}{dx}$ is the same term that appears in equation (A2). The last term of equation (A14) is the required additional term which takes into account the stretching actions, which occur for finite rotations of the flange, and permits the computation of the rotation ϕ .

Basic differential equation of torque for a compressed hinged flange which includes the effects of finite rotation.—The complete differential equation of torque which replaces equation (A2) and includes the last term of equation (A14) is

$$(GJ - \sigma I_p) \frac{d\phi}{dx} - EC_{BT} \frac{d^3\phi}{dx^3} + \frac{2}{15} E b^2 I_p \left(\frac{d\phi}{dx} \right)^3 = 0 \quad (\text{A15})$$

The constants of equation (A15) are

$$\left. \begin{aligned} J &= \frac{bt^3}{3} \\ I_p &= \frac{b^3t}{3} \\ C_{BT} &= \frac{b^3t^3}{36} \\ G &= \frac{E}{2(1+\mu)} \end{aligned} \right\} \quad (\text{A16})$$

Substituting equations (A16) into (A15) yields

$$\frac{d^3\phi}{dx^3} + \frac{12}{t^2} \left[\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} \right] \frac{d\phi}{dx} - \frac{8}{5} \left(\frac{b}{t} \right)^2 \left(\frac{d\phi}{dx} \right)^3 = 0 \quad (\text{A17})$$

A further simplification is effected by the use of

$$\left. \begin{aligned} \gamma_b &= b \frac{d\phi}{dx} \\ \xi &= \frac{x}{t} \\ m^2 &= 12 \left[\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} \right] \end{aligned} \right\} \quad (\text{A18})$$

The substitution of relations (A18) into equation (A17) gives the basic differential equation for a compressed hinged flange

$$\frac{d^2\gamma_b}{d\xi^2} + m^2\gamma_b - \frac{8}{5}\gamma_b^3 = 0 \quad (\text{A19})$$

SOLUTION OF THE BASIC DIFFERENTIAL EQUATION FOR FINITE ROTATION

The basic differential equation (A19) has the solution

$$\xi + \xi_0 = \pm \int \frac{d\gamma_b}{\sqrt{c^2 - m^2\gamma_b^2 + \frac{4}{5}\gamma_b^4}} \quad (\text{A20})$$

where c and ξ_0 are constants of integration. (The sign of

the radical must be chosen so as to keep $d\xi$ positive.) With the condition that $\gamma=0$ for $\xi=0$ ($\frac{d\phi}{dx}$ equals zero at the ends) and the substitutions

$$\left. \begin{aligned} g^2 &= \frac{m^2}{2} + \sqrt{\frac{m^4}{4} - \frac{4}{5}c^2} \\ h^2 &= \frac{m^2}{2} - \sqrt{\frac{m^4}{4} - \frac{4}{5}c^2} \end{aligned} \right\} \quad (\text{A21})$$

equation (A20) may be written

$$\xi = \int_0^{\gamma_b} \frac{d(\gamma_b/c)}{\sqrt{[1-g^2(\gamma_b/c)^2][1-h^2(\gamma_b/c)^2]}} \quad (\text{A22})$$

With new variables Ψ and k defined by

$$\left. \begin{aligned} \frac{1}{g} \sin \Psi &= \frac{\gamma_b}{c} \\ k &= \frac{h}{g} \end{aligned} \right\} \quad (\text{A23})$$

equation (A22) is transformed into

$$\xi = \frac{1}{g} \int_0^{\Psi} \frac{d\Psi}{\sqrt{1-k^2 \sin^2 \Psi}} \quad (\text{A24})$$

In order to determine the constant c , use is made of the condition that $\gamma_b=0$ for $\xi=\frac{L}{2t}$ or $x=\frac{L}{2}$. The upper limit for equation (A24) corresponding to $x=\frac{L}{2}$ must then be $\Psi=\pi$ in order to satisfy the first of equations (A23):

$$\begin{aligned} \frac{L}{2t} &= \frac{1}{g} \int_0^{\pi} \frac{d\Psi}{\sqrt{1-k^2 \sin^2 \Psi}} \\ \text{or} \quad g &= \frac{4t}{L} \int_0^{\pi/2} \frac{d\Psi}{\sqrt{1-k^2 \sin^2 \Psi}} \end{aligned} \quad (\text{A25})$$

In elliptic-function notation,

$$\left. \begin{aligned} \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1-k^2 \sin^2 \alpha}} &= K(k) \\ \int_0^{\Psi} \frac{d\alpha}{\sqrt{1-k^2 \sin^2 \alpha}} &= \text{sn}^{-1}(\sin \Psi) = \text{sn}^{-1}\left(\frac{g\gamma_b}{c}\right) \end{aligned} \right\} \quad (\text{A26})$$

Equation (A25) therefore may be written

$$g = \frac{4Kt}{L} \quad (\text{A27})$$

and equation (A24) becomes

$$\frac{4Kx}{L} = \int_0^{\Psi} \frac{d\alpha}{\sqrt{1-k^2 \sin^2 \alpha}} = \text{sn}^{-1}\left(\frac{g\gamma_b}{c}\right)$$

so that taking the elliptic sine of both sides gives

$$\gamma_b = \frac{c}{g} \text{sn}\left(\frac{4Kx}{L}\right) \quad (\text{A28})$$

The coefficient c/g is readily found from the definitions of g and h in equations (A21). From the first of these equations

$$h = \sqrt{m^2 - g^2} = \sqrt{12} \sqrt{\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} - \frac{g^2}{12}}$$

and from the second

$$\frac{c}{g} = \frac{\sqrt{5}}{2} h = \sqrt{15} \sqrt{\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} - \frac{g^2}{12}}$$

Making use of equation (A27) leads to the general solution

$$\gamma_b = \sqrt{15} \sqrt{\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} - \frac{1}{12} \left(\frac{4Kt}{L}\right)^2} \text{sn}\left(\frac{4Kx}{L}\right) \quad (\text{A29})$$

Another form of the solution which is sometimes convenient may be obtained by using a different expression for h . Since addition of equations (A21) gives

$$m^2 = g^2 + h^2 = g^2(1+k^2) = \left(\frac{4Kt}{L}\right)^2 (1+k^2) \quad (\text{A30})$$

it follows that

$$g = \frac{m}{\sqrt{1+k^2}}$$

and that

$$h = kg = \frac{km}{\sqrt{1+k^2}}$$

Hence,

$$\gamma_b = \frac{\sqrt{5}}{2} h \text{sn}\left(\frac{4Kx}{L}\right) = \frac{\sqrt{5}}{2} \frac{km}{\sqrt{1+k^2}} \text{sn}\left(\frac{4Kx}{L}\right) \quad (\text{A31})$$

With either equation (A29) or equation (A31) now available as an accurate expression for the fiber slope γ_b , it is possible, besides checking the known formula for the critical compressive stress, to write formulas also for the rotation at any station along the flange, for the middle-surface strain distribution along and across the flange, for the relation between hinge-line, average, and critical strains, and for the fractional shortening. The formulas will now be given in that order.

Check of critical compressive stress.—In order to show that equations (A29) or (A31) give the correct buckling stress at the start of the rotation ($\gamma_b=0$), the behavior of the elliptic function is considered as the rotation approaches zero.

The preceding section showed that the angle γ_b is proportional to h , and therefore to k . As k approaches zero, K approaches $\pi/2$, and the elliptic sine approaches the circular sine. Hence for loads only slightly above the critical, from equation (A29),

$$\gamma_b = \sqrt{15} \sqrt{\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} - \frac{1}{12} \left(\frac{2\pi t}{L} \right)^2} \sin \frac{2\pi x}{L}$$

At the critical load, $\gamma_b = 0$, and for loaded edges clamped,

$$(\epsilon_{av})_{\gamma_b=0} = \epsilon_{cr} = \frac{(t/b)^2}{2(1+\mu)} + \frac{1}{12} \left(\frac{2\pi t}{L} \right)^2 \quad (\text{A32})$$

This is the expression given as equation (1) in the body of the report. The critical compressive stress σ_{cr} is obtained by multiplying both sides of equation (A32) by the effective modulus in compression E_{sec} . Then

$$\sigma_{cr} = E_{sec} \epsilon_{cr} = E_{sec} \left[\frac{(t/b)^2}{2(1+\mu)} + \frac{1}{12} \left(\frac{2\pi t}{L} \right)^2 \right] \quad (\text{A33})$$

Since

$$m = g \sqrt{1+k^2} = \frac{4Kt}{L} \sqrt{1+k^2}$$

the general integral becomes

$$\begin{aligned} \phi &= \frac{\sqrt{5}}{2} \frac{t}{b} \left[\cosh^{-1} \frac{1}{\sqrt{1-k^2}} - \cosh^{-1} \frac{\sqrt{1-k^2} \operatorname{sn}^2(4Kx/L)}{\sqrt{1-k^2}} \right] & \left(\frac{1}{4} \leq \frac{x}{L} \leq 0 \right) \\ \phi &= \frac{\sqrt{5}}{2} \frac{t}{b} \left[\cosh^{-1} \frac{1}{\sqrt{1-k^2}} + \cosh^{-1} \frac{\sqrt{1-k^2} \operatorname{sn}^2(4Kx/L)}{\sqrt{1-k^2}} \right] & \left(\frac{3}{4} \leq \frac{x}{L} \leq \frac{1}{4} \right) \\ \phi &= \frac{\sqrt{5}}{2} \frac{t}{b} \cosh^{-1} \frac{1}{\sqrt{1-k^2}} & \left(\frac{x}{L} = \frac{1}{4} \text{ or } \frac{x}{L} = \frac{3}{4} \right) \\ \phi_{max} &= \sqrt{5} \frac{t}{b} \cosh^{-1} \frac{1}{\sqrt{1-k^2}} & \left(\frac{x}{L} = \frac{1}{2} \right) \end{aligned}$$

Variation of middle-surface strain over length and width of flange.—The middle-surface strain ϵ_x at any distance z from the hinge line was given in equation (A12) as

$$\epsilon_x = \epsilon_{av} + \frac{\gamma_b^2}{6} \left(1 - 3 \frac{z^2}{b^2} \right)$$

The slope of the free-edge fiber γ_b may now be inserted in this expression from either equation (A29) or equation (A31). Thus

$$\epsilon_x = \epsilon_{av} + \frac{5}{2} \left[\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} - \frac{1}{12} \left(\frac{4Kt}{L} \right)^2 \right] \left(1 - 3 \frac{z^2}{b^2} \right) \operatorname{sn}^2 \left(\frac{4Kx}{L} \right) \quad (\text{A36a})$$

or

$$\epsilon_x = \epsilon_{av} + \frac{5}{24} \frac{k^2 m^2}{1+k^2} \left(1 - 3 \frac{z^2}{b^2} \right) \operatorname{sn}^2 \left(\frac{4Kx}{L} \right) \quad (\text{A36b})$$

or

$$\epsilon_x = \frac{\left(\frac{t}{b} \right)^2}{2(1+\mu)} + \frac{m^2}{12} + \frac{5}{24} \frac{k^2 m^2}{1+k^2} \left(1 - 3 \frac{z^2}{b^2} \right) \operatorname{sn}^2 \left(\frac{4Kx}{L} \right) \quad (\text{A36c})$$

This is the expression given as equation (5) in the body of the report.

Rotation of any station along the flange.—By symmetry, the rotation of any station a distance x from either end is given by

$$\phi = \frac{L}{b} \int \gamma_b d \left(\frac{x}{L} \right) \quad (\text{A34})$$

and so is obtained by a simple integration of the fiber angle distribution along the length of the flange, subject to the condition that the rotation is zero at both ends of the flange.

If an analytical expression for ϕ is desired, either equation (A29) or its alternate (A31) may be integrated. Integration of equation (A31) gives

$$\begin{aligned} \phi &= \frac{\sqrt{5}}{2} \frac{km}{\sqrt{1+k^2}} \frac{L}{4bK} \int_0^{4Kx/L} \operatorname{sn} \left(\frac{4Kx}{L} \right) d \left(\frac{4Kx}{L} \right) \\ &= \frac{\sqrt{5}}{2} \frac{m}{\sqrt{1+k^2}} \frac{L}{4bK} \left[\cosh^{-1} \frac{1}{\sqrt{1-k^2}} - \cosh^{-1} \frac{\sqrt{1-k^2} \operatorname{sn}^2(4Kx/L)}{\sqrt{1-k^2}} \right] \end{aligned} \quad (\text{A35})$$

Relationships between hinge-line, average, and critical strains.—Along the hinge line $z=0$, and equations (A36) give

$$(\epsilon_x)_{z=0} = \epsilon_{av} + \frac{5}{2} \left[\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} - \frac{1}{12} \left(\frac{4Kt}{L} \right)^2 \right] \text{sn}^2 \left(\frac{4Kx}{L} \right)$$

or

$$(\epsilon_x)_{z=0} = \epsilon_{av} + \frac{5}{24} \frac{k^2 m^2}{1+k^2} \text{sn}^2 \left(\frac{4Kx}{L} \right)$$

Along the hinge line at $x = \frac{L}{4}$ and $x = \frac{3L}{4}$,

$$(\epsilon_x)_{z=0} = \frac{7}{2} \epsilon_{av} - \frac{5}{2} \left[\frac{(t/b)^2}{2(1+\mu)} + \frac{1}{12} \left(\frac{4Kt}{L} \right)^2 \right] \quad (\text{A37a})$$

or

$$(\epsilon_x)_{z=0} = \epsilon_{av} + \frac{5}{24} \frac{k^2 m^2}{1+k^2} \quad (\text{A37b})$$

Along the hinge line at $x=0$, $x = \frac{L}{2}$, and $x=L$,

$$(\epsilon_x)_{z=0} = \epsilon_{av}$$

Thus at the ends and at the middle the strain is uniformly distributed across the width of the hinged flanges.

Fractional shortening.—The fractional shortening of the flange δ/L (1/4 of its length is considered for convenience) is

$$\begin{aligned} e &= \frac{\delta}{L} \\ &= \frac{4}{L} \int_0^{L/4} (\epsilon_x)_{z=0} dx \\ &= \frac{1}{K} \int_0^K (\epsilon_x)_{z=0} d \left(\frac{4Kx}{L} \right) \\ &= \epsilon_{av} + \frac{5}{24} \frac{k^2 m^2}{1+k^2} \frac{1}{K} \int_0^K \text{sn}^2 \left(\frac{4Kx}{L} \right) d \left(\frac{4Kx}{L} \right) \\ &= \epsilon_{av} + \frac{5}{24} \frac{m^2}{1+k^2} \left(1 - \frac{\bar{E}}{K} \right) \end{aligned}$$

where $\bar{E} = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 \alpha} d\alpha$. From equation (A30)

$m = \frac{4Kt}{L} \sqrt{1+k^2}$, and by use of this value of m ,

$$e = \frac{\delta}{L} = \epsilon_{av} + \frac{10}{3} \left(\frac{t}{L} \right)^2 K(K - \bar{E}) \quad (\text{A38})$$

APPROXIMATE RELATIONSHIPS FOR POSTBUCKLING BEHAVIOR

The preceding relationships for the behavior of a hinged flange when compressed beyond the buckling load may be greatly simplified if the flange is long enough so that bending is negligible compared with the twist. Under such conditions the term $EC_{BT} \frac{d^3 \phi}{dx^3}$ in the differential equation (A15) may be neglected. The basic differential equation (A19) then reduces to a simple algebraic equation

$$m^2 \gamma_b - \frac{8}{5} \gamma_b^3 = 0$$

The fiber angle γ_b .—Solutions of the preceding equation are

$$\gamma_b = 0$$

and

$$\gamma_b = \pm \sqrt{\frac{5}{8}} m = \pm \sqrt{\frac{15}{2}} \sqrt{\epsilon_{av} - \epsilon_{cr}} \quad (\text{A39})$$

in which $\epsilon_{cr} = \frac{(t/b)^2}{2(1+\mu)}$, the term in length now being omitted.

The same quantities which were computed under "Elliptic Function Solution" may now be expressed once more in terms of the approximate solution for γ_b .

Rotation of flange.—The approximate rotation will be the integral of the approximate value of γ_b , or

$$\phi = \sqrt{\frac{15}{2}} \sqrt{\epsilon_{av} - \epsilon_{cr}} \left(\frac{x}{L} \right) \left(\frac{L}{b} \right) \quad \left(\frac{1}{2} \geq \frac{x}{L} \geq 0 \right)$$

and

$$\phi = \sqrt{\frac{15}{2}} \sqrt{\epsilon_{av} - \epsilon_{cr}} \left(1 - \frac{x}{L} \right) \left(\frac{L}{b} \right) \quad \left(1 \geq \frac{x}{L} \geq \frac{1}{2} \right)$$

A reference to figure 2 shows that the distribution of the angle ϕ is nearly linear for large rotations.

The maximum value of ϕ is $\frac{1}{2} \sqrt{\frac{15}{2}} \sqrt{\epsilon_{av} - \epsilon_{cr}} \left(\frac{L}{b} \right)$ or

$$\phi_{max} = 1.37 \frac{L}{b} \sqrt{\epsilon_{av} - \epsilon_{cr}} \quad (\text{A40})$$

A second approximation, which contains a small correction term to equation (A40), may be found from the relations

$$\lim_{k \rightarrow 1} \cosh^{-1} \frac{1}{\sqrt{1-k^2}} = \log \frac{2}{\sqrt{1-k^2}}$$

and

$$\lim_{k \rightarrow 1} K = \log \frac{4}{\sqrt{1-k^2}}$$

Since

$$\log \frac{2}{\sqrt{1-k^2}} = \log \frac{4}{\sqrt{1-k^2}} - \log 2$$

as $k \rightarrow 1$,

$$\cosh^{-1} \frac{1}{\sqrt{1-k^2}} = K - \log 2$$

The exact rotation at the middle of the column is given by equation (A35):

$$\phi_{max} = \sqrt{5} \frac{t}{b} \cosh^{-1} \frac{1}{\sqrt{1-k^2}}$$

therefore, as $k \rightarrow 1$,

$$\begin{aligned} \phi_{max} &= \sqrt{5} \frac{t}{b} (K - \log 2) \\ &= \sqrt{5} \frac{t}{b} \left(\frac{mL}{\sqrt{2} 4t} - \log 2 \right) \\ &= 1.37 \frac{L}{b} \sqrt{\epsilon_{av} - \epsilon_{cr}} - 1.55 \frac{t}{b} \end{aligned} \quad (\text{A41})$$

This corrective term $1.55 \frac{t}{b}$ is always a small part of ϕ_{max} .

Variation of middle-surface strain over width of flange.—The approximate strain distribution is obtained from equation (A12) by using the approximate value of γ_b from equation (A39):

$$\epsilon_x = \epsilon_{av} + \frac{5}{4} (\epsilon_{av} - \epsilon_{cr}) \left(1 - 3 \frac{z^2}{b^2} \right) \quad (\text{A42})$$

This result holds over most of the length of the flange but is in error near the ends and the middle where $\text{sn} \left(\frac{4Kx}{L} \right)$ has a value different from unity.

Relationship between the hinge-line, average, and critical strains.—Along the hinge line $z=0$, so that, approximately

$$(\epsilon_x)_{z=0} = \frac{9}{4} \epsilon_{av} - \frac{5}{4} \epsilon_{cr} \quad (\text{A43})$$

Fractional shortening.—The approximate shortening is

$$e = \frac{1}{L} \int_0^L \left(\frac{9}{4} \epsilon_{av} - \frac{5}{4} \epsilon_{cr} \right) dx = \frac{9}{4} \epsilon_{av} - \frac{5}{4} \epsilon_{cr} \quad (\text{A44})$$

and therefore is identical with ϵ_x along the hinge line $z=0$.

APPENDIX B

MAXIMUM STRENGTH OF A CRUCIFORM-SECTION COLUMN

The deformation theory of plasticity used here states that a relation exists between the stress intensity σ_i and the strain intensity e_i which is of the following form:

for loading (e_i increasing)

$$\sigma_i = E_{\text{sec}} e_i$$

for unloading (e_i decreasing)

$$d\sigma_i = E de_i$$

where

$$\sigma_i = \sqrt{\sigma_x^2 + \sigma_z^2 - \sigma_x \sigma_z + 3\tau^2}$$

$$e_i = \frac{2}{\sqrt{3}} \sqrt{\epsilon_x^2 + \epsilon_z^2 + \epsilon_x \epsilon_z + \frac{\gamma^2}{4}}$$

σ_x, ϵ_x stress and strain in the x -direction

σ_z, ϵ_z stress and strain in the z -direction

τ, γ shear stress and strain

In the case of a cruciform-section column compressed beyond the buckling stress σ_{cr} , the value of σ_x is the stress in the x -direction and is larger than σ_{cr} over most of the flange width. Also $\sigma_z = 0$, and with Poisson's ratio equal to $1/2$,

$$\epsilon_z = -\frac{1}{2} \epsilon_x; \text{ so that the fundamental stress-strain relation for}$$

increasing σ_i reduces to

$$\sqrt{\sigma_x^2 + 3\tau^2} = E_{\text{sec}} \sqrt{\epsilon_x^2 + \frac{\gamma^2}{3}}$$

in which

$$\sigma_x = E_{\text{sec}} \epsilon_x$$

$$\tau = \frac{E_{\text{sec}}}{3} \gamma$$

The value of ϵ_x at any point (x, z) of a cruciform flange is assumed from appendix A to be as in equation (A36c)

$$\epsilon_x = \frac{(t/b)^2}{2(1+\mu)} + \frac{m^2}{12} + \frac{5}{24} \frac{k^2 m^2}{1+k^2} \left(1 - 3 \frac{z^2}{b^2}\right) \sin^2 \left(\frac{4Kx}{L}\right)$$

where k^2 is a parameter lying between 0 and 1 which specifies the amount of twist,

$$K = \int_0^{\pi/2} \frac{d\alpha}{\sqrt{1 - k^2 \sin^2 \alpha}}$$

$$m^2 = 12 \left[\epsilon_{av} - \frac{(t/b)^2}{2(1+\mu)} \right] = K^2 (1 + k^2) \left(\frac{4t}{L} \right)^2$$

As soon as a value is assigned to k^2 corresponding to a certain amount of twist, the quantities K and m^2 are fixed and ϵ_x may be computed.

Over most of the length of the column, $\sin \left(\frac{4Kx}{L} \right) \approx 1$ and,

therefore, the variation of ϵ_x with x may be neglected by taking

$$\epsilon_x = \frac{(t/b)^2}{2(1+\mu)} + \frac{m^2}{12} + \frac{5}{24} \frac{k^2 m^2}{1+k^2} \left(1 - 3 \frac{z^2}{b^2}\right)$$

The shear strain γ arises from the twist $\frac{d\phi}{dx}$ of the flange after buckling and is proportional to the distance r away from the center line of the cross section:

$$\gamma = 2r \frac{d\phi}{dx}$$

However for insertion into the formula for strain intensity, a value of γ^2 is desired which is independent of r . Such a value may be obtained by taking the average value of γ^2 over the thickness. The mean value of γ^2 over the cross section is

$$\bar{\gamma}^2 = \frac{1}{t} \int_{-t/2}^{t/2} 4r^2 \left(\frac{d\phi}{dx} \right)^2 dr = \frac{t^2}{3} \left(\frac{d\phi}{dx} \right)^2 = \frac{\gamma_0^2}{3} \left(\frac{t}{b} \right)^2$$

From the theory of appendix A (equation (A31)),

$$\gamma_b = \frac{\sqrt{5}}{2} \frac{km}{\sqrt{1+k^2}}$$

over most of the section for which $\sin\left(\frac{4Kx}{L}\right) \approx 1$. Hence

$$\overline{\gamma^2} = \frac{5}{12} \left(\frac{t}{b}\right)^2 \frac{k^2 m^2}{1+k^2}$$

and thus the strain intensity $\sqrt{\epsilon_z^2 + \frac{\gamma^2}{3}}$ is completely de-

termined as soon as a value of the parameter k^2 is selected.

From the stress-strain relation the value of the stress intensity and of course E_{sec} is determined by the value of the strain intensity. (The elastic modulus E is used if the strain intensity is decreasing.) The stress σ_x may then be computed by the relation $\sigma_x = E_{sec}\epsilon_x$ as a function of the z -coordinate across the flange. The average value of σ_x across the width of the flange is then

$$\sigma_{av} = \frac{1}{b} \int_0^b \sigma_x dz$$

and is the average stress that would be determined from a testing machine at the value of k^2 selected.

In the actual calculations, the width b of the flange was divided into ten equal strips and the value of σ_{av} was found by a numerical summation. As the twist of the flange varies from zero to infinity, the parameter k^2 varies from zero to 1. The value of σ_{av} may be investigated as a function of k^2 and will have a maximum at some value of k^2 . This maximum value of σ_{av} multiplied by the total area gives the maximum load for the cruciform flange under consideration.

REFERENCES

1. Duberg, John E., and Wilder, Thomas W., III: Column Behavior in the Plastic Stress Range. *Jour. Aero. Sci.*, vol. 17, no. 6, June 1950, pp. 323-327.
2. Heimerl, George J.: Determination of Plate Compressive Strengths. NACA TN 1480, 1947.
3. Stowell, Elbridge Z.: A Unified Theory of Plastic Buckling of Columns and Plates. NACA Rep. 898, 1948. (Formerly NACA TN 1556.)
4. Ilyushin, A. A.: *Plastichnost*. OGIZ, Moskva-Leningrad, 1948, p. 115.
5. Wagner, Herbert: Torsion and Buckling of Open Sections. NACA TM 807, 1936.

REPORT 1030

INVESTIGATION OF SEPARATION OF THE TURBULENT BOUNDARY LAYER¹

By G. B. SCHUBAUER and P. S. KLEBANOFF

SUMMARY

An investigation was conducted on a turbulent boundary layer near a smooth surface with pressure gradients sufficient to cause flow separation. The Reynolds number was high, but the speeds were entirely within the incompressible flow range. The investigation consisted of measurements of mean flow, three components of turbulence intensity, turbulent shearing stress, and correlations between two fluctuation components at a point and between the same component at different points. The results are given in the form of tables and graphs. The discussion deals first with separation and then with the more fundamental question of basic concepts of turbulent flow.

INTRODUCTION

In 1944 an experimental investigation was begun at the National Bureau of Standards with the cooperation and financial assistance of the National Advisory Committee for Aeronautics to learn as much as possible about turbulent-boundary-layer separation. Considering that previous experimentation had been limited to mean speeds and pressures, it was decided that the best way to bring to light new information was to investigate the turbulence itself in relation to the mean properties of the layer. Since little was known about turbulent boundary layers in large adverse pressure gradients, the investigation was exploratory in nature and was pursued on the assumption that whatever kind of measurements that could be made on turbulence and turbulent processes would carry the investigation in the right direction.

The investigation was therefore long range, there being no natural stopping point as long as there remained unknowns and means for investigating them. The decision to stop came when it was decided that the more basic properties of turbulent motions, such as production, decay, and diffusion, which form the subject of modern theories, could better be investigated first without the effect of pressure gradient. The experimental work on separation was therefore halted after a certain fund of information had been obtained on turbulence intensity, turbulent shearing stress, correlation coefficients, and the scale of turbulent motions.

Use was made of the results from time to time as they could be made to serve a particular purpose. Certain of the results have appeared therefore in references 1 to 3. It is now felt that the results should be presented in their entirety for what they contribute to the separation problem and to the understanding of turbulent flow, even though they leave many questions unanswered.

The authors wish to acknowledge the active interest and support of Dr. H. L. Dryden during this investigation and the assistance given by Mr. William Squire in the taking of observations and the reduction of data.

SYMBOLS

x	distance along surface from forward stagnation point
y	distance normal to surface measured from surface
z	direction perpendicular to xy -plane
U	mean velocity in boundary layer
U_1	mean velocity just outside boundary layer
U_m	mean velocity just outside boundary layer at $x=17\frac{1}{2}$ feet, used as reference velocity
V	y -component of mean velocity in boundary layer
u, v, w	x -, y -, and z -components of turbulent-velocity fluctuations
u', v', w'	root-mean-square values of u, v , and w
ρ	density of air
ν	kinematic viscosity of air
p	pressure
q_1	free-stream dynamic pressure $\left(\frac{1}{2}\rho U_1^2\right)$
q_m	free-stream dynamic pressure at $x=17\frac{1}{2}$ feet $\left(\frac{1}{2}\rho U_m^2\right)$
τ	turbulent shearing stress $(-\rho\overline{uv})$
$\overline{uv}$	mean value of product of u and v
$C_{\tau 1}, C_{\tau m}$	coefficients of turbulent shearing stress $\left(C_{\tau 1}=\tau/\frac{1}{2}\rho U_1^2, C_{\tau m}=\tau/\frac{1}{2}\rho U_m^2\right)$
τ_o	skin friction
C_f	coefficient of skin friction $\left(\tau_o/\frac{1}{2}\rho U_1^2\right)$
δ	boundary-layer thickness
δ^*	boundary-layer displacement thickness $\left(\int_0^\infty \left(1-\frac{U}{U_1}\right)dy\right)$
θ	boundary-layer momentum thickness $\left(\int_0^\infty \frac{U}{U_1}\left(1-\frac{U}{U_1}\right)dy\right)$
H	boundary-layer shape parameter (δ^*/θ)
L_x, L_y	scales of turbulence

¹ Supersedes NACA TN 2133, "Investigation of Separation of the Turbulent Boundary Layer" by G. B. Schubauer and P. S. Klebanoff, 1950.

- R_y transverse correlation coefficient $(\overline{u_1 u_2} / u_1' u_2')$,
where subscripts 1 and 2 refer to positions y_1
and y_2)
- R_x longitudinal correlation coefficient $(\overline{u_1 u_2} / u_1' u_2')$,
where subscripts 1 and 2 refer to positions x_1
and x_2)

APPARATUS AND TEST PROCEDURE

The setup for the investigation was arranged with two things in mind: (1) The Reynolds number was to be as high as possible and (2) the boundary layer was to be thick enough to permit reasonably accurate measurements of all components of the turbulence intensity and shearing stress by hot-wire techniques which were known to be reliable. Since this required a large setup, the 40-foot open-air wind tunnel at the National Bureau of Standards was chosen, and a wall of airfoil-like section shown in figures 1 and 2 was constructed in the center of the test section. The wall was 10 feet high, extending from floor to ceiling, and was 27.9 feet long. It was constructed of $\frac{1}{4}$ -inch Transite on a wooden frame, and the surface on the working side was given a smooth finish by sanding and varnishing and, finally, waxing and polishing. The profile was chosen so that the adverse pressure gradient on the working side would be sufficient to cause separation and yet have sufficiently small curvature to make the pressure changes across the layer negligible.

Since the separation point was found to be very close to

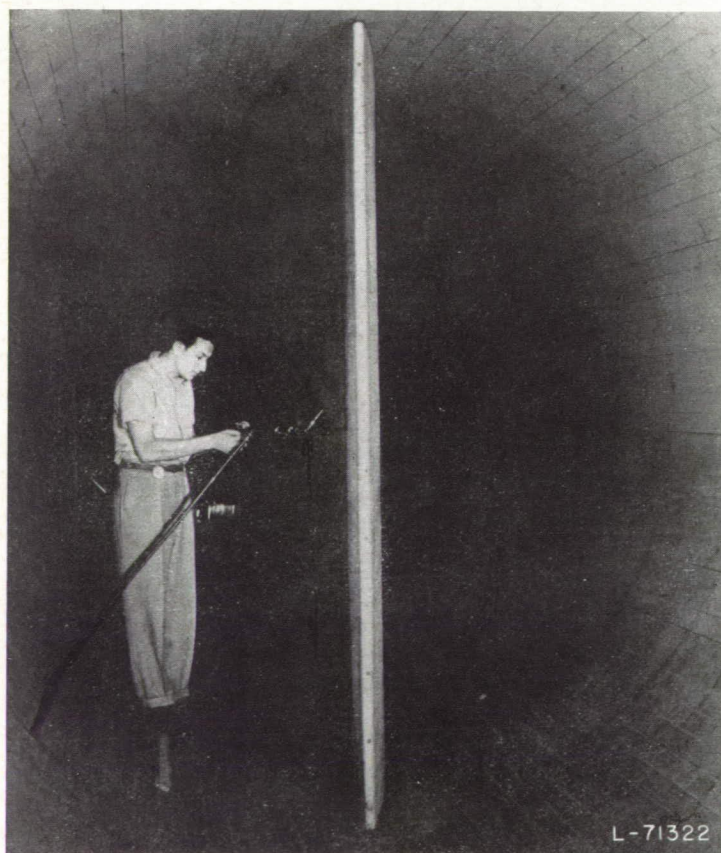


FIGURE 1.—Front view of "boundary-layer wall" in NBS 10-foot open-air wind tunnel.

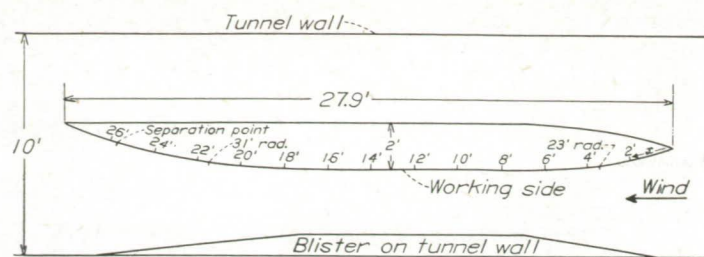


FIGURE 2.—Sectional drawing of "boundary-layer wall."

the trailing edge, a blister was constructed on the tunnel wall to move the separation point upstream to the location shown in figure 2. At the outset there was troublesome secondary flow from premature separation near the floor and, to a lesser degree, near the ceiling. A vent in the floor allowing air to enter the tunnel and blow away the accumulated dead air afforded a satisfactory remedy. The flow was then two-dimensional over the central portion of the wall from the leading edge to the separation point.

A steeply rising pressure, caused by the small radius of curvature of the leading edge and the induced angle of attack, produced transition about 2 inches from the leading edge. The boundary layer was therefore turbulent over practically the whole of the surface and, over the region of major interest, ranged in thickness from $2\frac{3}{8}$ inches at the $17\frac{1}{2}$ -foot position to 9 inches at the separation point. All measurements were made with a free-stream speed of about 160 feet per second at the $17\frac{1}{2}$ -foot position. The boundary-layer thickness at $17\frac{1}{2}$ feet was equivalent to that on a flat plate 14.3 feet long with fully turbulent layer and no pressure gradient, and the flat-plate Reynolds number corresponding to 160 feet per second was 14,300,000. The speed was always adjusted for changes in kinematic viscosity from day to day to maintain a fixed Reynolds number throughout the entire series of measurements. The turbulence of the free stream of the tunnel was about 0.5 percent.

All measurements were made at the midsection of the wall, where the flow most closely approximated two-dimensionality, and on the side labeled "Working side" in figure 2. While the measurements extended over a considerable period of time, there was no evidence from pressure and mean-velocity distributions that the geometry of the wall changed. There was, however, considerable scatter in the turbulence measurements from day to day, some of which was due to inherent inaccuracies associated with hot-wire measurements, and some of which may have been caused by actual changes in the flow. The results therefore do not lend themselves to a determination of differential changes in the x -direction with high accuracy. It was the intention to obtain results applicable to a smooth surface; therefore the surface was frequently polished and kept clean at all times. However, because of the texture of the Transite, the surface could not be given a mirrorlike finish equal to that of a metal surface.

Considerable emphasis was placed on the precise determination of the position of separation. A method was finally evolved whereby the line of separation and the direction of

the flow at the surface in the neighborhood of this line could be found. This consisted of pasting strips of white cloth on the surface with a starch solution. Small crystals of iodine were then stuck to the strips. Blue streaks on the starched cloth then showed the direction of the air flow. By this means separation could be located with an accuracy of ± 2 inches. Initially the line of separation was nowhere straight, but, after the removal of some of the reversed flow near the floor by the vent previously mentioned, the line was made straight for a distance of 2 feet in the center and was located 25.7 ± 0.2 feet from the leading edge.

The pressure distribution was measured with a static-pressure tube 0.04 inch in diameter, constructed according to the conventional design for such a tube. Mean dynamic pressure was obtained by adding a total-pressure tube of the same diameter but flattened on the end to form a nearly rectangular opening 0.012 inch wide.

The hot-wire equipment used in the investigation of turbulence has been fully described in reference 1, and it suffices here merely to call attention to the manner of operation and the performance of the equipment. The thick boundary layer made it possible to obtain essentially point measurements without having to construct hot-wire anemometers on a microscopic scale. The several types used are shown in figure 3. The $\frac{1}{16}$ -inch scale shows the high magnification of types A, B, C, and D. A complete holder is shown by E with the inch scale above. Heads of type A were used for measuring u' , those of type B or C were used for measuring turbulent shearing stresses, and those of type D were used for measuring v' and w' . In use the prongs pointed directly into the mean wind.

When the head of type C was used for measuring shearing stress, an observation of the mean-square signal from each of the wires was necessary. A similar pair of observations

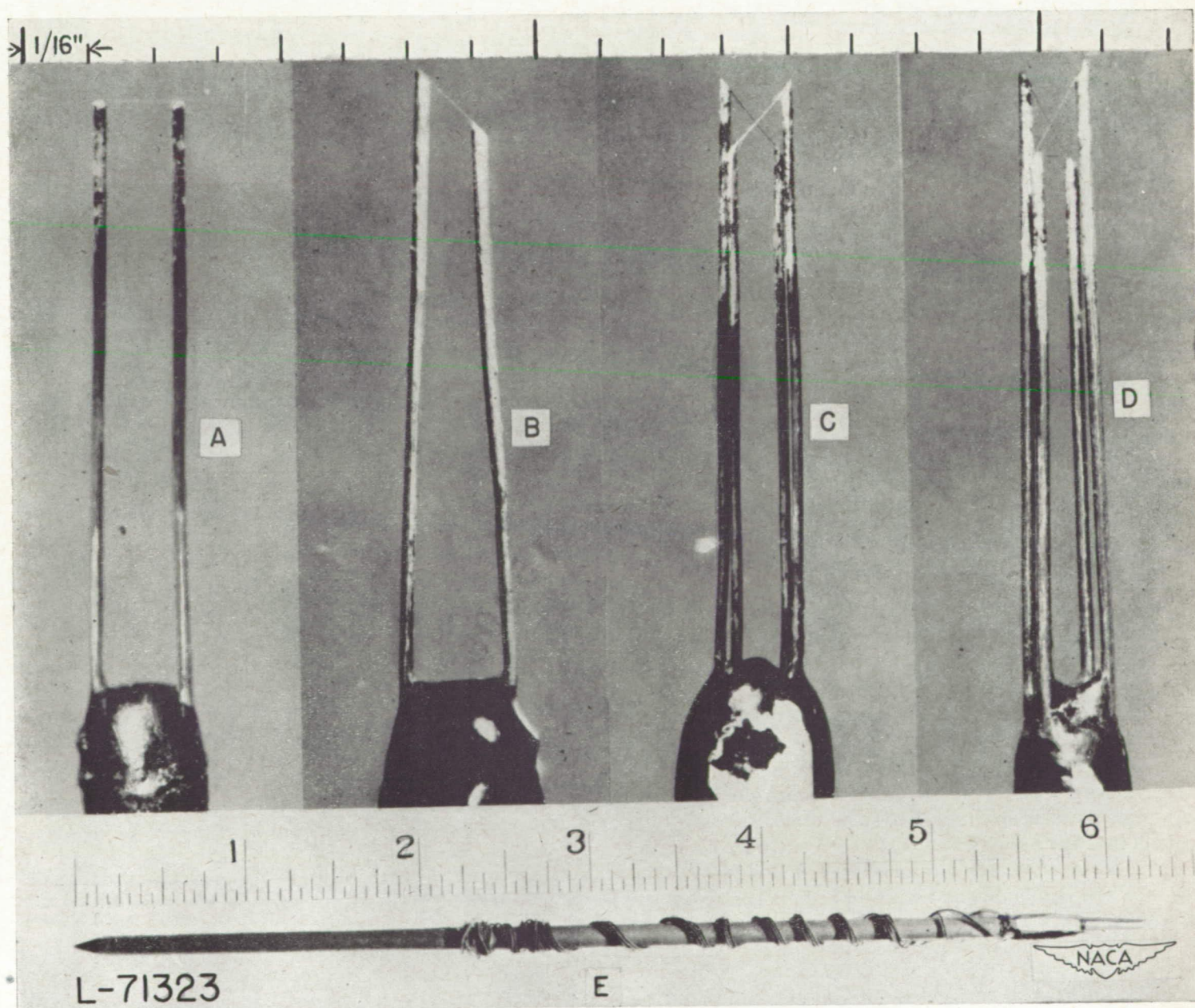


FIGURE 3.—Types of hot-wire anemometers and complete holder used in investigation

was necessary when using type B, but with only one wire the head had to be rotated through 180° . Since it was usually difficult to execute this rotation by remote control, most of the measurements of shearing stress were made with the head of type C.

The hot-wires themselves, shown at the tips of the prongs, were tungsten 0.00031 inch in diameter. Platinum wire could not be used because the air was taken into the tunnel from outdoors and platinum wires were broken by flying dirt particles. The diameter of 0.00031 inch was the smallest obtainable in tungsten at the time, and the length could not be reduced below about $\frac{1}{16}$ inch and still maintain the required sensitivity. In all cases the boundary-layer thickness was at least 25 times the wire length.

The uncompensated amplifier had a flat response from 2 to 5,000 cycles per second and an amplification decreasing above 5,000 cycles per second to about 50 percent at 10,000 cycles per second. The time constant of the wires ranged from 0.001 to 0.003 second, depending on operating conditions, and the over-all response of wire and amplifier could be made equal to that of the uncompensated amplifier by means of the adjustable compensation provided in the amplifier. However, with this relatively high time constant, the background noise level was high and had to be subtracted from the readings in order to obtain the true hot-wire signal.

The methods of determining u' , v' , w' , $\overline{uv}$, and $\overline{uv}/u'v'$ are fully described in reference 1. The determination of R_y and R_x involved the use of a pair of heads of type A, separated by known distances normal to the surface for R_y and along the tangent to the surface for R_x . The "sum-and-difference" method described in reference 4 was used, account being taken of the inequality of u' at the two wires and the differences in sensitivity.

The several measuring heads were mounted on various types of traversing equipment designed for convenience, rigidity, and a minimum of interference at the point where a measurement was being made.

TEST RESULTS

The results of the measurements are given in tables 1 to 8 and figures 4 to 14. Figures 15 to 22 repeat certain of the results to aid in the analysis.

The tabulation is made to present all of the detail contained in the measurements and to make the results readily available to any style of plotting that suits the reader's needs. Figures 4 to 14 are summary plots intended to show an over-all picture rather than detail.

PRESSURE DISTRIBUTION

The values given in table 1 and figure 4 were obtained from measurements of pressure with a small static-pressure tube placed $\frac{1}{4}$ inch from the surface at various positions along the midspan. The tube was also traversed in the y -direction, from which it was found that changes in pressure across the boundary layer were barely detectable in the region from $x=18$ to 23 feet and were not measurable elsewhere. The

pressure is therefore regarded as constant across the boundary layer, and all of the information on pressure gradient is given by the variation of q_1/q_m with x .

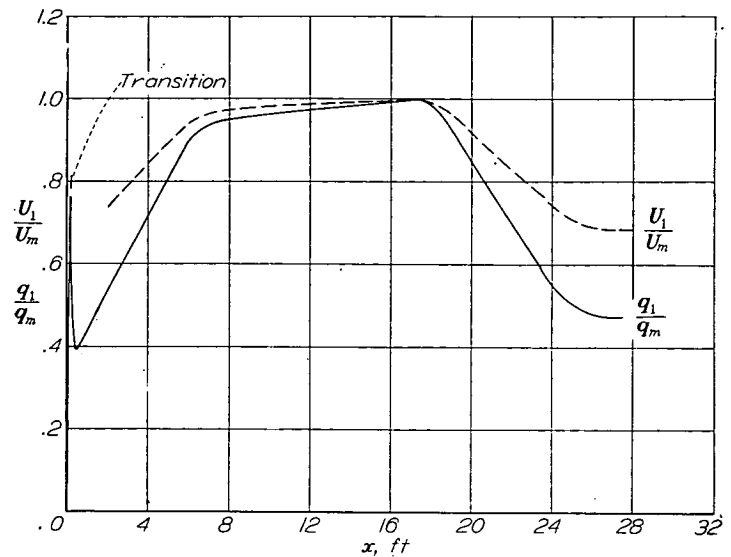


FIGURE 4.—Distribution of velocity and dynamic pressure just outside boundary layer.

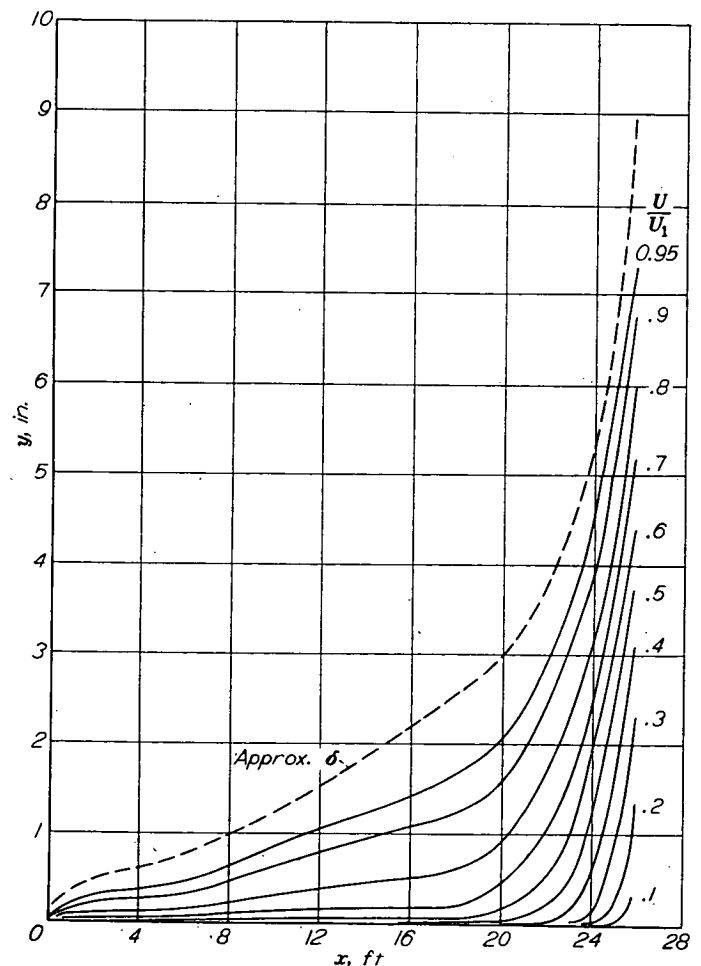


FIGURE 5.—Contour plot of mean velocities.

MEAN-VELOCITY DISTRIBUTION

Mean velocities were obtained from dynamic-pressure measurements made at various distances y . No correction was made for the effect of turbulence. The distributions of mean velocity are given in table 2 and summarized by the contour plot shown in figure 5. From these data were derived the values of δ^* , θ , and H given in table 3 and figure 6.

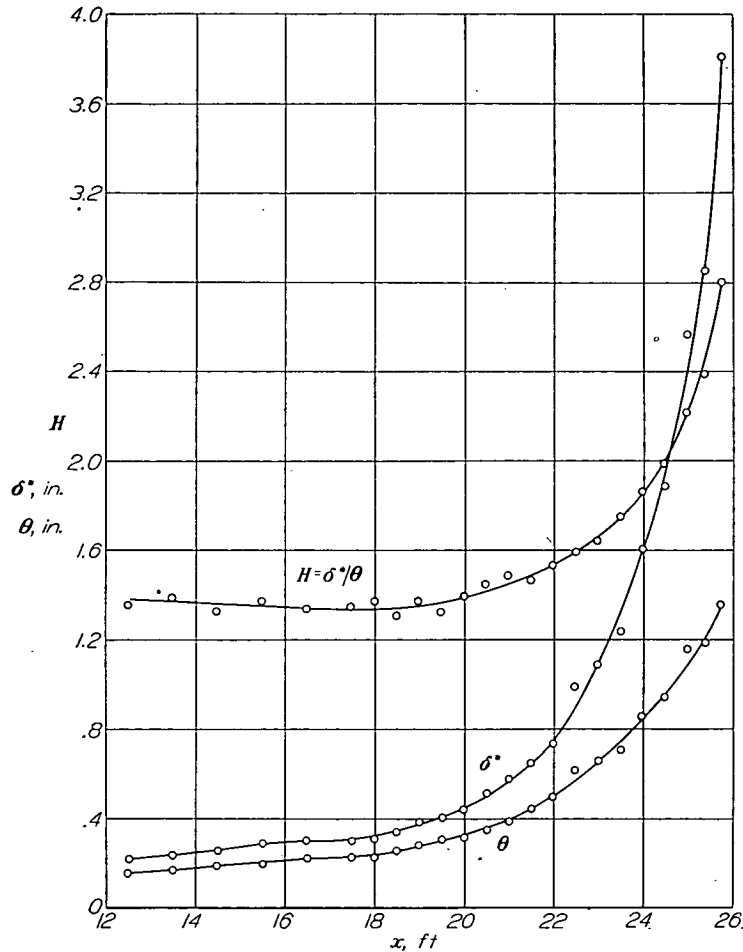
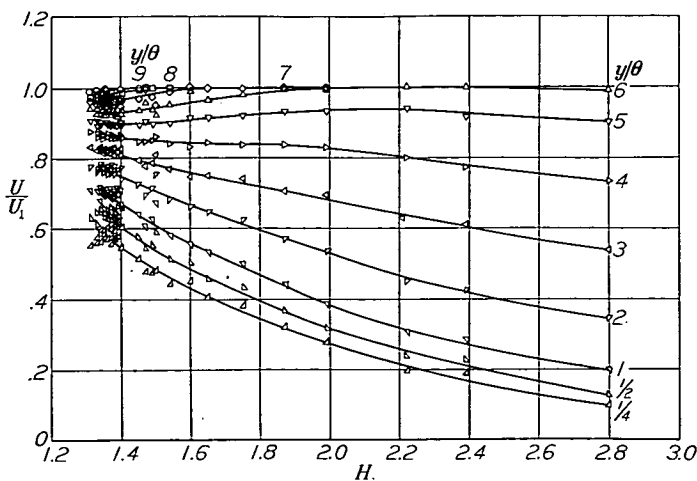
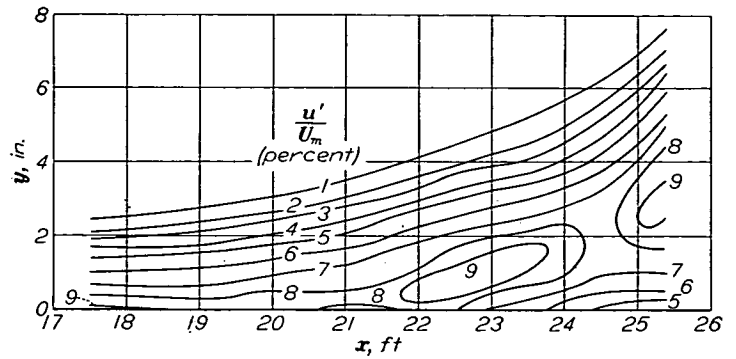
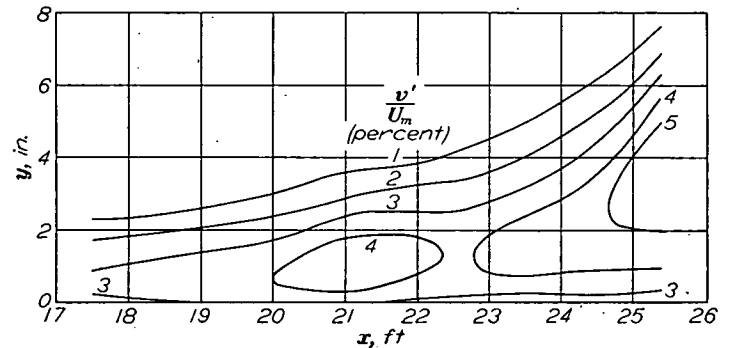
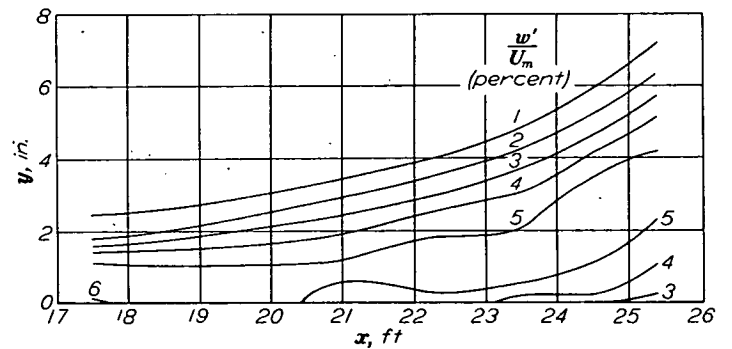
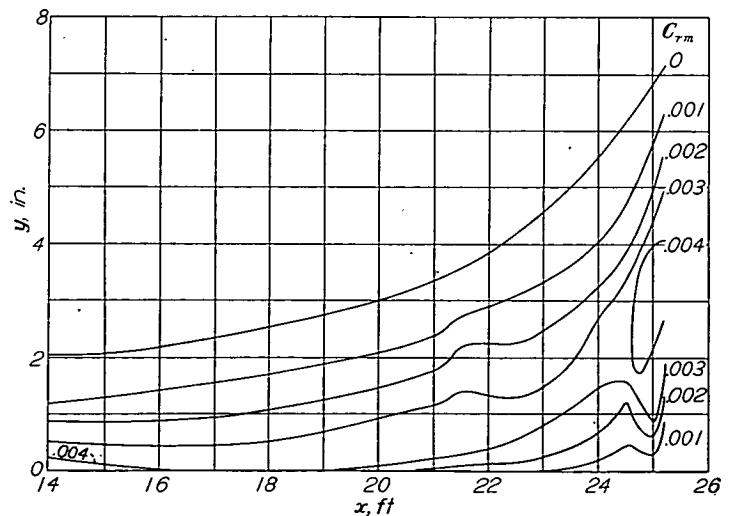


FIGURE 6.—Boundary-layer parameters.


 FIGURE 7.—Variation of U/U_1 with H for various values of y/θ .

 FIGURE 8.—Contour plot of u' .

 FIGURE 9.—Contour plot of v' .

 FIGURE 10.—Contour plot of w' .


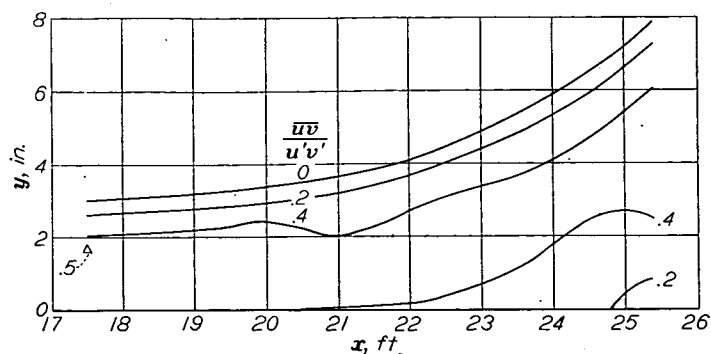
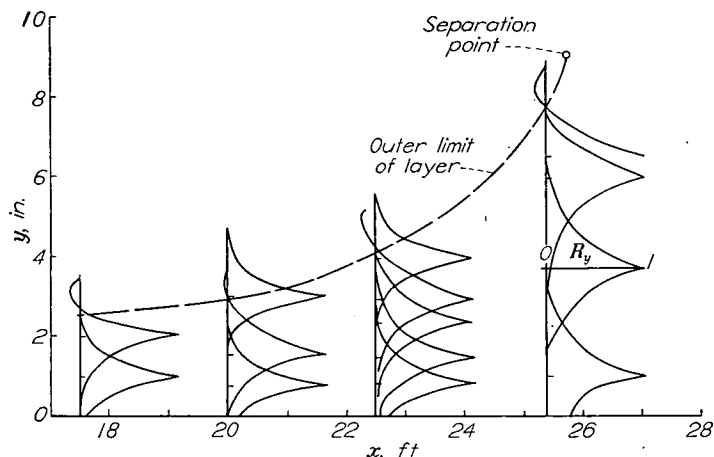
FIGURE 12.—Contour plot of $\overline{uv}$ -correlation coefficient.

FIGURE 13.—Transverse correlation coefficient.

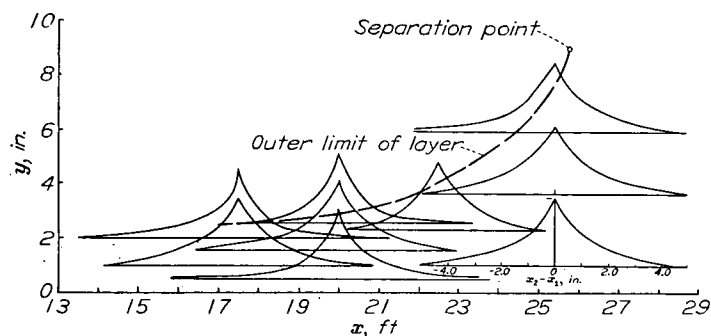


FIGURE 14.—Longitudinal correlation coefficient.

The distribution of mean velocity is plotted in figure 7 in the manner suggested by Von Doenhoff and Tetervin in reference 5. If H is a universal parameter specifying the boundary-layer profile, the curves of figure 7 should agree in all detail with those of figure 9 in reference 5. The agreement is good, although there are systematic differences slightly greater than the experimental dispersion.

TURBULENCE INTENSITIES

The turbulence intensities are given in table 4 in terms of u'/U_1 , v'/U_1 , and w'/U_1 . They are summarized in figures 8 to 10 in terms of u'/U_m , v'/U_m , and w'/U_m in order to show changes in the absolute magnitude of the fluctuations. As desired, u' , v' , and w' may be expressed in relation to any

of the mean velocities U , U_1 , or U_m by the aid of tables 1 and 2.

COEFFICIENT OF TURBULENT SHEARING STRESS AND $\overline{uv}$ -CORRELATION COEFFICIENT

The directly observed quantity $\overline{uv}$ has been expressed nondimensionally in terms of a coefficient of turbulent shearing stress

$$C_{\tau 1} = \frac{2\overline{uv}}{U_1^2}$$

$$C_{\tau m} = \frac{2\overline{uv}}{U_m^2}$$

The choice of coefficients is arbitrary, and $C_{\tau 1}$ is tabulated in table 5 while contour plots for $C_{\tau m}$ are given in figure 11. The choice of $C_{\tau m}$ for the figure was made because it was desired to show an over-all picture of variations in τ independent of variations in mean velocity.

The values of the correlation coefficient $\overline{uv}/u'v'$ are given in table 6 and figure 12.

CORRELATION COEFFICIENTS R_y AND R_z

The correlation coefficients R_y and R_z express the correlation between values of u at the same instant at two different points. This correlation between points separated by distances in the direction of the local normal to the surface is expressed by R_y , and the correlation between points separated by distances in the direction of the local tangent to the surface is expressed by R_z . These directions were normal and tangential to streamlines only when the local mean direction of the flow was tangent to the surface. Where the boundary layer was thickening rapidly, as near the separation point, the flow in the outer portion of the boundary layer had a greater radius of curvature than the surface and the direction was not tangent to the surface. In such regions, therefore, R_y and R_z do not conform strictly to the conventional definition of such coefficients.

Values of R_y are given in table 7 and values of R_z are given in table 8. Figures 13 and 14 show representative correlation curves in order to give an idea of the distances over which u is correlated compared with the boundary-layer thickness.

It will be noted that a correlation exists over much of the boundary-layer thickness. With the region near separation excluded, fluctuations at the center of the layer are related to those everywhere else in the same section. Under such conditions a small negative correlation is found between points in the layer and those outside, as shown in figure 13. Subsequent measurements in a boundary layer with approximately one-tenth of the free-stream turbulence have shown no effect of the free-stream turbulence on the magnitude of the negative correlation. An explanation of this negative correlation on the basis of continuity requirements is offered in reference 3.

From tables 7 and 8 one may calculate integral scales, defined by

$$L_y = \int_0^{\infty} R_y dy$$

$$L_x = \int_0^{\infty} R_x dx$$

These are not given here because it is felt that the qualitative concept of scale obtained from figures 13 and 14 conveys about as much physical significance to scale as is possible at present.

DISCUSSION OF RESULTS

MECHANICS OF SEPARATION

The separation point is defined as the point where the flow next to the surface no longer continues to advance farther in the downstream direction. This results from a failure of the medium to have sufficient energy to advance farther into a region of rising pressure. Certain characteristics of the mean flow serve as a guide to the imminence of separation. For example, the shape factor H can be expected to have a value greater than 2. In the present experiment H was found to have the value 2.7 at the separation point, comparing well with the value of 2.6 given in reference 5.

The empirical guides, however, give little insight into the physical factors involved. Separation is a natural consequence of the loss of energy in the boundary layer, and the burden of explanation rests rather with the question as to why separation does not occur at all times at a pressure minimum. At the surface the kinetic energy of the flow is everywhere vanishingly small. At a pressure minimum the potential energy is a minimum, and the air at the surface, having a vanishing amount of kinetic energy to draw upon, could never advance beyond a pressure minimum without receiving energy from the flow farther out. The necessary transfer is effected by the shearing stresses.

It is a well-known fact that viscous shearing stresses are so small that laminar flow can advance but a little distance beyond a pressure minimum. In contrast with this, turbulent shearing stresses can prevent separation entirely if the rate of increase of pressure is not too great. This emphasizes an important fact; namely, that when separation has not occurred, or has been delayed to distances well beyond the pressure minimum, as in the present experiment, viscous stresses play an insignificant role in the prevention or delay of separation.

Turbulent shearing stresses also determine the magnitude of shearing stresses in the laminar sublayer by forcing there a high rate of shear. This, in fact, gives boundary-layer profiles the appearance of near slip flow at the surface. Thus, turbulent stresses dominate all parts of the boundary layer. Viscous effects in the laminar sublayer and elsewhere still play an important role in determining the existing state of the turbulence. However, in dealing with the effects of turbulence, and not with the origin of turbulence, effects of viscosity can be neglected.

At the high Reynolds numbers of the present experiment the laminar sublayer was extremely thin and was never approached in any of the measurements. At the 17½-foot position at 0.1 inch from the surface the turbulent shearing stress was 190 times the viscous shearing stress. Considering the low order of magnitude of the viscous stresses compared with that of the turbulent stresses, the equations of motion may be closely approximated by including only the Reynolds stresses, and may be written

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{\partial \bar{u}^2}{\partial x} - \frac{\partial \bar{u}v}{\partial y} \quad (1)$$

$$U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{\partial \bar{u}v}{\partial x} - \frac{\partial \bar{v}^2}{\partial y} \quad (2)$$

While all terms in equations (1) and (2) have been measured, they have not been measured with sufficient accuracy to test the adequacy of the equations. The relative importance of the terms involving Reynolds stresses depends on location in the boundary layer. The normal stresses $\rho \bar{u}^2$ and $\rho \bar{v}^2$ are pressures and their gradients make merely small contributions to $\partial p / \partial x$ and $\partial p / \partial y$. Among the Reynolds stresses the shearing stress is the more important quantity and, accordingly, attention is devoted to it.

It is easy to see qualitatively on physical grounds how the shearing stress must be distributed across the boundary layer. The shearing stress is always in such a direction that fluid layers farther out pull on layers farther in. When the pressure is either constant or falling, all pull is ultimately exerted on the surface. Therefore the shearing stress must be at least as high at the surface as it is elsewhere, and it would be expected to be a maximum there, as it must fall to zero outside the boundary layer. When the pressure is rising, part of the pull must be exerted on the fluid near the surface that has insufficient energy of its own to advance to regions of higher pressure. In other words, the fluid in such layers must be pulled upon harder than it pulls upon the layer next nearer the surface. This means that the shearing stress must have a maximum away from the surface in regions of adverse pressure gradient.

Representative observed distributions are shown in figure 15. It will be seen that the maximum shear stress develops first near the surface and move progressively outward. The region between the surface and the maximum is receiving energy from the region beyond the maximum, the rate per unit volume at each point being $U \frac{\partial \tau}{\partial y}$. Thus the fall in the shearing stress toward the surface, producing a positive slope, is evidence that the shearing stress is acting to prevent separation. It is clear then that a falling to zero, as for example the curve at $x=25.4$ feet, is not the cause of separation. It is rather an indication that the velocity gradient is vanishing at the surface. This means that the velocity in the vicinity of the surface is vanishing and that a condition is developing in which no energy can be

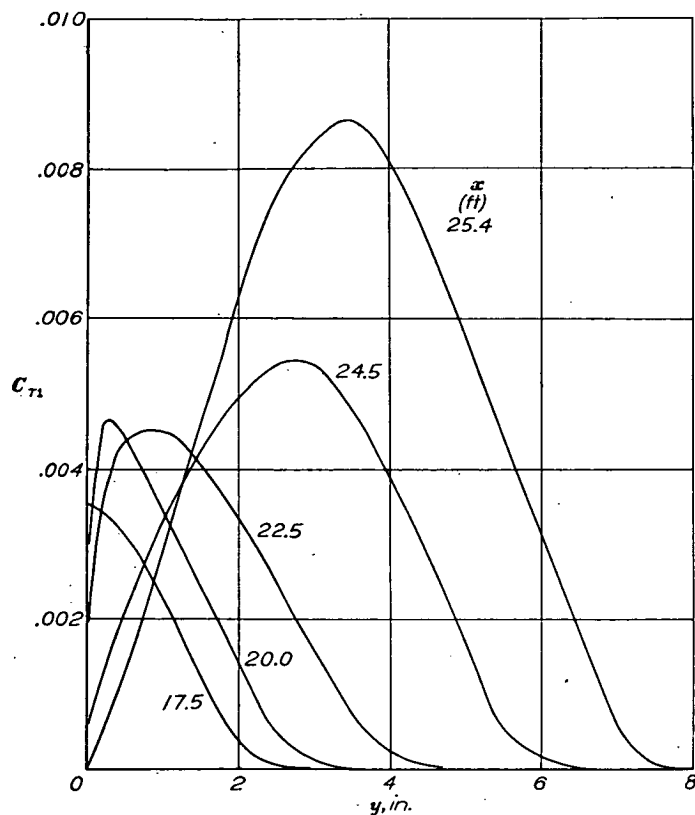


FIGURE 15.—Distribution of coefficient of turbulent shearing stress across boundary layer.

received. When this condition is fulfilled, the fluid can move no farther and separation has occurred.

The initial slope of the curves in figure 15 is given by equation (1), which becomes, when $y=0$:

$$\frac{\partial p}{\partial x} = \frac{\partial \tau}{\partial y} \quad (3)$$

A theory of the distribution of shearing stress based on the inner boundary conditions $\partial^2 \tau / \partial y^2 = 0$ and equation (3) and on the outer boundary conditions $\tau = 0$ and $\partial \tau / \partial y = 0$ at $y = \delta$ has been given by Fediaevsky (reference 6). The agreement between Fediaevsky's theory and experimental values from the present investigation was fair at the 17½-foot position and excellent at the 25-foot position, but elsewhere was poor. Two examples of the agreement are given in figure 16. The Fediaevsky theory, which defines merely how the curves shall begin and end, either loses control over the middle portion or ignores other controlling factors.

Since equation (3) specifies the initial slope, it is an aid in finding the skin friction by the method of extrapolating the distribution curves to $y=0$. The values found in this way are given in figure 17. As would be expected, the skin friction falls to zero at the separation point. The lack of agreement with values calculated by the Squire-Young formula (reference 7) is to be expected, as this formula does not include the effect of pressure gradient.

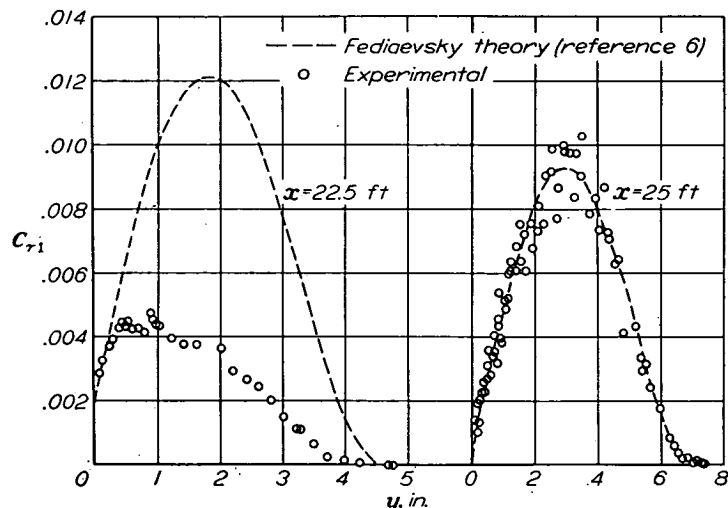


FIGURE 16.—Experimental values for coefficient of turbulent shearing stress compared with curves from Fediaevsky theory.

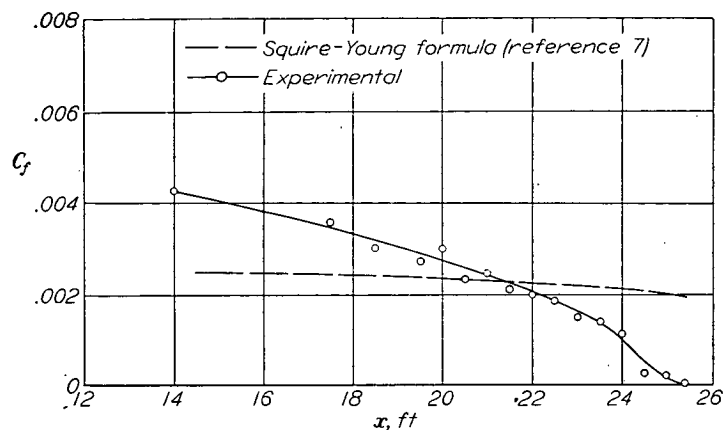


FIGURE 17.—Experimental values for coefficient of skin friction compared with values calculated with Squire-Young formula.

The foregoing discussion has simply described the shearing stress in the light of the present experiment and pointed out the role of shearing stress as an energy-transferring agent. While these phenomena are characteristic in every adverse pressure gradient, the form of the shearing stress and also the velocity profiles will be different for different pressure distributions. The present experiment gives merely one example.

ORIGIN OF TURBULENCE AND TURBULENT SHEARING STRESS

The discussion of origin of turbulence and turbulent shearing stress will be based on concepts that have superseded the older mixing-length theories. Unfortunately, experiments have not kept pace with ideas and the concepts have not yet been fully verified.

In recent years definite ideas have taken shape regarding the decay of turbulence. These stem from an observation made by Dryden (reference 8), namely, that the rates of decay of different frequency components in isotropic tur-

bulence require that the higher-frequency components gain energy at the expense of the lower-frequency components. It has now become generally accepted that decay involves a transfer of energy from larger eddies to smaller eddies by Reynolds stresses when the Reynolds number characteristic of the eddies is sufficiently high. This idea forms the physical basis for modern theories of isotropic turbulence (for example, references 9 to 15).

Information about turbulent flow points more and more to the conclusion that the concept is basic and may be carried over to shear flow. (See, for example, Batchelor's discussion of Kolmogoroff's theory, reference 9, and Townsend's discussion, reference 16.) The general idea may be expressed as follows: The highest Reynolds number is associated with the mean flow, and here the mean Reynolds stresses transfer energy to the flow system comprising the next smaller spatial pattern, for example, the largest eddies. This second system involves other Reynolds stresses which in turn transfer energy to smaller systems and so on through a spectrum of turbulence until the Reynolds number gets so low that the dissipation is completed by the action of viscosity alone. The evolution of heat by the action of viscosity is small for the larger systems and gets progressively greater as the systems get smaller and smaller, with a weighting depending on some Reynolds number characterizing the whole system, say, a Reynolds number based on the outside velocity and the boundary-layer thickness. The higher the Reynolds number the more is the action of viscosity confined to the high-frequency end of the spectrum. Thus at sufficiently high Reynolds numbers the action of viscosity is not only removed from the mean flow but also from all but the smaller-scale components of the turbulence. An exception must, of course, be made for the laminar sublayer, and the likelihood that this is a valid picture increases with distance from the surface.

These ideas then might be regarded as describing a tentative model of a turbulent boundary layer to be examined in the light of experiment. The model is, of course, conceived only in general outline and should not be assumed the same for all conditions.

The rate of removal of kinetic energy per unit volume from the mean flow by Reynolds stresses is given by:

$$\rho \left[\bar{u}^2 \frac{\partial U}{\partial x} + \bar{v}^2 \frac{\partial V}{\partial y} + \bar{u}v \left(\frac{\partial V}{\partial x} + \frac{\partial U}{\partial y} \right) \right] \quad (4)$$

This energy goes directly into the production of turbulence.

The term $\bar{u}v \frac{\partial U}{\partial y}$ will generally outweigh the others, but in order to see the relative magnitudes near separation the terms in expression (4) were calculated for the 24.5-foot position. The term $\bar{u}v \frac{\partial V}{\partial x}$ was found to be negligible. The other terms within the brackets together with their sum are

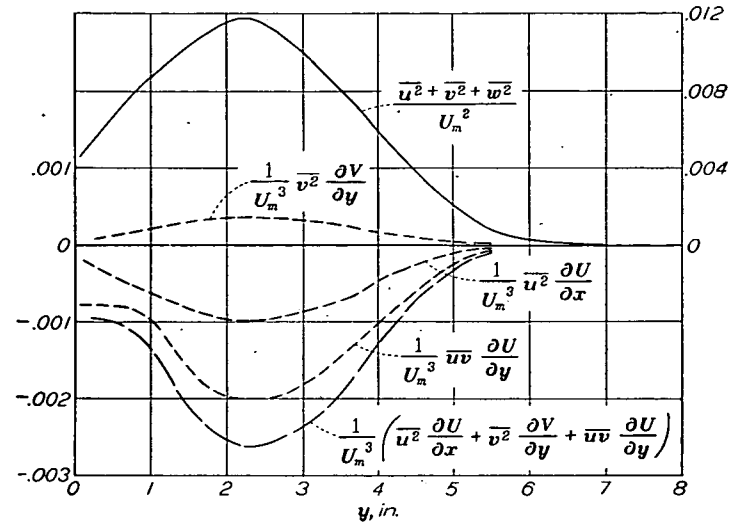


FIGURE 18.—Rate of production of turbulence and mean energy of turbulence at $x=24.5$ feet. Right ordinate scale to be used only for top curve.

shown in figure 18 divided by U_m^3 . It is seen that the term $\bar{u}v \frac{\partial U}{\partial y}$ is still the largest, and therefore remains the most important contributor to turbulence.

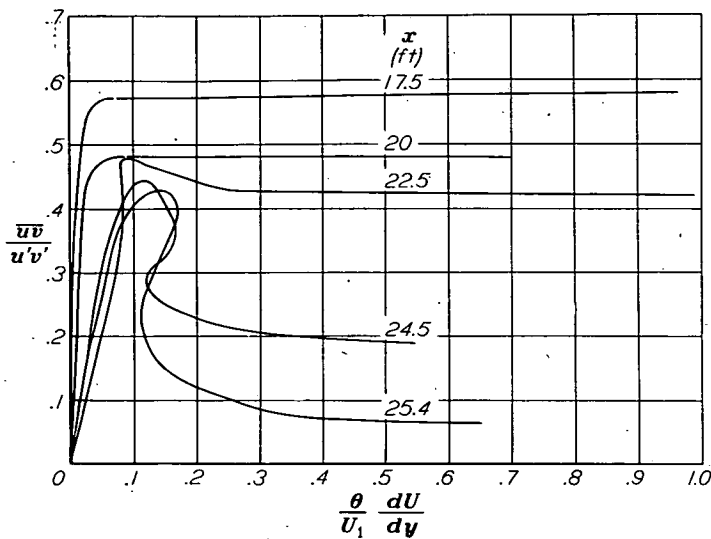
The distribution of turbulence energy is also given in figure 18. This shows a maximum energy content where the rate of production is the greatest; otherwise the comparison has no particular significance. Such coincidence is not required and is not found farther upstream. Data are not available for establishing the balance between production, diffusion, convection, and dissipation of turbulence energy.

It is clear that the turbulence exists because of the Reynolds stresses, and it is self-evident that the normal stresses $\rho \bar{u}^2$ and $\rho \bar{v}^2$ exist because of the turbulence, but the source of the shearing stress $\rho \bar{u}v$ is not apparent without further examination.

Since

$$\tau = -\rho \bar{u}v = -\rho \frac{\bar{u}v}{u'v'} u'v' \quad (5)$$

where $\bar{u}v/u'v'$ is the correlation coefficient, it is seen that τ depends on the correlation and intensity of u and v . If a flow is turbulent without a gradient in mean velocity, there can be no mean shearing stress and therefore no mean correlation between u and v . It is apparent then that a gradient is necessary to produce a correlation, and one might expect to find $\bar{u}v/u'v'$ proportional to dU/dy . From figure 12 it appears that $\bar{u}v/u'v'$ shows too little variation across the boundary layer to be proportional to the local value of the mean-velocity gradient. To apply a more direct test, $\bar{u}v/u'v'$ was plotted in figure 19 against the mean local gradient. Obviously $\bar{u}v/u'v'$ cannot be regarded as proportional to $(\theta/U_1) (dU/dy)$, and, what is more, it becomes independent of the local gradient for a wide range of values of $(\theta/U_1) (dU/dy)$.

FIGURE 19.—Relation between $\overline{uv}$ -correlation coefficient and local mean-velocity gradient.

Assuming the correctness of the concept of transfer of energy from larger to smaller flow regimes, it is seen that energy flows into turbulence mainly by way of the largest eddies, and it is then mainly these that account for the average shearing stress. Returning to figure 13, it is seen by the curves of R_v that the turbulent motions are correlated over much of the boundary-layer thickness up to the position $X=23$ feet and are still correlated over a considerable portion of the thickness at larger values of x . The extent of the R_v -correlation is roughly a measure of the extent of the largest eddies. This means that the correlation coefficient $\overline{uv}/u'v'$ arises from those components of the turbulence that extend over much of the boundary-layer thickness, and the correlation between u - and v -components of such a motion would be expected to depend on the mean-velocity gradient as a whole rather than upon the local gradient at any one point. Large mean gradients exist near the surface without producing correspondingly large correlation coefficients in the same locality, and it appears that the correlations here are very likely fixed by some over-all effect. If an over-all velocity gradient is represented at each position by U_1/U_m divided by δ , and this is used as the independent variable in figure 20 to cross-plot values of $\overline{uv}/u'v'$ taken from the flat portion of the curves in figure 19, a definite proportionality between these two quantities is found. This bears out the foregoing argument.

Figure 21 was originally prepared to test one of the equations of state in Nevzgljadov's theory (reference 17), which expresses the shearing stress as proportional to the turbulent energy per unit volume and the mean-velocity gradient. The theory is not supported by the results for the same reason as that mentioned in connection with figure 19. In fact, shearing stress per unit energy is much like the correlation coefficient and would be two-thirds of $\overline{uv}/u'v'$ if u' , v' , and w' were all equal. The similarity between figures 19 and 21 is therefore not surprising. The hairpin loops in the curves in these two figures apparently result from the distribution

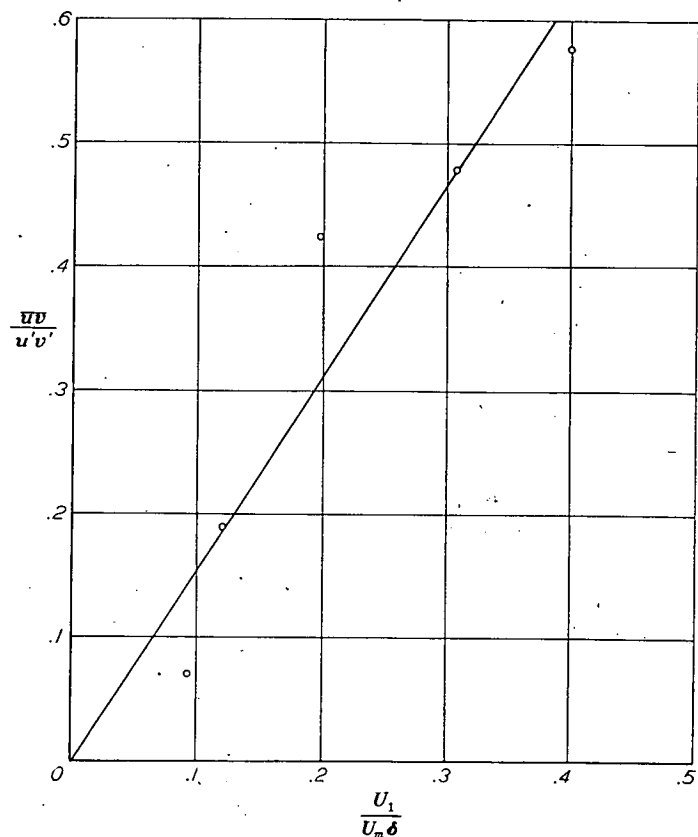
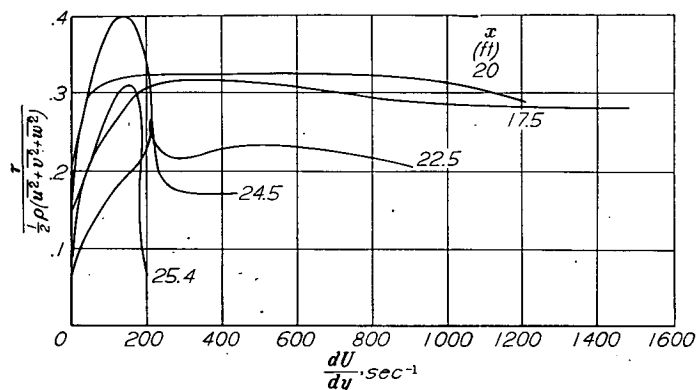
FIGURE 20.—Relation between $\overline{uv}$ -correlation coefficient near surface and general mean-velocity gradient.

FIGURE 21.—Relation between shearing stress per unit energy of turbulence and local mean-velocity gradient.

of shearing stress imposed by the adverse pressure gradient.

Figure 22 emphasizes the great difference between turbulent shear flow and laminar shear flow. In laminar flow the shearing stress is directly proportional to the local velocity gradient. In turbulent flow, shown in figure 22, the shearing stress may rise abruptly for scarcely any change in the local velocity gradient and again fall with increasing velocity gradient. This illustrates the difficulty of adopting the concepts of viscous flow in turbulent flow. The difference probably arises because turbulent phenomena, unlike molecular phenomena, are on a scale of space and velocity of the same order as that of the mean flow.

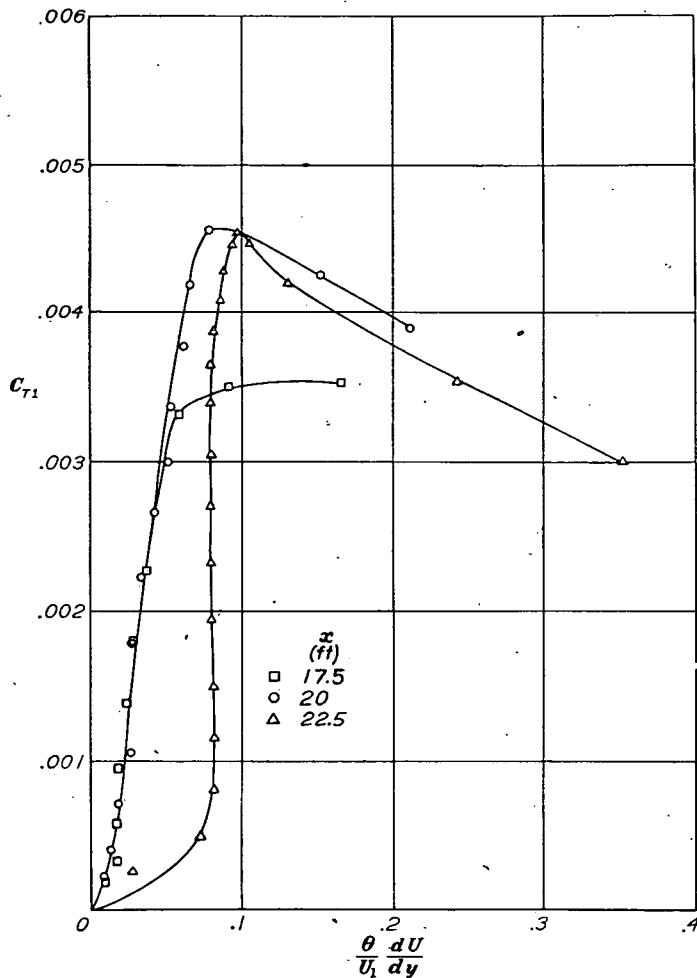


FIGURE 22.—Relation between coefficient of shearing stress and local mean-velocity gradient.

The tentative picture in the form of the model previously described is still speculative and probably oversimplified. It has, however, received support in the present experiment, perhaps as much as could be expected from over-all measurements embracing the entire frequency spectrum. Observations of these same quantities as a function of frequency would be much more informative, but unfortunately the experimental conditions in an open-air wind tunnel discouraged work of this sort. Other types of hot-wire measurements, such as those described by Townsend in reference 16, would be of as great value in probing for the true picture of a turbulent boundary layer as they were in bringing to light phenomena in the turbulent wake of a cylinder.

The present model is but an extension of the concepts required to explain the spectrum and decay of isotropic turbulence. However, in going from the relative simplicity of isotropic turbulence to boundary-layer turbulence many new factors are introduced. Distance from transition point, pressure gradient, curvature, and surface roughness doubtless affect details and may have profound influences. It must be left to future experiments and theory to fill in the gaps, and when this has been done perhaps the data given herein will have more meaning than they have at

present. It is with this thought in mind that the data are given in tables, in which form they are the more readily available for new uses.

CONCLUDING REMARKS

Certain measured characteristics of a separating turbulent boundary layer have been presented. The average characteristics are mean velocity, pressure, and the derived parameters, displacement thickness, momentum thickness, and shape factor. The turbulent characteristics comprise intensities, shearing stresses, transverse correlations, longitudinal correlations, and correlations between two fluctuation components at a point.

The results have been discussed, first, in connection with what they reveal about separation and, second, in connection with what they reveal about the nature of turbulent boundary layers. The modern concept of energy transfer through a spectrum was extended to the turbulent boundary layer. The resulting model of a turbulent boundary layer was supported by the results. This together with the support from theory and experiment in isotropic turbulence makes it appear that the model may be a very useful one for guiding future experiments.

It is seen that the investigation of separation of the turbulent boundary layer had to go beyond the mere investigation of separation. The real problem is the understanding of the mechanics of turbulent shear flow under the action of pressure gradient. The solution of this problem depends on the understanding of the mechanics of turbulence, and in this only rudimentary beginnings have been made.

NATIONAL BUREAU OF STANDARDS,
WASHINGTON, D. C., June 1, 1949.

REFERENCES

1. Schubauer, G. B., and Klebanoff, P. S.: Theory and Application of Hot-Wire Instruments in the Investigation of Turbulent Boundary Layers. NACA ACR 5K27, 1946.
2. Dryden, Hugh L.: Some Recent Contributions to the Study of Transition and Turbulent Boundary Layers. NACA TN 1168, 1947.
3. Dryden, Hugh L.: Recent Advances in the Mechanics of Boundary Layer Flow. Vol. I of Advances in Applied Mechanics, Richard von Mises and Theodore von Kármán, eds., Academic Press, Inc. (New York), 1948, pp. 2-40.
4. Dryden, Hugh L., Schubauer, G. B., Mock, W. C., Jr., and Skramstad, H. K.: Measurements of Intensity and Scale of Wind-Tunnel Turbulence and Their Relation to the Critical Reynolds Number of Spheres. NACA Rep. 581, 1937.
5. Von Doenhoff, Albert E., and Tetervin, Neal: Determination of General Relations for the Behavior of Turbulent Boundary Layers. NACA Rep. 772, 1943.
6. Fediaevsky, K.: Turbulent Boundary Layer of an Airfoil. NACA TM 822, 1937.
7. Squire, H. B., and Young, A. D.: The Calculation of the Profile Drag of Aerofoils. R. & M. No. 1838, British A. R. C., 1938.
8. Dryden, Hugh L.: Turbulence Investigations at the National Bureau of Standards. Proc. Fifth Int. Cong. Appl. Mech. (Sept. 1938, Cambridge, Mass.), John Wiley & Sons, Inc., 1939, pp. 362-368.

9. Batchelor, G. K.: Kolmogoroff's Theory of Locally Isotropic Turbulence. Proc. Cambridge Phil. Soc., vol. 43, pt. 4, Oct. 1947, pp. 533-559.
10. Batchelor, G. K., and Townsend, A. A.: Decay of Isotropic Turbulence in the Initial Period. Proc. Roy. Soc. (London), ser. A, vol. 193, no. 1035, July 21, 1948, pp. 539-558.
11. Batchelor, G. K.: The Role of Big Eddies in Homogeneous Turbulence. Proc. Roy. Soc. (London), ser. A, vol. 195, no. 1043, Feb. 3, 1949, pp. 513-532.
12. Von Kármán, Theodore: Progress in the Statistical Theory of Turbulence. Proc. Nat. Acad. Sci., vol. 34, no. 11, Nov. 1948, pp. 530-539.
13. Lin, C. C.: Note on the Law of Decay of Isotropic Turbulence. Proc. Nat. Acad. Sci., vol. 34, no. 11, Nov. 1948, pp. 540-543.
14. Heisenberg, W.: On the Theory of Statistical and Isotropic Turbulence. Proc. Roy. Soc. (London), ser. A, vol. 195, no. 1042, Dec. 22, 1948, pp. 402-406.
15. Kovasznay, Leslie S. G.: Spectrum of Locally Isotropic Turbulence. Jour. Aero. Sci., vol. 15, no. 12, Dec. 1948, pp. 745-753.
16. Townsend, A. A.: Local Isotropy in the Turbulent Wake of a Cylinder. Australian Jour. Sci. Res., ser. A, vol. 1, no. 2, 1948, pp. 161-174.
17. Nevzgliadov, V.: A Phenomenological Theory of Turbulence. Jour. Phys. (USSR), vol. 9, no. 3, 1945, pp. 235-243.

TABLE 1.—DISTRIBUTION OF DYNAMIC PRESSURE

x (ft)	q_1/q_m	x (ft)	q_1/q_m
0.05	0.218	13.0	0.977
.08	.547	13.5	.980
.12	.762	14.0	.991
.17	.694	14.5	.992
.21	.514	15.0	.988
.5	.396	15.5	.988
1.0	.442	16.0	.988
1.5	.488	16.5	.991
2.0	.534	17.0	.994
2.5	.576	17.5	1.000
3.0	.622	18.0	.988
3.5	.668	18.5	.966
4.0	.709	19.0	.927
4.5	.756	19.5	.890
5.0	.810	20.0	.852
5.5	.861	20.5	.813
6.0	.894	21.0	.777
6.5	.923	21.5	.740
7.0	.930	22.0	.697
7.5	.962	22.5	.659
8.0	.951	23.0	.625
8.5	.960	23.5	.589
9.0	.954	24.0	.558
9.5	.958	24.5	.529
10.0	.974	25.0	.507
10.5	.976	25.5	.493
11.0	.966	26.0	.484
11.5	.972	26.5	.478
12.0	.972	27.0	.475
12.5	.977	27.5	.472

TABLE 2.—MEAN-VELOCITY DISTRIBUTION NORMAL TO SURFACE

$x=0.5$ ft		$x=1.0$ ft		$x=1.5$ ft		$x=2.0$ ft		$x=2.5$ ft		$x=3.0$ ft		$x=3.5$ ft		$x=4.5$ ft	
y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1
0.01	0.552	0.01	0.609	0.01	0.612	0.01	0.589	0.01	0.584	0.01	0.588	0.01	0.576	0.01	0.582
.02	.645	.02	.671	.02	.660	.02	.634	.02	.630	.02	.642	.02	.620	.02	.641
.04	.725	.04	.727	.04	.714	.04	.693	.04	.690	.04	.695	.04	.682	.04	.693
.08	.814	.08	.795	.08	.784	.08	.762	.08	.777	.08	.752	.08	.737	.08	.743
.13	.899	.13	.865	.13	.856	.14	.828	.13	.815	.13	.804	.13	.790	.13	.788
.19	.959	.19	.924	.19	.903	.18	.884	.19	.867	.19	.866	.19	.847	.19	.844
.27	.991	.27	.975	.27	.950	.27	.940	.27	.915	.27	.911	.27	.894	.27	.892
.36	1.000	.36	.996	.36	.984	.36	.978	.36	.961	.36	.956	.36	.941	.36	.933
-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
-----	-----	.47	1.000	.46	.999	.46	.995	.46	.993	.46	.985	.46	.976	.46	.978
-----	-----	-----	-----	.58	1.002	.57	1.000	.58	.999	.57	1.000	.58	.998	.58	.991
-----	-----	-----	-----	-----	-----	-----	-----	.70	1.000	-----	-----	.70	1.000	.70	1.002

$x=5.5$ ft		$x=6.5$ ft		$x=7.5$ ft		$x=8.5$ ft		$x=9.5$ ft		$x=10.5$ ft		$x=11.5$ ft		$x=12.5$ ft	
y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1
0.01	0.565	0.01	0.554	0.01	0.536	0.01	0.521	0.01	0.504	0.01	0.507	0.01	0.511	0.01	0.504
.02	.617	.02	.596	.02	.561	.02	.538	.02	.536	.02	.532	.02	.524	.02	.533
.04	.670	.04	.653	.04	.617	.04	.604	.04	.577	.04	.594	.04	.563	.04	.566
.08	.722	.08	.707	.08	.669	.08	.665	.08	.646	.08	.652	.08	.623	.08	.636
.13	.778	.13	.761	.12	.715	.12	.718	.13	.693	.13	.693	.13	.668	.13	.680
.19	.836	.19	.811	.19	.763	.19	.761	.19	.729	.19	.733	.19	.708	.19	.716
.27	.874	.27	.851	.27	.810	.27	.802	.27	.764	.27	.764	.27	.738	.27	.751
.36	.920	.36	.898	.36	.852	.36	.844	.36	.801	.36	.801	.36	.770	.36	.782
.46	.956	.46	.935	.46	.897	.46	.884	.46	.850	.46	.837	.46	.806	.46	.815
.57	.982	.58	.963	.57	.939	.57	.930	.57	.889	.57	.889	.58	.841	.57	.844
.70	1.000	.70	.994	.70	.968	.70	.962	.70	.924	.70	.905	.71	.876	.70	.875
-----	-----	.84	.998	.84	.991	.84	.986	.84	.953	.84	.936	.84	.903	.84	.937
-----	-----	.99	1.002	.99	.998	.99	.997	.99	.984	.99	.968	1.32	.986	1.43	.979
-----	-----	-----	-----	1.15	1.000	1.15	1.003	1.15	.997	1.15	.986	1.69	.987	1.69	.990
-----	-----	-----	-----	-----	-----	-----	-----	1.32	.998	1.32	1.000	2.09	.997	2.08	.998
-----	-----	-----	-----	-----	-----	-----	-----	1.50	.998	-----	-----	2.51	.997	2.51	.998
-----	-----	-----	-----	-----	-----	-----	-----	1.88	1.000	-----	-----	2.95	1.002	2.94	1.002

$x=13.5$ ft		$x=14.5$ ft		$x=15.5$ ft		$x=16.5$ ft		$x=17.5$ ft		$x=18.0$ ft		$x=18.5$ ft		$x=19.0$ ft	
y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1	y (in.)	U/U_1
0.01	0.495	0.01	0.482	0.01	0.492	0.01	0.465	0.01	0.495	0.01	.0495	0.01	0.483	0.01	0.450
.02	.507	.04	.536	.02	.487	.02	.472	.02	.509	.02	.508	.02	.477	.02	.455
.04	.578	.07	.616	.04	.514	.04	.529	.04	.573	.04	.563	.04	.501	.04	.498
.08	.636	.12	.666	.07	.597	.07	.595	.08	.626	.07	.627	.07	.576	.07	.562
.13	.670	.18	.702	.12	.652	.12	.655	.12	.664	.12	.664	.13	.628	.12	.612
.19	.718	.26	.732	.19	.690	.19	.704	.19	.698	.19	.697	.19	.674	.19	.645
.27	.746	.35	.762	.26	.728	.26	.729	.26	.724	.26	.727	.27	.709	.26	.682
.36	.778	.45	.792	.35	.752	.35	.752	.36	.750	.35	.756	.36	.740	.35	.708
.46	.807	.57	.821	.46	.781	.45	.780	.46	.782	.45	.783	.46	.766	.46	.738
.58	.834	.69	.844	.57	.810	.57	.809	.57	.802	.57	.802	.57	.790	.57	.760
.71	.866	.98	.900	.70	.833	.70	.833	.70	.831	.70	.826	.70	.815	.70	.789
1.00	.925	1.31	.950	.98	.888	.98	.881	.99	.878	.99	.873	.99	.860	.84	.817
1.32	.968	1.68	.994	1.31	.936	1.31	.923	1.32	.927	1.32	.914	1.32	.924	.98	.846
1.70	.995	2.07	.996	1.68	.975	1.68	.965	1.69	.958	1.69	.961	1.69	.945	1.31	.887
2.09	.999	2.49	1.000	2.07	1.002	2.08	.990	2.08	.996	2.08	.992	2.08	.971	1.68	.930
2.52	.999	-----	-----	-----	-----	2.50	.996	2.50	.997	2.51	.996	2.51	1.000	2.07	.963
2.96	1.000	-----	-----	-----	-----	2.93	1.002	2.94	.999	2.94	1.002	-----	-----	2.49	.990
-----	-----	-----	-----	-----	-----	-----	-----	3.39	1.001	-----	-----	-----	-----	2.93	1.007

TABLE 2.—MEAN-VELOCITY DISTRIBUTION NORMAL TO SURFACE—Concluded

x=19.5 ft		x=20.0 ft		x=20.5 ft		x=21.0 ft		x=21.5 ft		x=22.0 ft		x=22.5 ft	
y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i
0.01	0.442	0.01	0.404	0.01	0.409	0.01	0.396	0.01	0.368	0.01	0.357	0.01	0.314
.02	.480	.04	.489	.02	.406	.02	.394	.02	.364	.02	.350	.02	.320
.04	.527	.07	.542	.04	.450	.05	.428	.04	.411	.05	.344	.05	.375
.08	.563	.14	.582	.09	.514	.11	.488	.09	.458	.09	.410	.09	.419
.13	.599	.18	.616	.15	.556	.15	.528	.15	.503	.15	.465	.15	.448
.19	.632	.26	.654	.22	.594	.22	.572	.22	.550	.22	.500	.22	.473
.27	.664	.35	.678	.31	.624	.31	.598	.31	.574	.31	.530	.31	.504
.36	.698	.45	.701	.42	.664	.42	.624	.42	.600	.42	.560	.42	.531
.46	.725	.57	.728	.54	.692	.54	.666	.54	.627	.54	.594	.54	.546
.58	.755	.70	.755	.68	.724	.68	.695	.68	.663	.68	.616	.68	.565
.71	.781	.99	.812	.83	.757	.83	.729	.83	.687	.83	.652	.83	.596
1.00	.834	1.32	.870	1.17	.809	1.18	.786	1.17	.747	1.17	.713	1.00	.631
1.33	.896	1.56	.909	1.56	.867	1.57	.854	1.56	.823	1.56	.780	1.57	.706
1.70	.922	2.08	.950	2.00	.925	2.00	.905	2.00	.882	2.00	.844	2.00	.768
2.10	.960	2.50	.982	2.47	.961	2.48	.950	2.47	.931	2.46	.890	2.47	.827
2.52	.985	2.94	.999	2.97	1.002	2.98	.995	2.97	.991	2.96	.951	2.97	.896
2.96	1.000	3.39	1.004	-----	-----	3.50	1.006	3.59	1.006	3.48	.988	3.49	.968
-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	4.01	1.003	4.03	1.002

x=23.0 ft		x=23.5 ft		x=24.0 ft		x=24.5 ft		x=25.0 ft		x=25.4 ft		x=25.77 ft	
y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i	y (in.)	U/U _i
0.01	0.276	0.01	0.249	0.01	0.204	0.01	0.174	0.01	0.112	0.01	0.100	0.01	0.076
.02	.282	.02	.274	.02	.216	.04	.209	.02	.116	.02	.088	.05	.084
.04	.329	.05	.319	.05	.230	.08	.231	.04	.121	.04	.120	.15	.102
.09	.374	.09	.346	.09	.273	.14	.251	.08	.112	.08	.137	.31	.088
.15	.389	.15	.361	.15	.297	.21	.274	.14	.157	.14	.149	.54	.122
.22	.419	.31	.423	.22	.319	.31	.287	.22	.189	.22	.186	.83	.130
.31	.460	.42	.443	.31	.337	.41	.303	.31	.200	.31	.188	1.18	.183
.42	.487	.54	.469	.42	.362	.54	.325	.41	.220	.41	.205	1.57	.225
.54	.508	.83	.523	.54	.389	.67	.356	.54	.235	.53	.213	2.01	.263
.68	.530	1.18	.576	.69	.415	.82	.373	.67	.253	.67	.235	2.24	.284
.83	.567	1.57	.648	.83	.442	1.17	.415	.82	.265	.82	.249	2.48	.332
1.07	.616	2.00	.725	1.18	.492	1.68	.499	1.16	.310	1.16	.283	2.73	.318

TABLE 3.—BOUNDARY-LAYER PARAMETERS

x (ft.)	δ^* (in.)	θ (in.)	H
0.5	0.039	0.026	1.50
1.0	.043	.032	1.35
1.5	.051	.037	1.38
2.0	.058	.042	1.38
2.5	.063	.047	1.34
3.0	.067	.048	1.40
3.5	.076	.055	1.38
4.5	.075	.054	1.39
5.5	.087	.064	1.36
6.5	.097	.072	1.35
7.5	.127	.093	1.37
8.5	.136	.099	1.37
9.5	.170	.123	1.38
10.5	.180	.133	1.35
11.5	.222	.163	1.36
12.5	.220	.162	1.36
13.5	.234	.168	1.39
14.5	.255	.192	1.33
15.5	.288	.208	1.38
16.5	.302	.226	1.34
17.5	.303	.225	1.35
18.0	.313	.229	1.37
18.5	.341	.261	1.31
19.0	.385	.282	1.37
19.5	.407	.307	1.33
20.0	.446	.319	1.40
20.5	.517	.357	1.45
21.0	.581	.390	1.49
21.5	.65	.443	1.47
22.0	.77	.501	1.54
22.5	.99	.62	1.60
23.0	1.09	.66	1.65
23.5	1.24	.71	1.75
24.0	1.61	.86	1.87
24.5	1.89	.95	1.99
25.0	2.57	1.16	2.22
25.4	2.85	1.19	2.39
25.77	3.81	1.36	2.80

TABLE 4.—TURBULENCE INTENSITIES

$x=17.5$ ft						$x=20.0$ ft					
y (in.)	u'/U_1	y (in.)	v'/U_1	y (in.)	w'/U_1	y (in.)	u'/U_1	y (in.)	v'/U_1	y (in.)	w'/U_1
0.05	0.088	0.06	0.029	0.10	0.058	0.05	0.091	0.11	0.047	0.06	0.056
.10	.092	.11	.029	.15	.058	.10	.088	.16	.045	.11	.056
.15	.090	.16	.030	.20	.062	.10	.088	.21	.044	.16	.056
.20	.085	.21	.031	.25	.054	.15	.096	.26	.046	.21	.059
.25	.082	.26	.031	.30	.059	.20	.096	.31	.046	.26	.057
.30	.082	.31	.031	.35	.061	.25	.096	.36	.047	.31	.061
.35	.081	.36	.031	.40	.059	.30	.093	.41	.048	.36	.058
.45	.074	.46	.031	.50	.058	.40	.087	.51	.048	.46	.059
.55	.073	.56	.031	.60	.059	.50	.086	.61	.049	.56	.057
.65	.069	.66	.030	.70	.059	.60	.088	.71	.047	.66	.058
.75	.067	.76	.031	.80	.055	.70	.083	.81	.046	.76	.058
.85	.066	.86	.030	.90	.055	.80	.089	.91	.046	.86	.059
.95	.064	.96	.029	1.00	.052	.90	.080	1.01	.047	.96	.054
1.05	.062	1.06	.029	1.10	.049	1.00	.077	1.11	.045	1.06	.054
1.25	.055	1.26	.026	1.30	.043	1.20	.069	1.31	.040	1.26	.051
1.45	.047	1.46	.023	1.50	.035	1.40	.063	1.51	.038	1.46	.047
1.65	.041	1.66	.021	1.70	.029	1.55	.062	1.71	.037	1.66	.042
1.85	.033	1.86	.017	1.90	.016	1.75	.061	1.91	.034	1.86	.039
2.05	.022	2.06	.013	2.10	.015	1.95	.064	2.11	.031	2.06	.036
2.25	.016	2.26	.010	2.30	.012	2.14	.043	2.31	.028	2.26	.029
2.45	.0095	2.46	.0080	2.50	.0094	2.24	.036	2.51	.024	2.46	.023
2.65	.0072	2.61	.0073	2.70	.0088	2.44	.028	2.71	.019	2.66	.018
2.85	.0061	2.81	.0059	-----	-----	2.63	.020	2.91	.015	2.86	.014
-----	-----	3.01	.0052	-----	-----	2.83	.013	3.11	.011	3.06	.011
-----	-----	3.21	.0045	-----	-----	3.03	.010	3.31	.0090	3.26	.0091
-----	-----	3.41	.0042	-----	-----	3.23	.0080	3.51	.0079	3.46	.0079
-----	-----	3.61	.0042	-----	-----	3.43	.0050	3.71	.0068	3.66	.0071
-----	-----	3.81	.0042	-----	-----	3.63	.0060	3.91	.0068	3.86	.0068
-----	-----	-----	-----	-----	-----	-----	-----	4.11	.0062	4.06	.0062
-----	-----	-----	-----	-----	-----	-----	-----	4.31	.0062	4.26	.0065
-----	-----	-----	-----	-----	-----	-----	-----	4.51	.0062	-----	-----

$x=21.0$ ft						$x=22.5$ ft					
y (in.)	u'/U_1	y (in.)	v'/U_1	y (in.)	w'/U_1	y (in.)	u'/U_1	y (in.)	v'/U_1	y (in.)	w'/U_1
0.01	0.082	0.05	0.039	0.06	0.051	0.05	0.096	0.06	0.034	0.10	0.056
.06	.091	.10	.042	.11	.054	.10	.103	.11	.037	.15	.058
.11	.095	.15	.044	.16	.055	.15	.108	.16	.038	.20	.061
.16	.097	.20	.046	.21	.057	.20	.108	.21	.040	.25	.062
.21	.095	.25	.048	.26	.058	.25	.113	.26	.041	.30	.063
.26	.096	.30	.048	.31	.058	.30	.114	.35	.042	.35	.060
.31	.095	.35	.049	.36	.058	.35	.114	.40	.042	.40	.064
.41	.092	.45	.053	.46	.061	.40	.116	.50	.043	.50	.063
.51	.088	.55	.052	.56	.061	.45	.117	.60	.042	.60	.068
.61	.084	.65	.053	.66	.062	.55	.117	.70	.044	.70	.070
.71	.089	.75	.054	.76	.062	.65	.114	.80	.045	.80	.071
.81	.084	.85	.052	.86	.058	.75	.116	.90	.044	.90	.072
.91	.084	.95	.054	.96	.058	.85	.117	1.00	.044	1.00	.070
1.01	.079	1.05	.053	1.06	.059	.95	.119	1.10	.045	1.10	.070
1.21	.077	1.25	.048	1.26	.056	1.05	.103	1.30	.045	1.30	.065
1.41	.073	1.45	.048	1.46	.055	1.25	1.09	1.45	.042	1.50	.063
1.61	.064	1.65	.048	1.66	.049	1.45	1.09	1.60	.042	1.70	.062
1.81	.063	1.85	.030	1.86	.047	1.65	1.00	1.80	.042	1.90	.065
2.01	.060	2.05	.042	2.06	.043	1.85	.097	2.00	.043	2.10	.058
2.21	.052	2.25	.042	2.26	.038	2.05	.089	2.20	.040	2.35	.056
2.41	.045	2.45	.032	2.46	.033	2.25	.083	2.40	.037	2.60	.051
2.61	.041	2.65	.028	2.66	.028	2.45	.078	2.60	.035	2.85	.044
2.81	.032	2.85	.024	2.86	.023	2.65	.075	2.80	.033	3.10	.038
3.01	.025	3.05	.022	3.06	.018	2.85	.063	3.00	.030	3.40	.032
3.21	.018	3.25	.017	3.26	.014	3.05	.059	3.20	.026	3.70	.024
3.41	.012	3.45	.014	3.46	.011	3.30	.052	3.40	.026	4.00	.017
3.91	.0049	3.65	.011	3.66	.0091	3.55	.040	3.60	.021	4.30	.012
4.01	.0045	3.85	.0092	3.86	.0071	3.80	.029	3.80	.018	4.60	.0095
4.21	.0043	4.05	.0078	4.06	.0063	4.05	.021	4.00	.014	4.90	.0076
4.41	.0040	4.25	.0078	4.26	.0061	4.30	.015	4.20	.012	5.20	.0067
-----	-----	-----	-----	4.46	.0062	4.55	.012	4.40	.0095	5.50	.0067
-----	-----	-----	-----	-----	-----	4.80	.010	4.60	.0076	-----	-----
-----	-----	-----	-----	-----	-----	5.05	.0091	4.80	.0064	-----	-----
-----	-----	-----	-----	-----	-----	5.30	.0090	5.00	.0055	-----	-----
-----	-----	-----	-----	-----	-----	5.55	.0089	5.05	.0051	-----	-----
-----	-----	-----	-----	-----	-----	-----	-----	5.25	.0049	-----	-----
-----	-----	-----	-----	-----	-----	-----	-----	5.45	.0043	-----	-----
-----	-----	-----	-----	-----	-----	-----	-----	5.65	.0043	-----	-----

TABLE 4.—TURBULENCE INTENSITIES—Concluded

$x=23.5$ ft						$x=24.5$ ft					
y (in.)	u'/U_1	y (in.)	v'/U_1	y (in.)	w'/U_1	y (in.)	u'/U_1	y (in.)	v'/U_1	y (in.)	w'/U_1
0.04	0.090	0.18	0.037	0.08	0.049	0.04	0.061	0.35	0.045	0.10	0.056
.09	.083	.26	.044	.13	.054	.09	.065	.40	.047	.14	.054
.14	.088	.31	.043	.18	.054	.14	.074	.45	.048	.19	.058
.19	.088	.36	.042	.23	.056	.19	.071	.50	.046	.29	.054
.24	.094	.46	.046	.28	.058	.24	.074	.55	.049	.39	.058
.29	.091	.56	.049	.33	.060	.29	.075	.65	.051	.49	.064
.34	.093	.66	.051	.48	.063	.34	.077	.75	.052	.59	.063
.44	.097	.76	.052	.58	.063	.44	.083	.85	.052	.69	.061
.54	.099	.86	.056	.68	.066	.54	.084	.95	.057	.79	.063
.64	.099	.96	.053	.78	.064	.64	.085	1.05	.054	.89	.063
.74	.104	1.06	.052	.88	.067	.74	.086	1.15	.059	.99	.062
.84	.103	1.16	.054	.98	.071	.84	.085	1.25	.057	.99	.062
.94	.104	1.36	.055	1.08	.073	.94	.077	1.29	.059	1.09	.069
1.04	.115	1.56	.053	1.28	.069	1.04	.097	1.35	.057	1.29	.070
1.24	.112	1.76	.055	1.48	.067	1.24	.098	1.49	.063	1.49	.070
1.44	.127	1.96	.053	1.68	.067	1.44	.098	1.55	.063	1.69	.075
1.64	.130	2.16	.053	1.88	.065	1.64	.098	1.69	.062	1.89	.075
1.84	.101	2.41	.056	2.08	.064	1.84	.099	1.89	.058	2.09	.077
2.04	.102	2.66	.048	2.33	.060	2.04	.105	2.09	.063	2.34	.079
2.22	.091	2.91	.047	2.58	.058	2.22	.112	2.34	.063	2.59	.077
2.54	.083	3.16	.039	2.83	.054	2.54	.106	2.59	.064	2.84	.075
2.72	.078	3.46	.036	3.08	.050	2.72	.097	2.84	.063	3.09	.074
3.04	.072	3.76	.034	3.38	.042	3.04	.097	3.09	.059	3.39	.065
3.34	.069	4.06	.028	3.68	.037	3.34	.091	3.39	.054	3.69	.061
3.64	.061	4.36	.022	3.98	.029	3.64	.086	3.69	.051	3.99	.056
3.94	.050	4.66	.017	4.28	.023	3.94	.081	3.99	.046	4.29	.052
4.24	.039	4.96	.014	4.58	.016	4.24	.066	4.29	.042	4.59	.043
4.54	.028	5.26	.010	4.88	.011	4.54	.057	4.59	.039	4.89	.036
4.84	.017	5.56	.0077	5.18	.0080	4.84	.046	4.89	.032	5.19	.029
5.14	.011	5.86	.0073	5.48	.0072	5.14	.042	5.19	.026	5.49	.020
5.44	.0080	6.16	.0065	5.78	.0060	5.44	.031	5.49	.022	5.79	.015
5.74	.0068	6.46	.0059	6.08	.0060	5.74	.022	5.79	.017	6.09	.011
6.04	.0063	6.76	.0056	-----	-----	6.04	.014	6.09	.013	6.39	.0079
6.34	.0062	-----	-----	-----	-----	6.34	.011	6.39	.011	6.69	.0068
-----	-----	-----	-----	-----	-----	6.64	.0080	6.69	.0082	6.99	.0068
-----	-----	-----	-----	-----	-----	6.94	.0067	6.99	.0080	7.29	.0066
-----	-----	-----	-----	-----	-----	7.24	.0066	7.29	.0078	-----	-----
-----	-----	-----	-----	-----	-----	-----	-----	7.59	.0071	-----	-----
-----	-----	-----	-----	-----	-----	-----	-----	7.89	.0064	-----	-----
-----	-----	-----	-----	-----	-----	-----	-----	8.19	.0064	-----	-----

$x=25.4$ ft					
y (in.)	u'/U_1	y (in.)	v'/U_1	y (in.)	w'/U_1
0.05	0.057	0.13	0.034	0.08	0.038
.15	.062	.13	.036	.18	.042
.30	.072	.13	.036	.28	.042
.40	.081	.16	.036	.38	.045
.50	.085	.22	.039	.48	.047
.60	.091	.42	.046	.58	.049
.70	.091	.52	.045	.68	.048
.80	.089	.62	.049	.78	.053
.90	.099	.72	.050	.88	.057
1.05	.101	.82	.055	.98	.055
1.25	.108	.92	.057	1.08	.054
1.45	.098	1.02	.056	1.28	.059
1.65	.108	1.12	.058	1.48	.060
1.85	.117	1.32	.065	1.68	.060
2.05	.122	1.52	.064	1.88	.072
2.30	.126	1.72	.065	2.08	.068
2.55	.131	1.92	.071	2.33	.069
2.80	.129	2.12	.069	2.58	.074
3.05	.127	2.37	.072	2.83	.073
3.30	.129	2.62	.078	3.08	.076
3.55	.132	2.87	.083	3.38	.075
3.80	.121	3.12	.081	3.68	.074
4.05	.119	3.42	.079	3.98	.075
4.35	.114	3.72	.082	4.28	.070
4.65	.105	4.02	.081	4.58	.066
4.95	.102	4.32	.079	4.88	.059
5.25	.094	4.62	.081	5.18	.057
5.55	.087	4.92	.073	5.48	.052
5.85	.076	5.22	.068	5.78	.049
6.15	.065	5.52	.057	6.08	.039
6.45	.050	5.82	.055	6.38	.035
6.75	.015	6.12	.046	6.68	.027
7.05	.034	6.42	.041	6.98	.022
7.35	.023	6.72	.035	7.28	.015
7.65	.016	7.02	.026	7.58	.011
7.95	.012	7.32	.020	7.88	.0092
8.25	.010	7.62	.015	8.18	.0079
8.55	.0082	7.92	.011	8.30	.0079
8.85	.0078	8.22	.0098	8.63	.0079
-----	-----	8.39	.0098	-----	-----

x=14 ft		x=17.5 ft		x=17.5 ft		x=18.5 ft		x=19.5 ft		x=20.0 ft	
y (in.)	C _{r1}	y (in.)	C _{r1}	y (in.)	C _{r1}	y (in.)	C _{r1}	y (in.)	C _{r1}	y (in.)	C _{r1}
0.16	0.0040	0.09	0.0037	0.08	0.0034	0.08	0.0041	0.06	0.0029	0.08	0.0037
.21	.0039	.15	.0037	.13	.0033	.20	.0040	.13	.0031	.14	.0038
.26	.0037	.24	.0031	.18	.0033	.32	.0033	.30	.0039	.19	.0046
.31	.0040	.36	.0032	.23	.0030	.42	.0022	.38	.0032	.24	.0045
.36	.0037	.50	.0031	.25	.0033	.62	.0033	.53	.0033	.29	.0047
.41	.0035	.67	.0027	.33	.0037	.77	.0026	.61	.0024	.34	.0045
.46	.0034	.86	.0024	.38	.0034	.91	.0028	.77	.00.4	.39	.0046
.51	.0035	1.00	.0021	.48	.0032	1.04	.0022	.90	.00.2	.49	.0045
.56	.0031	1.16	.0018	.55	.0030	1.21	.00.8	1.09	.0021	.59	.0041
.66	.0029	1.34	.0020	.65	.0030	1.40	.0016	1.29	.0016	.69	.0043
.76	.0025	1.52	.0011	.75	.0025	1.56	.0014	1.49	.0015	.79	.0039
.86	.0024	1.68	.0007	.88	.0025	1.86	.00094	1.71	.0013	.89	.0037
.96	.0020	1.87	.00054	.98	.0022	2.04	.00048	1.89	.00094	.99	.0034
1.06	.0016	2.01	.00028	1.08	.0023	2.25	.00024	2.09	.00066	1.19	.0031
1.16	.0014	2.14	.00016	1.28	.0019	2.54	.00004	2.22	.00045	1.39	.0027
1.36	.00067	2.27	.00009	1.48	.0014	2.84	0	2.42	.00031	1.59	.0022
1.56	.00026	2.54	.00002	1.65	.0010	-----	-----	2.65	.00008	1.79	.0018
1.76	.00006	2.74	.00001	1.88	.00053	-----	-----	2.89	.00002	1.99	.0016
1.96	.00002	-----	-----	2.05	.00025	-----	-----	3.09	0	2.19	.0011
2.06	.00001	-----	-----	2.28	.00011	-----	-----	-----	-----	2.39	.00063
2.26	0	-----	-----	2.48	.00005	-----	-----	-----	-----	2.59	.00043
-----	-----	-----	-----	2.68	.00003	-----	-----	-----	-----	2.61	.00027
-----	-----	-----	-----	2.88	.00002	-----	-----	-----	-----	2.81	.00022
-----	-----	-----	-----	3.08	.00001	-----	-----	-----	-----	2.81	.00015
-----	-----	-----	-----	3.28	0	-----	-----	-----	-----	3.01	.00007
-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	3.21	.00002
-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	3.41	0

x=20.5 ft		x=21.0 ft		x=21.0 ft		x=21.5 ft		x=22.0 ft		x=22.5 ft	
y (in.)	C _{r1}	y (in.)	C _{r1}	y (in.)	C _{r1}	y (in.)	C _{r1}	y (in.)	C _{r1}	y (in.)	C _{r1}
0.25	0.0030	0.08	0.0031	0.08	0.0040	0.10	0.0037	0.07	0.0028	0.08	0.0028
.30	.0031	.13	.0041	.13	.0049	.15	.0041	.12	.0032	.13	.0033
.35	.0031	.18	.0044	.18	.0048	.19	.0040	.17	.0035	.18	.0034
.40	.0032	.23	.0047	.23	.0053	.20	.0048	.22	.0041	.23	.0037
.45	.0032	.28	.0032	.28	.0052	.24	.0046	.27	.0041	.28	.0039
.55	.0029	.33	.0040	.33	.0054	.25	.0048	.32	.0037	.33	.0039
.65	.0029	.38	.0042	.38	.0054	.29	.0043	.37	.0040	.38	.0043
.75	.0029	.43	.0044	.43	.0052	.34	.0047	.42</			

TABLE 5.—COEFFICIENT OF TURBULENT SHEARING STRESS—Concluded

$x=23.0$ ft		$x=23.5$ ft		$x=24.0$ ft		$x=24.5$ ft		$x=24.5$ ft	
y (in.)	C_{τ}	y (in.)	C_{τ}	y (in.)	C_{τ}	y (in.)	C_{τ}	y (in.)	C_{τ}
0.08	0.0039	0.13	0.0026	0.05	0.0017	0.12	0.00079	0.10	0.0022
.13	.0026	.18	.0028	.10	.0026	.15	.0011	.15	.0021
.18	.0030	.23	.0029	.15	.0034	.17	.0016	.20	.0016
.23	.0032	.28	.0034	.20	.0031	.22	.0013	.25	.0015
.28	.0029	.33	.0035	.25	.0027	.27	.0017	.30	.0017
.33	.0033	.38	.0033	.30	.0040	.32	.0022	.35	.0018
.38	.0035	.43	.0035	.35	.0029	.37	.0020	.40	.0022
.48	.0040	.53	.0040	.40	.0039	.42	.0019	.50	.0020
.58	.0038	.63	.0046	.45	.0034	.47	.0022	.60	.0024
.68	.0042	.70	.0041	.50	.0039	.57	.0025	.70	.0027
.78	.0047	.73	.0044	.55	.0041	.67	.0020	.80	.0024
.88	.0047	.80	.0046	.65	.0045	.68	.0023	.90	.0023
.92	.0041	.83	.0055	.75	.0046	.77	.0031	1.00	.0031
.98	.0047	.90	.0048	.85	.0047	.78	.0021	1.10	.0034
1.02	.0048	1.00	.0042	.95	.0047	.88	.0027	1.30	.0035
1.22	.0047	1.20	.0047	1.05	.0048	1.08	.0031	1.50	.0047
1.42	.0051	1.40	.0049	1.25	.0048	1.28	.0031	1.70	.0038
1.62	.0044	1.60	.0052	1.45	.0051	1.48	.0043	1.90	.0050
1.82	.0046	1.80	.0053	1.65	.0057	1.68	.0045	2.10	.0057
2.02	.0045	2.00	.0049	1.85	.0072	1.88	.0046	2.30	.0050
2.22	.0045	2.06	.0037	2.05	.0065	2.08	.0047	2.50	.0052
2.42	.0037	2.20	.0043	2.25	.0065	2.22	.0048	2.70	.0053
2.62	.0031	2.26	.0039	2.50	.0047	2.28	.0047	2.90	.0061
2.82	.0026	2.46	.0040	2.75	.0035	2.42	.0051	3.10	.0066
3.02	.0022	2.66	.0035	3.00	.0047	2.62	.0043	3.40	.0045
3.27	.0019	2.86	.0031	3.25	.0040	2.82	.0052	3.70	.0051
3.52	.0014	3.11	.0026	3.50	.0032	3.12	.0051	4.00	.0042
3.77	.00088	3.36	.0023	3.61	.0038	3.42	.0053	4.30	.0031
4.02	.00044	3.61	.0019	3.86	.0025	3.72	.0052	4.60	.0040
4.27	.00023	3.86	.0014	4.16	.0017	4.02	.0038	4.90	.0019
4.52	.00010	4.11	.00093	4.46	.00087	4.32	.0032	5.20	.00090
4.70	.00005	4.36	.00043	4.76	.00037	4.60	.0023	5.40	.00061
4.77	.00004	4.61	.00021	5.06	.00022	4.62	.0021	5.67	.00029
4.95	.00002	4.86	.00014	5.36	.00013	4.90	.0023	5.77	.00027
5.20	.00001	5.11	.00005	5.62	0	5.20	.0015	5.97	.00010
5.45	.00001	5.18	0	5.82	0	5.50	.00079	6.11	.00004
-----	-----	5.38	.0001	-----	-----	5.80	.00042	6.27	.00004
-----	-----	-----	-----	-----	-----	6.10	.00011	6.49	.0001
-----	-----	-----	-----	-----	-----	-----	-----	6.79	0

$x=25.0$ ft		$x=25.0$ ft		$x=25.0$ ft		$x=25.4$ ft	
y (in.)	C_{τ}	y (in.)	C_{τ}	y (in.)	C_{τ}	y (in.)	C_{τ}
0.10	0.0011	0.17	0.0010	0.06	0.00077	0.18	0.00050
.15	.0020	.22	.0013	.11	.0014	.28	.00086
.20	.0024	.27	.0020	.21	.0019	.38	.00064
.25	.0021	.37	.0026	.31	.0023	.48	.0011
.30	.0023	.40	.0023	.31	.0021	.58	.00088
.35	.0026	.47	.0026	.41	.0025	.68	.0013
.40	.0036	.50	.0031	.51	.0036	.78	.0019
.45	.0037	.60	.0028	.61	.0034	.88	.0022
.50	.0039	.70	.0035	.71	.0041	.98	.0031
.60	.0045	.80	.0032	.81	.0045	1.08	.0031
.60	.0032	.88	.0053	.83	.0044	1.18	.0043
.70	.0051	.90	.0040	1.03	.0051	1.38	.0047
.80	.0043	.98	.0038	1.23	.0061	1.48	.0044
.90	.0042	1.08	.0049	1.43	.0068	1.58	.0049
1.00	.0063	1.13	.0060	1.51	.0076	1.68	.0040
1.21	.0065	1.18	.0052	1.71	.0073	1.88	.0059
1.31	.0070	1.28	.0064	1.91	.0075	2.08	.0067
1.40	.0071	1.43	.0061	2.11	.0074	2.28	.0071
1.51	.0069	1.58	.0064	2.31	.0076	2.48	.0077
1.71	.0077	1.73	.0061	2.51	.0092	2.68	.0093
1.91	.0062	1.93	.0068	2.71	.0077	2.88	.0071
1.93	.0087	2.13	.0081	2.91	.0100	3.08	.0071
2.04	.0072	2.33	.0091	3.11	.0098	3.28	.0084
2.13	.0091	2.53	.0099	3.31	.0098	3.48	.0086
2.23	.0087	2.73	.0087	3.51	.0104	3.68	.0088
2.53	.0087	2.98	.0089	3.71	.0085	3.88	.0078
2.83	.0102	3.23	.0084	3.96	.0084	4.13	.0072
3.13	.0102	3.48	.0091	4.07	.0074	4.38	.0077
3.43	.0090	3.73	.0079	4.23	.0087	4.62	.0063
3.65	.0060	4.03	.0081	4.32	.0072	4.88	.0065
3.73	.0078	4.33	.0071	4.57	.0062	5.18	.0054
3.95	.0054	4.63	.0064	4.82	.0041	5.48	.0040
4.25	.0046	4.93	.0056	5.12	.0040	5.78	.0037
4.55	.0041	5.23	.0044	5.42	.0029	6.08	.0030
4.85	.0033	5.53	.0033	5.72	.0020	6.38	.0015
5.15	.0031	5.83	.0032	6.12	.0012	6.68	.0013
5.45	.0022	6.13	.0025	6.42	.00062	6.72	.0012
5.75	.0011	6.43	.0018	6.72	.00021	6.98	.00066
6.02	.00066	6.73	.00082	7.02	.00010	7.02	.00043
6.05	.00076	6.97	.00038	7.32	0	7.23	.00017
6.36	.00007	7.27	.00023	-----	-----	7.52	.00009
-----	-----	7.57	.00013	-----	-----	7.72	.00002
-----	-----	7.87	.00007	-----	-----	7.92	0
-----	-----	8.17	.00005	-----	-----	-----	-----

TABLE 6.— $\overline{uv}$ -CORRELATION COEFFICIENT

$x=17.5$ ft		$x=20.0$ ft		$x=21.0$ ft	
y (in.)	$\overline{uv}/u'v'$	y (in.)	$\overline{uv}/u'v'$	y (in.)	$\overline{uv}/u'v'$
0.10	0.59	0.10	0.45	0.10	0.37
.25	.57	.25	.50	.25	.42
.50	.57	.50	.49	.50	.46
.75	.55	.75	.49	.75	.47
1.00	.55	1.00	.48	1.00	.45
1.25	.55	1.25	.47	1.25	.45
1.50	.53	1.50	.47	1.50	.43
1.75	.48	1.75	.46	1.75	.42
2.00	.42	2.00	.44	2.00	.41
2.25	.32	2.25	.42	2.25	.39
2.50	.21	2.50	.40	2.50	.37
2.75	.14	2.75	.35	2.75	.31
3.00	.04	3.00	.19	3.00	.27
---	---	3.25	.02	3.25	.18
---	---	---	---	3.50	.07

$x=22.5$ ft		$x=23.5$ ft		$x=24.5$ ft		$x=25.4$ ft	
y (in.)	$\overline{uv}/u'v'$	y (in.)	$\overline{uv}/u'v'$	y (in.)	$\overline{uv}/u'v'$	y (in.)	$\overline{uv}/u'v'$
0.10	0.42	0.25	0.39	0.25	0.22	0.25	0.11
.25	.44	.50	.39	.50	.25	.50	.15
.50	.45	1.00	.40	1.00	.30	1.00	.24
.75	.47	1.50	.43	1.50	.34	1.50	.31
1.00	.47	2.00	.44	2.00	.37	2.00	.36
1.25	.48	2.50	.42	2.50	.41	2.50	.38
1.50	.48	3.00	.41	3.00	.43	3.00	.40
1.75	.48	3.50	.40	3.50	.44	3.50	.41
2.00	.47	4.00	.36	4.00	.45	4.00	.42
2.25	.47	4.50	.30	4.50	.42	4.50	.42
2.50	.45	5.00	.21	5.00	.38	5.00	.42
2.75	.45	5.50	0	5.50	.30	5.50	.43
3.00	.42	---	---	6.00	.19	6.00	.42
3.25	.39	---	---	6.50	0	6.50	.36
3.50	.34	---	---	---	---	7.00	.26
3.75	.26	---	---	---	---	7.50	.18
4.00	.21	---	---	---	---	8.00	0
4.25	.12	---	---	---	---	---	---
4.50	.06	---	---	---	---	---	---
4.75	0	---	---	---	---	---	---

TABLE 7.—TRANSVERSE CORRELATION COEFFICIENT

x=17.5 ft				x=20.0 ft									
y ₁ =0.98 in.		y ₁ =2.02 in.		y ₁ =0.76 in.		y ₁ =1.01 in.		y ₁ =1.53 in.		y ₁ =2.03 in.		y ₁ =2.96 in.	
y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y
0.01	0.96	0.01	0.92	0.03	0.94	0.04	0.93	0.02	0.93	0.02	0.96	0.05	0.91
.07	.84	.02	.89	.08	.78	.09	.82	.07	.85	.07	.88	.10	.82
.12	.73	.07	.76	.13	.70	.14	.74	.12	.79	.12	.79	.15	.70
.22	.59	.11	.73	.18	.60	.19	.63	.17	.72	.17	.75	.20	.56
.32	.49	.17	.51	.23	.53	.24	.56	.22	.65	.22	.68	.25	.48
.42	.41	.21	.54	.33	.42	.34	.45	.32	.55	.32	.55	.30	.39
.62	.22	.27	.29	.43	.33	.44	.35	.42	.47	.42	.46	.35	.37
.82	.12	.31	.35	.53	.27	.54	.33	.52	.38	.52	.34	.40	.33
1.02	.068	.41	.21	.73	.16	.64	.23	.62	.32	.62	.30	.45	.29
1.22	.027	.47	.13	.93	.10	.74	.21	.72	.27	.82	.15	.50	.24
1.42	0	.61	.12	1.13	.054	.84	.18	.82	.21	.92	.032	.55	.24
1.62	0	.67	0	1.33	.023	.94	.11	.92	.15	1.02	0	.65	.18
1.82	.97	.81	-.040	1.43	0	1.04	.091	1.02	.10	1.02	.96	.75	.14
2.02	.89	.87	-.097	1.63	0	1.14	.105	1.22	.042	1.08	.78	.85	.088
2.22	.80	1.01	-.066	1.83	.093	1.24	.058	1.42	.040	1.13	.65	.95	.088
2.42	.64	1.07	0	2.03	.088	1.34	.018	1.62	-.020	1.23	.56	1.15	.073
2.62	.53	1.21	-.093	2.23	.078	1.44	0	1.82	0	1.33	.45	1.45	.067
2.82	.38	1.27	0	2.43	.068	1.64	.93	2.02	0	1.53	.32	1.75	0
3.02	.40	1.41	0	2.63	.51	1.84	.87	2.22	.90	1.73	.24	1.95	.89
3.22	.31	1.55	.95	2.83	.37	2.04	.76	2.42	.82	1.93	.14	2.15	.84
3.42	.24	1.69	.89	3.03	.34	2.24	.68	2.62	.75	2.13	.078	2.35	.80
3.62	.19	1.83	.79	3.23	.25	2.44	.59	2.82	.67	2.33	.044	2.55	.65
3.82	.16	1.97	.62	3.43	.17	2.64	.52	3.02	.53	2.53	.059	2.75	.47
4.02	.10	2.11	.52	3.63	.15	2.84	.40	3.22	.42	2.73	.042	2.95	.34
4.22	.058	2.25	.39	3.83	.13	3.04	.31	3.42	.33	2.93	.027	3.15	.18
4.42	---	2.39	.26	4.03	---	3.24	.27	3.62	.24	3.13	.011	3.35	.12
4.62	---	2.53	.21	4.23	---	3.44	.17	3.82	.17	3.33	.015	3.55	.052
4.82	---	2.67	.082	4.43	---	3.64	.12	4.02	.16	3.53	0	3.75	.033
5.02	---	2.81	.076	4.63	---	3.84	.083	4.22	.11	3.73	0	3.95	0
5.22	---	2.95	.026	4.83	---	4.04	.055	4.42	.10	3.93	---	4.15	---
5.42	---	3.09	.058	5.03	---	4.24	---	4.62	.044	4.13	---	4.35	---
5.62	---	3.23	.041	5.23	---	4.44	---	4.82	.011	4.33	---	4.55	---
5.82	---	3.37	.029	5.43	---	4.64	---	5.02	0	4.53	---	4.75	---
6.02	---	3.51	.028	5.63	---	4.84	---	5.22	---	4.73	---	4.95	---
6.22	---	3.65	0	5.83	---	5.04	---	5.42	---	4.93	---	5.15	---
6.42	---	3.79	---	6.03	---	5.24	---	5.62	---	5.13	---	5.35	---
6.62	---	3.93	---	6.23	---	5.44	---	5.82	---	5.33	---	5.55	---
6.82	---	4.07	---	6.43	---	5.64	---	6.02	---	5.53	---	5.75	---
7.02	---	4.21	---	6.63	---	5.84	---	6.22	---	5.73	---	5.95	---
7.22	---	4.35	---	6.83	---	6.04	---	6.42	---	5.93	---	6.15	---
7.42	---	4.49	---	7.03	---	6.24	---	6.62	---	6.13	---	6.35	---
7.62	---	4.63	---	7.23	---	6.44	---	6.82	---	6.33	---	6.55	---
7.82	---	4.77	---	7.43	---	6.64	---	7.02	---	6.53	---	6.75	---
8.02	---	4.91	---	7.63	---	6.84	---	7.22	---	6.73	---	6.95	---
8.22	---	5.05	---	7.83	---	7.04	---	7.42	---	6.93	---	7.15	---
8.42	---	5.19	---	8.03	---	7.24	---	7.62	---	7.13	---	7.35	---
8.62	---	5.33	---	8.23	---	7.44	---	7.82	---	7.33	---	7.55	---
8.82	---	5.47	---	8.43	---	7.64	---	8.02	---	7.53	---	7.75	---
9.02	---	5.61	---	8.63	---	7.84	---	8.22	---	7.73	---	7.95	---
9.22	---	5.75	---	8.83	---	8.04	---	8.42	---	7.93	---	8.15	---
9.42	---	5.89	---	9.03	---	8.24	---	8.62	---	8.13	---	8.35	---
9.62	---	6.03	---	9.23	---	8.44	---	8.82	---	8.33	---	8.55	---
9.82	---	6.17	---	9.43	---	8.64	---	9.02	---	8.53	---	8.75	---
10.02	---	6.31	---	9.63	---	8.84	---	9.22	---	8.73	---	8.95	---
10.22	---	6.45	---	9.83	---	9.04	---	9.42	---	8.93	---	9.15	---
10.42	---	6.59	---	10.03	---	9.24	---	9.62	---	9.13	---	9.35	---
10.62	---	6.73	---	10.23	---	9.44	---	9.82	---	9.33	---	9.55	---
10.82	---	6.87	---	10.43	---	9.64	---	10.02	---	9.53	---	9.75	---
11.02	---	7.01	---	10.63	---	9.84	---	10.22	---	9.73	---	9.95	---
11.22	---	7.15	---	10.83	---	10.04	---	10.42	---	9.93	---	10.15	---
11.42	---	7.29	---	11.03	---	10.24	---	10.62	---	10.13	---	10.35	---
11.62	---	7.43	---	11.23	---	10.44	---	10.82	---	10.33	---	10.55	---
11.82	---	7.57	---	11.43	---	10.64	---	11.02	---	10.53	---	10.75	---
12.02	---	7.71	---	11.63	---	10.84	---	11.22	---	10.73	---	10.95	---
12.22	---	7.85	---	11.83	---	11.04	---	11.42	---	10.93	---	11.15	---
12.42	---	7.99	---	12.03	---	11.24	---	11.62	---	11.13	---	11.35	---
12.62	---	8.13	---	12.23	---	11.44	---	11.82	---	11.33	---	11.55	---
12.82	---	8.27	---	12.43	---	11.64	---	12.02	---	11.53	---	11.75	---
13.02	---	8.41	---	12.63	---	11.84	---	12.22	---	11.73	---	11.95	---
13.22	---	8.55	---	12.83	---	12.04	---	12.42	---	11.93	---	12.15	---
13.42	---	8.69	---	13.03	---	12.24	---	12.62	---	12.13	---	12.35	---
13.62	---	8.83	---	13.23	---	12.44	---	12.82	---	12.33	---	12.55	---
13.82	---	8.97	---	13.43	---	12.64	---	13.02	---	12.53	---	12.75	---
14.02	---	9.11	---	13.63	---	12.84	---	13.22	---	12.73	---	12.95	---
14.22	---	9.25	---	13.83	---	13.04	---	13.42	---	12.93	---	13.15	---
14.42	---	9.39	---	14.03	---	13.24	---	13.62	---	13.13	---	13.35	---
14.62	---	9.53	---	14.23	---	13.44	---	13.82	---	13.33	---	13.55	---
14.82	---	9.67	---	14.43	---	13.64	---	14.02	---	13.53	---	13.75	---
15.02	---	9.81	---	14.63	---	13.84	---	14.22	---	13.73	---	13.95	---
15.22	---	9.95	---	14.83	---	14.04	---	14.42	---	13.93	---	14.15	---
15.42	---	10.09	---	15.03	---	14.24	---	14.62	---	14.13	---	14.35	---
15.62	---	10.23	---	15.23	---	14.44	---	14.82	---	14.33	---	14.55	---
15.82	---	10.37	---	15.43	---	14.64	---	15.02	---	14.53	---	14.75	---
16.02	---	10.51	---	15.63	---	14.84	---	15.22	---	14.73	---	14.95	---
16.22	---	10.65	---	15.83	---	15.04	---	15.42	---	14.93	---	15.15	---
16.42	---	10.79	---	16.03	---	15.24	---	15.62	---	15.13	---	15.35	---
16.62	---	10.93	---	16.23	---	15.44	---	15.82	---	15.33	---	15.55	---
16.82	---	11.07	---	16.43	---	15.64	---	16.02	---	15.53	---	15.75	---
17.02	---	11.21	---	16.63	---	15.84	---	16.22	---	15.73	---	15.95	---
17.22	---	11.35	---	16.83	---	16.04	---	16.42	---	15.93	---	16.15	---
17.42	---	11.49	---	17.03	---	16.24	---	16.62	---	16.13	---	16.35	---
17.62	---	11.63	---	17.23	---	16.44	---	16.82	---	16.33	---	16.55	---
17.82	---	11.77	---	17.43	---	16.64	---	17.02	---	16.53	---	16.75	---
18.02	---	11.91	---	17.63	---	16.84	---	17.22	---	16.73	---	16.95	---
18.22	---	12.05	---	17.83	---	17.04	---	17.42	---	16.93	---	17.15	---
18.42	---	12.19	---	18.03	---	17.24	---	17.62	---	17.13	---	17.35	---
18.62	---	12.33	---	18.23	---	17.44	---	17.82	---	17.33	---	17.55	---
18.82	---	12.47	---	18.43	---	17.64	---	18.02	---	17.53	---	17.75	---
19.02	---	12.61	---	18.63	---	17.84	---	18.22	---	17.73	---	17.95	---
19.22	---	12.75	---	18.83	---	18.04	---	18.42	---	17.93	---	18.15	---
19.42	---	12.89	---	19.03	---	18.24	---	18.62	---	18.13	---	18.35	---
19.62	---	13.03	---	19.23	---	18.44	---	18.82	---	18.33	---	18.55	---
19.82	---	13.17	---	19.43	---	18.64	---	19.02	---	18.53	---	18.75	---
20.02	---	13.31	---	19.63	---	18.84	---	19.22	---	18.73	---	18.95	---
20.22	---	13.45	---	19.83	---	19.04	---	19.42	---	18.93	---	19.15	---
20.42	---	13.59	---	20.03	---	19.24	---	19.62	---	19.13	---	19.35	---
20.62	---	13.73	---	20.23	---	19.44	---	19.82	---	19.33	---	19.55	---
20.82	---	13.87	---	20.43	---	19.64	---	20.02	---	19.53	---	19.75	---
21.02	---	14.01	---	20.63	---	19.84	---	20.22	---	19.73	---	19.95	---
21.22	---	14.15	---	20.83	---	20.04	---	20.42	---	19.93	---	20.15	---
21.42	---	14.29	---	21.03	---	20.24	---	20.62					

TABLE 7.—TRANSVERSE CORRELATION COEFFICIENT—Concluded

x=22.5 ft.										x=23.5 ft.									
y ₁ =0.85 in.		y ₁ =1.45 in.		y ₁ =2.32 in.		y ₁ =2.90 in.		y ₁ =3.92 in.		y ₁ =0.84 in.		y ₁ =1.54 in.		y ₁ =3.05 in.		y ₁ =3.71 in.			
y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y	y ₂ -y ₁ (in.)	R _y		
0.02	0.95	0.01	0.96	0.02	0.96	0.02	0.98	0.01	0.97	0.03	0.95	0.02	0.96	0.02	0.97	0.01	0.96		
.05	.87	.06	.89	.07	.84	.04	.91	.06	.87	.07	.83	.06	.83	.05	.88	.08	.85		
.10	.76	.11	.76	.12	.79	.09	.85	.11	.83	.12	.75	.11	.74	.10	.80	.13	.79		
.15	.68	.16	.69	.17	.71	.14	.78	.21	.69	.22	.64	.21	.63	.20	.67	.23	.65		
.20	.65	.21	.63	.22	.65	.24	.68	.31	.55	.32	.52	.31	.54	.30	.56	.33	.52		
.30	.49	.31	.53	.32	.56	.34	.59	.41	.37	.42	.42	.41	.41	.40	.48	.43	.50		
.40	.42	.41	.46	.42	.49	.44	.52	.61	.22	.52	.37	.61	.34	.60	.30	.53	.41		
.50	.33	.51	.37	.62	.32	.64	.35	.81	.12	.62	.28	.81	.23	.80	.23	.63	.29		
.60	.26	.61	.28	.82	.23	.84	.21	1.01	.055	.82	.22	1.01	.13	1.00	.19	.83	.18		
.80	.17	.81	.19	1.02	.15	1.04	.098	1.21	.097	1.02	.14	1.21	.10	1.20	.76	1.03	.079		
1.00	.11	1.01	.14	1.22	.094	1.24	.028	1.41	.061	1.22	.12	1.41	.084	1.40	.067	1.23	.053		
1.20	.093	1.21	.12	1.42	.11	1.44	-.076	1.61	0	1.42	.085	1.71	.048	1.60	.024	1.43	0		
1.40	.067	1.41	.083	1.62	0	1.64	-.051	1.81	.068	1.62	.045	2.01	.026	1.80	0	1.63	-.076		
1.60	.042	1.61	.045	1.80	0	1.84	-.14	2.01	.051	1.82	.049	2.31	.029	2.00	0	1.83	-.16		
1.80	.030	1.81	.051	2.00	.94	2.04	-.099	2.41	0	2.02	.015	2.61	0	2.00	.98	2.03	-.11		
2.00	.033	2.01	.028	2.20	.80	2.24	.92	2.41	.92	2.22	.016	2.91	0	2.00	.89	2.23	0		
2.20	0	2.21	0	2.40	.71	2.44	.88	2.60	.86	2.32	.017	3.11	.017	2.00	.81	2.44	.97		
2.40	0	2.51	0	2.60	.60	2.64	.82	2.80	.76	2.52	.018	3.31	.018	2.00	.77	2.64	.84		
2.60	0	2.61	.92	2.80	.51	2.84	.68	3.00	.60	2.82	0	3.51	.019	2.00	.67	2.84	.76		
2.80	.91	2.81	.79	3.00	.45	3.04	.55	3.20	.51	3.03	.97	3.71	.020	2.00	.59	3.04	.72		
3.00	.72	3.01	.73	3.20	.38	3.24	.46	3.40	.39	3.07	.87	3.91	.021	2.00	.52	3.24	.59		
3.20	.44	3.21	.62	3.40	.33	3.44	.38	3.60	.29	3.11	.82	4.11	.022	2.00	.44	3.44	.50		
3.40	.21	3.41	.49	3.60	.28	3.64	.33	3.80	.23	3.18	.59	4.31	.023	2.00	.34	3.64	.36		
3.60	.15	3.61	.43	3.80	.25	3.84	.28	4.00	.12	3.21	.36	4.51	.024	2.00	.26	3.84	.28		
3.80	.15	3.81	.40	4.00	.22	4.04	.25	4.20	.09	3.24	.26	4.71	.025	2.00	.19	4.04	.21		
4.00	.15	4.01	.37	4.20	.20	4.24	.22	4.40	.07	3.27	.19	4.91	.026	2.00	.14	4.24	.14		
4.20	.15	4.21	.34	4.40	.18	4.44	.20	4.60	.05	3.30	.18	5.11	.027	2.00	.11	4.44	.12		
4.40	.15	4.41	.31	4.60	.16	4.64	.18	4.80	.04	3.33	.17	5.31	.028	2.00	.08	4.64	.08		
4.60	.15	4.61	.28	4.80	.14	4.84	.16	5.00	.03	3.36	.16	5.51	.029	2.00	.05	4.84	.05		
4.80	.15	4.81	.25	5.00	.12	5.04	.14	5.20	.02	3.39	.15	5.71	.030	2.00	.02	5.04	.02		
5.00	.15	5.01	.22	5.20	.10	5.24	.12	5.40	.01	3.42	.14	5.91	.031	2.00	.01	5.24	.01		
5.20	.15	5.21	.19	5.40	.08	5.44	.10	5.60	.00	3.45	.13	6.11	.032	2.00	.00	5.44	.00		
5.40	.15	5.41	.16	5.60	.06	5.64	.08	5.80	.00	3.48	.12	6.31	.033	2.00	.00	5.64	.00		
5.60	.15	5.61	.13	5.80	.04	5.84	.06	6.00	.00	3.51	.11	6.51	.034	2.00	.00	5.84	.00		
5.80	.15	5.81	.10	6.00	.02	6.04	.04	6.20	.00	3.54	.10	6.71	.035	2.00	.00	6.04	.00		
6.00	.15	6.01	.07	6.20	.01	6.24	.02	6.40	.00	3.57	.09	6.91	.036	2.00	.00	6.24	.00		
6.20	.15	6.21	.04	6.40	.00	6.44	.01	6.60	.00	3.60	.08	7.11	.037	2.00	.00	6.44	.00		
6.40	.15	6.41	.01	6.60	.00	6.64	.00	6.80	.00	3.63	.07	7.31	.038	2.00	.00	6.64	.00		
6.60	.15	6.61	.00	6.80	.00	6.84	.00	7.00	.00	3.66	.06	7.51	.039	2.00	.00	6.84	.00		
6.80	.15	6.81	.00	7.00	.00	7.04	.00	7.20	.00	3.69	.05	7.71	.040	2.00	.00	7.04	.00		
7.00	.15	7.01	.00	7.20	.00	7.24	.00	7.40	.00	3.72	.04	7.91	.041	2.00	.00	7.24	.00		
7.20	.15	7.21	.00	7.40	.00	7.44	.00	7.60	.00	3.75	.03	8.11	.042	2.00	.00	7.44	.00		
7.40	.15	7.41	.00	7.60	.00	7.64	.00	7.80	.00	3.78	.02	8.31	.043	2.00	.00	7.64	.00		
7.60	.15	7.61	.00	7.80	.00	7.84	.00	8.00	.00	3.81	.01	8.51	.044	2.00	.00	7.84	.00		
7.80	.15	7.81	.00	8.00	.00	8.04	.00	8.20	.00	3.84	.00	8.71	.045	2.00	.00	8.04	.00		
8.00	.15	8.01	.00	8.20	.00	8.24	.00	8.40	.00	3.87	.00	8.91	.046	2.00	.00	8.24	.00		
8.20	.15	8.21	.00	8.40	.00	8.44	.00	8.60	.00	3.90	.00	9.11	.047	2.00	.00	8.44	.00		
8.40	.15	8.41	.00	8.60	.00	8.64	.00	8.80	.00	3.93	.00	9.31	.048	2.00	.00	8.64	.00		
8.60	.15	8.61	.00	8.80	.00	8.84	.00	9.00	.00	3.96	.00	9.51	.049	2.00	.00	8.84	.00		
8.80	.15	8.81	.00	9.00	.00	9.04	.00	9.20	.00	3.99	.00	9.71	.050	2.00	.00	9.04	.00		
9.00	.15	9.01	.00	9.20	.00	9.24	.00	9.40	.00	4.02	.00	9.91	.051	2.00	.00	9.24	.00		
9.20	.15	9.21	.00	9.40	.00	9.44	.00	9.60	.00	4.05	.00	10.11	.052	2.00	.00	9.44	.00		
9.40	.15	9.41	.00	9.60	.00	9.64	.00	9.80	.00	4.08	.00	10.31	.053	2.00	.00	9.64	.00		
9.60	.15	9.61	.00	9.80	.00	9.84	.00	10.00	.00	4.11	.00	10.51	.054	2.00	.00	9.84	.00		
9.80	.15	9.81	.00	10.00	.00	10.04	.00	10.20	.00	4.14	.00	10.71	.055	2.00	.00	10.04	.00		
10.00	.15	10.01	.00	10.20	.00	10.24	.00	10.40	.00	4.17	.00	10.91	.056	2.00	.00	10.24	.00		
10.20	.15	10.21	.00	10.40	.00	10.44	.00	10.60	.00	4.20	.00	11.11	.057	2.00	.00	10.44	.00		
10.40	.15	10.41	.00	10.60	.00	10.64	.00	10.80	.00	4.23	.00	11.31	.058	2.00	.00	10.64	.00		
10.60	.15	10.61	.00	10.80	.00	10.84	.00	11.00	.00	4.26	.00	11.51	.059	2.00	.00	10.84	.00		
10.80	.15	10.81	.00	11.00	.00	11.04	.00	11.20	.00	4.29	.00	11.71	.060	2.00	.00	11.04	.00		
11.00	.15	11.01	.00	11.20	.00	11.24	.00	11.40	.00	4.32	.00	11.91	.061	2.00	.00	11.24	.00		
11.20	.15	11.21	.00	11.40	.00	11.44	.00	11.60	.00	4.35	.00	12.11	.062	2.00	.00	11.44	.00		
11.40	.15	11.41	.00	11.60	.00	11.64	.00	11.80	.00	4.38	.00	12.31	.063	2.00	.00	11.64	.00		
11.60	.15	11.61	.00	11.80	.00	11.84	.00	12.00	.00	4.41	.00	12.51	.064	2.00	.00	11.84	.00		
11.80	.15	11.81	.00	12.00	.00	12.04	.00	12.20	.00	4.44	.00	12.71	.065	2.00	.00	12.04	.00		
12.00	.15	12.01	.00	12.20	.00	12.24	.00	12.40	.00	4.47	.00	12.91	.066	2.00	.00	12.24	.00		
12.20	.15	12.21	.00	12.40	.00	12.44	.00	12.60	.00	4.50	.00	13.11</							

TABLE 8.—LONGITUDINAL CORRELATION COEFFICIENT

$x_1=17.5$ ft				$x_1=20.0$ ft					
$y=0.97$ in.		$y=2.01$ in.		$y=0.47$ in.		$y=1.55$ in.		$y=2.56$ in.	
x_2-x_1 (in.)	R_z	x_2-x_1 (in.)	R_z	x_2-x_1 (in.)	R_z	x_2-x_1 (in.)	R_z	x_2-x_1 (in.)	R_z
0.07	0.95	0.08	0.88	0.03	0.89	0.07	0.89	0.05	0.95
.15	.84	.15	.83	.08	.85	.12	.83	.10	.88
.31	.78	.30	.72	.15	.79	.22	.76	.20	.82
.51	.70	.50	.53	.23	.71	.32	.69	.40	.65
.81	.53	.80	.44	.33	.70	.52	.63	.60	.50
1.11	.49	1.10	.29	.53	.59	.72	.56	.90	.44
1.51	.34	1.50	.17	.73	.48	.92	.48	1.20	.23
2.01	.23	2.00	.13	1.01	.38	1.12	.42	1.50	.18
2.51	.14	2.50	.10	1.33	.32	1.32	.36	2.00	.094
3.01	.12	3.00	.080	1.63	.27	1.53	.34	2.50	.077
3.51	.075	3.50	.049	2.03	.20	1.84	.25	3.00	.038
4.01	.058	4.00	0	2.43	.17	2.12	.22	3.50	.046
4.51	.038	4.50	.015	2.83	.12	2.62	.11	4.00	.022
5.01	0	5.00	.035	3.23	.085	3.02	.14	4.51	0
— .03	.97	5.50	0	3.63	.054	3.52	.071	5.00	0
— .08	.92	— .03	.81	4.03	.033	4.02	.028	— .04	.91
— .13	.88	— .08	.80	4.43	.050	4.52	0	— .06	.92
— .28	.74	— .28	.61	5.33	0	— .02	.97	— .11	.88
— .48	.68	— .48	.49	— .02	.98	— .05	.97	— .21	.77
— .78	.59	— .78	.44	— .05	.96	— .10	.95	— .41	.60
— 1.08	.49	— 1.08	.32	— .14	.82	— .20	.82	— .61	.51
— 1.48	.35	— 1.48	.28	— .20	.71	— .40	.71	— .91	.34
— 1.98	.23	— 1.98	.20	— .49	.60	— .60	.62	— 1.21	.24
— 2.48	.22	— 2.48	.15	— .79	.44	— .90	.51	— 1.51	.21
— 2.98	.14	— 2.98	.12	— 1.09	.33	— 1.20	.37	— 2.01	.12
— 3.48	.10	— 3.48	.13	— 1.49	.22	— 1.50	.32	— 2.51	.065
— 3.98	.043	— 3.98	.059	— 1.99	.15	— 2.00	.24	— 3.01	.022
— 4.44	.022	— 4.48	.045	— 2.49	.11	— 2.50	.15	— 3.51	0
— 4.98	0	— 4.98	.025	— 2.99	.059	— 3.00	.13	— 4.01	0
-----	-----	— 5.48	.025	— 3.49	.056	— 3.50	.10	-----	-----
-----	-----	-----	-----	— 3.99	.025	— 3.84	.10	-----	-----
-----	-----	-----	-----	— 4.69	.049	— 4.34	.071	-----	-----
-----	-----	-----	-----	— 5.19	.015	— 4.84	.028	-----	-----
-----	-----	-----	-----	— 5.69	0	— 5.34	0	-----	-----
-----	-----	-----	-----	— 6.19	0	-----	-----	-----	-----

$x_1=22.5$ ft		$x_1=24.5$ ft		$x_1=25.4$ ft					
$y=2.32$ in.		$y=3.01$ in.		$y=0.98$ in.		$y=3.66$ in.		$y=5.96$ in.	
x_2-x_1 (in.)	R_z	x_2-x_1 (in.)	R_z	x_2-x_1 (in.)	R_z	x_2-x_1 (in.)	R_z	x_2-x_1 (in.)	R_z
0.03	0.94	0.03	0.96	0.03	0.94	0.02	0.97	0.03	0.97
.08	.91	.08	.91	.10	.92	.07	.96	.09	.94
.16	.87	.15	.86	.17	.86	.14	.89	.16	.92
.31	.76	.30	.78	.32	.75	.29	.78	.31	.76
.51	.65	.50	.67	.52	.60	.49	.68	.51	.62
.81	.52	.80	.53	.82	.54	.79	.56	.81	.42
1.11	.38	1.10	.36	1.12	.48	1.09	.40	1.11	.50
1.51	.28	1.50	.33	1.52	.32	1.49	.37	1.51	.34
2.01	.20	2.00	.20	2.02	.26	1.99	.22	2.01	.26
2.51	.12	2.50	.16	2.52	.13	2.49	.17	2.51	.16
3.01	.040	3.00	.10	3.02	.19	2.99	.15	3.01	.10
3.51	.019	3.50	.055	3.52	.077	3.49	.11	3.51	.076
4.01	0	4.00	0	4.02	.034	3.99	.047	4.01	.039
— .03	.97	— .03	.97	4.52	0	4.49	0	4.51	0
— .07	.95	— .09	.94	— .03	.93	4.99	0	5.01	0
— .14	.91	— .17	.84	— .08	.92	— .03	.96	— .03	.99
— .29	.79	— .32	.76	— .28	.81	— .08	.94	— .07	.99
— .49	.65	— .52	.65	— .48	.67	— .15	.90	— .15	.92
— .79	.51	— .82	.55	— .78	.47	— .30	.82	— .30	.83
— 1.09	.38	— 1.12	.39	— 1.08	.45	— .50	.69	— .50	.75
— 1.49	.26	— 1.52	.28	— 1.48	.36	— .80	.60	— .80	.60
— 1.99	.13	— 2.02	.18	— 1.98	.26	— 1.10	.49	— 1.10	.51
— 2.49	.10	— 2.52	.13	— 2.48	.21	— 1.50	.39	— 1.50	.36
— 2.99	.055	— 3.02	.095	— 2.98	.16	— 2.00	.26	— 2.00	.23
— 3.49	0	— 3.52	.038	— 3.48	.10	— 2.50	.15	— 2.50	.15
-----	-----	— 4.02	.019	— 3.98	.15	— 3.00	.14	— 3.00	.10
-----	-----	— 4.52	0	— 4.48	.058	— 3.50	.11	— 3.50	.084
-----	-----	-----	-----	— 5.00	0	— 4.00	.085	— 4.00	.095
-----	-----	-----	-----	-----	-----	— 4.50	.018	— 4.50	.065
-----	-----	-----	-----	-----	-----	— 5.00	0	— 5.00	0

REPORT 1031

A STUDY OF THE USE OF EXPERIMENTAL STABILITY DERIVATIVES IN THE CALCULATION OF THE LATERAL DISTURBED MOTIONS OF A SWEEP-WING AIRPLANE AND COMPARISON WITH FLIGHT RESULTS¹

By JOHN D. BIRD and BYRON M. JAQUET

SUMMARY

An investigation was made to determine the accuracy with which the lateral flight motions of a swept-wing airplane could be predicted from experimental stability derivatives determined in the 6-foot-diameter rolling-flow test section and 6- by 6-foot curved-flow test section of the Langley stability tunnel. In addition determination of the significance of including the nonlinear aerodynamic effects of sideslip in the calculations of the motions was desired. All experimental aerodynamic data necessary for prediction of the lateral flight motions are presented along with a number of comparisons between flight and calculated motions caused by rudder and aileron disturbances.

In general, the agreement between the calculated and measured motions of the airplane considered was good when the effects of all control movements were taken into account. The greatest disagreement occurred at lift coefficients where Reynolds number effects on the experimental derivatives would be expected to be high, which for the case considered was for lift coefficients above about 0.8 when wing slots were used. The nonlinear effects of sideslip for this airplane were not very significant for the motions considered, which generally involved sideslip angles less than 10° .

INTRODUCTION

For the past few years, numerous investigations have been made in the Langley stability tunnel to determine the effects of geometric variables on the static-, rolling-, yawing-, and pitching-stability derivatives of various airplane configurations. (See references 1 to 4.) In the past, however, none of the experimental data have been compared with data obtained in flight to determine its relative worth. The purpose of the present report is to determine the applicability of the experimental stability derivatives to the prediction of the lateral disturbed motions of an airplane in flight. The equations used for calculating the motions are given in the appendix.

A $\frac{1}{8}$ -scale model of a swept-wing version of a conventional fighter airplane, which was selected because of the large amount of flight data available (see references 5 and 6), was tested in the 6-foot-diameter rolling-flow test section and 6- by 6-foot curved-flow test section of the Langley stability tunnel to determine all the stability derivatives which are

usually considered necessary to calculate the lateral motions arising from a disturbance caused by the rudder or the ailerons. Comparisons have been made between the flight and calculated lateral motions for a wide range of conditions in gliding flight.

A few calculations have been made to determine the effects of nonlinear variations of the aerodynamic forces and moments with the angle of sideslip.

COEFFICIENTS AND SYMBOLS

The results of the wind-tunnel tests are presented as standard NACA coefficients of forces and moments. Moment coefficients are referred to a center of gravity located at 21.8 percent of the mean aerodynamic chord. The wind-tunnel data and motion calculations are referred to the stability axes, which are a system of axes having their origin at the center of gravity and in which the Z -axis is in the plane of symmetry and perpendicular to the relative wind, the X -axis is in the plane of symmetry and perpendicular to the Z -axis, and the Y -axis is perpendicular to the plane of symmetry. The positive directions of the stability axes and of angular displacements of the airplane and control surfaces are shown in figure 1.

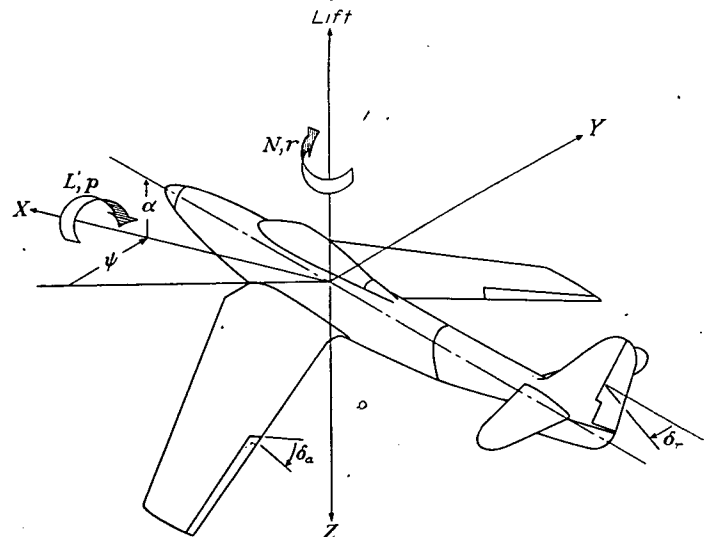


FIGURE 1.—System of stability axes. Arrows indicate positive forces, moments, and angular displacements.

¹ Supersedes NACA TN 2013, "A Study of the Use of Experimental Stability Derivatives in the Calculation of the Lateral Disturbed Motions of a Swept-Wing Airplane and Comparison with Flight Results" by John D. Bird and Byron M. Jaquet, 1950.

The coefficients and symbols are defined as follows:

C_L	lift coefficient ($Lift/qS$)
$C_{L_{max}}$	maximum lift coefficient
C_X	longitudinal-force coefficient (X/qS)
C_Y	lateral-force coefficient (Y/qS)
C_l	rolling-moment coefficient (L/qSb)
C_n	yawing-moment coefficient (N/qSb)
$C_{Y_\psi} = \frac{\partial C_Y}{\partial \psi}$	
$C_{l_\psi} = \frac{\partial C_l}{\partial \psi}$	
$C_{n_\psi} = \frac{\partial C_n}{\partial \psi}$	
$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$	
$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$	
$C_{Y_p} = \frac{\partial C_Y}{\partial \frac{pb}{2V}}$	
$C_{l_p} = \frac{\partial C_l}{\partial \frac{pb}{2V}}$	
$C_{n_p} = \frac{\partial C_n}{\partial \frac{pb}{2V}}$	
$C_{Y_r} = \frac{\partial C_Y}{\partial \frac{rb}{2V}}$	
$C_{l_r} = \frac{\partial C_l}{\partial \frac{rb}{2V}}$	
$C_{n_r} = \frac{\partial C_n}{\partial \frac{rb}{2V}}$	
I_{X_0}	moment of inertia about longitudinal principal axis
I_{Y_0}	moment of inertia about spanwise principal axis
I_{Z_0}	moment of inertia about normal principal axis
X	longitudinal force along X -axis
Y	lateral force along Y -axis
Z	normal force along Z -axis ($Lift = -Z$)
L	rolling-moment about X -axis
N	yawing-moment about Z -axis
$pb/2V$	wing-tip helix angle, radians
$rb/2V$	yawing-velocity parameter, radian measure
p	rolling angular velocity about X -axis
r	yawing angular velocity about Z -axis
v	linear velocity of airplane along Y -axis
V	free-stream velocity along X -axis
V_c	calibrated airspeed, based on sea-level density of air
β	angle of sideslip; $\beta = -\psi$ in wind-tunnel tests ($\tan^{-1} \frac{v}{V}$)
α_R	angle of attack of wing root chord line

α	angle of attack of thrust line ($\alpha_R - 1.2^\circ$)
ψ	angle of yaw, degrees
i_t	angle of incidence of stabilizer with respect to thrust line, positive when trailing edge is down
δ	control-surface deflection, measured in a plane perpendicular to hinge axis
Λ	angle of sweepback, degrees
q	free-stream dynamic pressure ($\frac{1}{2} \rho V^2$)
S	wing area
b	wing span
A	aspect ratio (b^2/S)
ρ	mass density of air
T	time
$T_{1/2}$	time to damp to half amplitude
P	period
l	tail length
Subscripts:	
a	aileron
r	rudder
f	flap
v	vertical tail

WIND-TUNNEL TESTS

APPARATUS AND MODEL

The experimental static-lateral-stability derivatives, rolling-stability derivatives, and yawing-stability derivatives were determined from tests conducted in the Langley stability tunnel in which rolling or curved flight is simulated by rolling or curving the air stream about a rigidly mounted model.

The tests were made on a conventional six-component balance system with the model mounted at the flight center of gravity which is at 21.8 percent of the mean aerodynamic chord of the wing.

The full-scale airplane (a swept-wing version of a conventional fighter) had the quarter-chord line of the wing, just outboard of the intake ducts, swept back 35° . Some of the pertinent airplane characteristics are given in table I. More details of the airplane may be obtained from references 5 and 6.

The $\frac{1}{9}$ -scale model shown in figure 2 and in the photographs of figures 3 to 6 was constructed of laminated mahogany, finished in clear lacquer, and all surfaces were highly polished. The model propeller had three metal blades set at an angle of 28° at the 0.75 radius. All propeller-on tests were made with windmilling propeller. The model wing had a removable leading edge so that slats of 0 percent, 40 percent, and 80 percent of the swept span could be used interchangeably to simulate those of the full-scale airplane. The top surfaces of the slats were cast to the contour of the airfoil and the slats were extended by means of metal brackets which also act as fences to reduce spanwise flow along the slot. A cross section through the slot and slat is shown in reference 5.

The wing had a plain trailing-edge flap with a chord of 15.1 percent of the wing chord measured perpendicular to the hinge axis. The gap was sealed for all tests. As in the case of the airplane the main wheels of the model were fixed for all tests; whereas the nose wheel and nose-wheel doors were removed for all flaps-up tests.

Shown in figure 2 are the two ventral fins tested on the model. The large ventral fin was used for all tests except a few with the 80-percent-span slot configuration for which the small ventral fin was used.

TEST AND TEST CONDITIONS

Trim tests.—Model trim lift coefficients of 0.33, 0.55, 0.76, and 0.95 were selected as representative of those obtained in flight tests. The angle of incidence of the horizontal tail was measured with respect to the thrust line.

In order to determine the trim angles of the horizontal tail for the previously mentioned lift coefficients, tests were made through the angle-of-attack range with the horizontal tail set at -5° , -3° , and 1° incidence. From these tests the trim angles of the horizontal tail were determined. (See table II.)

Static tests.—In order to determine the static-stability derivatives C_{l_p} , C_{n_p} , and C_{Y_p} , the model was tested at $\psi = \pm 5^\circ$ through an angle-of-attack range of $\alpha = -2^\circ$ to $\alpha = 23^\circ$ for the flaps up (trim $C_L = 0.33$) and $\alpha = -2^\circ$ to $\alpha = 18^\circ$ for the flaps down (trim $C_L = 0.76$) for each of the slot configurations. Tests were also made at all selected test trim lift coefficients through an angle-of-yaw range of $\psi = \pm 20^\circ$ to determine the variations of C_l , C_n , and C_Y with ψ for all slot configurations.

Tare tests were made for the 40-percent-span slot configuration (flaps up and down). The effect of the slots on the tares was assumed to be small; therefore, the tares for the 40-percent-span slot configuration were applied to all configurations. The Mach and Reynolds numbers for the tests were 0.17 and 1.01×10^6 , respectively.

Rolling-flow tests.—Tests were conducted in the 6-foot-diameter rolling-flow test section of the Langley stability tunnel, wherein rolling flight of the model was simulated by rotating the air stream. The model was mounted rigidly on a conventional support strut. Details of this test procedure are given in reference 1.

All slot configurations were tested through the angle-of-attack range with the flaps up (trim $C_L = 0.33$) and with the flaps down (trim $C_L = 0.76$) at helix angles $pb/2V$ of 0, ± 0.0253 , and ± 0.0757 radian. The slopes of C_l , C_n , and C_Y plotted against $pb/2V$ are the derivatives C_{l_p} , C_{n_p} , and C_{Y_p} . The 40-percent-span slot configuration (flaps up and down) was tested at the selected trim lift coefficients for the previously mentioned values of $pb/2V$ from $\psi = 0^\circ$ to $\psi = 20^\circ$ to determine the variation of C_{l_p} , C_{n_p} , and C_{Y_p} with ψ . The Mach number and Reynolds number for the rolling-flow tests were 0.17 and 1.01×10^6 , respectively.

Yawing-flow tests.—Yawing-flow tests were conducted in the 6- by 6-foot curved-flow test section of the Langley stability tunnel. In this section, curved flight is simulated

approximately by directing the air in a curved path about a fixed model.

All slot configurations were tested in curved flow through the angle-of-attack range with the flaps up (trim $C_L = 0.33$) and with the flaps down (trim $C_L = 0.76$) at values at $rb/2V$ of 0, -0.039 , -0.082 , and -0.108 . The slopes of C_l , C_n , and C_Y plotted against $rb/2V$ are the derivatives C_{l_r} , C_{n_r} , and C_{Y_r} .

The 40-percent-span slot configuration (flaps up and down) was tested through an angle-of-yaw range of $\psi = \pm 20^\circ$ for values of $rb/2V$ of 0, -0.039 , -0.082 , and -0.108 at the previously mentioned trim lift coefficients to determine the variation of C_{l_r} , C_{n_r} , and C_{Y_r} with ψ . The values presented herein are the average of the results at corresponding positive and negative angles of yaw.

The 40-percent-span slot configuration (flaps up) was tested at a trim lift coefficient of 0.33 through the angle-of-yaw range with the propeller off.

The yawing-flow tests were made at a Mach number of 0.13 and a Reynolds number of 0.8×10^6 .

CORRECTIONS

Approximate jet-boundary corrections based on methods derived for unswept wings were applied to the angle of attack, longitudinal-force coefficient, and rolling-moment coefficient, and a blocking correction of 1.01 was applied to the dynamic pressure.

Corrections for the effect of the support strut have been applied to C_X , C_L , C_l , C_n , C_Y , C_{l_p} , C_{n_p} , and C_{Y_p} . In rolling flow and curved flow, accurate tares were difficult to obtain; and, as a result, the derivatives C_{l_p} , C_{n_p} , C_{Y_p} , C_{l_r} , C_{n_r} , and C_{Y_r} are not corrected for the effects of the support strut.

The derivative C_{l_p} was corrected for the effective pitching velocity, which exists when the model is tested at an angle of yaw, by the following equation:

$$C_{l_p} = C_{l_p}' \cos \psi + f(A, \Lambda, \psi)$$

where C_{l_p}' is measured about the wind axis, and $f(A, \Lambda, \psi)$, which is small as compared with $C_{l_p}' \cos \psi$, is a function determined by use of the methods of reference 4. Corresponding effective pitching corrections were not applied to the derivatives C_{n_p} and C_{Y_p} .

A correction was also applied to the derivative C_{Y_r} to account for the error caused by the cross-tunnel static-pressure gradient which is associated with curved flow.

EXPERIMENTAL RESULTS

The experimental data are discussed briefly with reference to the effects of the slots and angle of yaw on the aerodynamic characteristics of the model, because the effects of these variables on the rotary derivatives have not been investigated extensively to date. The figures which present the results obtained in the present investigation are listed in table III.

The basic lift and longitudinal-force data of figures 7 and 8 are generally in good agreement with larger scale tests of another model of the same airplane, as given in reference 7.

The main effect of the slots is to extend the linear range of those stability derivatives which are largely contributed by the wing to higher lift coefficients in a manner similar to the effect of slots on the lift curve of a wing. (See figs. 7 to 10 and 12, 13, 15, and 16.) One significant effect of the slots is on the damping in yaw C_{n_r} , which increased as the slot span is increased. When the 80-percent-span slots are used C_{n_r} is increased (over that of the unslotted configuration) by about 25 percent at $C_L=0$. (See fig. 15 (a).) When the flaps are deflected (fig. 15 (b)), the effects of the slots on the yawing-stability derivatives are not as great as when the flaps are retracted.

A comparison of figures 11(a) to 11(f) shows that C_{n_y} varies to some extent through the yaw range, C_{Y_ψ} is approximately constant, and for any given lift coefficient C_{l_y} is

approximately constant between $\psi=-10^\circ$ and $\psi=10^\circ$. As the angle of yaw is increased (fig. 14 (a)), C_{Y_ψ} tends to decrease, C_{n_r} remains approximately constant, and C_{l_y} decreases slightly. The variations are similar when the flaps are deflected. (See fig. 14 (b).)

The tests for the determination of the variation of C_{Y_r} , C_{n_r} , and C_{l_r} with ψ were made for negative values of $rb/2V$ only. Results for positive values of $rb/2V$ and positive ψ were obtained by assuming that the model was essentially symmetrical about the XZ -plane and by utilizing the results for the corresponding opposite angles of yaw and $rb/2V$ with regard for signs. This procedure amounted to averaging the derivatives for corresponding positive and negative angles of yaw.

In general, as the angle of yaw is increased from 0° to 20° , the damping in yaw C_{n_r} is increased by about 15 percent, and C_{Y_r} and C_{l_r} are decreased slightly (fig. 17 (a)). Deflection of the flaps or removal of the propeller does not appreciably

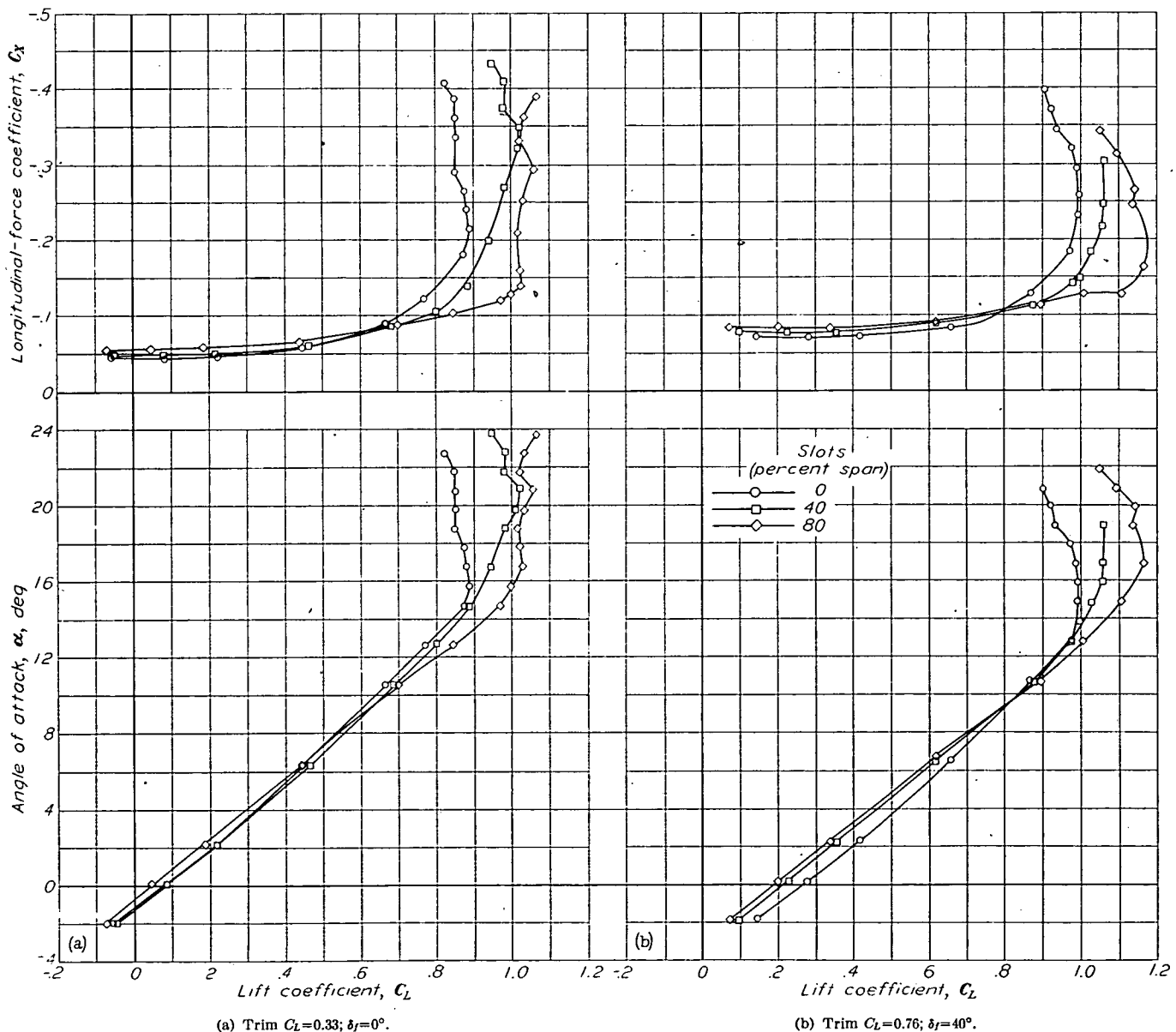


FIGURE 7.—Variation of longitudinal-force coefficient and angle of attack with lift coefficient for three slot configurations. Large ventral fin on; propeller on; $\psi=0^\circ$; $R=1.01 \times 10^6$.

change the variations of the yawing-stability derivatives with angle of yaw. (See fig. 17.)

Because C_{n_r} is largely dependent on the size of the vertical tail it is approximately true throughout the yaw range that

$$C_{n_r} = -\frac{2l_v}{b} C_{n_{\beta}}$$

Substitution of the proper values of the span, $C_{n_{\beta}}$, and the tail length l_v indicates that the trend of the variation of C_{n_r} with ψ shown in figure 17 is reasonable.

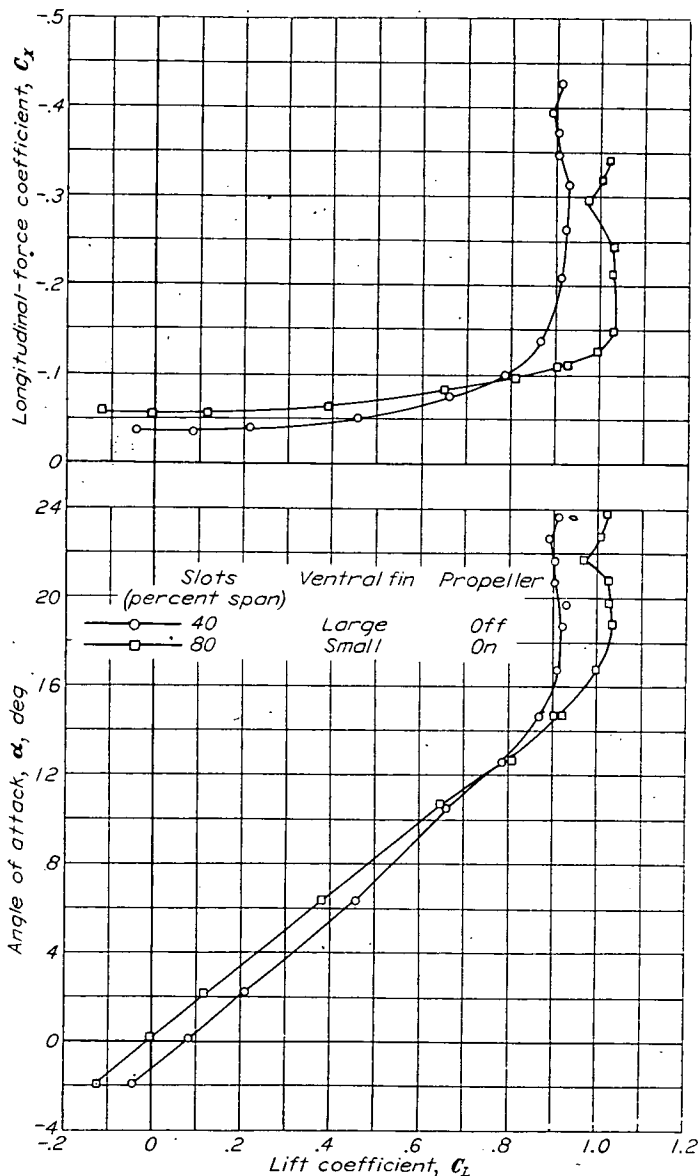


FIGURE 8.—Variation of longitudinal-force coefficient and angle of attack with lift coefficient for two slot configurations. Trim $C_L=0.33$; $\psi=0^\circ$; $\delta_r=0^\circ$; $R=1.01 \times 10^6$.

The relative constancy of C_{Y_p} , C_{l_p} , C_{n_p} , C_{Y_r} , C_{l_r} , and C_{n_r} with angle of yaw as indicated by figures 14 and 17 and the linearity of the curves of C_{Y_r} , C_{l_r} , and C_{n_r} plotted against ψ for angles of yaw up to approximately 10° (fig. 11) were factors which indicated that nonlinearities were not of first-order importance for this airplane in the calculation of motions involving reasonably small variations in ψ . Consequently, most of the motion calculations neglect the effect of ψ on the stability derivatives.

The results of unpublished tests of swept wings at Reynolds numbers to 8.0×10^6 in the Langley 19-foot pressure tunnel indicate that the linear part of the curve of C_{l_p} plotted against C_L is increased by an increase in Reynolds number. The curves of C_{l_p} against C_L given herein agree well with those obtained in flight (references 5 and 6) and in tests of a $\frac{1}{4.5}$ -scale model (reference 7) except at lift coefficients above $C_L=0.8$ where the magnitude of the present test values decreases for the slotted-wing configurations. The linear parts of the curves of the rolling- and yawing-stability derivatives are believed to be extended similarly to higher lift coefficients if the Reynolds number is increased.

MOTION CALCULATION METHODS

The lateral disturbed motions of the test airplane were calculated from the aerodynamic data obtained from the tests described in the section entitled "Wind-Tunnel Tests." The mass and dimensional characteristics of the airplane are given in table I, and a tabulation of the flight conditions for which lateral disturbed motions were calculated for comparison with the flight motions is given in table IV. Most of the calculations involved dynamic derivatives which were constant for a given lift coefficient as is usually employed in the theory of small disturbances used in lateral-stability calculations.

Solutions of the lateral equations of motion, given in the appendix, were obtained for unit step disturbances in roll and yaw by the method described in reference 8. The aileron and rudder deflections during the flight motion under investigation were then approximated by a series of step functions usually at $\frac{1}{2}$ -second intervals. The motion arising from the control movements was then calculated by

$$\beta_n, p_n, \text{ or } r_n = \sum_{n=0}^m \left[(\delta_{m-n})_{\text{rudder}} (B_n)_{\text{yaw}} + (\delta_{m-n})_{\text{aileron}} (B_n)_{\text{roll}} \right]$$

where B_n is the value of the unit solution caused by $\Delta C_n=1$ or $\Delta C_l=1$ at a time $T=kn$, and δ_{m-n} is the fraction of the unit disturbance applied by the rudder or aileron at a time $T=k(m-n)$. The rudder and aileron effectiveness were

obtained from reference 7. This procedure is essentially an approximate evaluation of Duhamel's integral and was considered sufficiently accurate for these calculations. Reference 8 gives a more exact graphical evaluation of this integral. The yawing moment caused by aileron deflection and the rolling moment caused by rudder deflection were not considered of enough significance to warrant their inclusion in these calculations. In some cases, however, these factors may be of greater significance.

In a few cases, unit solutions to the equations of motion were obtained by a Laplace transform procedure which has been adapted for use with automatic digital computers. The results were, of course, identical with those presented in this report.

Calculations of the lateral motions for a few cases employing a nonlinear variation of rolling-moment and yawing-moment coefficients with angle of sideslip and a variation of C_{n_r} with angle of sideslip were carried out by use of the Kutta three-eighths rule for solving the lateral equations of motion. (See reference 9.) All lateral-motion calculations were made on an automatic digital computing machine.

CALCULATED LATERAL MOTIONS

GENERAL

The flight records corresponding to the motions calculated for this report showed that the motions resulting from right and left control movements were not exactly of opposite magnitude. This result indicates that there was some asymmetry in the characteristics of the airplane, although the

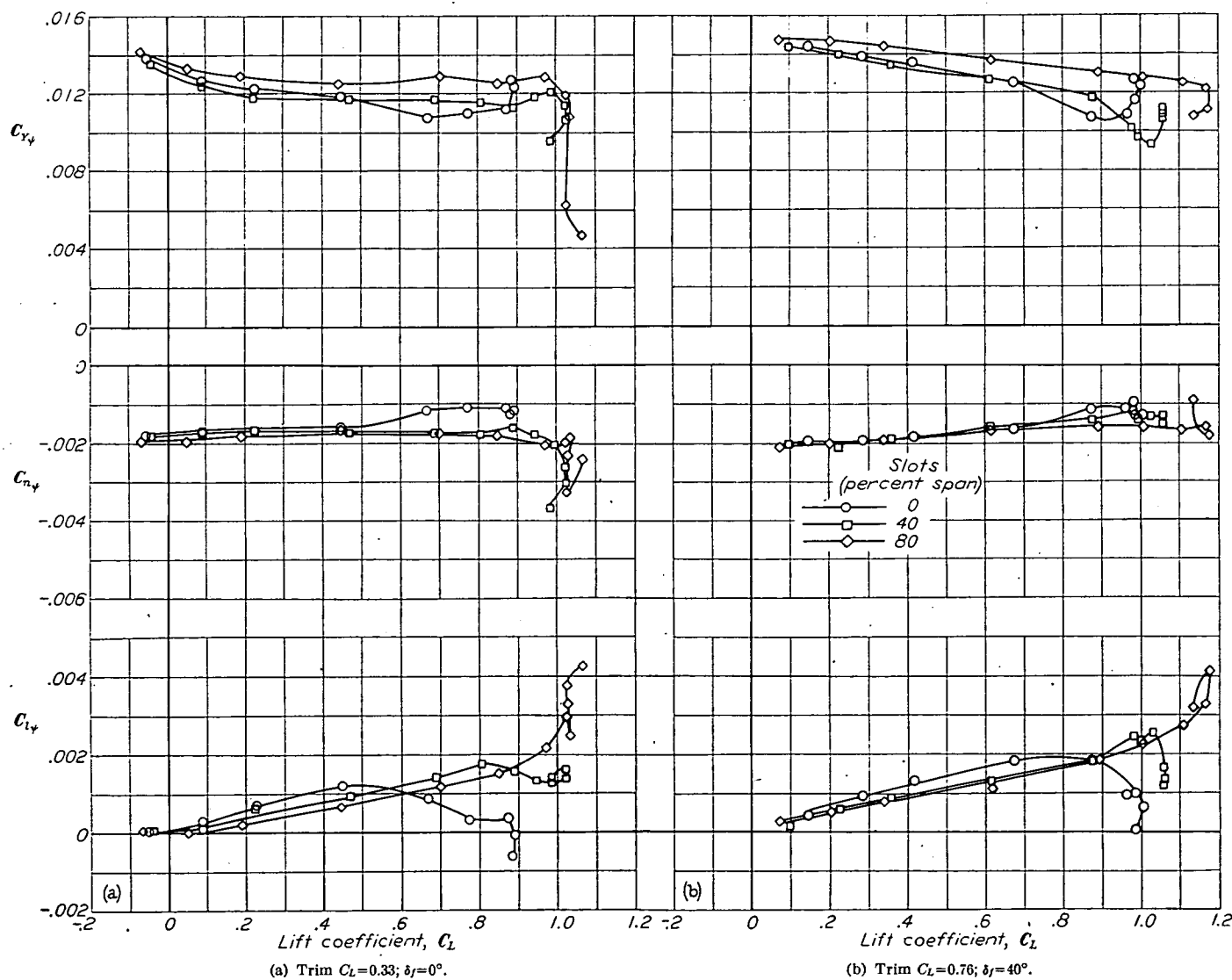


FIGURE 9.—Variation of lateral static-stability derivatives with lift coefficient for three slot configurations. Large ventral fin on; propeller on; $\psi = 0^\circ$; $R = 1.01 \times 10^6$.

wind-tunnel test results indicated no marked asymmetry in the model characteristics. Part of the differences in the flight motions to the right and left are undoubtedly due to variation in the thrust conditions. Although the flight tests were made under approximately zero thrust, no convenient means of obtaining this condition was available, and thus this requirement was left to the judgement of the pilot. Some small part of the difference in the motions to the left and right should be attributable to instrument error as the film record frequently contained considerable hash which, of course, tended to obscure the actual motion record. The differences between the motions to the right and left were resolved in the present report by presenting both records with the sign of one reversed so that the motions were superimposed. The shaded areas in the lateral-motion plots (figs. 18 to 23 and 25 to 35) represent the difference between the motions to the right and left. The flight motions corresponding to the calculated motions were obtained from references 5 and 6 and related tests. The figures which present the results of the lateral-motion calculations are listed in table III.

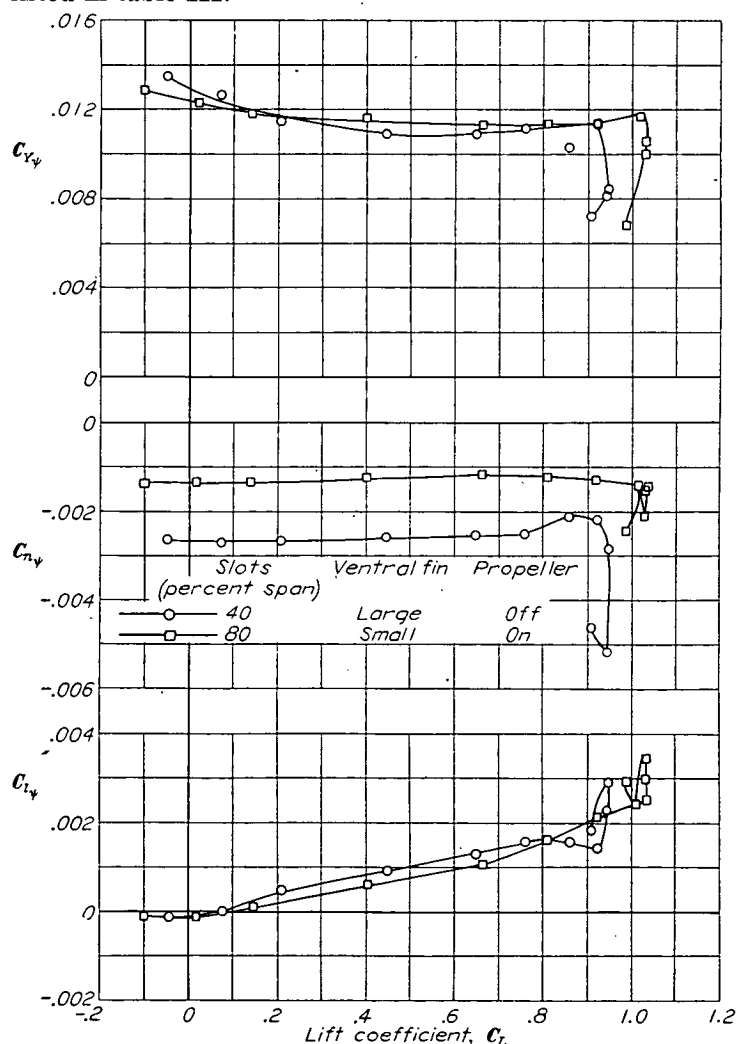


FIGURE 10.—Variation of lateral static-stability derivatives with lift coefficient for two slot configurations. Trim $C_L=0.33$; $\psi=0^\circ$; $\delta_f=0^\circ$; $R=1.01 \times 10^6$.

LATERAL OSCILLATIONS

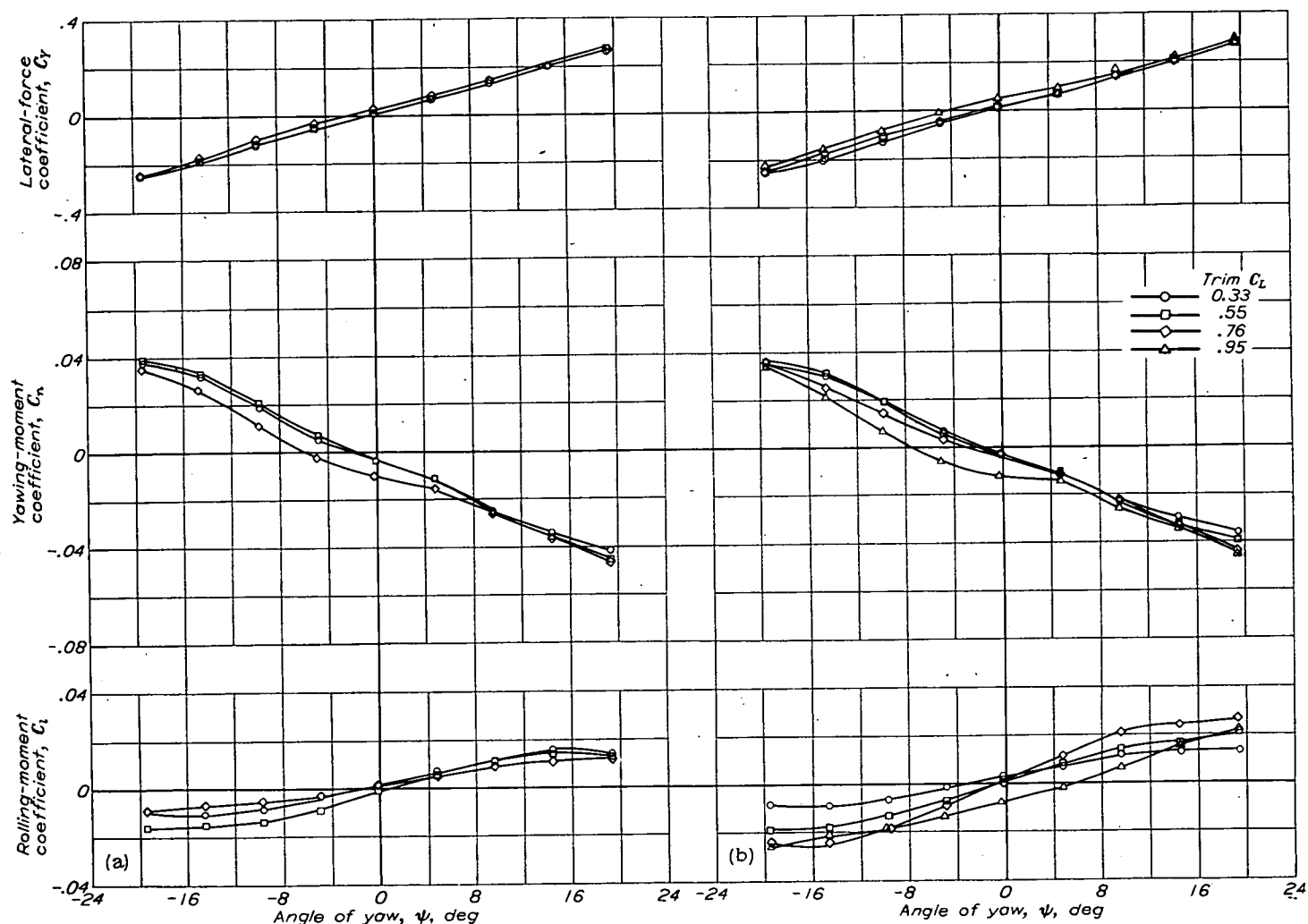
Lateral oscillations were initiated by abruptly deflecting the rudder of the airplane and returning it to neutral equally as rapid a moment later. Some aileron waggle caused by the floating tendency of the ailerons occurred and was accounted for in the calculations by the procedure given in the calculation methods.

The calculated and flight motions are generally in good agreement for all motions calculated (figs. 18 to 23) except for the condition at $C_L=0.977$. (See fig. 23.) A comparison of the flight and calculated periods and time to damp to half amplitude (fig. 24) also indicate fairly good agreement except for the motions at lift coefficients of 0.977 and 1.169.

The values of C_{l_p} used for the calculation of the lateral motions were obtained from the curves of C_l plotted against ψ (fig. 11) wherever possible rather than the curves of C_{l_p} obtained from tests made at $\psi=\pm 5^\circ$ (figs. 9 and 10) because this procedure is believed to be more accurate. Although the difference between the two methods is generally small, such is not always true; and in some cases the calculated rate of damping was found to be appreciably affected by the difference in C_{l_p} .

At the higher lift coefficients, the experimental stability derivatives deviate appreciably from their initial trends. This tendency previously has been referred to as a Reynolds number effect and is probably the cause for the lack of agreement at high lift coefficients between the flight and calculated results. References 5 and 6 which present flight tests for this airplane show no reduction in C_{l_p} up to the maximum test lift coefficient. The rotary derivatives of the airplane presumably behave similarly. Evidence of the deviation of the experimental stability derivatives from their true variation with lift coefficient is observed in figure 20 which presents results for a lateral oscillation occurring at $C_L=0.759$. The calculated result indicates an excessive response in sideslip, yaw, and roll.

A few additional calculations of the period and time to damp, made for other high lift coefficients in flight, indicate increasingly poor agreement between the flight and calculated times to damp to one-half amplitude with increasing lift coefficient. The relatively good agreement between the periods of the flight and calculated motions for all lift coefficients, however, indicates that the experimental values of $C_{n\beta}$ are fairly close to the correct values. The period of the motion is primarily a function of the directional-stability parameter $C_{n\beta}$. (See reference 10.) An extrapolation of the curves of the derivatives plotted against lift coefficient—which amounted to selecting the value of the derivative just previous to the break in the curves occurring at maximum lift coefficient—was employed for one case at $C_L=1.17$ but failed to yield a satisfactory result. A linear extrapolation of the curves in the region preceding the departure from the theoretical or linear trend is expected to be more satisfactory.

FIGURE 11.—Variation of C_Y , C_N , and C_L with angle of yaw. Large ventral fin on; propeller on; $R=1.01 \times 10^6$.

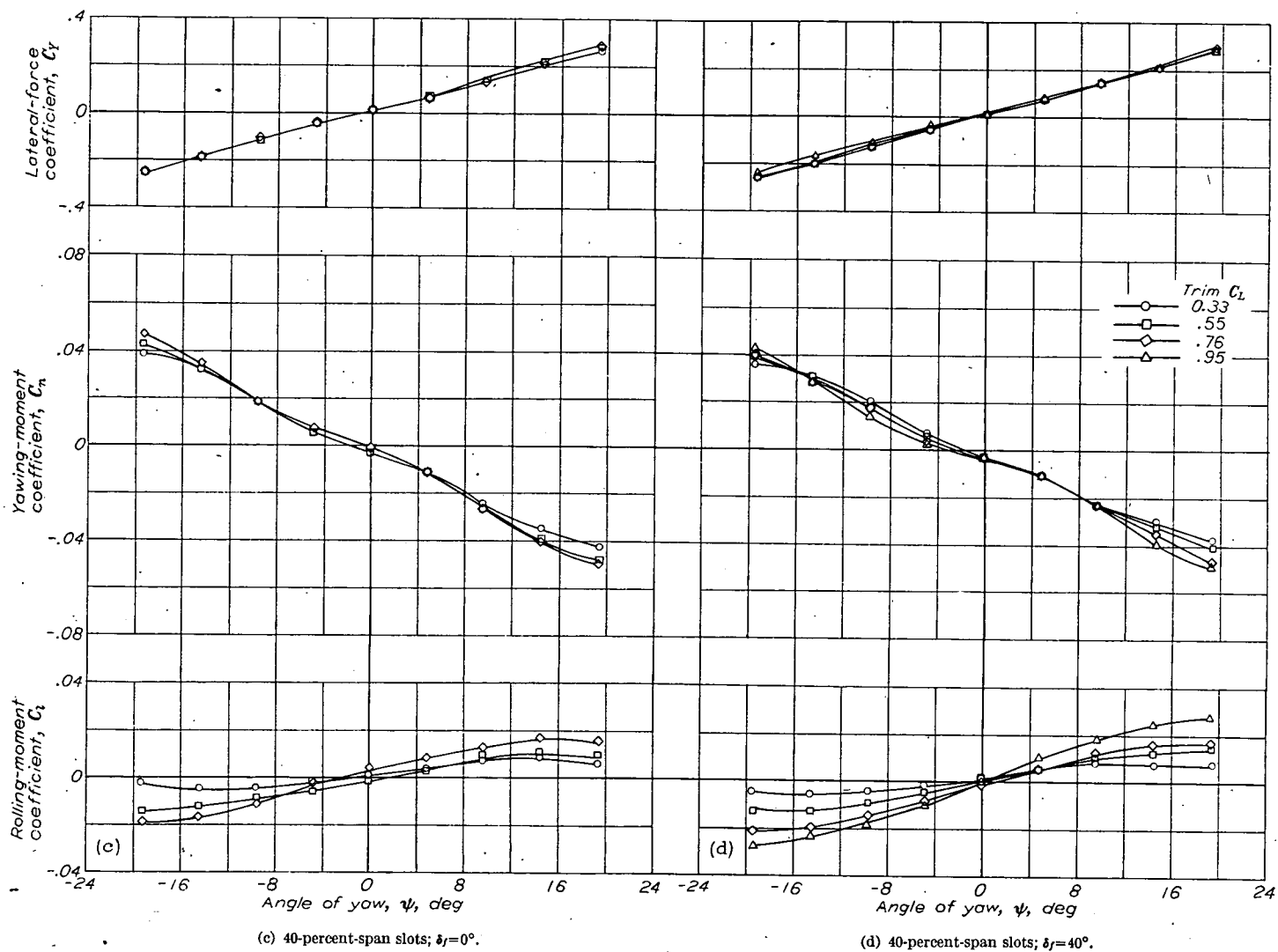


FIGURE 11.—Continued.

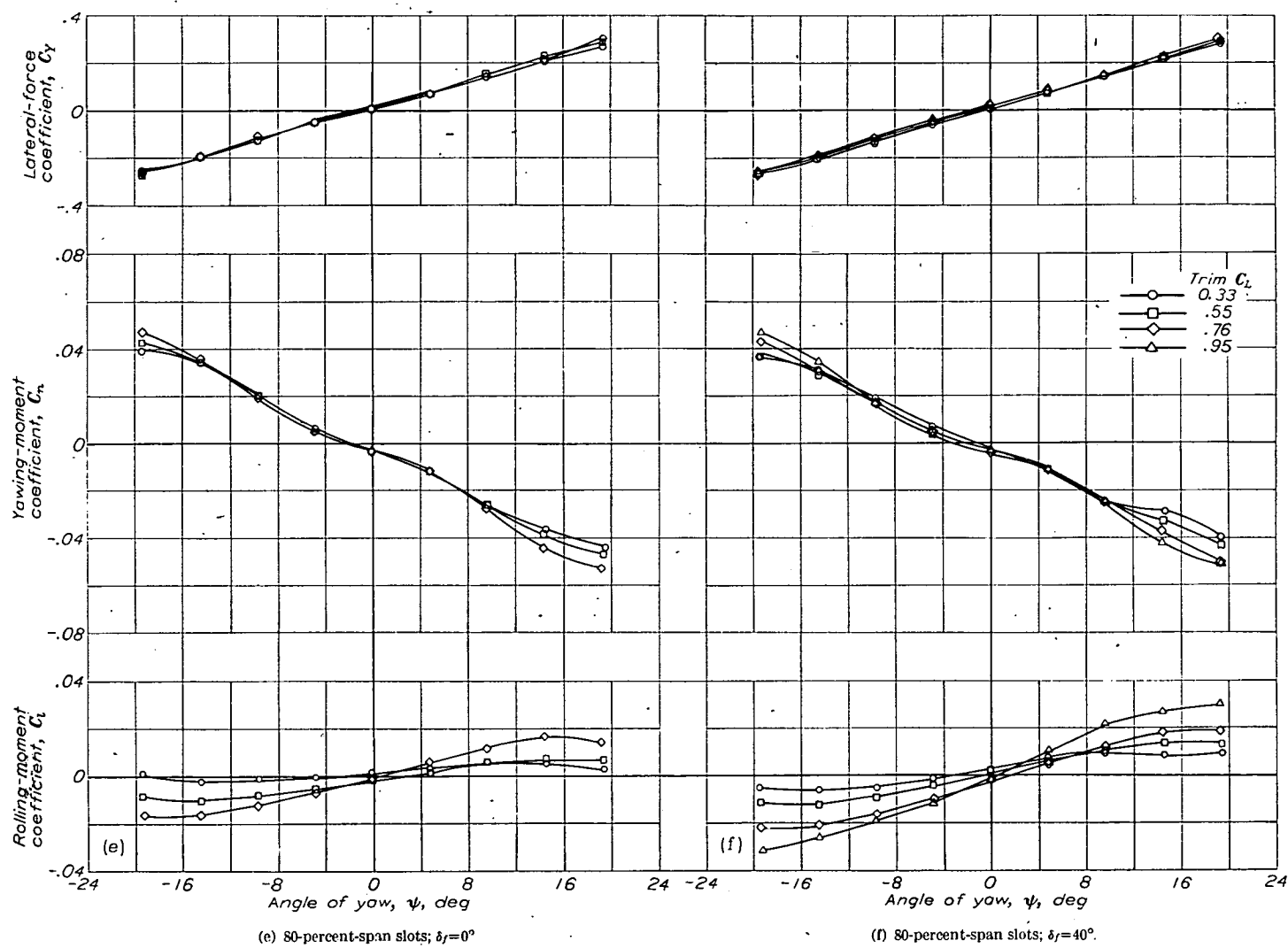


FIGURE 11.—Continued.

RUDDER KICKS

Rudder kicks were initiated by abruptly deflecting the rudder of the airplane and holding this deflection as the airplane responded. The records of the flight motions were short, usually covering 5 seconds. As in the case of the lateral oscillations discussed previously, some slight aileron waggle that occurred was accounted for in the calculated motions. The agreement between the flight and calculated motions is, in general, quite good (figs. 25 to 31). Agreement is best at the lower lift coefficients; however at a lift coefficient of 0.919 (fig. 28), good agreement is shown between flight and calculated angles of sideslip and rolling velocities.

The flight tests reported in reference 5 showed that at low lift coefficients the airplane rolled in response to a rudder kick as if the airplane had a decided negative dihedral

effect. Figure 31 indicates that this peculiar response was largely caused by the slight aileron waggle which occurred during the motion. In general, for all configurations investigated, it was found necessary to account for any slight aileron movements in order to predict satisfactorily the lateral motions of the airplane.

The motions at $C_L=0.794$ (fig. 30) give evidence of departure of the experimental derivatives from the true variation with lift coefficient. This result again is the Reynolds number effect previously discussed. The flight and calculated periods of the lateral oscillation caused by the rudder kicks are in very good agreement which indicates again that the values of C_{n_p} used in the calculations were reasonably accurate.

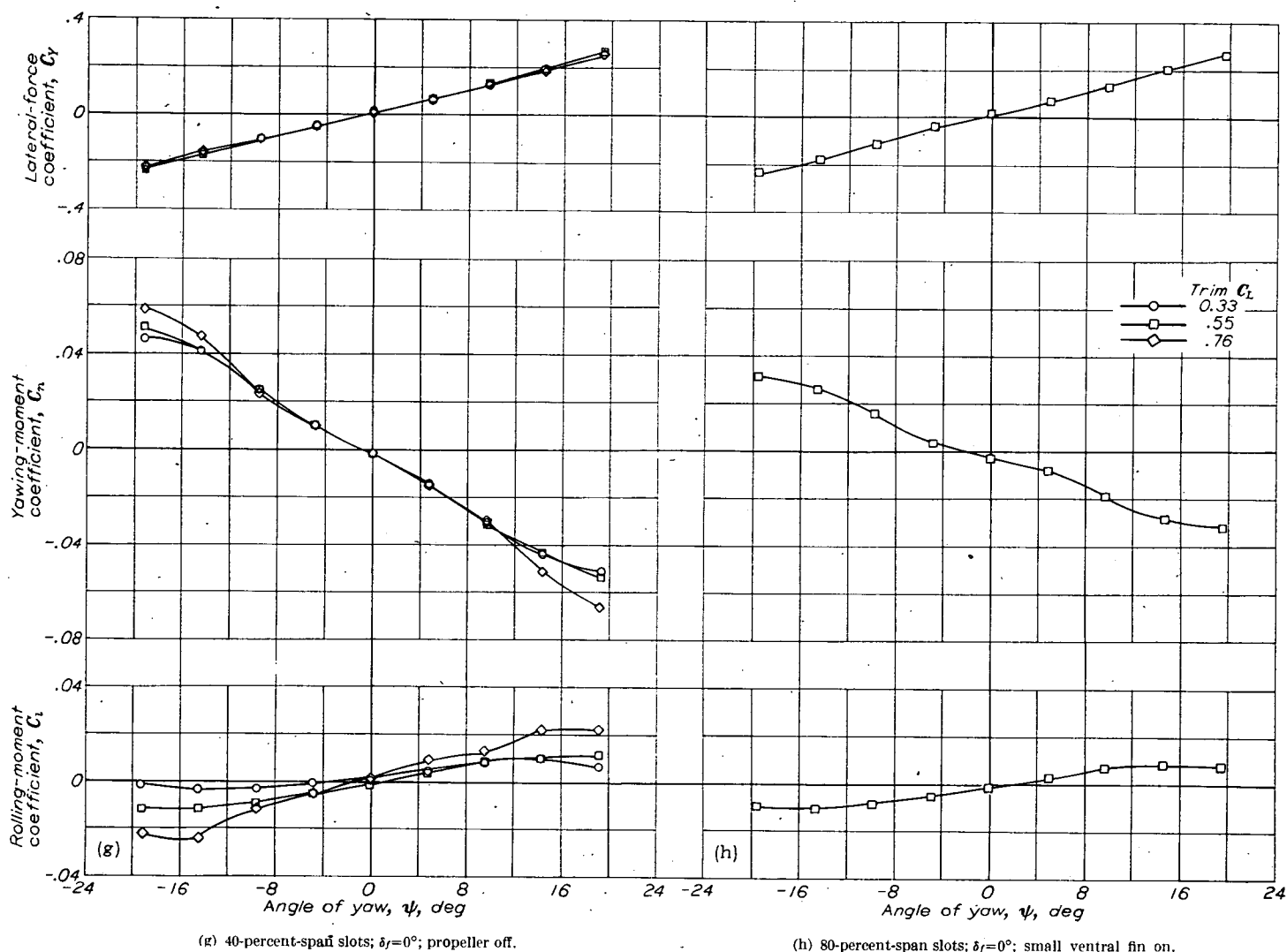
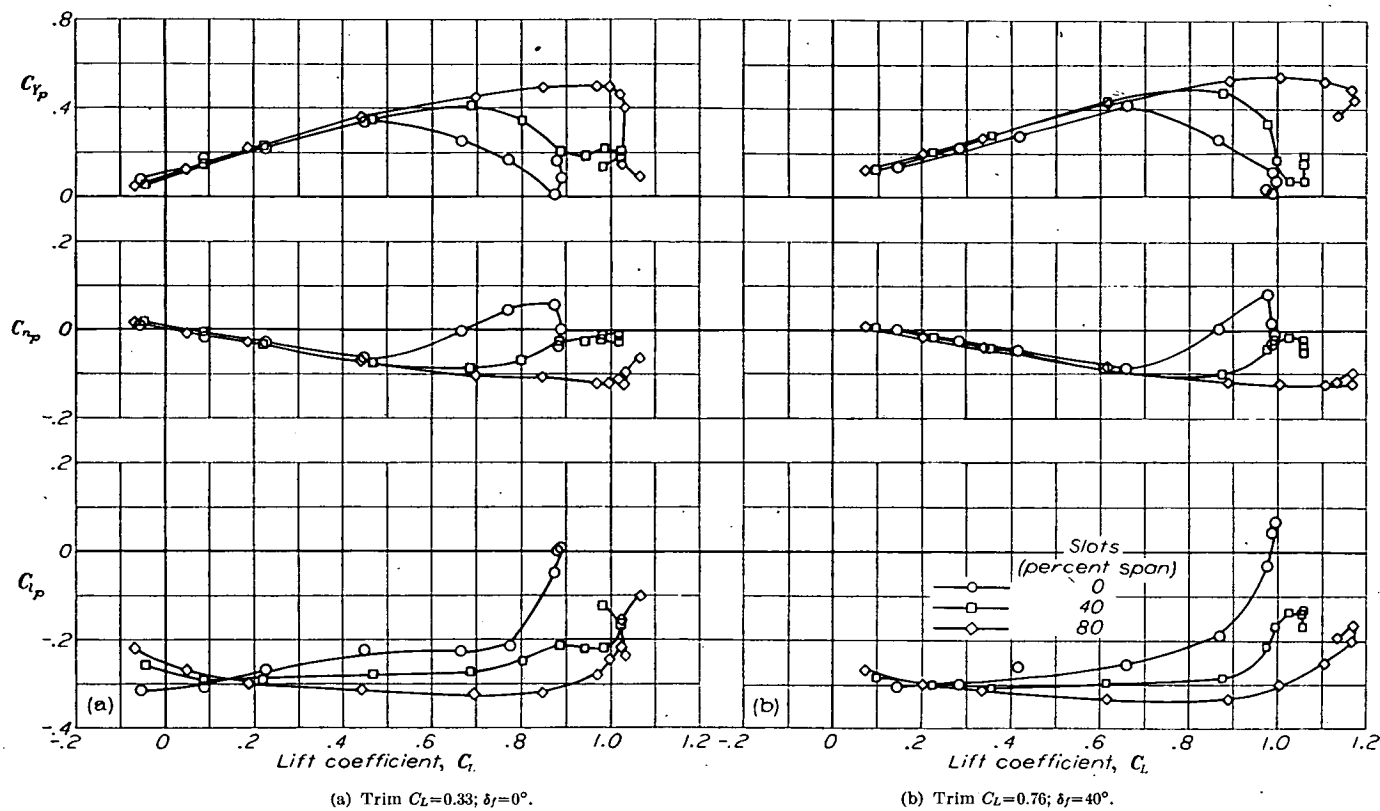
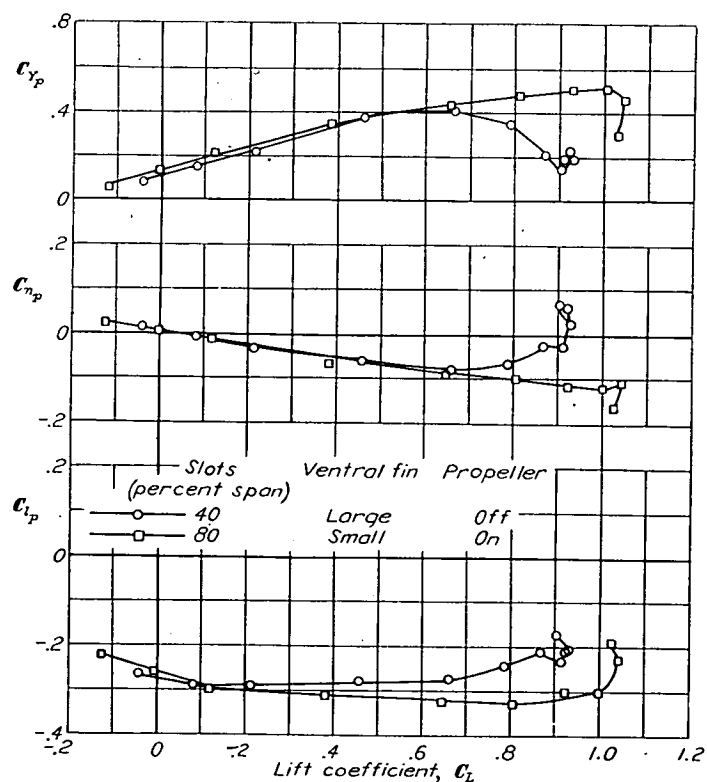
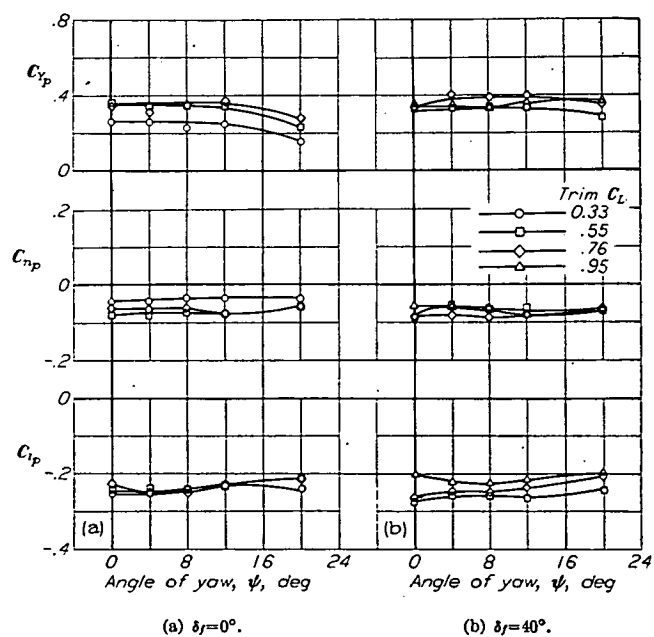


FIGURE 11.—Concluded.

FIGURE 12.—Variation of rolling-stability derivatives with lift coefficient for three slot configurations. $\psi=0^\circ$; $R=1.01 \times 10^4$.FIGURE 13.—Variation of rolling-stability derivatives with lift coefficient for two slot configurations. $\psi=0^\circ$; $\delta_f=0^\circ$; $R=1.01 \times 10^4$.FIGURE 14.—Variation of rolling-stability derivatives with angle of yaw. Large ventral fin on; propeller on; 40-percent-span slots; $R=1.01 \times 10^4$.

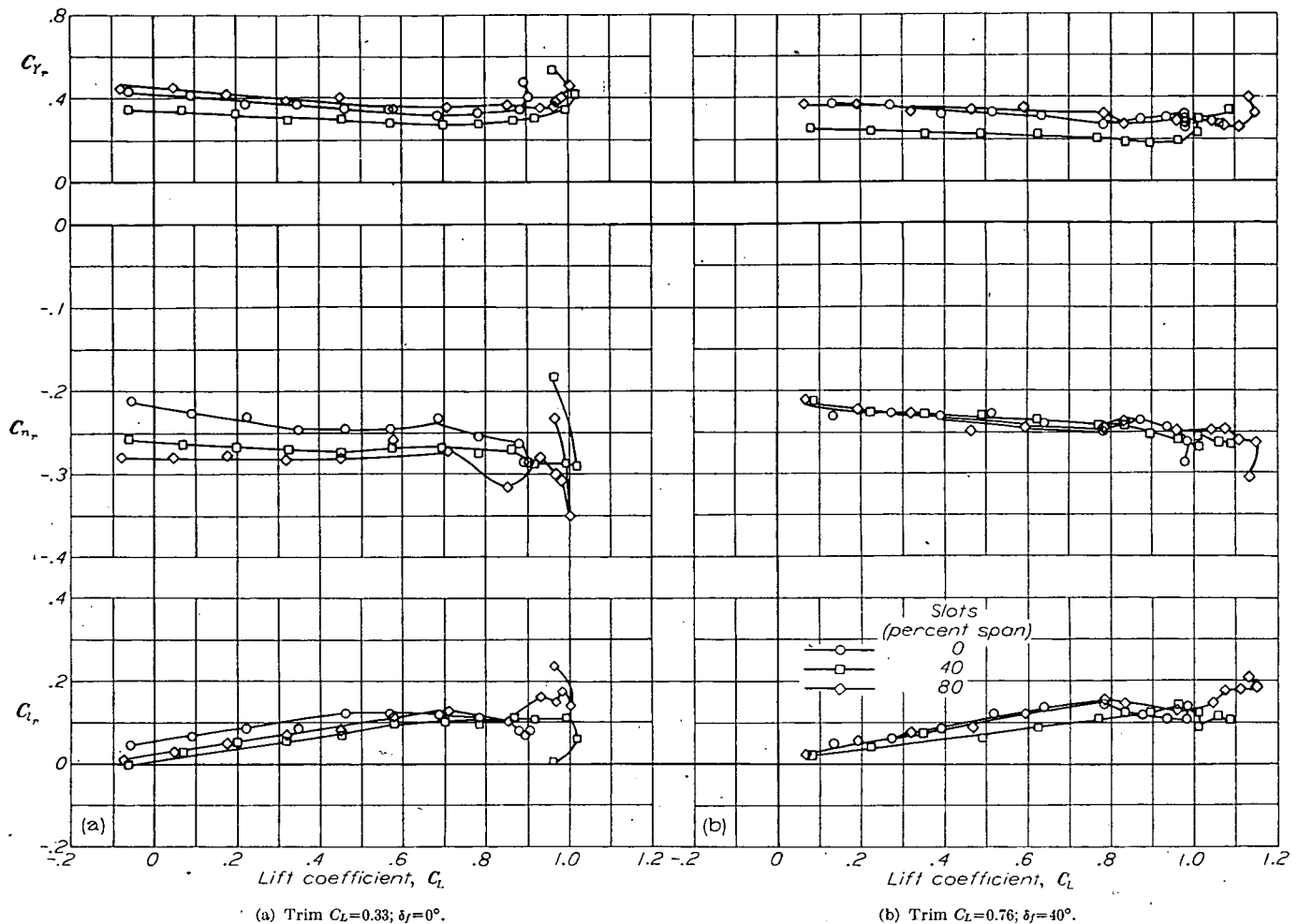


FIGURE 15.—Variation of yawing-stability derivatives with lift coefficient for three slot configurations. Large ventral fin on; propeller on; $\psi=0^\circ$; $R=0.8 \times 10^6$.

AILERON ROLLS

Aileron rolls were initiated by abruptly deflecting the aileron control of the airplane and holding the deflected position as the airplane responded. The rudder control was held as near neutral as possible. The records of motion were necessarily short because of the large angles of bank assumed by the airplane after a short period of time.

Comparison of the flight and calculated aileron rolls indicates fairly good agreement (figs. 32 to 35) except for the high-lift-coefficient condition without nose slots. The stability derivatives for the unslotted configuration show a departure at fairly low lift coefficient from the initial trend of the variation of the derivatives with lift coefficient. As previously mentioned, this result is not obtained at the flight Reynolds numbers. The calculations for this high lift coefficient were carried out with two sets of stability deriva-

tives. One set was obtained by a linear extrapolation of the curves of the derivatives plotted against lift coefficient and the second set, by selecting for the value of the derivative that value just previous to the final break in the curves of the derivatives plotted against lift coefficient. The second procedure was necessary because the flight lift coefficient was greater than the maximum experimental lift coefficient. The linear extrapolation produces the better result, but it can be seen that neither set of derivatives was very satisfactory although it would appear logical to believe, in view of the Reynolds number effects indicated previously, that the linear extrapolation of the derivatives would have given fairly good results. The aileron roll for the 80-percent-span slot configuration with flaps down at $C_L=1.169$ (fig. 35) shows fair agreement between the flight and calculated motions so that a beneficial effect of the slots in maintaining an unseparated

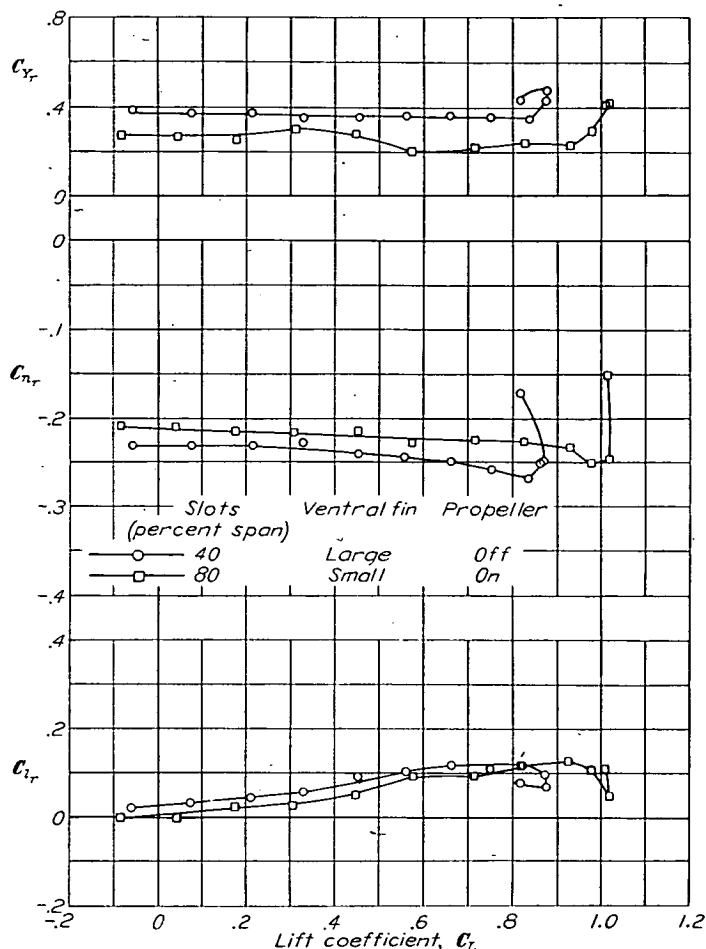


FIGURE 16.—Variation of yawing-stability derivatives with lift coefficient for two slot configurations. Trim $C_L = 0.33$; $\psi = 0^\circ$; $\delta_f = 0^\circ$; $R = 0.8 \times 10^6$.

flow over the wing to a fairly high lift coefficient is indicated.

The effects of elasticity of the wing were not considered in the calculations of the aileron rolls because of lack of knowledge of the flexibility of the wing. Rough estimates of the effect of elasticity on the rolling velocity for the results of figure 33 indicate that the discrepancies shown between flight and calculated results would be reduced by about 50 percent or more.

A comparison has been made of the maximum rolling velocities obtained by calculation for conditions duplicating those in flight, by flight tests, and by calculations for a coordinated maneuver in which the yawing velocity and angle of sideslip are maintained at zero. For this last case, the rolling velocity becomes

$$p = \frac{-\Delta C_l}{C_{l_p}} \frac{2V}{b}$$

The results are presented in the following table:

Figure	Maximum rolling velocities from figures			$\left \frac{\Delta C_l}{C_{l_p}} \right \frac{2V}{b}$
	Calculated	Flight (left)	Flight (right)	
32	0.62	0.44	---	2.67
33	.77	.60	0.53	1.11
34	.78	---	.67	1.13
35	.34	.33	.30	.68

The comparison indicates that a marked difference may be obtained in the maximum rolling velocity by eliminating the degrees of freedom of sideslip and yaw. This last method of calculation is used frequently in comparing the relative merits of various forms of ailerons.

NONLINEAR CALCULATIONS

Calculations, in which the curves of C_l and C_n against ψ , were represented not by a single slope but by a series of tangents to the curves and in which the variation of C_n with ψ was included, were made for two motions to illustrate the effect of these nonlinearities. The results are given in figures 36 and 37 along with the flight motion and the calculations made for linear slopes and constant damping derivatives. The effect of the aileron waggle was small and was taken to be the same as for the linear calculation. These figures indicate that for the angles of sideslip encountered in these motions little is to be gained by going to the additional effort required to make the calculations for the nonlinear case. The nonlinear calculations required approximately ten times as long to complete as the linear calculations. Motions at large angles of sideslip or motions for a configuration having more pronounced nonlinearities than for the airplane considered herein would undoubtedly show the nonlinearities to be of greater significance.

SUMMARY OF RESULTS

The investigation indicated that the lateral disturbed motions of the airplane considered herein may be generally calculated with good accuracy by the use of experimental-scale-model data up to lift coefficients where any appreciable effects of the difference in Reynolds number between the scale and flight conditions begin to be apparent. This effect is usually evidenced by a departure of the scale results from a gradual variation with lift coefficient and, for the case investigated, corresponds to a lift coefficient of 0.8 for the slotted-wing configurations. At higher lift coefficients the accuracy in calculating flight motions is progressively

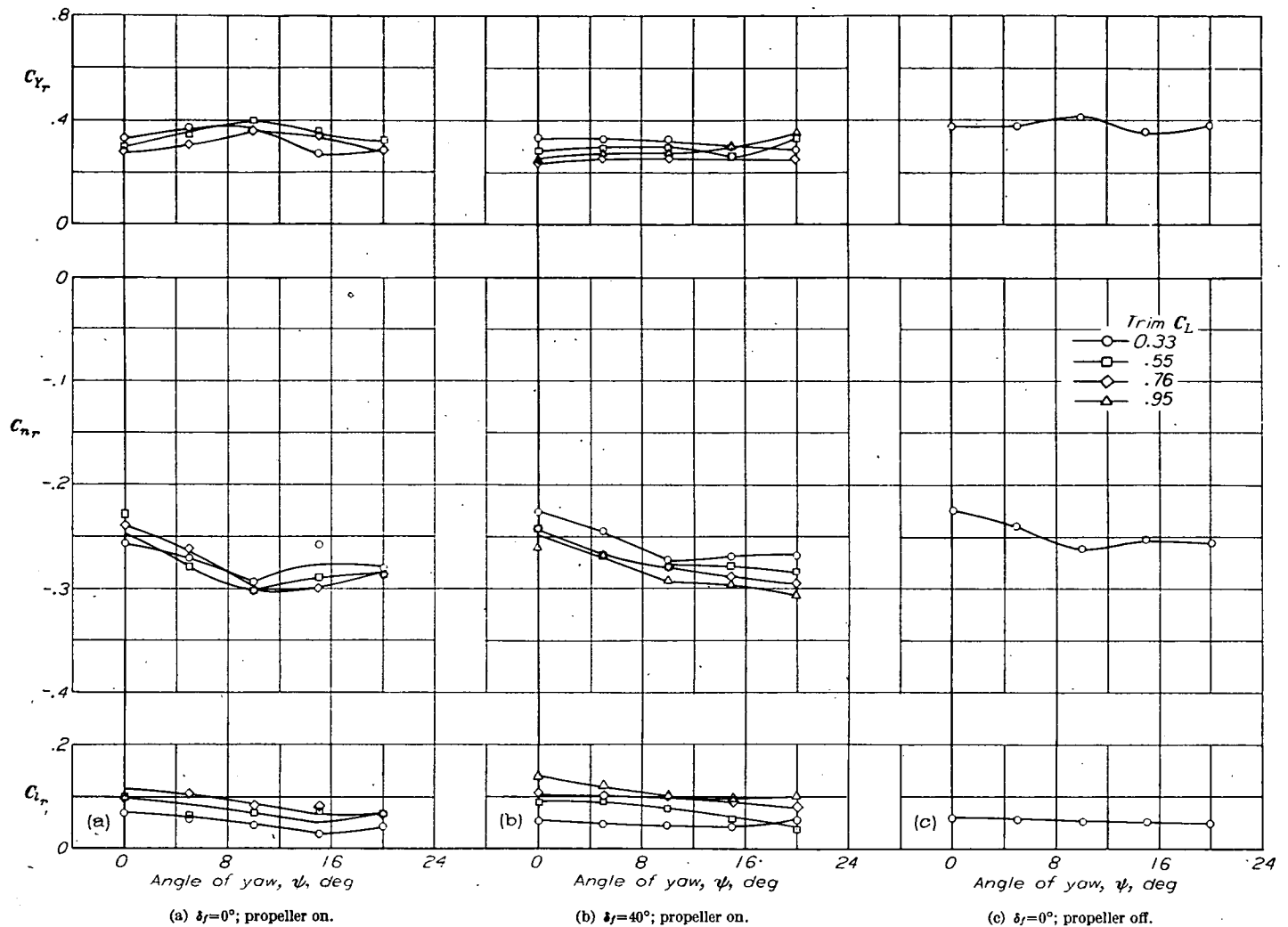


FIGURE 17.—Variation of yawing-stability derivatives with angle of yaw. Large ventral fin on; 40-percent-span slots; $R=0.8 \times 10^6$.

poorer, especially in the time to damp to half amplitude where errors as much as 100 percent or more may be encountered. For the airplane of this investigation, at these high lift coefficients, the calculated time to damp to half amplitude was generally higher than the flight value. In most cases the period of the motion could be calculated to within 5 percent of that obtained in the flight motion. In order to obtain good accuracy in predicting flight motions, proper consideration of all control movements must be made.

For most cases, the nonlinear variation of the directional-stability parameter, the effective dihedral parameter, and the variation of the damping in yaw with angle of yaw could

be neglected. If, however, the flight motions reach a sufficiently large amplitude for a long period of time, as in an aileron roll for example, greater accuracy in the calculated motions should result if the nonlinearities in the variation of yawing-moment coefficient, rolling-moment coefficient, and damping in yaw with angle of yaw are included in the calculations.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., November 1, 1949.

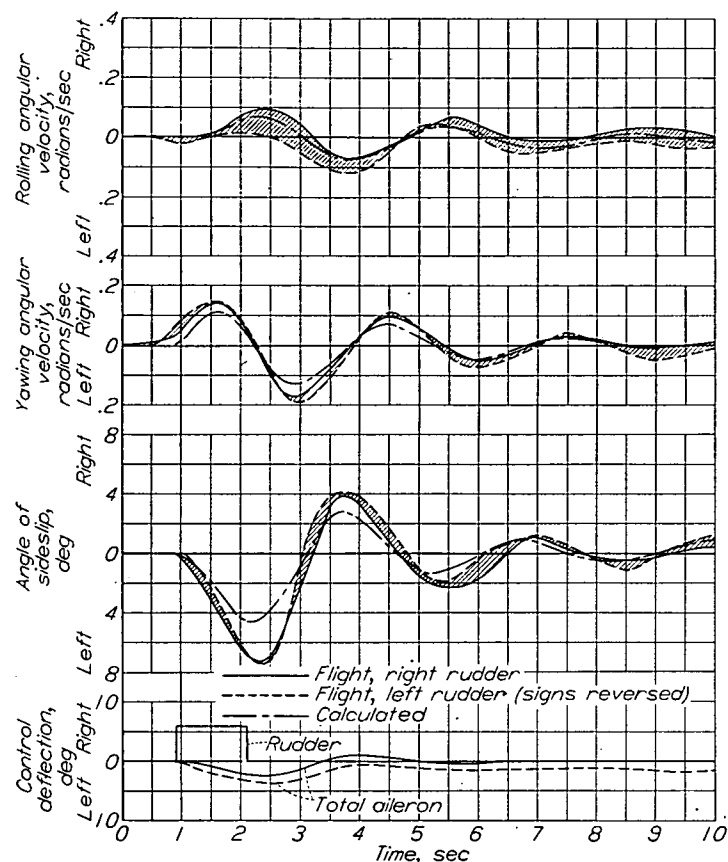


FIGURE 18.—Comparison of the flight and calculated lateral motions resulting from an abrupt deflection and release of the rudder. 40-percent-span slots; flaps up; $C_L=0.334$; $V_C=198$ miles per hour; engine idling.

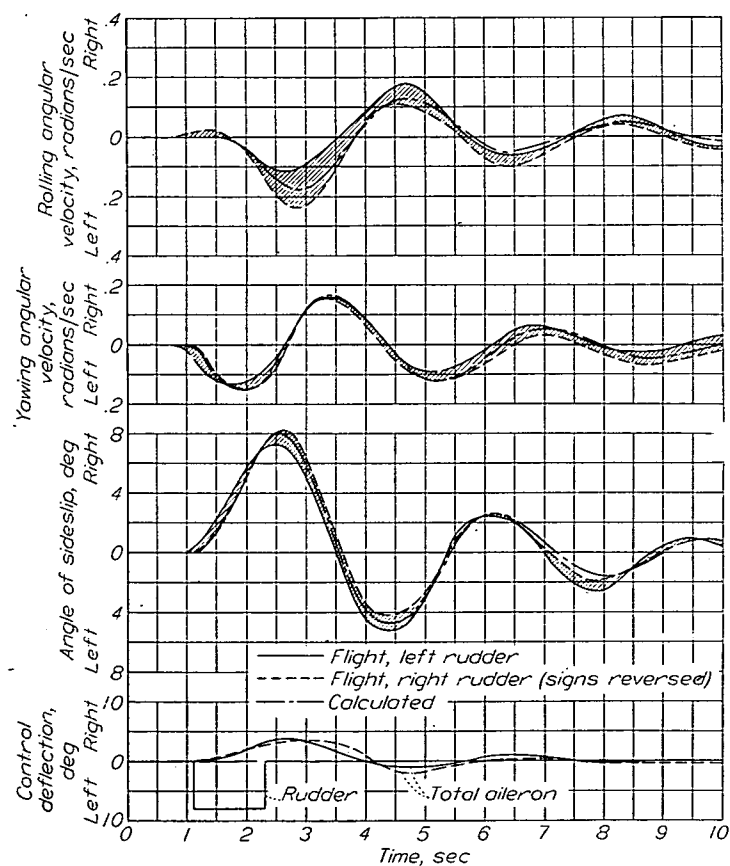


FIGURE 19.—Comparison of flight and calculated lateral motions resulting from an abrupt deflection and release of the rudder. 40-percent-span slots; flaps up; $C_L=0.551$; $V_C=156$ miles per hour; engine idling.

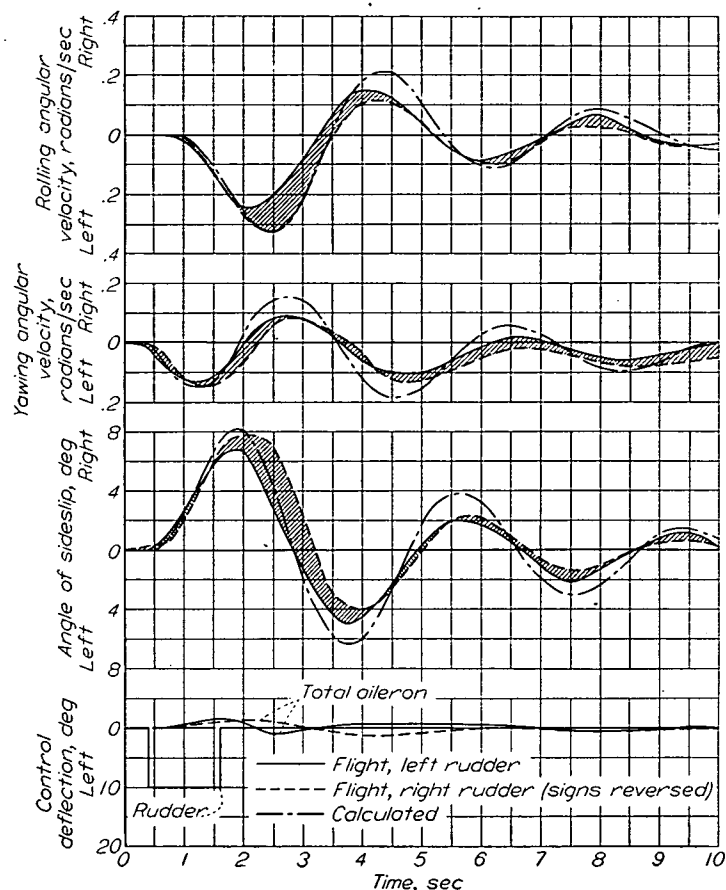


FIGURE 20.—Comparison of flight and calculated lateral motions resulting from an abrupt deflection and release of the rudder. 40-percent-span slots; flaps up; $C_L=0.759$; $V_C=136$ miles per hour; engine idling.

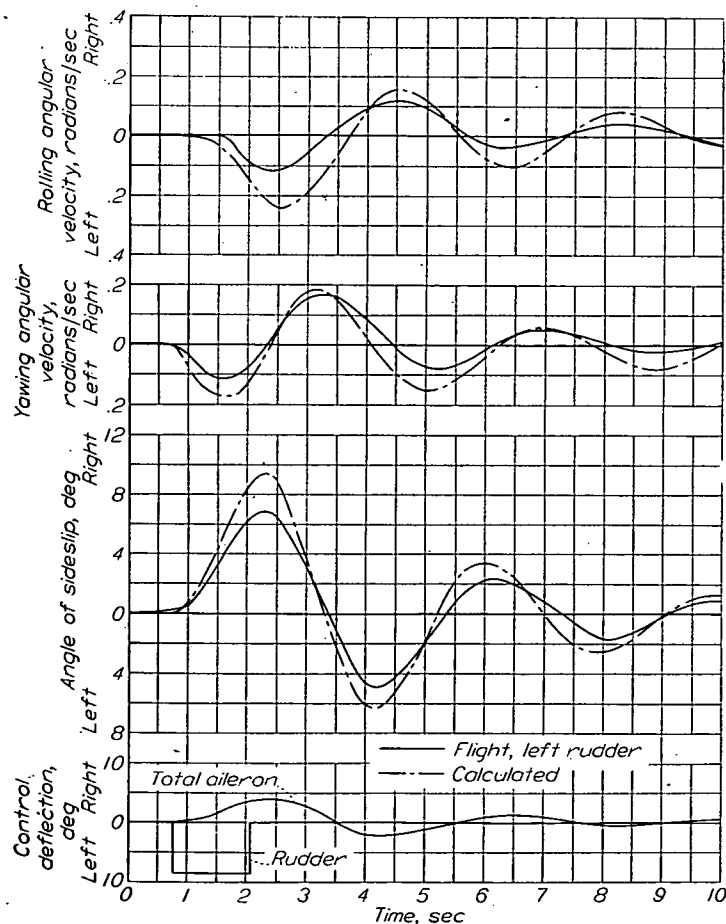


FIGURE 21.—Comparison of flight and calculated lateral motions resulting from an abrupt deflection and release of the rudder. 40-percent-span slots; flaps down; $C_L=0.524$; $V_C=160$ miles per hour; engine idling.

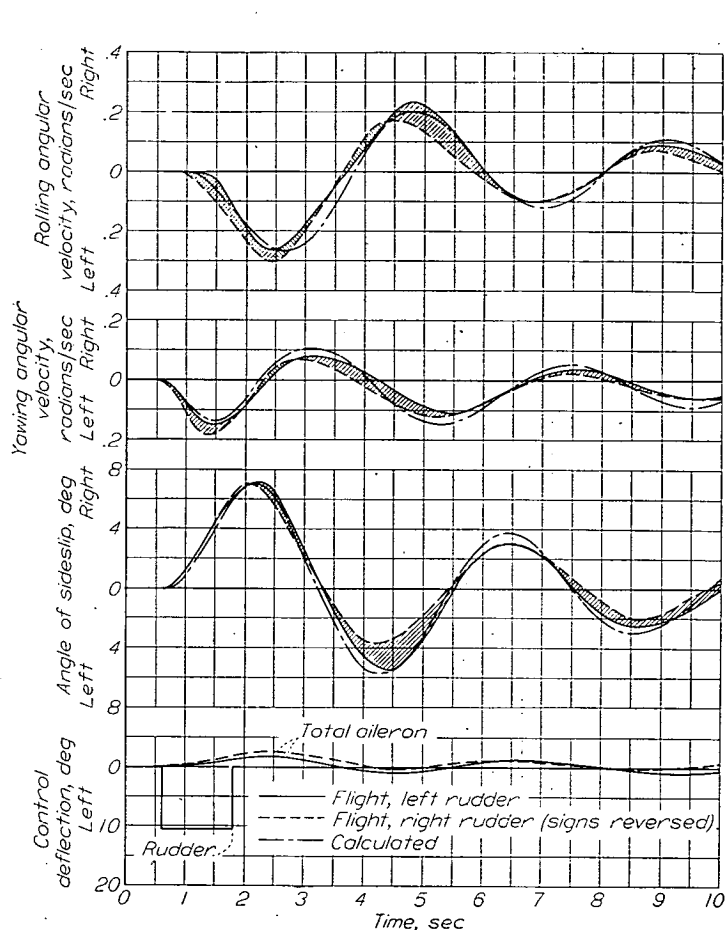


FIGURE 22.—Comparison of flight and calculated lateral motions resulting from an abrupt deflection and release of the rudder. 40-percent-span slots; flaps down; $C_L=0.801$; $V_C=128$ miles per hour; engine idling.

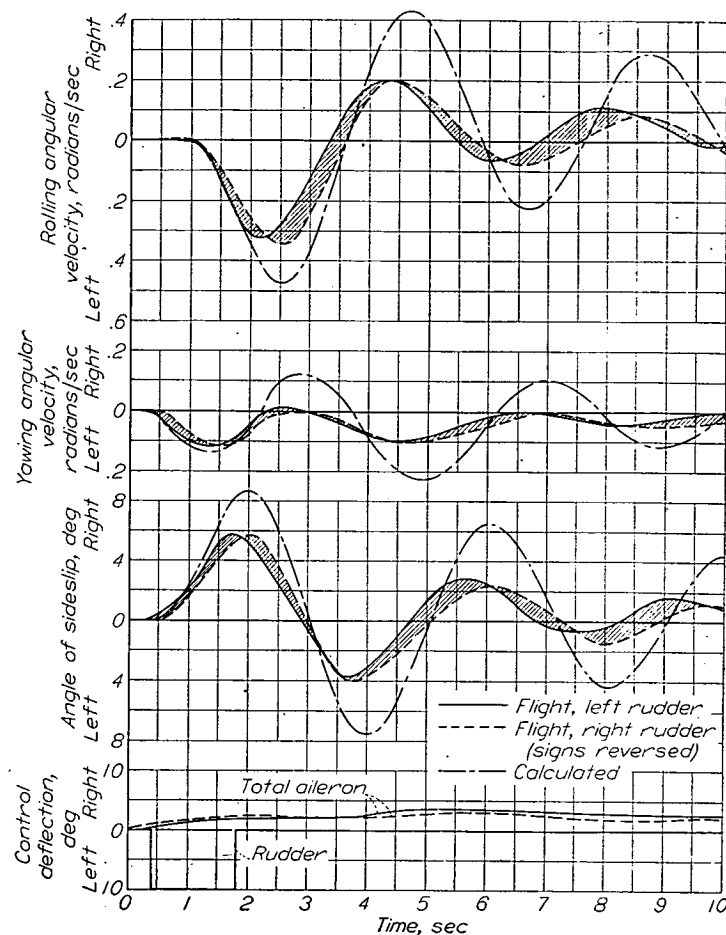


FIGURE 23.—Comparison of flight and calculated lateral motions resulting from an abrupt deflection and release of the rudder. 80-percent-span slots; flaps up; small ventral fin on; $C_L=0.977$; $V_C=120$ miles per hour; engine idling.

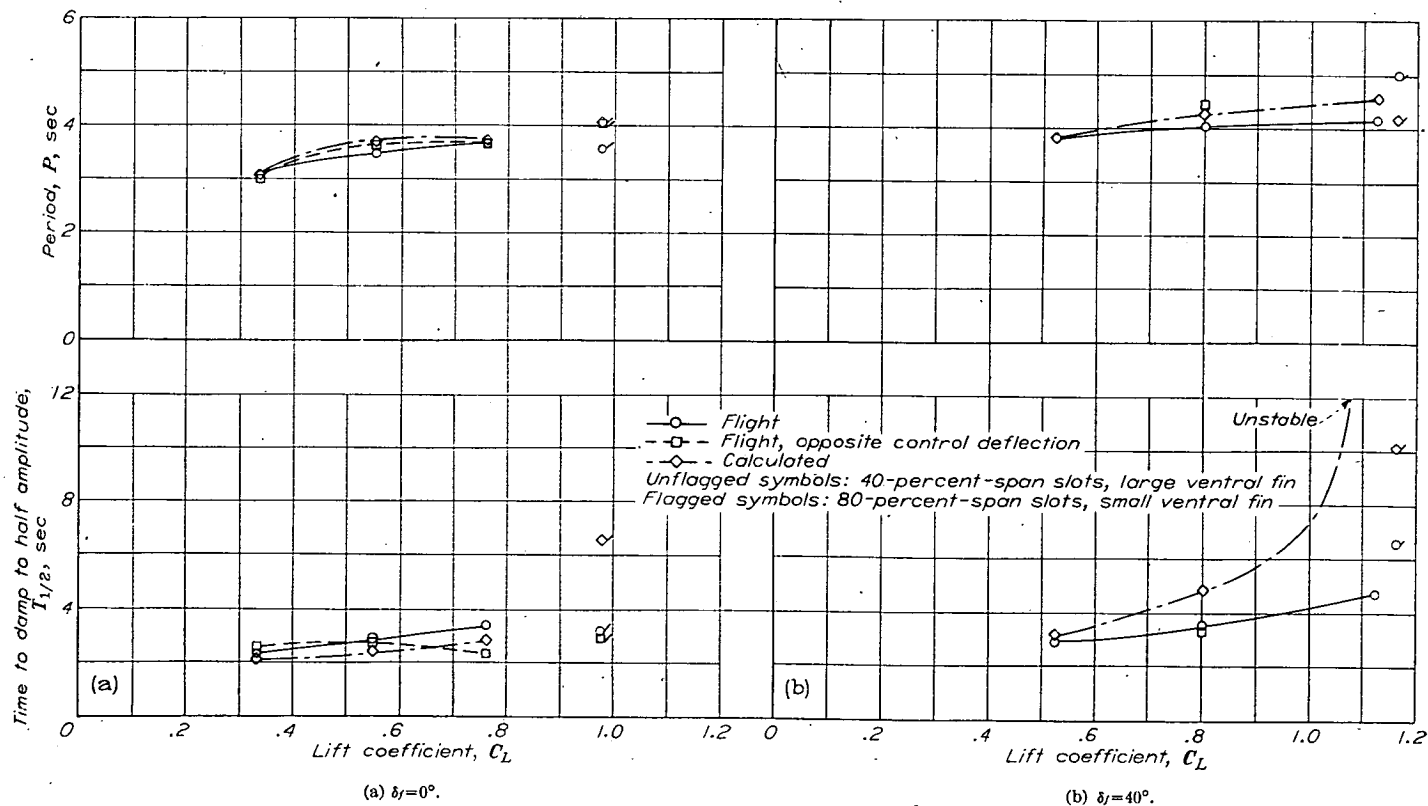


FIGURE 24.—Comparison of flight and calculated values of the period and time to damp to half amplitude.

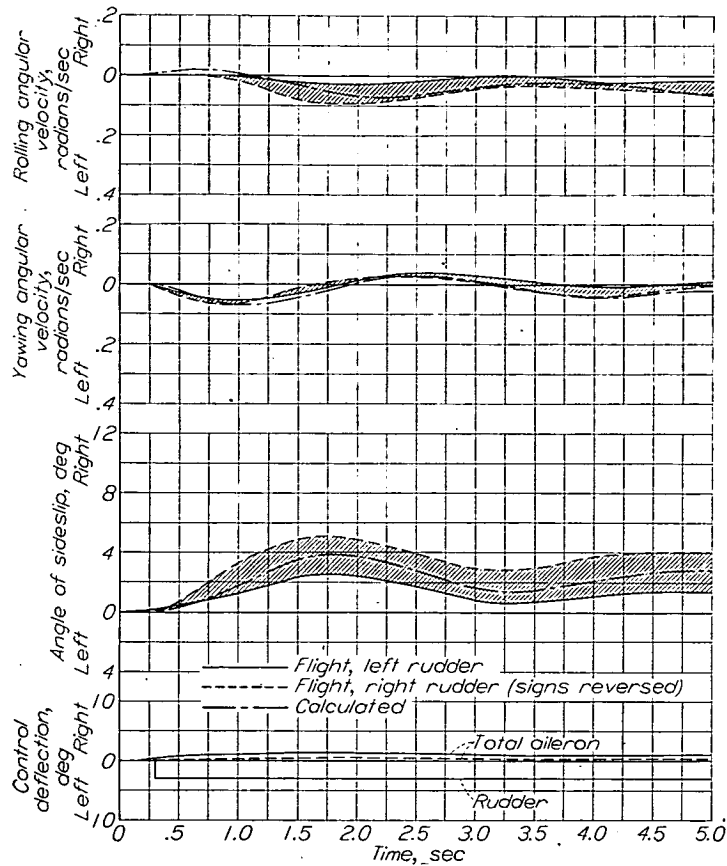


FIGURE 25.—Comparison of flight and calculated lateral motions resulting from a left rudder kick. 40-percent-span slots; flaps up; $C_L=0.341$; $V_C=198$ miles per hour; engine idling.

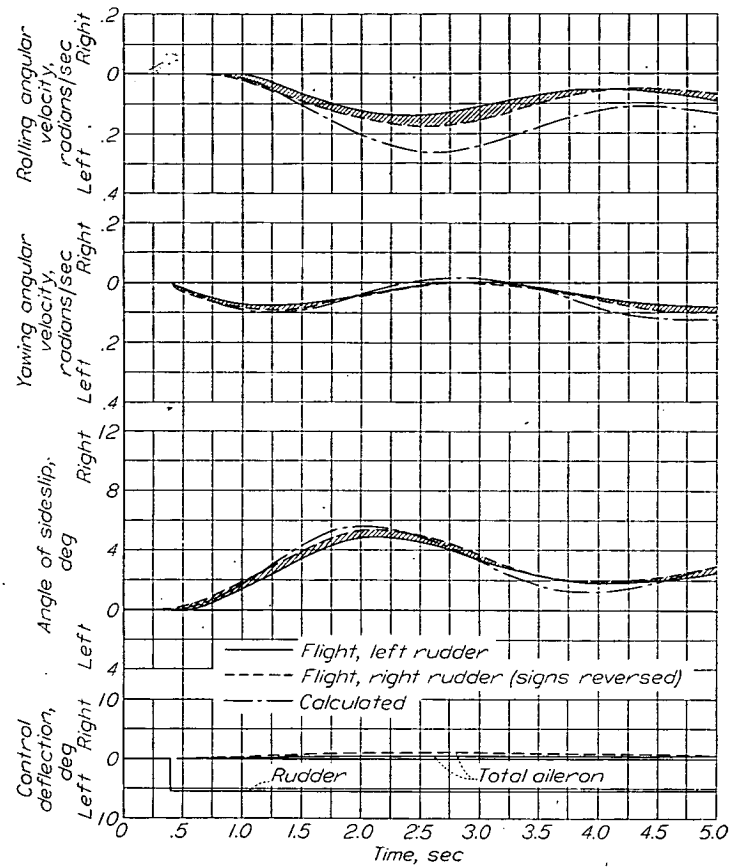


FIGURE 26.—Comparison of flight and calculated lateral motions resulting from a left rudder kick. 40-percent-span slots; flaps up; $C_L=0.556$; $V_C=160$ miles per hour; engine idling.

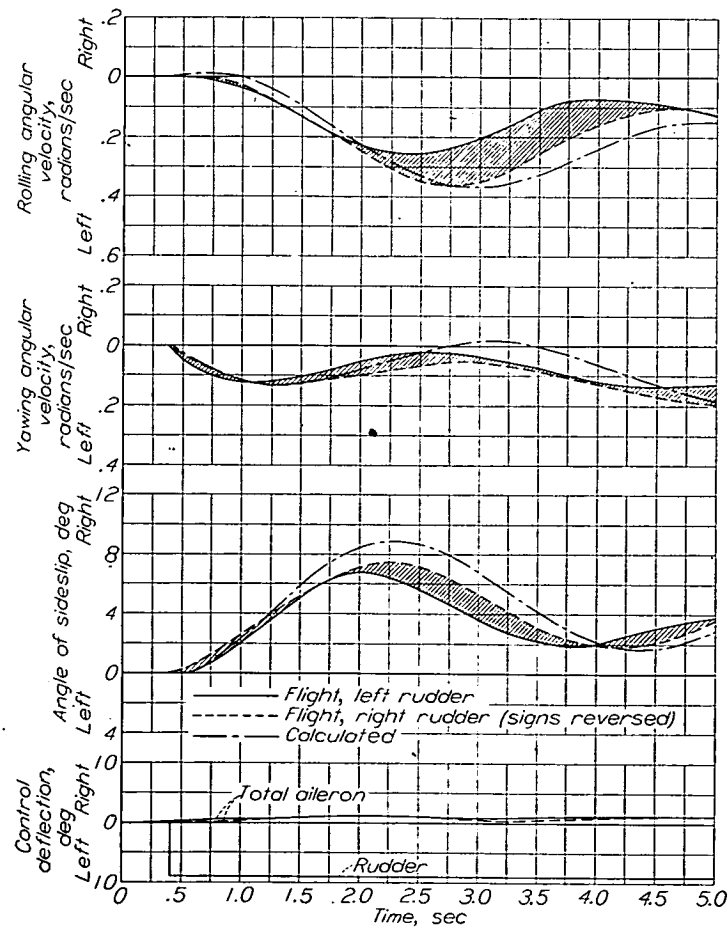


FIGURE 27.—Comparison of flight and calculated lateral motions resulting from a left rudder kick. 40-percent-span slots; flaps up; $C_L=0.764$; $V_C=133$ miles per hour; engine idling.

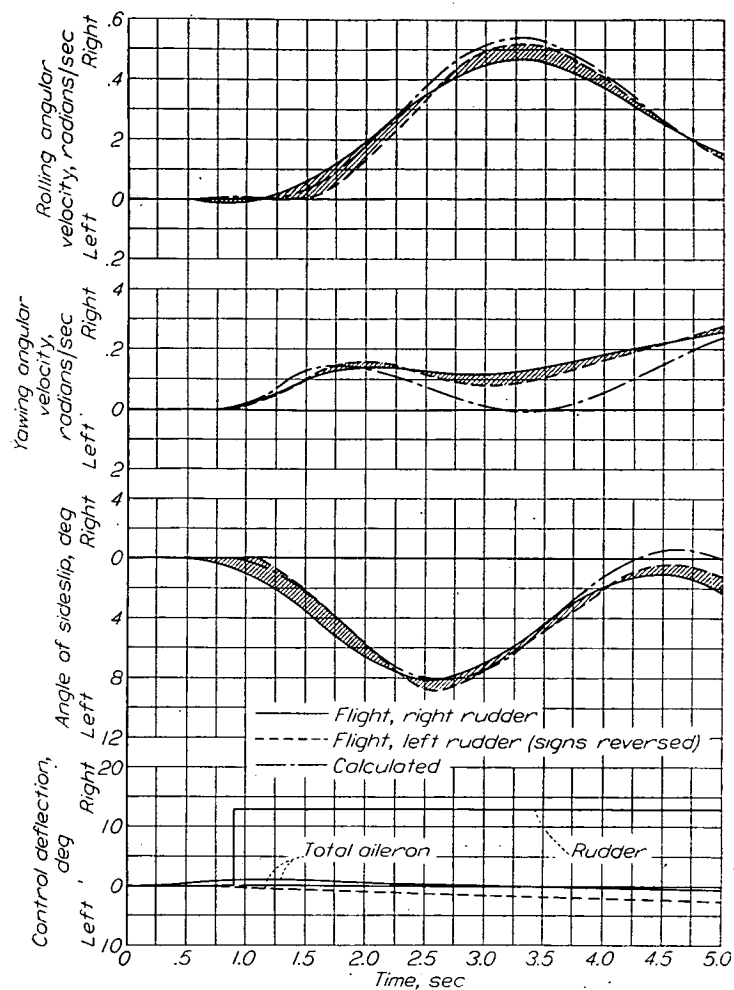


FIGURE 28.—Comparison of flight and calculated lateral motions resulting from a right rudder kick. 40-percent-span slots; flaps up; $C_L=0.919$; $V_C=120$ miles per hour; engine idling.

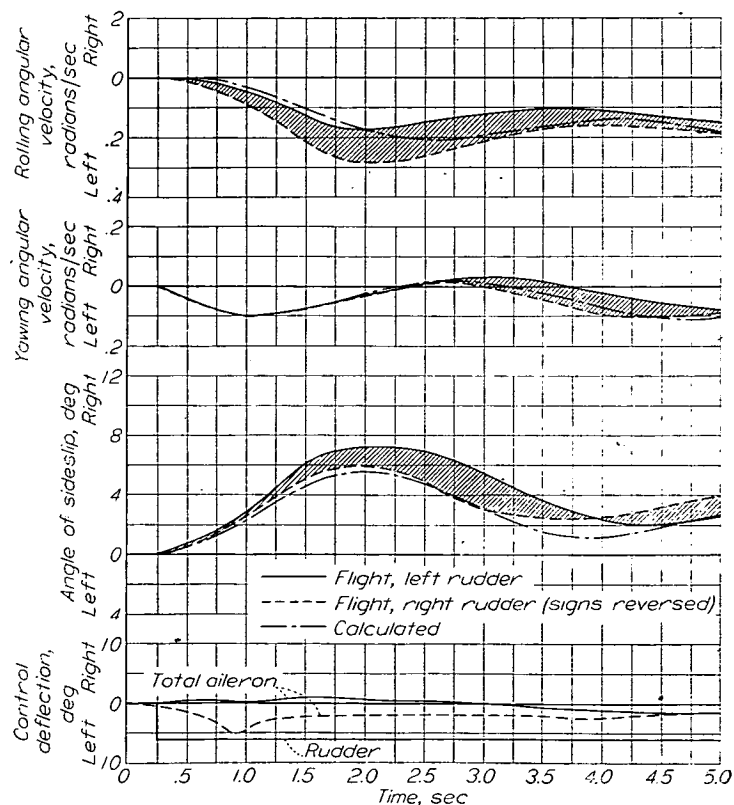


FIGURE 29.—Comparison of flight and calculated lateral motions resulting from a left rudder kick. 40-percent-span slots; flaps down; $C_L=0.537$; $V_C=157$ miles per hour; engine idling.

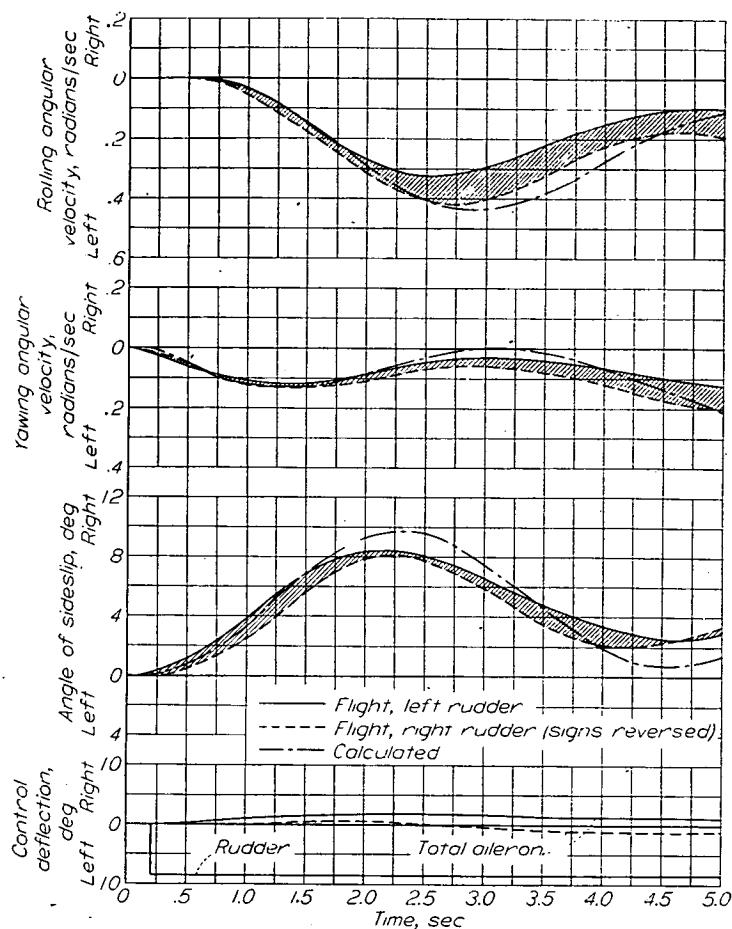


FIGURE 30.—Comparison of flight and calculated lateral motions resulting from a left rudder kick. 40-percent-span slots; flaps down; $C_L=0.794$; $V_C=130$ miles per hour; engine idling.

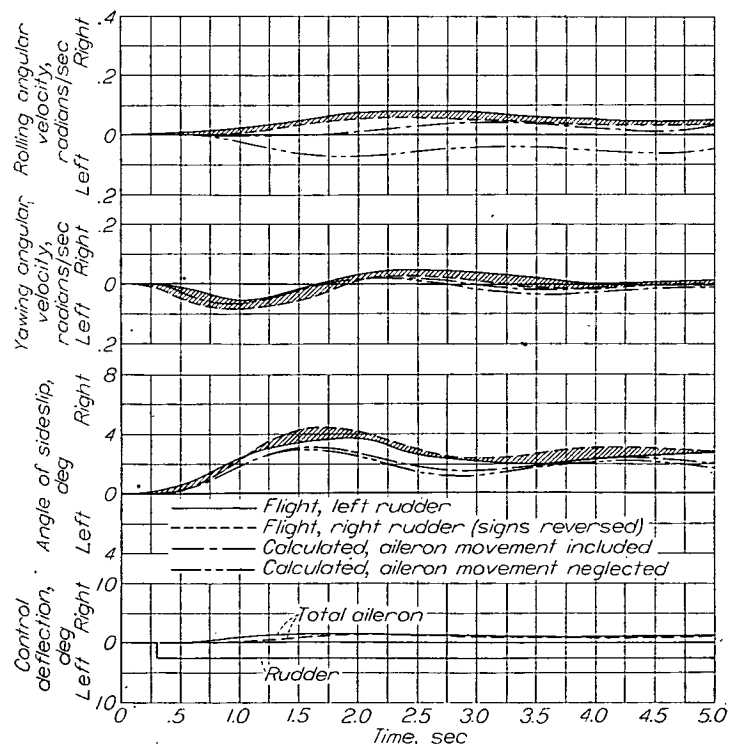


FIGURE 31.—Comparison of flight and calculated lateral motions resulting from a left rudder kick. 80-percent-span slots; flaps up; $C_L=0.278$; $V_C=228$ miles per hour; engine idling.

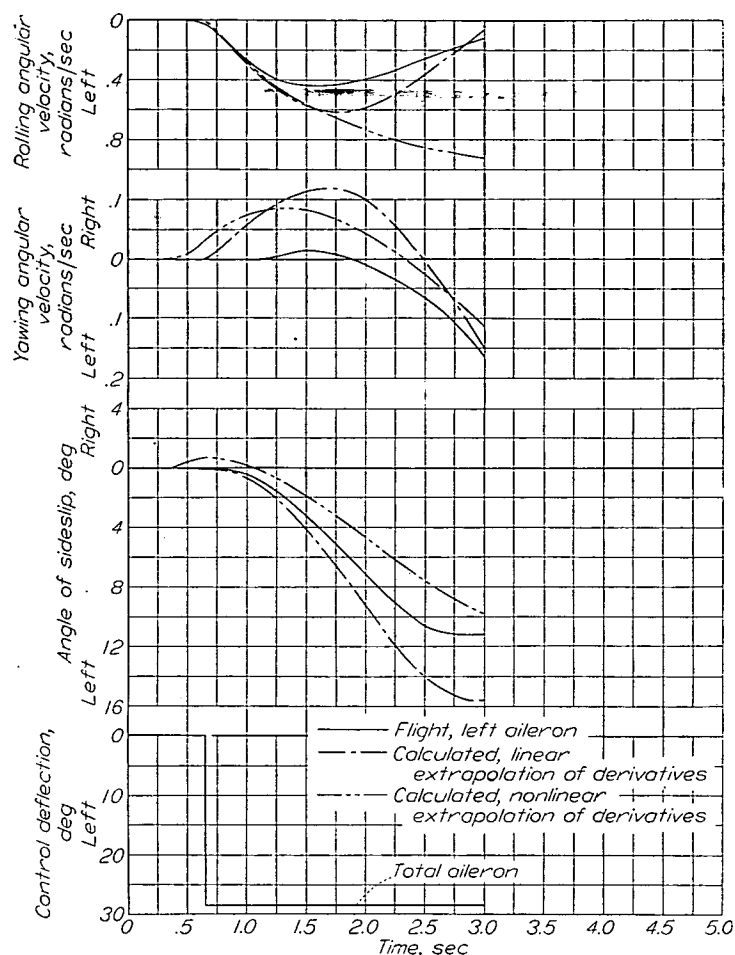


FIGURE 32.—Comparison of flight and calculated lateral motions resulting from an abrupt left aileron deflection. 0-percent-span slots; flaps up; $C_L=0.983$; $V_C=119$ miles per hour; engine idling.

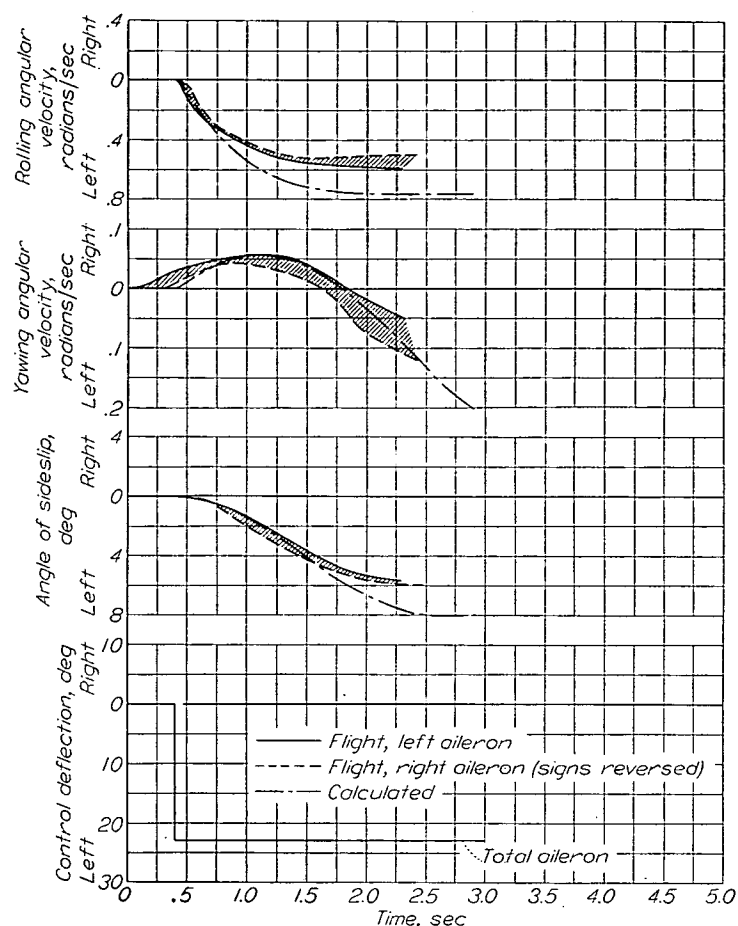


FIGURE 33.—Comparison of flight and calculated lateral motions resulting from an abrupt left aileron deflection. 40-percent-span slots; flaps up; $C_L=0.598$; $V_C=150$ miles per hour; engine idling.

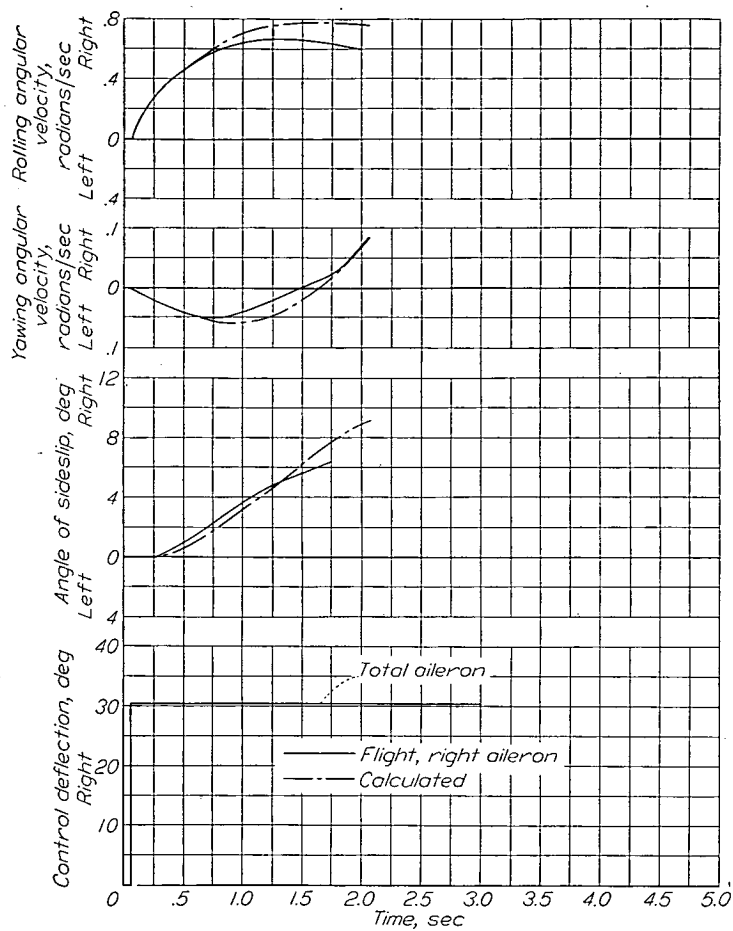


FIGURE 34.—Comparison of flight and calculated lateral motions resulting from an abrupt right aileron deflection. 40-percent-span slots; flaps down; $C_L=0.598$; $V_C=146$ miles per hour; engine idling.

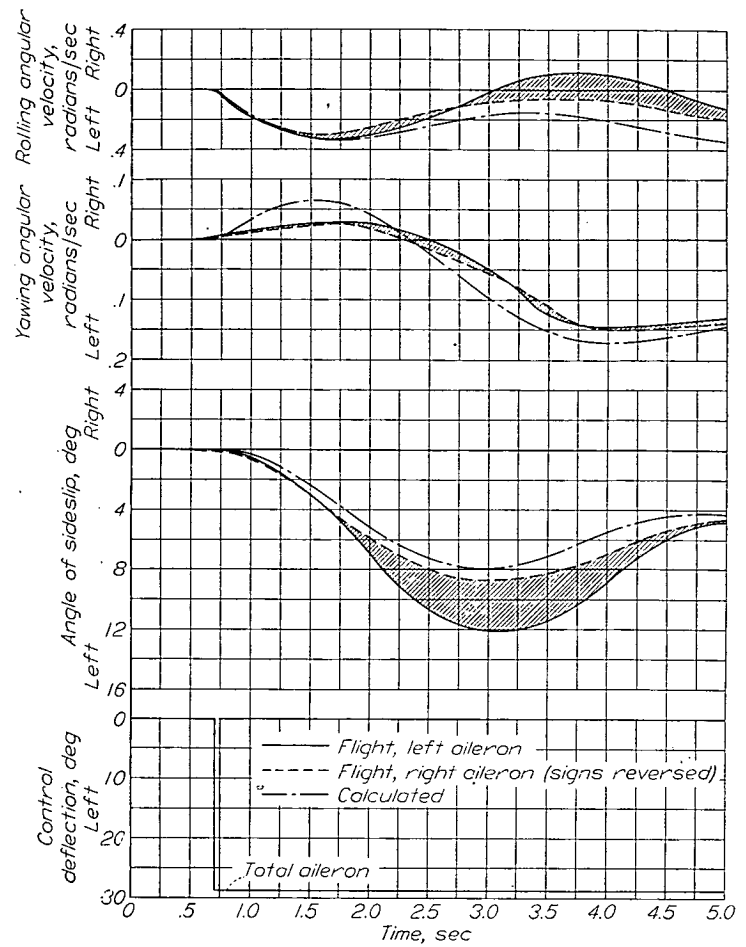


FIGURE 35.—Comparison of flight and calculated lateral motions resulting from an abrupt left aileron deflection. 80-percent-span slots; flaps down; $C_L=1.169$; $V_C=110$ miles per hour; engine idling.

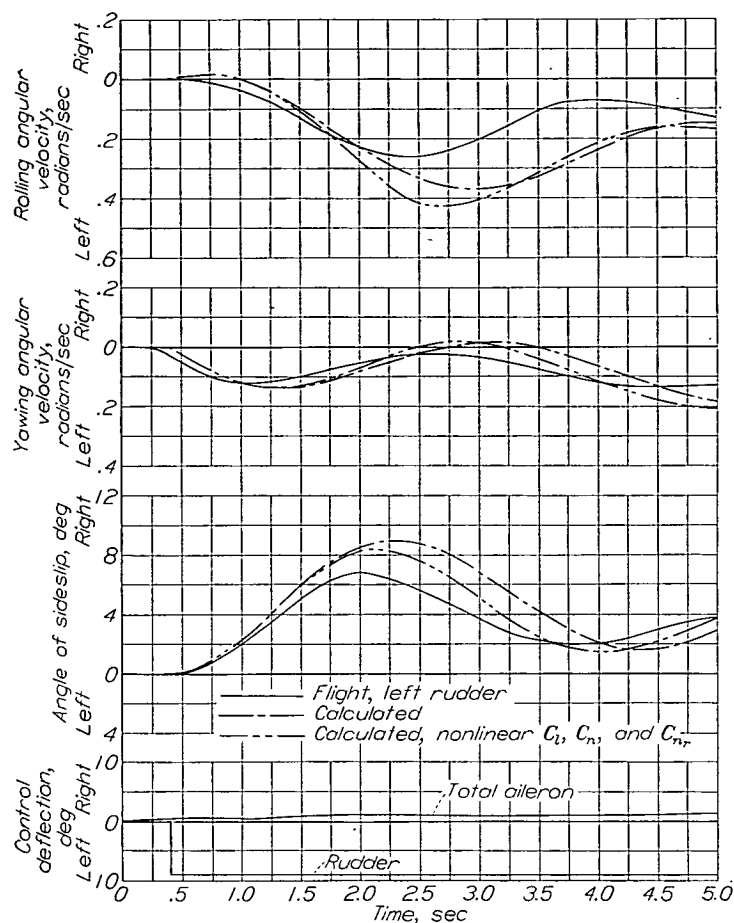


FIGURE 36.—Effect of nonlinear variation of C_l , C_n , and C_{n_r} with angle of yaw on the calculated lateral motions resulting from a left rudder kick. 40-percent-span slots; flaps up; $C_L=0.764$; $V_C=133$ miles per hour; engine idling.

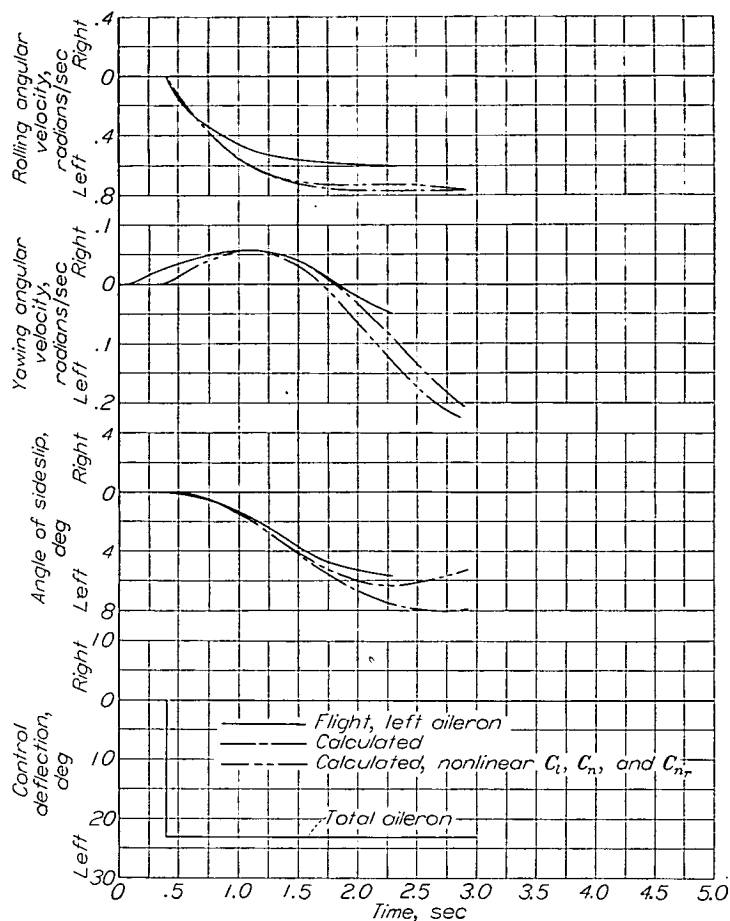


FIGURE 37.—Effect of nonlinear variation of C_l , C_n , and C_{n_r} with angle of yaw on the calculated lateral motions resulting from an abrupt left aileron deflection. 40-percent-span slots; flaps up; $C_L=0.598$; $V_C=150$ miles per hour; engine idling.

APPENDIX

LATERAL EQUATIONS OF MOTION

The lateral equations of motions used for most of the calculations of this report consider the fluctuation of the aerodynamic forces to be directly proportional to the component angular and translational velocities associated with the axis system as is customary in classical stability theory. The modification to these equations to account for the nonlinear effects of sideslip is discussed in the text. The equations used herein contain the necessary product of inertia and control terms and are given as follows:

Roll

$$C_{l_{\delta_r}} \delta_r + C_{l_{\delta_a}} \delta_a + C_{l_{\beta}} \beta + \frac{1}{2} C_{l_p} D\varphi + \frac{1}{2} C_{l_r} D\psi = 2\mu \left\{ \left[\left(\frac{k_{x_0}}{b} \right)^2 \cos^2 \eta + \left(\frac{k_{z_0}}{b} \right)^2 \sin^2 \eta \right] D^2 \varphi + \left[\left(\frac{k_{z_0}}{b} \right)^2 - \left(\frac{k_{x_0}}{b} \right)^2 \right] \cos \eta \sin \eta D^2 \psi \right\}$$

Yaw

$$C_{n_{\delta_r}} \delta_r + C_{n_{\delta_a}} \delta_a + C_{n_{\beta}} \beta + \frac{1}{2} C_{n_p} D\varphi + \frac{1}{2} C_{n_r} D\psi = 2\mu \left\{ \left[\left(\frac{k_{z_0}}{b} \right)^2 \cos^2 \eta + \left(\frac{k_{x_0}}{b} \right)^2 \sin^2 \eta \right] D^2 \psi + \left[\left(\frac{k_{z_0}}{b} \right)^2 - \left(\frac{k_{x_0}}{b} \right)^2 \right] \cos \eta \sin \eta D^2 \varphi \right\}$$

Sideslip

$$C_{Y_{\delta_r}} \delta_r + C_{Y_{\delta_a}} \delta_a + C_{Y_{\beta}} \beta + \frac{1}{2} C_{Y_p} D\varphi + \frac{1}{2} C_{Y_r} D\psi + C_L \varphi + C_L \psi \tan \gamma = 2\mu (D\beta + D\psi)$$

where

$$D = \frac{d}{ds}$$

$$s = \frac{TV}{b}$$

$C_{l_{\delta_r}}$	rate of change of rolling-moment coefficient with rudder deflection
$C_{l_{\delta_a}}$	rate of change of rolling-moment coefficient with aileron deflection
$C_{n_{\delta_r}}$	rate of change of yawing-moment coefficient with rudder deflection
$C_{n_{\delta_a}}$	rate of change of yawing-moment coefficient with aileron deflection
$C_{Y_{\delta_r}}$	rate of change of lateral-force coefficient with rudder deflection
$C_{Y_{\delta_a}}$	rate of change of lateral-force coefficient with aileron deflection

φ

angle of bank, radians

$$\mu = \frac{W}{\rho S b g}$$

W

weight of airplane

g

acceleration due to gravity

k_{x_0}

radius of gyration about principal longitudinal axis $\left(\sqrt{\frac{I_{x_0} g}{W}} \right)$

k_{z_0}

radius of gyration about principal normal axis $\left(\sqrt{\frac{I_{z_0} g}{W}} \right)$

γ

flight path angle, positive for climb

η

inclination of principal longitudinal axis of inertia with respect to flight path, positive when above flight path at nose

$$C_{Y_{\beta}} = \frac{\partial C_Y}{\partial \beta}$$

REFERENCES

1. MacLachlan, Robert, and Letko, William: Correlation of Two Experimental Methods of Determining the Rolling Characteristics of Unswept Wings. NACA TN 1309, 1947.
2. Queijo, M. J., and Jaquet, Byron M.: Calculated Effects of Geometric Dihedral on the Low-Speed Rolling Derivatives of Swept Wings. NACA TN 1732, 1948.
3. Goodman, Alex, and Brewer, Jack D.: Investigation at Low Speeds of the Effect of Aspect Ratio and Sweep on Static and Yawing Stability Derivatives of Untapered Wings. NACA TN 1669, 1948.
4. Toll, Thomas A., and Queijo, M. J.: Approximate Relations and Charts for Low-Speed Stability Derivatives of Swept Wings. NACA TN 1581, 1948.
5. Sjöberg, S. A., and Reeder, J. P.: Flight Measurements of the Stability, Control, and Stalling Characteristics of an Airplane Having a 35° Sweptback Wing without Slots and with 80-Percent-Span Slots and a Comparison with Wind-Tunnel Data. NACA TN 1743, 1948.
6. Sjöberg, S. A., and Reeder, J. P.: Flight Measurements of the Lateral and Directional Stability and Control Characteristics of an Airplane Having a 35° Sweptback Wing with 40-Percent-Span Slots and a Comparison with Wind-Tunnel Data. NACA TN 1511, 1948.
7. Lockwood, Vernard E., and Watson, James M.: Stability and Control Characteristics at Low Speed of an Airplane Model Having a 38.7° Sweptback Wing with Aspect Ratio 4.51, Taper Ratio 0.54, and Conventional Tail Surfaces. NACA TN 1742, 1948.
8. Jones, Robert T.: A Simplified Application of the Method of Operators to the Calculation of Disturbed Motions of an Airplane. NACA Rep. 560, 1936.
9. Levy, H., and Baggott, E. A.: Numerical Studies in Differential Equations. Vol. 1, Watts & Co. (London), 1934.
10. Zimmerman, Charles H.: An Analysis of Lateral Stability in Power-Off Flight with Charts for Use in Design. NACA Rep. 589, 1937.

TABLE I.—PERTINENT AIRPLANE DIMENSIONS AND CHARACTERISTICS

Mass characteristics:	
Normal gross weight, lb.....	8,700
Moments of inertia, ft-lb-sec ² :	
I_{x_0}	7,654
I_{y_0}	14,088
I_{z_0}	20,159
Principal axis (relative to thrust line at nose), deg.....	0.45 up
Center of gravity:	
Location on M.A.C., percent M.A.C.....	21.8
Relative to thrust line, percent M.A.C. below.....	14.8
Wing:	
Span, ft.....	33.6
Area, sq ft.....	250
Airfoil section (normal to L. E.):	
Root.....	Modified 66, 2x-116 ($a=0.6$)
Tip.....	Modified 66, 2x-216 ($a=0.6$)
M.A.C., ft.....	7.79
Leading edge of M.A.C. (ft behind L. E. root chord)....	3.27
Aspect ratio.....	4.51
Taper ratio.....	1.84:1.00
Dihedral, deg.....	0
Sweepback (at quarter-chord line), deg.....	35
Total area, plain sealed wing flaps, sq ft.....	12.52
Ailerons:	
Span (along hinge line, each), ft.....	8.75
Area (rearward of hinge center line, each), sq ft.....	6.51
Horizontal tail:	
Total area, sq ft.....	46.53
Vertical tail:	
Fin area (above horizontal tail), sq ft.....	13.47
Total rudder area, sq ft.....	10.26
Ventral fin area, sq ft:	
Large ventral.....	17.10
Small ventral (approx.).....	8.50

TABLE II.—TRIM ANGLES OF HORIZONTAL TAIL

Slots (percent span)	δ_f (deg)	Ventral fin	Propeller	Trim C_L	i_t (deg)
0	0	Large	On	.33 .55 .76	-0.10 -1.10 -2.75
0	40	Large	On	.33 .55 .76 .95	.90 -.60 -1.50 -2.25
40	0	Large	On and off	.33 .55 .76	-.60 -2.70 -4.60
40	40	Large	On	.33 .55 .76 .95	.75 -1.50 -3.20 -4.40
80	0	Large and small	On	.33 .55 .76	0 -2.40 -4.25
80	40	Large	On	.33 .55 .76 .95	.60 -1.40 -3.00 -4.20

TABLE III.—INDEX OF FIGURES

Wind-tunnel results:

Static stability characteristics:		Figures
Longitudinal characteristics.....		7 and 8
Lateral derivatives.....		9 and 10
Variations of C_l , C_n , and C_Y with ψ		11
Rolling derivatives:		
Derivatives at $\psi=0^\circ$		12 and 13
Variations of derivatives with ψ		14
Yawing derivatives:		
Derivatives at $\psi=0^\circ$		15 and 16
Variation of derivatives with ψ		17
Calculated and flight motions:		
Abrupt deflection and release of rudder.....		18 to 23
Summary of period and damping values.....		24
Rudder kicks.....		25 to 31
Aileron rolls.....		32 to 35
Nonlinear aerodynamic effects of sideslip.....		36 and 37

TABLE IV.—FLIGHT CONDITIONS FOR WHICH MOTIONS WERE CALCULATED

Figure	Slots (percent span)	Flaps (deg)	Ventral fin	V_e (mph)	C_L	Type of test
18	40	0	Large	198	0.334	Oscillation
19	40	0		156	.551	
20	40	0		136	.759	
21	40	40		160	.524	
22	40	40	Small	128	.801	Rudder kick
23	80	0		120	.977	
25	40	0		198	.341	
26	40	0		160	.556	
27	40	0	Large	133	.764	Aileron roll
28	40	0		120	.919	
29	40	40		157	.537	
30	40	40		130	.794	
31	80	0	Large	228	.278	Aileron roll
32	0	0		119	.983	
33	40	0		150	.508	
34	40	40		146	.598	
35	80	40		110	1.169	

REPORT 1032

A COMPARISON OF THEORY AND EXPERIMENT FOR HIGH-SPEED FREE-MOLECULE FLOW¹

By JACKSON R. STALDER, GLEN GOODWIN, and MARCUS O. CREAGER

SUMMARY

A comparison is made of free-molecule-flow theory with the results of wind-tunnel tests performed to determine the drag and temperature-rise characteristics of a transverse circular cylinder.

The model consisted of a 0.0031-inch-diameter cylinder constructed of butt-welded iron and constantan wires. Temperatures were measured at the center and both ends of the model. The cylinder was tested under conditions where the Knudsen number of the flow (ratio of mean-free-molecular path to cylinder radius) varied between 4 and 185. The molecular speed ratio (ratio of stream speed to the most probable molecular speed) varied from 0.5 to 2.3. In terms of conventional flow parameters, these test conditions correspond to a Mach number range from 0.55 to 2.75 and a Reynolds number range from 0.005 to 0.90.

The measured values of the cylinder center-point temperature confirmed the salient point of the heat-transfer analysis which was the prediction that an insulated cylinder would attain a temperature higher than the stagnation temperature of the stream.

Good agreement was obtained between the theoretical and the experimental values for the drag coefficient. As predicted, theoretically, the drag coefficient was independent of Knudsen (or Reynolds number) variation and depended primarily upon the molecular-speed ratio.

INTRODUCTION

The various phenomena connected with the high-speed flows of rarefied gases have recently been the subject of several analytic investigations (references 1, 2, 3, and 4). Interest in this phase of aerodynamics and thermodynamics has been aroused primarily because of the necessity of being able to compute forces on and heat transfer to missiles which may fly at great altitudes at high speed. In another analytic exploration of this field, simplified concepts of aerodynamic and thermodynamic phenomena occurring at high altitudes have been used by Whipple (reference 5) in an investigation of the properties of the upper atmosphere by means of measurements of the brightness and deceleration of meteors.

Experimental work in the field of high-speed rarefied gas flows has been practically nonexistent up to this point. The pioneer work of Epstein (reference 6) on the drag of spheres

in a rarefied gas was concerned only with very low-speed motions. Consequently, the purpose of the present investigation was twofold: (1) to obtain experimental data that could be compared with the predictions of an analytic treatment of aerodynamic and thermodynamic processes occurring at very high altitudes; and (2) as a means to this end, to develop a wind-tunnel and related instrumentation that would permit testing of models under simulated conditions of high-altitude high-speed flight. In connection with the second objective—the development of a wind tunnel and instrumentation—it should be pointed out that a low-density wind tunnel is being developed concurrently at the University of California and the authors wish to acknowledge the assistance they have received from the reports issued by this group, as well as the stimulus received from conversations with the University of California personnel.

At the beginning of the experimental work, it was decided that the initial model tests in the low-density wind tunnel should be confined to the free-molecule-flow regime despite the fact that this regime is probably of less technical importance than the slip-flow and intermediate regimes. This decision was reached because the free-molecule regime is susceptible to analysis; whereas the slip-flow and intermediate regimes have not yet been adequately treated (cf., reference 2). In view of the many uncertainties connected with the development of entirely new types of wind-tunnel instrumentation, it was thought to be essential to test in a field where the experimental work could be guided by a theoretical background.

The selection of the test body was governed both by the requirement that free-molecule flow be obtained and by the practical considerations arising from tunnel size and instrumentation. The former condition required that the significant body dimension be small compared to the mean-free-molecular path.

A cylinder with axis perpendicular to the stream was chosen as the body which most nearly fulfilled the requirements of the test. The radius of the cylinder could be made small compared with the mean-free path. In addition, since it was desired to measure both aerodynamic forces and body temperature, the cylinder had the virtue of being doubly symmetrical so that only drag force had to be measured. The problem of temperature measurement could

¹ Supersedes NACA TN 2244, "A Comparison of Theory and Experiment for High-Speed Free-Molecule Flow" by Jackson R. Stalder, Glen Goodwin, and Marcus O. Creager, 1950.

be solved by constructing the model from dissimilar metals so that the model itself formed a thermocouple.

The present investigation is a first step toward an attempt to gain an understanding of the comparatively new and unexplored field of high-speed low-density flows. The experiments reported herein were made to provide a comparison with some important results of free-molecule-flow theory and to facilitate the solution of problems connected with the design, operation, and instrumentation of a low-density wind tunnel. The tests were conducted in the Ames low-density wind tunnel.

NOTATION

A	surface area, square feet
c_x	components of total velocity, feet per second
c_y	
c_z	
C_D	total drag coefficient, dimensionless
C_{D_i}	impinging drag coefficient, dimensionless
C_{D_r}	re-emission drag coefficient, dimensionless
D	total drag force, pounds
E_i	rate of incident molecular energy, foot-pounds per second
E_r	rate of re-emitted molecular energy, foot-pounds per second
E_R	rotational energy, foot-pounds per molecule
E_t	rate of incident translational molecular energy on front side of body, foot-pounds per second
$f(s)$	dimensionless function of s defined by $f(s)=Z_1(s^2+3)+Z_2(s^2+\frac{7}{2})$
$f^\circ(s)$	dimensionless function of s defined by $f^\circ(s)=Z_1(s^2+2)+Z_2(s^2+\frac{5}{2})$
$g(s)$	dimensionless function of s defined by $g(s)=3(Z_1+Z_2)$
$g^\circ(s)$	dimensionless function of s defined by $g^\circ(s)=2(Z_1+Z_2)$
G	normal force, pounds
I_0	modified Bessel function of first kind and zero order
I_1	modified Bessel function of first kind and first order
J	mechanical equivalent of heat, 778 foot-pounds per Btu
k	Boltzmann constant, 5.66×10^{-24} foot-pounds per °F per molecule
K	Knudsen number (ratio of mean-free-molecular path to cylinder radius), dimensionless
l_x	direction cosines, dimensionless
l_y	
l_z	
L	length, feet
m	mass of one molecule, slugs
M	Mach number, dimensionless
n	number of molecules striking body, molecules per square foot, second
N	number of molecules per unit volume of gas

p	free-stream static pressure, pounds per square foot
r	cylinder radius, feet
R	universal gas constant, 1544 foot-pounds per pound-mole, °F absolute
s	molecular-speed ratio (ratio of stream mass velocity to most probable molecular speed), dimensionless
T	free-stream static temperature, °F absolute
T_0	tunnel stagnation temperature, °F absolute
T_c	cylinder temperature, °F absolute
T_w	surface temperature, °F absolute
T_t	test-section wall temperature, °F absolute
U	mass velocity, feet per second

U_x	components of mass velocity, feet per second
U_y	
U_z	
v_r	re-emission velocity, feet per second
v_m	most probable molecular speed, feet per second
W	molecular weight, pounds per pound-mole
x	Cartesian coordinates
y	
z	
Z_1	dimensionless function of s defined by $Z_1 = \pi e^{-s^2/2} I_0(s^2/2)$
Z_2	dimensionless function of s defined by $Z_2 = \pi s^2 e^{-s^2/2} [I_0(s^2/2) + I_1(s^2/2)]$
α	accommodation coefficient, dimensionless
β	reciprocal of most probable molecular speed, seconds per foot
Γ	gamma function
γ	ratio of specific heats, dimensionless
ϵ	emissivity, dimensionless
θ	angle of body element with respect to free stream, radians
ρ	density of free stream, slugs per cubic foot
σ	Stefan-Boltzmann constant, 3.74×10^{-10} foot-pounds per square foot, °F ⁴

$$\psi = 1 + \frac{1 + \frac{3\sqrt{\pi}}{2} s \sin \theta [1 + \operatorname{erf}(s \sin \theta)] e^{s^2 \sin^2 \theta}}{1 + \sqrt{\pi} s \sin \theta [1 + \operatorname{erf}(s \sin \theta)] e^{s^2 \sin^2 \theta}}$$

ψ' dimensionless group defined by

$$\psi' = 1 + \frac{1 - \frac{3\sqrt{\pi}}{2} s \sin \theta [1 - \operatorname{erf}(s \sin \theta)] e^{s^2 \sin^2 \theta}}{1 - \sqrt{\pi} s \sin \theta [1 - \operatorname{erf}(s \sin \theta)] e^{s^2 \sin^2 \theta}}$$

χ dimensionless group defined by

$$\chi = e^{-s^2 \sin^2 \theta} + \sqrt{\pi} s \sin \theta [1 + \operatorname{erf}(s \sin \theta)]$$

χ' dimensionless group defined by

$$\chi' = e^{-s^2 \sin^2 \theta} - \sqrt{\pi} s \sin \theta [1 - \operatorname{erf}(s \sin \theta)]$$

$\operatorname{erf}(a)$ error function $\left(\frac{2}{\sqrt{\pi}} \int_0^a e^{-x^2} dx \right)$, dimensionless

ANALYSIS

The general problem of the calculation of aerodynamic forces and heat transfer to a body in a free-molecule-flow field is amenable to solution by application of the simpler concepts of the kinetic theory of gases. The simplification arises from the assumption that the motions of molecules that strike the body are in no way affected by collisions with molecules that have struck the body and have returned to the main stream of gas. This basic postulate allows the molecular motions of the oncoming stream to be treated as having the classical Maxwellian velocity distribution superimposed on a uniform mass velocity. Hence, the severe complication of dealing with a nonuniform-gas is avoided. Of course, this assumption does not imply that returning molecules do not collide with oncoming molecules since collisions do occur, on the average, at a distance from the body equal to one mean free path. However, if this mean free-path length is large compared with the body dimensions, it can be seen from geometrical considerations that the probability is small that the impinging molecules that do collide with returning molecules will strike the body without first having had their original Maxwellian velocity restored by collisions with other impinging molecules. It would seem probable, also, that a high value of mass velocity compared with the random thermal velocity of the gas would favor free-molecule flow because the re-emitted molecules would tend to be swept along with the stream away from the zone of influence of the body.

In general, the mechanism of energy and momentum transfer for free-molecule flow are inseparable. In effect, this means that the drag on a body is affected by the surface temperature of the body. This state of affairs arises because of the fact that a portion of the momentum imparted to the body is due to the emission of molecules from the surface and the velocity of emission depends upon the temperature of the body. This phenomenon will be discussed in detail later.

Because of the assumption that the motions of the oncoming and re-emitted molecules can be treated independently, the computation of the drag on and heat transfer to a cylinder can be broken down into separate parts—the part due to impact by impinging molecules and a part due to re-emission of the molecules from the surface. By the use of this basic principle, it is possible to calculate the drag and heat-transfer characteristics of a transverse cylinder in a free-molecule-flow field.

HEAT TRANSFER TO A TRANSVERSE CYLINDER IN A FREE-MOLECULE-FLOW FIELD

The calculation of the heat-transfer or energy-exchange process between the cylinder and the gas is made by using the basic method described in reference 4. The general equations of reference 4 are expressed in Cartesian coordinate form and can be summarized as follows:

The translational molecular energy incident upon the front side of an elemental plane area dA inclined at an angle θ with respect to the stream mass velocity U is, for a monatomic gas,

$$dE_i = n \left(\frac{mU^2}{2} + \psi kT \right) dA \quad (1)$$

where n , the number of molecules striking unit surface area per second, is given by

$$n = \frac{Nv_m}{2\sqrt{\pi}} \chi \quad (2)$$

For the rear side of the elemental area, similar expressions apply.

$$dE_i' = n' \left(\frac{mU^2}{2} + \psi' kT \right) dA \quad (3)$$

$$n' = \frac{Nv_m}{2\sqrt{\pi}} \chi' \quad (4)$$

The total molecular energy incident upon the front and rear sides of dA is, for a monatomic gas, given by equations (1) and (3), respectively, since only translational energies are involved. However, for a diatomic gas, the expression for the incident energy contains an additional term due to the rotational component of the molecular internal energy, and this component must be added to the translational component. The rotational energy per molecule of a gas composed of rigid dumbbell molecules is

$$E_R = kT \quad (5)$$

Then, the total incident molecular energy on the front side of dA is, for a diatomic gas, given by the following expression:

$$dE_i = dE_i + nE_R dA = n \left[\frac{mU^2}{2} + (\psi + 1)kT \right] dA \quad (6)$$

and, for the rear side of dA ,

$$dE_i' = dE_i' + n'E_R dA = n' \left[\frac{mU^2}{2} + (\psi' + 1)kT \right] dA \quad (7)$$

The method of calculation of the energy transported from the elemental area by re-emitted molecules is shown in appendix A. The result derived therein can be applied to the calculation of the energy transport by a monatomic gas from the front side of the area by combining equations (A1) and (A12) to yield

$$dE_r = dE_i(1 - \alpha) + 2\alpha n k T_c dA \quad (8)$$

and, for the energy transport by a diatomic gas, as

$$dE_r = (dE_i + n k T dA)(1 - \alpha) + 3\alpha n k T_c dA \quad (9)$$

Parallel equations can be written for the rear side of the area, dA .

The net radiant energy exchange is (assuming gray-body radiation)

$$\epsilon \sigma (T_c^4 - T_i^4) dA \quad (10)$$

where T_i is the effective temperature of the surroundings to which the area dA is radiating. This expression implies that the body of which dA is a part is small in comparison with the surrounding volume.

The problem now becomes one of applying the foregoing equations, which were derived for a plane area placed at an

arbitrary angle with respect to the stream, to a specific body—in this case a cylinder with axis perpendicular to the stream velocity. For a cylinder of length L , the elemental area may be written as

$$dA = rLd\theta \quad (11)$$

$$2\alpha rL \left[\int_0^{\pi/2} n \left(\frac{mU^2}{2} + \psi kT \right) d\theta + \int_0^{\pi/2} n' \left(\frac{mU^2}{2} + \psi' kT \right) d\theta \right] - 4\alpha rLkT_c \left(\int_0^{\pi/2} n d\theta + \int_0^{\pi/2} n' d\theta \right) + 2\pi rL \left[Q - \epsilon \sigma (T_c^4 - T_i^4) \right] = 0 \quad (12)$$

For a diatomic gas, a similar equation applies

$$2\alpha rL \left\{ \int_0^{\pi/2} n \left[\frac{mU^2}{2} + (\psi + 1)kT \right] d\theta + \int_0^{\pi/2} n' \left[\frac{mU^2}{2} + (\psi' + 1)kT \right] d\theta \right\} - 6\alpha rLkT_c \left(\int_0^{\pi/2} n d\theta + \int_0^{\pi/2} n' d\theta \right) + 2\pi rL \left[Q - \epsilon \sigma (T_c^4 - T_i^4) \right] = 0 \quad (13)$$

The reduction of the integrals appearing in equations (12) and (13) is a straightforward but lengthy procedure. The methods used to evaluate these integrals are given in appendix B. The final results are written here, for a monatomic gas, as

$$\begin{aligned} & 2 \frac{T_c}{T} (Z_1 + Z_2) - \left[Z_1 (s^2 + 2) + Z_2 \left(s^2 + \frac{5}{2} \right) \right] + \\ & \frac{2\pi^{3/2}}{pv_m \alpha} \left[\epsilon \sigma (T_c^4 - T_i^4) - Q \right] = 0 \end{aligned} \quad (14)$$

and, for a diatomic gas, as

$$\begin{aligned} & 3 \frac{T_c}{T} (Z_1 + Z_2) - \left[Z_1 (s^2 + 3) + Z_2 \left(s^2 + \frac{7}{2} \right) \right] + \\ & \frac{2\pi^{3/2}}{pv_m \alpha} \left[\epsilon \sigma (T_c^4 - T_i^4) - Q \right] = 0 \end{aligned} \quad (15)$$

In order to facilitate computation, values of the dimensionless groups,

$$\begin{aligned} f(s) &= Z_1 (s^2 + 3) + Z_2 \left(s^2 + \frac{7}{2} \right), \\ g(s) &= 3(Z_1 + Z_2), \\ f^o(s) &= Z_1 (s^2 + 2) + Z_2 \left(s^2 + \frac{5}{2} \right), \end{aligned}$$

and

$$g^o(s) = 2(Z_1 + Z_2),$$

appearing in equations (14) and (15), are tabulated in the table of appendix B.

Equations (14) and (15) then completely describe energy transfer occurring between a transverse cylinder and its surroundings in terms of the cylinder temperature, static gas temperature and pressure, molecular-speed ratio, accommodation coefficient, the effective temperature of the surroundings to which the cylinder is emitting radiation, the emissivity of the cylinder surface, and the internal energy input to the cylinder. The equations are subject to the following restrictions:

1. The surface temperature of the cylinder is assumed to be constant, circumferentially.

2. The cylinder receives and emits radiant energy as a gray body; that is, the emissivity is independent of wave length. Also, the cylinder is assumed to be small compared to the surrounding volume.

3. The internal energy of the gas stream in which the cylinder is located has the value $(3/2) kT$ for a monatomic gas and $(5/2) kT$ for a diatomic gas.

Then, if internal energy input to the cylinder per unit area is denoted as Q (in the usual experimental arrangement this would be electrical energy), an equation can be written which expresses an energy balance on the cylinder. For a monatomic gas, the energy-balance equation is

DRAG ON A TRANSVERSE CYLINDER IN A FREE-MOLECULE-FLOW FIELD

The general problem of aerodynamic forces acting on bodies in free-molecule flow has been treated by several authors (references 1, 2, and 3). The basic method of calculation was identical in each case. The method consists of the assumption of a gas with a Maxwellian velocity distribution superimposed upon an arbitrary mass velocity. The number of molecules having specified velocities that impinge upon an elemental area inclined at an arbitrary angle to the stream mass velocity is computed and, from this calculation, the force due to the transmitted momentum is obtained. To this momentum due to molecular impingement, however, must be added the momentum imparted to the surface by molecules emitted from the surface. The magnitude and direction of the force imparted by emitted molecules depends, among other things, on the type of emission that is assumed to occur. A condensed discussion of the several possible types of surface emission can be found in reference 4. An extensive treatment of the subject is given in reference 7.

In order to determine the total aerodynamic force on a body of arbitrary shape, the differential momentum imparted to an element of the body surface is computed, as outlined in the preceding paragraph, and the total momentum obtained by integration over the body surface. Also, since the impinging and re-emitted momenta can be added linearly, the resulting total momentum can be calculated for any desired type of re-emission.

In this paper, completely diffuse scattering of the incident molecular stream from an insulated surface is assumed. By this assumption it is meant that (a) the molecules that are scattered from the surface are emitted with a Maxwellian distribution of speed, (b) the direction of molecular emission from the surface is controlled by the Knudsen cosine law (cf., references 7 and 8), (c) the body temperature is assumed to be uniform over the surface, and further (d) that conservation of number of molecules is maintained for each element of the body surface. These assumptions seem to be in accord with physical fact, as discussed in reference 6.

The specific problem of the aerodynamic drag on a transverse cylinder has been treated in references 1 and 3. The reason for repetition of the calculation here is that the results have been obtained in a compact closed form which is amenable to easy calculation and, further, that a solution for the re-emission drag is obtained which involves measurable physical quantities. Appendices A and C contain the

details of the calculations and only the final results are presented here. The results of the computations are as follows: The total drag acting on an insulated transverse circular cylinder of radius r and length L is given by

$$D = \frac{\rho U^2}{2} (2rL)(C_{D_i} + C_{D_r}) \quad (16)$$

where C_{D_i} and C_{D_r} are the drag coefficients associated with impinging and re-emitted momenta. The value of C_{D_i} is shown in appendix C to have the value

$$C_{D_i} = \frac{\pi^{1/2} e^{-s^2/2}}{s} \left\{ I_0(s^2/2) + \left(\frac{1+2s^2}{2} \right) \left[I_0(s^2/2) + I_1(s^2/2) \right] \right\} \quad (17)$$

and the value of C_{D_r} for an insulated cylinder is shown in appendix A to be, for a monatomic gas,

$$C_{D_r} = \frac{\pi^{3/2}}{4s} \sqrt{\frac{Z_1(s^2+2) + Z_2\left(s^2 + \frac{5}{2}\right)}{2(Z_1 + Z_2)}} \quad (18)$$

and, for a diatomic gas,

$$C_{D_r} = \frac{\pi^{3/2}}{4s} \sqrt{\frac{Z_1(s^2+3) + Z_2\left(s^2 + \frac{7}{2}\right)}{3(Z_1 + Z_2)}} \quad (19)$$

APPARATUS AND TEST METHODS

THE WIND TUNNEL

The Ames low-density wind tunnel, in which the present tests were conducted, is an open-jet nonreturn-type tunnel. A sketch showing the major components of the wind tunnel is shown in figure 1. The test gas, from a bottled source, is throttled into a settling chamber and then passes through an axially symmetric nozzle which discharges a free jet into an 18-inch cubical test chamber. The test chamber is connected through a surge chamber to four booster-type oil-diffusion pumps which continuously evacuate the system. The diffusion pumps are connected in parallel and are backed by a mechanical vacuum pump which discharges to the atmosphere. Valves are provided at the inlet to each diffusion pump in order to allow the pumping system to be isolated when the test chamber is opened to the atmosphere.

A traversing mechanism is installed in the test chamber which permits three degrees of motion relative to the gas stream. The mechanism is used to position models and instrumentation in the desired location within the gas stream.

The nozzles used in the tests were axially symmetric with an outlet diameter of 2 inches and were machined from clear plastic. The nozzle contours were determined by the method outlined in reference 9. No boundary-layer correction was applied to the nozzle contours.

After the system was evacuated, the flow rate of the test gas into the settling chamber was adjusted until the stagnation pressure, as measured by a McLeod gage, reached the desired value.

The static pressure in the test chamber was read on another McLeod gage and, if the resulting pressure ratio was that

required to establish flow at the particular Mach number desired for the run, the tunnel was considered ready for tests. In order to match tunnel conditions of static pressure and pressure ratio with those that existed at an earlier time, it was sometimes necessary to alter the amount of throttling at the pump inlets due to a slight variation of pumping speed from day to day.

The pressure level in the tunnel was varied by increasing or decreasing the flow of the test gas into the system. The static and stagnation pressures were again checked to determine if sufficient pressure ratio was available to establish flow at the desired Mach number. This procedure was somewhat complicated by the fact that at the lower pressure levels the boundary layer in the nozzle would tend to thicken, thus causing a decrease in the Mach number compared with that obtained at the higher pressure levels.

The wind tunnel could be operated through a range of test-section pressures of from 20 to 135 microns of mercury for a given nozzle and over a range of Mach numbers from 0.5 to 2.75 by using several nozzles.

INSTRUMENTATION AND THE TEST MODEL

The static pressures which existed in the wind tunnel were, in general, too low to be measured with a conventional manometer or Bourdon-type pressure gage; therefore, two types of gages widely used in high-vacuum work were utilized, namely, the McLeod and Pirani gages. For a complete description of these gages the reader is referred to any standard text on high-vacuum technique (e. g., reference 10); however, some of the special features of the gages used will be described.

The McLeod gage used as a primary standard in these tests had an initial volume of 146 milliliters connected to a closed tube of 1.95-millimeter diameter. This arrangement gave a scale length of 51.4 centimeters for a pressure range of 0 to 3000 microns. (For the sake of brevity, the term micron is used throughout as a unit of pressure rather than the term microns of mercury.) The diameter of the closed tube was measured by the mercury pellet method to within ± 0.01 millimeter and the difference in height of the mercury columns could be read to the nearest 0.1 millimeter by use of a micrometer scale. At a pressure of 100 microns, this gage could be read to within 1 percent with ease.

The gage was connected to the test chamber with an 18-millimeter-diameter glass tube to insure a short time lag between equalization of the system and the gage pressures. A dry-ice and acetone cold trap was provided between the gage and the test chamber to prevent condensable vapors from reaching the gage.

In general, it is difficult to obtain a completely independent method of checking the accuracy of pressure measurements made with a McLeod gage for pressures below about 1 millimeter of mercury. Above 1 millimeter of mercury a U-tube manometer, filled with a very low-vapor-pressure oil and having a high-vacuum system (of the order of 10^{-5} mm of Hg) connected to one leg, may be used. The McLeod gage used in the tests was checked in this manner and the agreement between the oil manometer and the McLeod gage was within ± 3 percent.

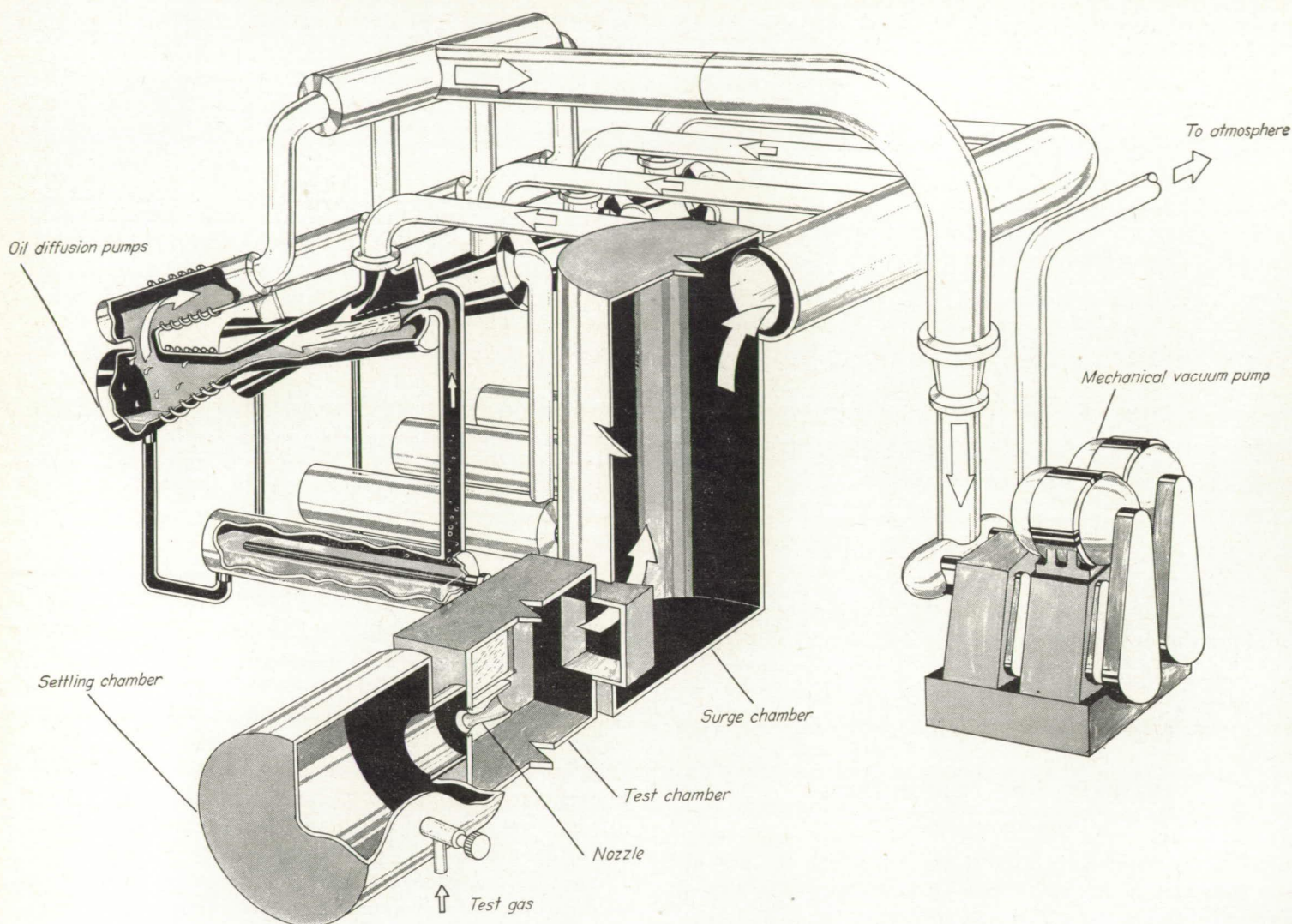


FIGURE 1.—Ames low-density wind tunnel.

An indication of the accuracy of the McLeod gage used in the tests in the pressure range below 200 microns was obtained by comparing the pressures indicated by the test McLeod gage with those obtained with a laboratory-type McLeod gage having a range of 0 to 200 microns, which has been calibrated against a laboratory standard McLeod gage at the manufacturer's plant. The agreement between the test gage and the laboratory-type gage was within ± 3 percent at a pressure of 100 microns.

The Pirani gage used in the tests consisted of an open glass tube, which contained a heated filament. The gas pressure in the tube determined the temperature and thus the resistance of the filament. A temperature-compensating tube of identical construction was sealed off at a very low pressure. The measuring and compensating tube filaments formed two legs of a Wheatstone bridge; and the unbalance of the Wheatstone bridge circuit, as indicated by a self-balancing potentiometer, was a measure of the pressure in the open leg of the gage. The Pirani gage was calibrated against the 0- to 3000-micron-range McLeod gage as a standard.

A drift of the calibration of the Pirani gage with time was noted and for this reason the Pirani gage was calibrated

before and after every test run. A typical calibration curve is shown in figure 2.

Examination of figure 2 reveals that the sensitivity of the Pirani gage used was nearly constant from a pressure of 10 microns to a pressure of 80 microns. Above about 200 microns this Pirani gage becomes too insensitive to be usable as a pressure-measuring instrument.

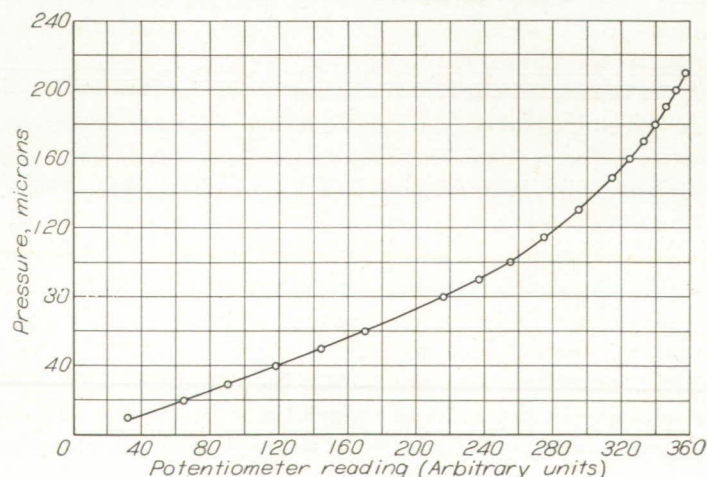


FIGURE 2.—Typical calibration of Pirani gage in nitrogen gas.

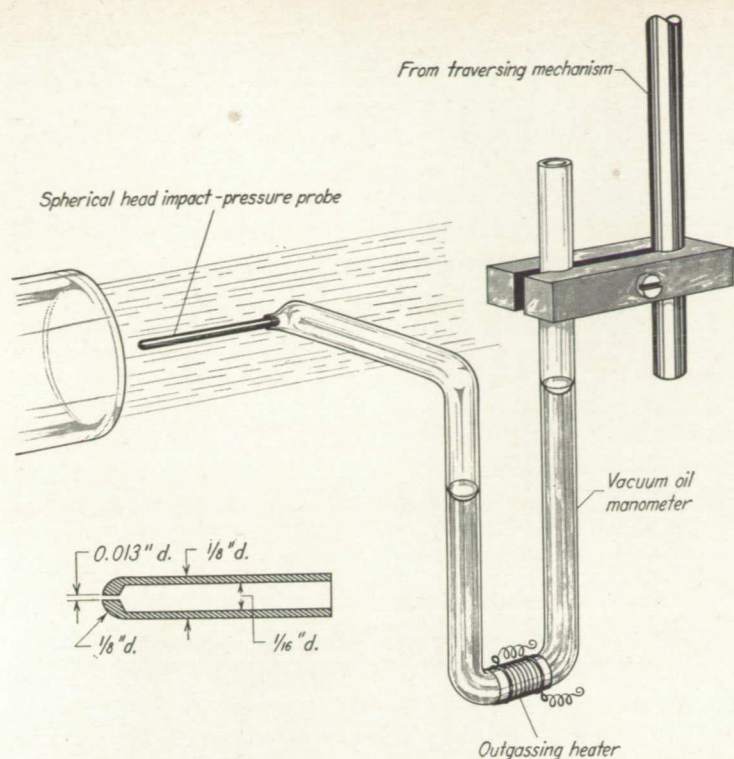


FIGURE 3.—Impact-pressure-measuring system.

Whenever a pressure-measuring device is connected to a vacuum system by a tube or small orifice, two troublesome effects are present—time lag and gassing. The time lag of the pressure gage is due to the fact that, as the pressure in the main part of the system is changed, a volume of gas must flow out of or into the pressure gage and its connecting tubing. At the low pressures encountered during the tests, this time lag may be of the order of hours if the gage volume and connecting tubing are not properly proportioned. Gassing, that is, gas being given up or adsorbed by the surface of the pressure gage and its connecting tubing, has the effect of causing the pressure gage to indicate a different pressure than that which exists in the main part of the system. Outgassing effects can be reduced materially by proper cleaning of the surface and by heating the surface under a vacuum to drive off water vapor and other materials which cling to ordinary surfaces.

Schaaf, Mann, and Cyr (references 11 and 12) have investigated this problem in considerable detail, and show that, if the assumption is made that gassing is a function of surface area only, there exists an optimum geometry of gage volume and connecting tubing which will result in minimum time lags. References 11 and 12 were used as guides in the design of the impact and static-pressure-measuring devices which were used in the tunnel calibration tests, and, in addition, the time lag of these pressure-measuring systems was measured.

The impact pressures encountered during the wind-tunnel-calibration tests were too high to be measured with a Pirani gage and, because of the difficulty of obtaining a suitable flexible tube which would have a small time lag, the McLeod gage could not be used. These impact pressures, therefore, were measured with a glass U-tube manometer filled with a low-vapor-pressure oil. The manometer was located inside

the test chamber with one leg of the manometer open and the other end fastened with a glass connection to a 1/8-inch-diameter spherical-head impact tube. Figure 3 shows a diagrammatic sketch of this arrangement. The difference in oil level between the two legs, as read by a coordinate-type cathetometer located outside the tunnel, was the difference between the test-chamber static pressure and the impact pressure. The test-chamber static pressure was measured by the 0- to 3000-micron McLeod gage described earlier.

The time lag of the system shown in figure 3 was measured by setting the pressure in the test chamber at approximately 3 millimeters of mercury and then evacuating the chamber to approximately 10 microns as quickly as possible. The time required for the oil level in the two legs to equalize was approximately 15 seconds. This indicates a very short time lag for the practical case as the rate of pressure change encountered in the wind-tunnel calibration was small compared to that measured during the time-lag test. The procedure used in the wind-tunnel-calibration tests was to continue to take readings of the oil level until steady state was established.

The static pressure of the gas stream from the nozzle was measured with a 0.10-inch-diameter ogival probe connected directly to a Pirani gage as shown in figure 4. Three 0.0225-inch-diameter holes spaced 120° apart were located at a point on the ogive where the pressure is equal to free-stream pressure as calculated by the method given in reference 13. The position along the probe where the surface pressure is equal to free-stream pressure varies somewhat with Mach number; however, this variation is

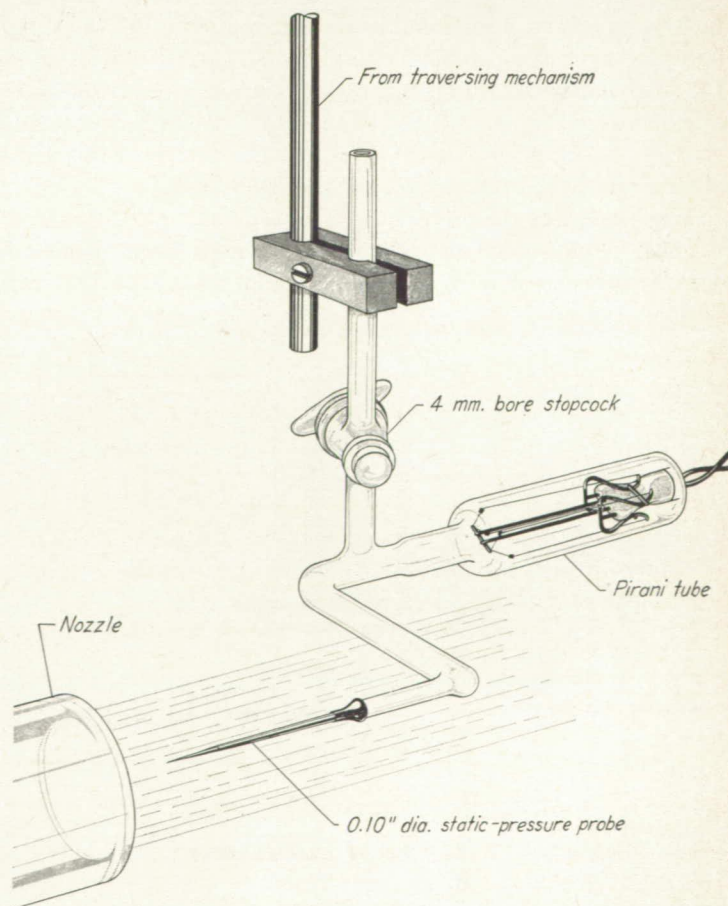


FIGURE 4.—Static-pressure-measuring system.

not large and static pressures measured by this probe agreed with pressures measured with a 4° cone.

No boundary-layer correction was applied to the ogive, or cone contour; consequently, it is possible that some error in the measurement of static pressure existed. Figure 5 shows the main dimensions of the static-pressure probe used.

In order that the Pirani gage shown in figure 4 could be calibrated against the McLeod gage, which was connected to the test chamber, a 4-millimeter-bore stopcock was glass-welded to the Pirani gage to allow calibration with a relatively large opening and attendant short time lag. As the stream static pressures were measured with the stopcock closed, the following tests were conducted to measure the order of magnitude of the time lag of the system. The test chamber was pressurized to 165 microns and suddenly evacuated to approximately 5 microns. The rate of pressure change as measured by the Pirani gage with the stopcock open and with the stopcock closed is shown in figure 6. The procedure was repeated with a rising pressure and the results are also shown in figure 6.

It can be seen from figure 6 that the two curves, representing the runs with the stopcock open and closed, are essentially congruent after 70 seconds for decreasing pressure and after 50 seconds for the increasing pressure runs. With the stopcock open, the time lag of the Pirani gage should be very small (references 11 and 12) and the curve of pressure as a function of time includes the time required for the pumping equipment to evacuate the test chamber, the time lag in the self-balancing potentiometer, and the thermal lag in the Pirani filaments. The curve of pressure as a function of time for the stopcock closed includes all these factors plus the additional time lag imposed by the small holes in the static probe. The pressure variation during the time-lag tests was much more severe than any encountered during the static survey tests; therefore it can be concluded that, by waiting for at least 2 minutes before recording data, all errors due to time lag would be eliminated.

The test model.—The model tested was a 0.0031-inch-diameter right-circular cylinder fabricated from iron and constantan wires and had an exposed length of $1\frac{1}{2}$ inch.

NOTE.—All dimensions given in inches.

x	d
0	0
0.05	0.019
.10	.036
.15	.051
.20	.064
.25	.075
.30	.084
.35	.091
.40	.096
.45	.099
.50	.100

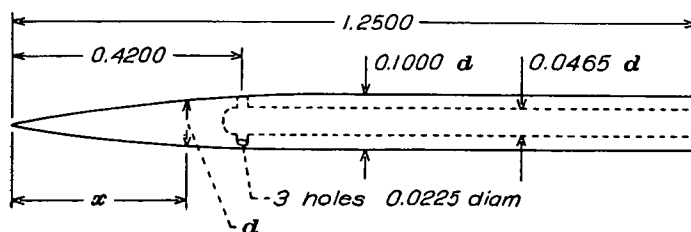


FIGURE 5.—Static-pressure probe used in the tests.

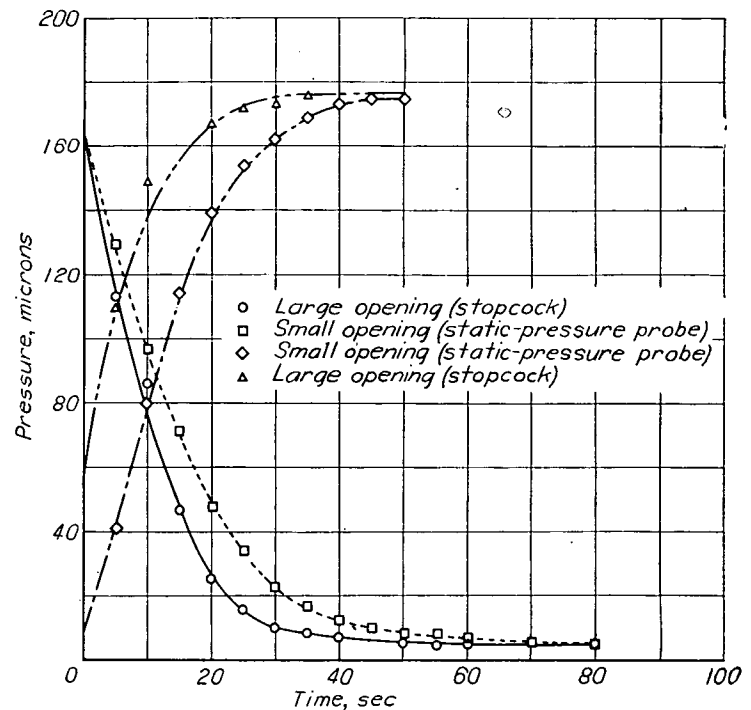


FIGURE 6.—Time response of the static-pressure-measuring system.

The iron and constantan wires were electrically butt-welded together to form a thermocouple junction. Figure 7 shows a microphotograph of the model. It can be seen that the welding operation did not appreciably alter the shape of the model. The butt-welded junction was located in the center of the model and a 0.0031-inch iron wire was silver soldered to the constantan side of the model at a distance of $\frac{1}{16}$ inch from the center junction. A 0.0031-inch-diameter constantan wire was similarly fastened to the iron side of the model. These two auxiliary thermocouple junctions were used to measure the temperature at the ends of the model.

The entire model was covered with fine-grain soot obtained from an acetylene flame in order that the emissivity would be constant over the total model length and so that an established value of emissivity of 0.95 could be used to evaluate the heat loss by radiation.

Thermocouple voltages from the center and end thermocouples were read on a null-type laboratory potentiometer having a least count of 0.01 millivolt.

The drag balance.—A specially constructed microbalance shown in figures 8 (a) and 8 (b) was used to measure the drag of the model tested. The balance consisted of a beam mounted upon an agate knife-edge which rested in a steel V-block, as shown diagrammatically in figure 9. The model was mounted between two arms extending down from either side of the beam. Electrical leads were transferred from the moving beam to the balance frame through mercury pools.

A drag force on the model caused an unbalance of the beam which was counteracted by a magnetic-restoring force produced by an armature located in a magnetic field produced by an alternating-current coil. An optical lever approximately 2 feet long was used to determine the zero position of the balance.

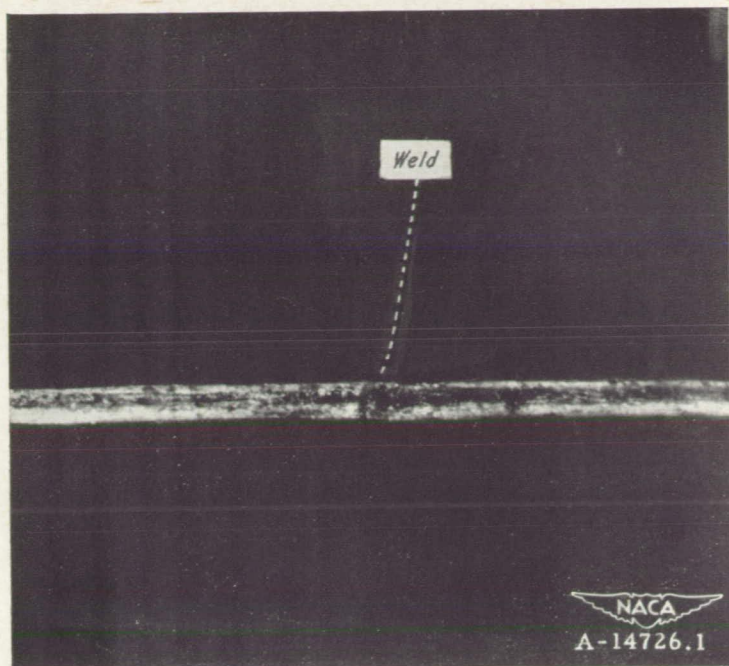


FIGURE 7.—Micro-photograph of model showing weld at junction of iron and constantan. Enlarged 60 times.

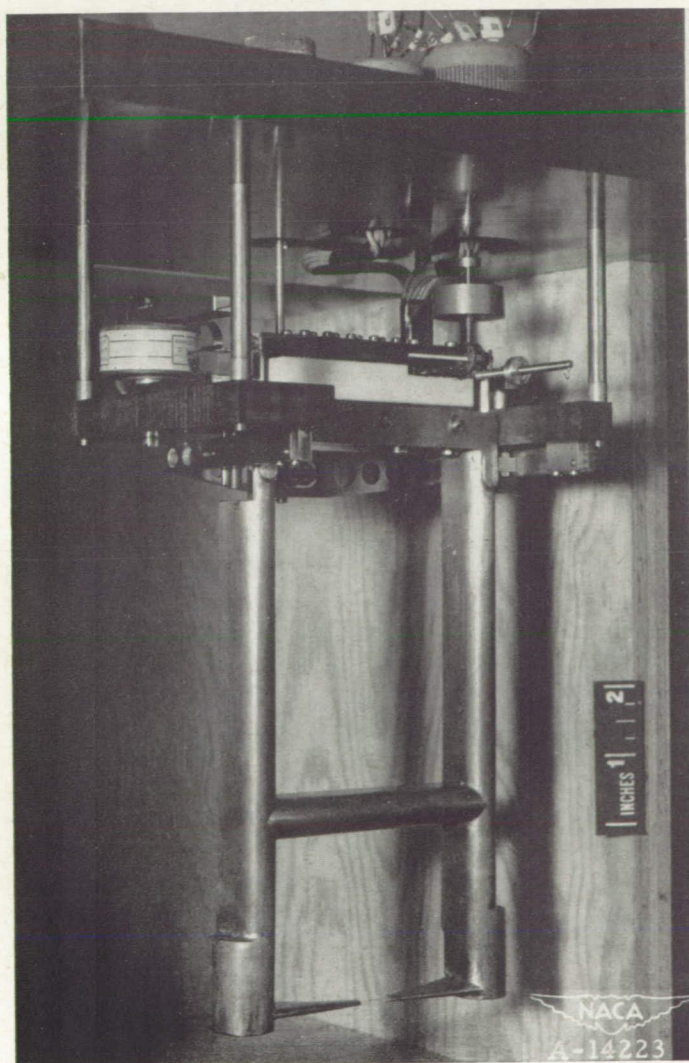
The magnitude of the drag force in terms of magnetizing current was determined by hanging known weights on a lever arm attached to the balance beam and measuring the current necessary to restore the balance to zero position. Figure 10 shows a typical calibration of the drag balance.

The model was mounted between the extension arms of the drag balance and was shielded over all but the center $1\frac{1}{2}$ inch, as shown in figures 8 (a) and 8 (b). A check was made on the effectiveness of the shielding on the supporting members. A small shield was mounted to the center main shield (extension arms) covering the center portion of the model (normally exposed). The completely shielded model was then placed in the gas stream. No drag forces were detected.

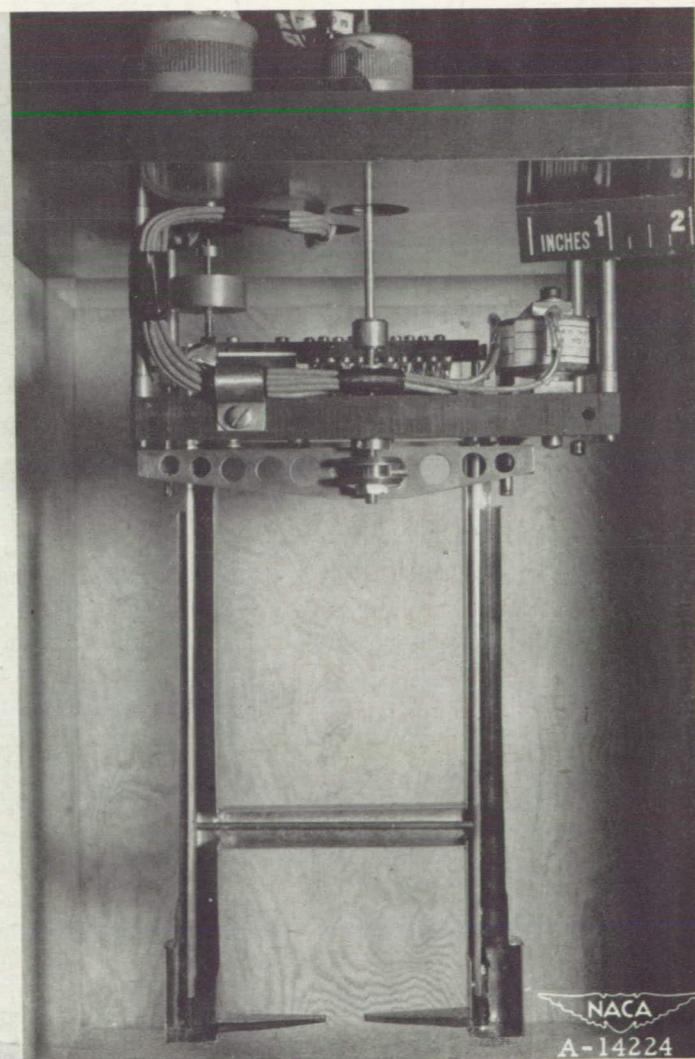
It was found that the balance exhibited an erratic calibration shift from day to day, which necessitated calibration of the balance before and after each drag determination.

CALIBRATION OF THE WIND TUNNEL

The purpose of the calibration of the wind tunnel was to determine the Mach number and static-pressure distributions of the stream in the plane at which the model was



(a) Front view.



(b) Rear view.

FIGURE 8.—Photograph of drag balance showing force shields.

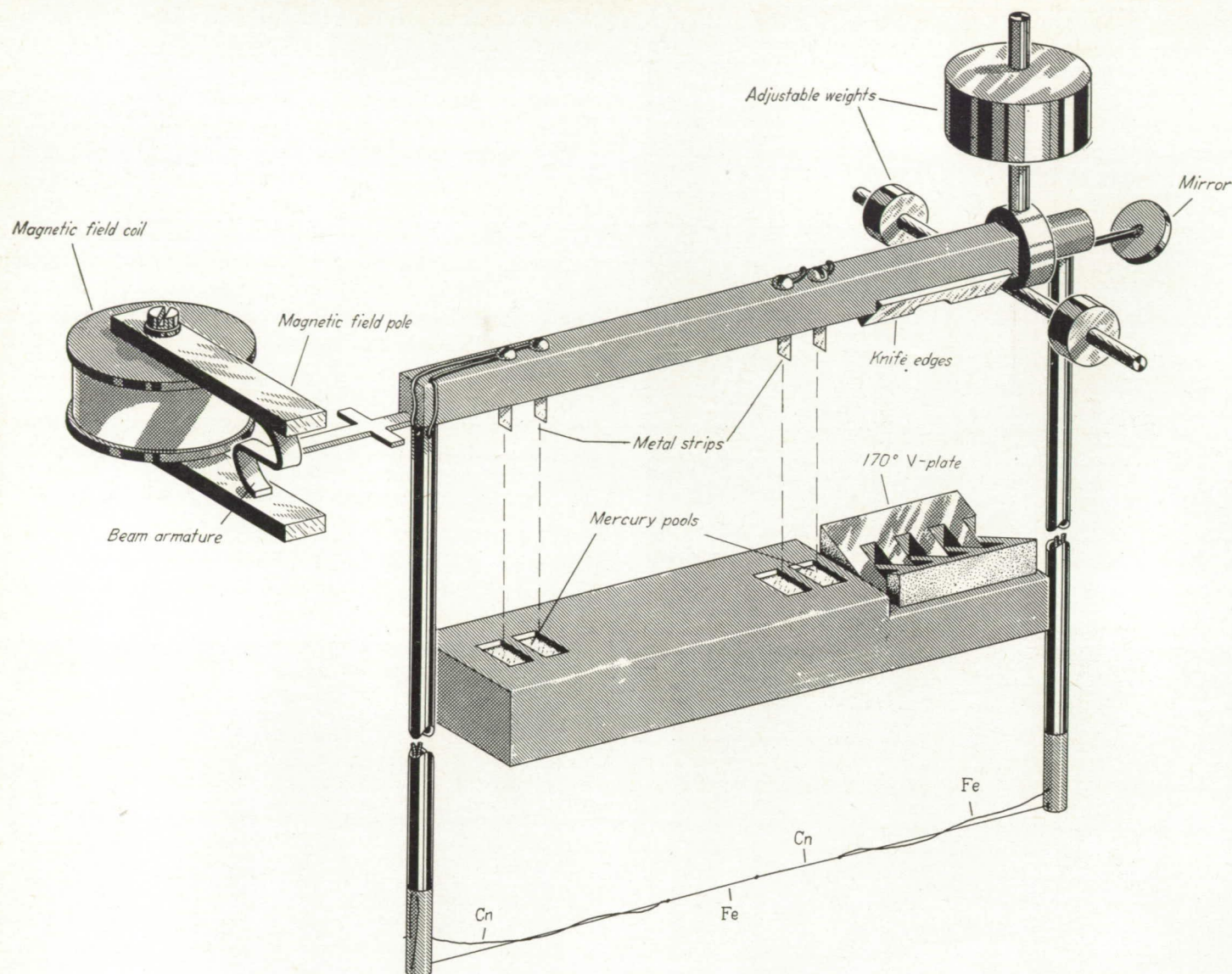


FIGURE 9.—Diagrammatic sketch of drag balance.

located— $\frac{1}{4}$ inch downstream of the exit plane of the various nozzles used.

The method used was to survey the stream with the impact- and static-pressure probes described earlier. These impact- and static-pressure probes, together with their pressure-measuring gages, were mounted 2 inches apart upon a common support from the traversing mechanism and readings were taken at $\frac{1}{16}$ -inch intervals from the center to the edge of the stream. This method of mounting allowed both the impact and static pressures to be measured during the course of one run, thus obviating the necessity of matching tunnel conditions on separate runs.

Table I lists the nozzles tested, their design Mach number, and the tunnel conditions for which the surveys were made.

The static pressure varied across the stream issuing from each of the nozzles tested. For most of the nozzles, the pressure was somewhat higher near the center portion of the stream than was the pressure of the quiescent gas in the test chamber. Certain of these nozzles, however, did

TABLE I.—WIND-TUNNEL-CALIBRATION DATA

Nozzle number	Design Mach number	Test gas	Test chamber static pressure (microns Hg)	Stagnation pressure (mm Hg)	Test ¹ Mach number
1	4.5	N ₂	71.0	2.7	2.49
		N ₂	121.0	5.0	2.78
		He	18.7	.70	.55
		He	51.5	2.2	2.08
		He	90.0	3.8	2.37
2	3.6	N ₂	70.0	1.2	2.07
		N ₂	117.0	2.2	2.25
		He	51.7	.95	1.57
		He	89.2	1.7	1.75
3	3.0	N ₂	71.0	0.90	1.85
		He	31.0	.35	.73
		He	45.0	.55	1.00
4	3.11	He	20.0	0.19	.89
		He	54.7	.55	1.67
		He	79.5	.70	1.68
		He	103.0	.90	1.73
		He	119.0	1.05	1.79
		He	137.0	1.20	1.83
5	2.64	He	20.0	0.17	.78
		He	41.6	.30	1.23
		He	61.6	.40	1.36
		He	77.8	.45	1.44

¹ Average Mach number over center half inch of stream.

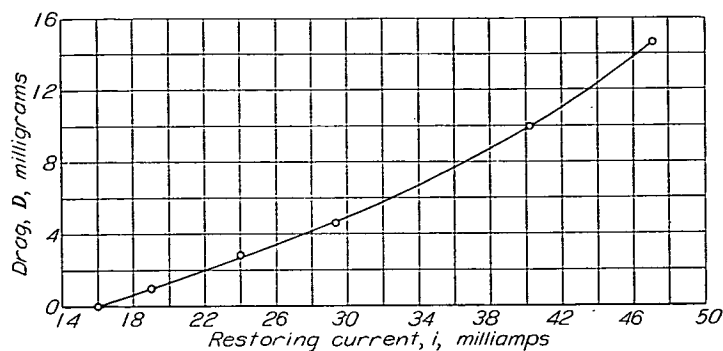


FIGURE 10.—Typical calibration curve for drag balance.

not show this general trend. In these nozzles the pressure first decreased, then increased, as the probe was moved from the nozzle wall to the center of the stream. Figures 11 and 12 show this effect. In general, the static-pressure variation across the stream was not large. The average change over the center $\frac{1}{2}$ inch of the stream amounted to 2 percent, while the maximum deviation was only 5 percent.

The impact-pressure surveys of all the nozzles had one common characteristic; that is, the impact pressure increased from a minimum value at the edge of the nozzle wall to a maximum value near the center of the stream. The area of the stream which had a constant, or nearly constant, impact pressure varied somewhat from nozzle to nozzle, and also to some extent with pressure level. The average variation of impact pressure over the center $\frac{1}{2}$ inch of the stream for all nozzles tested was 7 percent with a maximum variation

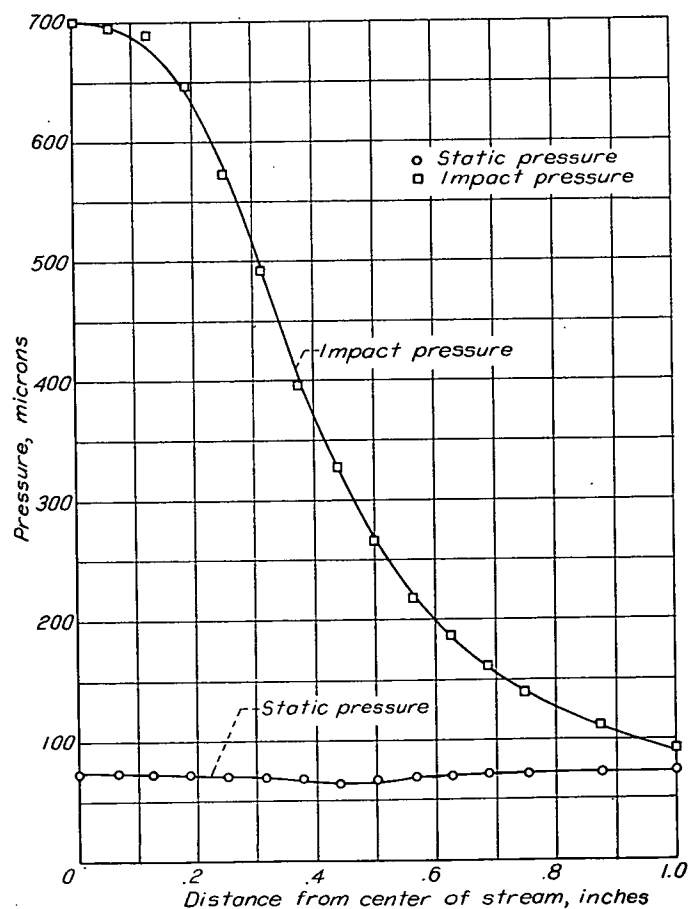


FIGURE 11.—Variation of impact and static pressure with nozzle radius for nozzle number 1 using nitrogen gas.

in some cases of 14 percent. Figures 11 and 12 show the variation of impact pressure across the stream for one nozzle using two gases, nitrogen and helium.

The Mach number of a gas stream can be computed from measured values of the stagnation and static pressure of the stream, provided that the flow is isentropic between the points of pressure measurement. However, if the static and impact pressures are measured at the same point in the stream, no assumption regarding the nature of the flow need be made in order to compute the local stream Mach number. For an inviscid flow, the Mach number can be calculated from Rayleigh's equation, the use of which involves the assumption of a normal shock wave ahead of the impact-pressure tube and isentropic compression of the gas in the subsonic flow behind the shock wave to the stagnation point on the impact-pressure tube. In a low-density flow, however, viscous effects are present in the flow about a stagnation region, thus invalidating the assumption of isentropic compression behind the normal shock wave. Chambré (reference 14) computed the viscous correction to be applied to spherical-head impact tubes in a low-density flow, and figure 13 shows the Mach number distribution of nozzle 1 as computed by the Rayleigh equation (inviscid) and as computed by Chambré's equation. The Mach numbers calculated from the stagnation and local static pressures were,

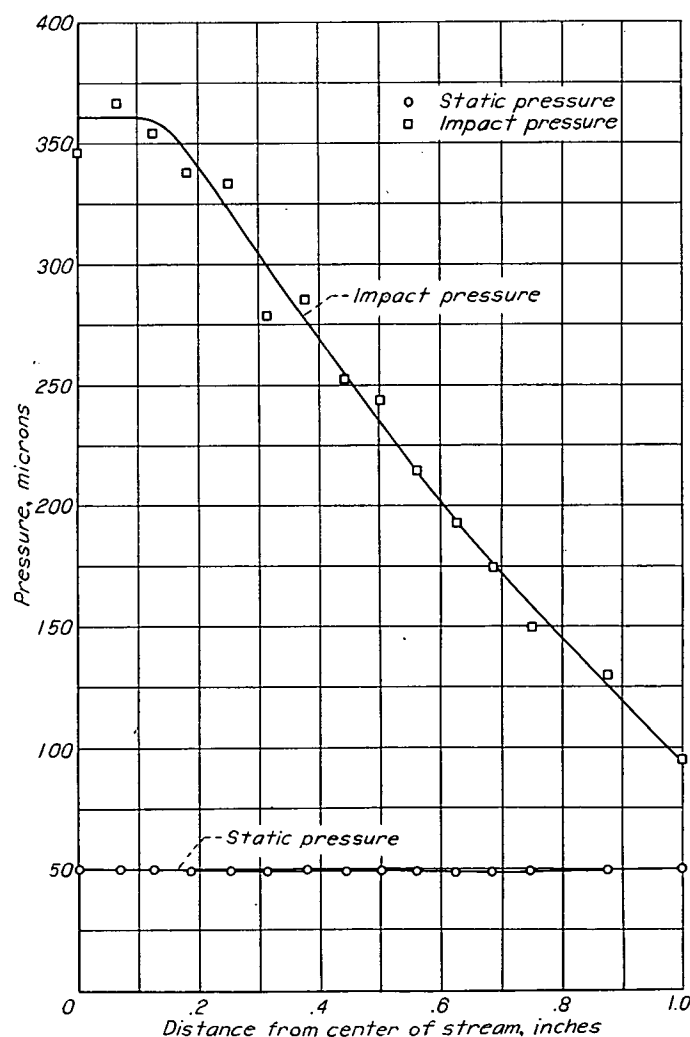


FIGURE 12.—Variation of impact and static pressure with nozzle radius for nozzle number 1 using helium gas.

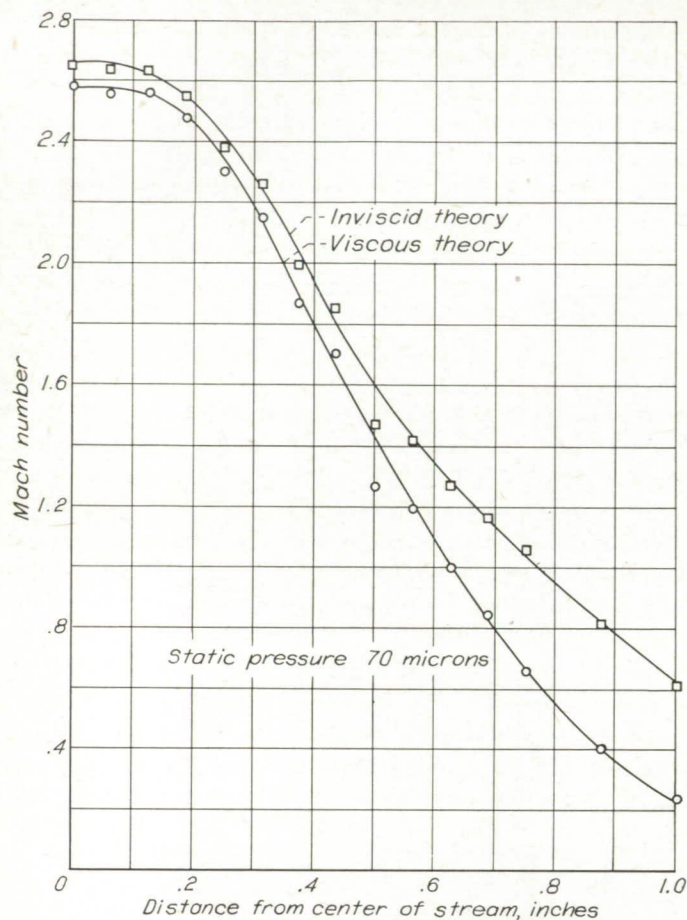


FIGURE 13.—Comparison of Mach number distribution in nozzle number 1 as calculated with and without viscous correction, test gas, nitrogen.

in all cases, higher than those computed using local values of impact and static pressure, thus indicating that the flow through the nozzle was nonisentropic due to viscous effects. Chambré's equation was used to compute Mach number from measured static and impact pressures for the tests described herein.

The variation of impact and static pressure across the stream resulted in a stream in which the Mach number was not constant. Figures 13 and 14 show this effect and are generally representative of all nozzles tested. If only the center $\frac{1}{2}$ inch of the nozzles is considered, the average variation in Mach number amounted to 5 percent; however, at certain pressure levels in some of the nozzles the variation was as much as 14 percent.

In order to determine if shock waves were present in the center $\frac{1}{2}$ inch of the gas stream, the nitrogen-glow-discharge technique of flow visualization was used to examine the flow qualitatively. This technique was originally described in reference 15. The method used to generate the glow discharge was to pass the test gas through a screen which was connected to a pulsed direct-current power supply. The screen was located at the entrance plane to the nozzle and was pulsed to a negative peak potential of 1000 volts. The pulsation frequency was 1000 cycles per second.

Figure 15 shows the gas stream from nozzle 1 at a static pressure of 114 microns. A 5° half-angle wedge was placed in the stream to generate a relatively weak shock wave which could be used as a yardstick to evaluate the strength of the nozzle shocks, if present. No shock waves were

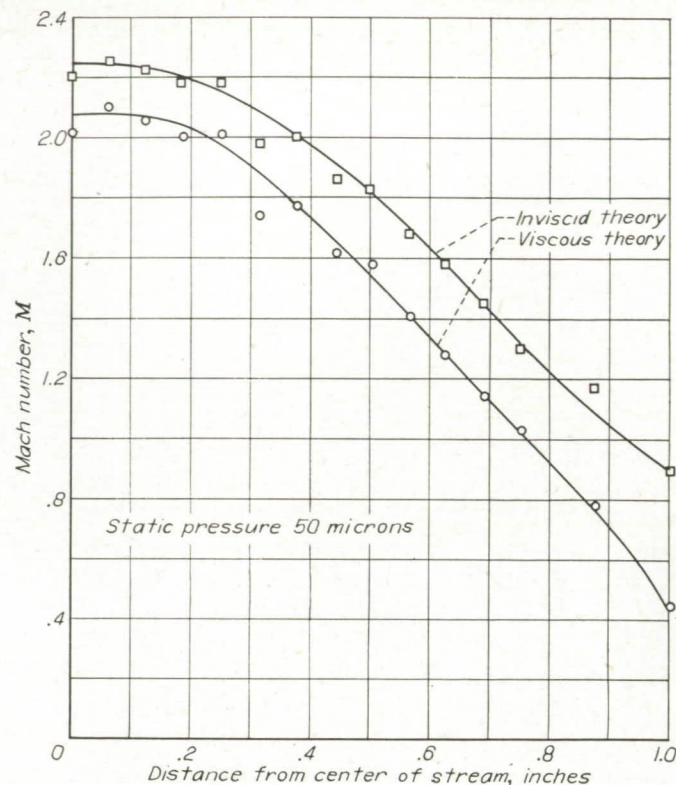


FIGURE 14.—Comparison of Mach number distribution in nozzle number 1 as calculated with and without viscous correction, test gas, helium.

observed in the gas stream at this pressure level if the wedge was removed. As the static-pressure level was lowered, the shock wave from the wedge became too indistinct to be photographed. Observation of the nitrogen glow did not indicate the presence of strong shock waves in the nozzles.

DRAW AND HEAT TRANSFER TESTS

The test procedure used in the drag and heat-transfer experiments was to calibrate the drag balance with the tunnel shut down and then to set tunnel conditions corresponding to those obtained during tunnel-calibration tests by matching the stagnation and test chamber static pressures. As extremely close control over tunnel conditions was difficult to attain, the pressure level during certain test runs

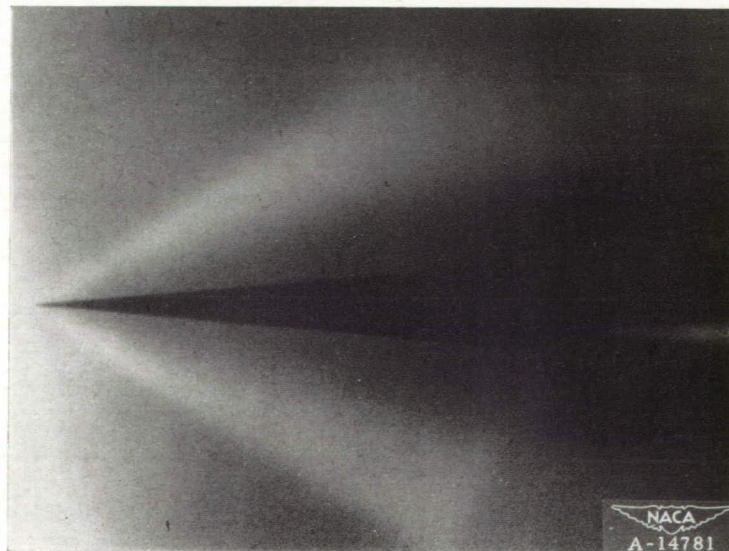


FIGURE 15.—Shock wave from 5° half-angle wedge at pressure of 114 microns as seen with nitrogen glow discharge.

varied approximately 5 percent from those obtained during the tunnel-calibration tests. Examination of the results of the tunnel-calibration tests showed that Mach number variation is small for small changes in the pressure level; therefore, the Mach number obtained during nozzle-calibration tests was applied to the heat-transfer and drag tests without correction. The Mach number used in plotting the data (which was converted numerically to molecular-speed ratio by means of the relation $s = M \sqrt{\gamma/2}$) was the average value obtained from a graphical integration of the Mach number distribution across the center $1\frac{1}{32}$ inch of the stream.

After stable operating conditions had been attained, the drag force, model temperatures, and other pertinent data were recorded. Table II lists the recorded data. It will be noted that, in the column listing the recorded drag force, two values occur on some of the runs. The different values were due to separate drag measurements during the course of one run or resulted from small shifts in the calibration of the drag balance as determined before and after each run. In the figures which are later presented, showing drag coefficients, the points correspond to those calculated using the average of the two values listed in table II.

To determine if the model surface conditions were changing with time, the model drag and temperature were measured at $\frac{1}{2}$ -hour intervals over a period of 6 hours for one particular test condition. Figures 16 and 17 show the results of these tests and it can be seen that the drag coefficient and temperature ratio remained essentially constant, thus indicating that the accommodation coefficient and tunnel conditions were constant over this period of time.

The reproducibility of the data was checked by repeating the above-described run on separate days. The reproducibility was found to be within the accuracy of the drag and temperature measurements described above.

RESULTS AND DISCUSSION

A DISCUSSION OF THE ASSUMPTIONS AND RESULTS OF THE THEORETICAL ANALYSIS

An examination of the heat-transfer equations (14) and (15) for a monatomic and a diatomic gas, respectively, reveals several interesting features. If an adiabatic case is considered, that is, if both internal-energy input to the cylinder and radiant-energy exchange are considered absent, it can be seen that the ratio of cylinder temperature to free-stream static temperature is a unique function of the molecular-

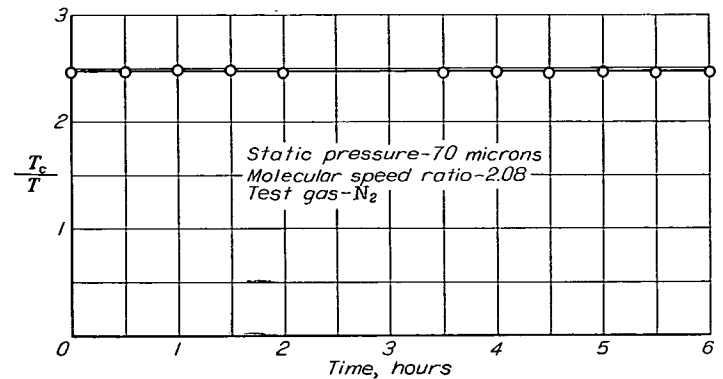


FIGURE 17.—Variation of the ratio of model to free-stream temperature, T_c/T with time.

speed ratio. The values of the temperature ratio computed from equations (14) and (15) are shown in figure 18. For comparison, curves of the temperature ratio for continuum flow are also shown. These latter curves were computed for the case where the total stream energy is recovered—in other words, the temperature recovery factor was considered to be unity. Several remarkable conclusions can be drawn from an examination of this figure. It can be seen that an insulated cylinder in a free-molecule-flow field will attain a higher temperature at a given value of molecular-speed ratio (or Mach number) than will be obtained in a continuum flow field. Since the continuum curves also represent the temperature ratio of a gas that has been accelerated adiabatically from rest to any value of molecular speed ratio, it may be inferred that an insulated cylinder located in a low-density wind tunnel under free-molecule-flow conditions will attain a temperature which exceeds the total temperature of the gas stream. This result, of course, is in direct contrast to the corresponding phenomenon which occurs under con-

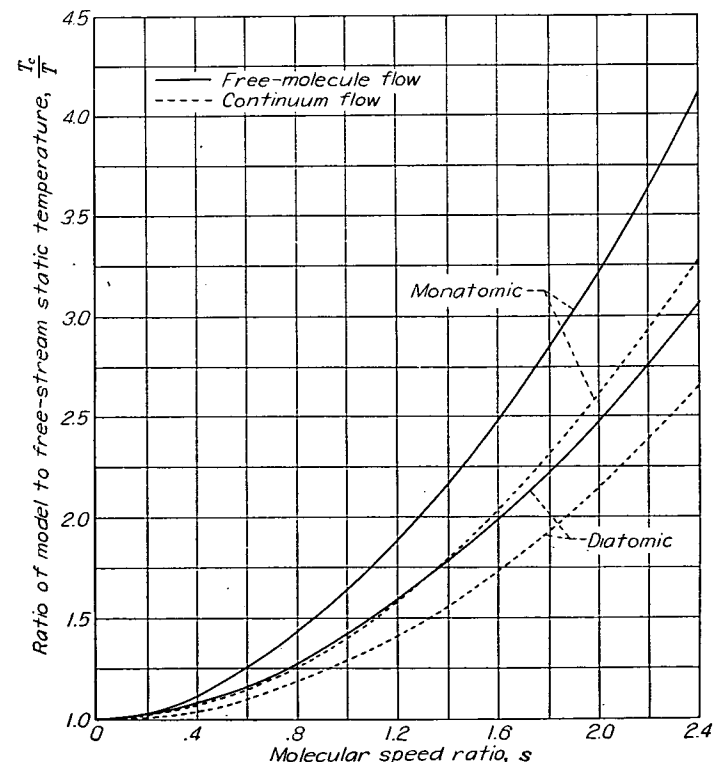


FIGURE 18.—Variation of the ratio of cylinder temperature to free-stream static temperature with molecular speed ratio for an adiabatic cylinder in free-molecule flow.

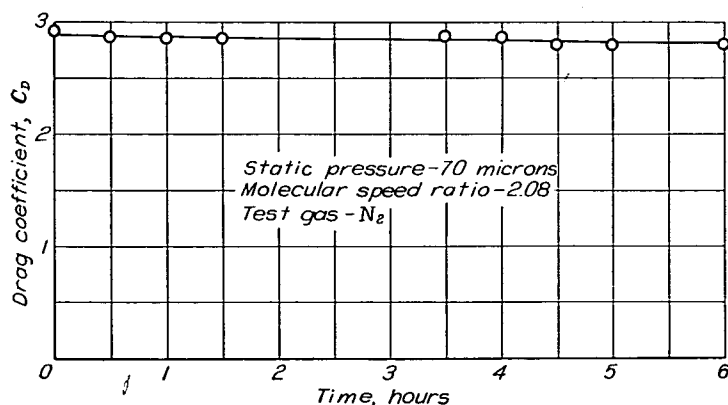


FIGURE 16.—Variation of measured drag coefficient with time.

tinuum-flow conditions where an insulated body can have, at most, a temperature equal to the stream stagnation temperature and normally does not even attain this temperature due to outward heat flow in the boundary layer. The anomaly can be explained, however, by a consideration, in the case of free-molecule flow, of the magnitudes of the incident and re-emitted molecular energy. The incident molecular energy is computed using the total velocity resulting from combination of the stream mass velocity and the random thermal velocities. When the total velocity term is squared there results a term, which is twice the scalar product of the vector mass velocity and the vector thermal velocity. This scalar product term effectively increases the apparent internal energy of the gas from the continuum value of $3/2 kT$ per molecule to a value varying from $2 kT$ to $5/2 kT$ depending upon the speed and orientation of the body. The apparent internal energy becomes $5/2 kT$ for high speeds and body angles of attack greater than zero. This is just equal to the internal plus potential energy per molecule of a cube of continuum gas. Therefore, the energy incident on the body in free-molecule flow becomes equal to that of a continuum for large values of the molecular speed ratio for surfaces inclined at angles of attack greater than zero. The molecular energy which is re-emitted from the surface is assumed to be equal to the energy of a stream issuing effusively from a Maxwellian gas, in equilibrium at some yet unspecified temperature, into a perfect vacuum. For the case of an insulated body, the temperature of this gas is that of the body. The energy of the re-emitted stream calculated in this manner is equal to $2 kT_w$ per molecule. The corresponding energy for a continuum gas at the same temperature is $5/2 kT_w$ per molecule if both thermal and potential energies are considered. For a given re-emitted stream temperature, a smaller amount of energy per molecule is transported from a body for the case of free-molecule flow than is transported in the case of continuum flow. It is clear that, if the same total amount of energy, namely the incident energy, is to be removed in each case, the effusive stream temperature and, hence, the body temperature for an insulated body must be higher for free-molecule flow than for continuum flow.

It is seen from figure 18, that, theoretically, a higher cylinder temperature is predicted for the case of a monatomic gas than for a diatomic gas. The explanation lies in the additional energy which the diatomic gas is able to remove from the cylinder by virtue of the internal energy component due to molecular rotation. Of course, the impinging diatomic gas transmits more energy to the cylinder due to the rotational energy component; however, the rotational component is a smaller fraction of the total impinging energy than of the total re-emitted energy.

An examination of the equations for the drag coefficient of an insulated cylinder (equations (17) and (18)) shows that the total drag coefficient, $C_D = C_{D_i} + C_{D_r}$, is a function only of the molecular-speed ratio. A plot of the total drag coefficient as a function of the molecular-speed ratio for an insulated cylinder is shown in figure 19. It is notable that the drag coefficient approaches infinity as the molecular-speed ratio approaches zero. Hence, if the expression for total drag is written in the conventional drag-coefficient

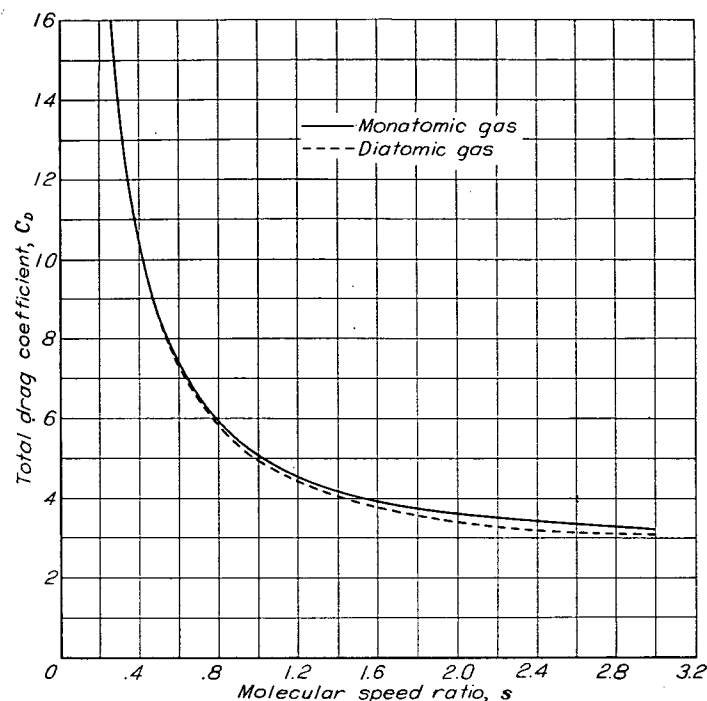


FIGURE 19.—The total drag coefficient for an insulated transverse cylinder in a free-molecule flow as a function of the molecular speed ratio, s .

form, it can be shown from equations (17) and (18) that C_D becomes inversely proportional to the speed ratio as s approaches zero. It is also notable that the drag coefficient approaches a constant value as the speed ratio becomes large. This means, simply, that the effect of the random thermal motions becomes negligible as the speed of the cylinder relative to the gas becomes high.

HEAT-TRANSFER TEST RESULTS

The salient point of the analysis concerning heat transfer to a cylinder in a free-molecule field was the prediction that an insulated model would attain a temperature in excess of the tunnel stagnation temperature. This result was observed in all runs. The maximum rise of the model center-point temperature above the tunnel stagnation temperature was 65° F , using nitrogen as the test gas, and 147° F , using helium as the test gas.

The results of the heat-transfer tests are shown in figures 20 and 21 in which the ratio of the cylinder-center temperature to the free-stream static temperature is plotted as a function of the molecular-speed ratio for the nitrogen and helium runs, respectively. The stream-static temperature was computed from the adiabatic relationship, using the measured value of Mach number (or molecular-speed ratio) corresponding to the nozzle pressure ratio as determined from the nozzle-calibration tests. The temperature-ratio curve for continuum flow assuming a temperature-recovery factor of unity is shown for comparison. It will be noted that in all cases the temperature ratio exceeded the maximum which could be attained under continuum-flow conditions. It can be seen that, in general, the test points indicate that the cylinder did not attain the full temperature rise that was predicted by the free-molecule analysis. This result may be ascribed to heat losses occurring from radiation and from conduction along the wire to the end supports.

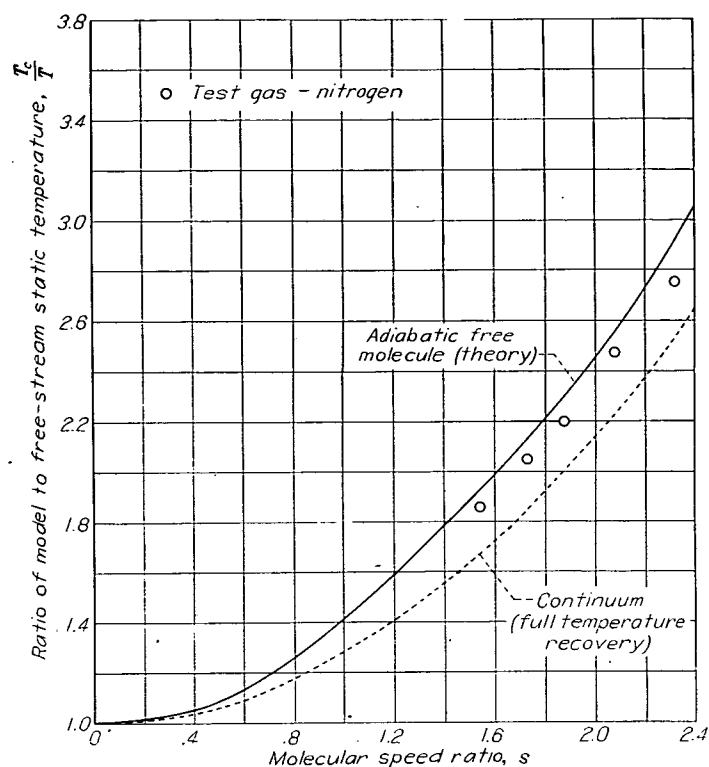


FIGURE 20.—Variation of the ratio of model temperature to free-stream static temperature with molecular speed ratio for a diatomic gas.

The fact that a large temperature gradient existed along the length of the wire due to end losses is clearly shown by the cylinder-end temperatures which are listed, together with the cylinder center-point temperature, in table II.

DRAG TEST RESULTS

The measured drag force on the cylinder was reduced to conventional dimensionless drag-coefficient form by means of the equation

$$C_D = \frac{D}{\gamma L r p M^2} = \frac{D}{2 L r p s^2} \quad (20)$$

The data points obtained from equation (20) are shown in figure 22. For comparison, the theoretical curves of drag coefficient as a function of molecular-speed ratio for an insulated cylinder, taken from figure 19, are shown as a solid line. It can be seen that agreement between theory and experiment is good considering the limitations of the theory and the difficulties attendant with the measurement of drag forces from 1 to 20 milligrams. The experimental values for the drag coefficient tend to fall below the theoretical values at molecular-speed ratios above about 1.0 and to be higher than the theoretical values at speed ratios below 1.0. The former characteristic may be due to a tendency toward specular reflection at the higher speed ratios while the latter characteristic, at subsonic values of speed ratio, may have been caused by interference with the flow caused by the shields at the ends of the model.

The major premise of the existence of free-molecule flow was borne out by several factors in addition to the good agreement between the calculated and experimental values

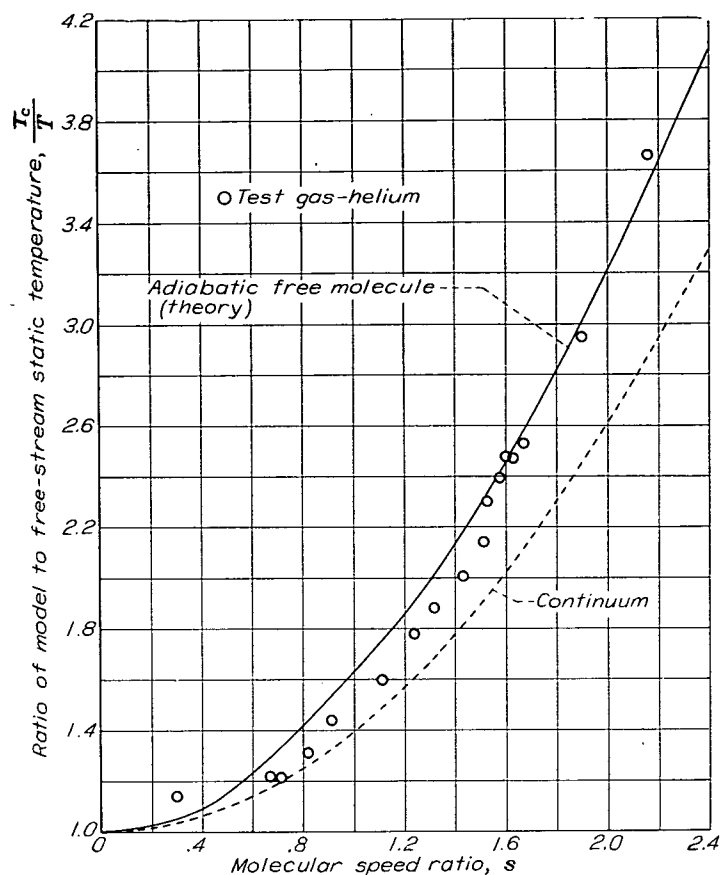


FIGURE 21.—Variation of the ratio of model temperature to free-stream static temperature with molecular speed ratio for a monatomic gas.

of drag coefficient. One of the basic assumptions of the drag analysis is that the drag coefficients are independent of Reynolds number or Knudsen number. In contrast, the continuum- and slip-flow regimes are characterized by the dependence of drag coefficients both upon Reynolds number and Mach number (cf., reference 16). Although it was not

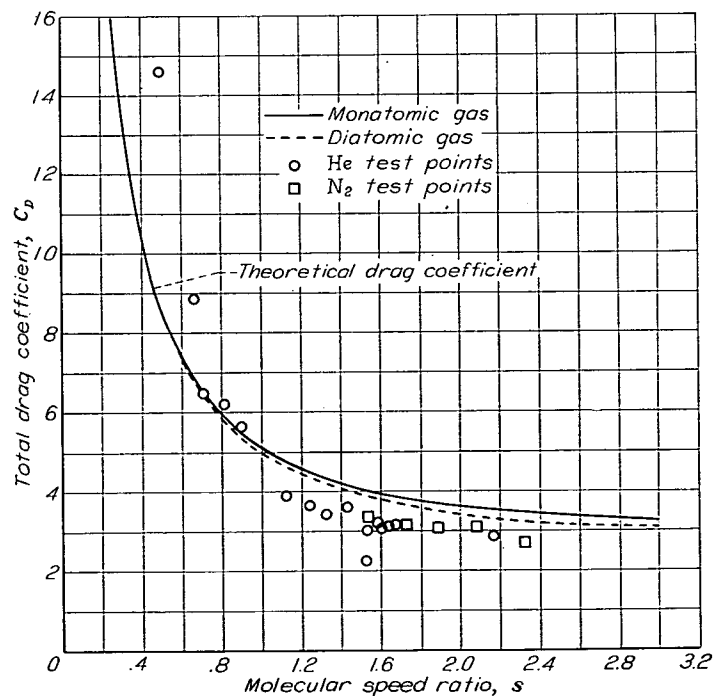


FIGURE 22.—Variation of total drag coefficient with molecular speed ratio.

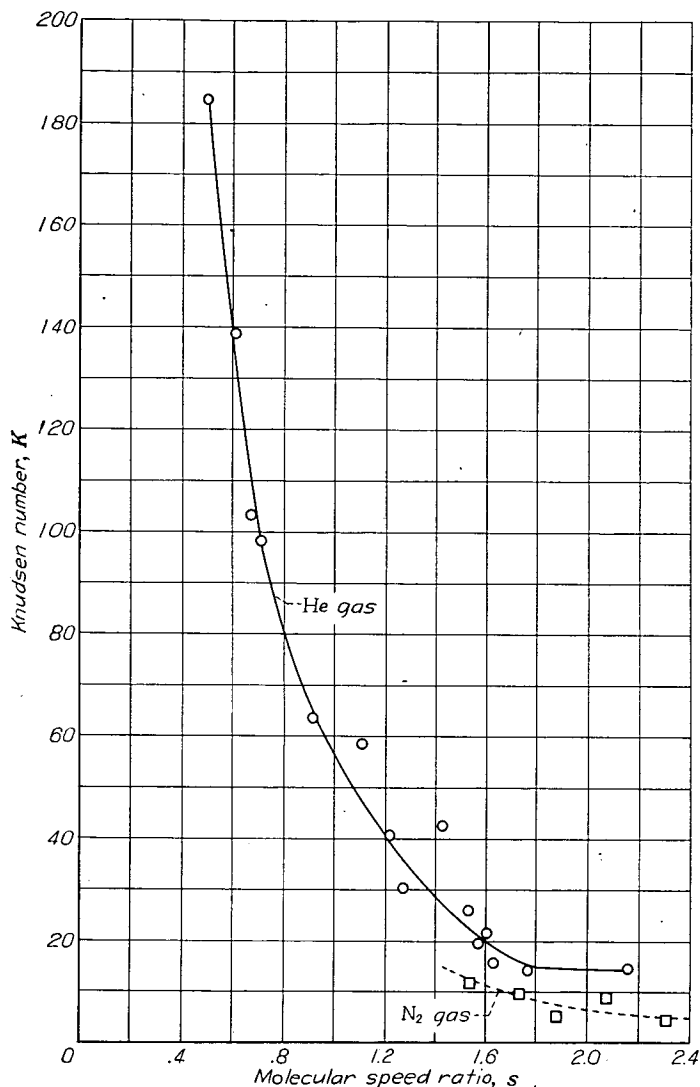


FIGURE 23.—The range of Knudsen numbers and molecular speed ratios covered in the tests. Knudsen number based upon model radius.

experimentally possible to vary the Knudsen number independently of the speed ratio for one test gas, the use of both nitrogen and helium gave overlapping values of speed ratio with an approximate 2:1 variation of Knudsen number. The variation of Knudsen number covered by the tests with molecular-speed ratio is shown in figure 23. No trend is noticeable in the drag coefficients obtained in each case; consequently, it can be concluded that free-molecule flow existed.

A comparison between measured drag coefficient and the drag coefficient calculated by methods as given by Lamb (reference 17) illustrates the inapplicability of continuum theory in the free-molecule regime. A sample calculation made by Lamb's continuum method for conditions of molecular-speed ratio equal to 0.5 and a Knudsen number of 185 (based on cylinder radius) gave a drag coefficient of 390. The corresponding experimental value was 14.6.

CONCLUSIONS

The following conclusions may be drawn from a comparison of the test results with free-molecule-flow theory shown in this report:

1. The salient point of the heat-transfer analysis, which was the prediction that an insulated cylinder would attain a temperature higher than the stagnation temperature of the stream, was confirmed by the test results.

2. Drag coefficients, calculated from free-molecule-flow theory, agreed well with the measured drag coefficients. As predicted theoretically, the measured drag coefficients were independent of the Reynolds and Knudsen numbers and depended primarily upon the molecular-speed ratio.

AMES AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., Sept. 12, 1950.

APPENDIX A

THE MOMENTUM AND ENERGY EXCHANGE PROCESSES ACCOMPANYING MOLECULAR EMISSION FROM A SURFACE

For the case of diffuse scattering of a stream of molecules from a surface, all directional history of the impinging stream is erased upon impact with the surface. The impinging stream is assumed to be scattered in a random manner which will be described in detail later. The energy and momentum carried by the scattered stream depends upon the velocity of emission which can be related to the energy (or temperature) level that the stream has. The effective temperature or energy of the scattered stream depends upon the efficiency of the energy-exchange process that occurs between the solid surface and the impinging stream. This efficiency can be described by introduction of the accommodation coefficient which is defined as

$$\alpha = \frac{E_i - E_r}{E_i - E_w} \quad (\text{A1})$$

where the symbols have the following definitions:

E_i rate of incident molecular energy

E_r rate of re-emitted molecular energy

E_w rate of re-emitted molecular energy that would be carried by the scattered stream if it were a stream issuing from a gas in equilibrium at the surface temperature, T_w

It is assumed that the emitted stream has a Maxwellian distribution of speed corresponding to a gas in equilibrium at an unspecified temperature, T_r . This fictitious gas has, also, an unspecified number of molecules per unit volume, N_r . Now, as shown in reference 8, page 62, the number of molecules scattered from unit surface area per second that have speeds in the range dv and that are moving in a direction that makes an angle lying in the range $d\Omega$ with the normal to the surface is given by

$$dn_r = 2\pi N_r A_r v^3 e^{-\beta_r v^2} \sin \Omega \cos \Omega dv d\Omega \quad (\text{A2})$$

Then, the normal component of force imparted to the surface by this group of molecules is

$$dG_r = 2\pi N_r A_r m v^4 e^{-\beta_r v^2} \sin \Omega \cos^2 \Omega dv d\Omega dA \quad (\text{A3})$$

and the translational energy carried by this group is

$$dE_r = \pi N_r A_r m v^5 e^{-\beta_r v^2} \sin \Omega \cos \Omega dv d\Omega dA \quad (\text{A4})$$

By integration of these expressions with respect to Ω and v , the following values are obtained for the total momentum and energy carried by the scattered stream from dA :

$$dG_r = 2\pi N_r A_r m dA \int_0^\infty v^4 e^{-\beta_r v^2} dv \int_0^{\pi/2} \sin \Omega \cos^2 \Omega d\Omega = \frac{N_r m}{4\beta_r^2} dA \quad (\text{A5})$$

$$dE_r = \pi N_r A_r m dA \int_0^\infty v^5 e^{-\beta_r v^2} dv \int_0^{\pi/2} \sin \Omega \cos \Omega d\Omega = \frac{N_r m}{2\sqrt{\pi}\beta_r^3} dA \quad (\text{A6})$$

In order to obtain a value for N_r in terms of known variables, the number of molecules scattered from the surface is equated to the number of molecules striking the surface. The number of molecules per second scattered from unit area is

$$n_r = 2\pi N_r A_r \int_0^\infty v^3 e^{-\beta_r v^2} dv \int_0^{\pi/2} \sin \Omega \cos \Omega d\Omega = \frac{N_r}{2\sqrt{\pi}\beta_r} \quad (\text{A7})$$

and the number of molecules striking unit area per second is shown in reference 4 to be given by

$$n = \frac{N\chi}{2\sqrt{\pi}\beta} \quad (\text{A8})$$

The value of N_r is obtained by equating n_r and n from equations (A7) and (A8), thus,

$$N_r = N\chi \frac{\beta_r}{\beta} \quad (\text{A9})$$

By substitution of this value of N_r in the momentum and energy equations, the following is obtained:

$$dG_r = \frac{Nm\chi}{4\beta\beta_r} dA \quad (\text{A10})$$

$$dE_r = \frac{Nm\chi}{2\sqrt{\pi}\beta\beta_r^2} dA \quad (\text{A11})$$

An identical analysis yields a value for E_w of

$$dE_w = \frac{Nm\chi dA}{2\sqrt{\pi}\beta\beta_w^2} = 2nkT_w dA \quad (\text{A12})$$

For the case of a diatomic gas where the expression for the re-emission energy per molecule contains an additional term having the value kT_w , which accounts for the rotational energy per molecule, the total re-emission energy has the value $3nkT_w dA$.

Equation (A10) describes the normal re-emission pressure on the front side of a unit inclined area. In order to obtain the net drag force (or pressure) in the direction of the mass velocity U , the pressure on the rear side of the inclined area must be taken into account. The computation of the pressure on the rear side of the area is made in a similar manner to the computation of the pressure on the front side of the area and leads to an expression

$$dG_r' = \frac{Nm\chi'}{4\beta\beta_r} dA \quad (\text{A13})$$

Then, the net drag force due to re-emission pressure in the direction of U , the mass velocity, is

$$dG_r - dG_r' = \frac{Nm \sin \theta}{4\beta\beta_r} (\chi - \chi') dA \quad (\text{A14})$$

Replacing χ and χ' by their values, equation (A14) becomes

$$dG_r - dG_r' = \frac{\sqrt{\pi} N m s \sin^2 \theta}{2\beta\beta_r} dA \quad (\text{A15})$$

This expression can be put into drag-coefficient form by use of the following substitutions: $Nm = \rho$, $s/\beta = U$. Equation (A15) then becomes

$$dG_r - dG_r' = \left(\frac{\rho U^2}{2}\right) \frac{\sqrt{\pi} \sin^2 \theta}{s_r} dA \quad (\text{A16})$$

where s_r is the molecular-speed ratio of the re-emitted stream. Then the re-emission drag coefficient, based on the total area of one side of the plane area, is

$$C_{D_r} = \frac{\sqrt{\pi} \sin^2 \theta}{s_r} \quad (\text{A17})$$

In order to obtain an expression for s_r which contains measurable or calculable physical quantities it is necessary to write an equation expressing an energy balance on the body in question. For the case of an insulated cylinder, if radiant energy exchange is neglected, only molecular energies are involved and the incident molecular energy can be equated to the re-emitted molecular energy. To write an energy balance equation for a transverse cylinder, it is assumed that the energy balance is maintained for the cylinder as a whole and not for individual elements of the cylinder. This means, essentially, that the cylinder has a uniform temperature, circumferentially. The total incident molecular energy on the cylinder is, for a monatomic gas,

$$E_i = \frac{2rL\rho}{4\sqrt{\pi}\beta^3} \left[\int_0^{\pi/2} \chi(s^2 + \psi) d\theta + \int_0^{\pi/2} \chi'(s^2 + \psi') d\theta \right] \quad (\text{A18})$$

The total molecular energy emitted from the cylinder surface is

$$E_r = \frac{2rL\rho}{2\sqrt{\pi}\beta\beta_r^2} \left(\int_0^{\pi/2} \chi d\theta + \int_0^{\pi/2} \chi' d\theta \right) \quad (\text{A19})$$

The values of the integrals are shown in appendix B to be

$$\begin{aligned} \int_0^{\pi/2} \chi d\theta &= \frac{Z_1 + Z_2}{2} + \sqrt{\pi} s \\ \int_0^{\pi/2} \chi' d\theta &= \frac{Z_1 + Z_2}{2} - \sqrt{\pi} s \\ \int_0^{\pi/2} \chi\psi d\theta &= Z_1 + \frac{5}{4} Z_2 + \frac{5}{2} \sqrt{\pi} s \\ \int_0^{\pi/2} \chi'\psi' d\theta &= Z_1 + \frac{5}{4} Z_2 - \frac{5}{2} \sqrt{\pi} s \end{aligned}$$

When E_i and E_r are equated from equations (A18) and (A19) and the values of the integrals are substituted, the following result is obtained:

$$\frac{1}{4\beta^3} \left[s^2(Z_1 + Z_2) + \frac{4Z_1 + 5Z_2}{2} \right] = \frac{1}{2\beta_r^2\beta} (Z_1 + Z_2) \quad (\text{A20})$$

Therefore,

$$\beta_r = \beta \sqrt{\frac{2(Z_1 + Z_2)}{Z_1(s^2 + 2) + Z_2\left(s^2 + \frac{5}{2}\right)}} \quad (\text{A21})$$

A similar analysis for the case of a diatomic gas where the incident molecular energy is

$$E_i = \frac{rL\rho}{2\sqrt{\pi}\beta^3} \left[(s^2 + 1) \int_0^{\pi/2} \chi d\theta + (s^2 + 1) \int_0^{\pi/2} \chi' d\theta + \int_0^{\pi/2} \chi\psi d\theta + \int_0^{\pi/2} \chi'\psi' d\theta \right] \quad (\text{A22})$$

and the re-emission energy is

$$E_r = \frac{3rL\rho}{2\sqrt{\pi}\beta\beta_r^2} \left(\int_0^{\pi/2} \chi d\theta + \int_0^{\pi/2} \chi' d\theta \right) \quad (\text{A23})$$

gives a value of β_r of

$$\beta_r = \beta \sqrt{\frac{3(Z_1 + Z_2)}{Z_1(s^2 + 3) + Z_2\left(s^2 + \frac{7}{2}\right)}} \quad (\text{A24})$$

Now, the total re-emission drag force acting on dA has been shown (cf., equation (A16)) to be

$$dG_r - dG_r' = \left(\frac{\rho U^2}{2}\right) \frac{\sqrt{\pi} \sin^2 \theta}{s_r} dA$$

To obtain the total re-emission drag on a cylinder, equation (A16) must be integrated over the cylinder surface, thus,

$$G_r - G_r' = \left(\frac{\rho U^2}{2}\right) (2rL) \frac{\sqrt{\pi}}{s_r} \int_0^{\pi/2} \sin^2 \theta d\theta = \left(\frac{\rho U^2}{2}\right) (2rL) \frac{\pi^{3/2}}{4s_r} \quad (\text{A25})$$

Then, the re-emission drag coefficient based on the projected cylinder area is

$$C_{D_r} = \frac{\pi^{3/2}}{4s_r} \quad (\text{A26})$$

The final equation for the re-emission drag coefficient for an insulated cylinder in a monatomic gas stream is obtained by using the relation $s_r = U\beta_r$ and taking β_r from equations (A21) and (A24).

$$C_{D_r} = \frac{\pi^{3/2}}{4s} \sqrt{\frac{Z_1(s^2 + 2) + Z_2\left(s^2 + \frac{5}{2}\right)}{2(Z_1 + Z_2)}} \quad (\text{A27})$$

and, for an insulated cylinder in a diatomic gas stream, as

$$C_{D_r} = \frac{\pi^{3/2}}{4s} \sqrt{\frac{Z_1(s^2 + 3) + Z_2\left(s^2 + \frac{7}{2}\right)}{3(Z_1 + Z_2)}} \quad (\text{A28})$$

APPENDIX B

HEAT TRANSFER TO A TRANSVERSE CIRCULAR CYLINDER IN FREE-MOLECULE FLOW

It has been shown (see Analysis) that an energy balance on a cylinder located in a stream of monatomic gas flowing with a velocity U perpendicular to the cylinder axis is expressed by

$$2\alpha r L \left[\int_0^{\pi/2} n \left(\frac{mU^2}{2} + \psi k T \right) d\theta + \int_0^{\pi/2} n' \left(\frac{mU^2}{2} + \psi' k T \right) d\theta \right] - 4\alpha r L k T_c \left(\int_0^{\pi/2} n d\theta + \int_0^{\pi/2} n' d\theta \right) + 2\pi r L [Q - \epsilon \sigma (T_c^4 - T_i^4)] = 0 \quad (B1)$$

Equation (B1) can be simplified by dividing through by $2\pi r L$ and replacing n and n' by their values from equations (2) and (4), thus,

$$\frac{N v_m \alpha}{4\pi^{3/2}} (mU^2 - 4kT_c) \left(\int_0^{\pi/2} \chi d\theta + \int_0^{\pi/2} \chi' d\theta \right) + \frac{\alpha k T N v_m}{2\pi^{3/2}} \left(\int_0^{\pi/2} \chi \psi d\theta + \int_0^{\pi/2} \chi' \psi' d\theta \right) - \epsilon \sigma (T_c^4 - T_i^4) + Q = 0 \quad (B2)$$

The integrals to be evaluated are

$$\int_0^{\pi/2} \chi d\theta \quad (B3)$$

$$\int_0^{\pi/2} \chi' d\theta \quad (B4)$$

$$\int_0^{\pi/2} \chi \psi d\theta \quad (B5)$$

$$\int_0^{\pi/2} \chi' \psi' d\theta \quad (B6)$$

If the value of χ in equation (B3) is replaced by its definition (from list of symbols), equation (B3) becomes

$$\int_0^{\pi/2} [e^{-s^2 \sin^2 \theta} + \sqrt{\pi} s \sin \theta + \sqrt{\pi} s \sin \theta \operatorname{erf}(s \sin \theta)] d\theta \quad (B7)$$

The first term of this integral can be evaluated by setting

$$s = 2x \text{ and } \cos 2\theta = z$$

Then, $d\theta = -\frac{dz}{2 \sin 2\theta}$ and the limits become

$$z = 1 \text{ when } \theta = 0$$

$$z = -1 \text{ when } \theta = \frac{\pi}{2}$$

Thus we can write

$$\int_0^{\pi/2} e^{-s^2 \sin^2 \theta} d\theta = \frac{e^{-z}}{2} \int_{-1}^1 \frac{e^{xz}}{\sqrt{1-z^2}} dz \quad (B8)$$

Now, by definition (cf., equation (11), p. 46, reference 18), the modified Bessel function of order n can be written as

$$I_n(x) = \frac{1}{\sqrt{\pi} \Gamma[n + (1/2)]} \left(\frac{x}{2} \right)^n \int_{-1}^1 e^{\pm xz} (1-z^2)^{n-1/2} dz \quad (B9)$$

Then

$$I_0(x) = \frac{1}{\sqrt{\pi} \Gamma(1/2)} \int_{-1}^1 \frac{e^{xz}}{\sqrt{1-z^2}} dz$$

and since

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

the first term of integral (B3) becomes

$$\frac{\pi}{2} e^{-s^2/2} I_0\left(\frac{s^2}{2}\right) \quad (B10)$$

The second term (cf., equation (B7)) is

$$\sqrt{\pi} s \int_0^{\pi/2} \sin \theta d\theta = \sqrt{\pi} s \quad (B11)$$

The third term of equation (B7) is

$$\sqrt{\pi} s \int_0^{\pi/2} \sin \theta \operatorname{erf}(s \sin \theta) d\theta$$

which can be integrated by parts. Let

$$\operatorname{erf}(s \sin \theta) = u$$

and

$$\sin \theta d\theta = dv$$

Then

$$du = \frac{2s}{\sqrt{\pi}} e^{-s^2 \sin^2 \theta} \cos \theta d\theta$$

and

$$v = -\cos \theta$$

Then, the integral becomes

$$\sqrt{\pi} s \left[-\cos \theta \operatorname{erf}(s \sin \theta) \right]_0^{\pi/2} + 2s^2 \int_0^{\pi/2} \cos^2 \theta e^{-s^2 \sin^2 \theta} d\theta \quad (B12)$$

The bracketed term of equation (B12) is zero, leaving the second term yet to be evaluated. Make the substitutions

$$\cos^2 \theta = \frac{1 + \cos u}{2}$$

$$\sin^2 \theta = \frac{1 - \cos u}{2}$$

$$s^2 = 2x$$

Then

$$d\theta = \frac{du}{2}$$

and, replacing limits

$$\theta = 0 \quad u = 0$$

$$\theta = \frac{\pi}{2} \quad u = \pi$$

there results

$$2s^2 \int_0^{\pi/2} \cos^2 \theta e^{-s^2 \sin^2 \theta} d\theta = \frac{s^2}{2} \int_0^{\pi} (1 + \cos u) e^{-z(1 - \cos u)} du$$

$$= \frac{s^2 e^{-z}}{2} \int_0^{\pi} e^{+z \cos u} du + \frac{s^2 e^{-z}}{2} \int_0^{\pi} \cos u e^{z \cos u} du \quad (\text{B13})$$

Integral (B13) becomes, when substituting $z = \cos u$ and changing limits,

$$\frac{s^2 e^{-z}}{2} \int_{-1}^1 \frac{e^{zz}}{\sqrt{1-z^2}} dz + \frac{s^2 e^{-z}}{2} \int_{-1}^1 \frac{z e^{zz}}{\sqrt{1-z^2}} dz$$

$$= \frac{\pi s^2}{2} e^{-s^2/2} \left[I_0\left(\frac{s^2}{2}\right) + I_1\left(\frac{s^2}{2}\right) \right] \quad (\text{B14})$$

which is obtained by integrating the second term of (B14) by parts and substituting equation (B9). Then, the final value for integral (B3) is

$$\int_0^{\pi/2} \chi d\theta = \frac{\pi}{2} e^{-s^2/2} I_0\left(\frac{s^2}{2}\right) + \sqrt{\pi} s + \frac{\pi s^2}{2} e^{-s^2/2} \left[I_0\left(\frac{s^2}{2}\right) + I_1\left(\frac{s^2}{2}\right) \right] \quad (\text{B15})$$

Now, integrals (B4), (B5), and (B6) contain only terms which can be evaluated by the methods shown above by which integral (B3) was evaluated. The results are

$$\int_0^{\pi/2} \chi' d\theta = \frac{\pi}{2} e^{-s^2/2} I_0\left(\frac{s^2}{2}\right) - \sqrt{\pi} s + \frac{\pi s^2}{2} e^{-s^2/2} \left[I_0\left(\frac{s^2}{2}\right) + I_1\left(\frac{s^2}{2}\right) \right] \quad (\text{B16})$$

$$\int_0^{\pi/2} \chi \psi d\theta = \pi e^{-s^2/2} I_0\left(\frac{s^2}{2}\right) + \frac{5}{2} \sqrt{\pi} s + \frac{5}{4} \pi s^2 e^{-s^2/2} \left[I_0\left(\frac{s^2}{2}\right) + I_1\left(\frac{s^2}{2}\right) \right] \quad (\text{B17})$$

$$\int_0^{\pi/2} \chi' \psi' d\theta = \pi e^{-s^2/2} I_0\left(\frac{s^2}{2}\right) - \frac{5}{2} \sqrt{\pi} s + \frac{5}{4} \pi s^2 e^{-s^2/2} \left[I_0\left(\frac{s^2}{2}\right) + I_1\left(\frac{s^2}{2}\right) \right] \quad (\text{B18})$$

In order to eliminate an excessive amount of writing, new variables are defined as follows:

$$Z_1 = \pi e^{-s^2/2} I_0\left(\frac{s^2}{2}\right) \quad (\text{B19})$$

$$Z_2 = \pi s^2 e^{-s^2/2} \left[I_0\left(\frac{s^2}{2}\right) + I_1\left(\frac{s^2}{2}\right) \right] \quad (\text{B20})$$

Values of the dimensionless variables Z_1 and Z_2 were computed from the tables of Bessel functions in reference 19. Then, equation (B21) is obtained by substitution of equations (B15) to (B20) into equation (B2).

$$\frac{N v_m \alpha}{4 \pi^{3/2}} (m U^2 - 4 k T_c) (Z_1 + Z_2) + \frac{\alpha k T N v_m}{2 \pi^{3/2}} \left(2 Z_1 + \frac{5}{2} Z_2 \right) - \epsilon \sigma (T_c^4 - T_i^4) + Q = 0 \quad (\text{B21})$$

If $m v_m^2/2 = k T$, $k T N = p$, and $s = U/v_m$ are substituted into equation (B21), the following equation results:

$$2 \frac{T_c}{T} (Z_1 + Z_2) - \left[Z_1 (s^2 + 2) + Z_2 \left(s^2 + \frac{5}{2} \right) \right] + \frac{2 \pi^{3/2}}{p v_m \alpha} [\epsilon \sigma (T_c^4 - T_i^4) - Q] = 0 \quad (\text{B22})$$

which is the final energy exchange equation for a monatomic gas. An identical treatment of equation (13) for a diatomic gas gives

$$3 \frac{T_c}{T} (Z_1 + Z_2) - \left[Z_1 (s^2 + 3) + Z_2 \left(s^2 + \frac{7}{2} \right) \right] + \frac{2 \pi^{3/2}}{p v_m \alpha} [\epsilon \sigma (T_c^4 - T_i^4) - Q] = 0 \quad (\text{B23})$$

The groups $f(s) = Z_1 (s^2 + 3) + Z_2 (s^2 + \frac{7}{2})$, $g(s) = 3 (Z_1 + Z_2)$, $f^0(s) = Z_1 (s^2 + 2) + Z_2 (s^2 + \frac{5}{2})$, and $g^0(s) = 2 (Z_1 + Z_2)$ are tabulated in the following table:

s	$f(s)$	$g(s)$	$f^0(s)$	$g^0(s)$
0.2	9.80271	9.61234	6.59860	6.40823
.4	10.94784	10.16410	7.55981	6.77607
.6	12.89399	11.04924	9.21092	7.36616
.8	15.69746	12.22282	11.62319	8.14855
1.0	19.43609	13.63288	14.89180	9.08859
1.2	24.20781	15.22758	19.13197	10.15172
1.4	30.12863	16.96068	24.47507	11.30712
1.6	37.32923	18.79408	31.06454	12.52939
1.8	45.95230	20.69872	39.05273	13.79915
2.0	56.14932	22.65358	48.59807	15.10239
2.2	68.07771	24.64392	59.86307	16.42928
2.4	81.89947	26.65990	73.01284	17.77327
2.6	97.77002	28.69470	88.21412	19.12980
2.8	115.88287	30.74382	105.63493	20.49588
3.0	136.37821	32.80398	125.44355	21.86932
3.2	159.43361	34.87300	147.80928	23.24867
3.4	185.21769	36.94914	172.90131	24.63276
3.6	213.89927	39.03112	200.88890	26.02075
3.8	245.64801	41.11800	231.94201	27.41200
4.0	280.63340	43.20904	266.23039	28.80603
4.2	319.02474	45.30354	303.92365	30.20236
4.4	360.99204	47.40108	345.19168	31.60072
4.6	406.70528	49.50126	390.20486	33.00084
4.8	456.33364	51.60366	439.13242	34.40244
5.0	510.04756	53.70808	492.14487	35.80539
5.2	568.01730	55.81428	549.41254	37.20952
5.4	630.41182	57.92200	611.10449	38.61467
5.6	697.40229	60.03114	677.39191	40.02076

APPENDIX C

THE DRAG DUE TO IMPINGING MOMENTUM ON A TRANSVERSE CYLINDER IN FREE-MOLECULE FLOW

The differential momentum imparted, by the impact of impinging molecules, to the front and rear sides of a body located in a free-molecule flow field is (cf., references 1, 2, and 3)

$$dG_i = \frac{\beta^3 m N}{\pi^{3/2}} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^{\infty} (l_x c_x + l_y c_y + l_z c_z) c_x e^{-\beta^2[(c_x - U_x)^2 + (c_y - U_y)^2 + (c_z - U_z)^2]} dc_x dc_y dc_z - \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_0^0 (l_x c_x + l_y c_y + l_z c_z) c_x e^{-\beta^2[(c_x - U_x)^2 + (c_y - U_y)^2 + (c_z - U_z)^2]} dc_x dc_y dc_z \right\} dA \quad (C1)$$

In equation (C1), l_x, l_y, l_z are the cosines of the angle between the coordinate axes and the direction of the momentum force. Now let θ be the angle of attack of the element, that is, the angle between the mass velocity U and the plane of the plate. Then the drag force is in the direction of U and the following relations can be derived:

$$\left. \begin{aligned} U_x &= U \sin \theta & l_x &= \sin \theta \\ U_y &= -U \cos \theta & l_y &= -\cos \theta \\ U_z &= 0 & l_z &= 0 \end{aligned} \right\} \quad (C2)$$

Then, integration of equation (C1) and substitution of the relations of equation (C2) yield

$$dG_i = \frac{\rho U^2}{2} \left[\frac{2}{\pi^{1/2}} \frac{e^{-s^2 \sin^2 \theta}}{s} + 2 \sin \theta \operatorname{erf}(s \sin \theta) \left(1 + \frac{1}{2s^2} \right) \right] dA \quad (C3)$$

It is remarkable that a purely analytic solution for this type flow leads to a simple equation of the type of (C3) in which the bracketed term is a dimensionless function of molecular speed ratio and which may be interpreted as an ordinary drag coefficient.

In order to obtain the drag force on a transverse cylinder due to impinging momentum, equation (C3), which was derived for an elemental plane area, must be integrated over the surface of the cylinder. To this end, dA is written as

$$dA = r L d\theta \quad (C4)$$

Then, for the cylinder, the impinging drag force is

$$G_i = 2rL \left(\frac{\rho U^2}{2} \right) \int_0^{\pi/2} \left[\frac{2e^{-s^2 \sin^2 \theta}}{\pi^{1/2}s} + 2 \sin \theta \operatorname{erf}(s \sin \theta) \left(1 + \frac{1}{2s^2} \right) \right] d\theta \quad (C5)$$

This integral can be evaluated by the methods shown in appendix B. The final value for the drag force due to impinging molecules is

$$G_i = 2rL \left(\frac{\rho U^2}{2} \right) \frac{\pi^{1/2} e^{-s^2/2}}{s} \left\{ I_0 \left(\frac{s^2}{2} \right) + \left(\frac{1+2s^2}{2} \right) \left[I_0 \left(\frac{s^2}{2} \right) + I_1 \left(\frac{s^2}{2} \right) \right] \right\} \quad (C6)$$

Thus, there results from equation (C6) an impinging drag coefficient based on the projected area of the cylinder, which has the value

$$C_{Di} = \frac{\pi^{1/2} e^{-s^2/2}}{s} \left\{ I_0 \left(\frac{s^2}{2} \right) + \left(\frac{1+2s^2}{2} \right) \left[I_0 \left(\frac{s^2}{2} \right) + I_1 \left(\frac{s^2}{2} \right) \right] \right\} \quad (C7)$$

REFERENCES

1. Ashley, Holt: Application of the Theory of Free Molecule Flow to Aeronautics. Inst. Aero. Sci., Preprint 164, 1949.
2. Tsien, Hsue-Shen: Superaerodynamics, Mechanics of Rarefied Gases. Jour. Aero. Sci., vol. 13, no. 12, Dec. 1946, pp. 653-664.
3. Heineman, M.: Theory of Drag in Highly Rarefield Gases, Communications on Pure and Applied Mathematics. Vol. 1, no. 3, Sept. 1948, pp. 259-273.
4. Stalder, Jackson R., and Jukoff, David: Heat Transfer to Bodies Traveling at High Speed in the Upper Atmosphere. NACA Rep. 944, 1949. (Formerly NACA TN 1682)
5. Whipple, Fred L.: Meteors and the Earth's Upper Atmosphere. Review of Modern Physics, vol. 15-16, Oct. 1943, pp. 246-264.
6. Epstein, Paul S.: On the Resistance Experienced by Spheres in Their Motion Through Gases. Physical Review, vol. 23, no. 6, June 1924, pp. 710-733.
7. Loeb, Leonard B.: Kinetic Theory of Gases. McGraw-Hill Book Co., Inc., N. Y., 2d ed., 1934.
8. Kennard, Earle H.: Kinetic Theory of Gases, With an Introduction to Statistical Mechanics. McGraw-Hill Book Co., Inc., N. Y., 1st ed., 1938.
9. Cronvich, Lester L.: A Numerical-Graphical Method of Characteristics for Axially Symmetric Isentropic Flow. Jour. Aero. Sci., vol. 15, no. 3, Mar. 1948, pp. 155-162.
10. Dushman, Saul: Scientific Foundations of Vacuum Technique. John Wiley and Sons, N. Y., 1949.
11. Mann, W. R., and Schaaf, S. A.: Vacuum Gage Time Lags Due to Outgassing. University of California, Dept. of Engineering, Rep. No. HE-150-26, Sept. 1947.
12. Schaaf, S. A., and Cyr, R. R.: Time Constants for Vacuum Gage Systems. University of California, Dept. of Engineering, No. HE-150-42, Mar. 1948.

13. Jones, Robert T., and Margolis, Kenneth: Flow Over a Slender Body of Revolution at Supersonic Velocities. NACA TN 1081, 1946.
14. Chambré, P. L.: The Theory of the Impact Tube in a Viscous Compressible Gas. University of California, Dept. of Engineering, Rep. No. HE-150-50, Nov. 1948.
15. Williams, Thomas W., and Benson, James M.: Preliminary Investigation of the Use of Afterglow for Visualizing Low-Density Compressible Flows. NACA TN 1900, 1949.
16. Schamberg, Richard: The Fundamental Differential Equations and the Boundary Conditions for High-Speed Slip-Flow and Their Application to Several Specific Problems. Thesis, GALCIT, 1947. (Thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy.)
17. Lamb, Horace: Hydrodynamics. Dover Publications, N. Y., 6th ed., 1945.
18. Gray, Andrew, and Mathews, G. B.: A Treatise on Bessel Functions and Their Applications to Physics. MacMillan and Co., Ltd., London, Eng., 1931.
19. Watson, G. N.: A Treatise on the Theory of Bessel Functions. The University Press, Cambridge, Eng., 1944.

REPORT 1033

COMPARISON BETWEEN THEORY AND EXPERIMENT FOR WINGS AT SUPERSONIC SPEEDS¹

By WALTER G. VINCENTI

SUMMARY

In this paper, a critical comparison is made between experimental and theoretical results for the aerodynamic characteristics of wings at supersonic flight speeds. As a preliminary, a brief, nonmathematical review is given of the basic assumptions and general findings of supersonic wing theory in two and three dimensions. Published data from two-dimensional pressure-distribution tests are then used to illustrate the effects of fluid viscosity and to assess the accuracy of linear theory as compared with the more exact theories which are available in the two-dimensional case. Finally, an account is presented of an NACA study of the over-all force characteristics of three-dimensional wings at supersonic speed. In this study, the lift, pitching moment, and drag characteristics of several families of wings of varying plan form and section were measured in the wind tunnel and compared with values predicted by the three-dimensional linear theory. The regions of agreement and disagreement between experiment and theory are noted and discussed.

INTRODUCTION

The aerodynamics of wings at supersonic flight speeds is currently the subject of much research and discussion. As a result of many recent investigations, based on the earlier work of Prandtl, Ackeret, Busemann, and von Kármán, the theory of the subject is well advanced, both as applied to airfoil sections in two-dimensional flow and to complete, three-dimensional wings. Experimental knowledge is, by contrast, considerably less extensive, particularly with regard to the three-dimensional case. There are, however, sufficient experimental data in hand to permit a reasonably systematic comparison between theory and experiment. It is the purpose of this paper to present such a comparison insofar as the current availability of experimental results will allow.

THEORETICAL CONSIDERATIONS

To provide background for those who are unacquainted with the fundamentals of supersonic wing theory, it may be useful to review briefly the assumptions and findings of work in this field. (For a more complete discussion of the theory

and a bibliography of pertinent references, the reader is referred to the Tenth Wright Brothers Lecture by Theodore von Kármán, reference 1.)

In the solution of problems in supersonic wing theory, the following assumptions are usually made concerning the flow field which surrounds the wing:

- (a) The fluid medium is continuous and homogeneous.
- (b) The fluid has the thermodynamic characteristics of a perfect gas with constant specific heats.
- (c) Viscosity and thermal conductivity are vanishingly small.
- (d) External forces (such as gravity) are negligible.

For flight at ordinary altitudes and air temperatures, the most drastic of these assumptions is that of vanishingly small viscosity and thermal conductivity. This assumption allows the effects of fluid friction and heat transfer to be disregarded except as they are necessary to explain the existence of shock waves and vortices within the flow field. The assumption thus retains the essential features of supersonic flow as it is known to occur away from the immediate vicinity of the wing surface. It results, however, in the omission of the friction drag and of any changes in pressure distribution caused by growth or separation of the boundary layer.

On the basis of the foregoing assumptions, it is possible to obtain explicit relations for the sudden changes in flow which occur across a shock wave as well as a differential equation for the gradual changes which take place in the regions between such waves. When expressed with the geometrical coordinates as the independent variables, the differential equation governing the flow in the region between shock waves is nonlinear. It is therefore difficult to apply rigorously to most problems of practical interest.

Fortunately, in the special case of an airfoil section in a two-dimensional supersonic stream, results can be obtained with a high degree of mathematical rigor despite the nonlinearity of the governing differential equation. For reasons of mathematical practicality, it has been usual to restrict the solutions to instances in which the local velocity in the flow field is everywhere supersonic. This limits the solutions to airfoils with a sharp leading edge and to angles of attack and free-stream Mach numbers such that the shock wave from the leading edge is attached to the airfoil and the flow on

¹ Paper presented at the Second International Aeronautical Conference, Institute of the Aeronautical Sciences and The Royal Aeronautical Society, New York City, May 24-27, 1949. Supersedes NACA TN 2100, "Comparison Between Theory and Experiment for Wings at Supersonic Speeds" by Walter G. Vincenti, 1950.

the downstream side of the wave is supersonic. (It has also been customary to neglect the rotation of the fluid particles which will exist aft of the leading-edge wave in those cases in which the wave is curved, although this approximation is not essential.) Within these restrictions, section characteristics can be calculated to a high degree of precision for sections of even appreciable thickness. The method of computation reduces in practice to a stepwise application of the known relations for the compression through a shock wave and for the expansion around a convex corner. The procedure has therefore been termed the "shock-expansion" method (see, for example, reference 2). For rapid calculations, more restricted methods, such as Ackeret's linear theory (references 3 and 4) and Busemann's second-order theory (references 5, 6, and 7), can be obtained by means of series approximation to the complete equations for the shock wave and the expansion.

In the more practical case of a complete, three-dimensional wing, the general mathematical problem is forbiddingly complex, and it is necessary to simplify the nonlinear differential equation at the outset in order to obtain a solution. To accomplish this, it is assumed that the local velocity at all points in the flow field differs only slightly in magnitude and direction from the velocity of the undisturbed stream. This implies, in effect, that the thickness, camber, and angle of attack of the wing are small. With this approximation, the complete, nonlinear differential equation reduces, through the omission of terms of higher than the first order in the flow disturbances, to a linear equation which can be solved by established mathematical methods. On the basis of this equation, an extensive body of theory has been formulated covering a wide range of practical wings. For the present it will suffice to mention certain general concepts and results of this theory. Examples of specific calculations will be presented in the course of the later discussion.

A fundamental result of the linear theory, well known by now, is the concept of the Mach cone. According to this concept, the effect of a given disturbance in a uniform supersonic stream is felt only within the interior of a circular cone with vertex located at the point of the disturbance and axis extending downstream parallel to the original flow. The geometry of the cone is determined by the requirement that the component of free-stream velocity normal to the surface of the cone is equal to the speed of sound in the undisturbed stream. It follows that the semivertex angle of the cone is a function of the free-stream Mach number only. These considerations apply not only to the effects of an isolated disturbance but to the region of influence of each disturbance in a distributed system as well.

The concept of the Mach cone has immediate implications with regard to the aerodynamic problems of three-dimensional wings. This is illustrated in figure 1, which shows certain features of the flow over three flat lifting surfaces of representative plan form. In the case of the rectangular plan form A, for example, it follows from the concept of the Mach cone that, to a first approximation, the effects of the finite span are confined to the regions of the wing lying within the cone from the leading edge of each tip. The flow over

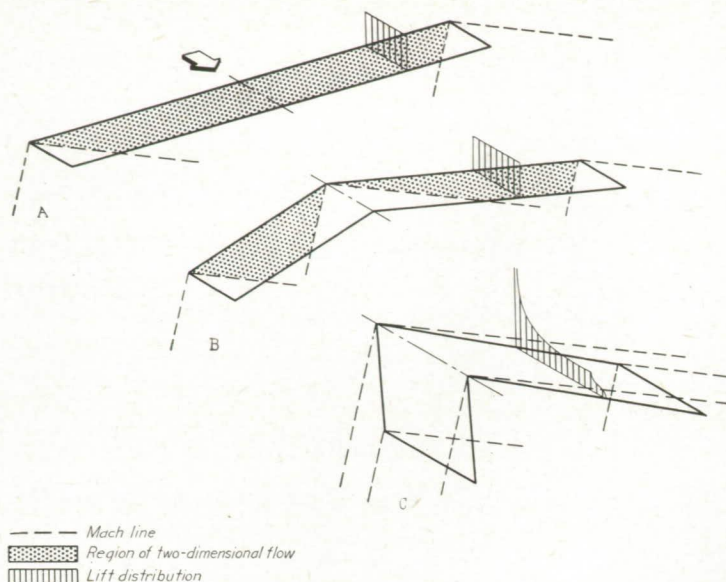


FIGURE 1.—Flat lifting surfaces in supersonic flow (linear theory).

the remainder of the wing (shown shaded) is identical with the two-dimensional flow over a wing of infinite span. On the moderately swept plan form B, the flow over the shaded regions is, by the same reasoning, unaffected by the presence of either the tips or root of the wing. Within these regions the flow can be treated as essentially two-dimensional by evaluating the velocity and the deflection angle in the direction normal to the leading edge. On the highly swept plan form C, all of the wing is within the fields of influence of the root and tips, and no regions of purely two-dimensional flow are to be expected.

Carrying these considerations a step farther, we may also examine the effect which the relationship between the plan form and the Mach cones has upon the chordwise lift distribution for the three wings. On both wings A and B, where the leading edge lies ahead of the Mach cones from the corners of the plan form, the Mach number of the component of free-stream velocity normal to the leading edge is greater than one. For reasons just examined, the lift distribution at the spanwise stations for which it is shown will be the same as the distribution over a flat lifting surface in a two-dimensional supersonic stream. Characteristic features of this distribution are that the intensity of lift at the leading edge is finite and has zero gradient in the chordwise direction. On plan form C, where the leading edge is swept behind the Mach cone, the Mach number of the flow component normal to the leading edge is less than one. It develops from the theory that in this case the lift distribution near the edge resembles the theoretical distribution predicted by linear theory for a flat lifting surface in a purely subsonic flow—that is, the lift intensity tends to an infinite value at the leading edge and drops off rapidly along the chord toward the trailing edge.

The foregoing differences in lift distribution provide one example of a general principle, the significance of which was first noted by R. T. Jones (reference 8). This principle, which arises throughout the study of wings by the linear theory, can be stated as follows: When the component of free-stream velocity normal to a wing element (i. e., lead-

ing edge, ridge line, or trailing edge) is greater than the speed of sound, the theoretical flow in the vicinity of the element has the essential character of the two-dimensional supersonic flow about an element of the same geometric type; similarly, when the velocity component normal to the element is less than the speed of sound, the theoretical local flow resembles that which prevails in the two-dimensional subsonic case. Because of the utility of this general result, it has become customary to describe the wing elements themselves as either "supersonic" or "subsonic." To determine which category an element occupies, it is obviously sufficient, as in figure 1, to note whether it is swept ahead of or behind the Mach cone. It is apparent that a wing element may change from one classification to the other as its orientation relative to the Mach cone is changed. This can be brought about by variation in either the free-stream Mach number or the geometry of the wing.

As a result of the inherent differences in the flow about supersonic and subsonic elements, theoretical calculations for three-dimensional wings indicate marked and interesting changes in the flight characteristics with changes in Mach number or wing geometry. By studying these effects, wing shapes can be found which afford optimum aerodynamic characteristics for a given flight condition. The results of such studies, indeed, provide a valuable guidance to the aircraft designer. In anticipation of the experimental results to be presented later, however, a word of caution is in order here. As exemplified in figure 1, the differences in theoretical pressure distribution between a supersonic and subsonic element may be characterized by large differences in chordwise pressure gradient. These differences may, in a real, viscous medium, give rise to corresponding differences in boundary-layer flow and hence to aerodynamic effects which are beyond the scope of the inviscid theory. As a result, the true variation of the wing characteristics with change in Mach number or wing geometry may be considerably different from that predicted by the theory. The later experimental results with regard to the drag of triangular wings supply an excellent example of such an effect.

In anticipation of the experimental data, it should also be pointed out that the concepts and results of the linear theory, based as they are upon the assumption of small disturbances, constitute only a first-order approximation to the truth even for the supposedly inviscid gas. When disturbances of appreciable magnitude are considered, the previous concept of a Mach cone traversing the entire flow field is no longer tenable. On the contrary, a given disturbance in a supersonic stream is then confined, not to the interior of a cone, but to the interior of some more complex surface whose shape and position depend upon the magnitude of the disturbance as well as upon other conditions in the general flow field. It follows that the regions of influence of a wing tip or wing root are not strictly as shown in figure 1, and the previous distinction between a supersonic and subsonic element cannot be applied without qualification. The ideas of the linear theory with regard to pressure propagation, therefore, should not be taken literally nor should deductions based upon them be accepted without reservation.

It is apparent from these brief theoretical considerations that calculations by the linear theory may be expected to fall short of the truth for two primary reasons. These are

- (a) the omission from the theory of all viscous phenomena, and
- (b) the theoretical assumption that the flow disturbances are small.

The importance of these approximations cannot be assessed at present from purely theoretical knowledge. Some insight is provided, however, by the available experimental results.

PRESSURE-DISTRIBUTION MEASUREMENTS IN TWO DIMENSIONS

It is desirable to begin the comparison between theory and experiment by examining some typical pressure-distribution results for an airfoil section in a two-dimensional supersonic stream. Because of the availability in the two-dimensional case of theories of greater accuracy than the linear theory, it is possible here to distinguish between the effects of viscosity and the effects of the terms neglected through the assumption of small disturbances.

A typical two-dimensional pressure distribution is given in figure 2, which shows the calculated and measured results for a 10-percent-thick, symmetrical, biconvex section at a Mach number of 2.13 and an angle of attack of 10° . The local pressure coefficient is plotted as a function of the chordwise position on the airfoil, positive values being plotted below the horizontal axis and negative values above. The theoretical pressure distributions given by the linear and shock-

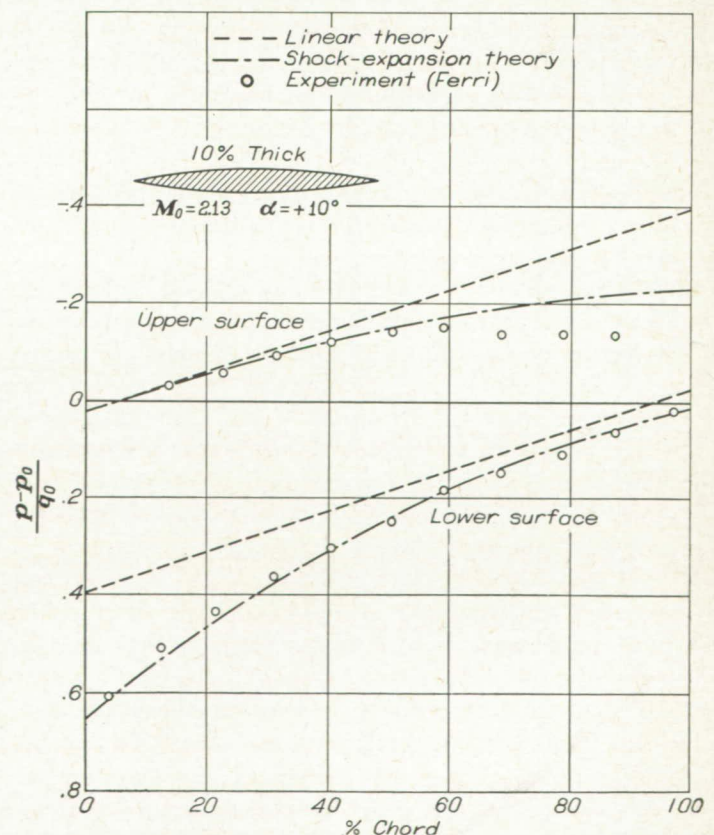


FIGURE 2.—Pressure distribution for symmetrical biconvex section.

expansion theories are shown by curves as noted. The individual circles indicate experimental points obtained from the results of Ferri (reference 9).

The data of figure 2 show that considerable accuracy is gained by going from the linear to the shock-expansion theory. Over most of the airfoil section, the linear theory predicts the correct sense for the pressure gradient, but the quantitative agreement between the curve given by this theory and the experimental points is poor compared with the excellent check given by the shock-expansion method. Over the rear 40 percent of the upper surface, neither of the theories agrees with the trend exhibited by experiment.

The discrepancy between the theoretical pressure distributions calculated by the linear and shock-expansion theories is of importance primarily for its effect upon the chordwise distribution of lift. Examination of figure 2 reveals that the total lift of the section, as approximated by the area between the curves for the upper and lower surfaces, is given almost identically by the two theories. This illustrates the fact that in the two-dimensional case the higher-order terms neglected in the linear theory have little effect upon the over-all lift of the section. They do, however, serve to concentrate the lift farther forward on the chord than the linear theory would predict. This effect is essentially a consequence of the airfoil thickness and diminishes as the thickness is reduced.

The failure of even the shock-expansion theory to predict the pressure variation over the rear part of the upper surface is due to shock-wave, boundary-layer interaction (reference 9). In the idealized, inviscid fluid, the two-dimensional flow over a lifting airfoil at supersonic speeds is characterized by an oblique compression wave originating on the upper surface at the trailing edge. In the real, viscous fluid, this flow pattern is modified by an interaction between the oblique wave and the viscous boundary layer on the airfoil surface. The boundary layer separates from the upper surface some distance forward of the trailing edge, with the formation of a weak compression wave at the separation point and a consequent increase in pressure between this point and the trailing edge. There is, as a result, a noticeable loss of lift over the rear of the airfoil.

The foregoing results, of course, imply certain deviations of the true aerodynamic coefficients from the curves predicted by the linear theory. For the reasons outlined, the higher-order pressure effects neglected in the linear theory have little influence upon the lift-curve slope, although they do result in a relatively forward shift of the center of pressure (or aerodynamic center). The interaction between the trailing shock wave and the viscous boundary layer acts both to decrease the lift-curve slope slightly and to displace the center of pressure still farther forward. Viscous friction, the effects of which are not visible in the pressure distribution, tends to increase the true drag relative to the calculated value, though this tendency is opposed here by the unpredicted increase in pressure near the trailing edge as the result of the shock-wave, boundary-layer interaction. All of these effects are apparent in the available force-test data for airfoils in two-dimensional flow (references 9 and 10). As will be seen, they are also observed in the results for three-

dimensional wings, at least for those cases in which the wing elements are predominantly supersonic.

FORCE TESTS IN THREE DIMENSIONS

The discussion to this point has been confined to theoretical considerations and to a comparison between theoretical and experimental results for a typical airfoil section in two-dimensional flow. The remainder of the paper will be concerned with a more general comparison between theory and experiment for complete, three-dimensional wings.

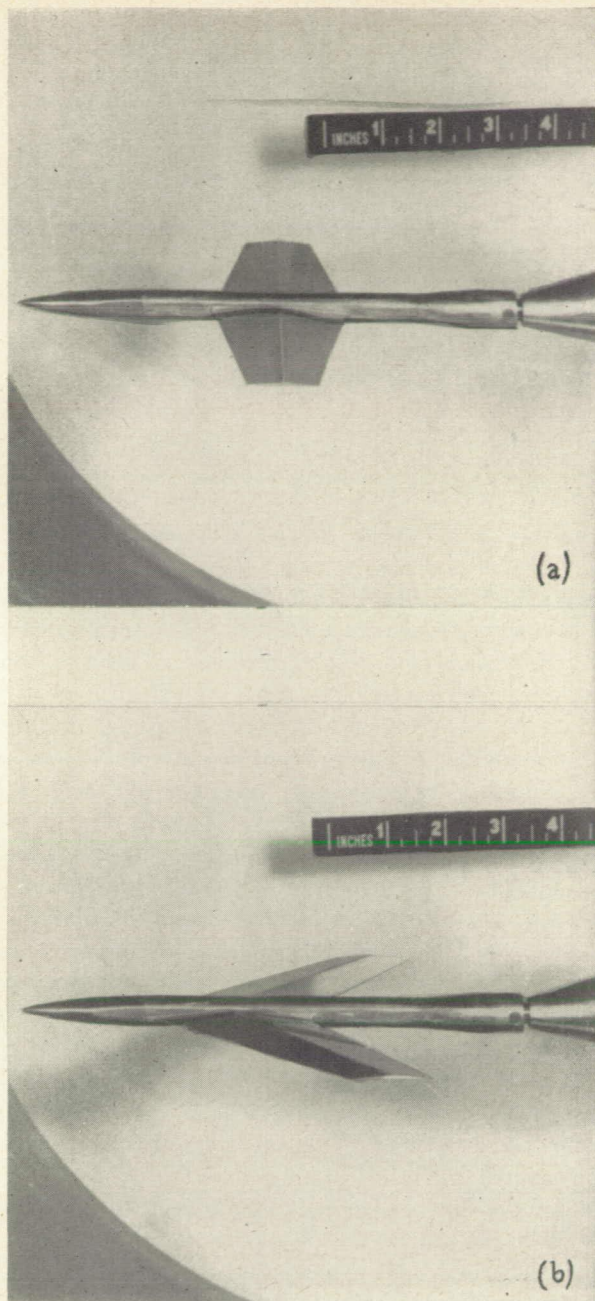
The results upon which this comparison is based were obtained in 1946 as part of an investigation of wing characteristics conducted at the Ames Aeronautical Laboratory of the NACA. The portion of the general investigation to be discussed here was concerned with force tests at supersonic speeds of approximately 30 wing models chosen to cover a wide range of geometric variables and to include examples with both supersonic and subsonic wing elements. The experimental work was performed in the Ames 1- by 3-foot supersonic wind tunnel No. 1, which is a continuous-flow, closed-return tunnel of approximately 10,000 horsepower.²

The wing models were supported in the wind tunnel on a slender body of revolution mounted directly ahead of a three-component, strain-gage balance as shown in figure 3. For the majority of the models, the airfoil section taken in the streamwise direction was a 5-percent-thick isosceles triangle, that is, a triangle with maximum thickness of 5 percent located at midchord. This cambered section was chosen primarily for ease of construction. The models were made of hardened, ground tool steel with the leading and trailing edges maintained sharp to less than a 0.001-inch radius, except for certain tests in which the leading edge was purposely rounded. The support body, which was the same for all models, was kept as small as possible consistent with the requirement that it could be used with a wide range of plan forms.

Because of the presence of the support body, the experimental results to be presented apply, strictly speaking, to wing-body combinations rather than to the wings alone. The theoretical curves are, on the other hand, for simple, isolated wings. A detailed examination of the interference problem indicates that, for the particular body used here, the effects of the body are small insofar as the lift and pitching moment are concerned. The influence on minimum drag may, however, be considerable. The measured values of the minimum drag coefficient must therefore be regarded as of primarily qualitative significance in comparison with theory.

Because of limitations of time and space, it is obviously impossible in a paper of this kind to discuss more than a small portion of the results obtained in the investigation. The data presented will therefore be chosen primarily for their value in illustrating certain general ideas or typical conclusions. This approach will result in the omission of

² As with most experimental investigations, many people contributed to the final results of the study. Particular credit is due, however, to Jack N. Nielsen, Milton D. Van Dyke, and Frederick H. Matteson, who participated in the analysis of the results, to Robert T. Madden, Richard Scherrer, and John A. Blackburn, who conducted the wind-tunnel tests, and to Albert G. Oswald, who was in charge of the wind-tunnel instrumentation.



(a) Unswept wing.
(b) Swept wing.

FIGURE 3.—Typical wing models mounted on support body in Ames 1-by 3-foot supersonic wind tunnel.

many interesting items dear to the heart of the experimentalist, but it is hoped that an adequate over-all picture of the significant results will emerge. In all of the figures presented, the aerodynamic coefficients will be referred to the plan form area of the wing, including that portion of the plan form enclosed by the support body. All of the results are for a free-stream Mach number of 1.53 and a test Reynolds number of 0.75 million based upon the mean geometric chord of the wing. Unless stated otherwise, it may be assumed that the results were obtained using models with the cambered, isosceles-triangle section previously described.

In the discussion of the results, it is convenient to consider first the lift and pitching moment, since these charac-

teristics depend primarily upon the distribution of normal pressure over the surface of the wing. The consideration of drag, which depends upon the frictional forces as well, will be deferred until later.

LIFT AND PITCHING MOMENT

According to the linear theory, the lift and pitching-moment curves for any given wing are each a straight line. At a given Mach number, the slope of the line depends solely upon the plan form of the wing and is independent of the camber and thickness. The intercept—that is, the angle of zero lift or the moment at zero lift—is a function of both the camber and the plan form, but is independent of the wing thickness. Only the slope of the curves will be discussed here, since this is the characteristic of greatest practical importance.

Lift-curve slope.—The nature of the agreement between theory and experiment with regard to the lift-curve slope for unswept wings is illustrated in figure 4. Here $dC_L/d\alpha$ is plotted as a function of aspect ratio for a series of four unswept wings having a common taper ratio of 0.5. The wing corresponding to each test point is indicated by a small sketch, which shows also the trace of the Mach cones from the forwardmost point of the wing. On this and later figures, the variation predicted by the linear theory is shown over as wide a range as is practicable on the basis of existing computational methods.

The agreement between theory and experiment in figure 4 is seen to be excellent over the entire range of aspect ratios. The exact coincidence for aspect ratios from 2 to 6 is, in fact, too good to be absolutely true. It appears likely that the secondary effects of viscosity and support-body interference, which must certainly be present in some degree, are completely compensating for these wings. The decrease in lift-curve slope observed both experimentally and theoretically at the low aspect ratios is caused by a loss of lift within the Mach cones which originate at the leading edge of the wing tips. As the aspect ratio is reduced, a greater and greater percentage of the plan form is included within these Mach

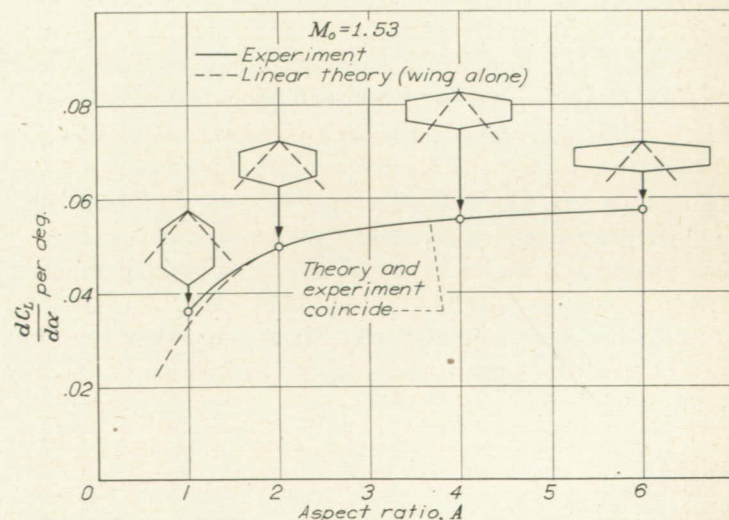


FIGURE 4.—Effect of aspect ratio on lift-curve slope.

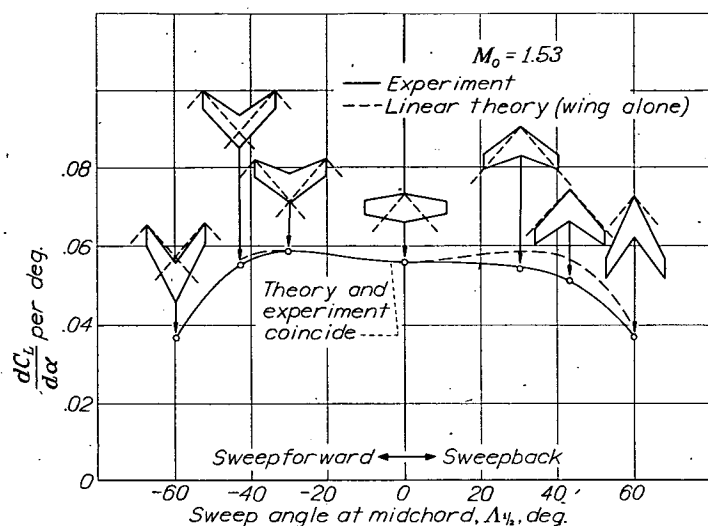


FIGURE 5.—Effect of sweep on lift-curve slope.

cones, with a resulting decrease in the lifting effectiveness of the wing.

The effect of wing sweep on the slope of the lift curve is illustrated in figure 5. Here $dC_L/d\alpha$ is shown as a function of the sweep angle at the midchord line for a series of seven wings also of taper ratio 0.5. The unswept wing of this series is identical with the aspect ratio 4 wing of the previous figure. In the design of the swept wings, the aspect ratio was made to decrease as the cosine of the angle of sweep, since wings of constant aspect ratio did not appear structurally feasible. The sweep angles were chosen to provide representative plan forms with both supersonic and subsonic leading and trailing edges. The wing of 43° sweepback was designed to have its leading edge coincident with the Mach cone, which has a sweep angle of 49.2° at the test Mach number of 1.53. Since the sweep angle of these wings is specified at the midchord line, a given swept-forward wing can be obtained from the corresponding swept-back wing by a simple reversal of the direction of motion.

The agreement between theory and experiment in figure 5 is almost exact over the range of sweep angles from 0° to 43° sweepforward, the forwardmost limit of the theoretical results. For all of the swept-back wings, the experimental slopes fall consistently below the theoretical values by from 8 to 10 percent. In both the swept-back and swept-forward direction, the experimental results exhibit a marked reduction in $dC_L/d\alpha$ as the edges of the plan form are swept increasingly farther behind the Mach cone. This trend is predicted by the theoretical curve in the swept-back case and would undoubtedly be confirmed for the swept-forward wings if complete theoretical results were available.³ It is interesting to note, incidentally, that the 43° swept-back wing, which has its leading edge coincident with the Mach cone, shows no departure from the general trend of the experimental results.

³ For the range of sweep angles from 43° to 60° sweepback, the shape of the theoretical curve is somewhat approximate. Strictly speaking, small discontinuities in the slope of the curve would be expected at approximately 43° and 55° where the leading edge and trailing edge of the plan form coincide, respectively, with the Mach cone. No attempt was made to determine these discontinuities, the theoretical curve being faired smoothly through the available calculated points.

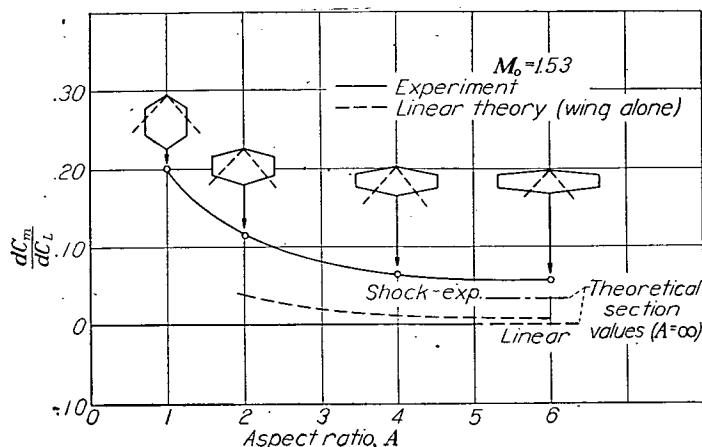


FIGURE 6.—Effect of aspect ratio on moment-curve slope.

For the range of sweep angles between $\pm 43^\circ$, the theoretical curve of figure 5 is exactly symmetrical about the vertical axis. This means that, within this range, the theoretical lift-curve slope of a plan form of the present series is unchanged by a reversal of the direction of motion. Similar result has been obtained by several authors for other, more general classes of wings (see, for example, references 11 and 12), though the limits of generality have not, to the writer's knowledge, been completely established.⁴ The observed departure of the experimental results from the theoretical symmetry may be due to differences in aeroelastic deformation between corresponding swept-forward and swept-back wings or to asymmetry in the effects of other secondary factors such as viscosity and support body interference.

To summarize, we may say that the agreement between experiment and linear theory with regard to the lift-curve slope of three-dimensional wings is satisfactory for most practical purposes. In view of the situation previously observed in the two-dimensional case, however, it cannot be assumed that agreement in the integrated lift implies complete agreement in the details of the lift distribution.

Moment-curve slope.—Further indication that the details of the flow over the wings are, as in the two-dimensional case, somewhat different from the predictions of the linear theory is given by the pitching-moment data. Figure 6 shows the moment-curve slope as a function of aspect ratio for the series of unswept wings previously discussed. The moment coefficient is here taken about the centroid of plan-form area, with the mean aerodynamic chord as the reference length. The moment-curve slope is thus an approximate measure of the displacement of the aerodynamic center of the wing forward of the centroid of area, expressed as a fraction of the mean aerodynamic chord.

It can be seen from figure 6 that the linear theory predicts a progressively forward displacement of the aerodynamic center as the aspect ratio is reduced. As in the case of the

⁴ Since the present paper was written, the theoretical result observed here has been established with complete generality with regard to plan form by Clinton E. Brown of The Langley Aeronautical Laboratory of the NACA. (See Brown, Clinton E.: The Reversibility Theorem for Thin Airfoils in Subsonic and Supersonic Flow. NACA TN 1944, 1949.) According to Brown's proof, which is based upon previous work by Max M. Munk, the theoretical lift-curve slope of a given wing is, to the first order, invariant with respect to a reversal of the direction of motion, irrespective of the Mach number or shape of the plan form.

lift-curve slope, this variation is due to the loss of lift which occurs over the rear portion of the wing within the Mach cones from the tips. The trend of the experimental values is in agreement with the theoretical curve, but the forward displacement is uniformly greater than the theory predicts. The reason for this discrepancy becomes apparent if we imagine the wing series of figure 6 to be extended to indefinitely high aspect ratios. In the limit of infinite aspect ratio, the flow over the wing would be purely two-dimensional, and the theoretical characteristics would be simply those of the wing section. For the present isosceles-triangle section, the values of dC_m/dC_L given by the linear and shock-expansion theories are as indicated by the two horizontal lines to the right. The theoretical curve for the finite-span wings, of course, approaches the linear section value as an asymptote. If only nonviscous effects were important in the experiments, the measured curve would be expected to approach the section value predicted by the shock-expansion method. The fact that it seems to approach an asymptote above this latter value is consistent with the occurrence of shock-wave, boundary-layer interaction near the supersonic trailing edge as previously observed in the two-dimensional results (fig. 2). We may thus infer that the discrepancy between experiment and linear theory over the entire range of aspect ratios is due to a combination of both higher-order pressure effects and fluid viscosity.

The effect of sweep on the moment-curve slope is shown in figure 7 for the same series of wings used before. It is apparent that here experiment and theory agree neither quantitatively nor qualitatively. For the unswept wing, the observed discrepancy can be accounted for as explained in connection with figure 6. The disagreement in the variation with angle of sweep is, however, difficult to reconcile on the basis of present knowledge. In general, the effects of boundary-layer separation may be expected to have a major influence on the moment characteristics of swept wings, particularly in those cases in which the wing elements are predominately subsonic. The possible importance of the higher-order pressure effects should not be overlooked, however. It can be shown from quite general considerations that the calculation by the linear theory of the aerodynamic-center position for any given wing is subject to a possible error of the same order of magnitude as the percent thickness of the airfoil section. For this reason, the development of a rea-

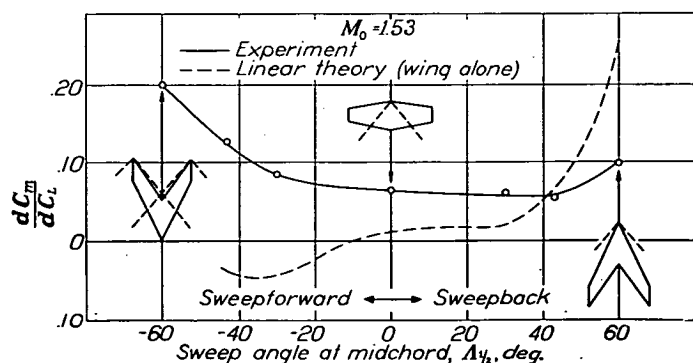


FIGURE 7.—Effect of sweep on moment-curve slope.

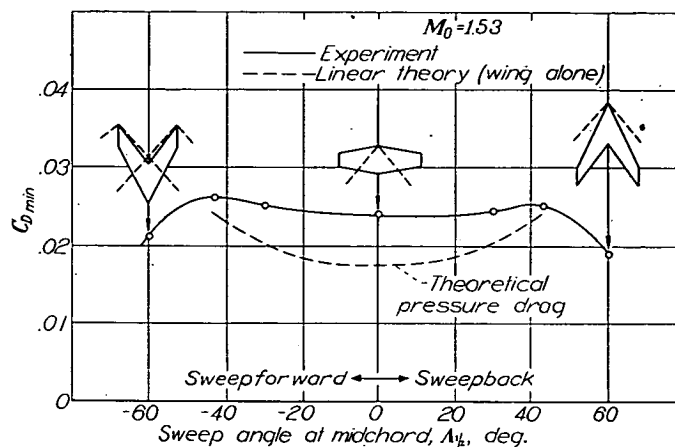


FIGURE 8.—Effect of sweep on minimum drag.

sonably general, second-order wing theory may prove essential to a complete understanding of the pitching-moment problem.

DRAG

The calculation of wing drag by the linear theory leads to a parabolic curve of drag versus lift. The value of the minimum drag coefficient depends, for a given Mach number, upon the thickness, camber, and plan form of the wing, while the lift coefficient at which the minimum occurs is a function of the camber and plan form. The rise in drag as the lift coefficient departs from that for minimum drag depends, according to the linear theory, upon the geometry of the plan form only.

Minimum drag.—A typical illustration of the effect of change in plan form on the minimum drag is given in figure 8, which shows the variation in minimum drag coefficient for the previous series of swept wings. The theoretical curve shown is for the pressure drag only—that is, no attempt has been made to estimate the skin friction. Because of the mathematical complications introduced by camber when the edges of the wing are subsonic, it was not practicable here to extend the theoretical curve beyond 43° in either direction. Within these limits, the theoretical drag increases with increasing sweep. Extension of the curve to higher angles of sweep would be expected to show a marked decrease in the calculated drag, similar to the well-known results for uncambered wings swept behind the Mach cone.

The experimental curve of figure 8 follows the general trend indicated by theory. As the sweep increases from zero in either direction, the measured drag first rises to a maximum in the vicinity of the Mach cone and then decreases markedly with further increase in sweep. The large decrease in drag obtained by sweeping the wing behind the Mach cone has been observed by numerous investigators and need not be enlarged upon here. What is more interesting in the present results is the failure of the experimental values to rise as rapidly as does the theoretical curve in the lower range of sweep angles. For the wings of 0° and $\pm 30^\circ$ sweep, the displacement of the experimental points above the theoretical curve is consistent with a reasonable allowance for skin friction and support-body interference. For the wings of $\pm 43^\circ$ sweep, however, the experimental values are almost

coincident with the theoretical. This result suggests that the linear theory may be overly pessimistic regarding wing drag when the Mach number normal to the wing elements is near unity. Support for this conjecture is found in the work of Hilton and Pruden (reference 10), who report a similar situation in two-dimensional tests of an airfoil section at moderately supersonic speeds. It is likely that in both instances the results are due to transonic effects which are beyond the scope of the linear theory.

The symmetry of the curves of figure 8 is also worthy of note. It has been shown by several authors (see, for example, references 1 and 12) that, to the order of accuracy of the linear theory, the minimum pressure drag of a wing of any plan form is unchanged by a reversal of the direction of motion, provided the wing section is without camber. For cambered wings, the corresponding drag theorem is probably less general with regard to plan form, though, as in the case of the lift-curve slope, the limits of generality have not been defined. For the present wings, reversibility is readily proven over the range of sweep angles between $\pm 43^\circ$. As a result, the theoretical curve of figure 8 is, like the corresponding curve for $dC_L/d\alpha$ in figure 5, exactly symmetrical over this interval. In spite of the theoretical result, however, the almost perfect symmetry of the experimental curve of figure 8 comes as somewhat of a surprise. It might be expected that secondary differences between corresponding swept-forward and swept-back wings would cause an asymmetry here akin to that observed in the experimental values of lift-curve slope.

The most interesting results with regard to drag, however, are concerned with the effects of thickness distribution on the minimum drag of triangular wings. At about the time the present study was beginning, theoretical results by Puckett appeared (reference 13) which indicated that the minimum pressure drag of an uncambered triangular wing with a subsonic leading edge could be held to a relatively low value by proper location of the position of maximum thickness. To check these results, two triangular wings of aspect ratio 2 were included in the present study. Both wings had an uncambered double-wedge section with a thick-

ness ratio of 5 percent. In one case the maximum thickness was located at midchord, in the other at a position 20 percent of the chord aft of the leading edge.

The findings for these wings are summarized in figure 9, which shows the theoretical and experimental values of the minimum drag coefficient plotted as a function of the position of maximum thickness. The curve of theoretical pressure drag, which is representative of Puckett's results, is divided into two parts by a sharp break in slope, located in this instance at 42 percent of the chord. For points to the right of this break, the ridge line defined by the position of maximum thickness is supersonic, and the flow around the ridge resembles the supersonic flow around a convex corner. Under these conditions, there is little pressure recovery over the rear of the wing, and the drag is relatively high. For points to the left of the break, the ridge line is subsonic, and the local flow is of the characteristically subsonic type. Under these conditions, the pressure recovery over the rear of the wing is considerable, and the drag is correspondingly reduced. For the wings under consideration, the net result of moving the maximum thickness forward from the 50-percent to the 20-percent station is to reduce the computed pressure-drag coefficient from 0.0092 to 0.0054. Unfortunately, the measured values of the minimum drag, indicated by the two small circles, do not follow the theoretical trend. The apparent effect of the forward displacement is, in fact, to increase the drag slightly.

When this result was first noted, the experimental data were suspected of being in error. Repeated tests, however, gave identical results. It was next thought that support-body interference might be to blame. Estimates indicated, however, that such interference could hardly account for the large difference in the increments by which the measured total drag exceeded the computed pressure drag for the two wings. Consideration of the friction drag finally supplied the key to a possible explanation. To examine this possibility, curves of theoretical total drag were computed on the basis of the skin-friction coefficients corresponding to completely laminar and completely turbulent flow in the boundary layer. When this was done, it was found, as is apparent in the figure, that the experimental point for the wing with maximum thickness at 50 percent fell midway between the two resulting curves, while that for the wing with maximum thickness at 20 percent was slightly above the curve for completely turbulent flow. This suggested that the failure of the experimental points to follow the trend of the theoretical pressure drag might be due to a difference in the extent of laminar boundary-layer flow on the two wings.

To check this hypothesis, the liquid film method developed by Gray of the R.A.E. for the indication of transition at subsonic speeds (reference 14) was adapted for use in a supersonic stream. This method depends upon the fact that the rate of evaporation of a film of liquid on the surface of a model is, on the average, greater where the boundary layer is turbulent than where it is laminar. In applying this principle at the Ames Laboratory, the model is first coated with

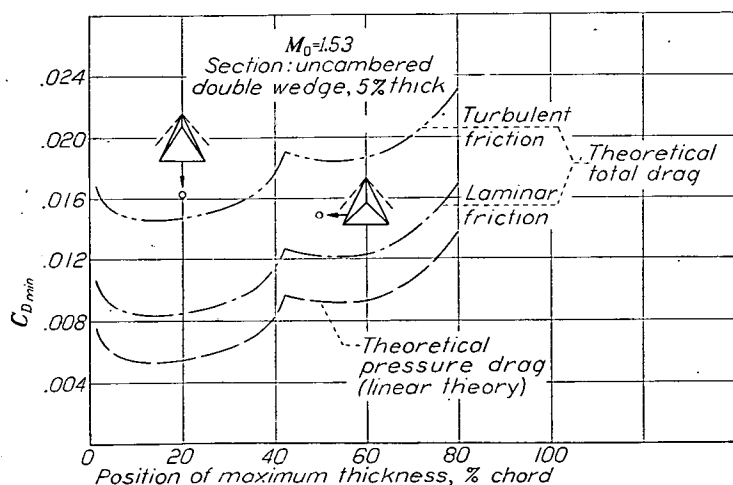
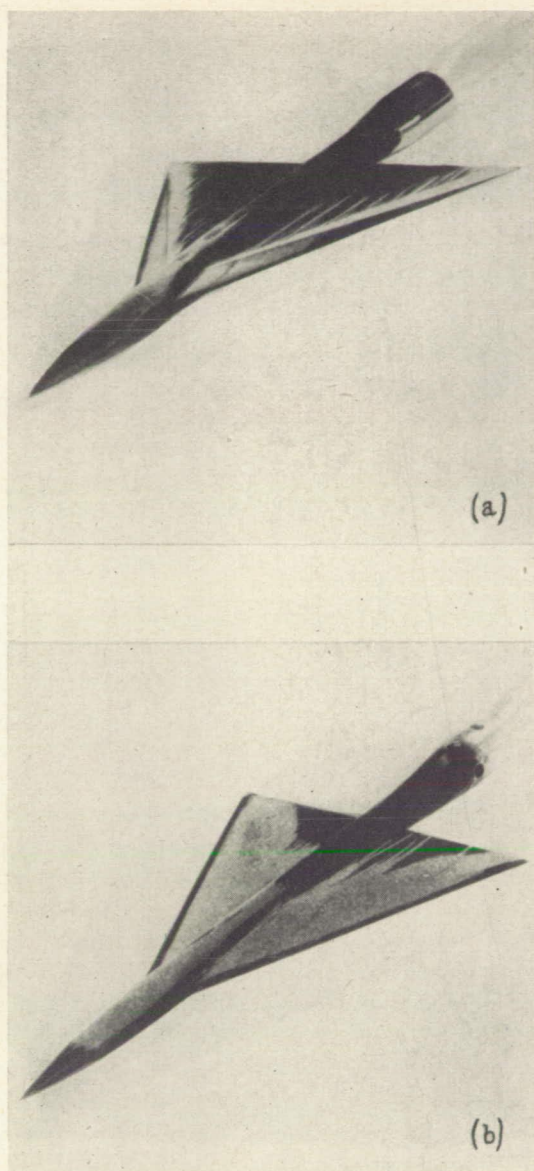


FIGURE 9.—Effect of position of maximum thickness on minimum drag of triangular wings.



(a) Maximum thickness at 20-percent chord.
(b) Maximum thickness at 50-percent chord.

FIGURE 10.—Results of liquid-film tests on triangular wings at zero lift. Section: uncambered double wedge, 5-percent thick, $M_0 = 1.53$.

flat black lacquer and then, immediately prior to installation in the tunnel, with a liquid mixture containing glycerin. A run is then made at the desired test condition for a sufficient time to allow the liquid to evaporate completely in the turbulent region but remain moist over most of the laminar area. Upon removal from the tunnel, the model is dusted with talcum powder, which adheres to the laminar but not to the turbulent area, thus increasing the contrast for photographic purposes and providing a clear indication of the extent of the two types of boundary-layer flow.

The results of liquid film tests of the two triangular wings at zero lift are shown in figure 10. For the wing with maximum thickness at midchord, the region of turbulent flow, which appears as the dark region on the model, constitutes only about half of the surface area aft of the ridge line. For the wing with maximum thickness displaced forward, the turbulent region occupies almost all of the considerably

larger area which is aft of the ridge line on this wing.⁵ These results were repeated many times during the numerous tests necessary to perfect the liquid-film technique. Examination of calculated pressure distributions for the two wings shows in each case excellent correlation between the experimentally determined region of turbulent flow and the calculated region of adverse pressure gradient. Because of the effects of support-body interference, it is not possible to make a decisive comparison between the measured values of total drag and theoretical values calculated on the basis of the observed areas of laminar and turbulent flow. The evidence of the liquid-film tests, however, leaves little doubt as to the primary reason why forward displacement of the maximum thickness fails to produce the reduction in minimum drag predicted by the inviscid, linear theory.

The foregoing result has important implications with regard to the degree of drag reduction possible at supersonic speeds through the use of sweepback. The relatively high pressure drag of an unswept wing at speeds above the speed of sound is a direct result of an absence of pressure recovery over the rear of the wing. The high pressure drag is thus associated with a chordwise pressure gradient which is, for the most part, favorable to the boundary-layer flow. The reduction of pressure drag by means of sweepback depends, on the other hand, upon the presence of an appreciable pressure recovery, or in other words, upon the existence of a region of adverse gradient. If the region of such gradient occupies the major portion of the wing, then, as was seen in the case of the triangular wing with thickness forward, the detrimental effects upon the skin friction may more than offset the gains in pressure drag. This suggests that it may be desirable here, as in the case of the subsonic, low-drag airfoil, to look for wing shapes which have their pressure recovery confined to a relatively small part of the wing area. Wings of this type may, in fact, prove more practical at supersonic than at subsonic speeds, since there is indication (reference 15) that the boundary-layer phenomena at the higher speeds may be more conducive to long runs of laminar flow.

Drag rise and lift drag ratio.—The final question to be discussed is that of the variation in drag with change in lift. As previously mentioned, the theoretical curve of drag versus lift is, for any given wing, parabolic in shape. The rise in drag as the lift coefficient departs from that for minimum drag depends, for a given Mach number, on the wing plan form only and is independent of the camber and thickness. The shape of the theoretical parabola for a given wing is thus identical with that for a flat lifting surface of the same plan form as the wing in question.

In the case of a plan form with a supersonic leading edge, the determination of the rise of the theoretical parabola is relatively simple. In this case, which is exemplified by plan forms A and B of figure 1, the local pressure on the flat lifting surface is everywhere finite. The variation in drag with change in lift can thus be found by simple integration of

⁵The white streaks extending back into the otherwise dark turbulent area are streamers of excess liquid blown back from the laminar region. These streamers may at times be used as a valuable indication of the direction of flow within the boundary layer, particularly on highly swept wings.

the pressures acting on the top and bottom of the surface. For all of the wings of the present study having a supersonic leading edge, the shape of the drag curve given by the theoretical calculation shows good agreement with experiment.

In the case of a wing with a subsonic leading edge, the theoretical problem is more complex. In this case, exemplified by plan form C of figure 1, there is a singularity—that is, an infinite value—in the theoretical lift intensity at the leading edge of the equivalent flat surface. The effect of this singularity is to produce a finite suction force on the leading edge in the direction opposite to the free stream. This force—sometimes referred to simply as “leading-edge suction”—reduces the rise of the theoretical drag parabola below what it would be if only the pressures on the top and bottom of the wing were considered. Actually, of course, the details of the flow about the leading edge must, in any real case, be considerably different from the representations of the linear theory, since an infinite lift intensity is obviously impossible. It does not follow, however, that the theoretical forward force at the leading edge will not exist. The situation here is much the same as that encountered at the leading edge of an airfoil section in two-dimensional, incompressible flow. In this latter case, it is known, both from experiment and from the indications of more refined calculations, that the elementary theory gives an accurate prediction of the leading-edge suction within certain limits of angle-of-attack and leading-edge radius. The range of applicability of the linear theory as applied to swept wings at supersonic speeds must similarly be established by careful theoretical and experimental investigation.

The results of the present study are not, in general, conclusive with regard to the conditions necessary for the attainment of the theoretical force at the subsonic edge. The data for the triangular wings, however, do offer some possibly significant findings. These are illustrated in figure 11, which shows the effects of change in wing section upon the drag due to lift for the triangular wings previously

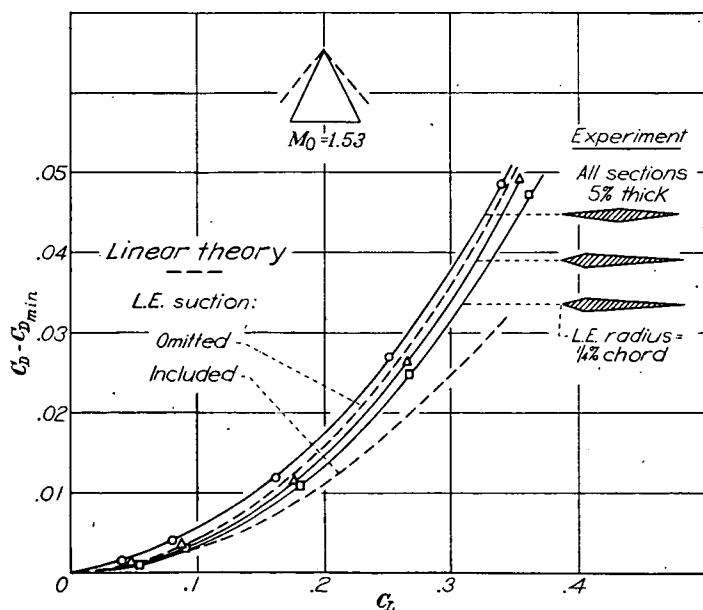


FIGURE 11.—Effect of wing section on drag rise of triangular wings.

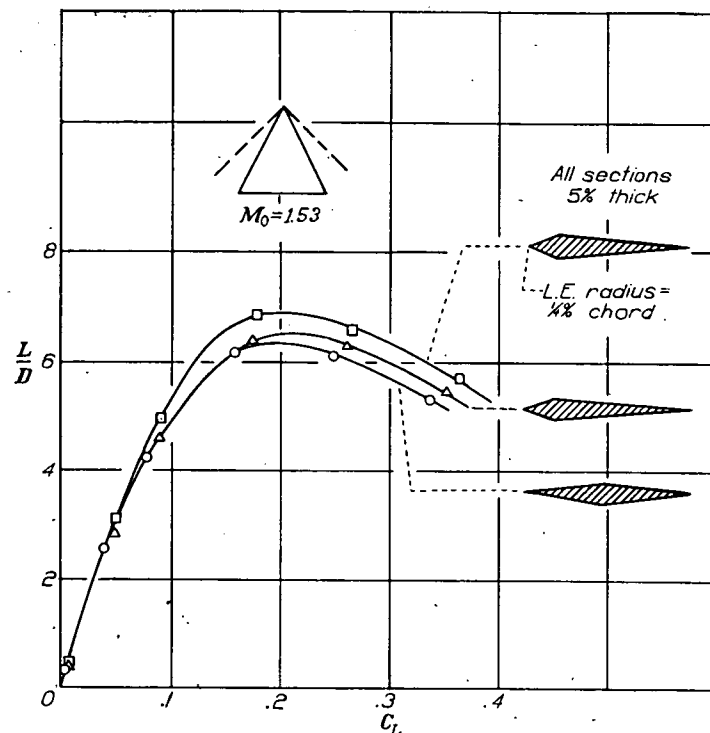


FIGURE 12.—Effect of wing section on lift-drag ratio of triangular wings.

discussed. The two theoretical curves show the calculated drag rise with the leading-edge suction both included and omitted. For the wing with maximum thickness at midchord, the experimental curve is slightly above the theoretical curve with leading-edge suction omitted. This is as might be expected for a sharp-edged wing, the slight increase above the upper theoretical curve being due possibly to an increase in friction drag with increasing lift or to support-body interference. Moving the maximum thickness forward on the wing to the 20-percent-chord position resulted in a slight reduction in drag despite the retention of a sharp leading edge. This gain may be due either to the attainment of leading-edge suction as a result of the larger leading-edge wedge angle on this wing or to a change in the variation of friction drag with lift. In an attempt to bring the drag rise of the second wing down to the values indicated by the complete theory, the edge of this wing was rounded to a radius of 0.25 percent of the chord, which is of the same order of magnitude as the radius of an NACA low-drag section of comparable thickness ratio. This rounding of the leading edge afforded some benefit, the resulting experimental values being approximately midway between the two theoretical curves. Additional rounding—to a 0.50-percent radius over the entire span and then to a still greater value over the outer half—had no further effect.

The influence of the foregoing changes on the experimental curves of lift-drag ratio is shown in figure 12. The wing with maximum thickness at midchord has a value of $(L/D)_{max}$ of about 6.3. When the maximum thickness is moved forward to the 20-percent-chord station, the decrease in drag rise apparent in figure 11 more than outweighs the slight increase in minimum drag observed in figure 9. As a result, the maximum lift-drag ratio increases slightly.

Rounding the leading edge of the second wing, while reducing the drag rise as previously noted, does not alter the minimum drag. As a consequence, the maximum lift-drag ratio is increased to approximately 6.8. These results suggest that the aerodynamic gains predicted on the basis of the theoretical leading-edge suction can be at least partially realized in practice. The determination of the optimum profile shape for this purpose may, however, involve considerable detailed research.

It is interesting for contrast with the foregoing results to point out the detrimental effects at the test Mach number of rounding the leading edge on an unswept wing. In tests of an unswept, untapered wing of aspect ratio 4, rounding the leading edge to a radius of 0.25 percent of the chord resulted in a 27-percent increase in minimum drag and a consequent reduction in maximum lift-drag ratio from 6 to about 5.5. The rise in the drag curve was unaffected by the modification.

CONCLUDING REMARKS

The foregoing results represent only a small contribution to the body of experimental and theoretical knowledge now being accumulated concerning the characteristics of wings at supersonic speeds. As is the case with most measurements of over-all forces, the data of the present study raise more questions than they answer. Detailed and patient investigations of pressure distribution and boundary-layer flow are required to develop a rational explanation for many of the observed phenomena. Several major problems have not been discussed here at all, including the important question of the adequacy of the Kutta condition to describe the real flow at a highly swept, subsonic trailing edge. There is sufficient to be done, indeed, to keep many investigators occupied for years to come.

AMES AERONAUTICAL LABORATORY,

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., May 3, 1950.

REFERENCES

1. von Kármán, Theodore: Supersonic Aerodynamics—Principles and Applications. Jour. of the Aero. Sci., vol. 14, no. 7, July 1947, pp. 373-402.
2. Ivey, H. Reese, Stickler, George W., and Schuettler, Alberta: Charts for Determining the Characteristics of Sharp-Nose Airfoils in Two-Dimensional Flow at Supersonic Speeds. NACA TN 1143, 1947.
3. Ackeret, J.: Air Forces on Airfoils Moving Faster than Sound. NACA TM 317, 1925.
4. Taylor, G. I.: Applications to Aeronautics of Ackeret's Theory of Aerofoils Moving at Speeds Greater than that of Sound. R. & M. No. 1467, British A.R.C., 1932.
5. Busemann, A., and Walchner, O.: Airfoil Characteristics at Supersonic Speeds. R.T.P. Translation No. 1786, British Ministry of Aircraft Production.
6. Lock, C. N. H.: Examples of the Application of Busemann's Formula to Evaluate the Aerodynamic Force Coefficients on Supersonic Aerofoils. R. & M. No. 2101, British A.R.C., 1949.
7. The Staff of the Ames 1- by 3-foot Supersonic Wind Tunnel Section: Notes and Tables for Use in the Analysis of Supersonic Flow. NACA TN 1428, 1947.
8. Jones, Robert T.: Wing Plan Forms for High-Speed Flight. NACA TN 1033, 1946.
9. Ferri, Antonio: Experimental Results with Airfoils Tested in the High-Speed Tunnel at Guidonia. NACA TM 946, 1940.
10. Hilton, W. F., and Pruden, F. W.: Subsonic and Supersonic High Speed Tunnel Tests of a Faired Double Wedge Aerofoil. R. & M. No. 2057, British A.R.C., 1943.
11. Lagerstrom, P. A., Wall, D., and Graham, M. E.: Formulas in Three-Dimensional Wing Theory. Douglas Aircraft Company Rep. No. SM11901, Aug. 1946.
12. Hayes, Wallace D.: Reversed Flow Theorems in Supersonic Aerodynamics. North American Aviation Rep. No. AL-755, Aug., 1948.
13. Puckett, Allen E.: Supersonic Wave Drag of Thin Airfoils. Jour. of Aero. Sci., vol. 13, no. 9, Sept. 1946, pp. 475-484.
14. Gray, W. E.: A Simple Visual Method of Recording Boundary Layer Transition (Liquid Film). Tech. Note Aero 1816, R.A.E., Aug. 1946.
15. Lees, Lester: The Stability of the Laminar Boundary Layer in a Compressible Fluid. NACA Rep. 876, 1947.

REPORT 1034

INVESTIGATION OF SPOILERAILERONS FOR USE AS SPEED BRAKES OR GLIDE-PATH CONTROLS ON TWO NACA 65-SERIES WINGS EQUIPPED WITH FULL-SPAN SLOTTED FLAPS¹

By JACK FISCHEL and JAMES M. WATSON

SUMMARY

A wind-tunnel investigation was made to determine the characteristics of spoiler ailerons used as speed brakes or glide-path controls on an NACA 65-210 wing and an NACA 65₂-215 wing equipped with full-span slotted flaps. Several plug-aileron and retractable-aileron configurations were investigated on the two wing models with the full-span flaps retracted and deflected. Tests were made at various Mach numbers between 0.13 and 0.71.

The results of this investigation have indicated that the use of plug or retractable ailerons, either alone or in conjunction with wing flaps, as speed brakes or glide-path controls is feasible and very effective. In an illustrative example, the estimated time required for descent of a high-performance airplane from 40,000 feet was reduced from 12.3 minutes to 3.3 minutes. The plug and retractable ailerons investigated, when used as speed brakes, had only a small effect on the wing pitching moments. The rolling effectiveness of the ailerons will not be impaired by such use and should be as good as the effectiveness when the ailerons are projected in normal manner from the retracted position.

INTRODUCTION

One of the less obvious but nevertheless important needs of the high-performance military and commercial aircraft currently in use or in the design stage is that of utilizing suitable devices as aerodynamic speed brakes or glide-path controls, or both. Speed brakes and glide-path controls are beneficial for aircraft under various normal or emergency operating conditions, such as: a rapid descent from high altitude while airplane speed is being limited, landing on short runways over obstacles, reducing speed rapidly to increase the firing efficiency of fighter aircraft, and so forth. For the high-performance aircraft, the use of full-span slotted flaps and spoiler lateral-control devices would be particularly beneficial for providing high lift for landing and take-off as well as adequate lateral control. In order to obviate the necessity of including additional devices on the airplane, the use as speed brakes of spoiler ailerons, either alone or in conjunction with slotted flaps, was reported in reference 1 and was shown to be satisfactory. By means of suitable linkage, the slotted flaps can be deflected, and the spoiler ailerons on

both wing semispans can be projected equally above the wing to act as speed brakes or glide-path controls; and in either the neutral or an extended position, the ailerons can at the same time be operated differentially by movement of the control stick to provide lateral control.

The lateral control characteristics of various spoiler ailerons on unswept wings have been presented previously (for example, see references 2 to 7); however, the aerodynamic characteristics of these ailerons pertaining to their use as speed brakes or glide-path controls have seldom been presented.

In order to provide some information on the characteristics of plug and retractable ailerons when used as speed brakes or glide-path controls, the incremental values of lift, drag, and pitching-moment coefficients obtained at various aileron projections and flap deflections during the investigations of references 3 to 5 are presented herein. These data were obtained through a large angle-of-attack range on semispan wings having NACA 65-210 and NACA 65₂-215 airfoil sections. The investigation was performed in the Langley 7- by 10-foot tunnels at various Mach numbers between 0.13 and 0.71. Complete lift, drag, and pitching-moment data of these aforementioned wings with ailerons neutral have been presented in references 5 and 8. Data illustrating the rolling effectiveness of the ailerons when used as speed brakes or glide-path controls and a discussion pertaining to the application of the incremental lift, drag, and pitching-moment data to aircraft are presented herein.

COEFFICIENTS AND SYMBOLS

C_L	lift coefficient $\left(\frac{\text{Twice lift of semispan model}}{qS}\right)$
C_D	drag coefficient (D/qS)
C_m	pitching-moment coefficient $(M_p/qS\bar{c})$
Δ	increment caused by aileron projection
C_l	rolling-moment coefficient (L/qSb)
c	local wing chord, feet
$\bar{c}$	wing mean aerodynamic chord, 2.86 feet $\left(\frac{2}{S} \int_0^{b/2} c^2 dy\right)$
b	twice span of each semispan model, 16 feet
y	lateral distance from plane of symmetry, feet

¹ Supersedes NACA TN 1933, "Investigation of Spoiler Ailerons for Use as Speed Brakes or Glide-Path Controls on Two NACA 65-Series Wings Equipped with Full-Span Slotted Flaps" by Jack Fischel and James M. Watson, 1949.

S	twice area of each semispan model, 44.42 square feet
D	twice drag of semispan model, pounds
L	rolling moment, resulting from aileron projection, about plane of symmetry, foot-pounds
M_p	twice pitching moment of semispan model about 35-percent root-chord station
q	free-stream dynamic pressure, pounds per square foot $\left(\frac{1}{2} \rho V^2\right)$
V	free-stream velocity, feet per second
V_i	indicated airspeed, miles per hour
ρ	mass density of air, slugs per cubic foot
α	angle of attack with respect to chord plane at root of model, degrees
δ_f	flap deflection, measured between wing chord plane and flap chord plane (positive when trailing edge of flap is down), degrees
M	Mach number (V/a)
R	Reynolds number
a	speed of sound, feet per second

CORRECTIONS

All data presented are based on the dimensions of each complete wing.

The test data have been corrected for jet-boundary effects according to the methods outlined in reference 9. The Glauert-Prandtl transformation (reference 10) has been utilized to account for effects of compressibility on these jet-boundary corrections. Blockage corrections were applied to the test data by the methods of reference 11.

MODEL AND APPARATUS

The right-semispan-wing models investigated with spoiler ailerons (figs. 1 to 6) were mounted in either the Langley 300 MPH 7- by 10-foot tunnel or the Langley high-speed 7- by 10-foot tunnel with their root sections adjacent to one of the vertical walls of the tunnel, the vertical wall thereby serving as a reflection plane. Two wings were used for this investigation: one wing embodied NACA 65-210 airfoil sections and the other wing embodied NACA 65₂-215 airfoil sections. The wings were constructed with the same plan-form dimensions (figs. 1, 3, and 5) and each wing had an aspect ratio of 5.76; a taper ratio of 0.57, and had neither twist nor dihedral. The NACA 65₂-215 wing was constructed with two trailing-edge sections which were used alternately for tests of the plain-wing configuration and for tests of the wing configuration with flaps (fig. 6). The NACA 65-210 wing was equipped with two trailing-edge sections—one to accommodate the basic plug-aileron and retractable-aileron configurations (fig. 2) and the other to accommodate the circular-plug-aileron configurations (fig. 4); each trailing-edge section had a cut-out to accommodate the flap in the retracted position ($\delta_f=0^\circ$). A more detailed

description of the construction and mounting of the models is presented in references 3, 4, 5, and 8.

A 0.25c slotted flap which extended from the wing root section to the 95-percent-semispan station was used on both semispan wings in this investigation. This flap was originally designed and constructed to conform to the contour of the NACA 65-210 wing and was used in all investigations on that wing (references 3, 4, and 8). Because of its availability and satisfactory aerodynamic characteristics, this flap was also used in the flap-deflected wing configurations tested on the NACA 65₂-215 wing (reference 5). The positions of the flap with respect to the wing at the various deflections investigated with each aileron configuration are shown in figures 1, 3, and 6. These positions were found to be optimum, aerodynamically, for each flap deflection (references 5 and 8).

Each of the various aileron configurations investigated had a span of 49.2 percent of the wing semispan and was fabricated from duralumin or steel sheet in five equal spanwise segments (figs. 1 to 6). The basic plug ailerons and retractable ailerons on the NACA 65-210 wing had $\frac{3}{16}$ -inch perforations which removed about 9 percent of the original aileron area (reference 3). On the NACA 65₂-215 wing, identical ailerons of varying projection were used in tests of both the plug-aileron and retractable-aileron configurations; these ailerons were fastened to the upper surface of the wing at the 0.70c station (fig. 6). Although these ailerons were not projected out of the wing profile (from the neutral position) as they would be in a practical airplane installation (for example, the configurations on the NACA 65-210 wing), the configurations investigated are believed to simulate practical airplane installations and to provide aerodynamic data representative of these installations. (See fig. 6.)

TESTS

All tests of the basic plug-aileron, the basic retractable-aileron, and the thin-plate circular-plug-aileron configurations on the NACA 65-210 wing model were performed in the Langley high-speed 7- by 10-foot tunnel. All tests of the double-wall circular-plug-aileron configuration on the NACA 65-210 wing model and of the two aileron configurations on the NACA 65₂-215 wing model were performed in the Langley 300 MPH 7- by 10-foot tunnel.

With the flap retracted or deflected, the aerodynamic characteristics of each wing-aileron configuration were determined at various aileron projections and for several angles of attack. Tests were made at Mach numbers between 0.13 and 0.71 (with corresponding Reynolds numbers of 2.6×10^6 to 11.6×10^6 , based on the wing mean aerodynamic chord of 2.86 ft). Negative aileron projections indicate that the ailerons were extended above the wing upper surface.

The average variation of Reynolds number with Mach number for all tests is shown in figure 7.

RESULTS AND DISCUSSION

EFFECT OF AILERON BRAKES ON WING AERODYNAMIC CHARACTERISTICS

Incremental data of lift, drag, and pitching-moment coefficients obtained at various aileron projections with the four aileron configurations investigated on the NACA 65-210 wing and with the flap retracted and deflected are presented in figures 8 to 15. Corresponding data obtained with the two aileron configurations on the NACA 65₂-215 wing are presented in figures 16 to 23.

Incremental lift coefficient ΔC_L .—The incremental values of lift coefficient generally became more negative with increase in aileron projection for all aileron configurations and flap conditions. The data obtained with the basic retractable aileron, however, showed inconsistent trends of reversed or positive values of ΔC_L for small aileron projections at various angles of attack and Mach numbers with the flap deflected. (See figs. 11, 22, and 23.) This phenomenon is usually exhibited by retractable ailerons with the flap deflected. A comparison of the present data with the rolling-moment data of references 3 to 5 shows, as anticipated, that the aforementioned effects paralleled the rolling effectiveness of the ailerons; that is, the rolling effectiveness increased when ΔC_L became more negative.

In general, the incremental values of lift coefficient increased negatively with increases in the Mach and Reynolds numbers and with increase in the flap deflection for all configurations. In most cases, increases in the angle of attack produced a small or inconsistent effect on the values of ΔC_L produced by almost all the ailerons; the thin-plate circular-plug aileron on the NACA 65-210 wing was the only configuration for which ΔC_L became more negative with increased angle of attack (figs. 12 and 13).

The plug ailerons on both wings usually produced slightly larger negative values of ΔC_L than the retractable ailerons. This effect is consistent with the fact that the rolling effectiveness usually observed for the plug aileron is greater than that for the retractable aileron (references 3 and 5). A comparison of the plug-aileron and retractable-aileron data obtained on the two wings also shows that more negative values of ΔC_L were generally obtained on the thicker wing. (See figs. 8 to 11, 16, 19, 20, and 23.)

Incremental drag coefficient ΔC_D .—The incremental drag data of figures 8 to 23—which are based on the values of drag coefficient measured on the wings at approximately a constant angle of attack (at the values of α and C_L shown in the figures for zero aileron projection)—exhibit certain trends that accompanied the lift changes discussed in the section entitled "Incremental lift coefficient ΔC_L ." In most cases, the incremental values of drag coefficient increased with increase in aileron projection; however, at large values of lift coefficient with the flap deflected, an opposite trend was exhibited over a part of the projection range. The values of ΔC_D exhibited a negligible or inconsistent variation with increase in Mach and Reynolds numbers, except possibly at low negative values of C_L with flap retracted (figs. 8 and 10).

In general, the values of ΔC_D became considerably larger (or more positive) as the flap was deflected at a constant value of lift coefficient; however, an increase in the angle of attack and lift coefficient in any flap configuration generally caused a decrease in the values of ΔC_D . This decrease in ΔC_D became more pronounced with increase in aileron projection and with deflection of the flap and, in some instances, particularly at the higher lift coefficients with the flap deflected, the values of ΔC_D became negative. An analysis of the data shows that these trends result from the smaller positive increment in profile drag and the larger reduction in induced wing drag produced by projection of the ailerons as the angle of attack and lift coefficient increased.

For all practical purposes, however, some of the aforementioned changes in ΔC_D —particularly the decreases in the values of ΔC_D with increase in C_L , and the negative values of ΔC_D —would probably never be realized by an airplane in flight. The loss in lift resulting from projection of the aileron brakes on the airplane would probably have to be restored by an increase in the wing angle of attack to retain constant lift and avoid excessive accelerations and sinking speeds. This increase would result in approximately a constant induced drag and an increase in the total wing drag because of the larger profile drag resulting from the higher wing angle of attack and the projected aileron brakes. In order to illustrate the changes in ΔC_D obtained at constant lift coefficient for various aileron projections over the lift-coefficient range, some of the data of figures 8 to 23 have been analyzed and plotted as shown in figure 24. The values of ΔC_D resulting from aileron projection, with angle of attack varied to maintain constant lift coefficient, increased with increase in aileron projection and flap deflection and, for a given aileron projection, ΔC_D was usually fairly constant over the lift-coefficient range. In addition, the incremental drag values were negligibly affected by changes in Mach and Reynolds numbers.

The plug and retractable ailerons produced approximately similar values of ΔC_D on each wing model, but the two circular-plug ailerons generally produced the highest values of ΔC_D on the NACA 65-210 wing model. The data also show that more positive values of ΔC_D were usually obtained on the NACA 65₂-215 wing than on the NACA 65-210 wing at corresponding aileron projections and lift coefficients; however, the large changes in ΔC_D observed for the plug ailerons on the NACA 65₂-215 wing at small projections (figs. 16 to 19, and 24) result principally from the sudden opening of the plug slot rather than from the projection of the aileron alone, as was discussed for the lateral-control investigation reported in reference 5.

Incremental pitching-moment coefficient ΔC_m .—In general, the values of ΔC_m obtained at various aileron projections became more negative (or less positive) with increase in angle of attack in all flap conditions and became more negative with increase in aileron projection in the flap-retracted condition. Changes in the Mach and Reynolds

numbers generally had a negligible or an inconsistent effect on the values of ΔC_m obtained in all flap conditions, and the values of ΔC_m usually became less negative (or more positive) with increase in flap deflection. Because all values of ΔC_m were fairly small, however, the incremental wing pitching moments would probably be easily trimmed on an airplane, regardless of flap condition or aileron configuration.

EFFECT OF AILERON BRAKES ON AIRCRAFT PERFORMANCE

In order to illustrate the utility and one of the advantages to be gained from the changes in lift and drag produced by spoiler ailerons when used as glide-path controls on an airplane, the descent characteristics from an altitude of 40,000 feet of a typical high-performance airplane with and without glide-path controls were computed and are presented in figure 25. Unpublished wind-tunnel data obtained on the model of a high-performance propeller-driven airplane having four engines were used to determine the characteristics of the basic airplane. This airplane had a wing loading of 63 pounds per square foot, a wing aspect ratio of 10.18, and a wing taper ratio of 0.43; in addition, an effective thrust of zero in the flap-retracted condition was assumed. The glide-path controls assumed for the airplane were plug ailerons projected 8 percent chord above both wing panels, and the incremental data used for the glide-path controls were taken from data obtained on the NACA 65₂-215 wing (fig. 16). Although the NACA 65₂-215 wing had a lower aspect ratio than that of the assumed airplane, the lift coefficients at which the assumed airplane flew, in the illustrative example, were low enough to minimize differences in the induced drag and, hence, the incremental drag resulting from aileron projection. For other cases, however, particularly at high lift coefficients, differences in aspect ratio may cause appreciable drag differences which should be considered in a performance analysis. The airplane descent was assumed to start at an altitude of 40,000 feet and a Mach number of 0.7, and the airplane maintained this Mach number until an indicated airspeed of 450 miles per hour was reached. This indicated airspeed was then maintained for the remainder of the descent to sea level.

As can be seen from figure 25, projection of the plug aileron on both wing panels of the airplane decreased the time required to descend from 40,000 feet to sea level from 12.3 to 3.3 minutes. Also a decrease in the horizontal distance required to reach sea level of approximately 73 miles was effected (a straight line of descent was assumed). This saving of time and distance in descending from high altitudes would be particularly important for an emergency condition, such as failure in the cabin pressurization, and also for normal operating conditions, such as at the termination of a long-distance flight at the most efficient altitude (reference 12).

The illustrative example presented in the foregoing discussion was computed with the assumption that the airplane angle of attack was varied to maintain the proper lift coefficient with the flap retracted. This method of operation, however, may not be the most effective one. Deflection of the flap and ailerons simultaneously to provide the necessary lift coefficient at a constant angle of attack—and, at the same time, to increase the drag—or deflection of the ailerons with

the flaps deflected may result in larger decreases in the time and distance required to reach sea level than are shown in the illustrative example. However, other problems, such as downwash fluctuations in the region of the tail plane and excessive flap loads possibly encountered at high Mach number, may complicate or prevent such means of operation. For the illustrative example (wherein a -8-percent-chord aileron projection was employed), in order to maintain a constant α approximately a 10° deflection of the full-span flap would probably be required for simultaneous operation of flap and ailerons as glide-path controls. In general, the simultaneous use of the flaps and the aileron brakes will probably depend on the variation of lift and pitching moment desired for particular maneuvers, such as fighter combat maneuvers, in which only a drag increment is desired.

In addition to their action as glide-path controls, spoiler-aileron brakes provide the added advantage of decreasing wing bending moments by moving the spanwise center of loading inboard on the wings, as shown in figure 26. This effect is particularly important and beneficial for airplanes during descent at high speeds and lessens the possibility of structural failure during this maneuver.

Projection of the ailerons with the flaps deflected in a landing approach would also substantially aid the airplane in landing over high obstacles and on short landing fields and would appreciably decrease the length of landing run. The use of spoiler ailerons as speed brakes to limit or reduce airplane speed in a dive or to reduce airplane speed rapidly in order to increase firing efficiency of fighter aircraft is also feasible, as is apparent from the data previously presented.

Another functional advantage obtainable with spoiler-aileron brakes is the possible use of the ailerons as a gust-alleviation device. Because of the relatively greater adverse effects on passenger riding comfort and wing structural loads of gusts at high speeds, an automatic spoiler-aileron gust-alleviation system should be given due consideration.

Although a comparison of the characteristics of the spoiler-aileron speed brakes discussed herein with the characteristics of other brake devices (such as those of references 13 to 15) is not presented, several advantages of the aileron brakes are readily apparent. These advantages include: The variable braking control permitted by the aileron brakes as compared with the inflexibility of control of some of the other devices; the use of spoiler-aileron brakes would obviate the necessity of including separate braking devices on an airplane; the aileron brakes may be used, retracted into the wing, and immediately used again, but a parachute brake can be used only once before disposal or repacking (on the ground) and is inflexible in control; and also, the spoiler-aileron device would not adversely affect the effectiveness of adjoining wing controls, whereas other devices may (reference 15). In addition, unlike reversible-pitch propeller brakes (reference 16), spoiler-aileron brakes may be used on aircraft having diverse propulsive systems and would obviate any complexity involved in the use of propeller brakes on conventionally powered aircraft. The projected ailerons, when used as glide-path controls or speed brakes, probably would not cause severe tail buffeting inasmuch as the ailerons are placed on the outboard part of the wing near the tip, and the

wake formed by them would be outboard of the tail surfaces. Few data are available, however, concerning any induced effects of the aileron brakes on the wing downwash or on fluctuations in the downwash, and further investigation of such effects may be desirable.

ROLLING CHARACTERISTICS OF THE SPOILER AILERONS USED AS SPEED BRAKES OR GLIDE-PATH CONTROLS

In references 3 to 5, the rolling effectiveness of the plug and retractable ailerons on the NACA 65-210 and NACA 65₂-215 wings was shown to be very satisfactory for normal operation from the retracted aileron position. In order to illustrate the rolling effectiveness of these ailerons from a projected position—that is, when they are used as glide-path controls or speed brakes—some of the data previously presented in references 3 to 5 have been replotted, with zero rolling moment corresponding to some finite aileron projection on both semispans of a complete wing, and are presented in figures 27 to 29. The values of lift coefficient and angle of attack listed on these figures are those obtained with the ailerons in the raised position on both wing panels.

These data indicate that the rolling effectiveness produced by the plug and retractable ailerons from a projected-aileron neutral position was very satisfactory, particularly for the flap-deflected condition. For normal operation from the retracted aileron position, an aileron control-stick differential providing approximately equal up and down projections will probably be required for the plug ailerons; whereas a differential providing large up projections and little or no down projections for lateral control may be required for the retractable ailerons. However, when the ailerons are also used as speed brakes and glide-path controls, any extreme aileron control-stick differential normally employed for lateral control (such as that for the retractable ailerons) would probably have to change as the brakes project on both semispan wings, so that an aileron control-stick linkage allowing approximately equal up and down projections would be obtained for moderate brake projections on both wing panels.

CONCLUSIONS

A wind-tunnel investigation was made to determine the characteristics of plug and retractable ailerons used as speed brakes or glide-path controls on an NACA 65-210 and an NACA 65₂-215 wing equipped with full-span slotted flaps. The investigation was performed at various Mach numbers from 0.13 to 0.71. The results of the investigation led to the following conclusions:

1. The time for descent and distance for descent from high altitudes and wing bending moments can be greatly reduced by use of spoiler ailerons as brakes.
2. When used as speed brakes or glide-path controls, the rolling effectiveness of plug and retractable ailerons need not be impaired as compared with the effectiveness of the ailerons from the fully retracted position.
3. The incremental values of drag coefficient ΔC_D produced by projection of the ailerons on both wing panels of a complete wing generally became more positive with increase in aileron projection and flap deflection and were inconsistently or negligibly affected by changes in Mach

number. In addition, the ailerons generally produced larger increments of drag on the thicker wing model.

4. The increment in lift coefficient ΔC_L produced by projection of the ailerons generally became more negative with increase in aileron projection, flap deflection, and Mach and Reynolds numbers.

5. In general, the incremental values of pitching-moment coefficient ΔC_m produced by projection of the ailerons were fairly small, varied only slightly with changes in angle of attack, Mach number, aileron projection, or flap deflection and were about the same on both wing models.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., June 3, 1949.

REFERENCES

1. Rogallo, F. M.: Aerodynamic Characteristics of a Slot-Lip Aileron and Slotted Flap for Dive Brakes. NACA ACR, April 1941.
2. Fischel, Jack, and Ivey, Margaret F.: Collection of Test Data for Lateral Control with Full-Span Flaps. NACA TN 1404, 1948.
3. Fischel, Jack, and Schneider, Leslie E.: High-Speed Wind-Tunnel Investigation of an NACA 65-210 Semispan Wing Equipped with Plug and Retractable Ailerons and a Full-Span Slotted Flap. NACA TN 1663, 1948.
4. Fischel, Jack: Wind-Tunnel Investigation of an NACA 65-210 Semispan Wing Equipped with Circular Plug Ailerons and a Full-Span Slotted Flap. NACA TN 1802, 1949.
5. Fischel, Jack, and Vogler, Raymond D.: High-Lift and Lateral Control Characteristics of an NACA 65₂-215 Semispan Wing Equipped with Plug and Retractable Ailerons and a Full-Span Slotted Flap. NACA TN 1872, 1949.
6. Deters, Owen J., and Russell, Robert T.: Investigation of a Spoiler-Type Lateral Control System on a Wing with Full-Span Flaps in the Langley 19-Foot Pressure Tunnel. NACA TN 1409, 1947.
7. Fischel, Jack, and Tamburello, Vito: Investigation of Effect of Span, Spanwise Location, and Chordwise Location of Spoilers on Lateral Control Characteristics of a Tapered Wing. NACA TN 1294, 1947.
8. Fischel, Jack, and Schneider, Leslie E.: High-Speed Wind-Tunnel Investigation of High Lift and Aileron-Control Characteristics of an NACA 65-210 Semispan Wing. NACA TN 1473, 1947.
9. Swanson, Robert S., and Toll, Thomas A.: Jet-Boundary Corrections for Reflection-Plane Models in Rectangular Wind Tunnels. NACA Rep. 770, 1943.
10. Göethert, B.: Plane and Three-Dimensional Flow at High Subsonic Speeds. NACA TM 1105, 1946.
11. Thom, A.: Blockage Corrections in a Closed High-Speed Tunnel. R. & M. No. 2033, British A.R.C., 1943.
12. Shevell, Richard S.: Operational Aerodynamics of High-Speed Transport Aircraft. Jour. Aero. Sci., vol. 15, no. 3, March 1948, pp. 133-143.
13. Purser, Paul E., and Turner, Thomas R.: Wind-Tunnel Investigation of Perforated Split Flaps for Use as Dive Brakes on a Tapered NACA 23012 Airfoil. NACA ARR, Nov. 1941.
14. Purser, Paul E., and Turner, Thomas R.: Aerodynamic Characteristics and Flap Loads of Perforated Double Split Flaps on a Rectangular NACA 23012 Airfoil. NACA ARR, Jan. 1943.
15. Toll, Thomas A., and Ivey, Margaret F.: Wind-Tunnel Investigation of a Rectangular NACA 2212 Airfoil with Semispan Ailerons and with Nonperforated, Balanced Double Split Flaps for Use as Aerodynamic Brakes. NACA ARR L5B17, 1945.
16. Stone, Irving: Reversible Props Brake Plane in Dive. Aviation Week, Dec. 27, 1948, pp. 22 and 25.

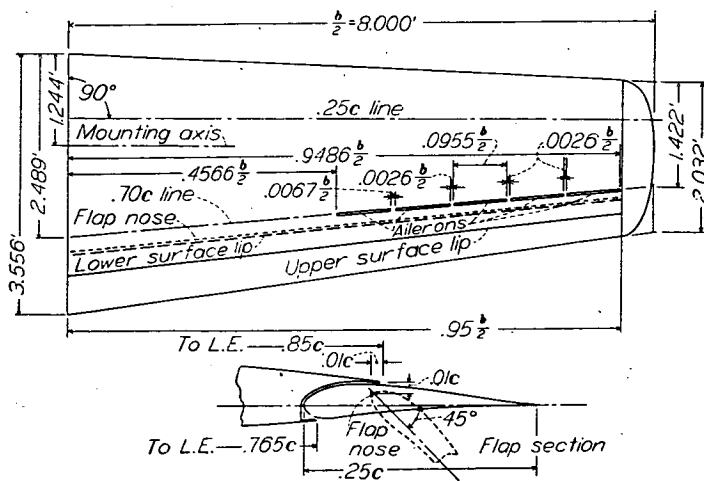


FIGURE 1.—Plan-form drawing of NACA 65-210 semispan wing model equipped with basic plug and retractable ailerons and a full-span slotted flap.

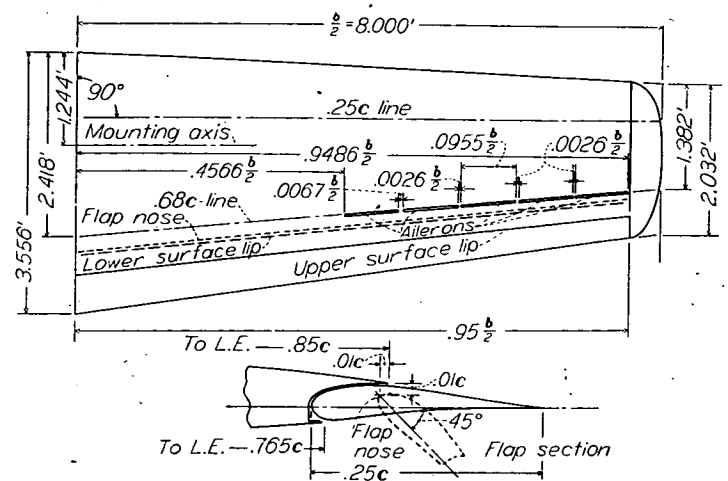
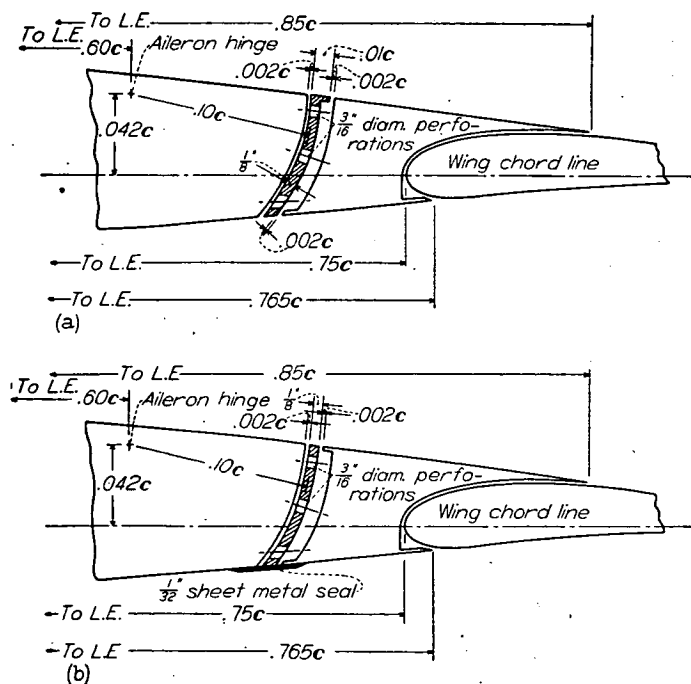
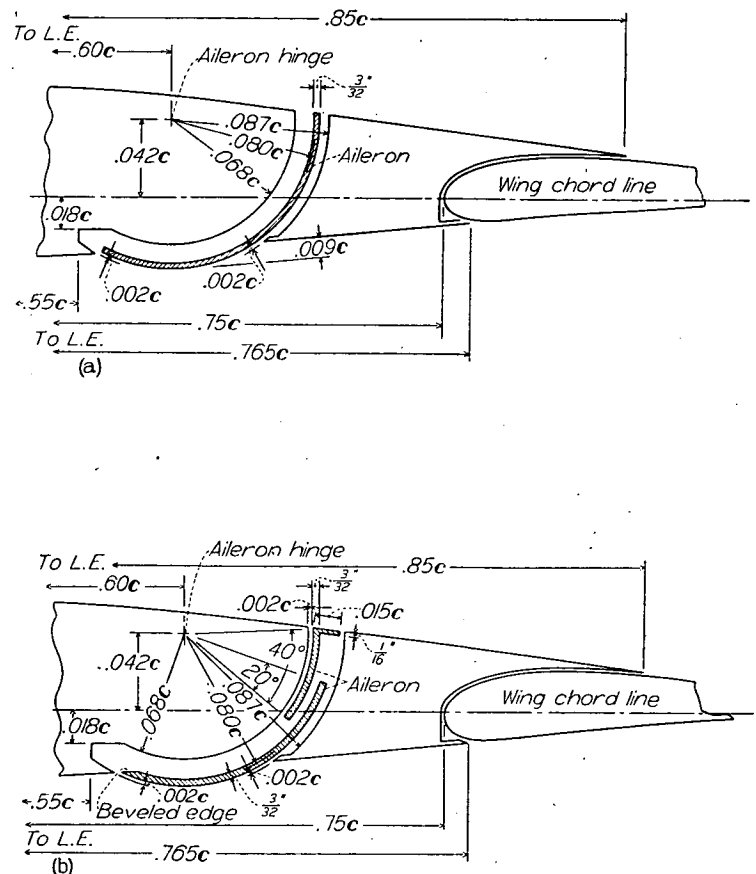


FIGURE 3.—Plan-form drawing of NACA 65-210 semispan wing model equipped with circular plug ailerons and a full-span slotted flap.



(a) Basic plug-aileron configuration.
(b) Basic retractable-aileron configuration.

FIGURE 2.—Section drawings of basic plug-aileron and retractable-aileron configurations investigated on NACA 65-210 wing model.



(a) Thin-plate circular-plug-aileron configuration.
(b) Double-wall circular-plug-aileron configuration.

FIGURE 4.—Section drawings of thin-plate and double-wall circular-plug-aileron configurations investigated on NACA 65-210 wing model.

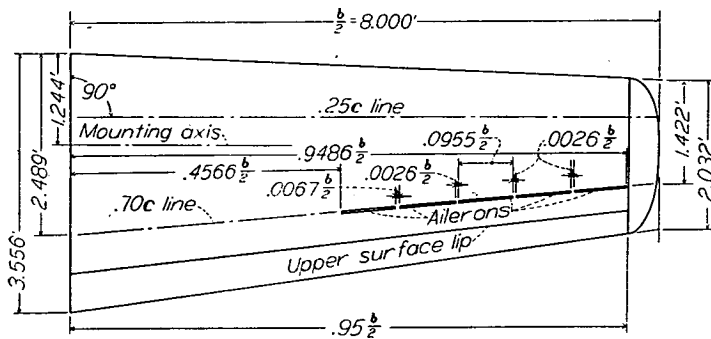
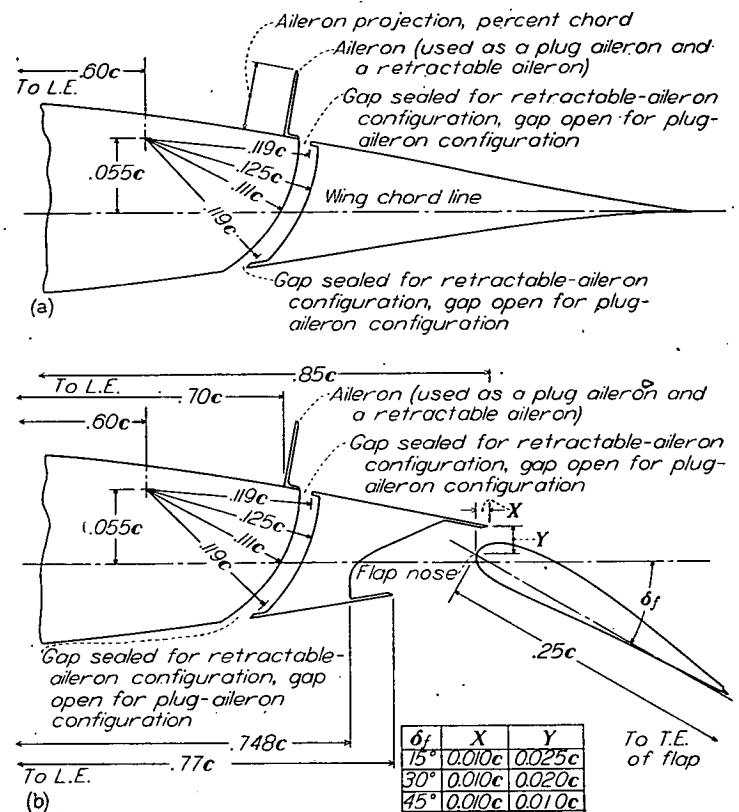


FIGURE 5.—Plan-form drawing of NACA 65-215 semispan wing model equipped with plug and retractable ailerons and a full-span slotted flap.



(a) Aileron configuration on the plain wing.

(b) Aileron configuration with flap deflected.

FIGURE 6.—Section drawings of plug-aileron and retractable-aileron configurations investigated on NACA 65-215 wing model.

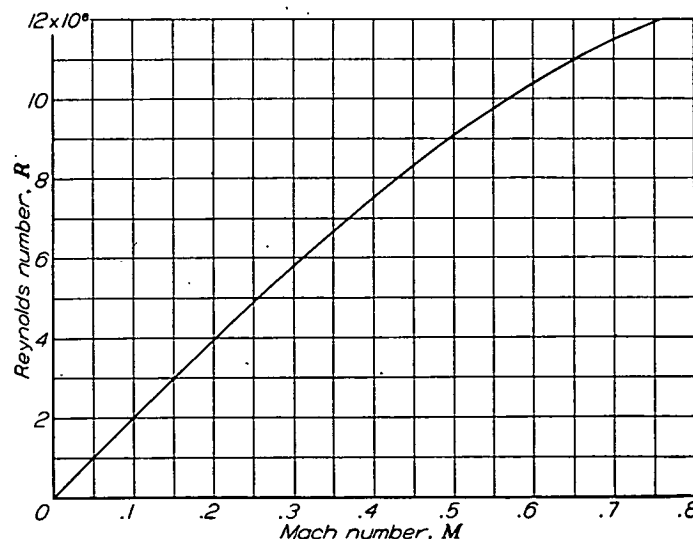


FIGURE 7.—Average variation of Reynolds number with Mach number. Reynolds number is based on wing mean aerodynamic chord of 2.86 feet.

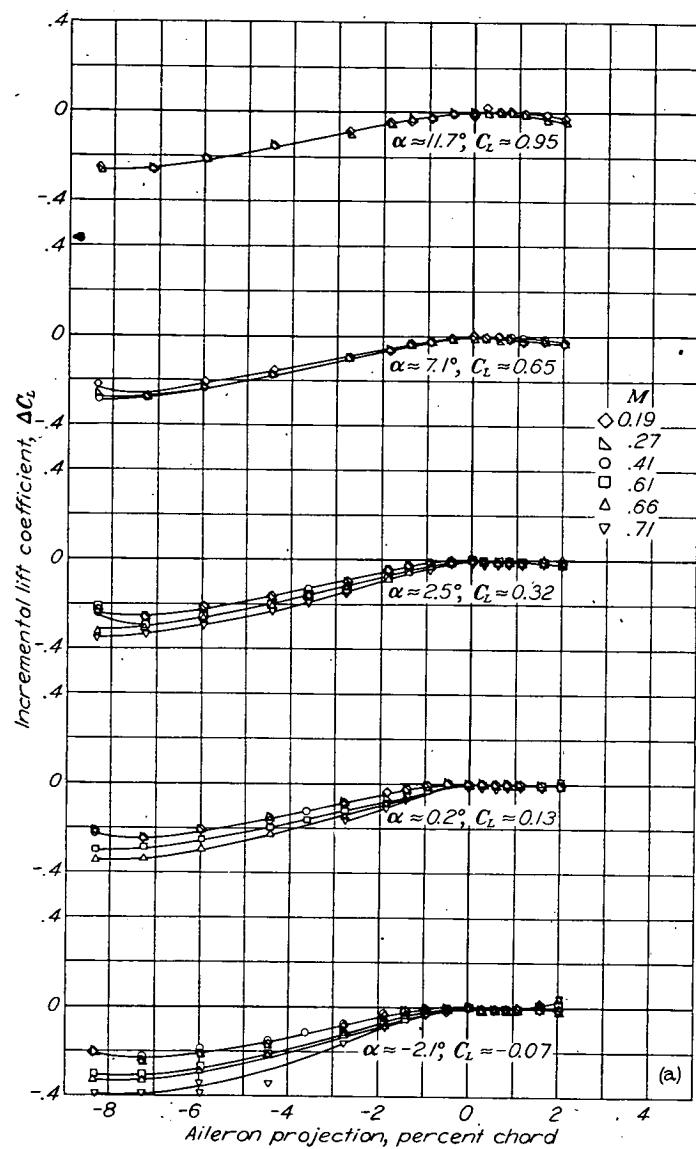


FIGURE 8.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the basic plug aileron on both semispans of the NACA 65-210 wing. Flap retracted.

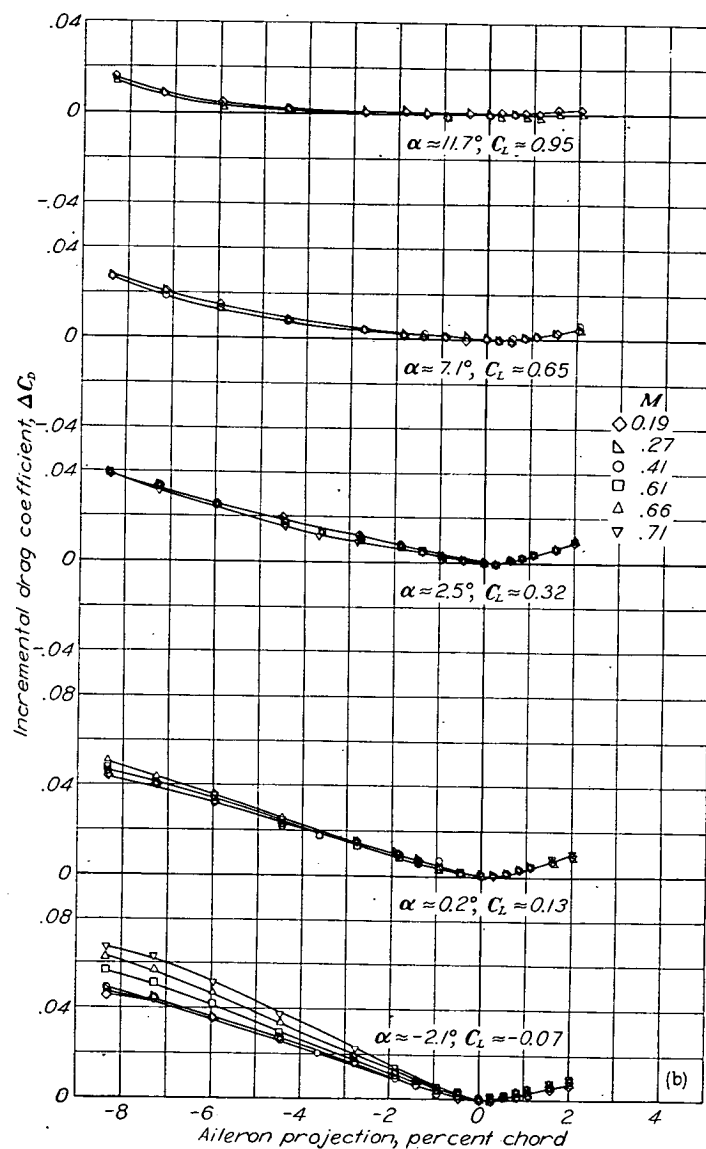
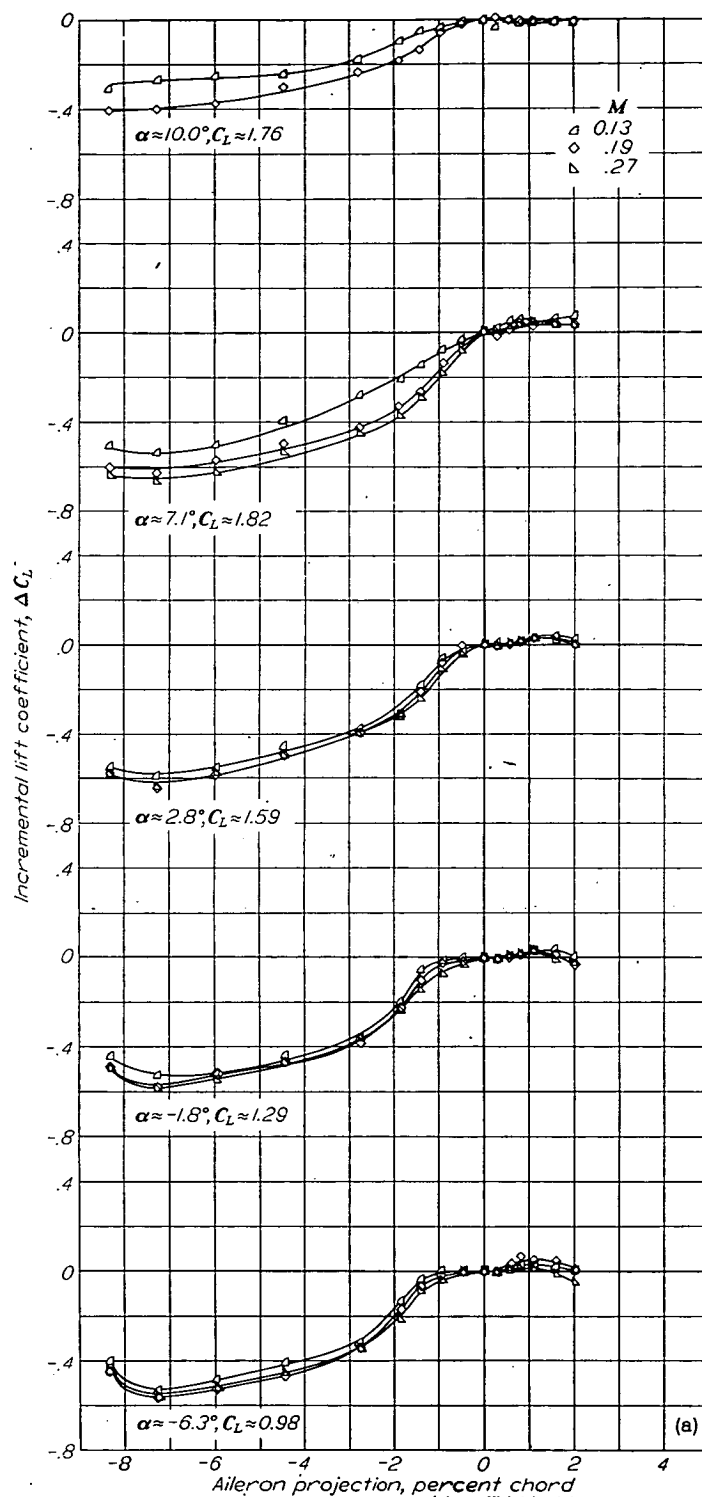
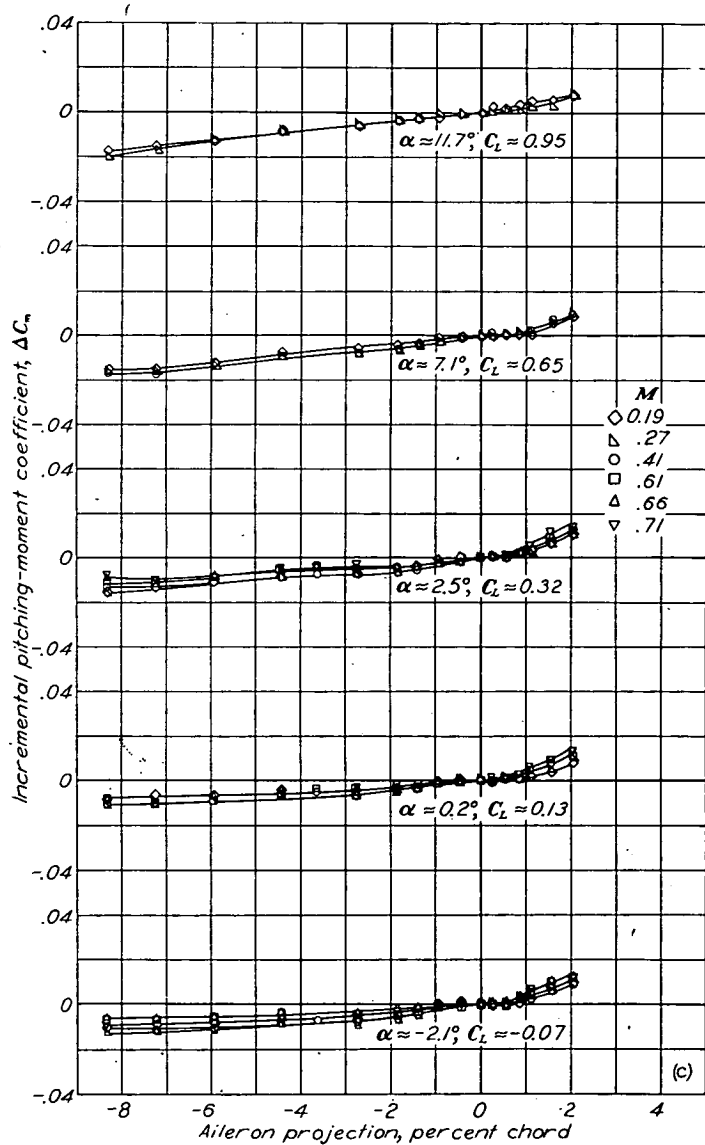


FIGURE 8.—Continued.



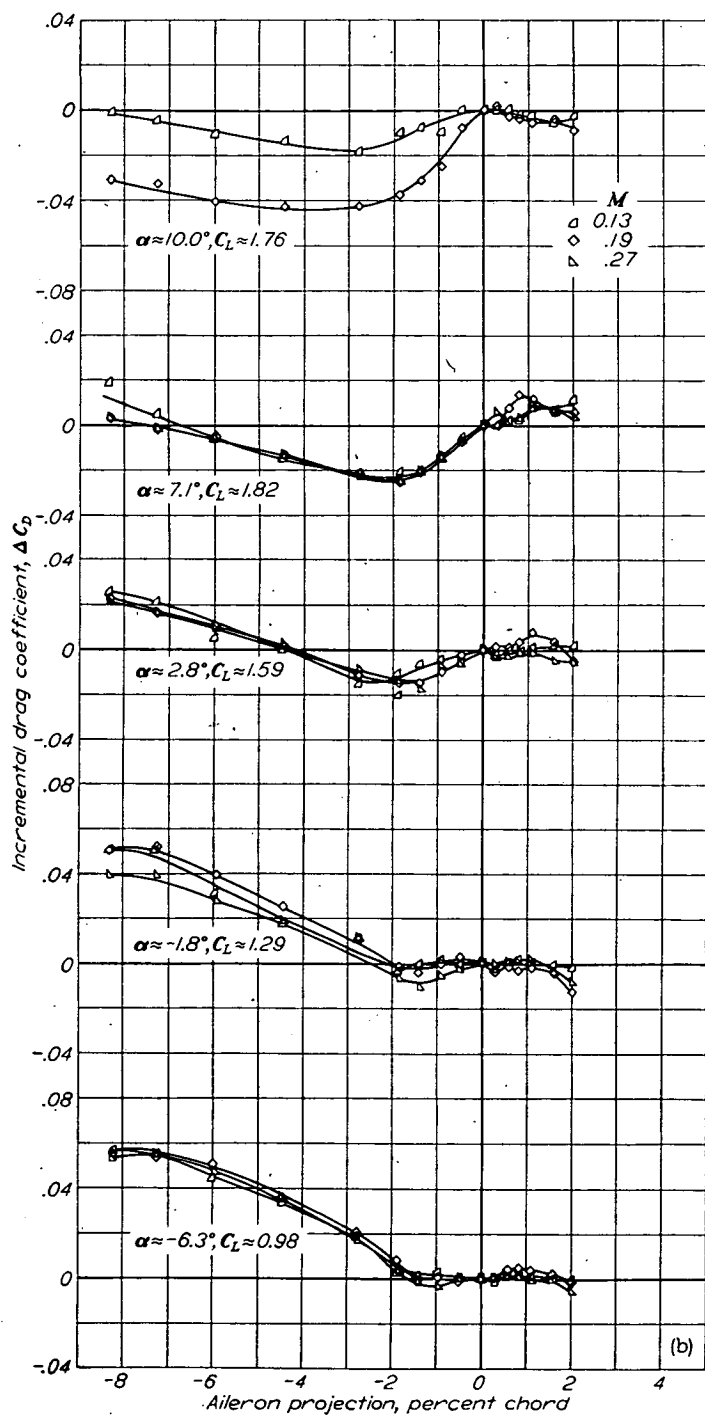


FIGURE 9—Continued.

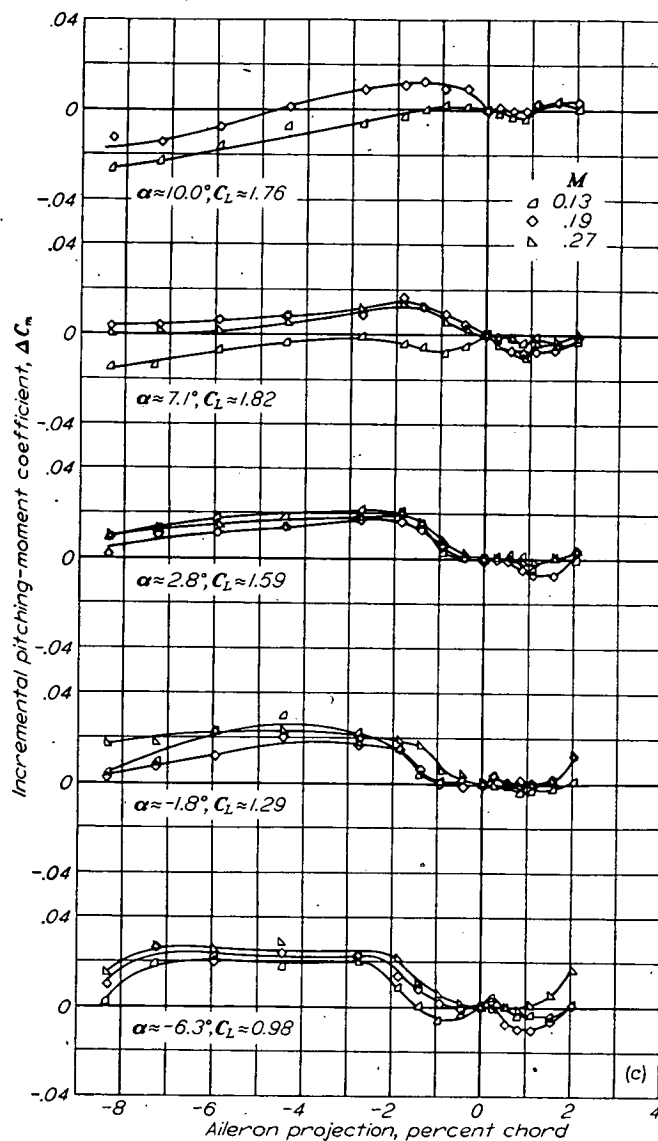


FIGURE 9.—Concluded.

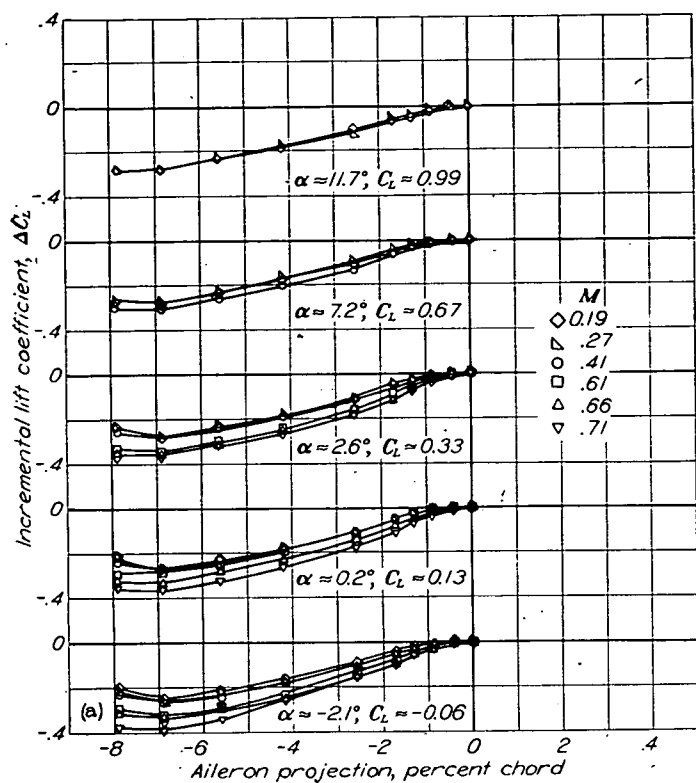


FIGURE 10.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the basic retractable aileron on both semispans of the NACA 65-210 wing. Flap retracted.

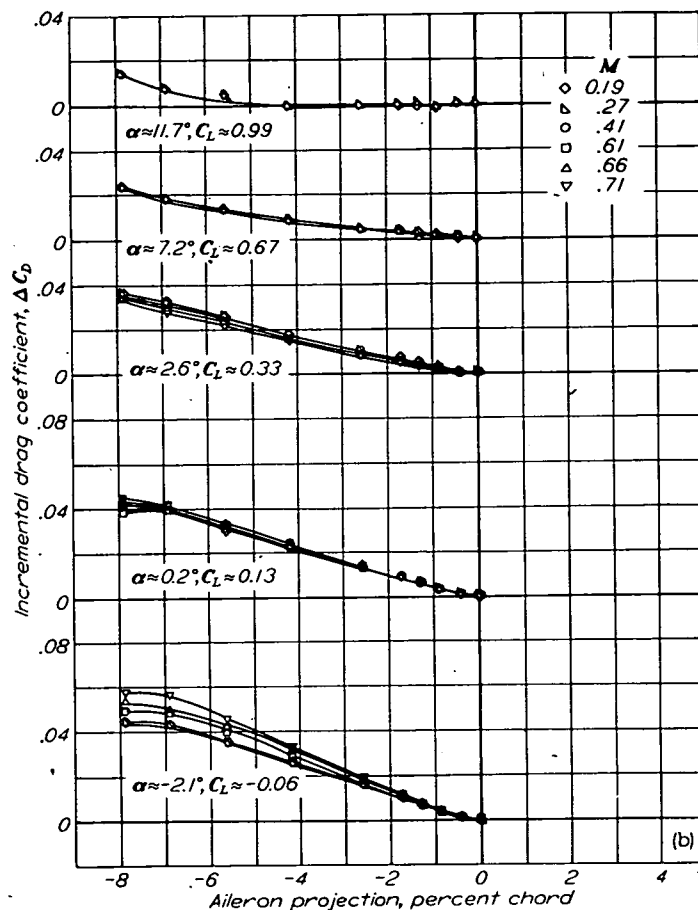


FIGURE 10.—Continued

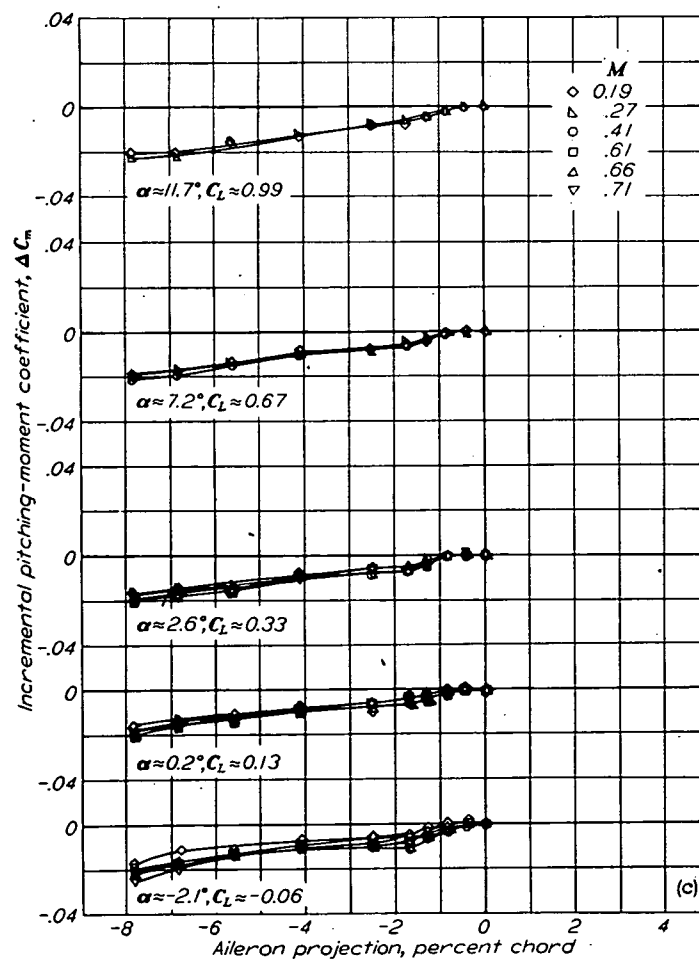


FIGURE 10.—Concluded.

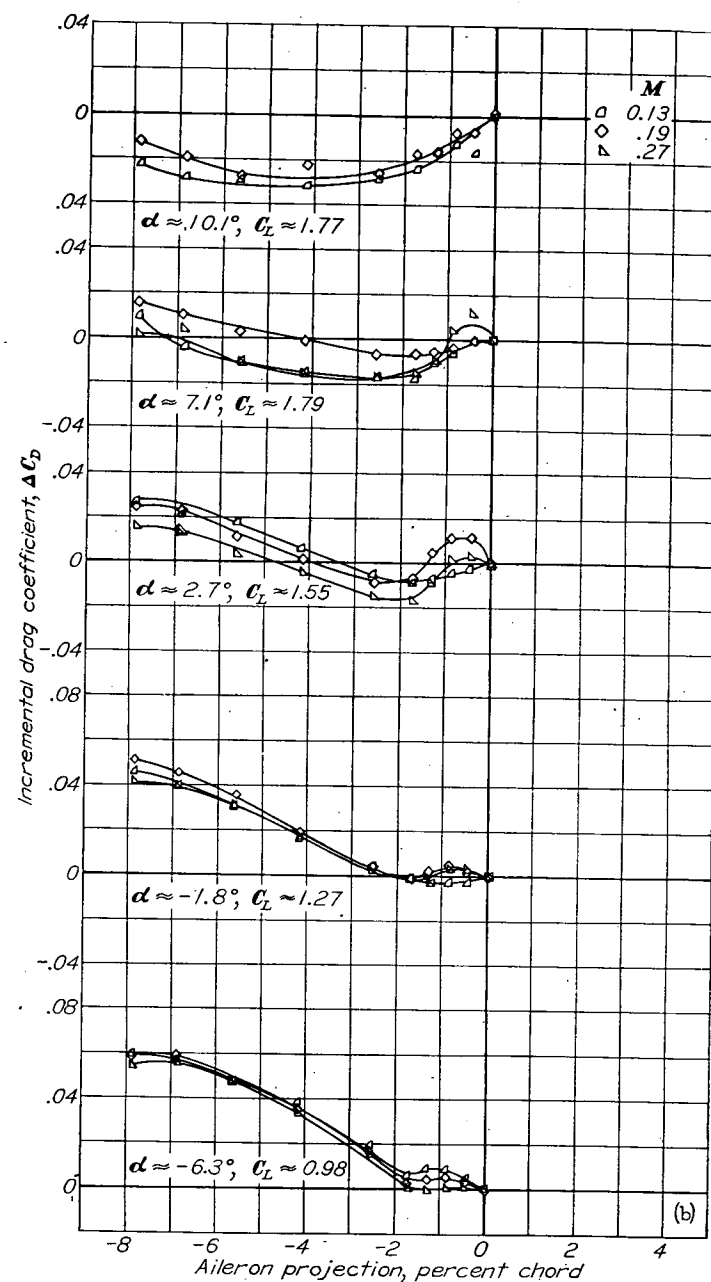
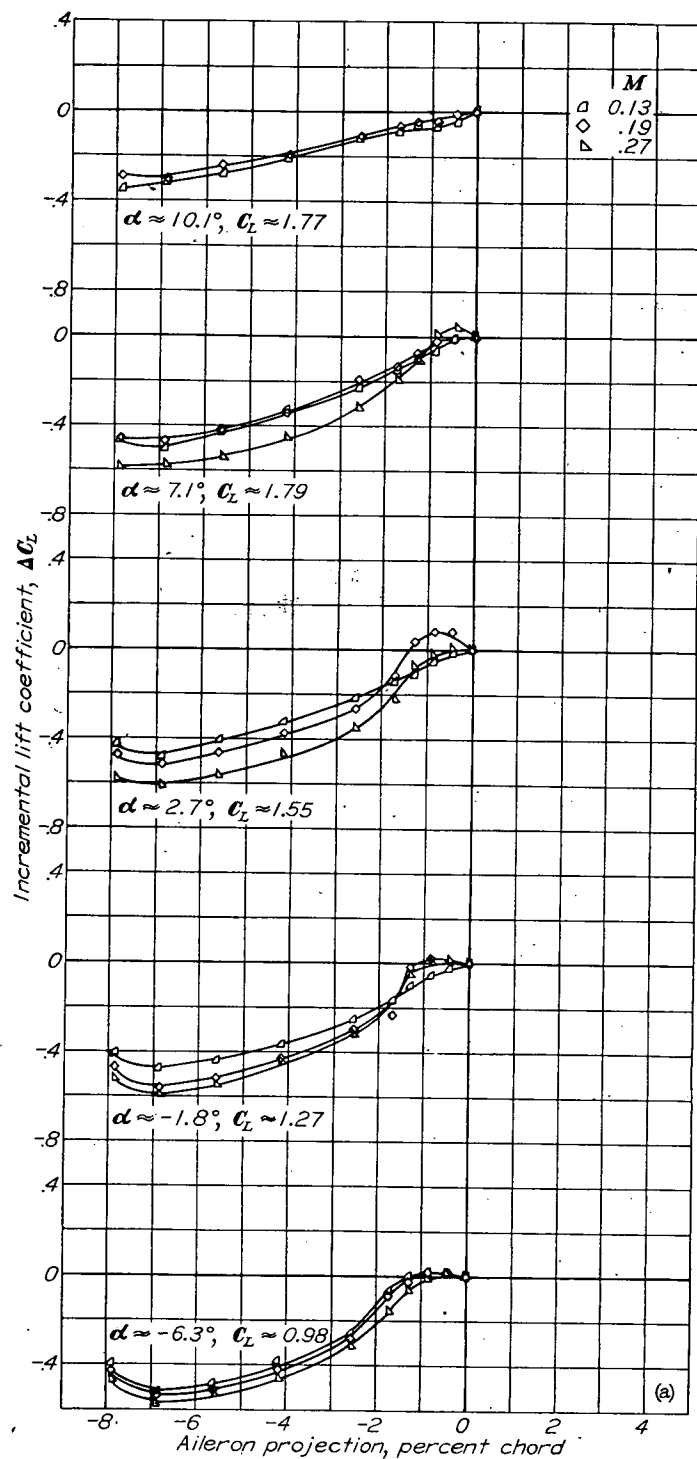


FIGURE 11.—Continued.

FIGURE 11.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the basic retractable aileron on both semispans of the NACA 65-210 wing. Flap deflected 45° .

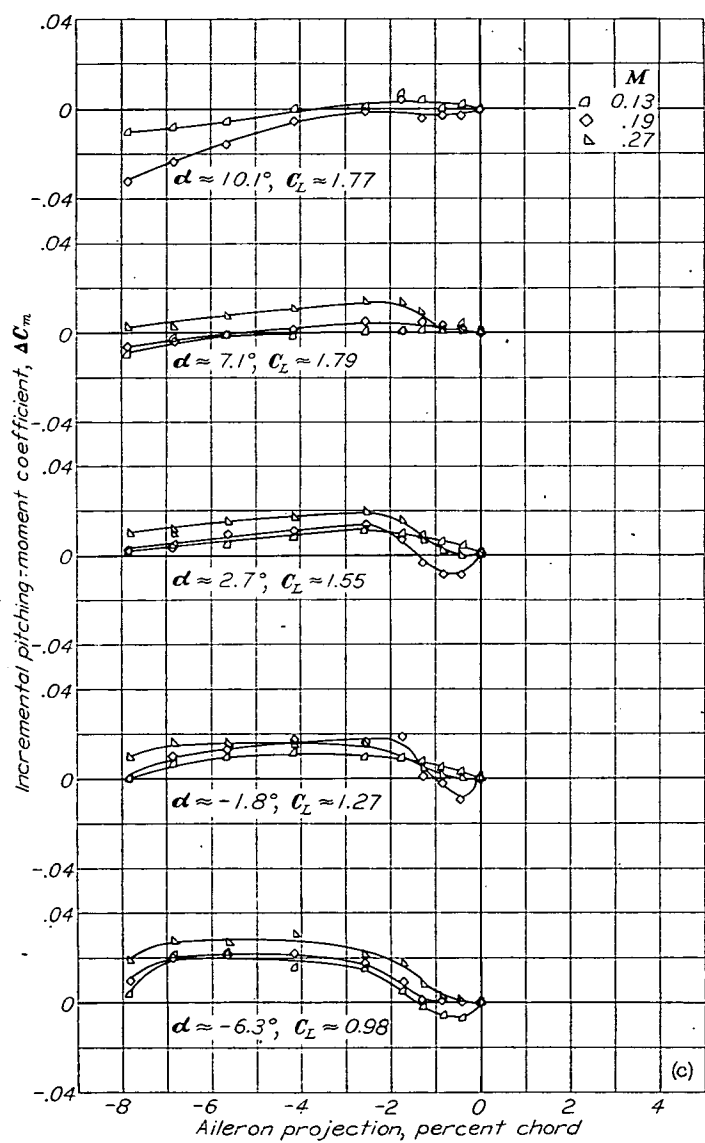


FIGURE 11.—Concluded.

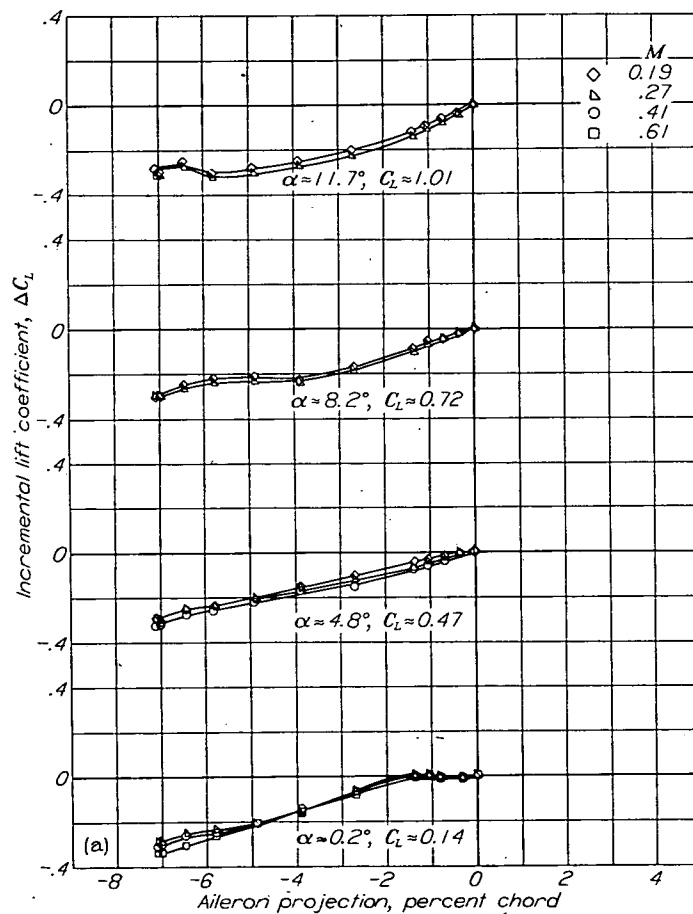


FIGURE 12.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the thin-plate circular plug aileron on both semispans of the NACA 65-210 wing. Flap retracted.

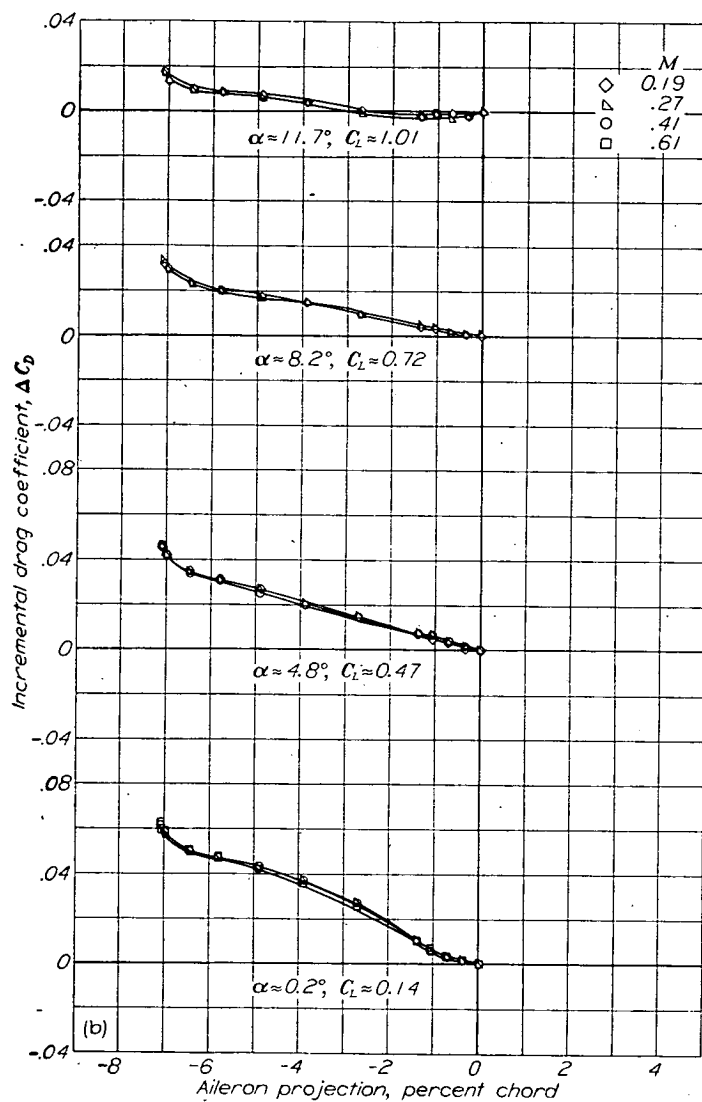


FIGURE 12.—Continued.

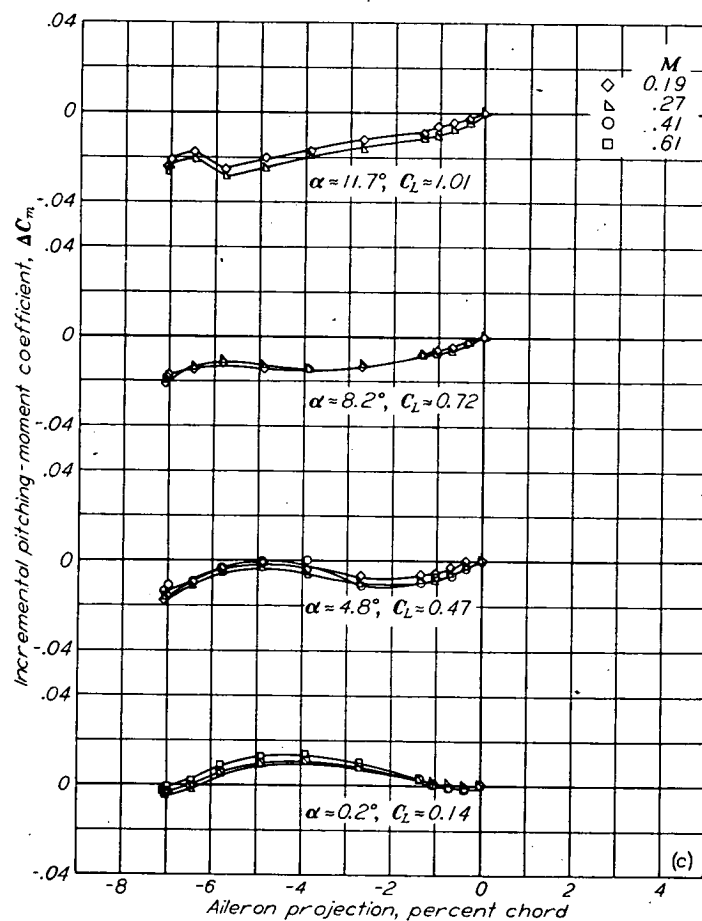


FIGURE 12.—Concluded.

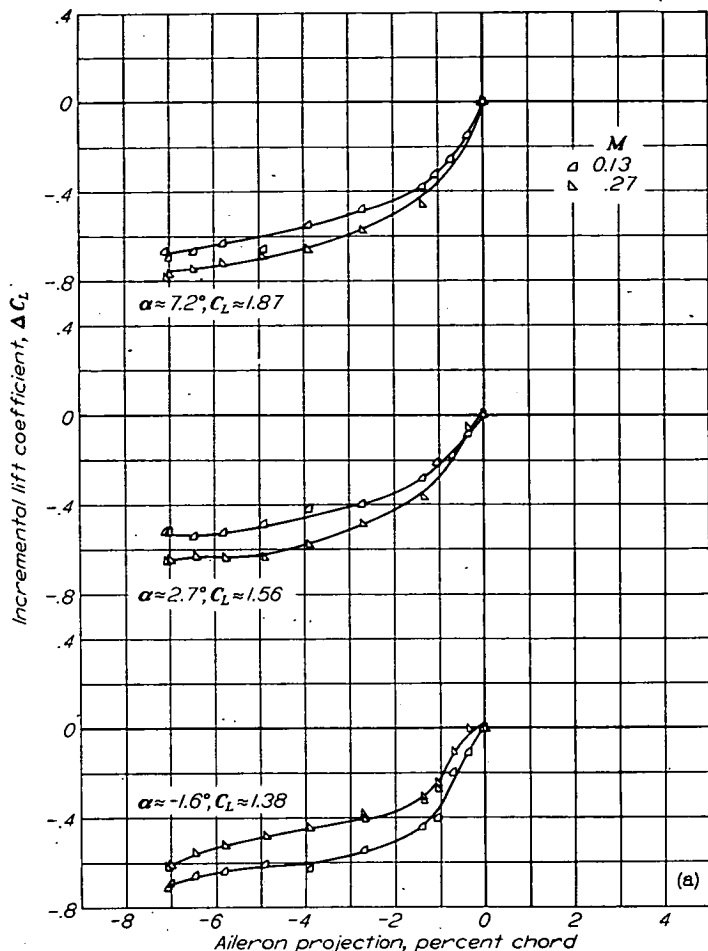


FIGURE 13.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the thin-plate circular plug aileron on both semispans of the NACA 65-210 wing. Flap deflected 45° .

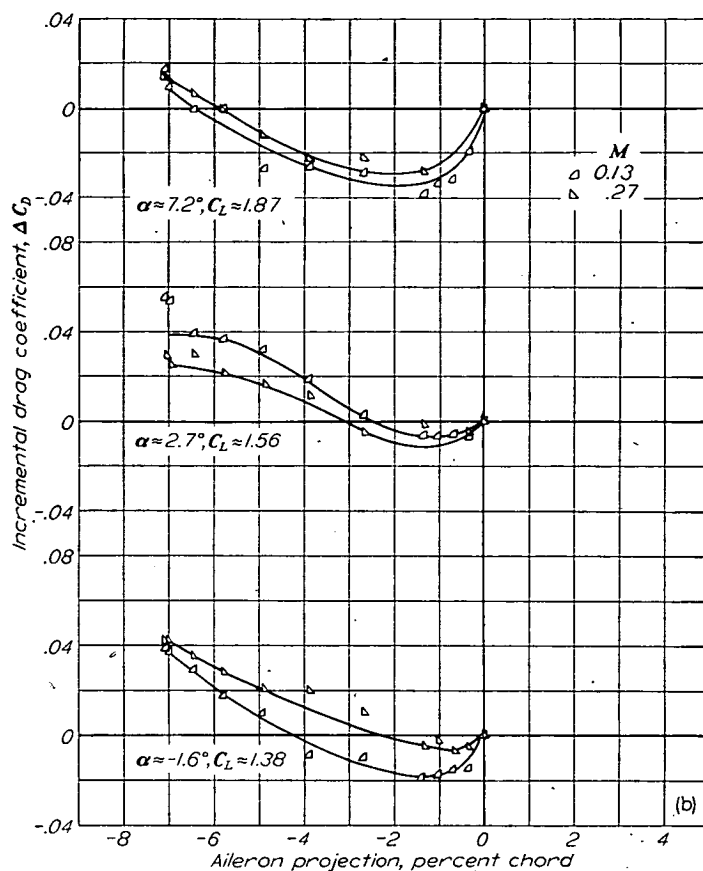


FIGURE 13.—Continued.

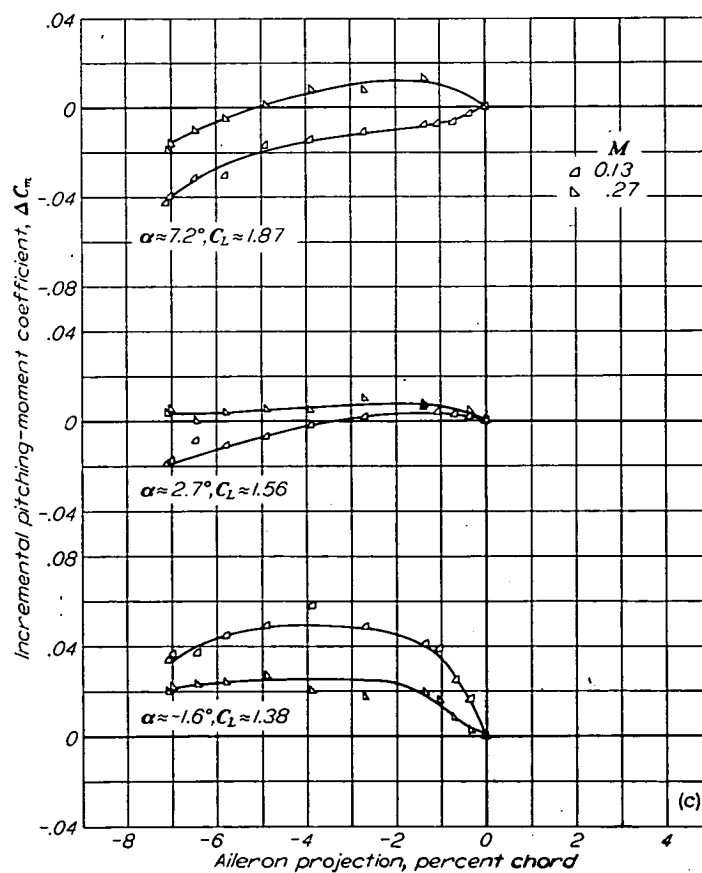


FIGURE 13.—Concluded.

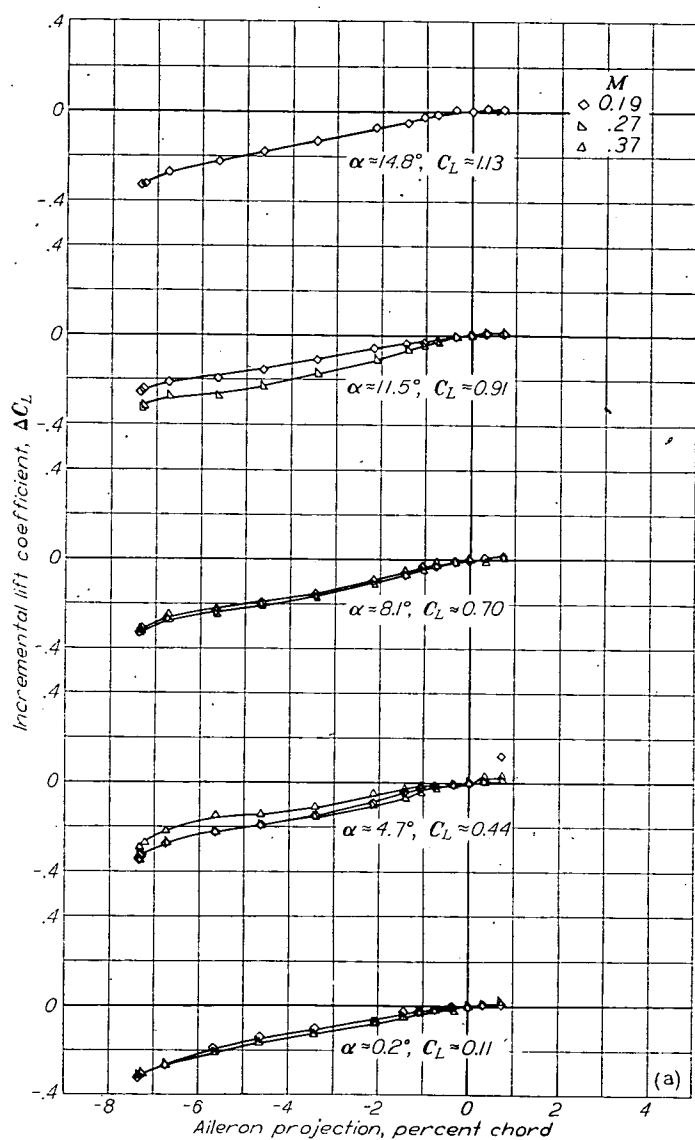


FIGURE 14.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the double-wall circular plug aileron on both semispans of the NACA 65-210 wing. Flap retracted.

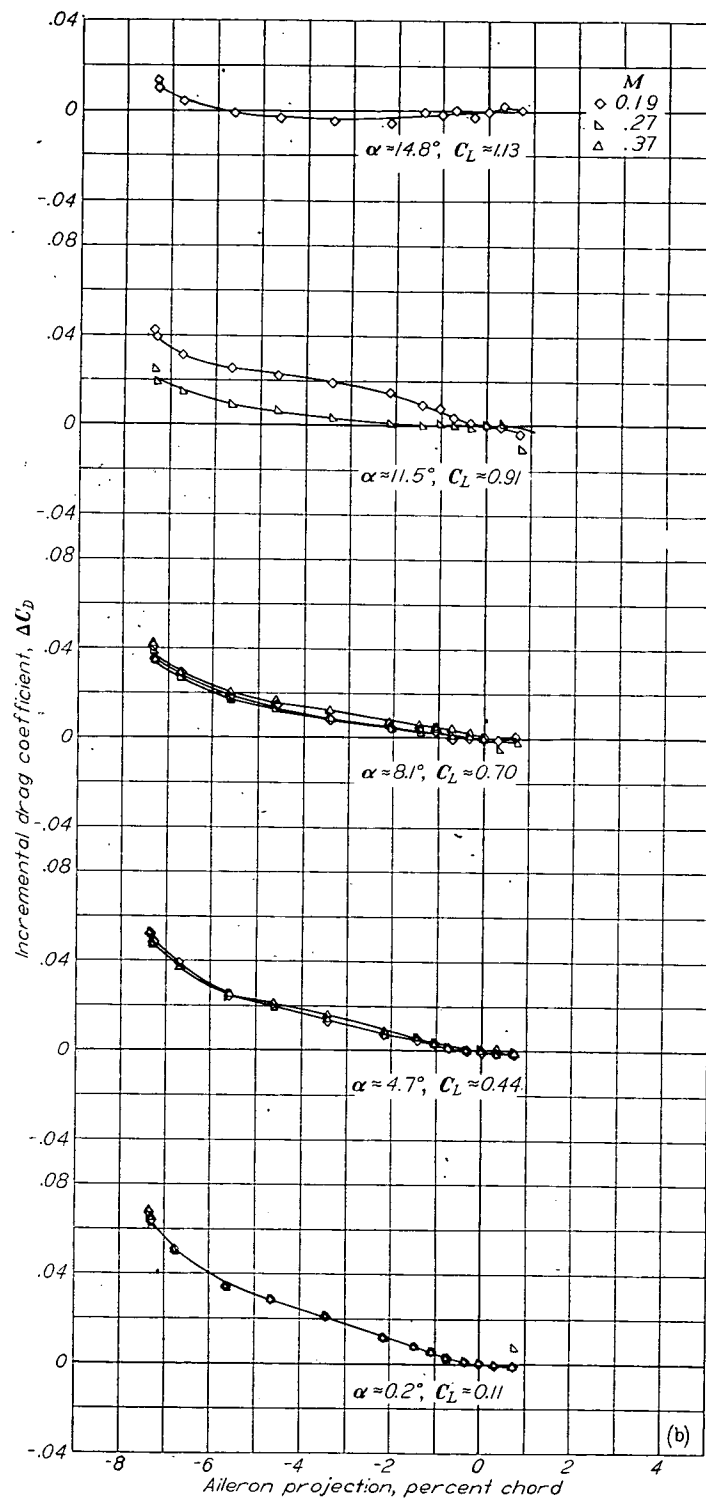


FIGURE 14—Continued.

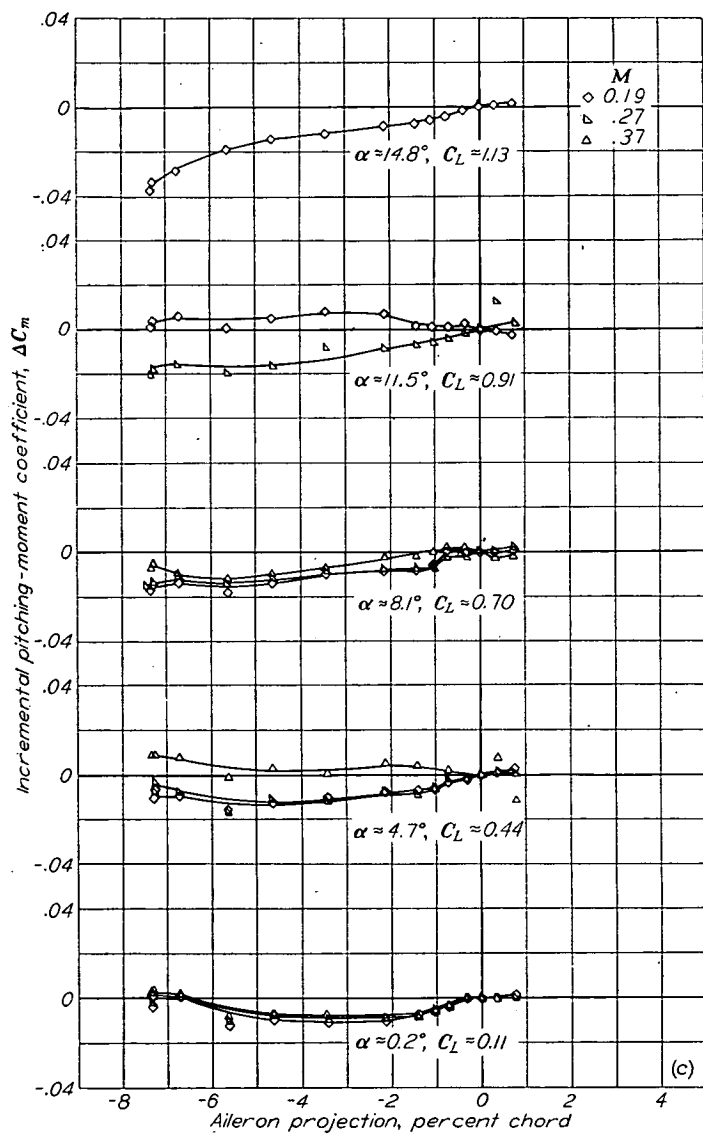
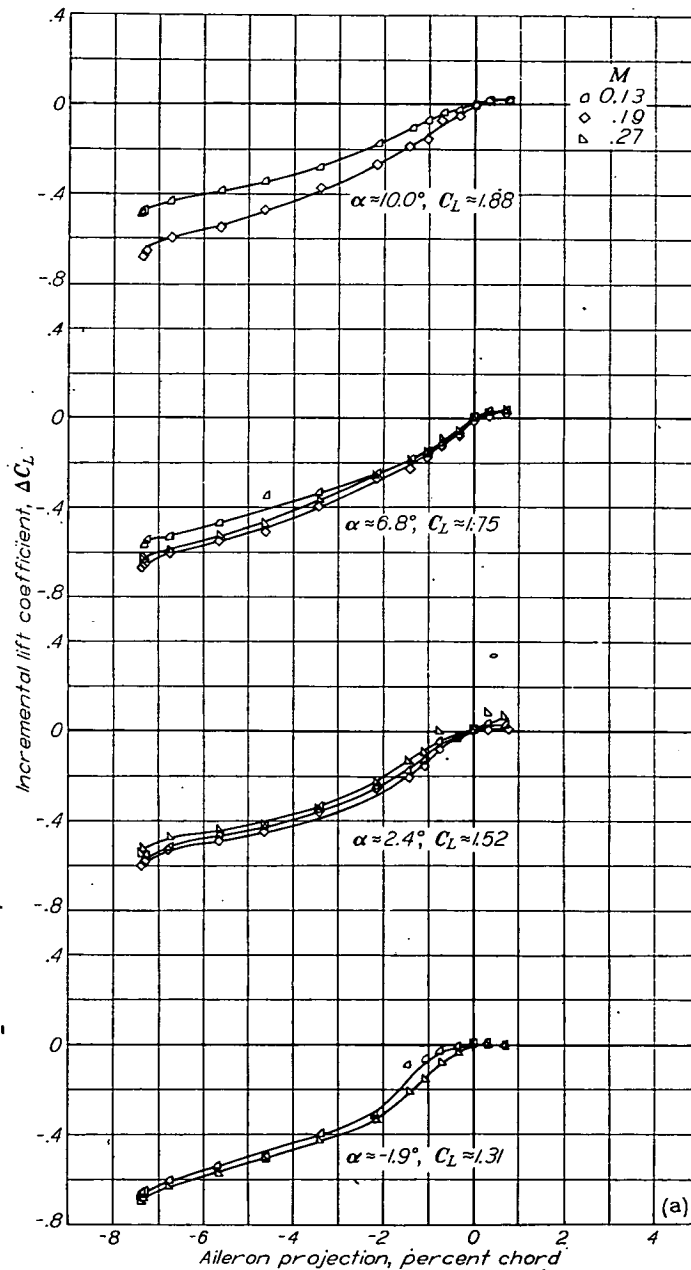


FIGURE 14.—Concluded.


 FIGURE 15.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the double-wall circular plug aileron on both semispans of the NACA 65-210 wing. Flap deflected 45° .

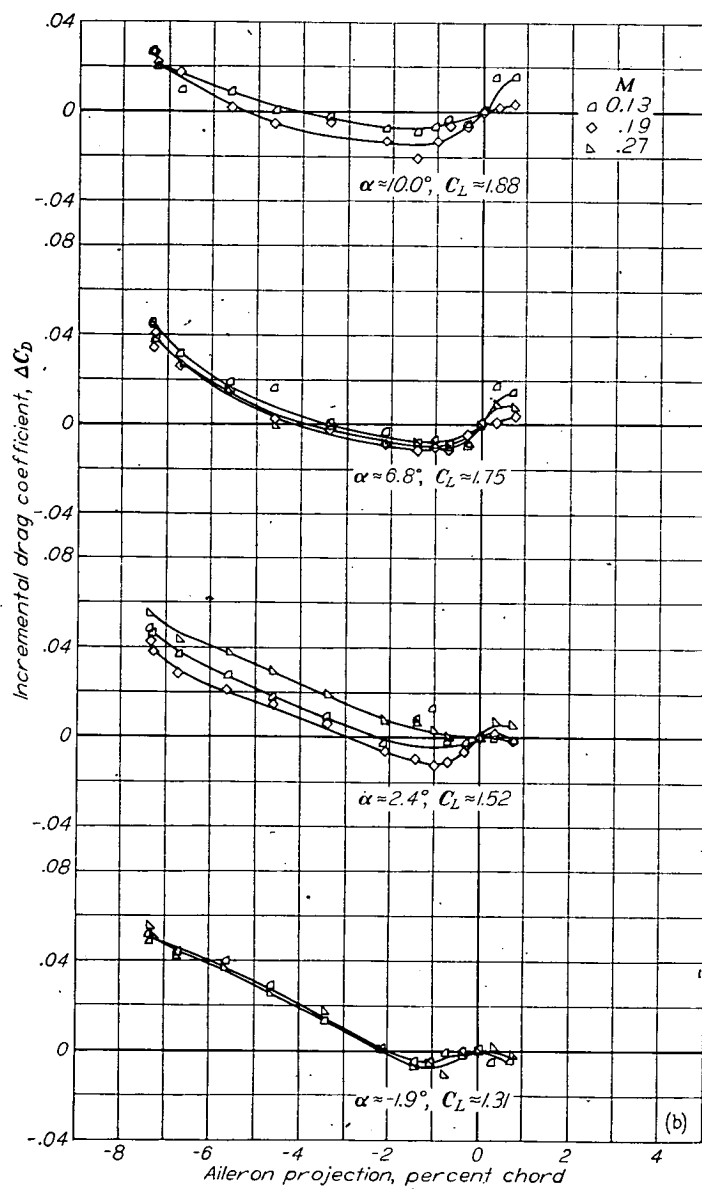


FIGURE 15.—Continued.

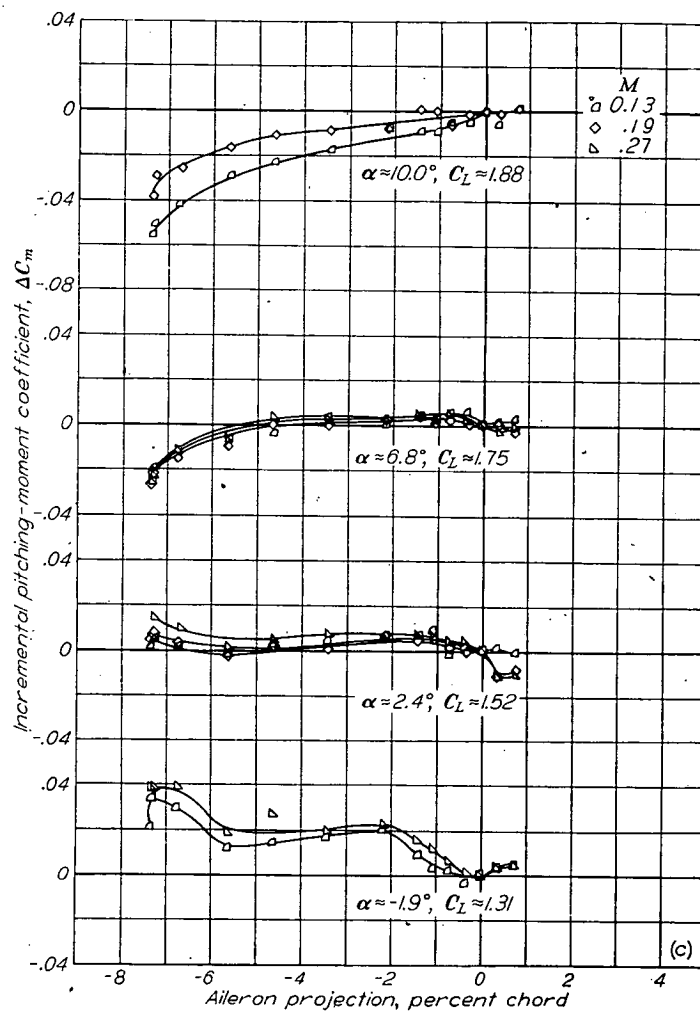


FIGURE 15.—Concluded.

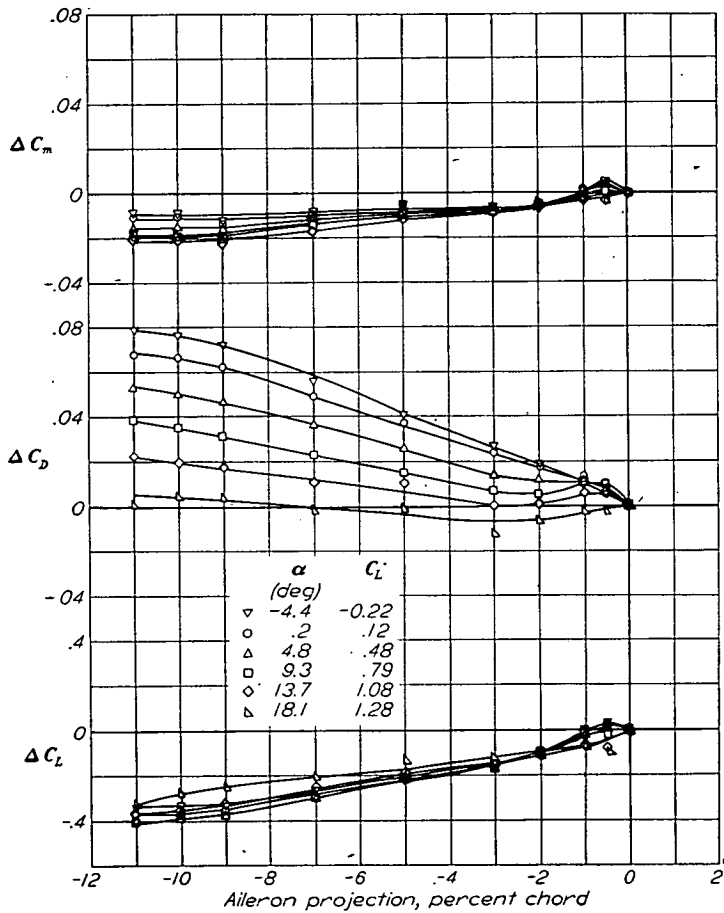


FIGURE 16.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the plug aileron on both semispans of the NACA 65-215 wing. Plain wing; $M=0.19$.

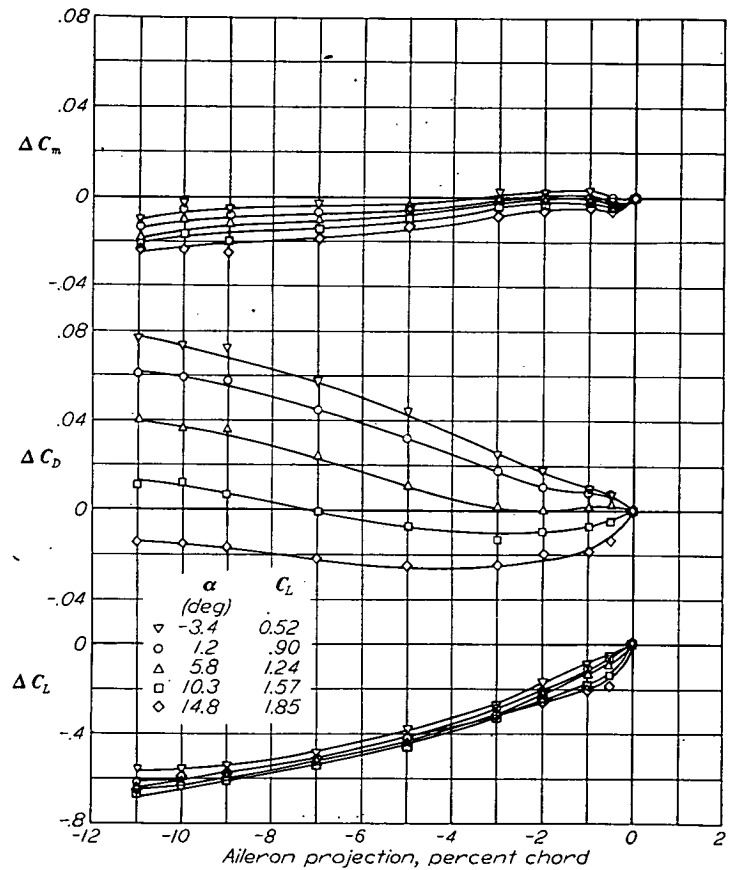


FIGURE 17.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the plug aileron on both semispans of the NACA 65-215 wing. Flap deflected 15°; $M=0.13$.

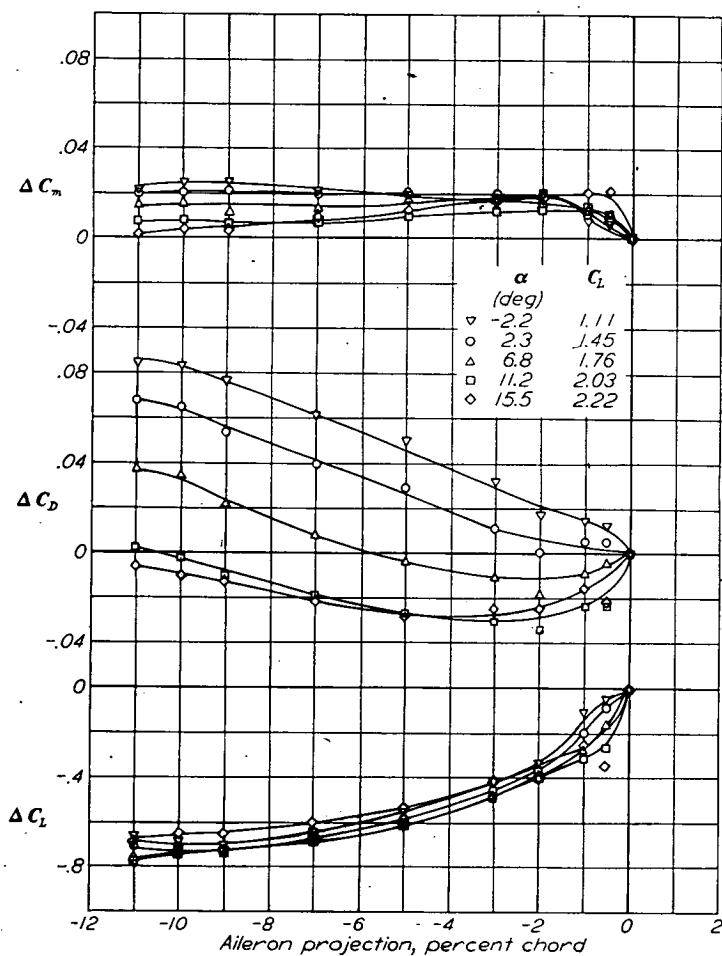


FIGURE 18.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the plug aileron on both semispans of the NACA 65-215 wing. Flap deflected 30°; $M=0.13$.

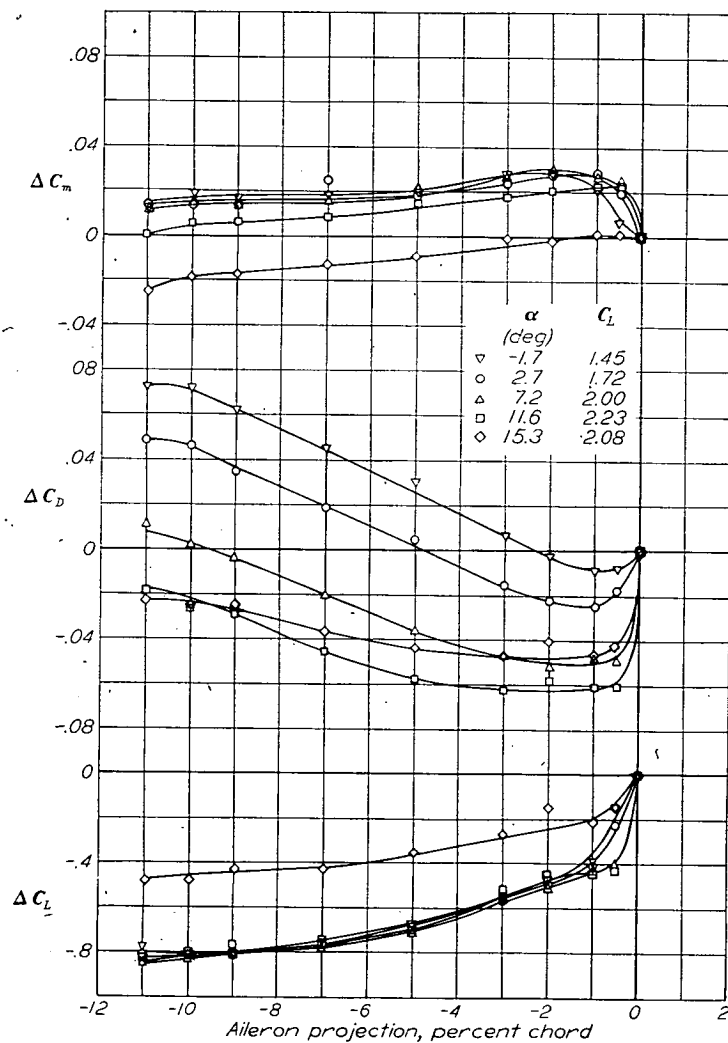


FIGURE 19.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the plug aileron on both semispans of the NACA 65-215 wing. Flap deflected 45°; $M=0.13$.

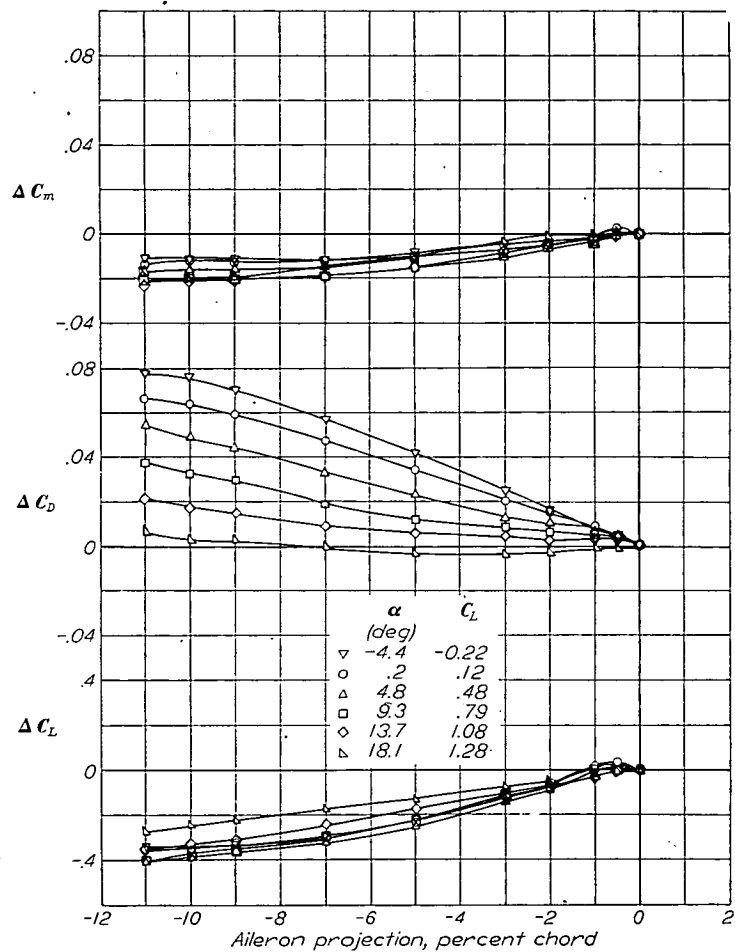


FIGURE 20.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the retractable aileron on both semispans of the NACA 65-215 wing. Plain wing; $M=0.19$.

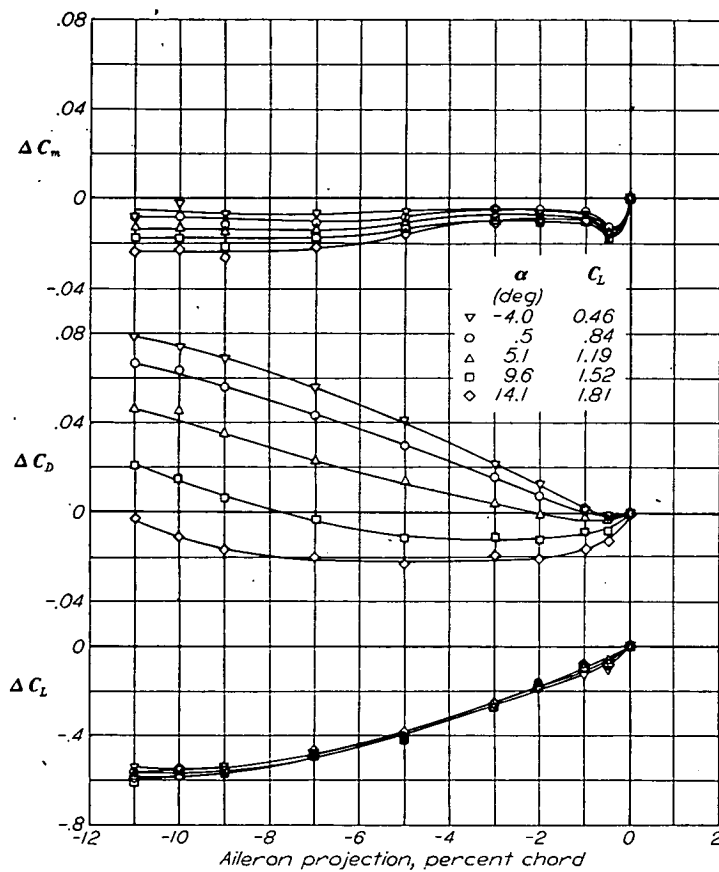


FIGURE 21.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the retractable aileron on both semispans of the NACA 65-215 wing. Flap deflected 15°; $M=0.13$.

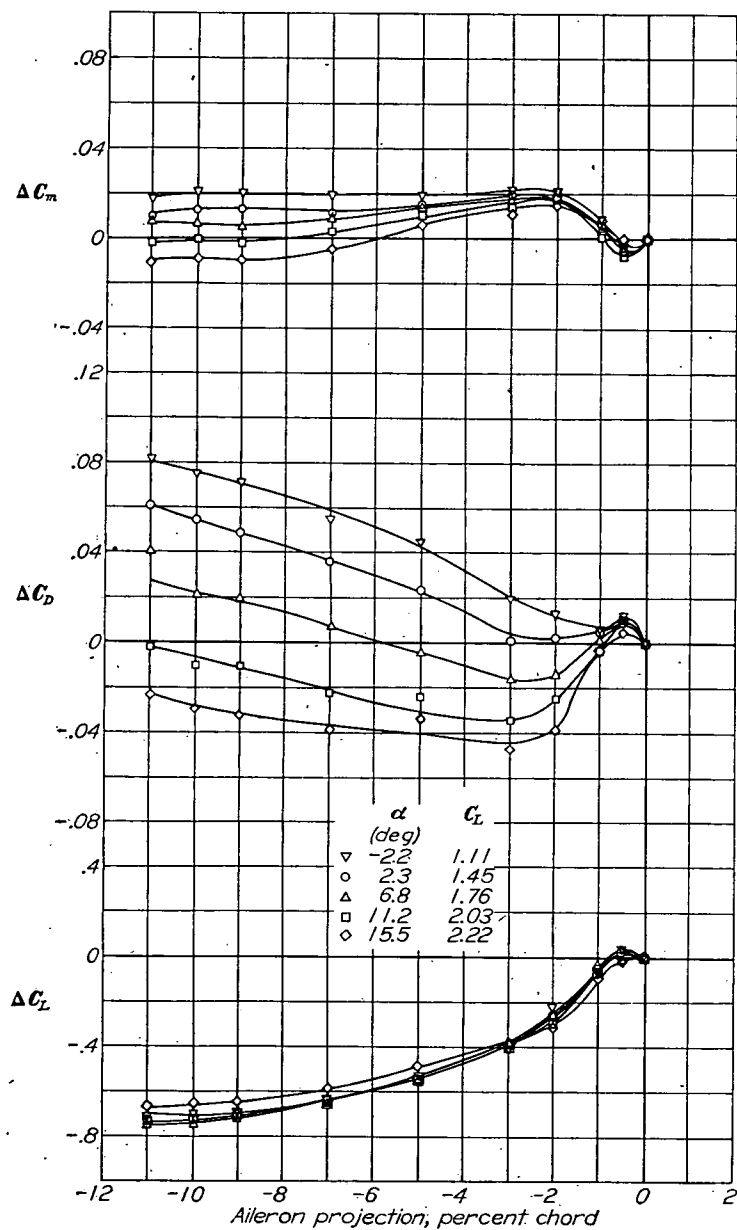


FIGURE 22.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the retractable aileron on both semispans of the NACA 65-215 wing. Flap deflected 30°; $M=0.13$.

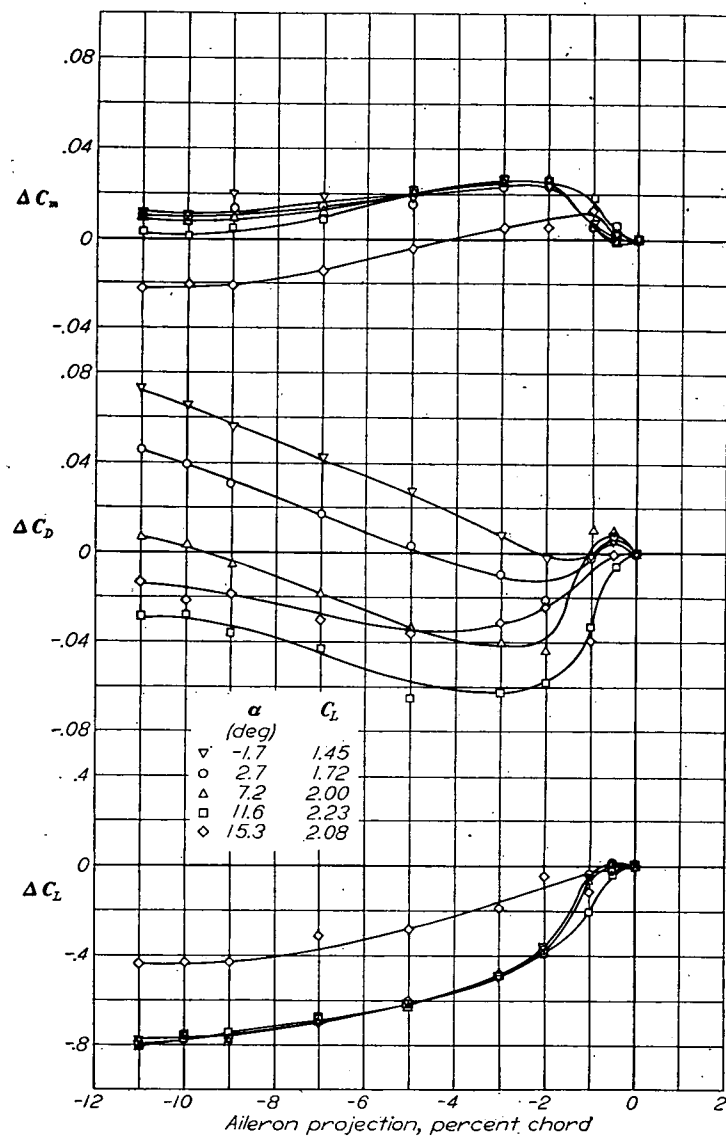


FIGURE 23.—Incremental values of lift, drag, and pitching-moment coefficients obtained by projection of the retractable aileron on both semispans of the NACA 65-215 wing. Flap deflected 45°; $M=0.13$.

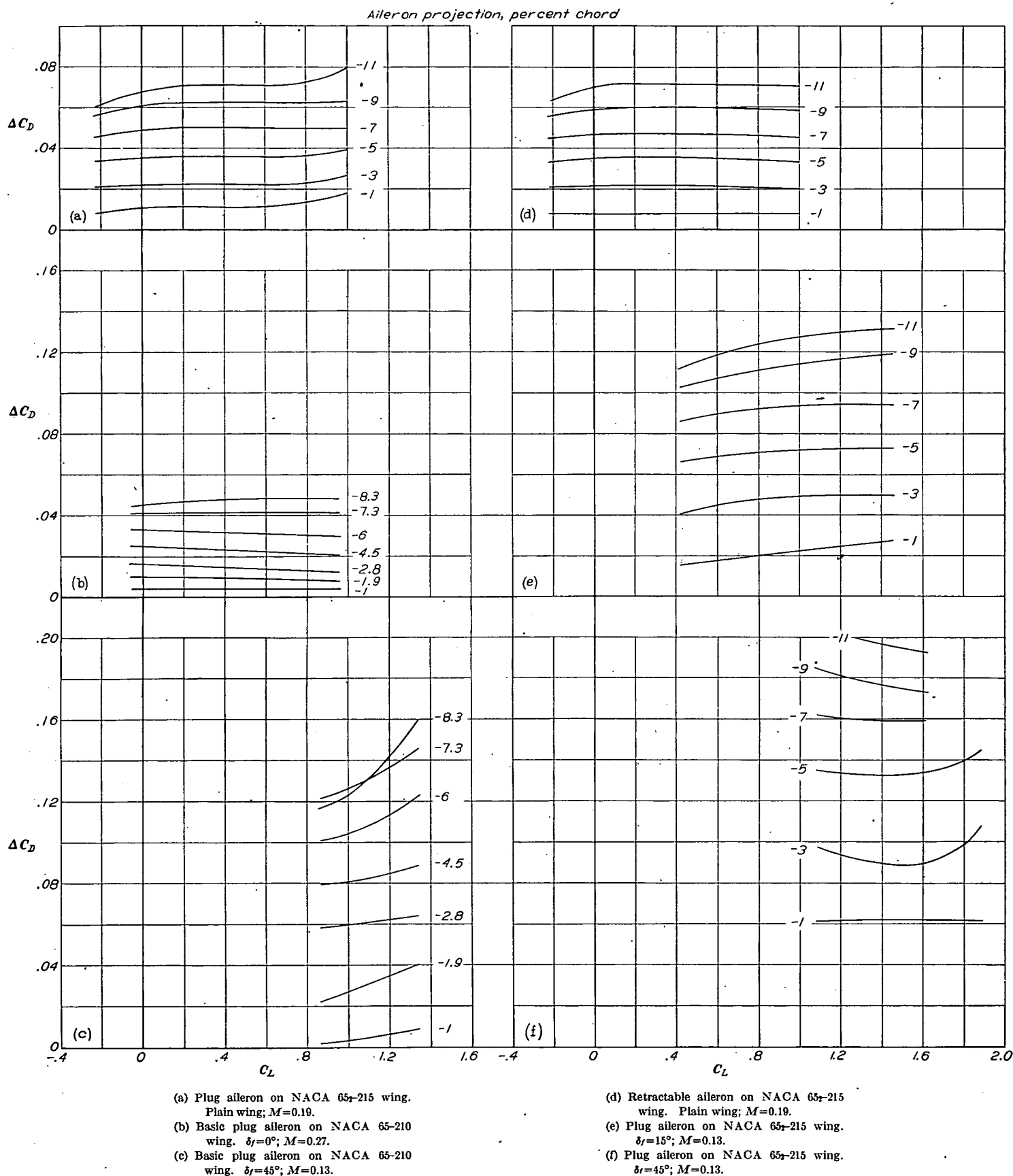


FIGURE 24.—Incremental values of drag coefficient obtained at constant lift coefficient for several of the wing aileron configurations investigated.

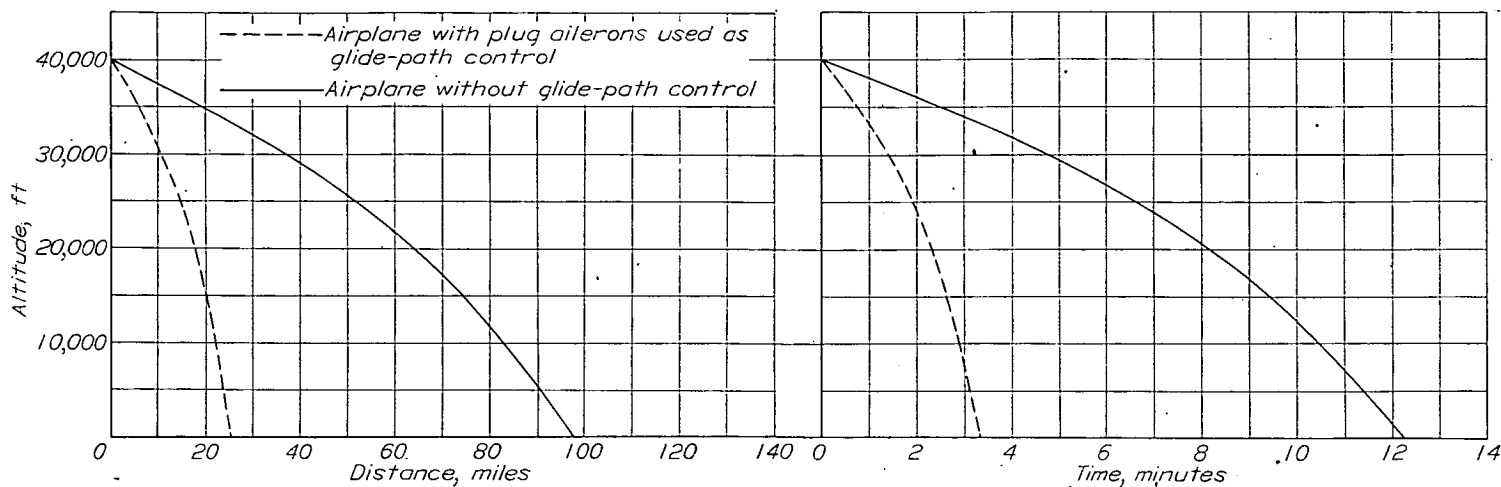


FIGURE 25.—Comparison of estimated elapsed time and ground distance covered during descent of a typical airplane from an altitude of 40,000 feet with and without a glide-path control. Plug ailerons raised to 8-percent-chord projection on both semispan wings used as glide-path control. (Assumed airplane conditions: wing loading, 63 lb/sq ft; wing aspect ratio, 10.18; wing taper ratio, 0.43; effective propeller thrust, 0; $\delta_f = 0^\circ$; $M = 0.7$ until $V_f = 450$ mph is attained.)

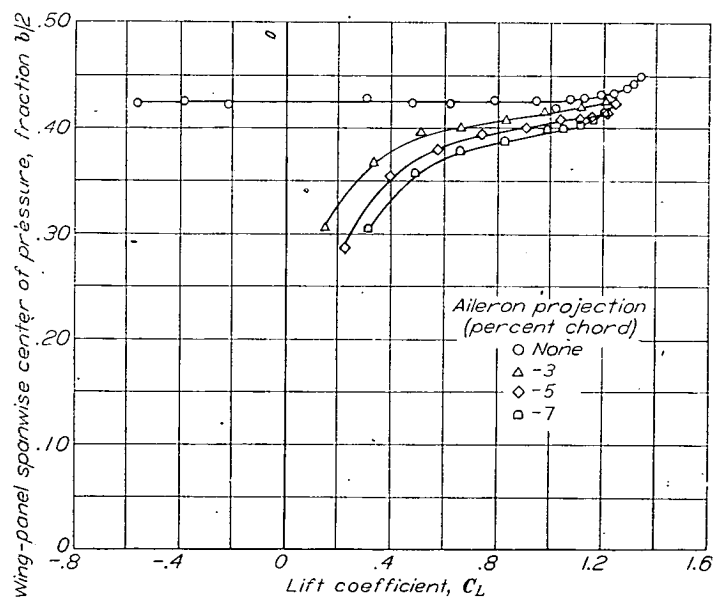


FIGURE 26.—Effect of aileron projection on wing-panel spanwise center of pressure of the NACA 65-215 wing with retractable ailerons. Plain wing; $M = 0.19$.

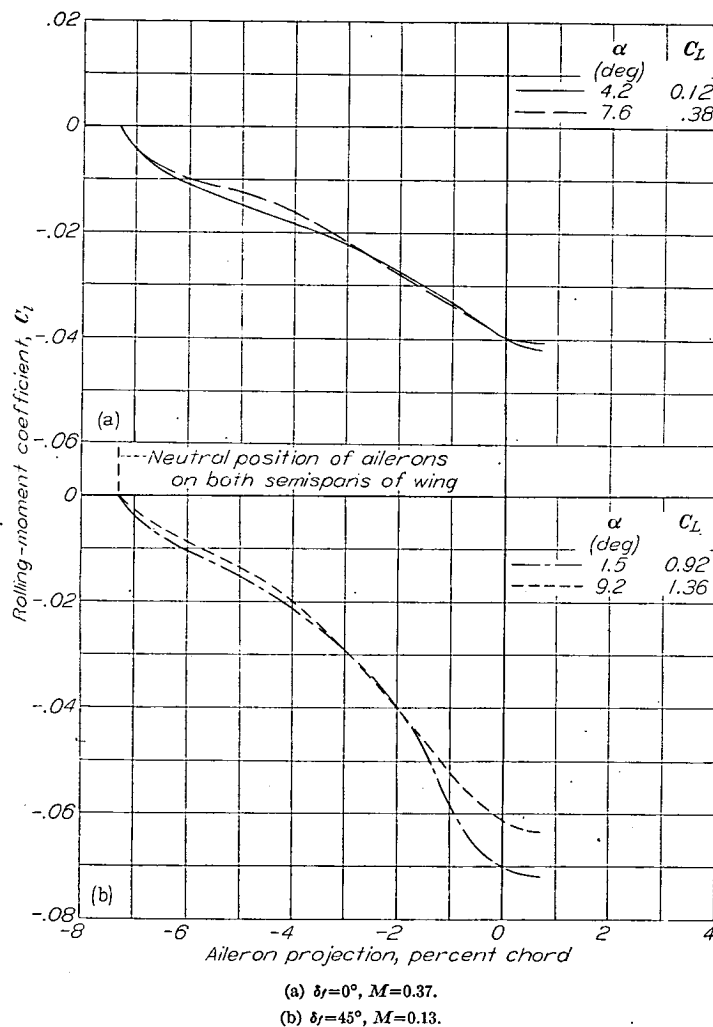


FIGURE 27.—Variation of rolling-moment coefficient of complete wing with projection of double-wall circular plug aileron on one semispan of NACA 65-210 wing. Neutral position of ailerons: -7.3-percent-chord projection on both semispans of a complete wing. (See reference 4.)

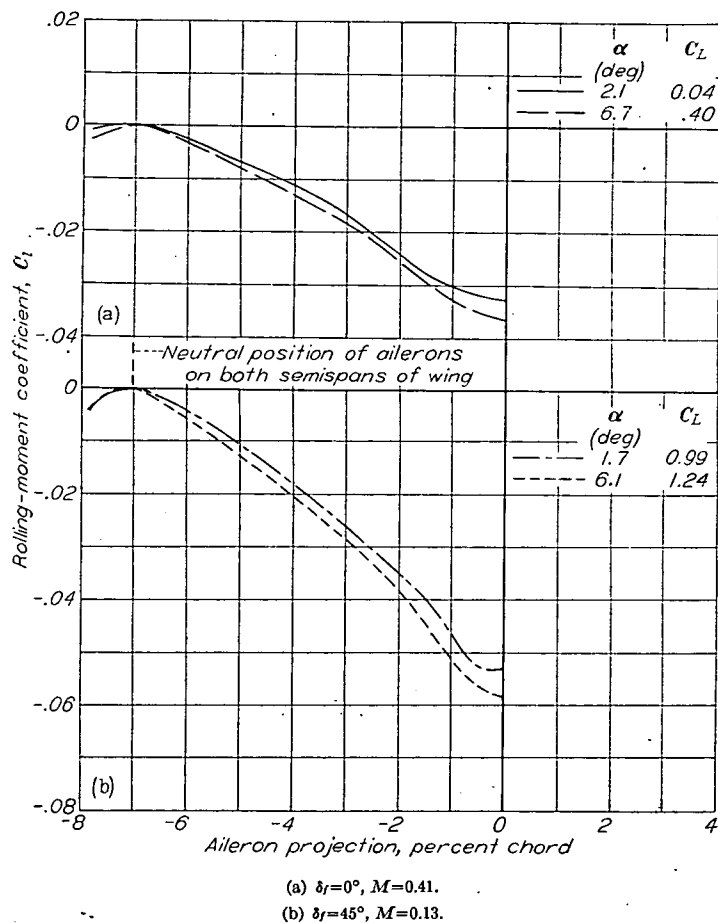


FIGURE 28.—Variation of rolling-moment coefficient of complete wing with projection of basic retractable aileron on one semispan of NACA 65-210 wing. Neutral position of ailerons: -7.0-percent-chord projection on both semispans of a complete wing. (See reference 3.)

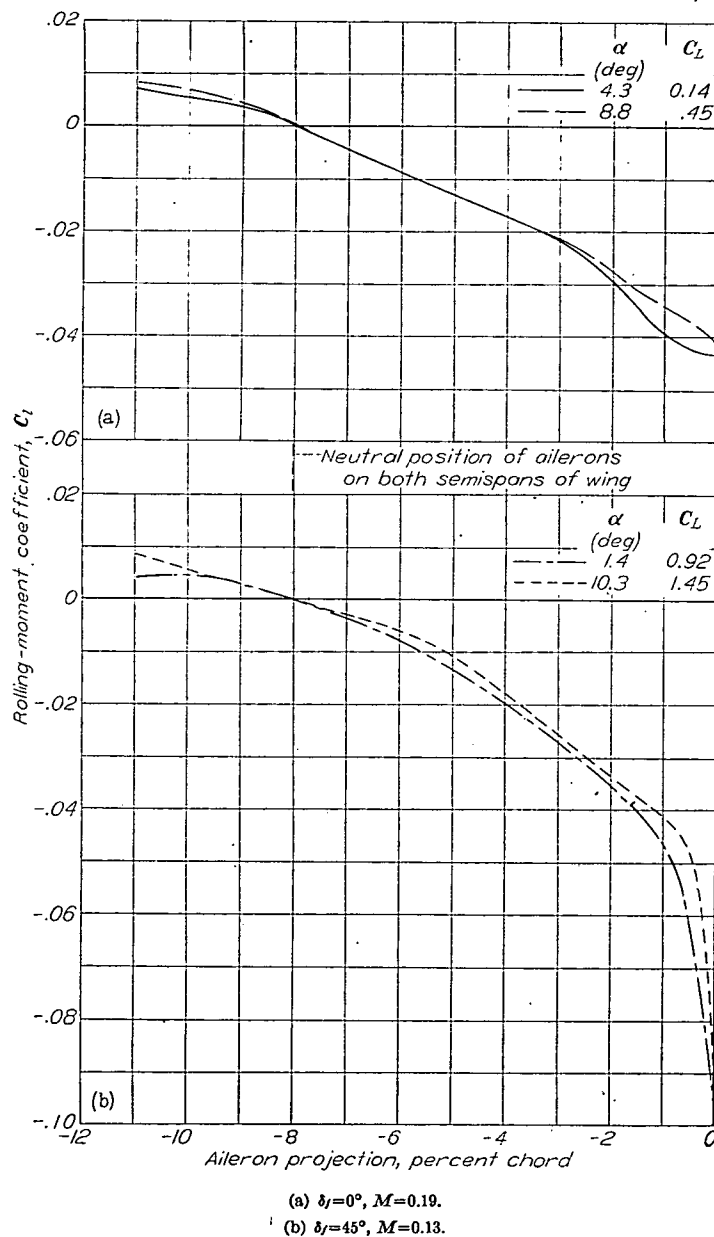


FIGURE 29.—Variation of rolling-moment coefficient of complete wing with projection of plug aileron on one semispan of NACA 65-215 wing. Neutral position of ailerons: -8.0 percent-chord projection on both semispans of a complete wing. (See reference 5.)

REPORT 1035

ANALYSIS OF MEANS OF IMPROVING THE UNCONTROLLED LATERAL MOTIONS OF PERSONAL AIRPLANES¹

By MARION O. MCKINNEY, Jr.

SUMMARY

A theoretical analysis has been made of means of improving the uncontrolled motions of personal airplanes. The purpose of this investigation was to determine whether such airplanes could be made to fly uncontrolled for an indefinite period of time without getting into dangerous attitudes and for a reasonable period of time (1 to 3 min) without deviating excessively from their original course.

The results of this analysis indicated that the uncontrolled motions of a personal airplane could be made safe as regards spiral tendencies and could be greatly improved as regards maintenance of course without resort to an autopilot. The only way to make the uncontrolled motions completely satisfactory as regards continuous maintenance of course, however, is to use a conventional type of autopilot.

Theoretical analysis indicated that, although most present-day personal airplanes possess a slight degree of positive spiral stability, they can easily get into dangerous attitudes and deviate excessively from their original course in uncontrolled flight because of out-of-trim moments and insufficient spiral stability. In order to insure even reasonably satisfactory uncontrolled motions, these out-of-trim moments must be almost entirely eliminated by trimming the airplane in flight and by keeping control-system friction low or using some mechanical system to provide positive centering of the controls. Spiral stability can be increased by increasing tail length or increasing the vertical-tail area and dihedral angle simultaneously, or by increasing all of these factors simultaneously, without adversely affecting the flying qualities of the airplane.

INTRODUCTION

The problem of making a personal airplane fly uncontrolled for an indefinite period of time without getting into dangerous attitudes and for a reasonable period of time (1 to 3 min) without excessive change in heading has attracted considerable interest. Personal airplanes, when flown by inexperienced pilots or without the proper instruments, may get into dangerous attitudes during periods of blind flying. They may also wander off course while the pilot is busy with maps and navigation problems or is otherwise occupied so that he does not concentrate on flying the airplane. An analysis has been made therefore to determine means of improving the uncontrolled motions of a personal airplane. Although an airplane may possess sufficient stability to

insure its return to the original flight attitude following a disturbance such as a gust, it cannot be expected to return to its original heading with respect to the compass without the application of corrective control. If an autopilot is not used to supply this control action, the problem then becomes one of making the airplane safe in uncontrolled flight and reducing the deviation from course to a minimum.

SYMBOLS

All forces and moments are referred to the stability system of axes which is defined as an orthogonal system having its origin at the center of gravity and the Z-axis in the plane of symmetry and perpendicular to the relative wind (positive direction downward), the X-axis in the plane of symmetry and perpendicular to the Z-axis (positive direction forward), and the Y-axis perpendicular to the plane of symmetry (positive direction to right).

S	wing area, square feet
S_t	vertical tail area, square feet
b	wing span, feet
l	distance from airplane center of gravity to vertical-tail center of pressure, feet
m	mass of airplane, slugs
ρ	air density, slugs per cubic foot
V	airspeed, feet per second
q	dynamic pressure, pounds per square foot $\left(\frac{1}{2} \rho V^2\right)$
β	angle of sideslip, degrees except where otherwise noted
r	yawing angular velocity, radians per second
p	rolling angular velocity, radians per second
C_L	lift coefficient (Lift/ qS)
C_Y	lateral-force coefficient (Lateral force/ qS)
C_l	rolling-moment coefficient (Rolling moment/ qSb)
C_n	yawing-moment coefficient (Yawing moment/ qSb)
$C_{Y\beta}$	variation of lateral-force coefficient with angle of sideslip in radians ($\partial C_Y / \partial \beta$)
$C_{l\beta}$	variation of rolling-moment coefficient with angle of sideslip, per degree except where otherwise noted ($\partial C_l / \partial \beta$)
$C_{n\beta}$	variation of yawing-moment coefficient with angle of sideslip, per degree except where otherwise noted ($\partial C_n / \partial \beta$)

¹Supersedes NACA TN 1997, "Analysis of Means of Improving the Uncontrolled Lateral Motions of Personal Airplanes" by Marion O. McKinney, Jr., 1949.

C_{l_r}	variation of rolling-moment coefficient with yawing-angular-velocity factor $\left(\partial C_{l_r} / \partial \frac{rb}{2V}\right)$
C_{n_r}	variation of yawing-moment coefficient with yawing-angular-velocity factor $\left(\partial C_{n_r} / \partial \frac{rb}{2V}\right)$
C_{l_p}	variation of rolling-moment coefficient with rolling-angular-velocity factor $\left(\partial C_{l_r} / \partial \frac{pb}{2V}\right)$
C_{n_p}	variation of yawing-moment coefficient with rolling-angular-velocity factor $\left(\partial C_{n_r} / \partial \frac{pb}{2V}\right)$
$C_{L_{\alpha_t}}$	slope of lift curve of vertical tail
k_X	radius of gyration about X-axis, feet
k_Z	radius of gyration about Z-axis, feet
μ	relative-density factor ($m/\rho S b$)

CALCULATIONS

Two types of calculations were performed in the present investigation: calculations of spiral-stability boundaries and calculations of the motions of several configurations of a hypothetical personal airplane for several disturbances. The characteristics of the basic airplane, which is fairly representative of present-day two-place personal airplanes, are given in references 1 and 2 and were determined by averaging the characteristics of several personal airplanes. The various modified configurations include changes in the dihedral angle, vertical-tail area, and tail length for improving the uncontrolled motions of the conventional personal-airplane configuration. The results of the calculations apply directly only to the hypothetical personal airplane which had a wing loading of 9.25 pounds per square foot and a span of 32 feet. The results can be applied fairly well to specific personal airplanes, however, by dividing the values of time by $\frac{10.5}{b} \sqrt{\frac{W}{S}}$ where $\frac{W}{S}$ and b are the wing loading and span, respectively, of the specific airplane.

The spiral-stability boundaries were calculated by the method presented in reference 3 which states that, for level flight, neutral spiral stability occurs when

$$C_{l_\beta} C_{n_r} = C_{n_\beta} C_{l_r}$$

The values of the stability derivatives C_{n_r} , C_{n_β} , and C_{l_r} used in the boundary calculations are given in table I. The derivative C_{l_β} was treated as the dependent variable. The

TABLE I.—VALUES OF STABILITY DERIVATIVES USED IN CALCULATIONS OF SPIRAL-STABILITY BOUNDARIES

C_{n_β}	C_{n_r}		C_L	C_{l_r}	
	Normal tail arm (0.466)	Long tail arm (0.926)		Ailerons fixed	Ailerons free
0	-0.039	-0.078	0.20	0.050	-----
.001	-.092	-.184	.35	.086	0.030
.002	-.145	-.290	.40	.100	-----
.003	-.199	-.398	.60	.150	-----
.004	-.252	-.505	.80	.200	-----
.005	-.305	-.610			

spiral-stability boundaries were calculated with the assumption that the value of C_{n_β} was increased by increasing the vertical-tail area so that the value of $-C_{n_r}$ increased as C_{n_β} increased.

The rolling and yawing motions of the airplane following various control and gust disturbances were calculated by using the equations of motion presented in reference 3. The applied disturbing moments used in the calculations are given in table II and the stability derivatives used in the motion calculations are presented in table III. The lift coefficient of 0.35 is fairly representative of the lift coefficient of personal airplanes at cruising speed, and the lift coefficient of 1.8 represents the maximum lift coefficient of the airplane with flaps down.

RESULTS AND DISCUSSION

A personal airplane may get into a dangerous attitude or deviate excessively from its original course in uncontrolled flight as far as its lateral characteristics are concerned for two reasons: it may be out of trim, or it may be spirally unstable.

TABLE II.—CONDITIONS FOR WHICH MOTIONS WERE CALCULATED

Type of disturbance	C_L	C_l	C_n
50° total aileron deflection	1.80	0.040	-0.008
50° total aileron deflection	.35	.060	-.002
1° total aileron deflection	.35	.002	-.00005
1° rudder deflection	.35	0	.001
Rolling gust	.35	.004	0

TABLE III.—VALUES OF STABILITY DERIVATIVES USED IN CALCULATIONS OF MOTIONS

$[\mu = 3.76, k_X = 0.150b, k_Z = 0.183b]$

(a) $C_L = 0.35$

Derivative	Configuration						
	1	1A	2	3	4	5	6
$^1C_{Y\beta}$	-0.274	-0.274	-0.274	-0.510	-0.274	-0.274	-0.510
$^1C_{l_\beta}$	-.067	-.067	-.137	-.180	-.067	-.137	-.180
$^1C_{n_\beta}$	.064	.064	.064	.172	.064	.064	.172
C_{l_p}	-.425	-.425	-.425	-.425	-.425	-.425	-.425
C_{n_p}	-.022	-.022	-.022	-.022	-.022	-.022	-.022
C_{l_r}	.086	.030	.086	.086	.086	.086	.086
C_{n_r}	-.097	-.194	-.097	-.196	-.194	-.194	-.392

(b) $C_L = 1.80$

Derivative	Configuration					
	1	2	3	4	5	6
$^1C_{Y\beta}$	-0.808	-0.808	-1.044	-0.808	-0.808	-1.044
$^1C_{l_\beta}$	-.130	-.259	-.143	-.130	-.259	-.143
$^1C_{n_\beta}$	.086	.086	.194	.086	.086	.194
C_{l_p}	-.442	-.442	-.442	-.442	-.442	-.442
C_{n_p}	-.074	-.074	-.074	-.074	-.074	-.074
C_{l_r}	.442	.442	.442	.442	.442	.442
C_{n_r}	-.220	-.220	-.319	-.317	-.317	-.517

$^1\beta$ in radians.

Several factors are involved in eliminating out-of-trim moments and spiral stability may be increased by several means. The present analysis therefore is divided into two parts for convenience in discussion. The first part treats the uncontrolled motions of a conventional personal airplane and means of improving these motions without changing the geometric configuration of the airplane. The second part treats the uncontrolled motions of various configurations modified geometrically to improve the spiral stability.

CONVENTIONAL AIRPLANE CONFIGURATION

Spiral stability.—An airplane must be spirally stable if it is to fly uncontrolled without diverging from its original attitude. The first step in an analysis of means of improving the uncontrolled lateral motions, therefore, is to determine whether present-day personal airplanes are spirally stable. An indication of whether such airplanes are spirally stable can be obtained from figure 1. This figure shows calculated spiral-stability boundaries for a hypothetical personal airplane at various lift coefficients with fixed controls as functions of the directional-stability parameter $C_{n\beta}$ and the effective-dihedral parameter $-C_{l\beta}$. An airplane for which the point on the chart would be on the right side of the boundary is spirally stable; whereas one for which the point would be on the left side of the boundary is spirally unstable. The crosshatched region indicates the position in which points for most present-day personal airplanes would be located on the chart. The lift coefficient corresponding to the cruising speed was determined for several personal airplanes from published performance specifications and was found to be between 0.25 and 0.35. The data presented in figure 1 indicate therefore that most present-day personal airplanes possess a slight degree of positive spiral stability for the cruising-flight condition for which good uncontrolled behavior is most desired.

The significance of a slight degree of positive spiral stability is illustrated by the calculated motion of the hypothetical personal airplane following a disturbance by a rolling gust. The location of the point representing this airplane relative to the spiral-stability boundary is shown in figure 2 (a) where the conventional personal airplane with controls fixed is designated configuration 1. The motion of this airplane following a disturbance by a mild rolling gust ($\frac{pb}{2V} = 0.01$ for 1 sec) is presented in figure 3 where the variations of the angles of bank and heading with time are shown. This figure shows that the gust caused the airplane to bank about 5° and that the airplane slowly returned toward 0° bank. As a result of this bank, the airplane turned considerably off its original course. This motion represents about as poor behavior as could be expected of present-day personal airplanes since the directional stability of the hypothetical airplane ($C_{n\beta} = 0.00115$) is higher than that of most personal airplanes and the cruising lift coefficient of the hypothetical airplane ($C_L = 0.35$) is as high as that of any present-day personal airplane. Most personal airplanes, particularly those with relatively high performance, would be expected

to return toward 0° bank more rapidly and turn less in response to the same disturbance than configuration 1 since they would probably be more spirally stable than this hypothetical airplane.

The effect on spiral stability of freeing the ailerons is illustrated by the calculations for the hypothetical personal airplane with ailerons free, designated configuration 1A. The following assumptions were made for these calculations: that the value of $-C_{l\beta}$ was not affected by the freeing of the ailerons, that the airplane had an NACA 4412 airfoil section with Frise ailerons, and that no friction was present in the aileron control system to prevent the ailerons from floating freely. These ailerons have a strong up-floating tendency so that, as the airplane turns with the stick free, the aileron on the faster-moving wing deflects up and the aileron on the slower-moving wing deflects down. This movement of the ailerons tends to roll the airplane out of the turn. In effect, this movement of the ailerons causes the value of C_{lr} to be lower with stick free than with stick fixed and thereby shifts the spiral-stability boundary as shown for configuration 1A in figure 2 (b). This effect of freeing the ailerons on the motion resulting from a rolling-gust disturbance is illustrated in figure 3 which shows that with the ailerons free (configuration 1A) the airplane returns toward 0° bank more rapidly and does not turn so far off the course as with the ailerons fixed. The airfoil section and the aileron balance assumed for configuration 1A give about as much up-floating tendency as can be expected without resort to some such device as downwardly deflected tabs on the ailerons to provide additional up-floating tendency; therefore, the difference between the motions for configurations 1 and 1A represents the maximum that can be expected from freeing the ailerons unless additional up-floating tendency is provided.

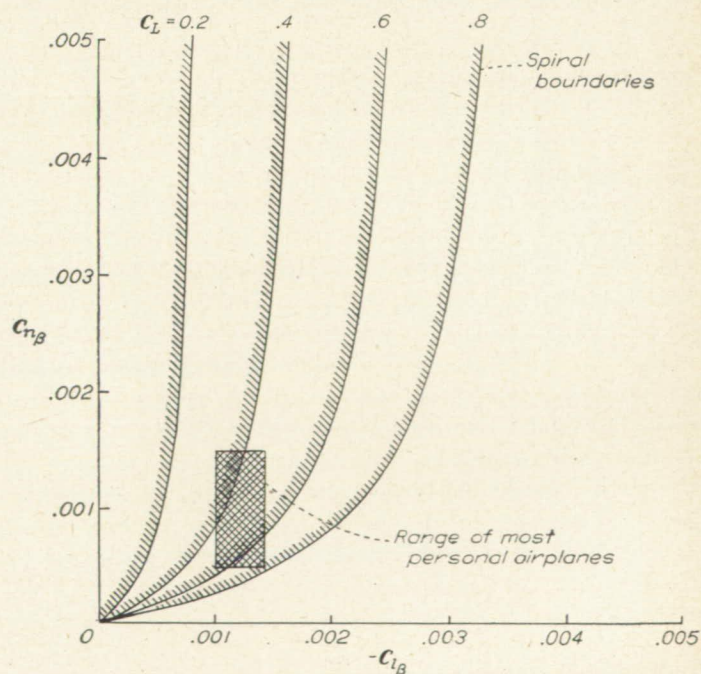


FIGURE 1.—Spiral-stability boundaries for a personal airplane with fixed controls.

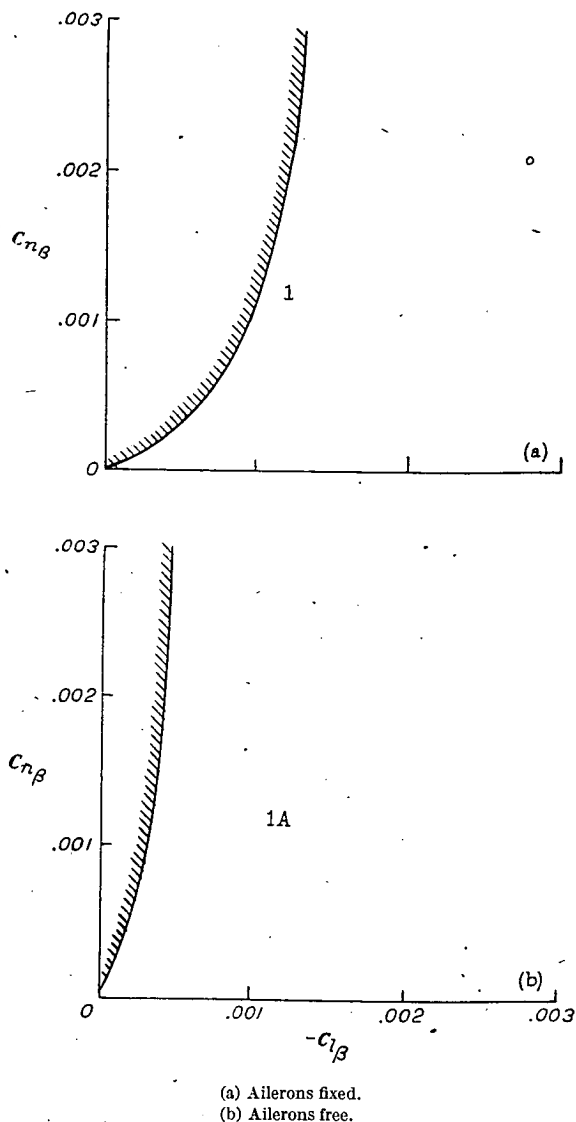


FIGURE 2.—Spiral stability of the conventional personal airplane for configurations 1 and 1A.

The effect of freeing the rudder can be ascertained from an analysis of the equation for neutral spiral stability ($C_{l\dot{\beta}}C_{n\dot{\gamma}} = C_{n\dot{\beta}}C_{l\dot{\gamma}}$). Freeing the rudder changes the values of $C_{n\dot{\beta}}$ and $C_{n\dot{\gamma}}$ in approximately the same ratio so that freeing the rudder has almost no effect on spiral stability.

The analysis has shown that most present-day personal airplanes are spirally stable with the controls fixed and will return toward 0° bank following a disturbance although they will have changed heading somewhat and that these airplanes are just as spirally stable or even more spirally stable with controls free than with controls fixed. It is known, however, that in uncontrolled flight most personal airplanes tend to deviate from their original attitude and not return. Since this characteristic, therefore, cannot usually be attributed to spiral instability it must result from out-of-trim moments.

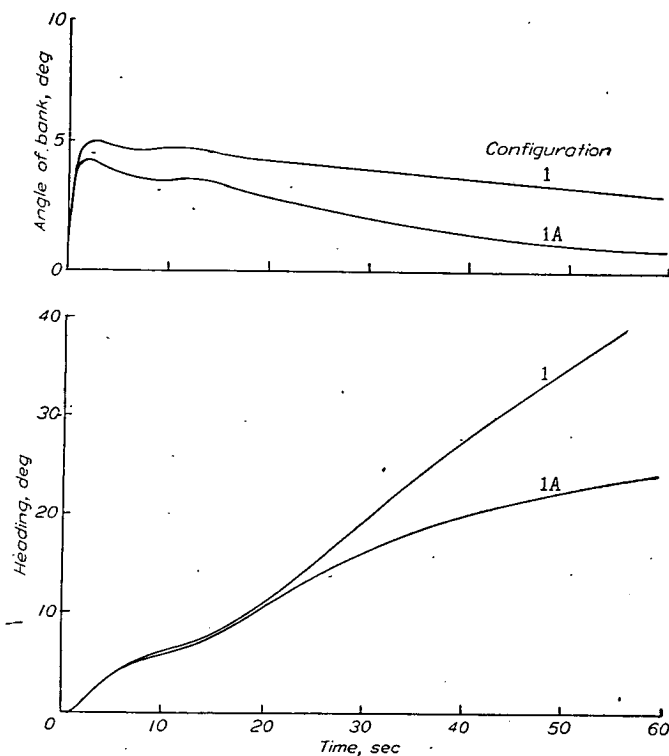


FIGURE 3.—Motions of the conventional personal airplane resulting from a mild gust disturbance (gust strength $\frac{pb}{2V} = 0.01$ for 1 sec) for the two configurations shown in figure 2.

Out-of-trim moments.—Almost all personal airplanes are out of trim in roll and yaw to a certain extent because of improper rigging, change of trim with power, absence of trim tabs, and control-system friction which prevents proper centering of the controls. As pointed out previously, an airplane has no stability of course. An airplane which is out of trim cannot, therefore, be reasonably expected to fly uncontrolled for an appreciable period of time without considerable change in heading. Trim tabs, or some other means of trimming the airplane in flight should be considered essential, therefore, if the airplane is to fly uncontrolled for a reasonable period of time without excessive change in heading. Control-system friction will tend to hold the controls in the proper position after the pilot has trimmed the airplane with the stick and the rudder pedals, but it will cause considerable trouble that tends to offset this one good characteristic. For example, friction keeps the controls from centering after they are deflected by a gust or other disturbance. Friction also obscures the feel of the controls, a condition which is always objectionable, particularly since the pilot cannot center the controls without the aid of instruments under blind-flying conditions.

The effect of out-of-trim moments on the uncontrolled motions of the hypothetical personal airplane is shown in figures 4 and 5. Figure 4 shows the variation of bank and heading with time when the ailerons are 1° out of trim and figure 5 shows similar motions for the case of 1° out-of-trim

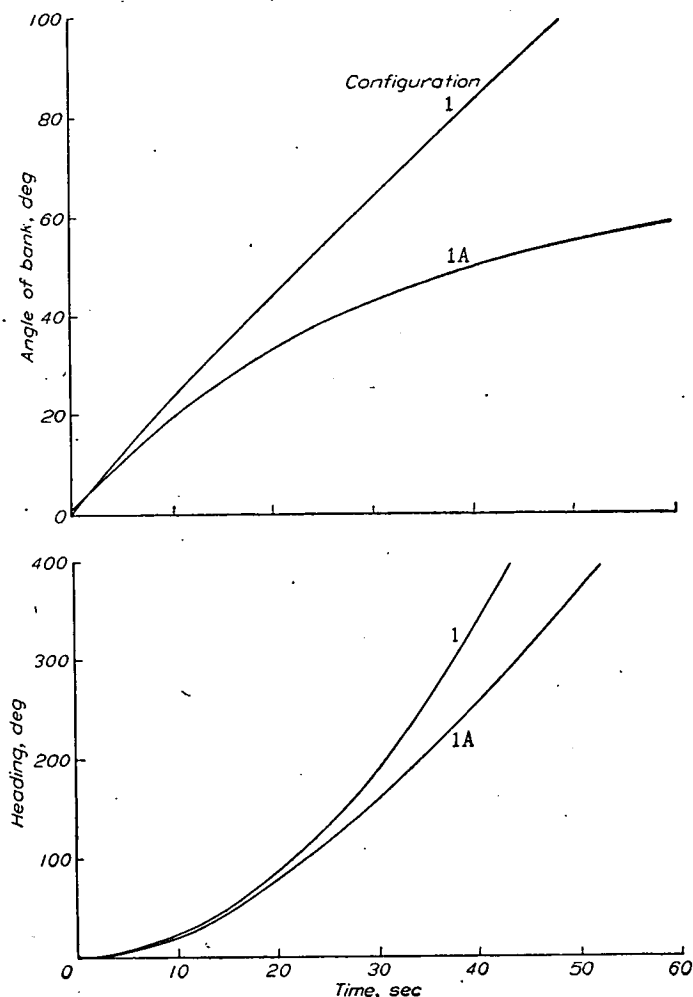


FIGURE 4.—Motions of the conventional personal airplane resulting from 1° out-of-trim aileron deflection for the two configurations shown in figure 2.

rudder deflection. The calculated motions presented in these figures show that, if either the ailerons or rudder are out of trim, the airplane will bank and turn at a fairly rapid rate with no tendency to return to its original attitude as regards either bank or heading. This is true either with the ailerons fixed (configuration 1) where the airplane has slight positive spiral stability or with ailerons free (configuration 1A) where the airplane has considerably more spiral stability. It is apparent from these calculated motions that almost no out-of-trim moments can be tolerated so that, in addition to providing some means of trimming the airplane in flight, the effect of control-system friction in holding the controls deflected and obscuring the feel of the controls must be eliminated.

As pointed out in reference 4, the allowable limits for control-system friction cannot be set at the present time. Vibration of the airplane may relieve, to a certain extent, the effect of friction in holding the controls deflected so that the allowable limits for control friction cannot be determined

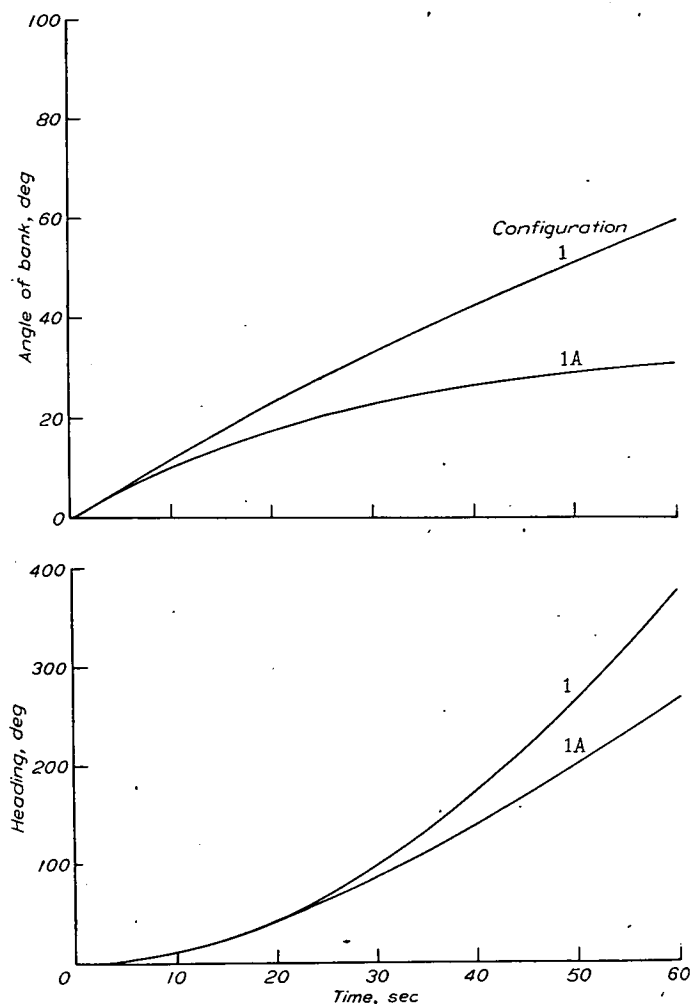


FIGURE 5.—Motions of the conventional personal airplane resulting from 1° out-of-trim rudder deflection for the two configurations shown in figure 2.

solely from static considerations of the aerodynamic and frictional hinge moments. Some special flight research work is required to establish an upper limit for the allowable friction as regards proper centering of the ailerons and rudder to prevent an airplane from getting into dangerous attitudes or deviating excessively from its original course in uncontrolled flight.

If keeping the friction forces in the control system low enough is found to be too difficult to be practical for a personal airplane, some mechanical device might be employed that would eliminate the effect of friction without necessitating the elimination of the friction. One such device, the effect of which is being studied experimentally in flight tests at the Langley Aeronautical Laboratory, is illustrated schematically in figure 6. This device consists essentially of preloaded springs that provide positive centering for the controls since at any deflection they provide a restoring force which is greater than the static-friction force in the control system. Since these springs would cause a

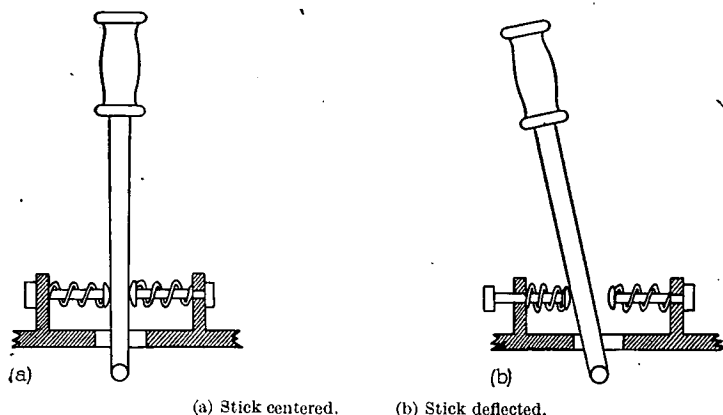


FIGURE 6.—Schematic diagram of a device for positively centering the controls.

nonlinear control-force gradient through zero deflection which might be annoying to the pilot at times, means for engaging and disengaging the centering device at will might be required. Because of the stretch in the control system, such a device might preferably be installed at the ailerons and rudder rather than on the control stick or rudder pedals as indicated by the sketch. This device could also be used to trim the airplane if the preload in the springs is greater than the control forces required for trim.

If out-of-trim moments are eliminated by trimming the airplane in flight and by eliminating the effect of control-system friction in preventing the controls from centering properly, a conventional personal airplane should be fairly safe as regards the ability to fly uncontrolled for indefinite periods of time without getting into a dangerous attitude and should be fairly satisfactory as regards the ability to fly uncontrolled for reasonable periods of time without excessive deviations from its original course. A considerable improvement in the uncontrolled motions of a personal airplane may, however, be obtained by modifying it to obtain greater spiral stability.

MODIFIED AIRPLANE CONFIGURATION

Several means are available for modifying a conventional personal airplane so as to increase its spiral stability. These methods are fairly obvious from examination of the relation from reference 3 which shows that an airplane is spirally stable when

$$C_{l\beta}C_{n_r} > C_{n\beta}C_{l_r}$$

This expression indicates that spiral stability can be increased by increasing the values of $-C_{l\beta}$ and $-C_{n_r}$ or by reducing the values of $C_{n\beta}$ and C_{l_r} . The value of $-C_{l\beta}$ can be increased by increasing the dihedral angle without appreciably affecting the other stability derivatives. The values of $C_{n\beta}$ and $-C_{n_r}$ are both functions of the vertical-tail size and tail length as shown by the following approximate equations:

$$C_{n\beta} \approx \frac{S_t}{S} \frac{l}{b} C_{L_{a_t}}$$

and

$$-C_{n_r} \approx 2 \frac{S_t}{S} \left(\frac{l}{b}\right)^2 C_{L_{a_t}}$$

where the principal assumption is that the airplane has about zero $C_{n\beta}$ without a vertical tail. These two equations indicate that $C_{n\beta}$ and $-C_{n_r}$ can be varied either simultaneously or independently by adjusting the vertical-tail area and tail length. The value of C_{l_r} cannot be changed greatly for the controls-fixed condition by changes to the geometry of the airplane. As previously mentioned, however, C_{l_r} can be varied considerably with the ailerons free by adjusting the aileron floating characteristics.

As pointed out in reference 4, the design conditions for increasing spiral stability often conflict with other factors known to be essential in the attainment of satisfactory flying qualities. When an airplane is modified so as to improve its uncontrolled motions by increasing its spiral stability, the effect of these changes on its flying qualities should be considered. The discussion of the effects of modifications to the conventional personal airplane is presented in two parts: the effect on flying qualities and the effect on the uncontrolled motions.

Effect of modifications on flying qualities.—Experience has shown that increasing the dihedral angle of an airplane so as to increase its spiral stability causes its flying qualities to become less satisfactory since the rolling velocity in an aileron roll tends to reverse. With most personal airplanes, however, some increase in dihedral angle can be effected without causing the flying qualities to become unsatisfactory. Experience has also shown that the spiral stability of most personal airplanes should not be increased by reducing the size of the vertical tail (reducing $C_{n\beta}$) since this change would result in unsatisfactory flying qualities in the form of excessive sideslip in aileron rolls. Analysis of figure 1 indicates that the spiral stability of a personal airplane can be increased by increasing the vertical-tail area and dihedral angle simultaneously so as to maintain the same ratio of $C_{n\beta}$ to $C_{l\beta}$ as that of the original airplane. This change can be made without sacrificing controllability. The effect of increasing the tail length and reducing the tail size of a personal airplane so as to increase the damping in yaw without increasing $C_{n\beta}$ has not been definitely determined. Flight experience with models has indicated, however, that increasing the tail length will not have an adverse effect on controllability.

On the basis of this analysis several modified configurations of the hypothetical conventional personal airplane were chosen for a more detailed analysis of flying qualities and uncontrolled motions. These configurations are indicated by the sketches of figure 7 and by the spiral stability charts of figure 8. Configuration 1 represents the conventional personal airplane which is used as a basis for comparison. Configuration 2 represents an airplane with an increase in dihedral angle of about 6° from configuration 1. Configuration 3 incorporates an increase in dihedral angle of 10° and an increase in vertical-tail area of about 2.5 times that of configuration 1. Configuration 4 represents an airplane having twice the tail length and half the tail area of the conventional personal airplane. Configuration 5 represents

a combination of the high dihedral of configuration 2 and the tail length and tail area of configuration 4. Configuration 6 incorporates a simultaneous increase in dihedral angle

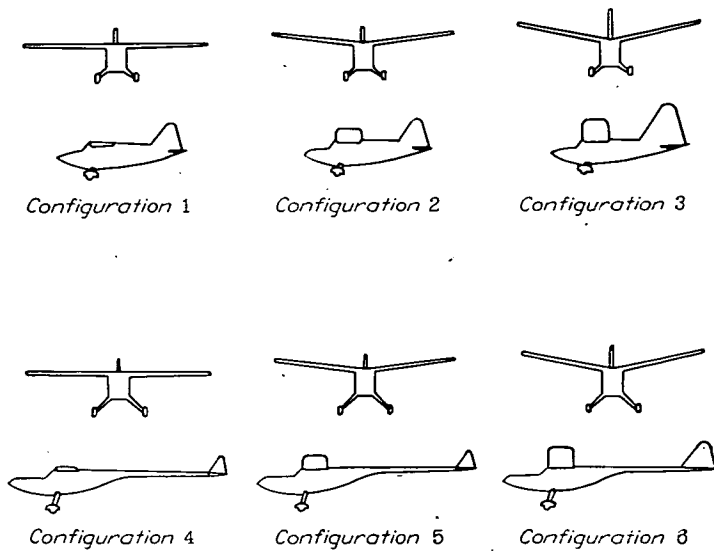
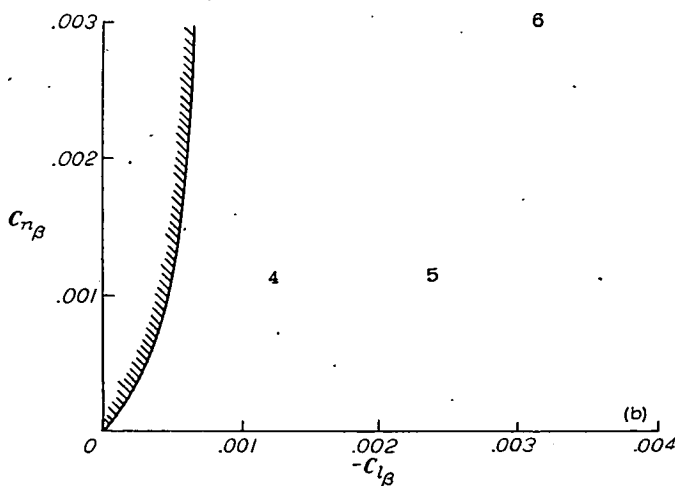
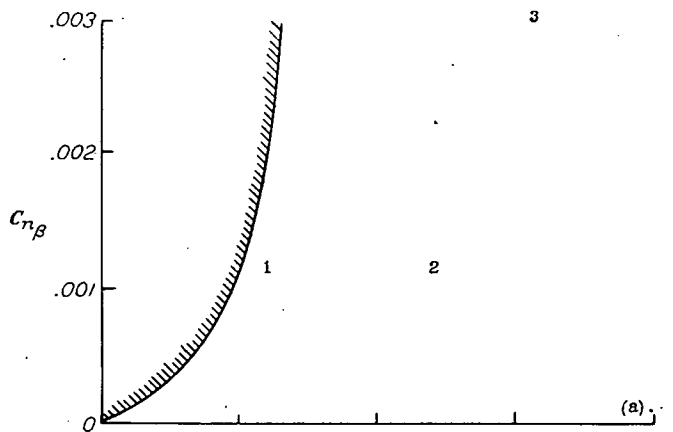


FIGURE 7.—Sketches of the conventional personal airplane (configuration 1) and the modified airplanes (configurations 2 to 6).

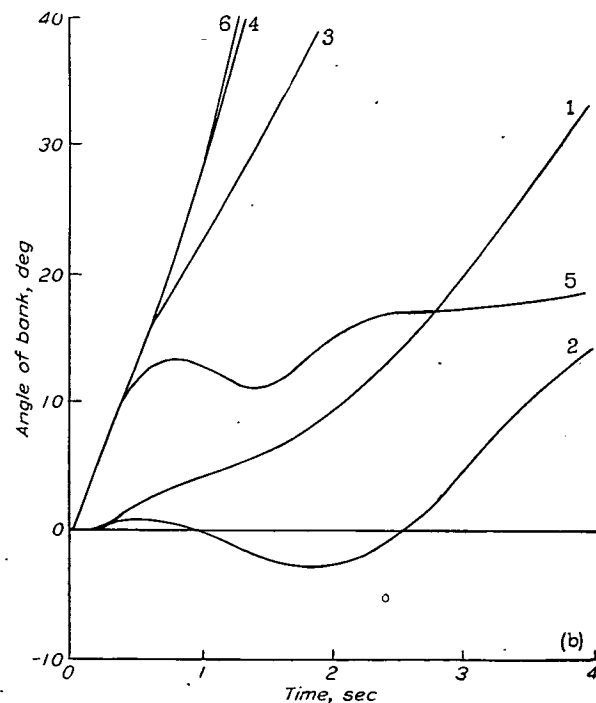
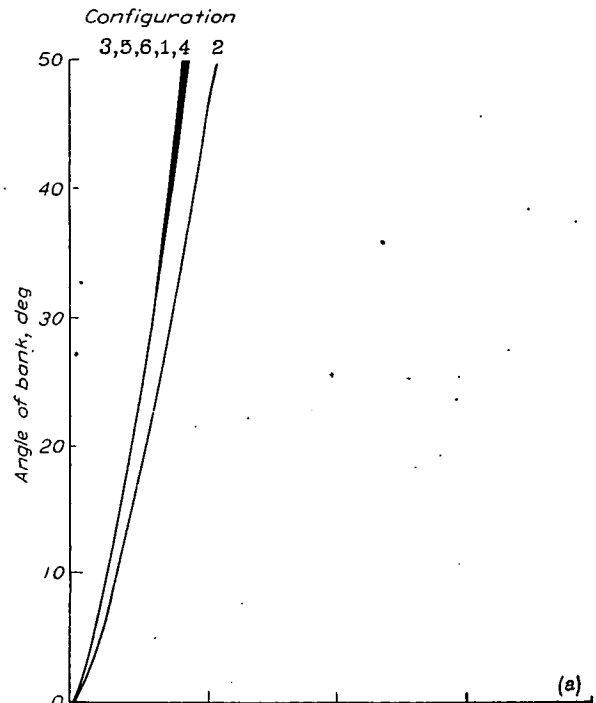


(a) $l=0.466$.
(b) $l=0.926$.

FIGURE 8.—Spiral stability of the conventional personal airplane (configuration 1) and the modified airplanes (configurations 2 to 6).

of 10° and an increase in vertical-tail area of about 2.5 times that of configuration 4.

The effects of these various modifications on controllability are shown in figure 9 by the calculated rolling motions resulting from 50° total aileron deflection. Figure 9 (a) shows that the controllability in cruising flight is not greatly affected by any of the modifications to the conventional personal-airplane configuration. Figure 9 (b), however, shows that



(a) $C_L=0.35$.
(b) $C_L=1.80$.

FIGURE 9.—Rolling motions resulting from 50° total aileron deflection for the six configurations shown in figures 7 and 8.

increasing the dihedral angle (configurations 1 to 2 or 4 to 5) has a pronounced adverse effect on the controllability. The flying qualities for configuration 2 are unsatisfactory since the rolling velocity reverses. Increasing the damping in yaw (configurations 1 to 4 or 2 to 5) or increasing the dihedral angle and vertical-tail area simultaneously (configurations 1 to 3 or 4 to 6) improves the controllability of the airplane at high lift coefficients.

Effect of modifications on uncontrolled motions.—The effects of the various modifications on the uncontrolled motions of the personal airplane are shown in figures 10 to 12. These figures show that all the modifications for increasing the spiral stability of the airplane improved its uncontrolled motions. In response to a rolling gust (fig. 10) the modified configurations returned toward 0° bank more rapidly and did not turn as far off course as the original airplane configuration. In response to an out-of-trim aileron or rudder deflection (figs. 11 and 12) the modified configurations did not bank as far or turn off course as fast as the original airplane configuration. These data indicate that, if a personal airplane is modified so as to increase its spiral stability, larger out-of-trim moments can be tolerated than on the conventional configuration.

Increasing the dihedral angle alone (configurations 1 to 2 or 4 to 5) is the least effective method of improving the uncontrolled motions of a personal airplane since the motions resulting from an out-of-trim rudder deflection are about the

same with the high dihedral as with the original dihedral. (See figs. 10 to 12.) Since increasing the dihedral alone also causes the controllability to become less satisfactory, this method of increasing the spiral stability does not appear to be very satisfactory. Changing the tail length of personal airplanes very much is probably not very practical because of the greater landing-gear length required when the tail length is increased. The most practical method of increasing the spiral stability of a personal airplane so as to improve its uncontrolled motions appears to be to increase its dihedral angle and vertical-tail area simultaneously (configurations 1 to 3 and 4 to 6), so as to keep the ratio of C_{n_β} to C_{l_β} about the same, and to use as great a tail length as is practical. As pointed out previously, it is also possible to improve the uncontrolled motions still more for the control-free condition by increasing the upward-floating tendency of the ailerons provided there is no friction to prevent the ailerons from floating freely. This change probably would not affect the controllability of the airplane.

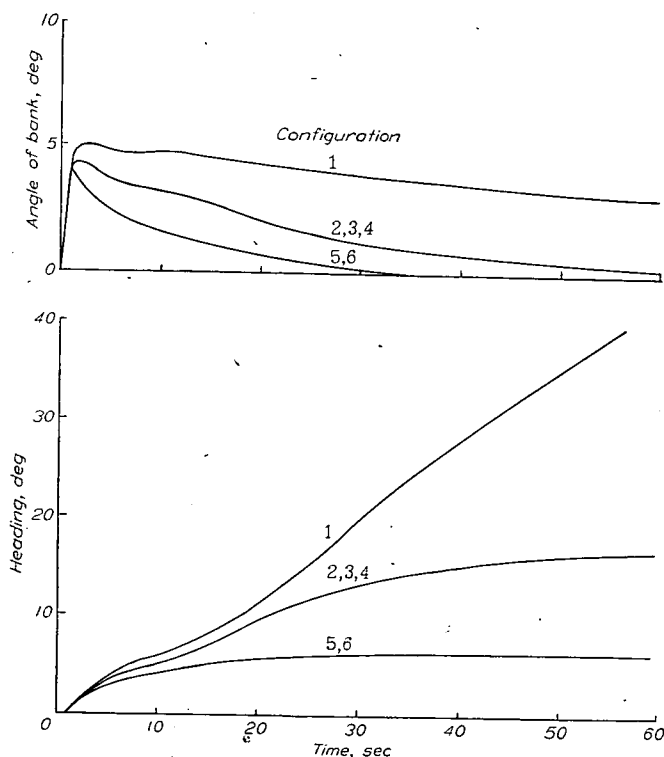


FIGURE 10.—Motions resulting from a mild gust disturbance (gust strength $\frac{pb}{2V} = 0.01$ for 1 sec) for the six configurations shown in figures 7 and 8.

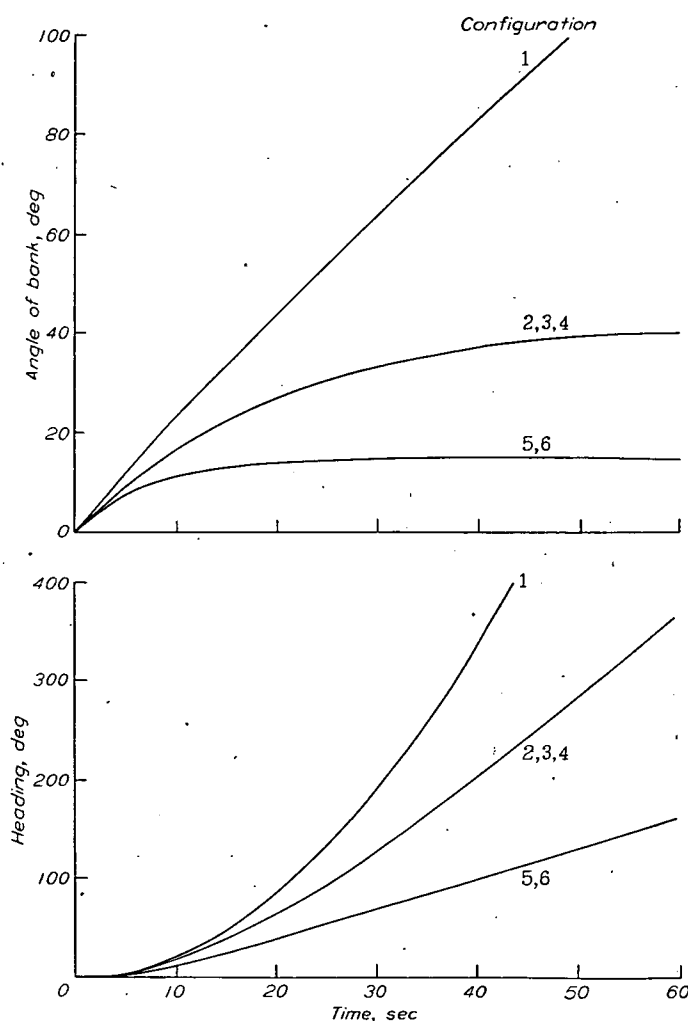


FIGURE 11.—Motions resulting from 1° out-of-trim aileron deflection for the six configurations shown in figures 7 and 8.

GENERAL CONSIDERATIONS REGARDING MAINTENANCE OF COURSE

As pointed out previously, an airplane has no stability of course and consequently cannot be expected to return to its original course after a disturbance unless a conventional type of autopilot is used. This fact is illustrated in figures 3 and 10 where it is shown that there is a change of heading after a gust disturbance even for very spirally stable configurations. Continuous maintenance of course cannot, therefore, be obtained without an autopilot. Fairly good maintenance of course over a reasonably long period of time should be possible, however, without an autopilot if the airplane is spirally stable and stays in trim. From the theory of random motion, the deviation from course due to random gust disturbances would be expected to average out to no deviation over an infinite period of time. For any finite period of time, however, the deviations from course due to random gusts would tend to add up to no deviation but

would not be expected to add up to exactly zero deviation. Because of this tendency for the deviations caused by random gusts to cancel out, the deviation from course over a reasonably long period of time would be expected to be fairly small.

CONCLUSIONS

An analysis of the uncontrolled motions of personal airplanes has shown that a personal airplane can be made safe as regards spiral tendencies and its uncontrolled motions as regards maintenance of course can be greatly improved without resort to an autopilot. The only way to make the uncontrolled motions completely satisfactory as regards continuous maintenance of course, however, is to use a conventional type of autopilot.

Theoretical analysis has indicated that most present-day personal airplanes possess a slight degree of positive spiral stability but can easily get into dangerous attitudes and deviate excessively from their original heading in uncontrolled flight because of out-of-trim moments and insufficient spiral stability. In order to insure even reasonably satisfactory uncontrolled motions, these out-of-trim moments must be eliminated or at least reduced to very small magnitudes. Some means of trimming the airplane in flight is necessary, therefore, and the effect of control-system friction in preventing proper centering of the controls by the pilot or by the aerodynamic forces must be almost entirely eliminated by having very low friction or by having some mechanical device that will provide positive centering of the controls. Increasing the spiral stability will also improve the uncontrolled motions of personal airplanes. An increase in spiral stability for personal airplanes can be obtained by increasing tail length or increasing the vertical-tail area and dihedral angle simultaneously, or by increasing all of these factors simultaneously, without adversely affecting the flying qualities of the airplane.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., June 22, 1949.

REFERENCES

1. Weick, Fred E., and Jones, Robert T.: The Effect of Lateral Controls in Producing Motion of an Airplane as Computed from Wind-Tunnel Data. NACA Rep. 570, 1936.
2. Jones, Robert T.: The Influence of Lateral Stability on Disturbed Motions of an Airplane with Special Reference to the Motions Produced by Gusts. NACA Rep. 638, 1938.
3. Zimmerman, Charles H.: An Analysis of Lateral Stability in Power-Off Flight with Charts for Use in Design. NACA Rep. 589, 1937.
4. Gilruth, R. R.: Requirements for Satisfactory Flying Qualities of Airplanes. NACA Rep. 755, 1943. (Formerly NACA ACR, April 1941.)

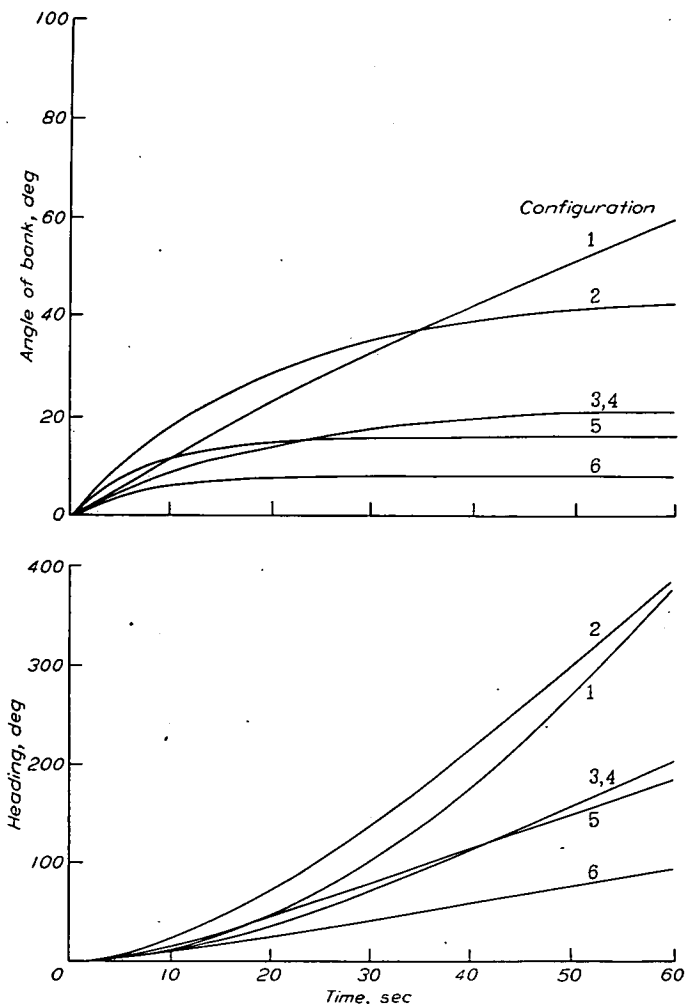


FIGURE 12.—Motions resulting from 1° out-of-trim rudder deflection for the six configurations shown in figures 7 and 8.

REPORT 1036

EXPERIMENTAL INVESTIGATION OF THE EFFECTS OF VISCOSITY ON THE DRAG AND BASE PRESSURE OF BODIES OF REVOLUTION AT A MACH NUMBER OF 1.5¹

By DEAN R. CHAPMAN and EDWARD W. PERKINS

SUMMARY

Tests were conducted to determine the effects of viscosity on the drag and base pressure of various bodies of revolution at a Mach number of 1.5. The models were tested both with smooth surfaces and with roughness added to evaluate the effects of Reynolds number for both laminar and turbulent boundary layers. The principal geometric variables investigated were afterbody shape and length-diameter ratio. For most models, force tests and base pressure measurements were made over a range of Reynolds numbers, based on model length, from 0.6×10^6 to 5.0×10^6 . Schlieren photographs were used to analyze the effects of viscosity on flow separation and shock-wave configuration near the base and to verify the condition of the boundary layer as deduced from force tests. The results are discussed and compared with theoretical calculations.

The results show that viscosity effects are large and depend to a great degree on the body shape. The effects differ greatly for laminar and turbulent flow in the boundary layer, and within each regime depend upon the Reynolds number of the flow. Laminar flow was found up to a Reynolds number of 6.5×10^6 and may possibly exist to higher values.

The flow over the afterbody and the shock-wave configuration near the base are shown to be very much different for laminar than for turbulent flow in the boundary layer. The base pressure on bodies with boattailing is much higher with the turbulent layer than with the laminar layer, resulting in a negative base drag in some cases. The total drag characteristics at a given Reynolds number are affected considerably by the transition to turbulent flow. The foredrag of bodies without boattailing and of boattailed bodies for which the effects of flow separation are negligible can be calculated with reasonable accuracy by adding the skin-friction drag based upon the assumption of the low-speed friction characteristics to the theoretical wave drag.

For laminar flow in the boundary layer the effects of varying the Reynolds number were found to be large, approximately doubling the base drag in many cases. The total drag of the bodies without boattailing varied about 20 percent over the Reynolds number range investigated. For turbulent flow in the boundary layer, however, variations in Reynolds number had only a small effect on base drag and total drag.

INTRODUCTION

The effects of viscosity on the aerodynamic characteristics of bodies moving at low subsonic speeds have been known for many years and have been evaluated by numerous investigators. The effects of viscosity at transonic speeds have been investigated to a limited extent, and significant effects on the flow over airfoils have been reported by Ackeret (reference 1) and Liepmann (reference 2). The relative thoroughness of these two investigations has furnished a good start toward a satisfactory evaluation and understanding of the effects of viscosity in transonic flow fields. However, little is known about viscous effects at supersonic speeds.

The experiments reported in references 3, 4, and 5 have succeeded in evaluating the magnitude of the skin friction for supersonic flows in pipes and on rotating surfaces, but not for flow over a slender body or an airfoil.² Reference 6 contains a small amount of data on the effects of Reynolds number on the drag of a sphere and a circular cylinder; however, these data are not applicable to aerodynamic shapes which are practical for supersonic flight.

It has been sometimes assumed that the effects of viscosity are small and need be considered only when determining the magnitude of skin friction. In reviewing past data for the effects of viscosity it was found that in many reports, such as references 7 and 8, the model size was not stated, thereby rendering the calculation of Reynolds number and the evaluation of such tests quite difficult. Preliminary tests made during 1945 in the Ames 1- by 3-foot supersonic wind tunnel No. 1, which is a variable-pressure tunnel, showed a relatively large effect of Reynolds number on the drag of bodies of revolution. The results of this cursory investigation were not reported because the magnitude of support interference was not known and because certain inaccuracies in the balance measurements were known to exist in the data taken at low tunnel pressures. An investigation of wing-body interaction at supersonic speeds has been conducted subsequently and the results presented in reference 9. Because of the support interference and the balance inaccuracies noted at low pressures the data presented therein on the effect of

¹ Supersedes NACA RM A7A31a, "Experimental Investigation of the Effects of Viscosity on the Drag of Bodies of Revolution at a Mach Number of 1.5" by Chapman, Dean R., and Perkins, Edward W., 1947. The principal results of various investigations conducted subsequent to 1947 which either pertain to or supplement the experiments described herein are indicated in footnotes.

² Subsequent to 1947 several investigations of skin friction have been conducted which indicate that at the supersonic Mach number of the present investigation the laminar and turbulent skin-friction coefficients for a flat plate are about 3- and 12-percent lower, respectively, than for low-speed flow at the same Reynolds number.

Reynolds number on the drag of smooth bodies are not sufficiently accurate throughout the range of Reynolds numbers for direct application to the conditions of free flight.

Since the effects of viscosity were known to be relatively large at the outset of this investigation, the purpose of the present research was made twofold. The primary purpose was to develop an understanding of the mechanism by which viscosity alters the theoretical inviscid flow over bodies of revolution at supersonic speeds, and the secondary purpose was to determine the magnitude of these effects for the particular bodies investigated. The experiments were conducted during 1946.

APPARATUS AND TEST METHODS

WIND TUNNEL AND INSTRUMENTATION

A general description of the wind tunnel and the principal instrumentation used can be found in reference 9. Included therein is a description of the strain-gage balance system employed for measuring aerodynamic forces and the schlieren apparatus which forms an integral part of the wind-tunnel equipment. In order to obtain accurate data at low as well as high tunnel pressures, a more sensitive drag gage was used in the present investigation than in the investigation of reference 9; however, all other details of the balance system were the same.

The tunnel total pressure, the static reference pressure in the test section, and the pressure in the air chamber of the balance housing were observed on a mercury manometer. Because the difference between the base pressure and the static reference pressure in the test section was ordinarily too small (only 0.5 cm. of mercury at low tunnel pressures) to be accurately read from a mercury manometer, a supplementary manometer using a fluid of lower specific gravity was employed. Because of its lower vapor pressure and its property of releasing little or no dissolved air when exposed to very low pressures, dibutyl phthalate, having a specific gravity of approximately 1.05 at room temperatures, was used as an indicating fluid in this manometer instead of the conventional light manometer fluids such as water and alcohol.

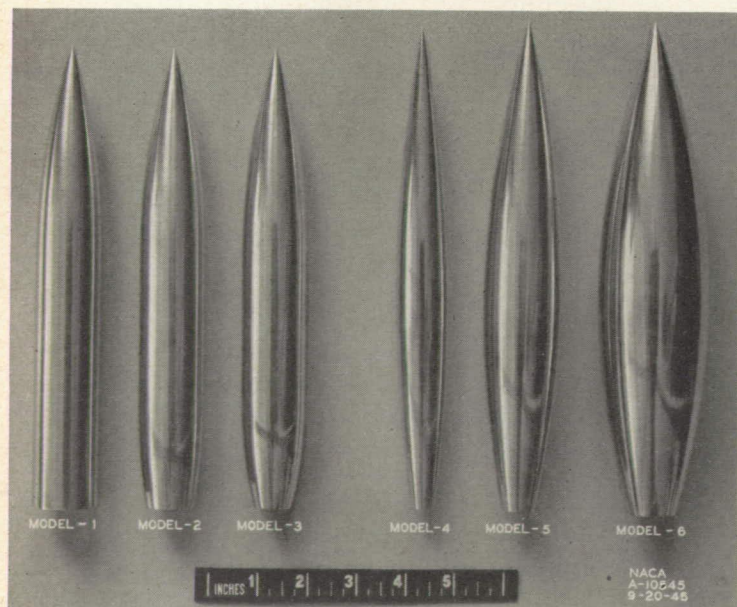
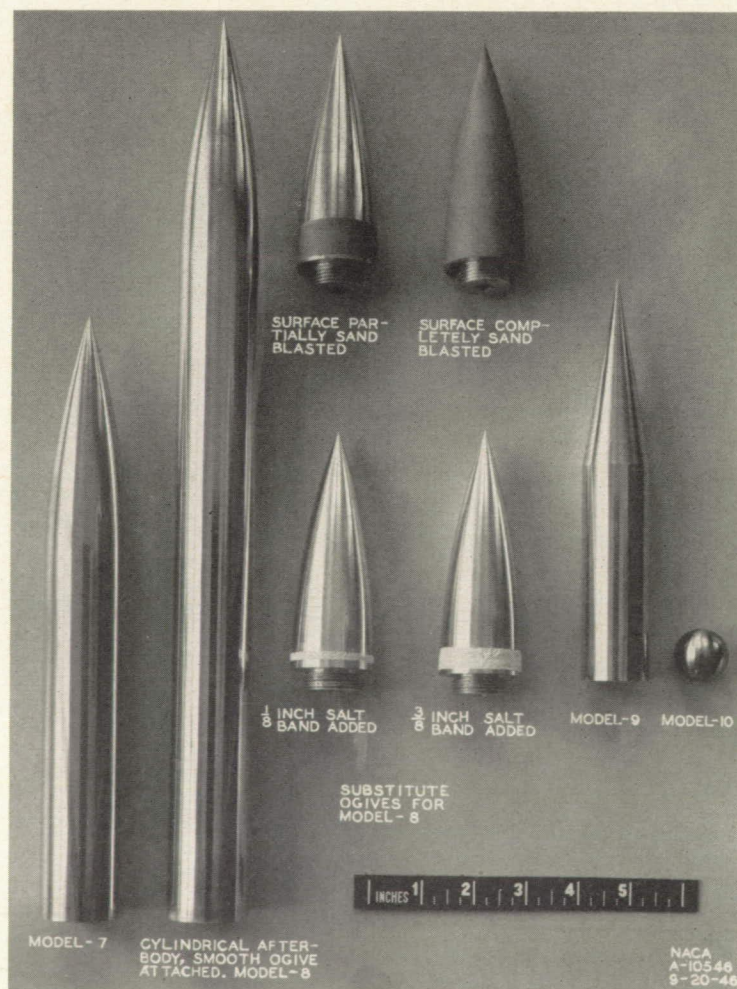


FIGURE 1.—Principal body shapes investigated.

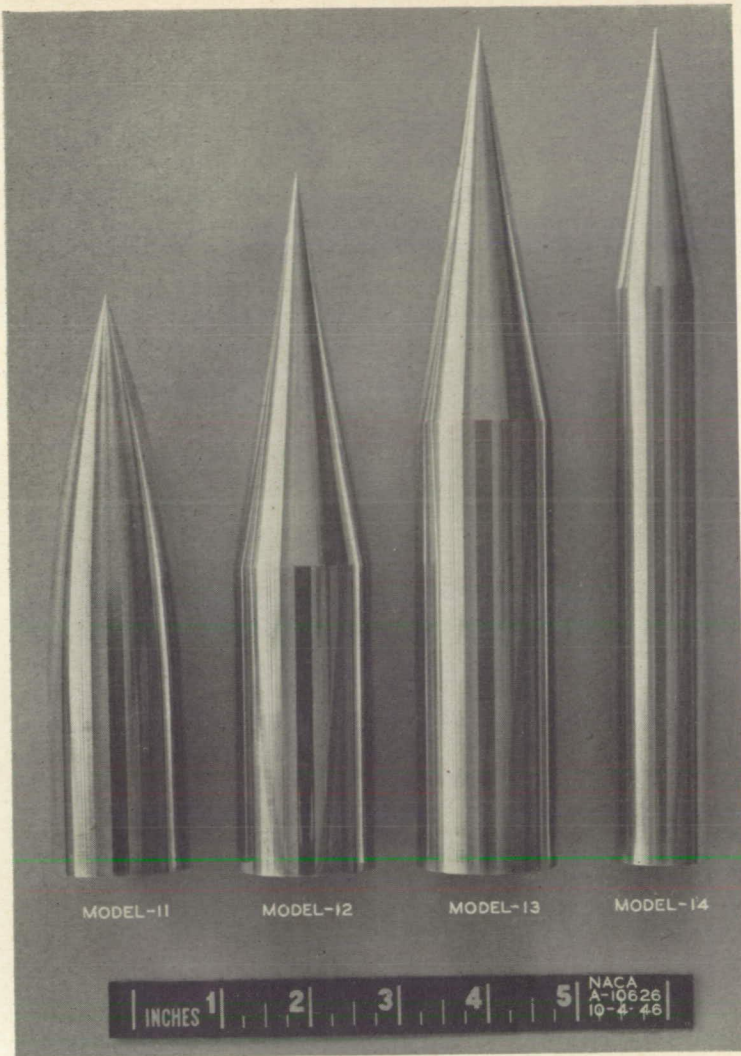
MODELS AND SUPPORTS

Photographs of the models, which were made of aluminum alloy, are shown in figures 1 and 2, and their dimensions are given in figure 3. Models 1, 2, and 3 were each formed of a 10-caliber ogive nose followed by a short cylindrical section; they differ from one another only in the amount of boat-tailing. The shape of the ogive was not varied in this investigation because the flow over it is not affected appreciably by viscosity. Models 4, 5, and 6, which differ from one another only in thickness ratio, were formed by parabolic arcs with the vertex at the position of maximum thickness. For convenience, some of the more important geometric properties of models 1 through 6 are listed in the following table:

Model	Frontal area A (sq in.)	Nose half angle (deg)	Length diameter ratio l/D	Base-area ratio A_B/A
1-----	1.227	18.2	7.0	1.00
2-----	1.227	18.2	7.0	.558
3-----	1.227	18.2	7.0	.348
4-----	.866	11.3	8.8	.191
5-----	1.758	15.9	6.2	.186
6-----	3.426	21.8	4.4	.187

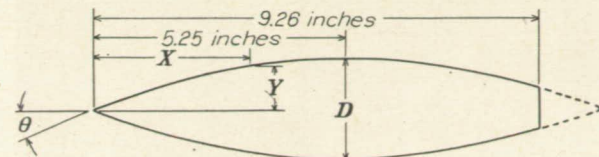
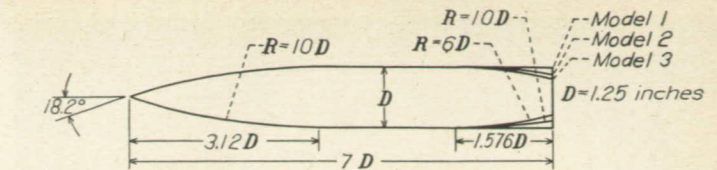


(a) Models used for boundary-layer tests and for comparison tests with other investigations.
FIGURE 2.—Special-purpose models.



(b) Models used to evaluate effect of length-diameter ratio on base pressure.
FIGURE 2.—Concluded.

In addition to the above-mentioned models, several other bodies were tested for certain specific purposes. Thus, models 7 and 8 were made unusually long so that the skin friction would be a large portion of the measured drag, thereby enabling the condition of the boundary layer to be deduced from force tests. Various substitute ogives, shown in figure 2 (a), were made interchangeable with the smooth ogive that is shown attached to the cylindrical afterbody of model 8. These ogives were provided with different types and amounts of roughness and could be tested either alone or with the long cylindrical afterbody attached. When the ogives were tested alone, a shroud of the same diameter as the ogive was used to replace the cylindrical afterbody. Model 9, a body with a conical nose, and model 10, a sphere of 1-inch diameter, were tested in order to compare the results of the present investigation with existing theoretical calculations and with the results of other experimental investigations. Models 11, 12, 13, and 14 were constructed to determine the effects of the length-diameter ratio for a fixed shape of afterbody. In all cases when a smooth surface was desired, the models were polished before testing to

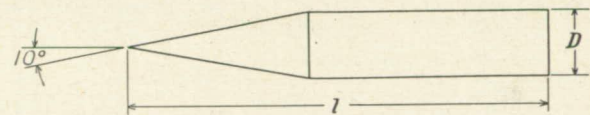
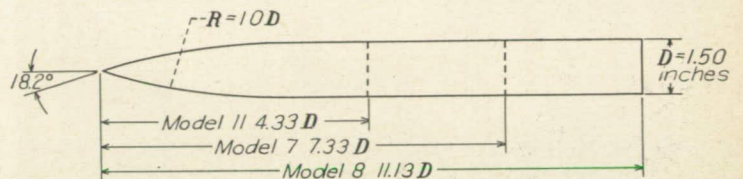


$$\text{Model 4, } Y=2.1 \left[\frac{X}{10.5} - \left(\frac{X}{10.5} \right)^2 \right], \theta=11.3^\circ, D=1.05 \text{ inches.}$$

$$\text{Model 5, } Y=3.0 \left[\frac{X}{10.5} - \left(\frac{X}{10.5} \right)^2 \right], \theta=15.9^\circ, D=1.50 \text{ inches.}$$

$$\text{Model 6, } Y=4.2 \left[\frac{X}{10.5} - \left(\frac{X}{10.5} \right)^2 \right], \theta=21.8^\circ, D=2.10 \text{ inches.}$$

(a) Boattailed bodies.



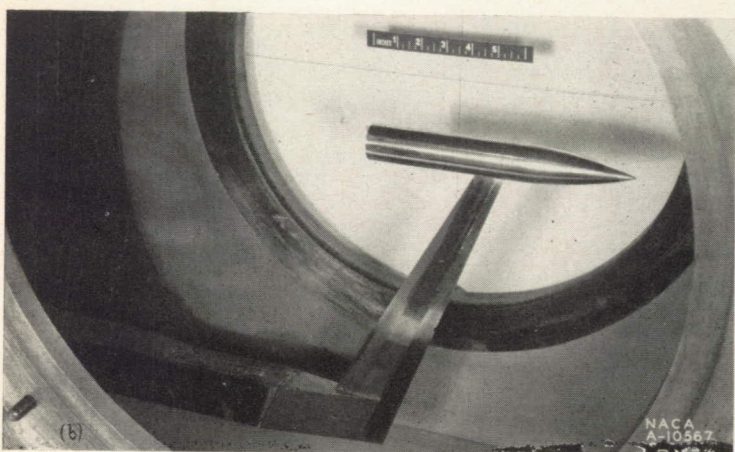
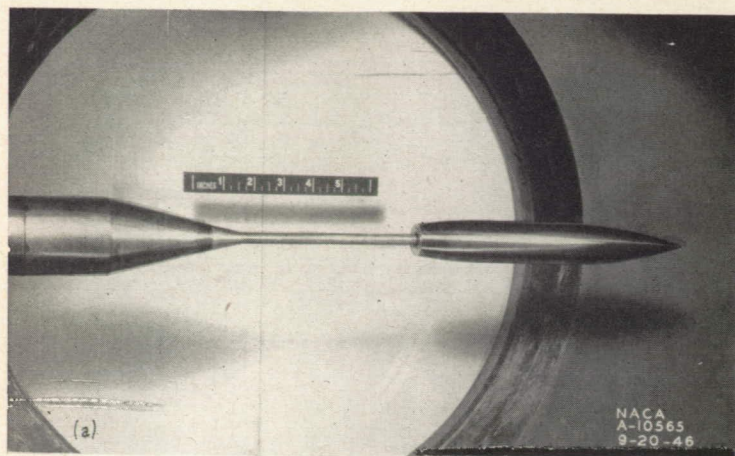
$$\begin{aligned} \text{Model 9, } l=7.5 \text{ inches, } D=1.25 \text{ inches.} \\ \text{Model 12, } l=7.5 \text{ inches, } D=1.50 \text{ inches.} \\ \text{Model 13, } l=9.0 \text{ inches, } D=1.50 \text{ inches.} \\ \text{Model 14, } l=9.0 \text{ inches, } D=1.00 \text{ inches.} \end{aligned}$$

(b) Models with cylindrical afterbodies.

FIGURE 3.—Model dimensions.

obtain a surface as free from scratches and machining marks as possible.

The models were supported in two different ways: by a rear support and by a side support, as shown in figure 4. The rear support used in the majority of the cases consisted of a sting which supported the model and attached to the balance beam. A thin steel shroud enclosed the sting and thereby eliminated the aerodynamic tare forces. Use of the rear support allowed force data, base pressure data, and schlieren photographs to be taken simultaneously. The side support which attached to the lower side of the model consisted of a 6-percent-thick airfoil of straight-side segments and 7° semiwedge angle at the leading and trailing edges. The side support was used to determine the effects of the axial variation in test-section static pressure on base pressure, and, in conjunction with a dummy rear support, to evaluate the effects of support interference.



(a) Rear support.
(b) Side support.

FIGURE 4.—Typical model installations.

TEST METHODS

The tests were conducted at zero angle of attack in a fixed nozzle designed to provide a uniform Mach number of approximately 1.5 in the test section. For the positions occupied by the different models, the free-stream Mach number actually varied from 1.49 to 1.51. This is somewhat lower than the Mach number of the tests reported in reference 9, which were conducted farther downstream in the test section.

Before and after each run precautions were taken to test the pressure lines for leaks and the balance system for friction or zero shift. Each run was made by starting the tunnel at a low pressure, usually 3 pounds per square inch absolute, and taking data at different levels of tunnel stagnation pressure up to a maximum of 25 pounds per square inch absolute. Because of the lag in the manometer system, approximately 15 minutes at low pressures and 5 minutes at high pressures were allowed for conditions to come to equilibrium. The over-all variation in Reynolds number based on body length ranged from about 0.10×10^6 to 9.4×10^6 . The specific humidity of the air usually was maintained below 0.0001 pound of water per pound of dry air, and in all cases was below 0.0003.

In general, each body was tested with a polished surface and then later with roughness added to fix transition. As illustrated in figure 2 (a), several different methods of fixing transition on a body in a supersonic stream were tried. The

usual carborundum method employed in subsonic research was not used because of the danger of blowing carborundum particles into the tunnel-drive compressors. The method finally adopted was to cement a $\frac{1}{8}$ -inch-wide band of particles of table salt around the body. This method proved successful at all but the very low Reynolds numbers. On models 1, 2, 3, and 12, roughness was located $\frac{1}{8}$ inch downstream of the beginning of the cylindrical section. On models 4, 5, and 6 the roughness was placed 4.5 inches from the nose and on model 8, $\frac{1}{8}$ inch upstream of the beginning of the cylindrical afterbody. Models 7, 9, 10, 11, 13, and 14 were tested in the smooth condition only.

RESULTS

REDUCTION OF DATA

The force data included in this report have been reduced to the usual coefficient form through division by the product of the free-stream dynamic pressure and the frontal area of the body. In each case, conditions just ahead of the nose of a model are taken as the free-stream conditions.

The measurements of the pressure on the base of each model are referred to free-stream static pressure and made dimensionless through division by the free-stream dynamic pressure. Thus, the base pressure coefficient is calculated from the equation

$$P_b = \frac{p_b - p_\infty}{q_\infty} \quad (1)$$

where

- P_b base pressure coefficient
- p_b pressure acting on the base
- p_∞ free-stream static pressure
- q_∞ free-stream dynamic pressure

The dynamic pressure is calculated from the isentropic relationships. A small experimentally determined correction is applied for the loss in total pressure due to condensation of water vapor in the nozzle. The Reynolds number is based upon the body length and is calculated from the isentropic relationships using Sutherland's formula for the variation of viscosity with the temperature of the air.

It is convenient to consider the force due to the base pressure as a separate component of the total drag. Accordingly, the base drag is referred to the frontal area and in coefficient form is given by

$$C_{D_b} = -P_b \left(\frac{A_b}{A} \right) \quad (2)$$

where

- C_{D_b} base drag coefficient
- A_b area of base
- A frontal area of the body

The foredrag is defined as the sum of all drag forces that act on the body surface forward of the base. Hence, the foredrag coefficient is given by

$$C_{D_F} = C_D - C_{D_b} \quad (3)$$

where C_D is the total drag coefficient and C_{D_F} the foredrag coefficient. The concept of foredrag coefficient is useful for several reasons. It is the foredrag that is of direct importance to the practical designer when the pressure acting on the base of a body is altered by a jet of gases from a power plant. Considering the foredrag as an independent compo-

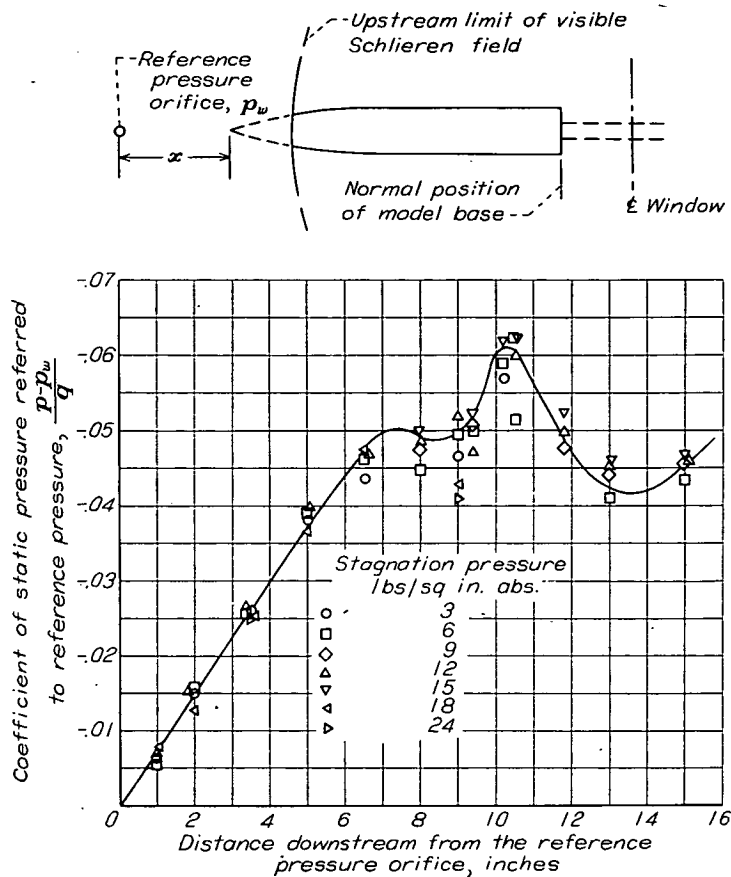
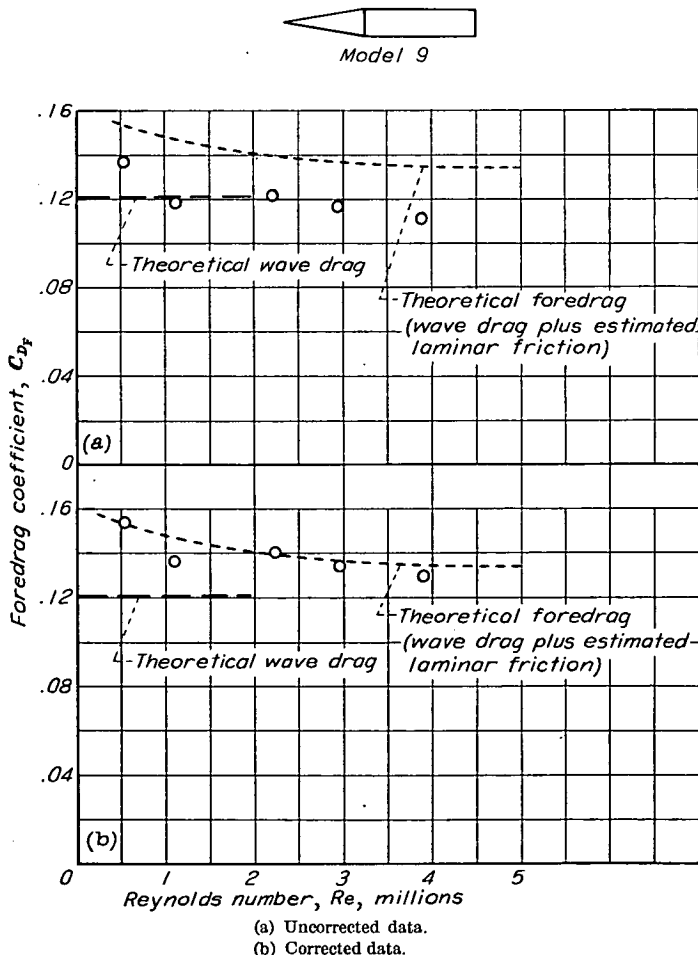
FIGURE 5.—Axial variation of the static pressure in the test section of the $M=1.5$ nozzle.

FIGURE 6.—Comparison of the foredrag coefficient of model 9 with and without corrections applied for the axial variation of the test-section static pressure.

nent of the total drag greatly simplifies the drag analysis of a given body. Finally, the foredrag, as will be explained later, is not affected appreciably by interference of the rear supports used in the investigation.

Since the nozzle calibration with no model present showed that the static pressure along the axis of the test section was not constant (fig. 5), the measured coefficients have been corrected for the increment of drag or pressure resulting from the axial pressure gradient. A detailed discussion of this correction is presented in appendix A, and the experimental justification is shown in figures 6 and 7.

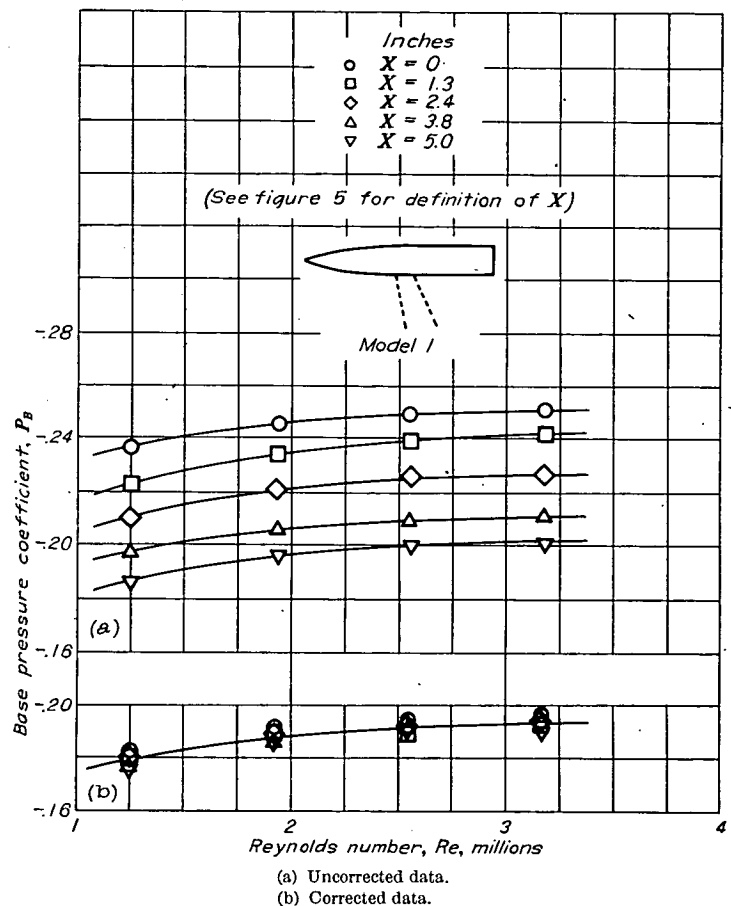


FIGURE 7.—Comparison of base pressure coefficients of model 1 measured at various positions along the tunnel axis, with and without corrections applied for the variation of test-section static pressure.

PRECISION

The table which follows lists the total uncertainty that would be introduced into each coefficient in the majority of the results if all of the possible errors that are known to exist in the measurement of the forces and pressures and the determination of free-stream Mach number and gradient corrections were to accumulate. Actually the errors may be expected to be partially compensating, so the probable inaccuracy is about half that given in the table. The sources and estimated magnitudes of the probable errors involved are considered at greater length in appendix B. The values in the following table are for the lowest and highest tunnel pressures and vary linearly in between. The table does not apply to data that are presented in figures 9 (b), 13, and 14. It also does not apply to models 4, 5, and 6 in figures 23 (a) and 29 (a) where the possible variation in the balance cali-

bration constant may increase the limits of error as discussed in appendix B.

Coefficient	Maximum value of error at lowest pressure	Maximum value of error at highest pressure
Total drag	$\pm (2.4\% \text{ plus } 0.004)$	$\pm (1.1\% \text{ plus } 0.004)$
Foredrag	$\pm (1.6\% \text{ plus } 0.004)$	$\pm (0.6\% \text{ plus } 0.004)$
Base pressure	$\pm (1.6\% \text{ plus } 0.005)$	$\pm (0.5\% \text{ plus } 0.005)$
Base drag	$\pm [0.8\% \text{ plus } 0.005 (A_b/A)]$	$\pm [0.5\% \text{ plus } 0.005 (A_b/A)]$

EFFECTS OF SUPPORT INTERFERENCE

Previous to the present investigation an extensive series of tests was conducted to determine the body shape and support combinations necessary to eliminate or evaluate the support interference. Based upon the results obtained, a summary of which appears in appendix C, it is believed that all the drag data presented herein for the models tested in the smooth condition are free from support interference effects with the exception of the data shown in figure 27. Also, for the models tested with roughness, the foredrag data are free from interference effects. However, an uncertainty in the base pressure coefficient exists which may vary from a minimum of ± 0.005 to a maximum of ± 0.015 for the different bodies. As a result, the base drag coefficients and total drag coefficients for the same test conditions are subject to a corresponding small uncertainty.

THEORETICAL CALCULATIONS

Although at present no theoretical method is available for calculating the base pressure and hence the total drag of a body, several methods are available which provide an excellent theoretical standard to which the experimental measurements of foredrag can be compared. In this report the theoretical foredrag is considered to be the sum of the theoretical wave drag for an inviscid flow and the incompressible skin-friction drag corresponding to the type of boundary layer that exists on the body.

A typical pressure distribution for the theoretical inviscid flow over one of the boattailed bodies tested in this investigation is shown in figure 8. For purposes of comparison the pressure distribution as calculated by the linear theory of von Kármán and Moore is included in this figure.

The wave drag of the cone-cylinder bodies was obtained from the theoretical flow over cones (references 10 and 11).³ The wave drag for the ogive-cylinder bodies was calculated by the method of characteristics for rotationally symmetric supersonic flow as given in references 12 and 13. In accordance with the theoretical results of reference 14, the fluid rotation produced by the very small curvature of the head shock wave was neglected. For moderate supersonic Mach numbers this procedure is justified experimentally in reference 8, where the theoretical calculation using the method of characteristics as presented in reference 12 are shown to be in excellent agreement with the measured pressure distributions for ogives with cylindrical afterbodies.

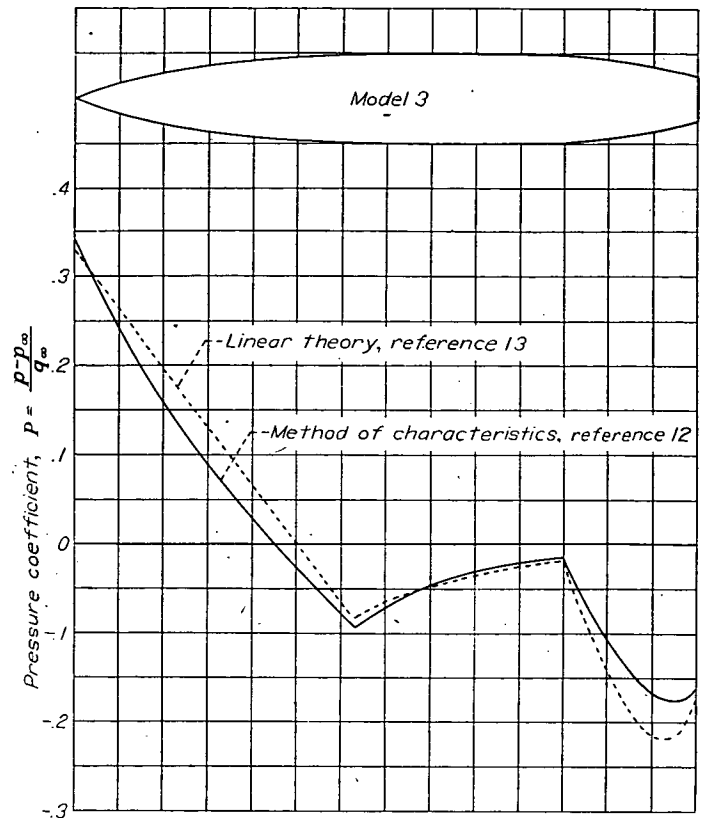


FIGURE 8.—Typical pressure distribution for a boattailed body at 1.5 Mach number.

An estimation of the skin-friction drag in any given case requires a knowledge of the condition of the boundary layer. The method used herein for laminar flow is as follows:

$$C_{D_f} = C_{f_{lam}} (A_F/A) \quad (4)$$

where

- C_{D_f} skin-friction drag coefficient for the model at the Reynolds number, Re , based on the full length of the model
- $C_{f_{lam}}$ low-speed skin-friction coefficient for laminar boundary-layer flow over a flat plate at Re
- A_F wetted area of the model forward of the base
- A frontal area of the model

For the models with roughness added it is assumed that the disturbance of the boundary layer resulting from the salt band was sufficient to cause transition to a turbulent boundary layer to occur at the band. The skin-friction drag is estimated by means of the equation

$$C_{D_f} = C'_{f_{lam}} \left(\frac{A_{lam}}{A} \right) + C_{f_{turb}} \left(\frac{A_F}{A} \right) - C'_{f_{turb}} \left(\frac{A_{lam}}{A} \right) \quad (5)$$

where

- $C'_{f_{lam}}$ low-speed skin-friction coefficient for laminar boundary-layer flow at the effective Reynolds number, Re' , based on the length of the model from the nose to the point where the salt band was added
- A_{lam} wetted area of that portion of the model forward of the salt band

³ In the original publication of the present investigation (1947) numerical values for the wave drag of the cones were based on the graphs of references 10 and 11. For the present report, however, slightly different numerical values are used which are based on more recent tabulated values of the surface pressure on cones in supersonic flow.

$C_{f_{turb}}$ low-speed skin-friction coefficient for turbulent boundary-layer flow over flat plate at the Reynolds number Re , based on the full length of the model

$C_{f'_{turb}}$ low-speed skin-friction coefficient for turbulent boundary-layer flow at the effective Reynolds number Re'

This method of calculation presumes that the fixed roughness was of such a nature as to cause the turbulent boundary-layer flow downstream of the point where the roughness was added to be the same as would have existed had the boundary-layer flow been turbulent all the way from the nose of the body.

DISCUSSION

FLOW CHARACTERISTICS

Before analyzing the effects of viscosity on the drag of the bodies of revolution, it is convenient to consider qualitatively the effects on the general characteristics of the observed flow. In so doing it is advantageous to consider first the condition of the boundary layer characterized by whether it is laminar or turbulent and then the effect of variation in Reynolds number on flow separation for each type of boundary layer.

Once the effects of the Reynolds number and the condition of the boundary layer on flow separation are known, the observed effects on the shock-wave configuration at the base of the model are easily explained. Likewise, once the effects on flow separation and shock-wave configuration are known, the resulting effects of viscosity on the foredrag, base drag, and total drag are easily understood.

Condition of the boundary layer.—Since results observed at transonic speeds (references 1 and 2) have shown that the general flow pattern about a body depends to a marked degree on the type of boundary layer present, it might be expected that the boundary-layer flow at supersonic speeds also may be of primary importance in determining the over-all aerodynamic characteristics of a body. Consequently, the determination of the extent of the laminar boundary layer under normal test conditions is of fundamental importance.

In an attempt to determine the highest Reynolds number at which laminar flow exists on models tested in this investigation, a relatively long polished body (model 7) was tested from a low pressure up to the highest tunnel pressure obtainable. In this case, the diameter of the shroud which enclosed the rear support sting was made the same as the diameter of the body. The foredrag measurements on this model are shown in figure 9 (a). Since the skin friction is a relatively large portion of the measured foredrag, the condition of the boundary layer can be deduced from these force tests. The data indicate that the boundary layer on this body was still laminar up to the highest obtainable Reynolds number of 6.5×10^6 . The computed foredrag data used for comparison are obtained by adding a laminar or turbulent skin-friction coefficient based on low-speed characteristics to the experimental wave drag of the ogive nose. This latter is determined by subtracting from the ogive foredrag coefficients the low-speed laminar skin-friction coefficients for the smooth ogive at the higher Reynolds numbers where the error, resulting from the assumption of the low-speed co-

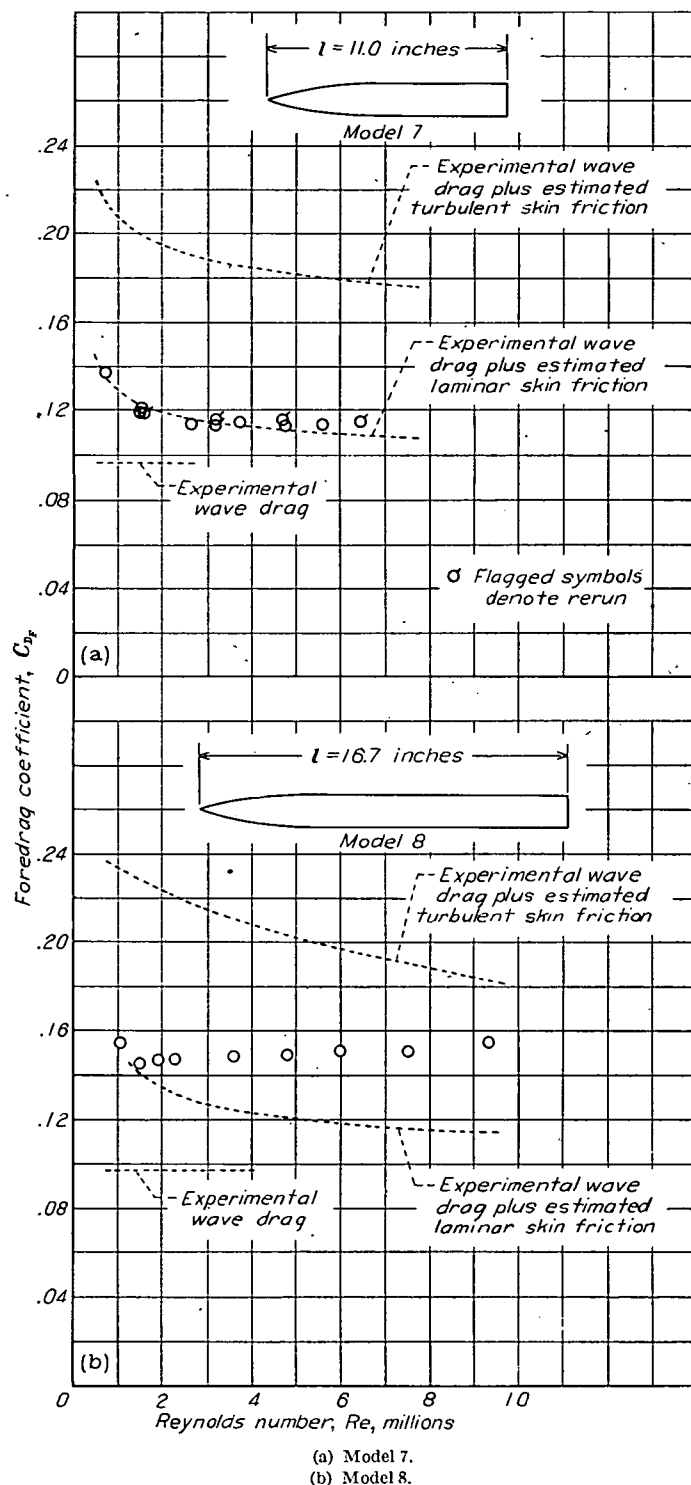


FIGURE 9.—Variation of foredrag coefficient with Reynolds number for models with long cylindrical afterbodies.

efficients, is a small percent of the deduced wave drag. (The theoretical wave drag, based on the theoretical pressure distribution from the method of characteristics, is approximately 5 percent higher than the experimental wave drag.) Schlieren photographs from which the condition of the boundary layer may be observed are shown in figure 10. They confirm the previous finding by showing that transition does not occur on the body, but begins a short distance downstream from the base of the model, as indicated by arrow 1 in the photograph.

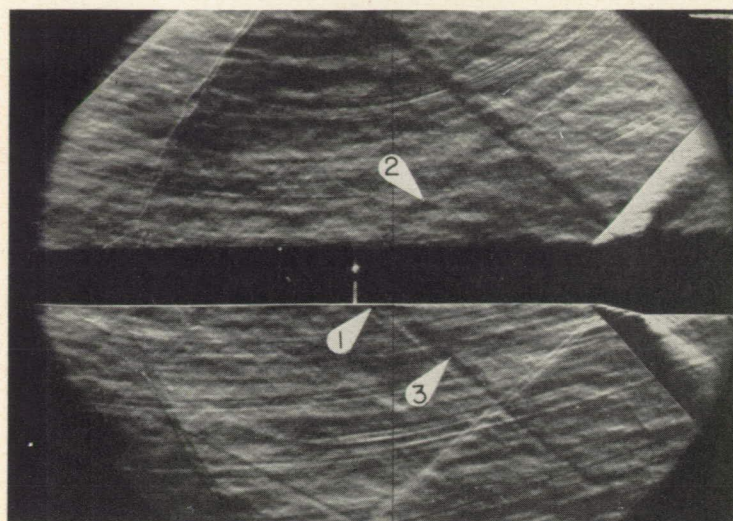
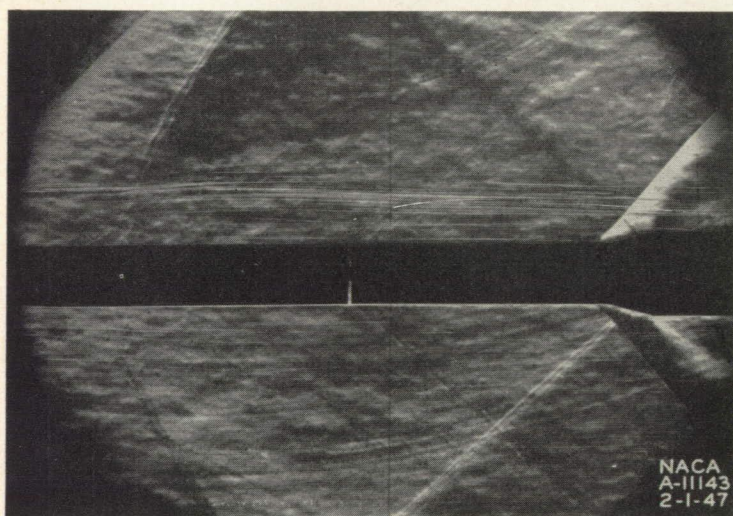
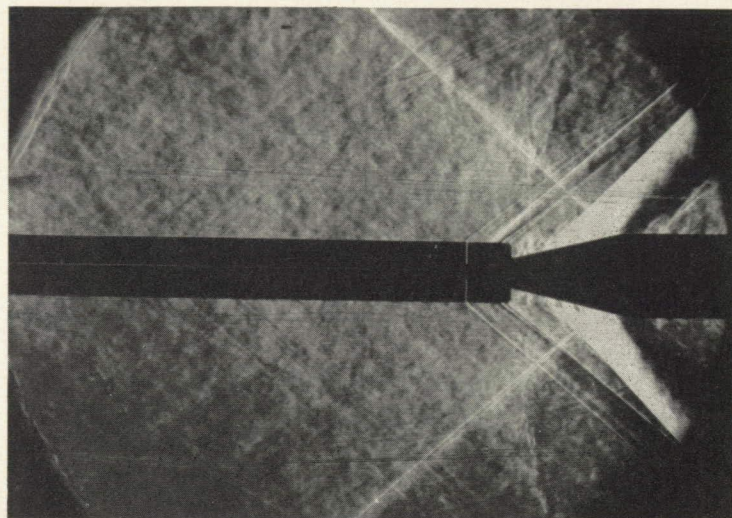
(a) $Re = 3.7 \times 10^6$.(b) $Re = 6.5 \times 10^6$.

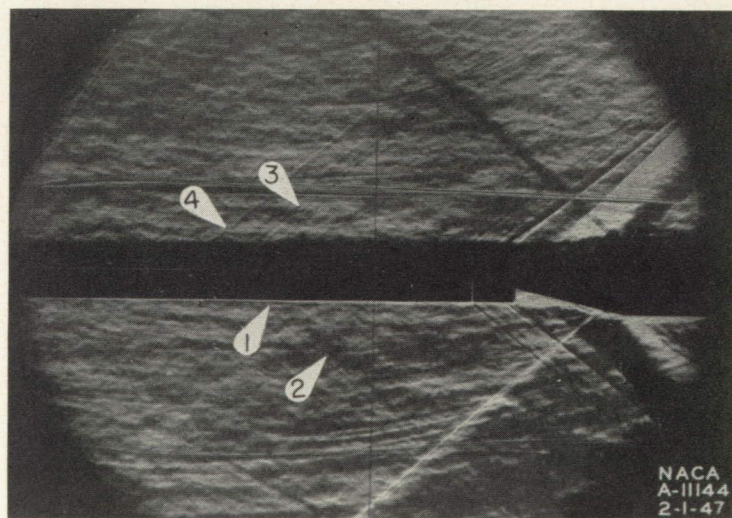
FIGURE 10.—Schlieren photographs showing laminar flow over the cylindrical afterbody of model 7 at two values of the Reynolds number. Knife edge horizontal.

A close examination of the photographs in figure 10 reveals that the beginning of transition (arrow 1) is located at the same point on the support shroud as the waves (arrows 2 and 3) which originate from a disturbance of the boundary layer. It was found by measurements on the schlieren photographs that the point of origin of these waves coincided with the intersection of the shroud and the reflected bow wave. This suggests that transition on the shroud is being brought about prematurely by the reflected bow waves. Additional evidence that this is not natural transition is obtained in noting from figure 10 that the point where transition begins does not move with a change in Reynolds number. If the model were longer than a critical length, which is about 11 inches for the conditions of the present tests, these reflected waves would strike the model somewhere on the afterbody and premature transition would be expected to affect the results. Figure 9 (b) shows the results of the measurements of fore-drag on a 16.7-inch body (model 8), which is considerably longer than the critical length. These force data confirm the above conjecture by clearly indicating a partially turbulent boundary layer on the body even at Reynolds numbers

as low as 2×10^6 . The schlieren photographs of the flow over this body are presented in figure 11. It is seen that, in this case also, the transition to turbulent flow (arrow 1) is located at the same point as the waves (arrows 2 and 3) originating from the disturbance of the boundary layer by the reflected bow wave. Similarly, an additional small wave (arrow 4) can be traced back to a disturbance of the boundary layer caused by a shock wave originating from an imperfect fit of the glass windows in the side walls.



(a) Knife edge vertical.



(b) Knife edge horizontal.

FIGURE 11.—Schlieren photograph showing premature transition on the cylindrical afterbody of model 8. Reynolds number 9.35×10^6 .

Although the maximum possible extent of laminar flow that may be expected on bodies of revolution cannot be determined on the basis of the present tests because of this interference from the reflected shock waves, the foregoing results show that, under the conditions of these tests, a laminar boundary layer exists over the entire surface of a smooth model about 11 inches long up to at least 6.5×10^6 Reynolds number. In comparison to the values normally encountered at subsonic speeds, a Reynolds number of 6.5×10^6 at first appears to be somewhat high for maintenance of laminar flow over a body, unless the pressure decreases in the direction

of the flow over the entire length of the body. The pressure distribution over model 7, shown in figure 12, has been determined by superimposing the pressure distribution which exists along the axis of the nozzle with no model present upon the theoretical pressure distribution calculated for model 7 by the method of characteristics. The resulting pressure distribution shows that the pressure decreases considerably along the ogive, but actually increases slightly along the cylindrical afterbody.

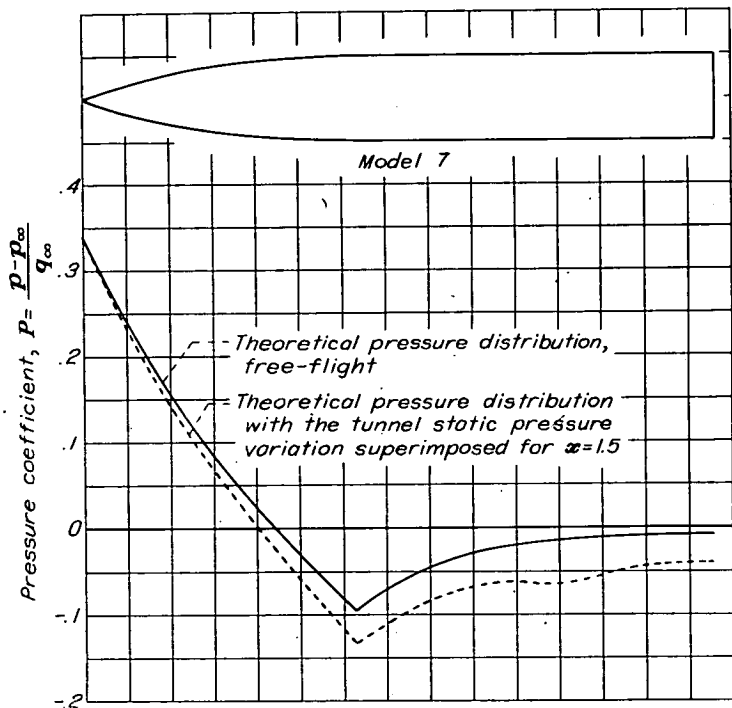


FIGURE 12.—Theoretical pressure distribution over the surface of model 7 at zero angle of attack and 1.5 Mach number.

An increase in the stability of the laminar boundary layer with an increase in Mach number has been indicated by the analysis of reference 15. With a given body shape, for which the pressure distribution changes with Mach number, an increase in stability with increasing Mach number has also been indicated for subsonic flows by the results of references 6 and 16 as well as by the experimental data given for airfoils in reference 15. The theoretical work of Lees (reference 17), however, indicates that the Reynolds number for neutral stability of laminar flow over an insulated flat plate decreases with an increase in Mach number.

It appears from the results of the present tests that any shock waves which originate from imperfections in the nozzle walls and disturb the boundary layer on a body can bring about transition prematurely. This may have some bearing on the results of the supersonic wind-tunnel tests conducted in the German wind tunnels at Kochel, since shock waves, ordinarily numbering about 15, are readily visible in various schlieren photographs. (See reference 18, for example.)

In order to cause the laminar boundary layer to become turbulent in the present investigation, an artifice such as adding roughness was necessary. In a supersonic stream, however, the addition of roughness to a body also increases

the wave drag. The magnitude of the wave drag due to roughness was determined by testing with full diameter shrouding and no afterbody attached, first the smooth ogive, and then the ogives with various amounts and kinds of roughness added (fig. 2 (a)). The corresponding foredrag measurements are shown in figure 13. These data illustrate that little additional drag is attributable to roughness at the low Reynolds numbers where the boundary layer is relatively thick, but that an appreciable amount of wave drag is attributable to the roughness at the higher Reynolds numbers. For all subsequent results presented, the amount of drag caused by the artificial roughness has been subtracted from the measured data taken for the bodies tested with transition fixed. In order to calculate the amount of drag caused by the roughness for models of diameters different from the ogives tested, it was assumed that for any model the increment in drag coefficient attributable to the drag of the artificial roughness was inversely proportional to the diameter of the model at the station at which the roughness was applied.

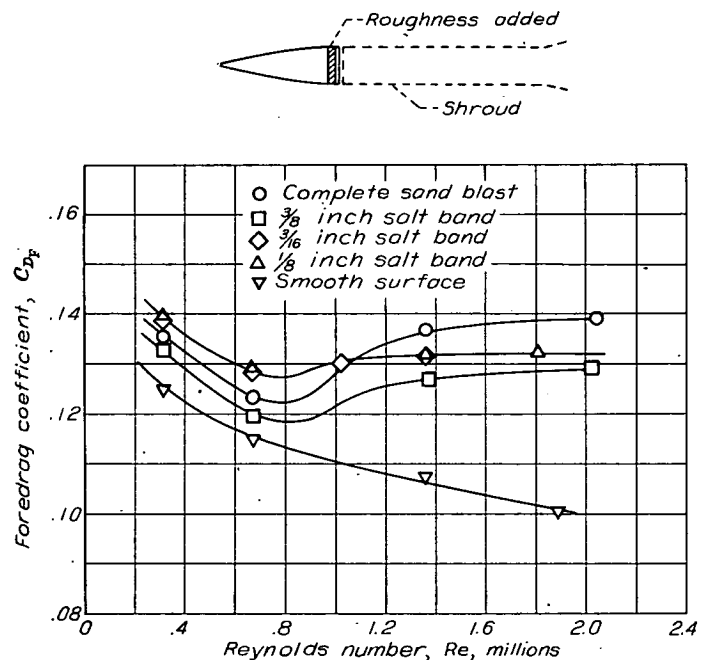


FIGURE 13.—Variation of foredrag coefficient with Reynolds number of the ogives with varying degrees of roughness added.

The foredrag measurements of model 8, which consists of a cylindrical afterbody with any one of the interchangeable ogives directly attached, are presented in figure 14. These data, from which the drag increment due to the added roughness has been subtracted as noted previously, show that the degree of roughness produced by sand blasting the surface of the ogive is insufficient to cause transition at low Reynolds numbers; whereas, the roughness produced by the $\frac{3}{16}$ -inch- or the $\frac{3}{8}$ -inch-wide salt band caused transition at all Reynolds numbers.

A vivid illustration of the turbulent character of the boundary layer on those bodies with roughness added is given by the schlieren photographs in figure 15. The boundary layer is best seen in the photograph taken with the knife edge horizontal. A comparison of these photographs with

those of laminar boundary layers (fig. 10, for example) illustrates how the condition of the boundary layer is apparent from schlieren photographs.

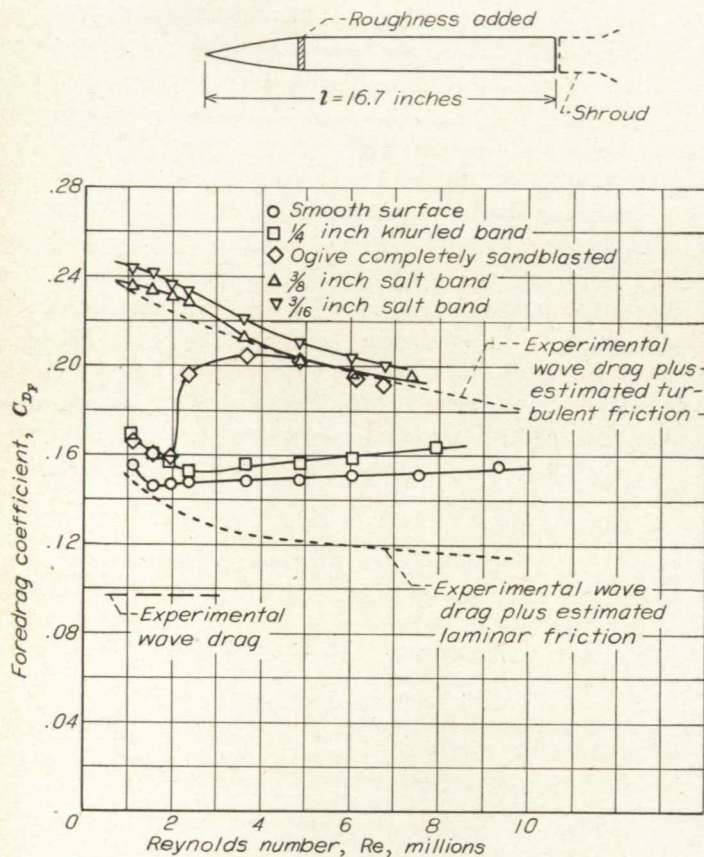
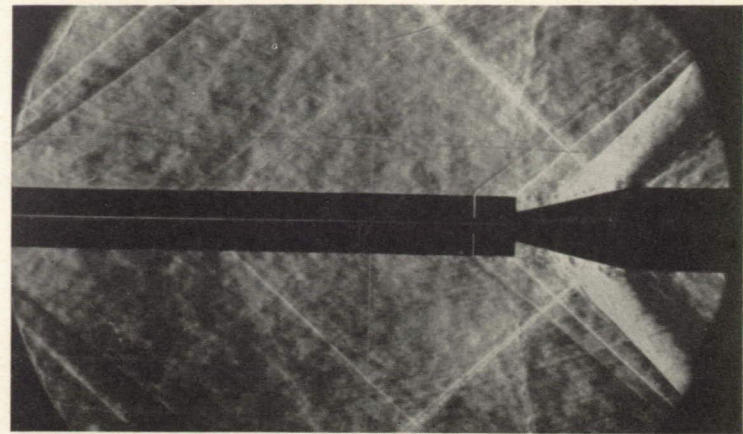


FIGURE 14.—Variation of foredrag coefficient with Reynolds number for model 8 with various amounts of roughness.

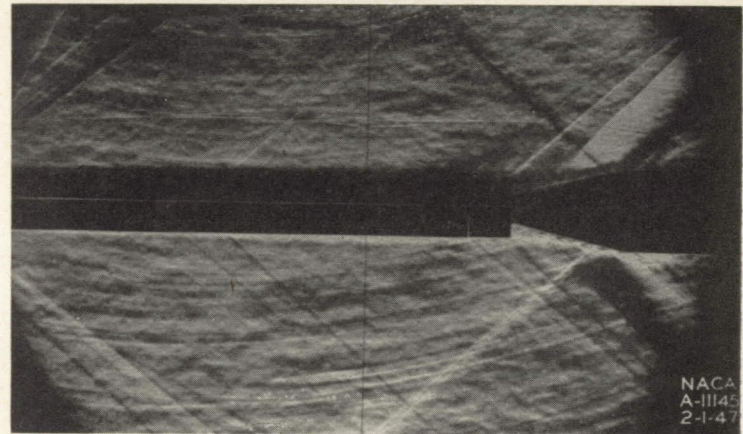
Flow Separation.—Changes in flow separation brought about by changing the boundary-layer flow from laminar to turbulent alter the effective shape of the body, the shock-wave configuration, and also the drag. It is therefore essential to consider the effects on flow separation of both the condition of the boundary layer and the Reynolds number.

The location and degree of separation of the laminar boundary layer for the boattailed bodies tested in the smooth condition varied noticeably with the Reynolds number of the flow. The schlieren photographs of model 6 in figure 16 are typical of this effect. Additional photographs, presented in figure 17, illustrate the same phenomena in the flow over models 2, 3, and 10, each at two different Reynolds numbers. In each case, as the Reynolds number of the flow is increased, the separation decreases, the convergence of the wake increases, and the trailing shock wave moves forward.

Separation of an apparently laminar boundary layer at supersonic speeds has been pointed out previously by Ferri in reference 19 for the two-dimensional flow over the surface of curved airfoils. The schlieren photographs of Ferri indicated that a shock wave formed at the point of laminar separation. On the other hand, the schlieren pictures of the flow fields for the bodies of revolution tested in the



(a) Knife edge vertical.



(b) Knife edge horizontal.

FIGURE 15.—Schlieren photographs of model 8 with transition fixed. Reynolds number 7.2×10^6 .

present investigation, show no definite shock wave accompanying separation except for the sphere (fig. 17) in which case the shock wave is very weak. It may be concluded, therefore, that a separation of the laminar boundary layer is not necessarily accompanied by a shock wave at supersonic speeds. The same conclusion for transonic flows has been drawn in reference 2.

In order to analyze more closely the details of the flow separation, the pressure distribution along the streamline just outside of the separated boundary layer was calculated for several flow conditions over models 3 and 6. The calculations were made using the method of characteristics and by obtaining the contour of the streamline just outside the separated boundary layer from enlargements of the schlieren photographs. Typical results from these calculations for model 3 are presented in figure 18. It is seen that the pressure on the outside of the boundary layer is approximately constant downstream of the point of separation as is characteristic along the boundary of a dead-air region. The pressure along the line of separation can be expected to be approximately equal to that in the dead-air region, and hence, equal to the base pressure. A comparison of the calculated values of the average pressure in the dead-air region with the measured values of the base pressure for several

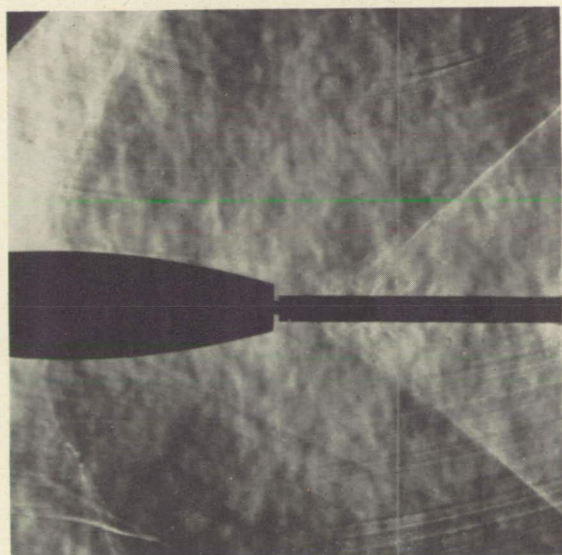
conditions of flow over models 3 and 6 is given in the following table:

Model	Reynolds number	Calculated pressure coefficient of dead-air region	Measured base pressure coefficient
3-----	0.6×10^6	-0.06	-0.06
3-----	2.0×10^6	-.11	-.12
6-----	$.6 \times 10^6$	-.10	-.11
6-----	1.5×10^6	-.13	-.13

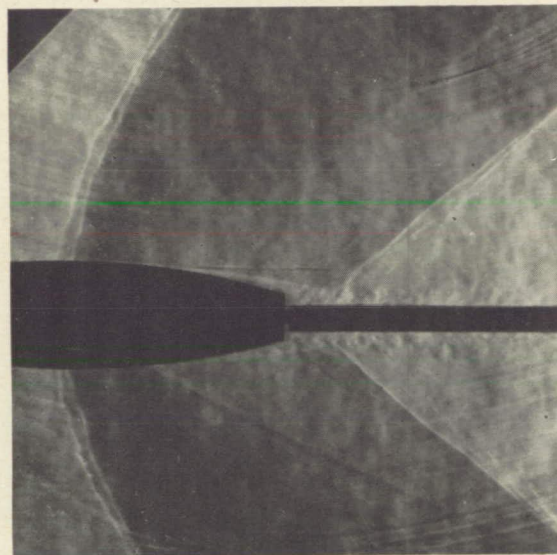
The preceding results indicate that under certain conditions the base pressure for laminar flow over highly boattailed bodies is directly related to the separation phenomenon which occurs forward of the base. This suggests that, if a means can be found to control the separation, the base pressure also can be controlled.

The theoretical pressure distributions on models 4 and 5 are similar to the pressure distribution on model 6, which is shown in figure 19. In each case the pressure in inviscid flow would decrease continually along the direction of flow upstream of the observed position of laminar separation. For subsonic flow this condition ordinarily would be termed favorable and separation would not be expected. Further research on this subject appears necessary in order to gain a satisfactory understanding of the observed results.

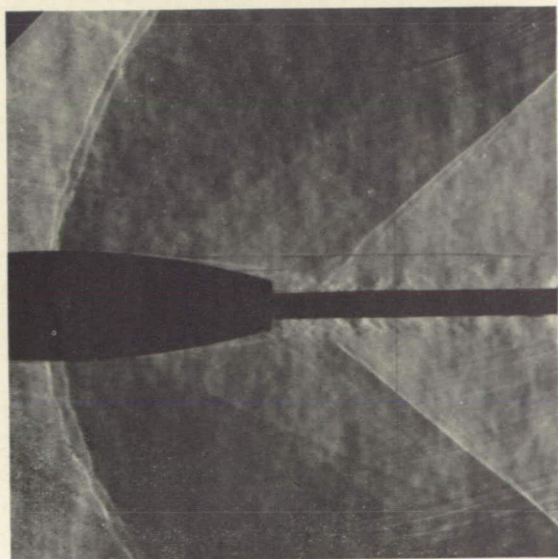
The findings of previous investigations of low-speed flows indicate that if a boundary layer which is normally laminar over the afterbody is made turbulent by either natural or artificial means, the resistance to separation is increased greatly. The tests on models 2, 3, 4, 5, and 6 with roughness added show clearly that this is also the case in supersonic flows. The two schlieren photographs presented in figure 20 were taken of model 6 with and without roughness added and are typical of this effect. A comparison of the two



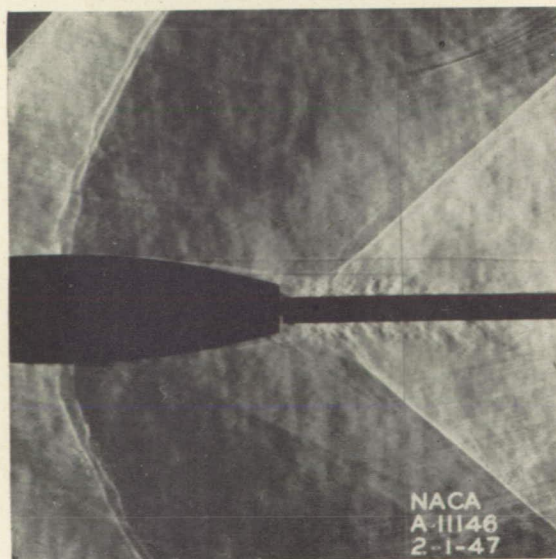
$Re = 0.58 \times 10^6$.



$Re = 0.87 \times 10^6$.



$Re = 1.1 \times 10^6$.



$Re = 1.4 \times 10^6$

FIGURE 16.—Schlieren photographs showing the effect of Reynolds number on laminar separation for model 6. Knife edge vertical.

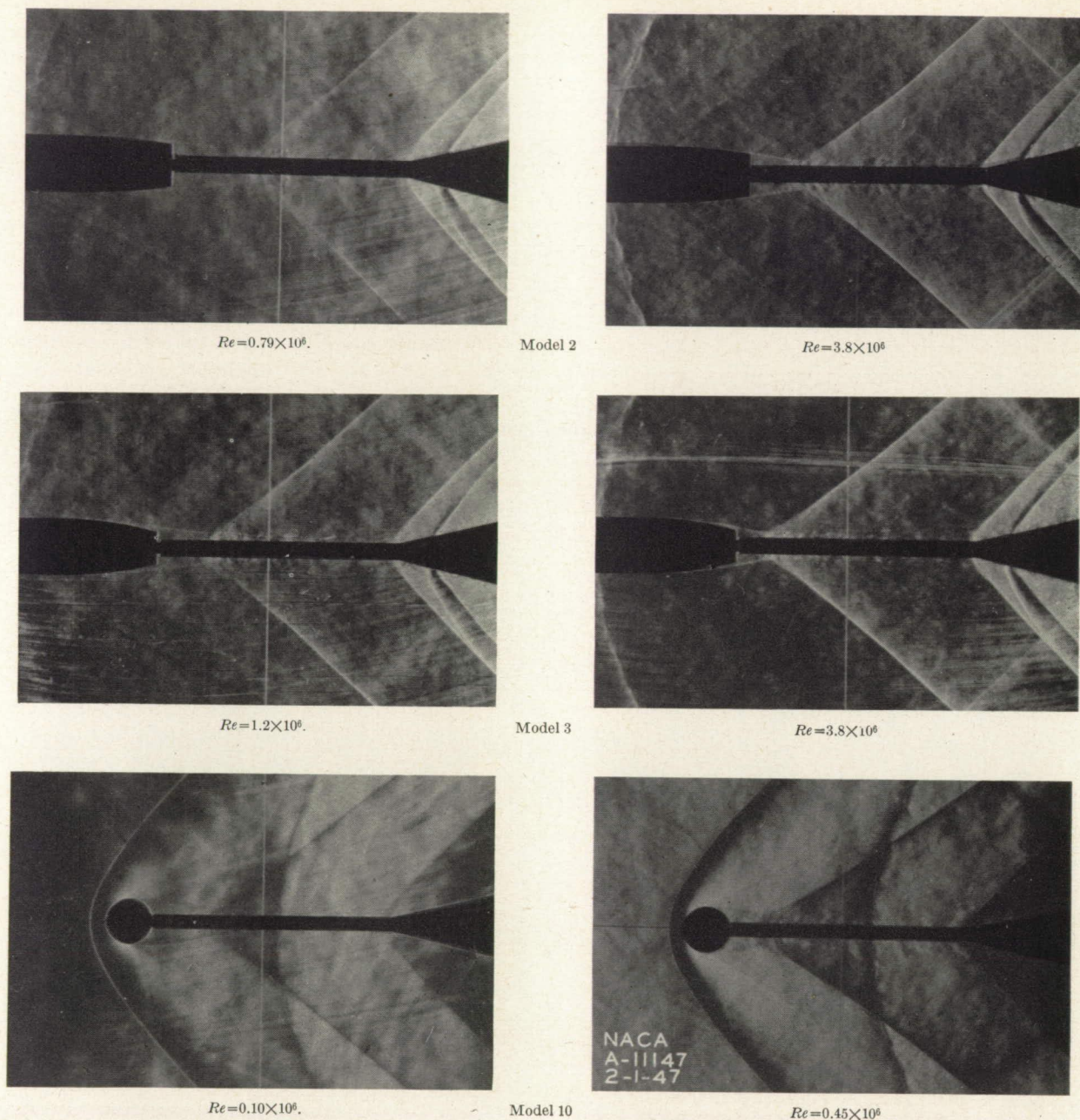


FIGURE 17.—Schlieren photographs showing the effect of Reynolds number on laminar separation for models 2, 3 and 10. Knife edge vertical.

photographs shows that, without roughness added, separation occurs near the point of maximum thickness, but if transition is fixed ahead of this point the separation point moves downstream close to the base.

Shock-wave configuration.—It is to be expected that the changes in flow separation due to changes in the condition of the boundary layer and in the Reynolds number of the flow will bring about changes in the shock-wave configuration at the base of a body. The schlieren photographs of figures 16 and 17, which show how the laminar separation decreases and the convergence of the wake increases as the Reynolds number is increased, also show that these phenomena are accompanied by a forward motion of the trailing shock wave. In general, as long as the boundary layer is laminar, the trailing shock wave moves forward as the Reynolds number increases, but no major change in the shock-wave configuration takes place.

The shock-wave configuration on a boattailed body with a turbulent boundary layer, however, is very much different from the configuration with a laminar layer, as is illustrated by the schlieren photographs of model 6, shown in figure 20. Such configuration changes due to the transition to turbulent boundary-layer flow correlate quite well with the angle β that the tangent to the surface just ahead of the base makes with the axis of symmetry. Figure 21 shows the changes in shock-wave configuration for models 1 through 6 arranged in order of increasing angle β . It is seen that, on the boattailed bodies with a small angle β , the transition to a turbulent boundary layer is accompanied by the appearance of a weak shock wave originating at the base of the body (models 4 and 2). For bodies with larger boattail angles (model 5) the strength of this wave, hereafter termed the "base shock wave," increases until it is approximately as strong as the original trailing shock wave. For even larger boattail angles,

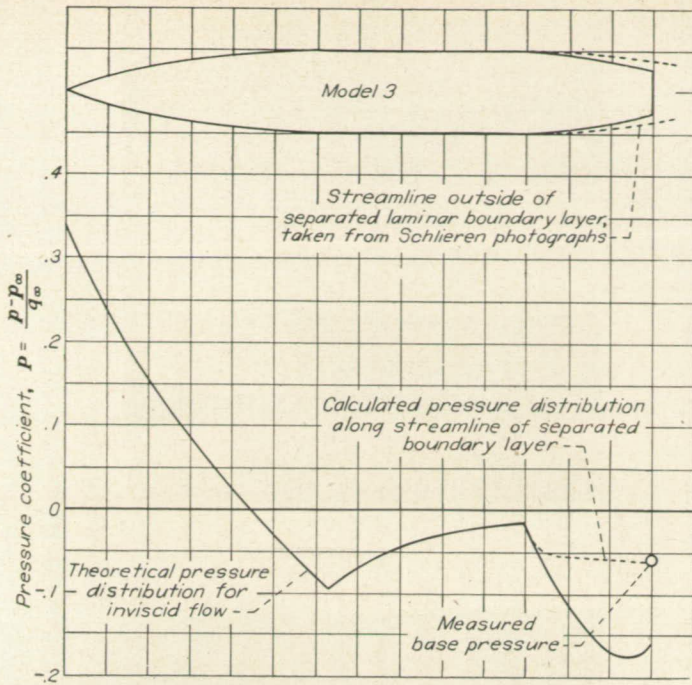


FIGURE 18.—Calculated pressure distribution for model 3 at a Reynolds number of 0.6×10^6 .

Apparent points of laminar separation occur in this region as determined from Schlieren photographs in figure 16 ($Re = 0.6 \times 10^6$ to $Re = 1.4 \times 10^6$)—

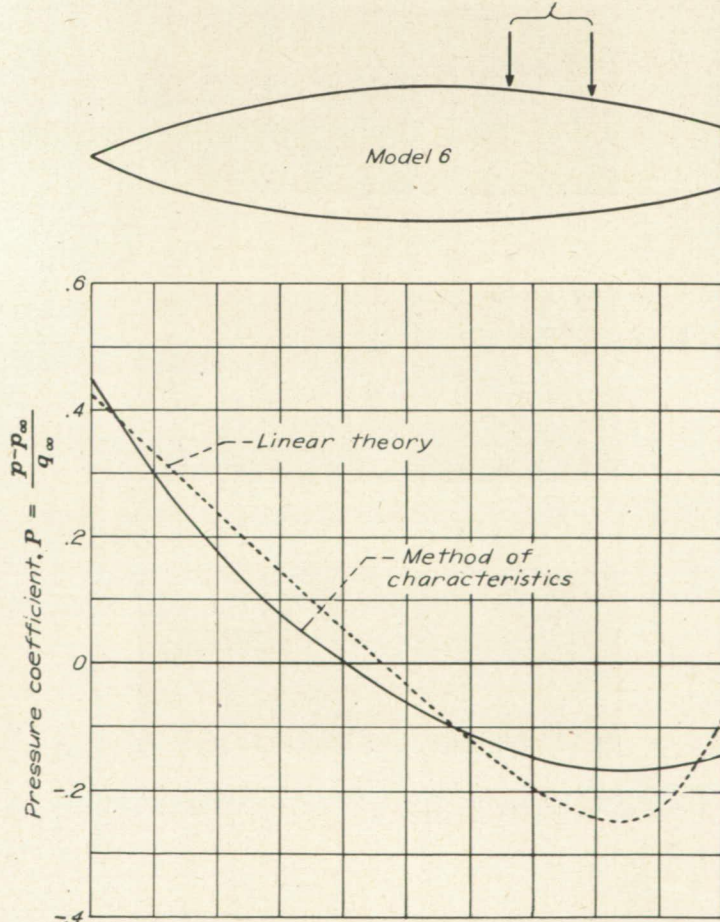
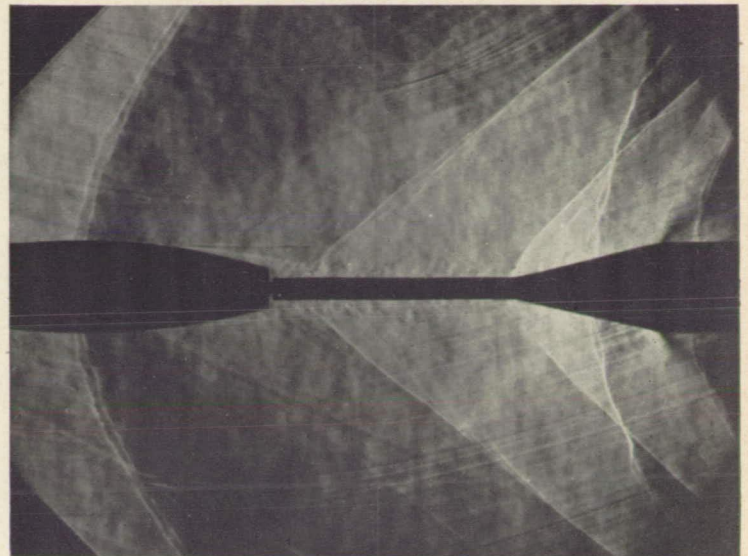
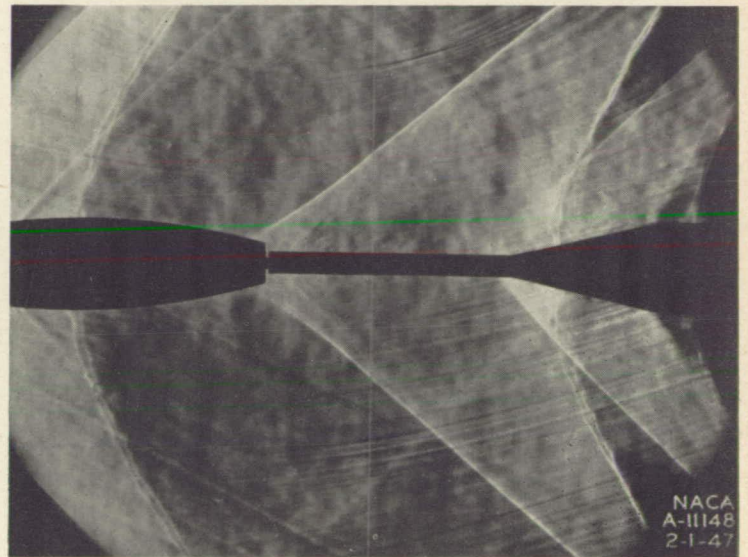


FIGURE 19.—Calculated pressure distribution on model 6.

⁴ Subsequent experiments with turbulent flow on model 3 at higher Mach numbers have shown that the base shock wave also exists at a Mach number of 2.0, but virtually disappears at a Mach number of 3.0 and higher.



(a) Laminar boundary layer, $Re = 0.87 \times 10^6$.

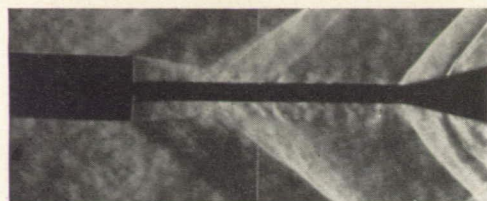


(b) Turbulent boundary layer, $Re = 0.87 \times 10^6$.

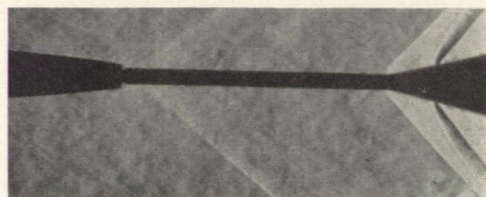
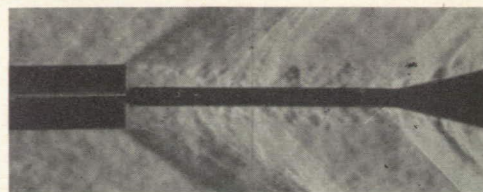
FIGURE 20.—Schlieren photographs of model 6 illustrating the effect on flow separation of the condition of the boundary layer.

the base shock wave becomes more distinct, and eventually is the only appreciable shock wave existing near the base of the body (models 3 and 6).⁴ In such a case, the compression through the base shock wave occurs forward of the base. This, as will be shown later, greatly increases the base pressure and decreases the base drag.

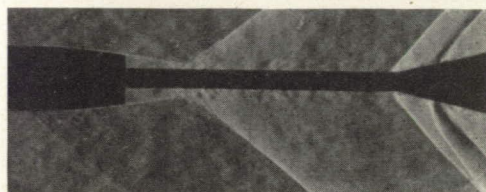
Compared to the phenomena observed with a laminar boundary layer (fig. 16), changes in the Reynolds number for a body with a turbulent boundary layer do not alter the shock-wave configuration to any significant extent, because the turbulent layer, even at low Reynolds numbers, ordinarily does not separate. This fact is evident in figure 22, which shows the schlieren photographs of model 3 at different Reynolds numbers with roughness added. No apparent change in the flow characteristics takes place as the Reynolds number is increased. With a turbulent boundary layer,



Model 1
 $Re = 3.8 \times 10^6$
 $\beta = 0^\circ$



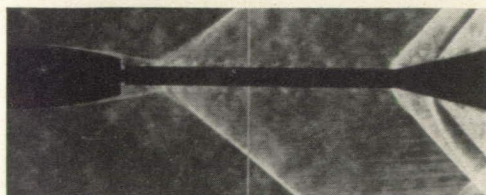
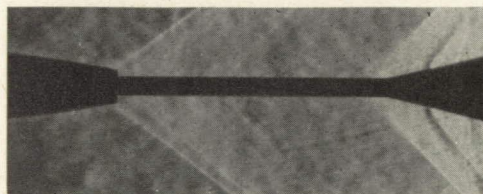
Model 4
 $Re = 4.0 \times 10^6$
 $\beta = 8.55^\circ$



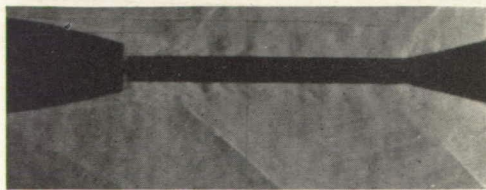
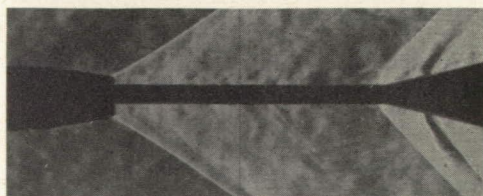
Model 2
 $Re = 3.8 \times 10^6$
 $\beta = 9.08^\circ$



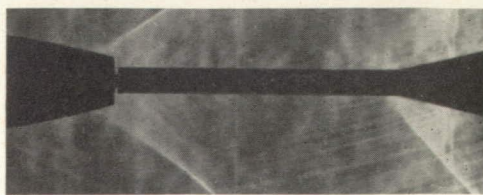
Model 5
 $Re = 2.7 \times 10^6$
 $\beta = 12.13^\circ$



Model 3
 $Re = 3.8 \times 10^6$
 $\beta = 15.25^\circ$



Model 6
 $Re = 1.1 \times 10^6$
 $\beta = 16.75^\circ$



Laminar

Turbulent

FIGURE 21.—Schlieren photographs showing the effect of turbulent boundary layer on shock-wave configuration at base of models 1, 2, 3, 4, 5 and 6. Knife edge vertical.

therefore, the effect on base drag of varying the Reynolds number may be expected to be much less than with a laminar layer.

ANALYSIS OF THE DRAG DATA

The qualitative effects of viscosity on flow separation and on shock-wave configuration, which have been discussed in the preceding sections, provide the physical basis for understanding the effects of varying the Reynolds number and changing the condition of the boundary layer on the drag coefficients of the various bodies tested.

Foredrag.—The foredrag coefficients of models 1 through 6 with laminar flow in the boundary layer are shown in figure 23 (a) as a function of the Reynolds number. These data show that, over the Reynolds number range covered in the tests, the foredrag of model 1 decreases about 20 percent, while that of model 6 increases about 15 percent. The foredrag of the other bodies does not change appreciably.

The reason the effects of Reynolds number vary consid-

erably with different body shapes is clearly illustrated by a comparison of the measured foredrags with the theoretical foredrags. In figure 24 (a) the theoretical and measured values of foredrag are compared for model 1, which has no boattailing, and for model 3, which is typical of the boat-tailed models. From this comparison, it is seen that, as previously noted for other models without boattailing, the theoretical and experimental foredrags for model 1 are in good agreement. The decrease in foredrag with increasing Reynolds number for the bodies without boattailing is due entirely to the decrease in skin-friction coefficient. For model 3, which has considerable boattailing, the curves of figure 24 (a) show that the theoretical and experimental foredrags agree only at high Reynolds numbers. At the low Reynolds numbers the measured foredrags are lower than the theoretical values because of the separation of the laminar boundary layer as previously illustrated by the schlieren photographs in figures 16 and 17. With sep-

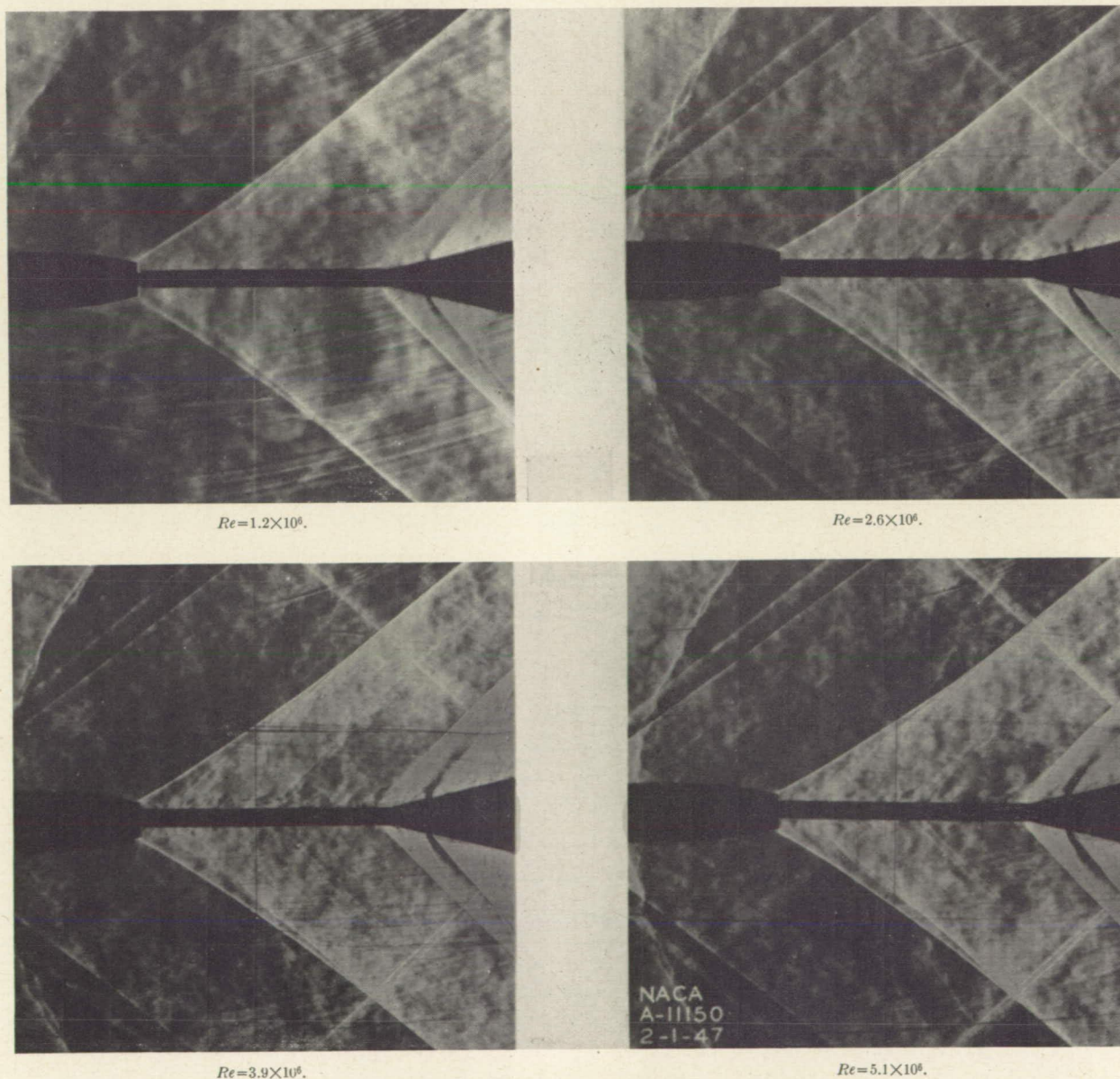


FIGURE 22.—Schlieren photographs showing the absence of any effect of Reynolds number on the flow over the afterbody of model 3 with roughness added. Knife edge vertical.

aration, the flow over the boattail does not follow the contour of the body, and the pressure in the accompanying dead-air region is higher than it would be if the separation did not occur (fig. 18). This makes the actual foredrag lower than the theoretical value for a flow without separation. At the higher Reynolds numbers, the separation is negligible and the flow closely follows the contour of the body; hence, the theoretical and experimental foredrags agree. The reason for the approximately constant foredrag of models 2, 3, 4, and 5, therefore, is that the changes due to skin friction and flow separation are compensating. For model 6 with a smooth surface, the foredrag shown in figure 23 (a) rises rather rapidly at low Reynolds numbers because the separation effects for this relatively thick body (fig. 16) more than compensate for the changes in skin friction due to the variation of the Reynolds number.

Figure 23 (b), which shows the foredrag coefficients of model 1 through 6 with roughness added, indicates that the foredrag for all the bodies decreases as the Reynolds number increases above a Reynolds number of 1.75×10^6 . This is to be expected, since with the change to turbulent boundary layer and consequent elimination of separation, the only factor remaining to influence the foredrag coefficients is the decrease of skin-friction coefficients with increase in Reynolds number. Below a Reynolds number of 1.75×10^6 , however, the foredrag of all the models except model 1 increases with increasing Reynolds number. The cause of this somewhat puzzling behavior is apparent upon closer examination of the data.

Figure 24 (b) shows a comparison of the theoretical fore-

drags with the experimental values for models 1 and 3 with roughness added. The theoretical value for skin-friction drag was calculated assuming laminar flow up to the location of the roughness, and turbulent flow behind it. This value of drag was added to the theoretical wave drag to obtain the theoretical foredrag. It is seen from figure 24 (b) that for model 1 the curves of theoretical and experimental foredrag have the previously indicated trend of decreasing drag with increasing Reynolds number over the entire range. However, for model 3, which is typical of the boattailed bodies, the measured foredrag at low Reynolds numbers falls considerably below the theoretical value in the manner previously noted. The reason for this is evident from an examination of the schlieren photographs shown in figure 25, which were taken of the flow over models 3 and 6 with roughness added. They show that at the low Reynolds numbers a flow separation similar to that observed for the smooth body (fig. 16) occurs, and the resulting shock-wave configuration is characteristic of the configuration for a laminar boundary layer rather than that for a turbulent boundary layer. It appears that, at the low Reynolds numbers, the amount of roughness added does not cause transition far enough upstream of the point for laminar separation so that the free stream can provide the boundary layer with the necessary additional momentum to prevent separation. The portions of the drag curves in which the desired transition was not realized are shown dotted over the region in which separation was apparent from the schlieren pictures. For model 1, the schlieren photographs showed that at the low Reynolds numbers the amount of roughness added was

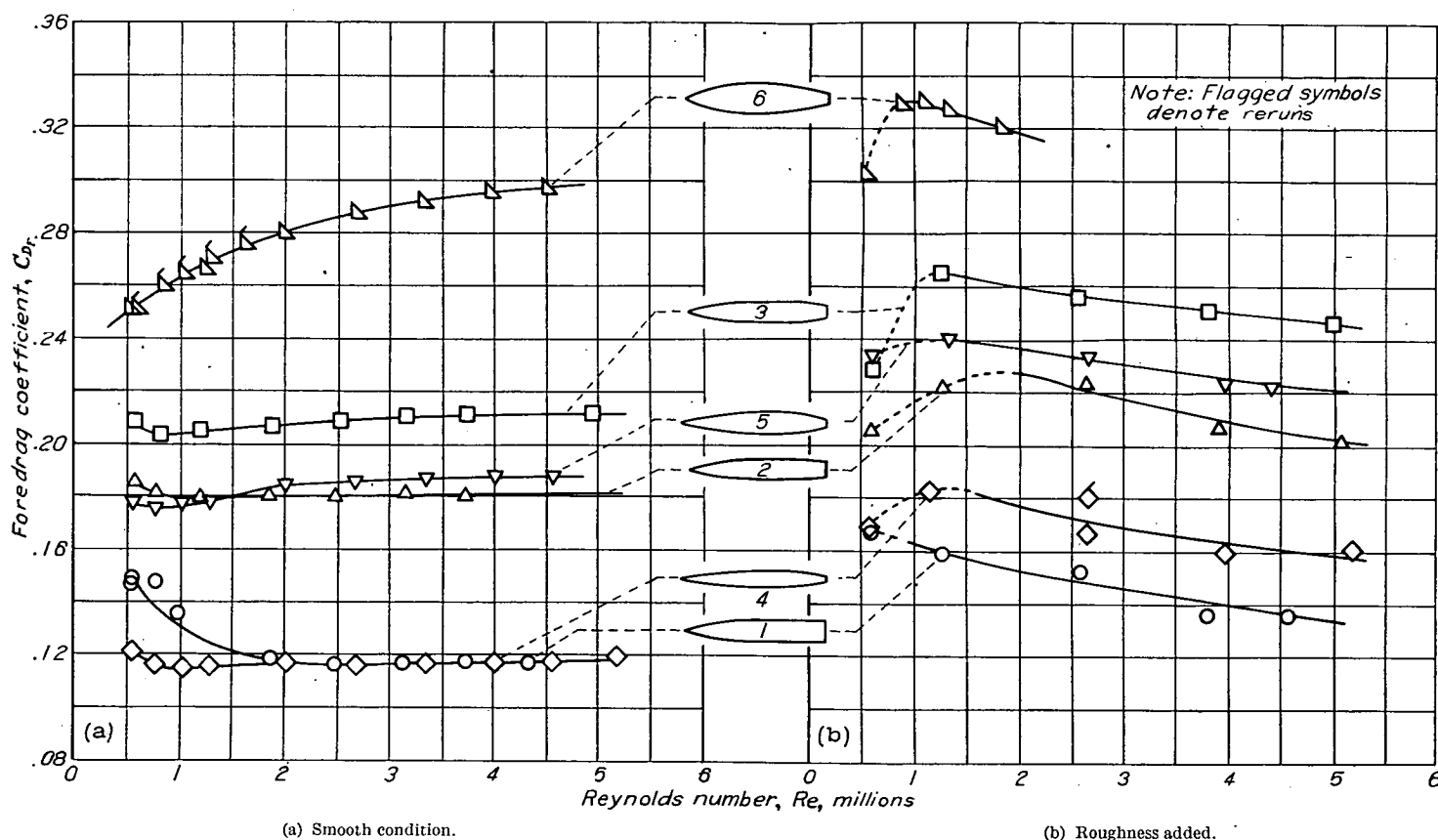
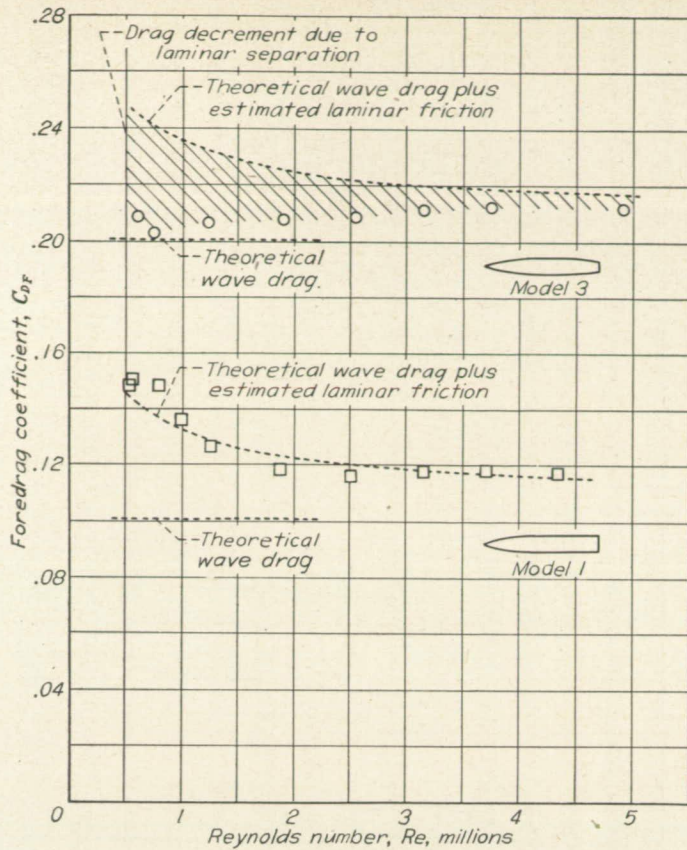
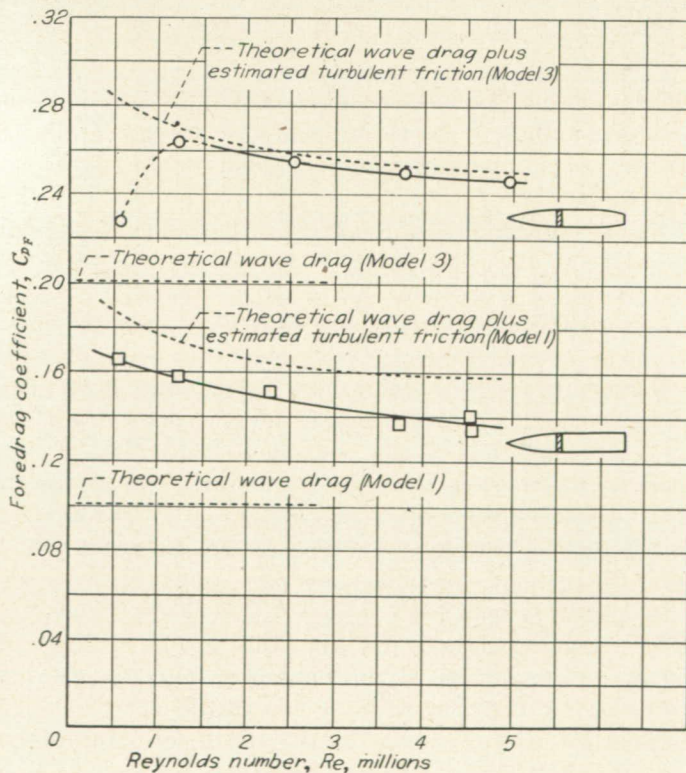


FIGURE 23.—Variation of foredrag coefficient for models 1, 2, 3, 4, 5 and 6 in the smooth condition and with roughness added.



(a) Smooth condition.



(b) Roughness added.

FIGURE 24.—Comparison of theoretical and experimental foredrag coefficients for models 1 and 3.

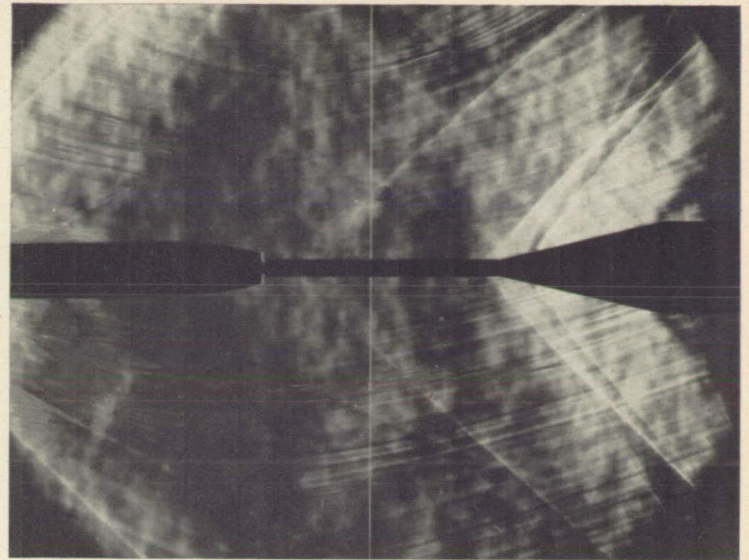
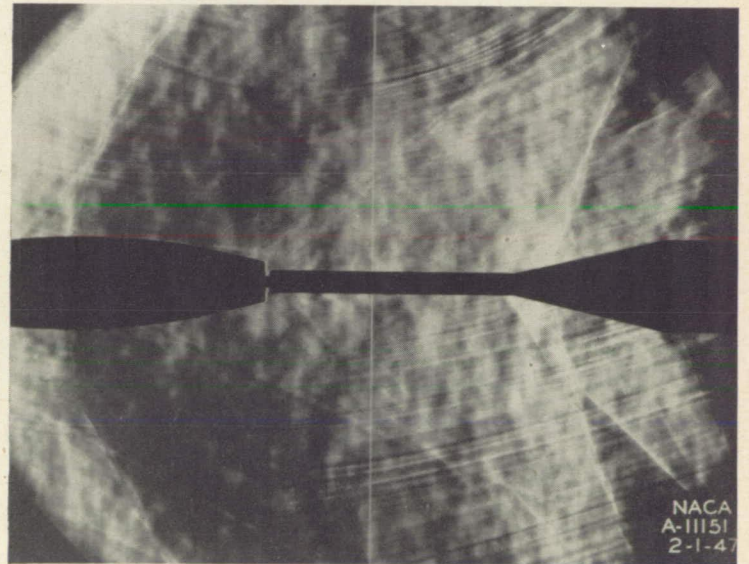
(a) Model 3, $Re = 0.58 \times 10^6$.(b) Model 6, $Re = 0.62 \times 10^6$.

FIGURE 25.—Schlieren photographs at low Reynolds numbers of models 3 and 6 with roughness added. Knife edge vertical.

sufficient to effect transition some distance ahead of the base, although not immediately aft of the roughness.

The agreement between the experimental and the theoretical results obtained by the use of equations (4) and (5) indicates that, at a Mach number of 1.5 and in the range of Reynolds numbers covered by this investigation, the familiar low-speed skin-friction coefficients can be used with fair approximation to estimate drag due to skin friction at supersonic speeds.

A comparison of the curves of figures 23 (a) and 23 (b) shows that for a given body at a given value of the Reynolds number the foredrag with roughness added is consistently higher than the corresponding foredrag of the smooth-surfaced body. In the general case, this over-all increase in foredrag is attributable both to the increase in the skin-friction drag of the body and to the change in flow separation with

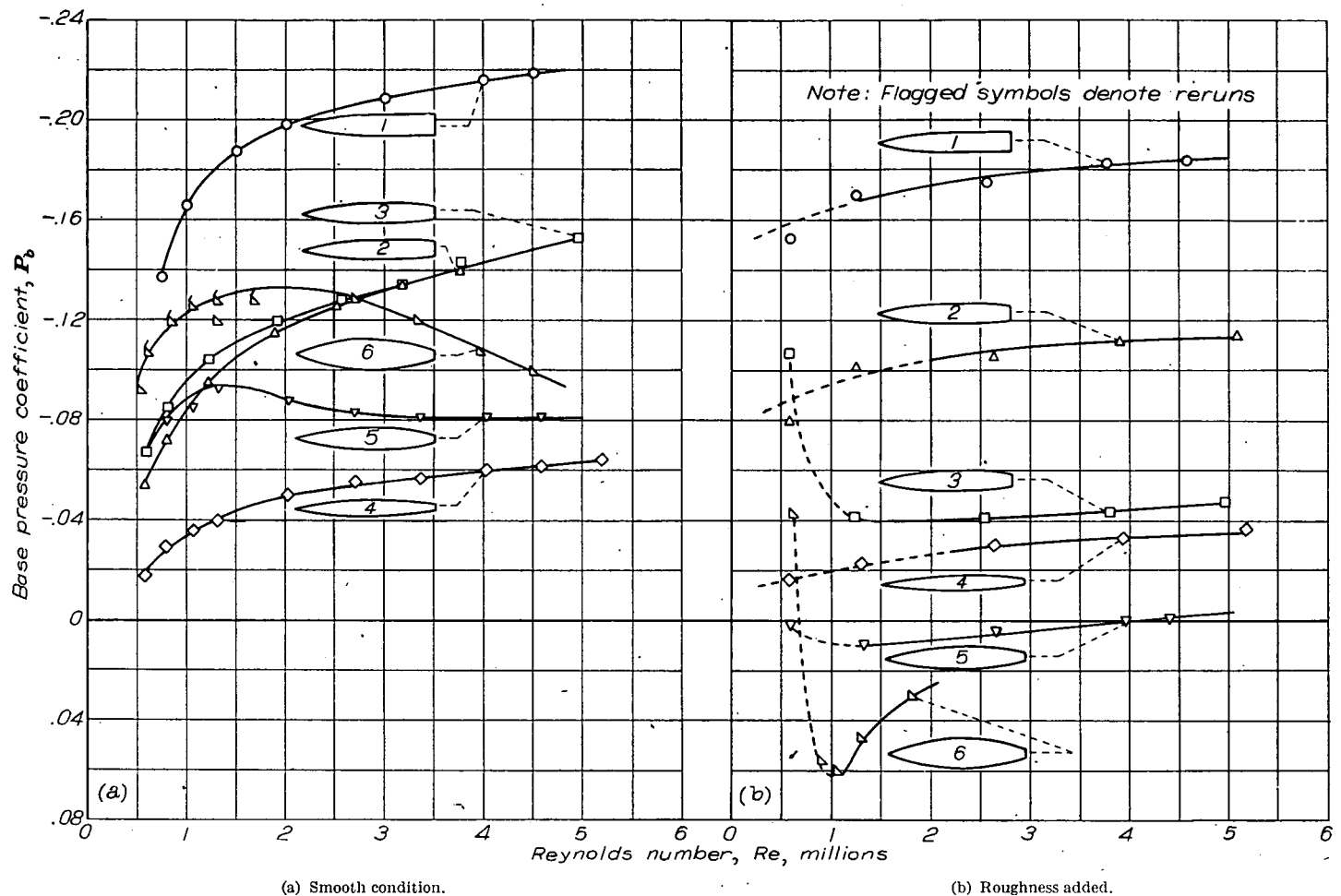


FIGURE 26.—Variation of base-pressure coefficient with Reynolds number for models 1, 2, 3, 4, 5 and 6 in the smooth condition and with roughness added.

consequent increase in the pressure drag of the boattail. For model 1, which has no boattailing, the increase in skin friction is the sole factor contributing to the increase in foredrag.

Base pressure and base drag.—Figure 26 (a) shows the base pressure coefficients plotted as a function of the Reynolds number for models 1 through 6, each with a smooth surface. It is evident from the data in this figure that the effects of Reynolds number on base pressure for a body with a laminar boundary layer are quite large. In the range of Reynolds numbers covered, the base pressure coefficient of model 1 increases about 60 percent, and the coefficients of models 2, 3, and 4 more than double. The thicker bodies with boattailing, models 5 and 6, do not exhibit such large changes in base pressure coefficient, for the coefficients apparently reach a maximum at a relatively low Reynolds number, and then decrease with further increase in the Reynolds number.

The base pressure coefficients for models 1 through 6 with roughness added are shown in figure 26 (b). Here again, the portions of the curves which correspond to the low Reynolds number region wherein transition was not completely effected are shown as dotted lines. Model 1 exhibits the lowest base pressure and model 6 the highest; in this latter case the base pressure is even higher than the free-stream static pressure. The physical reason for this is evident from the schlieren photograph at the bottom of figure 20, which shows that a compression through the shock wave occurs just ahead of the base of model 6. Except for the large changes in pressure

coefficient at low Reynolds numbers where the desired transition was not effected, the variation of base pressure coefficient with Reynolds number is relatively small for the bodies with roughness added.

From a comparison of the curves for the bodies with roughness added to the corresponding curves for the smooth-surfaced bodies, it is evident that a large change in the base pressure coefficient is attributable to the change in the condition of the boundary layer. In general, the base pressures for bodies with roughness added are considerably higher than the corresponding base pressures for the smooth-surfaced bodies. In the case of the boattailed bodies the physical reason for this increase in the base pressure is the appearance of the base shock wave, as shown in figure 21. For model 1, which has no boattailing, the mixing action and greater thickness of the turbulent boundary layer are probably responsible for the observed increase.

The foregoing data show that the effects of Reynolds number and condition of the boundary layer on the base pressure of a body moving at supersonic speeds depend considerably upon the shape of the afterbody. In order to ascertain whether the effects of viscosity also depend upon the length-diameter ratio for a fixed shape of afterbody, some models of different length-diameter ratios were tested and the data presented in figures 27 (a) and 27 (b) which show the variation of base pressure coefficient with Reynolds number. The data presented in this figure are not free of

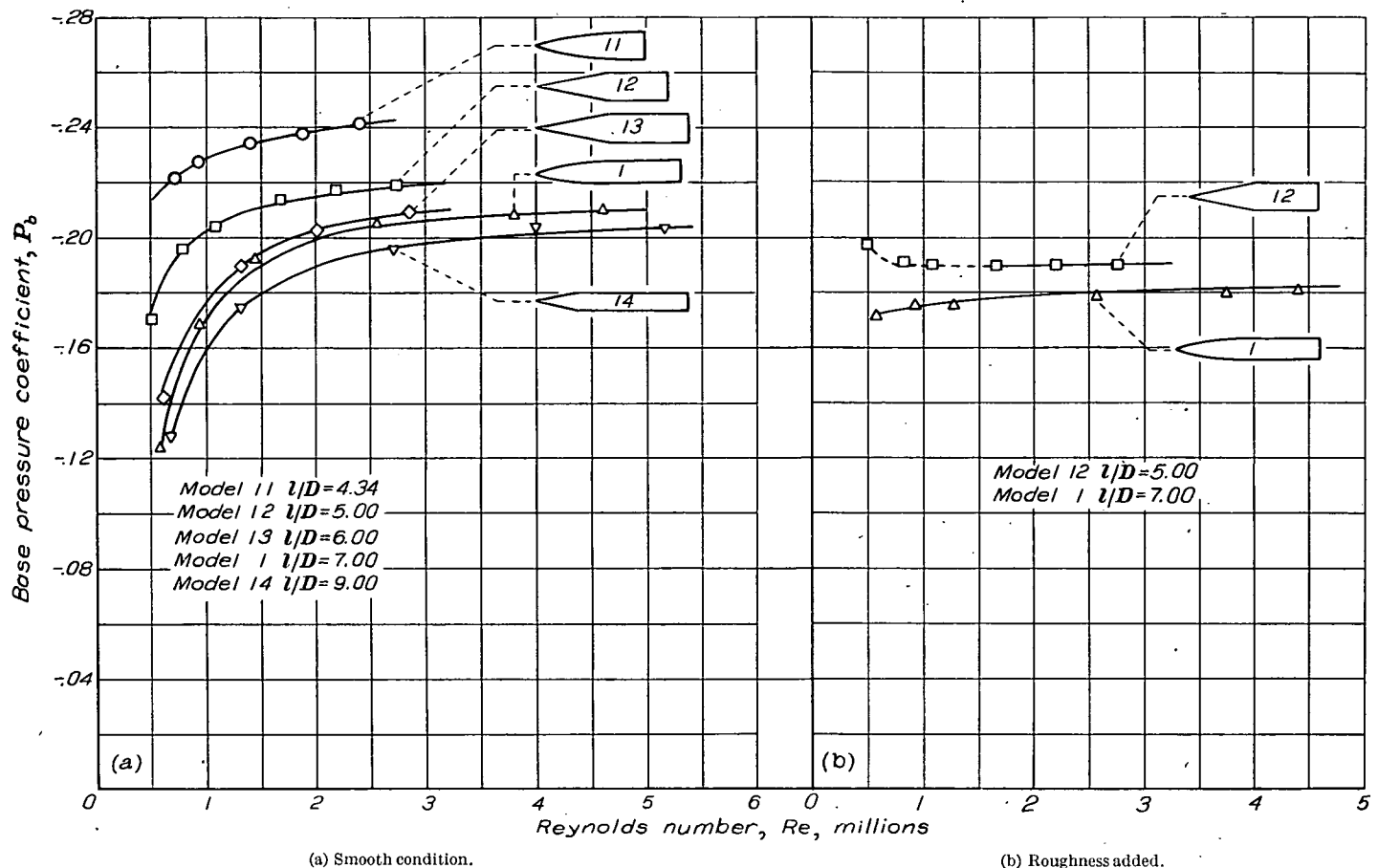


FIGURE 27.—Variation of base-pressure coefficient with Reynolds number for bodies without boattailing but with different length-diameter ratios.

support interference. From these data it is apparent that the effects of viscosity on the base pressure increase with the length-diameter ratio of the body.⁵ It is to be noted that the base pressure increases as the length-diameter ratio increases. This is somewhat at variance with the results of reference 20, which showed an effect, but not a systematic one, of length-diameter ratio on the base pressure of bodies without boattailing.

The base drag coefficient can be obtained from the base pressure coefficient of the models by using equation (2). The base drag coefficients for the smooth-surfaced bodies are presented in figure 28 (a) and for the bodies with roughness added in figure 28 (b). These curves are, of course, similar to the corresponding curves of base pressure coefficient given in figures 26 (a) and 26 (b). In this form the ordinates can be added directly to the foredrag coefficients of figure 23 to obtain the total drag coefficient of a given body. It is seen that the contribution of the base pressure to the total drag is very small for models with large amounts of boattailing, such as models 3, 4, 5, and 6.

Total drag.—The total drag coefficients for models 1 through 6 with smooth surfaces are shown in figure 29 (a) as a function of Reynolds number. These data show that the drag coefficients of both models 1 and 2 with a laminar boundary layer increase a little over 20 percent from the lowest to the highest value of Reynolds number obtained in

the tests. The other models exhibit somewhat smaller changes. The data presented in figures 23 and 28 indicate that the principal effect controlling the variation of total drag with Reynolds number for laminar flow in the boundary layer is the effect of Reynolds number on the base drag of the bodies. For the special case of highly boattailed bodies, however, this effect is of little relative importance because the base drag is a small part of the total drag. In such cases, the over-all variation of drag coefficient is due almost entirely to the variation of foredrag with Reynolds number.

Figure 29 (b) shows the total drag coefficients plotted as a function of the Reynolds number for models 1 through 6 with roughness added. Again, the portions of the curves that are shown dotted represent the Reynolds number region in which the amount of roughness added is insufficient to cause complete transition. All the curves have approximately the same trend, the over-all effect on the drag coefficients being about 15 percent or less for the various bodies.

A comparison of the curves of total drag for bodies with roughness added to the corresponding curves for bodies with smooth surfaces shows an interesting phenomenon. At the higher Reynolds numbers the drag of models 1 and 6 is actually decreased slightly by the addition of roughness, in spite of the corresponding increase in skin-friction drag. The reason is, of course, that the base drags are very much

⁵ Similar experiments conducted at a Mach number of 2.0 have shown the same trend. Both sets of data show reasonable correlation with the ratio of boundary-layer thickness to base diameter.

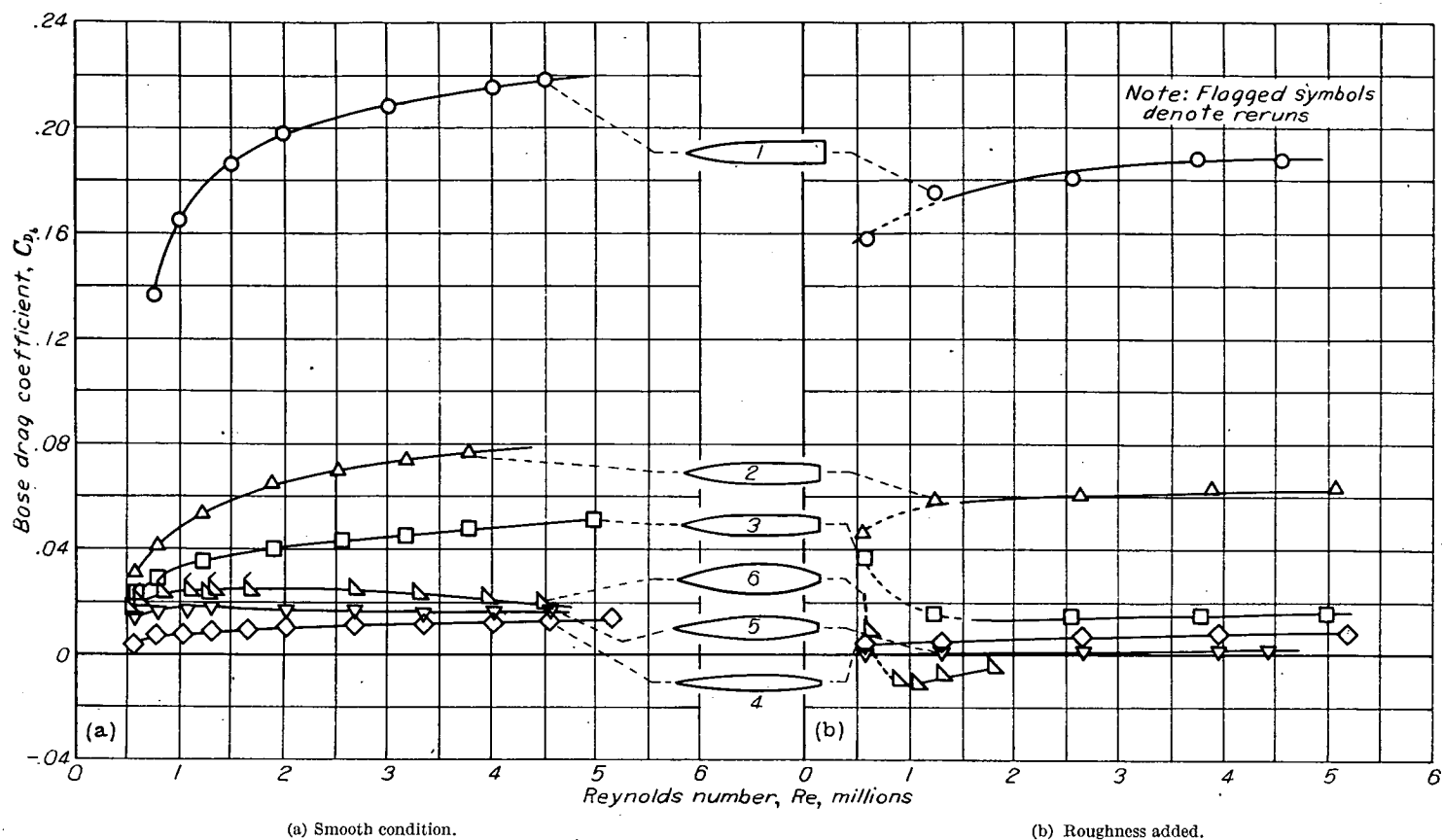


FIGURE 28.—Variation of base-drag coefficient with Reynolds number for models 1, 2, 3, 4, 5 and 6 in the smooth condition and with roughness added.

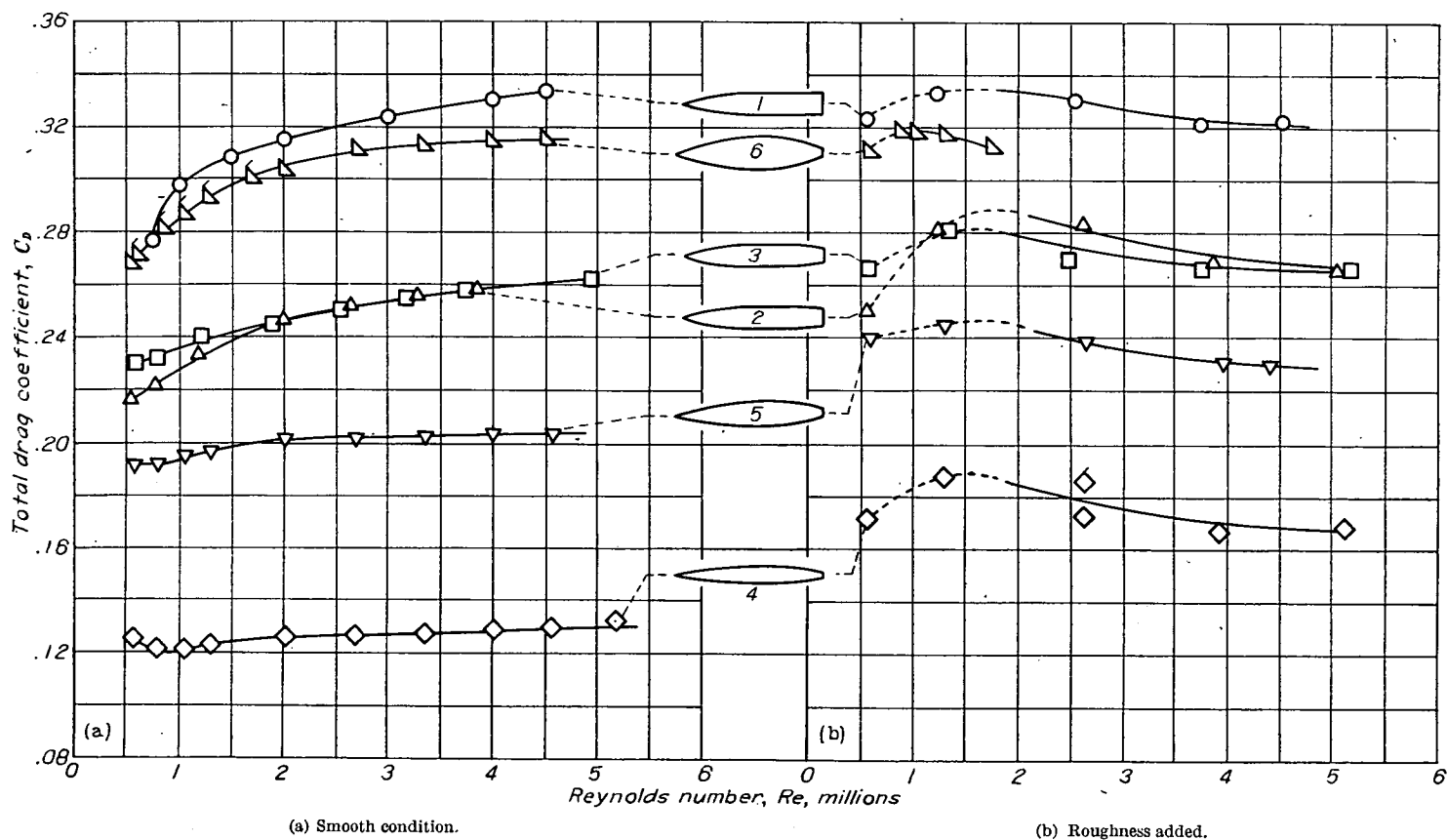


FIGURE 29.—Variation of total drag coefficient with Reynolds number for models 1, 2, 3, 4, 5 and 6 in the smooth condition and with roughness added.

lower for the turbulent boundary layer than for the laminar. The drag coefficients of the other bodies (models 2, 3, 4, and 5) are somewhat higher with roughness added, because the increase in friction drag of the turbulent boundary layer is greater than the decrease in base drag.

The importance of always considering both the Reynolds number of the flow and condition of the boundary layer is illustrated by the total drag characteristics of model 2. For example, if model 2 were tested with a turbulent boundary layer at a Reynolds number of 2×10^6 , the drag would be about 35 percent higher than if tested with a laminar boundary layer at a Reynolds number of 0.5×10^6 . Although discrepancies as large as these have not been reported as yet in the drag data from different supersonic wind tunnels, certain consistent differences, varying from about 5 to 25 percent, have been reported (reference 18) in the drag data of similar projectiles tested in the Gottingen and the Kochel tunnels. Although in reference 18 the discrepancies between the two tunnels were attributed only to the variation in skin friction with Reynolds number, it appears from the results of the present investigation that such discrepancies are attributable primarily to differences in flow separation and base pressure.

A comparison of the effects of viscosity for pointed bodies with the effects for a blunt body shows clearly that body shape must be considered, and that conclusions about viscosity effects based upon tests of blunt bodies may be completely inapplicable to the aerodynamic shapes which are suitable for supersonic flight. For example, in the case of a sphere at 1.5 Mach number with an over-all Reynolds number variation of from 7.5×10^4 to 9.0×10^5 , the agreement between the drag data from Gottingen (reference 7), Peenemunde (reference 18), and the present wind tunnel is within 1 percent of the values measured for free-flight (references 7 and 21). It is evident that the effects of viscosity on the drag of a sphere are quite different from the effects on the pointed bodies tested in this investigation.

CONCLUSIONS

The conclusions which follow apply for a Mach number of 1.5 and at Reynolds numbers based upon model length up to about 5×10^6 for bodies of revolution similar to the ones tested.

1. The effects of viscosity differ greatly for laminar and turbulent flow in the boundary layer, and within each regime depend upon the Reynolds number of the flow and the shape of the body.

2. Laminar flow was found on the smooth bodies up to a Reynolds number of 6.5×10^6 and may possibly exist to considerably higher values.

3. A comparison between the test results for laminar and for turbulent flow in the boundary layer at a fixed value of the Reynolds number shows that:

- (a) The resistance to separation with turbulent flow in the boundary layer is much greater.

- (b) The shock-wave configuration near the base of boat-tailed bodies is markedly different for the two types of boundary layer flow.
- (c) The foredrag coefficients with turbulent boundary layer ordinarily are higher.
- (d) The base pressure on boattailed bodies is much higher with the turbulent boundary layer.
- (e) The total drag is usually higher with the turbulent boundary layer.

4. For laminar flow in the boundary layer the following effects were found:

- (a) The laminar boundary layer separates forward of the base on all boattailed bodies tested, and the position of separation varies noticeably with Reynolds number. Laminar separation is not necessarily accompanied by a shock wave originating from the point of separation. On many of the models the pressure in an inviscid flow would continually decrease in the direction of the flow upstream of the separation point.
- (b) The trailing shock wave moves forward slightly as the Reynolds number is increased, but no significant change takes place in the shock-wave configuration near the base.
- (c) With increasing Reynolds numbers, the foredrag coefficients increase for highly boattailed bodies and decrease for bodies without boattailing. For moderately boattailed bodies the variation of the foredrag coefficient with Reynolds number is relatively small.
- (d) The base pressure changes markedly with Reynolds number. For bodies with the same afterbody shape, the base pressure also depends upon the length-diameter ratio of the body.
- (e) Total drag varies considerably with the Reynolds number, changing more than 20 percent for several of the models.

5. For turbulent flow in the boundary the following effects were found:

- (a) Separation does not ordinarily occur upstream of the base except for highly boattailed bodies.
- (b) The shock-wave configuration near the base does not change noticeably as the Reynolds number changes.
- (c) The foredrag coefficients decrease slightly as the Reynolds number is increased.
- (d) The base pressure changes very little with changing Reynolds number.
- (e) The total drag decreases as the Reynolds number is increased.

AMES AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., *January 31, 1947.*

APPENDIX A

VARIATION OF TEST-SECTION STATIC PRESSURE

Since the static pressure with no model present varied along the axis of the test section as shown in figure 5, it was necessary to apply a correction to the measured coefficients to account for the increment in drag or pressure resulting from this axial pressure gradient. Although the axial variation of test-section static pressure is not monotonic, the pressures at the downstream end of the test section are uniformly lower than the pressures of the upstream end where the nose of a model is ordinarily placed. This means that the actual pressure exerted at a given point on a body is lower than it would be if the ambient pressure gradient were zero as it is in free flight. The gradient corrections are calculated on the assumption that the magnitude of the pressure exerted at an arbitrary point on the body in the tunnel is lower than it would be if no gradient were present by an increment equal to the amount which the static pressure decreases (with no model present) from the position of the model nose to the position of the arbitrary point. At the Mach number of the present tests it is not necessary to include the corresponding axial variation of dynamic pressure in the corrections since it varies only ± 0.2 percent from the mean test-section value used in all calculations. The corrections to the measured coefficients of model 1 located 2.5 inches downstream from the reference pressure orifice, for example, amount to ± 0.012 in foredrag coefficient and -0.026 in base-drag coefficient; the corresponding percentages of the uncorrected coefficients of foredrag and base pressure are 12 and 15, respectively.

Because the gradient correction is relatively large in the present tests an experimental justification of such theoretical

corrections is in order. The validity of the corrections as applied to foredrag is confirmed by tests on model 9, which consists of a conical nose with a 20° included angle and a short cylindrical afterbody. The theoretical foredrag of this body, which is equal to the sum of the wave and friction drags can be easily determined as a function of Reynolds number. The wave drag of the conical nose is given by the calculations of Taylor and Maccoll (references 10 and 11). The frictional drag can be estimated using the low-speed laminar skin-friction coefficients, since the boundary layer was completely laminar over this model. A comparison of the corrected and uncorrected foredrag with the theoretical foredrag is shown in figure 6. The corrected foredrag coefficients are seen to be in good agreement with the theoretical values; whereas the uncorrected data fall below the wave drag at high tunnel pressures. This latter condition, of course, represents an impossible situation for a body without boattailing.

In order to check experimentally the validity of the corrections as applied to the measured base pressure, model 1 was tested on the side support at five different positions along the axis of the test section. Because the support system remained fixed relative to the body, the interference of the support is the same in each case, hence, any discrepancies in the measured base pressures at the various positions are attributable only to the pressure gradient along the tunnel axis. Figure 7 shows that the uncorrected base pressure data taken at the five different positions differ by about 25 percent, but the corresponding five sets of corrected data fall within about ± 1.5 percent of their mean, thus confirming the validity of the correction.

APPENDIX B

PRECISION OF DATA

The accuracy of the results presented can be estimated by considering the possible errors that are known to be involved in the measurement of the forces and pressures, and in the determination of the free-stream Mach number and gradient corrections.

The force measurements are subject to errors from shifts in the balance zero due to temperature effects and also from a shift in the calibration constant. The zero shift, which is less than ± 1 percent of the force data at low pressures and less than ± 0.2 percent at high pressures, was checked periodically by running the tunnel through the complete temperature range with no force applied to the balance. In the majority of cases the variation of the balance calibration constant, which was checked before and after each series of tests, permitted a possible deviation of ± 0.3 percent in the force data. All data presented in figures 9 (b), 13, 14, and the data for models 4, 5, and 6 in figures 23 (a) and 29 (a) were obtained during a period between two consecutive balance calibrations for which the constant differed by 6.4 percent. A comparison of the data obtained during this period with theoretical results and with the results of subsequent reruns of some of the same models indicates that the change in balance calibration occurred before the data in question were obtained. The results in the afore-mentioned figures were therefore computed on the basis of the later calibration. It is estimated that the maximum error in the balance calibration constant for these results is at worst no greater than $+0.3$ to -3.0 percent.

The pressure data, including the dynamic pressure, are subject to small errors resulting from possible inexact readings of the mercury manometers. The base pressure data are also subject to an additional error resulting from the small variation in the specific gravity of the dibutyl phthalate indicating fluid. At the most, these sources can cause an

error in the total and foredrag coefficients of about ± 0.3 percent, and in the base-drag coefficient of about ± 0.8 percent. The error in dynamic pressure due to the uncertainty in the free-stream Mach number is negligible, since the isentropic relation for the dynamic pressure as a function of Mach number is near a maximum at a Mach number of 1.5. For slender bodies of revolution the variation of the force coefficients with Mach number is quite small; hence, errors resulting from the variation of free-stream Mach number from 1.49 to 1.51 are negligible.

On the basis of the data presented in figures 6 and 7, it is estimated that for all tunnel pressures the uncertainty in the gradient corrections to total drag, foredrag, and base pressure coefficients can cause at the most an error in these coefficients of ± 0.004 , ± 0.004 , and ± 0.005 , respectively. It should be noted that in the table on precision, presented in the section on results, this source of error, which is independent of tunnel pressure, is expressed as an increment and not as a percentage of the measured coefficient.

Previous investigations have shown that an uncertainty may be introduced in supersonic wind-tunnel data if the humidity of the tunnel air is very high. To determine the effects of this variable in the present investigation, the specific humidity was varied from the lowest values (approximately 0.0001) to values approximately 20 times those normally encountered in the tests. Drag and base pressure measurements were taken on a body with a conical head and also on a sphere. The results showed no appreciable effect of humidity over a range much greater than that encountered in the present tests, provided the variation in test-section dynamic pressure with the change in humidity was taken into account in the reduction of the data. It is believed, therefore, that the precision of the results presented in this report is unaffected by humidity.

APPENDIX C

EFFECT OF SUPPORT INTERFERENCE

A knowledge of the effects of support interference upon the data in question is essential to an understanding of its applicability to free-flight conditions. Previous to the present investigation an extensive series of tests were conducted to determine the body shape and support combinations necessary to evaluate the support interference.

In general, it was found that for the models tested in the smooth condition (laminar boundary layer) the effect of the rear supports used in the present investigation was negligible for the boattailed models 2 and 3 and was appreciable only in the base pressure measurements for model 1. For model 1, combinations of rear support and side support were used to evaluate the effect of the rear support on the base pressure. The evaluation was made on the assumption of no mutual interference between the rear support and side support and was checked by the use of two different combinations of side support and rear support. The data indicate that the assumption is justified within the limits of the experimental accuracy and that the corrected, interference-free base pressures deduced by this method differ only slightly from those measured with the side support alone.

For the bodies with roughness added (producing a turbulent boundary layer) a complete investigation of the support interference was not made; consequently, a definite quantitative evaluation of the interference effects for each body in this condition cannot be given. From the data that were obtained it has been found that the foredrag is not affected appreciably by the presence of the supports used in the present investigation, but that a small amount of interference is evident in the base pressure coefficient which may vary from a minimum of ± 0.005 to a maximum of ± 0.015 for the different bodies. This uncertainty in the base pressure coefficient results in a correspondingly small uncertainty in the base drag coefficient and in the total drag coefficient.

REFERENCES

1. Ackeret, J., Feldmann, F., and Rott, N.: Investigations of Compression Shocks and Boundary Layers in Gases Moving at High Speed. NACA TM 1113, 1947.
2. Liepmann, H. W.: Further Investigations of the Interaction of Boundary Layer and Shock Waves in Transonic Flow. Jour. Aero. Sci., vol. 13, no. 12, Dec. 1946, pp. 623-637.
3. Theodorsen, Theodore, and Regier, Arthur: Experiments on Drag of Revolving Disks, Cylinders and Streamline Rods at High Speeds. NACA Rep. 793, 1944. (Formerly NACA ACR L4F16)
4. Keenan, Joseph H., and Neumann, Ernest P.: Friction in Pipes at Supersonic and Subsonic Velocities. NACA TN 963, 1945.
5. Frössel, W.: Flow in Smooth Straight Pipes at Velocities Above and Below Sound Velocity. NACA TM 844, 1938.
6. Ferri, Antonio: Influenza del Numero di Reynolds ai Grandi Numeri di Mach. Ministero del Aeronautica, Direzione Superiore Studi e delle Esperienze, Atti di Guidonia N. 67-68-69, Mar. 10, 1942, pp. 49-92.
7. Walchner, O.: Systematische Geschossmessungen im Windkanal. Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht 139, Teil 1, Oct. 1941, pp. 29-37. (Available as NACA TM 1122)
8. Bach, F.: Druckverteilungsmessungen an Geschossmodellen. Zentrale für Wissenschaftliches Berichtswesen der Luftfahrtforschung, Berlin-Adlershof. UM 6057, Mar. 1945.
9. Van Dyke, Milton D.: Aerodynamic Characteristics Including Scale Effect of Several Wings and Bodies Alone and in Combination at a Mach Number of 1.53. NACA RM A6K22, 1946.
10. Maccoll, J. W.: The Conical Shock Wave Formed by a Cone Moving at a High Speed. Proc. of the Royal Soc. of London, Ser. A, Vol. 159, Apr. 1, 1937, pp. 459-472.
11. Taylor, G. I., and Maccoll, J. W.: The Air Pressure on a Cone Moving at High Speeds. Proc. of the Royal Soc. of London, Ser. A, Vol. 139, Feb. 1, 1933, pp. 278-311.
12. Sauer, R.: Charakteristikenverfahren für Räumliche Achsensymmetrische Überschallströmungen. Zentrale für Wissenschaftliches Berichtswesen, Berlin, F. B. No. 1269, Aug. 14, 1940. (Available as NACA TM 1133)
13. Sauer, R.: Theoretische Einführung in die Gasdynamik. Berlin, Springer, 1943. (Reprinted by Edwards Bros., Ann Arbor, Mich., 1945.)
14. Tollmein, W., and Schäfer, M.: Rotationssymmetrische Überschallströmungen. Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht 139, Teil 2, Oct. 1941, pp. 5-15.
15. Allen, H. Julian, and Nitzberg, Gerald E.: The Effect of Compressibility on the Growth of the Laminar Boundary Layer on Low-Drag Wings and Bodies. NACA TN 1255, 1947.
16. Matt, H.: Hochgeschwindigkeitsmessungen an Rund- und Profilstangen verschiedener Durchmesser. Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht 156, Oct. 1942, pp. 109-113.
17. Lees, Lester: The Stability of the Laminar Boundary Layer in a Compressible Fluid. NACA Rep. 876, 1947. (Formerly NACA TN 1860)
18. Lehnert, R.: Systematische Messungen an neun einfachen Geschossformen im Vergleich zu Messungen der AVA-Göttingen. Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht 139, Teil 2, 1941, pp. 31-34.
19. Ferri, Antonio: Experimental Results with Airfoils Tested in the High-Speed Tunnel at Guidonia. NACA TM 946, 1940.
20. Erdmann, S.: Widerstandsbestimmung Von Kegeln und Kugeln aus der Druckverteilung bei Überschallgeschwindigkeit. Lilienthal-Gesellschaft für Luftfahrtforschung, Bericht 139, Teil 2, Oct. 1941.
21. Charters, A. C., and Thomas, R. N.: The Aerodynamic Performance of Small Spheres from Subsonic to High Supersonic Velocities. Jour. Aero. Sci., vol. 12, No. 4, Oct. 1945, pp. 468-476.

REPORT 1037

GENERAL METHOD AND THERMODYNAMIC TABLES FOR COMPUTATION OF EQUILIBRIUM COMPOSITION AND TEMPERATURE OF CHEMICAL REACTIONS¹

By VEARL N. HUFF, SANFORD GORDON, and VIRGINIA E. MORRELL

SUMMARY

A rapidly convergent successive approximation process is described that simultaneously determines both composition and temperature resulting from a chemical reaction. This method is suitable for use with any set of reactants over the complete range of mixture ratios as long as the products of reaction are ideal gases. An approximate treatment of limited amounts of liquids and solids is also included. This method is particularly suited to problems having a large number of products of reaction and to problems that require determination of such properties as specific heat or velocity of sound of a dissociating mixture.

The method presented is applicable to a wide variety of problems that include (1) combustion at constant pressure or volume; and (2) isentropic expansion to an assigned pressure, temperature, or Mach number. Tables of thermodynamic functions needed with this method are included for 42 substances for convenience in numerical computations.

INTRODUCTION

The theoretical performance of propulsion systems having high combustion temperatures can be calculated on the assumption that chemical equilibrium exists among the products of reaction. The equilibrium composition and the temperature for a system of N products of reaction are determined by the simultaneous solution of at least $N+1$ equations involving dissociation, mass balance, and energy or entropy balance. This calculation becomes increasingly difficult as N increases.

Numerous methods for solving these equations may be found in the literature that provide a successive approximation or trial-and-error process for determining the composition at an assumed temperature and pressure. Examples of these methods are found in references 1 to 4. When it is desired to find the temperature of a system in equilibrium, with a parameter such as entropy or enthalpy assigned, the composition is usually computed at a sequence of temperatures that either converge to the correct temperature or are spaced to permit interpolation to obtain the correct temperature.

A rapidly convergent successive approximation process that determines composition at an assigned temperature or that simultaneously determines both composition and temperature for assigned values of another parameter, such as enthalpy or entropy, was developed at the NACA Lewis laboratory during 1948 and is presented herein. This proc-

ess also permits computation of the partial derivatives required to compute such thermodynamic properties as specific heat and velocity of sound corresponding to chemical equilibrium. The equations are derived that are required for solution of the following cases: (1) combustion at constant pressure or volume; and (2) isentropic expansion to an assigned pressure, temperature, or Mach number. Examples are given for (1) constant-pressure adiabatic combustion; (2) isentropic expansion to an assigned pressure; and (3) isentropic expansion to an assigned Mach number.

This method is particularly suitable for problems having a large number of products of reaction and for problems that require determination of partial derivatives. Although it is possible, at least in special cases, to devise a procedure that involves less numerical computation, the method presented is applicable in a wide variety of cases and its numerical application to a given process is always simple and essentially the same for all reactions.

Tables of thermodynamic functions are needed for computing equilibrium compositions and temperature of chemical reactions. Tables containing the functions specific heat at constant pressure C_p° , sensible enthalpy $H_T^\circ - H_0^\circ$, and molar entropy S_T° exist for at least part of the desired temperature range for most of the substances of interest in the analysis of aircraft-propulsion systems. Several special functions are required for convenient use with the method described herein; tables were therefore prepared, from January to June 1949, that contain, in addition to C_p° , $H_T^\circ - H_0^\circ$, and S_T° , assigned values of enthalpy H_T° and values of $\log K$ and $\frac{-\Delta H^\circ}{RT}$ (logarithm of equilibrium constant and enthalpy change divided by gas constant times temperature, respectively, for reaction of formation of a substance from its elements in atomic gas state).

The data selected from various sources or computed by the NACA have been smoothed, interpolated to every 100°, and extended to 6000° K. A high degree of self-consistency has been maintained in the temperature range from 1000° to 6000° K by computing from specific-heat data the values of the other functions and retaining, in general, more decimal places than are significant. Interpolation formulas are given that permit computation of self-consistent values for all the functions at any temperature between 1000° and 6000° K.

¹ Supersedes NACA TN 2113, "General Method for Computation of Equilibrium Composition and Temperature of Chemical Reactions" by Vearl N. Huff and Virginia E. Morrell, 1950, and NACA TN 2161, "Tables of Thermodynamic Functions for Analysis of Aircraft-Propulsion Systems" by Vearl N. Huff and Sanford Gordon, 1950.

GENERAL METHOD

The thermodynamic state following a specific process, such as combustion at constant pressure, can be determined from an appropriate combination of the following equations: (a) dissociative equilibrium; (b) conservation of mass; (c) conservation of energy; (d) pressure; and (e) entropy. Equations (a) and (b) are used to specify chemical equilibrium and, when used with any two of the remaining equations, define a process.

The successive approximation procedure presented herein for finding the simultaneous solution of a specific combination of equations (a) to (e) consists of the following steps:

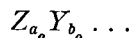
(1) Estimates of composition and temperature are made and used in simple equations to compute the values of error parameters, which indicate inconsistency among the estimates of composition and temperature. (These estimates need not be based on previous experience, but for rapid convergence it is desirable that they be close to the final values.)

(2) A set of linear simultaneous correction equations is given that determine a new composition and a new temperature.

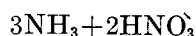
(3) The new composition is used to compute new values of the error parameters and step (2) is repeated until the desired accuracy is obtained.

EQUATIONS FOR DISSOCIATION, MASS, PRESSURE, AND VOLUME

The substances entering a reaction process will be designated the reactants and can be represented by the equivalent formula



where the subscripts $a_o, b_o, \dots$ are proportional to the total number of atoms of the elements $Z, Y, \dots$, respectively, contained in a quantity of the entering substance at the initial conditions. (A complete list of symbols is included in appendix A.) For example, the reactants for a rocket combustion process using 3 moles of ammonia (NH_3) for fuel and 2 moles of nitric acid (HNO_3) for an oxidant are

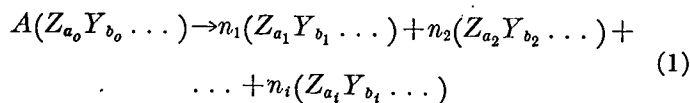


An equivalent formula would be



where the atoms hydrogen, nitrogen, and oxygen, may be represented by Z, Y , and X , respectively, and 11, 5, and 6 by a_o, b_o , and c_o , respectively. The weight of the equivalent formula M_r can be computed in the usual way and would be 177.128. (If desirable, the quantity of substance in the equivalent formula may be chosen to correspond to a specified value of M_r . For example, if M_r is to be one gram, the preceding values would be divided by 177.128.)

The reaction under consideration can be written



where n_i is the number of moles of the i th molecule or atom. The subscripts $a_i, b_i, \dots$, which can take on only positive

integral values or zero, denote the number of $Z, Y, \dots$ atoms in the i th molecule. For example, if Z, Y , and X again represent hydrogen, nitrogen, and oxygen, respectively, the values of a_i, b_i , and c_i for a water molecule H_2O would be 2, 0, and 1, respectively. It is assumed that the products of reaction are contained by a volume V numerically equal to the gas constant R times the absolute temperature T so that for ideal gases

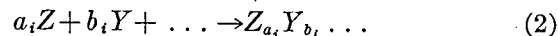
$$p_i = n_i$$

During the solution of the problem, it is necessary to determine the number of formula weights of the reactants A that are required to balance the reaction given by equation (1). Products of reaction in the gas phase are assumed to be ideal gases that form ideal mixtures and each condensed phase is assumed to have a partial pressure of zero, even when finely divided and suspended in the gas. For solids and liquids therefore

$$p_i = 0$$

As an approximation, the following assumptions are also made: Each condensed product is insoluble in all others; the fugacity of each condensed phase is equal to 1 atmosphere; the total volume occupied by the liquids and solids is negligible with respect to the volume occupied by the gases; and the liquid and solid particles have the same temperature and flow velocity as the gases.

Dissociation equations.—For simplicity of nomenclature and presentation, the equations for dissociation can be written in terms of the atomic gas as



The corresponding equation for the equilibrium constant K_i of gaseous molecules is

$$K_i = \frac{p_i}{p_z^{a_i} p_y^{b_i} \dots} \quad (3)$$

For liquid or solid molecules, assuming the fugacity of each condensed phase is equal to 1 atmosphere,

$$K_i = \frac{1}{p_z^{a_i} p_y^{b_i} \dots} \quad (4)$$

where $p_z, p_y, \dots$ are the partial pressures of the $Z, Y, \dots$ atoms in equation (1), respectively. The equilibrium constants can also be expressed in terms of the free-energy changes $(\Delta F_T^\circ)_i$ across the dissociation reactions represented by equation (2) or

$$\ln K_i = \left(\frac{-\Delta F_T^\circ}{RT} \right)_i \quad (5)$$

Because the trial composition may not correspond to that at chemical equilibrium, variables δ_i are conveniently defined so that for gaseous molecules (logarithms to the base 10 are used)

$$\delta_i = \log p_i - a_i \log p_z - b_i \log p_y - \dots - \log K_i \quad (6)$$

and for liquid or solid molecules

$$\delta_i = -a_i \log p_z - b_i \log p_r - \dots - \log K_i \quad (7)$$

where K_i is defined by equation (5). The value of each δ_i must approach zero when the solution to the problem is found. Application of equation (6) or (7) to each product of reaction will result in one equation for each molecule considered since for atoms, δ_i is identically zero.

Mass-balance equations.—A mass-balance equation stating the conservation of atomic type can be written for each chemical element present.

$$\begin{aligned} a &= \frac{1}{A} \sum_i a_i n_i \\ b &= \frac{1}{A} \sum_i b_i n_i \\ &\dots = \dots \end{aligned} \quad (8)$$

where $a, b, \dots$ are the number of gram atoms of substance $Z, Y, \dots$ per equivalent formula required to form the products of reaction. A trial composition generally leads to values of $a, b, \dots$ that differ from the desired values of $a_o, b_o, \dots$ but the difference will vanish when the correct composition is found.

Total-pressure equation.—The total pressure P is the sum of the partial pressures

$$P = \sum_i p_i \quad (9)$$

For a process with an assigned pressure, the value of P must approach the assigned value P_o as the solution of the problem is found.

Constant volume.—For processes that occur at constant volume, the density of the mixture is constant. The density ρ is defined as

$$\rho = \frac{AM_r}{V} = \frac{AM_r}{RT} \quad (10)$$

For a reaction process with an assigned density, the value of ρ must approach the assigned value ρ_o as the solution of the problem is found.

COMBUSTION AT CONSTANT PRESSURE

For given initial conditions, the temperature and the composition following a combustion process are to be found. When chemical energy is included in the enthalpy of each substance, the enthalpy of the products of reaction following an adiabatic combustion must be equal to the enthalpy of the reactants at the initial conditions. An arbitrary base may be adopted for assigning absolute values to the enthalpy of various substances because only differences are measurable. The base used to compute values of enthalpy H_T° was selected to produce positive values for all molecular types entering a combustion process in order to avoid a possible source of difficulty that might occur in the recommended

method of adjustment when a logarithm of a negative number (or zero) might be required.

Enthalpy of fuel and oxidant.—The enthalpy at initial conditions of the amount of fuel and oxidant corresponding to the equivalent formula $Z_{a_o} Y_{b_o} \dots$ is denoted by h_o and is given by the expression

$$h_o = n_f (H_T^\circ)_f + n_g (H_T^\circ)_g \quad (11)$$

where n_f and n_g are the number of moles of fuel and oxidant, respectively, corresponding to the equivalent formula $Z_{a_o} Y_{b_o} \dots$ and $(H_T^\circ)_f$ and $(H_T^\circ)_g$ are the molar enthalpies of the fuel and the oxidant, respectively, at the initial conditions. The molar enthalpy H_T° is defined by the equation

$$H_T^\circ = \int_0^T C_p^\circ dT + H_0^\circ$$

where C_p° is the molar specific heat at constant pressure, and H_0° is the chemical energy of the substance at a temperature of $0^\circ K$. Values of H_T° for several fuels and oxidants are presented with the tables of thermodynamic functions.

Enthalpy of products of reaction.—The enthalpy of the products of reaction per equivalent formula can be conveniently represented by a variable h that is given by the equation

$$h = \frac{1}{A} \sum_i (H_T^\circ)_i n_i \quad (12)$$

When enthalpy is assigned, (for example, with adiabatic combustion) the difference between h and the assigned value h_o must vanish when the correct values of n_i, A , and T are found. If heat were lost (nonadiabatic combustion), the value of h_o would be accordingly reduced.

Equations for constant-pressure combustion.—The equations defining the constant-pressure combustion are:

Type	Number of equations
Dissociative equilibrium...	1 for each molecular type.
Conservation of mass....	1 for each chemical element.
Constant pressure.....	1.
Conservation of energy...	1.

These equations are to be solved simultaneously for the variables n_i, A , and T ($p_i = n_i$ for gases).

Correction equations.—Since the preceding equations are not all linear, it is usually not feasible to find a direct solution. The Newton-Raphson method for solving nonlinear simultaneous equations (reference 5) is well suited to this type of computation. This method can be illustrated by a simple example. If Q_1 and Q_2 are functions of q and r ,

$$Q_1 = f_1(q, r)$$

$$Q_2 = f_2(q, r)$$

By taking estimated values, for example q_o and r_o , each function may be expanded in a Taylor's series about the point (q_o, r_o) and when derivatives of higher order than the first are neglected

$$\Delta Q_1 = \frac{\partial Q_1}{\partial q} \Delta q + \frac{\partial Q_1}{\partial r} \Delta r$$

$$\Delta Q_2 = \frac{\partial Q_2}{\partial q} \Delta q + \frac{\partial Q_2}{\partial r} \Delta r$$

The desired changes ΔQ_1 and ΔQ_2 can be computed; if the partial derivatives can be numerically evaluated, solving for the approximate changes in q and r to effect simultaneously the desired changes in both Q_1 and Q_2 is comparatively simple because the equations are linear.

If each of the functions δ_i , a , b , . . . , P , and h given by equations (6) to (9) and (12) is expanded in a Taylor's series about an estimated set of values of the variables and terms involving derivatives of order higher than the first are neglected, the following set of simultaneous linear correction equations results:

For gaseous products

$$x_i - a_i x_Z - b_i x_Y - \dots - q_i x_T = -\delta_i \quad (13)$$

For solid or liquid products

$$-a_i x_Z - b_i x_Y - \dots - q_i x_T = -\delta_i \quad (14)$$

For all products

$$\begin{aligned} \sum_i a_i n_i x_i - A \Delta x_A &= \delta_a \\ \sum_i b_i n_i x_i - A b x_A &= \delta_b \end{aligned} \quad (15)$$

$$\dots = \dots$$

$$\sum_i p_i x_i = \delta_P \quad (16)$$

$$\sum_i h_i' x_i - A h x_A + T C' x_T = \delta_h \quad (17)$$

where the correction variables and the error parameters may be defined in the logarithmic form

$$x_i = \Delta \log n_i = \Delta \log p_i$$

$$x_Z, x_Y, \dots = x_i \text{ for atoms}$$

$$x_A = \Delta \log A$$

$$x_T = \Delta \log T$$

$$-\delta_i = \Delta \delta_i$$

$$\delta_a = A a \log \frac{a_o}{a}$$

$$\delta_b = A b \log \frac{b_o}{b}$$

$$\dots = \dots$$

$$\delta_P = P \log \frac{P_o}{P}$$

$$\delta_h = A h \log \frac{h_o}{h}$$

and where $q_i = \left(\frac{\Delta H}{RT} \right)_i = \frac{\partial \log K_i}{\partial \log T}$, $h_i' = (H_T^o)_i n_i$, $C' = \sum_i (C_p^o)_i n_i$

The solution to the set of simultaneous equations relates the value of the r^{th} estimate to the $(r+1)^{\text{th}}$ estimate as follows:

$$\begin{aligned} \log (n_i)_{r+1} &= \log (n_i)_r + x_i \\ \log (A)_{r+1} &= \log (A)_r + x_A \\ \log (T)_{r+1} &= \log (T)_r + x_T \end{aligned} \quad (18)$$

The expansion in the Taylor's series has been carried out in the logarithmic form because this form has been found to result in rapid convergence over a wide range of conditions and avoids the possibility of computing negative partial pressures. If the expansion is carried out in powers of $x_i = \frac{\Delta n_i}{n_i}$ or $x_i = n_i \Delta \left(\frac{1}{n_i} \right)$ the same correction equations result as for the logarithmic variables except for the definitions of the correction variables and error parameters. Quite satisfactory results have been obtained by taking $x_i = \frac{\Delta n_i}{n_i}$ when x_i is positive and $x_i = n_i \Delta \left(\frac{1}{n_i} \right)$ when x_i is negative.

MATRIX CONSTRUCTION AND REDUCTION

A coefficient matrix is a scheme of detached coefficients of a set of linear equations that are to be solved simultaneously. An augmented matrix is identical to a coefficient matrix except that the constants are included. Equations (13) to (17) constitute such a set of equations for the simultaneous determination of the variables x_i , x_A , and x_T .

Construction.—Because of the large number of zeros occurring in the matrix, a considerable saving in effort can be made by proper arrangement of the order of the rows and the columns. The following arrangement provides a partly symmetrical matrix that has been found to be among the easiest to evaluate as long as the products of reaction are principally gaseous and the dissociation constants are expressed in terms of the atomic species:

The order of the columns should be—

- (a) x_i of gaseous molecules
- (b) x_i of atoms
- (c) x_i of liquid and solid products
- (d) x_A
- (e) x_T
- (f) Constant terms of equations

The order of the rows is—

- (a) Dissociation equations in same order as gaseous molecules in columns
- (b) Mass-balance equations in order of atoms in columns
- (c) Dissociation equations for solid and liquid products in same order as solid and liquid in columns
- (d) Total-pressure equation
- (e) Heat-balance equation in combustion calculation

The augmented matrix of equations (13) to (17) arranged in this recommended order is shown in figure 1.

Equation	Gaseous molecules			Atoms			Solids or liquids					
	x_1	x_2	...	x_Z	x_Y	...	x_N	x_A	x_T	Const		
(13)	1	0	0	$-a_1$	$-b_1$	...	0	0	0	$-q_1$	$-\delta_1$	
	0	1	0	$-a_2$	$-b_2$	...	0	0	0	$-q_2$	$-\delta_2$	
	0	0	...	...	...	...	0	0	0	...	...	
a	$a_1 n_1$	$a_2 n_2$	...	n_Z	0	0	...	$a_N n_N$	$-Aa$	0	δ_a	
b	$b_1 n_1$	$b_2 n_2$	...	0	n_Y	0	...	$b_N n_N$	$-Ab$	0	δ_b	
	...	...	...	0	0	...	...	...	0	...	...	
N	0	0	0	...	...	...	0	0	0	q_N	δ_N	
P	p_1	p_2	...	p_Z	p_Y	0	0	0	0	0	δ_P	
h	h_1'	h_2'	...	h_Z'	h_Y'	...	...	h_N'	$-Ah$	TC'	δ_h	

FIGURE 1.—General matrix of correction equations for adiabatic combustion at assigned pressure. Equation (13), dissociation of gaseous molecules; equation (15), mass balance; equation (14), dissociation of solids or liquids; equation (16), pressure; equation (17), heat balance.

Solution.—One of the best methods of solving simultaneous linear equations is given by Crout (reference 6). With this method, an auxiliary matrix is constructed from an original augmented matrix by a simple routine. This auxiliary matrix is of the order equal to the original matrix. The solution for the set of equations can be obtained by a process of back substitution in the auxiliary matrix.

For convenience, the order of the matrix is reduced before the Crout method is applied. A matrix arranged as recommended can be partitioned so that a unit matrix $[U_m]$ of the order (m, m) appears in the upper left corner, where m is equal to the number of types of gaseous molecule. The original augmented matrix can then be written

$$\begin{bmatrix} U_m & \alpha_1 \\ \alpha_2 & \alpha_3 \end{bmatrix} \quad (19)$$

When the Crout method is applied to the original augmented matrix, the Crout auxiliary matrix can be expressed as

$$\begin{bmatrix} U_m & \alpha_1 \\ \alpha_2 & \alpha_4 \end{bmatrix} \quad (20)$$

where $[U_m]$, $[\alpha_1]$, and $[\alpha_2]$ are identical to the corresponding submatrices of the original matrix. By observing the operations involved in the construction of the Crout auxiliary matrix, $[\alpha_4]$ is shown to be identical to the auxiliary matrix of the augmented matrix $[\alpha_5]$ defined by

$$[\alpha_5] = [\alpha_3] - [\alpha_2][\alpha_1] \quad (21)$$

For computation, equation (21) is written

$$[\alpha_5] = [\alpha_2][\alpha_3] \begin{bmatrix} -\alpha_1 \\ U_k \end{bmatrix} \quad (22)$$

where $[U_k]$ is a unit matrix of order equal to the number of columns of $[\alpha_1]$. The numerical solution is then obtained by carrying out the matrix multiplication indicated in equation (22) to find $[\alpha_5]$. The Crout auxiliary matrix $[\alpha_4]$

	Gaseous molecules			Atoms			Solids or liquids					
	x_1	x_2	...	x_Z	x_Y	...	x_N	x_A	x_T	Const		
a	$a_1 n_1$	$a_2 n_2$	...	n_Z	0	0	...	$a_N n_N$	$-Aa$	0	δ_a	
b	$b_1 n_1$	$b_2 n_2$	...	0	n_Y	0	...	$b_N n_N$	$-Ab$	0	δ_b	
	...	...	...	0	0	...	...	...	0	...	...	
N	0	0	0	a_N	b_N	...	0	0	0	q_N	δ_N	
P	p_1	p_2	...	p_Z	p_Y	...	0	0	0	0	δ_P	
h	h_1'	h_2'	...	h_Z'	h_Y'	...	...	h_N'	$-Ah$	TC'	δ_h	

(a) Submatrix $[\alpha_2; \alpha_3]$ taken from lower portion of figure 1.

	Gaseous molecules			Atoms			Solids or liquids					
	x_1	x_2	...	x_Z	x_Y	...	x_N	x_A	x_T	Const		
a	a_1	a_2	...	1	0	0	0	0	0	0		
b	b_1	b_2	...	0	1	0	0	0	0	0		
	...	...	...	0	0	...	0	0	0	0		
N	0	0	0	0	0	0	...	0	0	0		
	0	0	0	0	0	0	0	1	0	0		
q	q_1	q_2	...	0	0	0	0	0	0	1	0	
δ	δ_1	δ_2	...	0	0	0	0	0	0	0	1	

(b) Submatrix $\begin{bmatrix} -\alpha_1 \\ U_k \end{bmatrix}$ transposed ($-\alpha_1$) taken from figure 1).

FIGURE 2.—General form of submatrices of correction equations for adiabatic combustion at assigned pressure.

is constructed from $[\alpha_5]$. The values of the variables $x_{m+1}, \dots, x_{N+2}$ are found from $[\alpha_4]$ by the process of back substitution given by Crout. The values of the remaining variables are found by the matrix equation

$$\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = -[\alpha_1] \begin{bmatrix} x_{m+1} \\ \vdots \\ x_{N+2} \\ -1 \end{bmatrix} \quad (23)$$

For illustration, the submatrices $[\alpha_1]$, $[\alpha_2]$, and $[\alpha_3]$ were taken from figure 1 and used to construct figure 2. The submatrix $[\alpha_2; \alpha_3]$ corresponds to equations (15), (14), (16), and (17) and is shown in figure 2 (a). The transposed matrix of $\begin{bmatrix} -\alpha_1 \\ U_k \end{bmatrix}$ is shown in figure 2 (b); that is, the columns have been tabulated as rows with the first column at the top.

COMBUSTION AT CONSTANT VOLUME

The procedure given for finding the composition and the temperature of a combustion process at constant pressure can be applied to combustion at constant volume with the following changes:

(a) The correction equation for pressure is replaced by a correction equation for density obtained from equation (10)

$$x_A - x_T = \log \frac{\rho_o}{\rho} \quad (24)$$

(b) The correction equation for conservation of energy must be written in terms of internal energy E_T° and thus becomes

$$\sum_i (E_T^\circ)_i n_i x_i - A e x_A + T \sum_i (C_v^\circ)_i n_i x_T = A e \log \frac{e_o}{e} \quad (25)$$

where

$$e = \frac{1}{A} \sum_i (E_T^\circ)_i n_i$$

e_o is the assigned internal energy per equivalent formula at initial given conditions, and C_v° is the molar specific heat at constant volume. Substitution of these two equations in the matrix of figures 1 and 2 (a) will permit the composition and the temperature to be found for assigned values of density and internal energy. The application of this method to constant-volume combustion, which, for example, is involved in reciprocating engines and pulse-jet engines, has not been made at the Lewis laboratory.

ISENTROPIC EXPANSION TO ASSIGNED PRESSURE OR TEMPERATURE

Assigned pressure.—The calculation of temperature and equilibrium composition of the products of reaction following isentropic expansion to a fixed pressure involves the simultaneous solution of dissociation, conservation-of-mass, pressure, and entropy-balance equations.

For the reaction of equation (1), the dissociation, conservation of mass, and pressure equations (6) to (9) can again be applied. For the conditions following an isentropic expansion, the entropy s of the products of combustion per equivalent formula after expansion must be equal to the entropy s_o of the products of combustion per equivalent formula before expansion.

$$s_o = \left\{ \frac{1}{A} \sum_i [n_i (S_T^\circ)_i - R p_i \ln p_i] \right\}_{\text{combustion conditions}} \quad (26)$$

where $(S_T^\circ)_i$ is the absolute entropy of the product i at standard conditions. This formula is applicable to ideal solids and liquids, assuming $p_i = 0$, as long as their volume is negligible. After the expansion takes place, the entropy per equivalent formula is given by the expression

$$s = \left\{ \frac{1}{A} \sum_i [n_i (S_T^\circ)_i - R p_i \ln p_i] \right\}_{\text{exit conditions}} \quad (27)$$

Whereas equation (26) is, of course, evaluated at combustion-chamber temperature and pressure, equation (27) is evaluated for exit temperature and pressure. As the solution of the problem is found by successive adjustment of estimated quantities, the value of s approaches s_o .

In the adjustment of the values of n_i , A , and T , the correction equations (13) to (16), which have been derived from equations (6) to (9), can be applied. In addition, the fol-

lowing correction equation for entropy can be written from equation (27):

$$\sum s'_i x_i - A s x_A + C' x_T = \delta_s \quad (28)$$

where

$$\delta_s = A s \log \frac{s_o}{s}$$

$$s'_i = (S_T^\circ)_i n_i - R p_i (1 + \ln p_i)$$

The row matrix of equation (28) shown in figure 3 may be substituted in place of the **h** rows of figures 1 and 2 (a) and the computation carried out as in the combustion calculation.

Equation											
s	(28)	s'_1	s'_2	...	s'_r	s'_r	...	...	s'_N	$-A s$	C'
										δ_s	

FIGURE 3.—Row matrix to be substituted in place of **h** row in figure 1 and in figure 2 (a) for isentropic expansion to assigned pressure. Equation (28), entropy balance.

Assigned temperature.—For the computation of data for enthalpy-entropy diagrams and for other practical computations, it is often necessary to find the exit pressure and composition as a function of exit temperature. The procedure required is the same as that described for isentropic expansion to an assigned pressure except that, in addition to substituting the **s** row in place of the **h** row, the pressure equation (**p** row) and the temperature column (x_T) are dropped from the matrix of figure 1; accordingly, the **p** row and x_T column are dropped from figure 2 (a) and the **q** row from figure 2 (b).

ISENTROPIC EXPANSION TO LOCAL VELOCITY OF SOUND

The theoretical velocity of sound that includes the effect of dissociation can be computed at any point in a nozzle with a modification of the matrix previously derived to obtain the correction quantities.

Velocity of sound.—The velocity of sound u can be defined as

$$u^2 = \left(\frac{\partial P}{\partial \rho} \right)_s \quad (29)$$

where the subscript s denotes the condition of constant entropy. The total differential of pressure dP can be found from equation (9).

$$dP = \sum_i d p_i \quad (30)$$

and the total differential of density $d\rho$ can be found from equation (10).

$$d\rho = \frac{M_r}{RT} dA - \frac{A M_r}{RT^2} dT \quad (31)$$

Then

$$\frac{dP}{d\rho} = \frac{\sum_i d p_i}{\frac{M_r}{RT} dA - \frac{A M_r}{RT^2} dT} = \frac{\sum_i p_i \frac{d \log p_i}{d \log T}}{\frac{A M_r}{RT} \left(\frac{d \log A}{d \log T} - 1 \right)}$$

Therefore

$$u^2 = \left(\frac{\partial P}{\partial \rho} \right)_s = \frac{RT \sum_i p_i D_i}{A M_r (D_A - 1)} \quad (32)$$

where

$$D_i = \left(\frac{\partial \log n_i}{\partial \log T} \right)_s$$

$$D_A = \left(\frac{\partial \log A}{\partial \log T} \right)_s$$

This expression will permit evaluation of u^2 , provided the values of the partial derivatives D_i and D_A are found for conditions of chemical equilibrium and for an isentropic process. If the value of T is in degrees Kelvin and p_i in atmospheres the value of 8.3144×10^7 for R will give u in centimeters per second. The conditions of chemical equilibrium and constant entropy are introduced by writing the total differentials of equations (6) to (8) and (27). The total differential of these equations expressed in logarithmic variables and divided by $d \log T$ can be written, for gaseous products,

$$\frac{d \log p_i}{d \log T} - a_i \frac{d \log p_z}{d \log T} - b_i \frac{d \log p_r}{d \log T} - \dots - q_i = \frac{d \delta_i}{d \log T} \quad (33)$$

for liquid and solid products,

$$-a_i \frac{d \log p_z}{d \log T} - b_i \frac{d \log p_r}{d \log T} - \dots - q_i = \frac{d \delta_i}{d \log T} \quad (34)$$

and for all products of reaction,

$$\sum_i a_i n_i \frac{d \log n_i}{d \log T} - Aa \frac{d \log A}{d \log T} = Aa \frac{d \log a}{d \log T}$$

$$\sum_i b_i n_i \frac{d \log n_i}{d \log T} - Ab \frac{d \log A}{d \log T} = Ab \frac{d \log b}{d \log T}$$

$$\dots - \dots = \dots \quad (35)$$

$$\sum_i s_i' \frac{d \log n_i}{d \log T} - As \frac{d \log A}{d \log T} + C' = As \frac{d \log s}{d \log T} \quad (36)$$

If $d \log s$ is taken as 0, s is a constant; if $d \log a$, $d \log b$, . . . , and $d \delta_i$ are taken as 0, mass is constant, atomic types are conserved, and rate of change in composition corresponds to constant values of δ_i . With these assumptions the partial derivatives D_i and D_A may be substituted for the total derivatives in equations (33) to (36). The augmented matrix formed from these equations may be partitioned in a manner similar to the combustion matrix. The resulting submatrices are shown in figure 4 with the sign reversed. When D_i and D_A are determined by means of the matrices shown in figure 4, the velocity of sound can be calculated from equation (32). This equation can be applied to mixtures of liquid and solid products in equilibrium as long as their volume is negligible compared with the volume of the gas mixture and provided the liquid and solid particles move in velocity and temperature equilibrium with the gas.

Specific heat.—The molar specific heat at constant pressure of a mixture in equilibrium may be found from equation (12) as follows:

$$C_p = \frac{A}{n} \left(\frac{\partial h}{\partial T} \right)_p = \frac{1}{nT} \left[\sum_i (H_i^\circ) n_i \left(\frac{\partial \log n_i}{\partial \log T} \right)_p - Ah \left(\frac{\partial \log A}{\partial \log T} \right)_p + TC' \right] \quad (37)$$

	$-D_1$	$-D_2$		$-D_z$	$-D_r$			$-D_N$	$-D_A$
a	$a_1 n_1$	$a_2 n_2$		0	0	0	0	$a_N n_N$	$-Aa$
b	$b_1 n_1$	$b_2 n_2$		0	0	0	0	$b_N n_N$	$-Ab$
.....				0	0	0	0		0
N	0	0	0	a_N	b_N		0	0	q_N
s	s_1'	s_2'		s_z'	s_r'		0	s_N'	$-As$

(a) Submatrix $\begin{bmatrix} a_2 & a_3 \end{bmatrix}$.

a	a_1	a_2		1	0	0	0	0	0
b	b_1	b_2		0	1	0	0	0	0
.....				0	0		0	0	0
N	0	0	0	0	0	0		0	0
.....	0	0	0	0	0	0	0	1	0
q	q_1	q_2		0	0	0	0	0	1

(b) Submatrix $\begin{bmatrix} -a_i \\ -u_k \end{bmatrix}$ transposed.

FIGURE 4.—General form of submatrices of equations for partial derivatives at constant entropy.

where $n = \sum n_i$. Equation (30) can be written as

$$\sum_i p_i \frac{d \log n_i}{d \log T} = \frac{P d \log P}{d \log T} \quad (38)$$

If $d \log P$ is taken as 0, the pressure is constant; therefore, when equation (38) is substituted in the matrix of figure 4 in place of equation (36), the values of $\left(\frac{\partial \log n_i}{\partial \log T} \right)_p$ and $\left(\frac{\partial \log A}{\partial \log T} \right)_p$ can be found. These values can then be substituted in equation (37) to evaluate C_p° .

Isentropic expansion to assigned Mach number.—According to the law of conservation of energy the sum of the enthalpy and the kinetic energy of a certain quantity of gas at any point in a nozzle is constant. If this sum per equivalent formula at any point l is denoted by a parameter h^* , then

$$h^* = \left[h + \frac{1}{2} M_l^2 v^2 / J \right]_l \quad (39)$$

where v is the velocity of flow of the gas, J is a dimensional constant, and the subscript l indicates that the variables are evaluated at point l in the nozzle. The Mach number M of the flow is

$$M = \frac{v}{u} \quad (40)$$

Equations (32), (39), and (40) may be combined to give

$$h^* = \frac{\sum_i (H_i^\circ) n_i}{A} + \frac{M^2 \dot{R} T \sum_i p_i D_i}{2A(D_A - 1)} \quad (41)$$

where the value of R becomes 1.98718 (cal/(mole) (°K)). As the solution of the problem is found by successive adjustments of the estimated quantities, h^* approaches h_o .

If equation (41) is expanded in a manner similar to that used to obtain equation (17) and if the differentials of derivatives are assumed to be negligible, the correction equation becomes

$$\sum_i h_i'' x_i - Ah^* x_A + T C'' x_T = \delta_h^* \quad (42)$$

where

$$h_i'' = h_i' + \frac{M^2 R T p_i D_i}{2(D_A - 1)}$$

$$\delta_h^* = Ah^* \log \frac{h_o}{h^*}$$

$$C'' = \sum_i \left[n_i (C_s)_i + \frac{M^2 R p_i D_i}{2(D_A - 1)} \right]$$

Equation (42), together with equations (13) to (15) and (28), constitute the correction equations for the isentropic expansion to an assigned Mach number. The coefficients of these equations form the submatrices shown in figure 5.

In order to carry out the numerical computations, values of n_i , A , and T are estimated for the assigned conditions; the values of D_i and D_A are obtained by means of the submatrices of figure 4, and used to compute the numerical values of the elements of the bottom row of figure 5(a). The submatrices of figure 5 are then used to compute the values of the corrections to n_i , A , and T . This process can be repeated until the assigned conditions are satisfied.

	x_1	x_2	-----	x_Z	x_Y	---	---	x_N	x_A	x_T	
a	$a_1 n_1$	$a_2 n_2$	-----	n_Z	0	0	---	$a_N n_N$	$-Aa$	0	δ_a
b	$b_1 n_1$	$b_2 n_2$	-----	0	n_Y	0	---	$b_N n_N$	$-Ab$	0	δ_b
-----	-----	-----	-----	0	0	-----	-----	-----	-----	0	-----
-----	0	0	0	-----	-----	-----	0	0	0	-----	-----
N	0	0	0	a_N	b_N	-----	0	0	0	q_N	δ_N
s	s_1'	s_2'	-----	s_Z'	s_Y'	-----	-----	s_N'	$-As$	C'	δ_s
h*	h_1''	h_2''	-----	h_Z''	h_Y''	-----	-----	h_N''	$-Ah^*$	TC''	δ_{h^*}

(a) Submatrix $\begin{bmatrix} \alpha_2 & \alpha_3 \end{bmatrix}$.

	α_1	α_2	...	1	0	0	0	0	0	0	0
a	a_1	a_2	...	0	1	0	0	0	0	0	0
b	b_1	b_2	...	0	0	...	0	0	0	0	0
...	...	...	...	0	0	0	...	0	0	0	0
N	0	0	0	0	0	0	0	1	0	0	0
...	0	0	0	0	0	0	0	0	1	0	0
q	q_1	q_2	...	0	0	0	0	0	0	1	0
δ	δ_1	δ_2	...	0	0	0	0	0	0	0	1

(b) Submatrix $\begin{bmatrix} -\alpha_1 \\ -\frac{\alpha_1}{U_k} \end{bmatrix}$ transposed.

FIGURE 5.—General form of submatrices of correction equations for isentropic expansion to assigned Mach number.

Throat area of supersonic nozzle.—The process of isentropic expansion to a local Mach number of 1 is particularly interesting in the determination of the throat area of a nozzle having greater than critical pressure ratio. By assuming that the flow is isentropic and that chemical equilibrium is maintained throughout the expansion process, the flow velocity v at the throat must be equal to the velocity of sound u at the throat. The values n_i , A , T , and u can be found for a Mach number of 1 by use of the procedure given.

The throat area t can be calculated from the equation

$$\frac{t}{m} = \frac{RT}{AM_u} \quad (43)$$

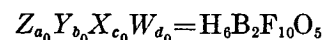
where m is the mass flow per second. If T is in degrees Kelvin and u is in centimeters per second, R equal to 82.0567 (cm³) (atm) / (°K) (mole) will give t/m in (cm²) (sec)/(gm). This equation can be applied to mixtures of liquid or solid phases in equilibrium provided that the volume occupied by the liquid and the solid phases is negligible compared with that of the gas phase and that the particles of liquid and solid are in thermal and velocity equilibrium with the gas phase.

EXAMPLE OF COMBUSTION OF DIBORANE WITH OXYGEN BIFLUORIDE

The calculation of equilibrium temperature and composition of the reaction of 1 mole of diborane (B₂H₆) with 5 moles of oxygen bifluoride (OF₂) is illustrated in this example for processes of

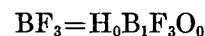
- constant-pressure adiabatic combustion
- isentropic expansion to 1 atmosphere
- isentropic expansion to the local velocity of sound

An equivalent formula of these reactants is



and $a_0=6$, $b_0=2$, $c_0=10$, and $d_0=5$.

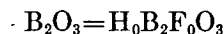
The following gaseous products will be considered as the products of reaction: boron trifluoride BF₃, boron trioxide B₂O₃, boron fluoride BF, boron hydride BH, boron oxide BO, diatomic boron B₂, hydrogen H₂, water vapor H₂O, hydroxyl radical OH, hydrogen fluoride HF, oxygen O₂, fluorine F₂, atomic hydrogen H, atomic boron B, atomic fluorine F, and atomic oxygen O. No liquids or solids are included. If the products are numbered in the order given, they can be identified in the terminology of equation (1) as follows:



and therefore

$$a_1=0, b_1=1, c_1=3, \text{ and } d_1=0$$

Similarly,



and

$$a_2=0, b_2=2, c_2=0, \text{ and } d_2=3$$

All values of a_i , b_i , c_i , and d_i for this problem, together with the thermodynamic properties used, are listed in table I. Although these thermodynamic values and the enthalpies of B_2H_6 and of OF_2 have since been revised, and therefore do not correspond to the values listed in the thermodynamic tables presented in a later section, they are adequate for the purpose of this example. The enthalpy values used are

$$(H_{298.16}^\circ)_{\text{liquid } B_2H_6} = 570.149 \text{ kilocalories per mole}$$

$$(H_{128.3}^\circ)_{\text{liquid } OF_2} = 67.077 \text{ kilocalories per mole}$$

The enthalpy of the amount of fuel and oxidant at initial conditions corresponding to the equivalent formula is, from equation (11),

$$h_o = 570.149 + 5(67.077) = 905.534 \frac{\text{kilocalories}}{\text{equivalent formula}} \quad (44)$$

The values of a_i , b_i , c_i , d_i , and h_o are constant for all parts of this example.

COMBUSTION PROCESS

The adiabatic combustion process was assumed to occur at a constant pressure of 20.4 atmospheres.

First estimate.—From previous computations or from simple calculations with equilibrium constants, estimating reasonable values for the composition and the temperature is usually possible. This procedure is recommended inasmuch as close estimates reduce the number of trials that must be made. In order to show that an arbitrary composition which is not based on probable final values of the composition

can be used, however, the first estimates for this example for n_i and A have been taken equal to 1 mole and a temperature of 4000° K. The possibility of divergence is discussed in a later section. All estimated quantities will be used with three decimal places to distinguish them from numbers that are always integers.

Evaluation of submatrices.—The numerical values of the elements of the submatrices shown in figures 2(a) and 2(b) can now be computed and are shown in figure 6. The steps are as follows:

1. The values of a_i , b_i , c_i and d_i are entered in rows **a**, **b**, **c**, and **d** of figure 6(b) and a 1 is entered on each square of the diagonal of $[U_k]$ according to figure 2(b).

2. Values of $q_i = \left(\frac{\Delta H}{RT}\right)_i$ from tables of thermodynamic functions are entered in row **q** of figure 6(b). In this case they are obtained from table I.

3. The values of the elements of the δ row of figure 6 (b) may be computed from equation (6) for gaseous products

$$\delta_i = \log p_i - a_i \log p_H - b_i \log p_B - c_i \log p_F - d_i \log p_O - \log K_i$$

The values of $\log K_i$ are obtained from tables of thermodynamic properties, in this case table I. Because all molecules and atoms are estimated to be 1, their logarithms are 0 so that in this case

$$\delta_i = -\log K_i$$

4. The estimated values of n_i are entered in row **p** of figure 6 (a). In case liquids or solids are present their value will be zero.

Gaseous molecules												Atoms							
2BF_3	2B_2O_3	2BF	2BH	2BO	2B_2	2H_2	2H_2O	2OH	2HF	2O_2	2F_2	2H	2B	2F	2O	2A	2T	Const	
a	0	0	0	1.000	0	0	2.000	2.000	1.000	1.000	0	0	1.000	0	0	0	-8.000	0	-0.999
b	1.000	2.000	1.000	1.000	1.000	2.000	0	0	0	0	0	0	0	1.000	0	0	-9.000	0	-5.879
c	3.000	0	1.000	0	0	0	0	0	1.000	0	2.000	0	0	1.000	0	0	-8.000	0	0.775
d	0	3.000	0	0	1.000	0	0	1.000	1.000	0	2.000	0	0	0	0	1.000	-9.000	0	-2.297
p	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	0	0	1.688
h	.721	2.334	2.629	3.569	2.527	5.720	.995	.577	.765	.320	.373	.960	1.051	3.177	.826	.794	-27.346	6.446	-13.125

(a) Submatrix $\begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix}$.

a	0	0	0	1	0	0	2	2	1	1	0	0	1	0	0	0	0	0	0
b	1	2	1	1	1	2	0	0	0	0	0	0	0	1	0	0	0	0	0
c	3	0	1	0	0	0	0	0	0	1	0	2	0	0	1	0	0	0	0
d	0	3	0	0	1	0	0	1	1	0	2	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
q	-62.075	-80.593	-17.288	-8.300	-18.183	-7.989	-13.939	-29.209	-13.603	-19.674	-15.313	-8.705	0	0	0	0	0	1	0
δ	-5.695	-5.109	-1.634	2.611	-1.033	2.763	0.406	0.347	0.167	-1.894	0.380	3.137	0	0	0	0	0	0	1

(b) Submatrix $\begin{bmatrix} -\alpha_1 \\ -U_i \end{bmatrix}$ transposed.

FIGURE 6.—Numerical example of submatrices of correction equations for adiabatic combustion of diborane and oxygen bifluoride after first estimate of n_i , A , and T .

5. The values of the elements in rows **a**, **b**, **c**, and **d** of figure 6 (a), except columns x_A , x_T , and constant, are obtained by multiplying the value of n_i by the value of the corresponding element of the respective row in figure 6 (b). For example, the entries in the first column are $0 \times 1.000 = 0$, $1 \times 1.000 = 1.000$, $3 \times 1.000 = 3.000$ and $0 \times 1.000 = 0$.

6. Values of the elements of row **h** of figure 6 (a), except columns x_A , x_T , and constant, are obtained by multiplying the value of n_i by the value of $(H_T^\circ)_i$ from tables of thermodynamic functions, in this case table I. For example, the entry in the first column is $72,172 \times 1.000 = 72,172$. All values in row **h** have been divided by 10^5 .

7. The values of the elements of column x_A figure 6 (a) are obtained by summing elements to the left in each row and writing the negative of the total in column x_A except for row **p** where the value is zero.

8. The value of the x_T column (fig. 6 (a)) is zero except for the **h** row where the value is $T \sum_i (C_p^\circ)_i n_i$. The values of $(C_p^\circ)_i$ are obtained from tables of thermodynamic functions, in this case table I.

9. Values of the constant column for figure 6 (a) for all rows except row **p** are found as follows: The value already entered in the x_A column for row **a** is $-Aa$. With the estimated value of $A = 1.000$

$$a = \frac{Aa}{A} = \frac{8.000}{1.000} = 8.000$$

$$\delta_a = Aa \log \frac{a_o}{a}$$

$$= 8.000 \log \frac{6}{8.000} = -0.999$$

the values of δ_b , δ_c , δ_d , and δ_h are found in a similar manner.

x_H	x_B	x_F	x_O	x_A	x_T	Const
12.000	1.000	1.000	3.000	-8.000	-127.873	1.391
1.000	13.000	4.000	7.000	-9.000	-283.010	-16.322
1.000	4.000	16.000	0	-8.000	-240.597	-13.564
3.000	7.000	0	17.000	-9.000	-333.400	-17.383
8.000	9.000	8.000	9.000	0	-294.871	-3.866
8.854	28.736	7.861	12.414	-27.346	-454.757	-7.663

(a) Matrix $[\alpha_2]$ obtained from matrix multiplication $[\alpha_2] \alpha_3 \begin{bmatrix} -\alpha_1 \\ U_k \end{bmatrix}$.

12.000	0.08333	0.08333	0.2500	-0.6667	-10.656	0.1159
1.000	12.917	0.3032	0.5226	-0.6451	-21.085	-1.273
1.000	3.917	14.729	-0.1560	-0.3263	-10.004	-0.5902
3.000	6.750	-2.297	12.364	-0.2745	-14.727	-0.8487
8.000	8.333	4.867	3.395	13.210	-4.857	0.8731
8.854	27.998	-1.366	-4.644	-5.102	173.652	0.1544

(b) Matrix $[\alpha_4]$ (Crout's auxiliary matrix of $[\alpha_3]$).

x_H	x_B	x_F	x_O	x_A	x_T
1.299	0.9290	1.222	1.459	0.1232	0.1544

(c) Values of corrections (Crout's final matrix).

FIGURE 7. Numerical example of the solution of correction equations by matrix methods.

10. The constant column of row **p** is found as follows: The sum of the elements of row **p** is the pressure $P = 16.000$; δ_P is computed from the formula

$$\delta_P = P \log \frac{P_o}{P}$$

$$\delta_P = 16.000 \log \frac{20.4}{16.000} = 1.688$$

The matrix multiplication $[\alpha_2] \alpha_3 \begin{bmatrix} -\alpha_1 \\ U_k \end{bmatrix}$ will result in the matrix $[\alpha_5]$ shown in figure 7 (a). The steps of this multiplication are shown in standard textbooks such as reference 7. Crout's auxiliary matrix corresponding to $[\alpha_5]$ may then be constructed and is shown in figure 7 (b) and the values of x_H , x_B , x_F , x_O , x_A , and x_T are shown in figure 7 (c). The values of the remaining functions are computed with the aid of equation (23). The solution is found to be

$$\begin{array}{ll} x_{BF_3} = 0.7056 & x_{OH} = 0.4907 \\ x_{B_2O_3} = -1.100 & x_{HF} = 1.377 \\ x_{BF} = 1.116 & x_{O_2} = 0.1737 \\ x_{BH} = 1.665 & x_{F_2} = -2.037 \\ x_{BO} = 0.6135 & x_H = 1.299 \\ x_{B_2} = -2.139 & x_B = 0.9290 \\ x_{H_2} = 0.03982 & x_F = 1.222 \\ x_{H_2O} = -0.7999 & x_O = 1.459 \\ x_A = 0.1232 & x_T = 0.1544 \end{array}$$

These values are to be applied to the initial estimates for n_i , A , and T according to the equation

$$(\log n_i)_{\text{second estimate}} = (\log n_i)_{\text{first estimate}} + x_i \quad (45)$$

For example, the second estimate of n_{BF_3} would be

$$(\log n_{BF_3})_{\text{second estimate}} = \log 1.000 + 0.7056$$

$$(n_{BF_3})_{\text{second estimate}} = 5.077$$

The second estimates of n_i , A , and T are then used to set up new submatrices according to the procedure described. The process is repeated until the desired accuracy has been obtained. For this example, six approximations were required to give the following final values of n_i , A , and T :

$$\begin{array}{ll} n_{BF_3} = 2.6593 & n_{OH} = 0.6785 \\ n_{B_2O_3} = 0.1235 & n_{HF} = 7.1456 \\ n_{BF} = 0.1936 & n_{O_2} = 0.9210 \\ n_{BH} = 0.0001 & n_{F_2} = 0.0003 \\ n_{BO} = 0.1669 & n_H = 1.7694 \\ n_{B_2} = 0 & n_B = 0.0577 \\ n_{H_2} = 0.1271 & n_F = 1.3043 \\ n_{H_2O} = 0.0627 & n_O = 5.1903 \\ A = 1.6622 & T = 4775.5^\circ \text{K} \end{array}$$

Discussion of convergence.—In order to demonstrate the convergence of the process with large errors in the first estimate, the example of the combustion of diborane and fluorine oxide was solved by using 1 mole of each product, a value of 1 for A , and a temperature of 4000° K for the first estimate. Because these first estimates were made without regard for the probable final values, large errors were present in the second approximation and six approximations were required to eliminate the error. The convergence is shown in terms of the parameters a , b , c , d , P , h , and ϵ in the following table where ϵ is defined as

$$\epsilon = \sum_i \left| \log k_i \right| + \left| \log \frac{a_o}{a} \right| + \left| \log \frac{b_o}{b} \right| + \left| \log \frac{c_o}{c} \right| + \left| \log \frac{d_o}{d} \right| + \left| \log \frac{P_o}{P} \right| + \left| \log \frac{h_o}{h} \right|$$

RESULTS OF APPROXIMATIONS								
Parameter	First estimate	Trial number						Desired value
		1	2	3	4	5	6	
a	8	36.840	7.005	6.286	6.079	6.002	6.000	6.000
b	9	23.346	11.605	2.653	2.325	2.008	2.000	2.000
c	8	51.540	24.082	13.104	10.541	10.016	10.000	10.000
d	9	29.641	11.954	33.600	5.240	5.022	5.000	5.000
P	16	125.485	38.000	52.434	21.416	20.436	20.400	20.400
h	2734.615	12,055.015	2090.090	2909.950	965.968	912.368	905.594	905.534
ϵ	26.892	5.861	4.092	2.505	.537	.011	.002	0

This method has been used in routine computation for several years without encountering a divergent case in a practical problem. At least for special cases when temperature is assigned, the process will converge for all values of the first estimates. Divergence is known to occur for certain cases where temperature is used as a variable when the first estimate of temperature and composition is sufficiently in error. Although no mathematical analysis has been made to determine the theoretical limits of convergence, the process appears to be satisfactory for practical computation.

Special treatment would be required if divergence is encountered. Obtaining convergence should be possible by a sufficiently close new estimate of composition and temperature. This procedure is recommended when it is feasible but other procedures can be devised, depending on the individual case.

ISENTROPIC EXPANSION TO FIXED PRESSURE

The temperature and the composition of the products of reaction following an isentropic-expansion ratio of 20.4 at chemical equilibrium were also computed for the products of reaction of this example. The value of s_o is found from equation (26) by using the final values of each constituent of the adiabatic combustion and the absolute entropy values corresponding to the final combustion temperature. The calculated value of s_o was 763.476 calories per °K per mole.

First estimates.—The number of approximations necessary

for a complete calculation can be considerably reduced if the initial estimate is based on previous experience. The final values of n_i and A determined for the combustion process of this example can therefore be the basis for this first estimate.

Because the expansion ratio is 20.4, the four largest components can be estimated to be 1/20.4 of their combustion value.

$$\begin{aligned} n_{\text{BF}_3} &= 0.1304 \\ n_{\text{HF}} &= 0.3503 \\ n_{\text{H}} &= 0.0867 \\ n_{\text{O}} &= 0.2544 \\ A &= 0.0815 \end{aligned}$$

For convenience of presentation, the temperature was estimated to be 4000° K so that the values of table I could be used again. The remaining products can be estimated from the dissociation equations by setting $\log k_i = 0$. For example, p_{F} would be determined with the assumed values of p_{HF} and p_{H} from equation (6) and table I ($p_i = n_i$)

$$\begin{aligned} 0 &= \log 0.3503 - \log 0.0867 - \log p_{\text{F}} - 1.8944 \\ \log p_{\text{F}} &= -0.45556 + 1.06198 - 1.8944 \\ &= -1.28798 \end{aligned}$$

$$p_{\text{F}} = 0.0515$$

Similarly, p_{B} can be estimated with the assumed values of p_{BF_3} and p_{F}

$$\begin{aligned} 0 &= \log 0.1304 - \log p_{\text{B}} - 3 \log 0.0515 - 5.6953 \\ \log p_{\text{B}} &= -0.88472 + 3.86394 - 5.6953 \\ p_{\text{B}} &= 0.0019 \end{aligned}$$

If this procedure is followed for all the remaining constituents, the following list of first estimates can be made:

$$\begin{aligned} n_{\text{BF}_3} &= 0.1304 & n_{\text{OH}} &= 0.0150 \\ n_{\text{B}_2\text{O}_3} &= 0.0078 & n_{\text{HF}} &= 0.3503 \\ n_{\text{BF}} &= 0.0043 & n_{\text{O}_2} &= 0.0269 \\ n_{\text{BH}} &= 0 & n_{\text{F}_2} &= 0 \\ n_{\text{BO}} &= 0.0053 & n_{\text{H}} &= 0.0867 \\ n_{\text{B}_2} &= 0 & n_{\text{B}} &= 0.0019 \\ n_{\text{H}_2} &= 0.0029 & n_{\text{F}} &= 0.0515 \\ n_{\text{H}_2\text{O}} &= 0.0009 & n_{\text{O}} &= 0.2544 \\ A &= 0.0815 & T &= 4000^\circ \text{K} \end{aligned}$$

Construction of submatrices.—The construction of the submatrices may now be carried out and is shown in figure 8. The steps are the same as for the combustion example except for steps 6 to 9, which are different because the enthalpy equation has been replaced with the entropy equation.

The values of the elements of row s of figure 8(a) are obtained from the expression

$$s'_i = n_i [(S_T^\circ)_i - 1.98718 - 4.57565 \log p_i]$$

The values of $(S_T^\circ)_i$ are obtained from tables of thermodynamic data, in this case table I. For example, the entry in the first column is computed to be

$$\begin{aligned} (s_{\text{BF}_3})' &= 0.1304 (105.951 - 1.98718 - 4.57565 \log 0.1304) \\ &= 14.0848 \end{aligned}$$

7. The values of the entries in the x_A column of figure 8(a) are obtained in the same manner as for figure 6(a) except for the **s** row where the sum of the elements of the **p** row times 1.98718 is added to the sum of the elements of the **s** row and entered in column x_A .

8. The value of the entries in the x_T column is zero except for the **s** row where it is $\sum_i n_i (C_p^\circ)_i$. The values of $(C_p^\circ)_i$ are obtained from tables of thermodynamic data, in this case table I.

9. The value of δ_s is found in a manner similar to δ_a .

ISENTROPIC EXPANSION TO MACH NUMBER OF 1

The temperature and the composition of the products of reaction following an isentropic expansion to the local velocity of sound was computed for the products of reaction considered in this example, assuming chemical equilibrium. The value of s_0 is the same as that found for the isentropic expansion to 1 atmosphere.

First estimate.—For simplicity, the same first estimates of 1 mole, 1, and 4000° K, for n_i , A , and T , respectively, were again made.

Construction of submatrices.—The submatrices corresponding to figure 5 may be constructed and are shown in figure 9.

The submatrices corresponding to figure 4 are first constructed. The steps are the same as for figure 8 except that row **p** and the constant column are omitted. The matrix

multiplication may then be carried out and the values of the partial derivatives D_i and D_A computed in a manner similar to the computation in the combustion example. These values of D_i and D_A together with $(H_T^\circ)_i$ are used to calculate the elements of row **h*** of figure 9(a) except columns x_A , x_T , and constant. For example, when the values of $D_{BF_3}=12.990$ and $D_A=19.039$ are used, the value of

$$(h_{BF_3})'' = (h_{BF_3})' + \frac{M^2 R T p_{BF_3} D_{BF_3}}{2(D_A - 1)} \text{ becomes } 75,034 = 72,172 + \frac{1 \times 1.98718 \times 4000 \times 1.000 \times 12.990}{2(19.039 - 1)} \text{ (cal)}$$

All values in row **h*** have been divided by 10^5 .

The value of the element in row **h***, column x_A is the sum of the elements to the left. The element in row **h***, column x_T is given by

$$TC'' = T \sum_i \left[n_i (C_p^\circ)_i + \frac{M^2 R p_i D_i}{2(D_A - 1)} \right]$$

and the value of the constant column is obtained as in the previous examples. Matrix multiplication that was carried out for the determination of D_i and D_A values may now be extended by an additional row and column and the value of x_i , x_A , and x_T found as in the previous examples. These values may then be used to obtain the second estimates for n_i , A , and T and the computation repeated until the desired accuracy has been obtained.

	Gaseous molecules												Atoms				x_A	x_T	Const
	x_{BF_3}	$x_{B_2O_3}$	x_{BF}	x_{BH}	x_{BO}	x_{B_2}	x_{H_2}	x_{H_2O}	x_{OH}	x_{HF}	x_{O_2}	x_{F_2}	x_H	x_B	x_F	x_O			
a	0	0	0	0	0	0	0.0058	0.0018	0.0150	0.3503	0	0	0.0867	0	0	0	-0.4596	0	0.01238
b	0.1304	0.0156	0.0043	0	0.0053	0	0	0	0	0	0	0	0	0.0019	0	0	-1.1575	0	0.00235
c	0.3912	0	0.0043	0	0	0	0	0	0	0.3503	0	0	0	0	0.0515	0	-0.7973	0	0.00760
d	0	0.0234	0	0	0.0053	0	0	0.0009	0.0150	0	0.0538	0	0	0	0	0.2544	-0.3528	0	0.02208
p	0.1304	0.0078	0.0043	0	0.0053	0	0.0029	0.0009	0.0150	0.3503	0.0269	0	0.0867	0.0019	0.0515	0.2544	0	0	0.02595
s	14.0848	0.9704	0.3559	0	0.4137	0	0.1751	0.0760	1.0552	21.4213	2.0438	0	3.7435	0.1140	2.8395	13.2827	-62.4405	8.464	0.09447

(a) Submatrix $\begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix}$.

a	0	0	0	1	0	0	2	2	1	1	0	0	1	0	0	0	0	0	0
b	1	2	1	1	1	2	0	0	0	0	0	0	0	1	0	0	0	0	0
c	3	0	1	0	0	0	0	0	0	1	0	2	0	0	1	0	0	0	0
d	0	3	0	0	1	0	0	1	1	0	2	0	0	0	0	1	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
q	-62.075	-80.593	-17.288	-8.300	-18.183	-7.989	-13.939	-29.209	-13.603	-19.674	-15.313	-8.705	0	0	0	0	0	1	0
δ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

(b) Submatrix $\begin{bmatrix} -\alpha_1 \\ -U_1 \end{bmatrix}$ transposed.

FIGURE 8.—Numerical example of submatrices of correction equations for isentropic expansion to 1 atmosphere for the reaction of diborane with oxygen bifluoride after first estimate of n_i , A , and T .

	Gaseous molecules												Atoms				x_A	x_T	Const
	x_{BF_3}	$x_{B_2O_3}$	x_{BF}	x_{BH}	x_{BO}	x_{B_2}	x_{H_2}	x_{H_2O}	x_{OH}	x_{HF}	x_{O_2}	x_{F_2}	x_H	x_B	x_F	x_O			
a	0	0	0	1.000	0	0	2.000	2.000	1.000	1.000	0	0	1.000	0	0	0	-8.000	0	-0.999
b	1.000	2.000	1.000	1.000	1.000	2.000	0	0	0	0	0	0	0	1.000	0	0	-9.000	0	-5.879
c	3.000	0	1.000	0	0	0	0	0	0	1.000	0	2.000	0	0	1.000	0	-8.000	0	0.755
d	0	3.000	0	0	1.000	0	0	1.000	1.000	0	2.000	0	0	0	0	1.000	-9.000	0	-2.297
s	103.964	114.773	71.917	59.425	67.633	68.593	49.067	70.471	62.002	59.067	68.796	68.826	38.319	47.562	49.562	49.492	-103.944	161.162	-163.234
h*	0.750	2.364	2.672	3.623	2.569	5.778	1.035	0.624	0.813	0.347	0.427	1.026	1.086	3.215	0.869	0.839	-28.035	-7.113	-13.760

(a) Submatrix $\begin{bmatrix} \alpha_2 & \alpha_3 \end{bmatrix}$.

a	0	0	0	1	0	0	2	2	1	1	0	0	1	0	0	0	0	0	0
b	1	2	1	1	1	2	0	0	0	0	0	0	0	1	0	0	0	0	0
c	3	0	1	0	0	0	0	0	0	1	0	2	0	0	1	0	0	0	0
d	0	3	0	0	1	0	0	1	1	0	2	0	0	0	0	1	0	0	0
e	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
q	-62.075	-80.593	-17.288	-8.300	-18.183	-7.989	-13.939	-29.209	-13.603	-19.674	-15.313	-8.705	0	0	0	0	0	1	0
δ	-5.695	-5.109	-1.634	2.611	-1.033	2.763	0.406	0.347	0.167	-1.894	0.380	3.137	0	0	0	0	0	0	1

(b) Submatrix $\begin{bmatrix} -\alpha_1 \\ -\alpha_2 \\ -\alpha_3 \\ -\alpha_4 \end{bmatrix}$ transposed.

FIGURE 9.—Numerical example of submatrices of correction equations for isentropic expansion to local velocity of sound for reaction of diborane and oxygen bifluoride after first estimate of n_i , A and T .

TABLES OF THERMODYNAMIC PROPERTIES

Tables of thermodynamic data, completed June 1949, are presented for the following substances:

A	Al(g)	B	C	Cl	F	H	e ⁻	Li	N	O
	Al(s)	B ₂	CO	Cl ₂	F ₂	H ₂	F ⁻	LiF	N ₂	O ₂
	Al(liq)	BF	CO ₂	ClF		HCl	Li ⁺	LiH	NO	OH
	AlO	BF ₃				HF				
	Al ₂ O ₃ (g)	BH				H ₂ O				
	Al ₂ O ₃ (s)	BO								
	Al ₂ O ₃ (liq)	B ₂ O ₃ (g)								
		B ₂ O ₃ (s)								
		B ₂ O ₃ (liq)								

These tables are taken from NACA TN 2161 except that the values for BF have been revised. Many of the data in the tables are based upon estimated vibrational frequencies or insufficient spectroscopic or thermochemical data to provide accurate data at high temperatures. Nevertheless, the data are considered sufficiently accurate for engineering evaluations of performance of aircraft propulsion systems until better data become available.

PREPARATION OF TABLES

The values of enthalpy and entropy below 1000° K and the values of specific heat at all temperatures were based upon data taken from the literature or calculated by NACA from spectroscopic data or estimated fundamental frequencies. The calculations were made by use of the accurate summation method described in reference 8 or by the use of the tables prepared by F. J. Krieger of Douglas Aircraft Company, Inc. based upon a harmonic oscillator. The

values of enthalpy and entropy above 1000° K were computed from the specific-heat data and these values were then used to compute the values of the remaining functions.

The thermodynamic functions computed by NACA are based upon the fundamental constants from reference 9 and are given in terms of the thermochemical calorie defined as 4.18400 absolute joules (reference 10).

Specific heat.—The specific-heat data were interpolated and extrapolated when necessary to obtain values of C_p° at 298.16° K and every 100° from 300° to 6000° K. In many cases these C_p° data were smoothed by the following method: Values of the first differences of C_p° for 100° K intervals δC_p° were plotted against temperature and a smooth curve drawn. New values of C_p° were then computed from the values of δC_p° read from the curve. In some cases the new C_p° values were tabulated to more decimal places than the original data. Care was taken to see that the new C_p° values were within about 1 or 2 units in the last tabulated place of the reference data in all but a few cases in which the reference data were irregular.

In order to minimize the labor required to integrate C_p° to obtain the other functions, a linear variation of C_p° was assumed by use of the equation

$$C_p^\circ = c_1 + c_2 T \quad (46)$$

where c_1 and c_2 are constants evaluated for each 100° temperature interval above 1000° K.

The maximum difference between a smooth function representing C_p° and the series of 50 straight-line segments represented by equation (46) is usually less than 0.005 percent at any temperature. In a few cases near 1000° K the error approaches 0.05 percent.

Enthalpy and entropy.—The data for enthalpy and entropy below 1000° K were taken from the literature or computed at the Lewis laboratory and when necessary interpolated to give the values at 298.16° K and every 100° from 300° to 1000° K.

The values above 1000° K were obtained by integration of equation (46) for C_p° using the constants for each 100° K temperature interval.

The value of the change of enthalpy δH_T° for a temperature change $\delta T = T - T_1$ is given by

$$\delta H_T^\circ = \int_{T_1}^T C_p^\circ dT = \bar{C}_p^\circ \delta T \quad (47)$$

where $\bar{C}_p^\circ$ is given by

$$\bar{C}_p^\circ = (C_p^\circ)_1 + \frac{c_2}{2} \delta T \quad (48)$$

and $(C_p^\circ)_1$ is the value of C_p° corresponding to the temperature T_1 . The corresponding change in entropy δS_T° is given by

$$\delta S_T^\circ = \int_{T_1}^T \frac{C_p^\circ}{T} dT = c_1 \delta \ln T + c_2 \delta T \quad (49)$$

where $\delta \ln T$ is given by

$$\delta \ln T = \ln T - \ln T_1 \quad (50)$$

Values of enthalpy and entropy for each 100° above 1000° K were obtained by accumulatively adding to the values at 1000° K the changes of enthalpy and entropy computed for each 100° interval by means of equations (47) and (49).

The values of enthalpy and entropy were computed to more decimal places than are tabulated and then rounded. Equations (47) and (49) may therefore occasionally yield values that differ by one unit in the last tabulated place of enthalpy and entropy because of rounding. Inconsistencies from this source are unavoidable and are not considered in the following discussion.

The representation of C_p° from 1000° to 6000° K by means of 50 straight-line segments permitted computation of self-consistent values of enthalpy and entropy but lead to values slightly different from those that would have resulted from a more laborious integration of a smooth C_p° function. For example, the values of enthalpy at 6000° K differ from those obtained by applying Simpson's one-third rule by 0.0045, 0.0012, 0.0038, and 0.0031 percent for H_2O , H_2 , CO_2 , and BF_3 , respectively.

In a few cases, discrepancies exist between the reference values of enthalpy and the values of enthalpy given herein that cannot be accounted for by the error resulting from the method of integration used. From an analysis of the values in the reference tables, these discrepancies appear to be caused by a combination of small inconsistencies and round-

ing errors in the references. The maximum discrepancy noted in enthalpy occurred in H_2O and was less than 0.25 percent of the value of $H_T^\circ - H_0^\circ$.

Enthalpy H_T° .—For convenience of computation, tables of enthalpy H_T° , the sum of the sensible enthalpy $H_T^\circ - H_0^\circ$ and chemical energy at 0° K H_0° , were prepared. An arbitrary base may be adopted for assigning absolute values to the enthalpy of various substances inasmuch as only differences in enthalpy are measurable. The base used in these tables was selected to obtain positive values for H_T° of the substances commonly used as rocket and ram-jet propellants and occurring in the products of combustion; it is shown in the following table:

Base substance	Phase	Temperature (°K)	Enthalpy assigned, H_T° (kcal/mole)
A	Gas	0	0
AlF ₃	Crystal	298.16	0
BF ₃	Gas	0	0
CO ₂	Gas	0	0
Cl ₂	Gas	298.16	10
HF	Gas	0	0
H ₂ O	Crystal	0	0
LiF	Gas	298.16	80
N ₂	Crystal	0	0
O ₂	Crystal	0	0
e ⁻	Gas	0	60

In determining the value of H_T° to be assigned to a substance having a known heat of formation, it is convenient to use the values of H_T° assigned to the elements as shown in the following table:

Element	Phase	Enthalpy assigned, H_T° (kcal/mole)	
		0° K	298.16° K
A	Gas	0	1.4812
Al	Crystal	233.6251	234.6951
B	Crystal	-----	173.3793
C	Graphite	91.9274	92.1790
Cl ₂	Gas	7.8061	10.0000
F ₂	Gas	60.9562	63.0699
H ₂	Gas	67.4169	69.4407
Li	Crystal	-----	132.2250
N ₂	Gas	1.6992	3.7715
O ₂	Gas	2.0362	4.1109
e ⁻	Gas	60.0000	61.4812

For example, if the value of enthalpy H_T° to be assigned to H_2O (liq) is to be determined at 298.16° K, the reaction of formation would be



and ΔH_f° is defined as

$$\Delta H_f^\circ = (H_T^\circ)_{H_2O} - (H_T^\circ)_{H_2} - \frac{1}{2} (H_T^\circ)_{O_2} \quad (52)$$

therefore,

$$(H_T^\circ)_{H_2O} = \Delta H_f^\circ + (H_T^\circ)_{H_2} + \frac{1}{2} (H_T^\circ)_{O_2} \quad (53)$$

With the use of the value $\Delta H_f^\circ = -68.3174$ (kcal/mole),

$$\begin{aligned} (H_{298.16}^\circ)_{H_2O(l)} &= -68,317.4 + 69,440.7 + \frac{1}{2}(4110.9) \\ &= 3178.75 \text{ (cal/mole)} \end{aligned}$$

For convenience, the values of H_T° thus assigned to a number of compounds have been computed with the aid of data from references 8 and 10 to 23 and are listed in table II. The energy of gas imperfections has been included in computing the values of H_T° assigned to the liquid phase of ammonia, *n*-butane, chlorine, hydrogen, and water.

$-\Delta H^\circ/RT$ and $\ln K$.—From the values of H_T° and S_T° obtained as previously described, the values of $-\Delta H^\circ/RT$ and $\ln K$ were computed for the reaction of formation of each substance from its elements in the atomic gas state. For example, the reaction of formation of H_2O is



From the definitions of ΔH° and $\ln K$

$$\frac{\Delta H^\circ}{RT} = \frac{(H_T^\circ)_{H_2O} - 2(H_T^\circ)_H - (H_T^\circ)_O}{RT} \quad (55)$$

and

$$\ln K = \frac{(S_T^\circ)_{H_2O} - 2(S_T^\circ)_H - (S_T^\circ)_O}{R} - \frac{(H_T^\circ)_{H_2O} - 2(H_T^\circ)_H - (H_T^\circ)_O}{RT} \quad (56)$$

As shown by equations (3) and (4), K may be expressed in terms of partial pressures. For example, for gaseous H_2O ,

$$K = \frac{p_{H_2O}}{p_H^2 p_O}$$

The values of $\ln K$ have been converted to $\log K$ in the tables for convenience.

INTERPOLATION OF TABLES

Interpolation formulas are given that permit computation of self-consistent values of the thermodynamic functions at temperatures intermediate to those tabulated. Linear interpolation is recommended for simplicity, however, when a high degree of self-consistency is not required. Interpolation formulas such as those of Newton or Lagrange will give values near the self-consistent value. Inasmuch as the tables are based on linear variations in C_p° , linear interpolation yields self-consistent results for the C_p° function. An example of the values obtained by the interpolation formulas given and by linear interpolation is shown for each function.

Interpolation of specific heat.—The value of C_p° for any temperature T is given by

$$C_p^\circ = (C_p^\circ)_1 + \frac{\delta T}{T_2 - T_1} [(C_p^\circ)_2 - (C_p^\circ)_1] \quad (57)$$

where $(C_p^\circ)_1$ and $(C_p^\circ)_2$ are the tabular values corresponding to the tabular temperatures T_1 and T_2 between which T lies and $\delta T = T - T_1$. For example, the value of C_p° for H_2O at $1573.4^\circ K$ is computed to be

$$\begin{aligned} C_p^\circ &= 11.134 + \frac{73.4}{100} (11.343 - 11.134) \\ &= 11.287 \text{ (cal/(mole) } (^\circ K)) \end{aligned}$$

Interpolation of enthalpy.—The value of H_T° for any temperature T is given by

$$H_T^\circ = (H_T^\circ)_1 + \bar{C}_p^\circ \delta T \quad (58)$$

where $(H_T^\circ)_1$ is the value listed at T_1 and where

$$\bar{C}_p^\circ = \frac{(C_p^\circ)_1 + C_p^\circ}{2}$$

For example, for H_2O at $1573.4^\circ K$

$$\bar{C}_p^\circ = \frac{11.134 + 11.287}{2} = 11.211 \text{ (cal/(mole) } (^\circ K))$$

$$H_T^\circ = 25,202.3 + 11.211 \times 73.4 = 26,025.2 \text{ (cal/mole)}$$

By linear interpolation,

$$H_T^\circ = 26,027.2 \text{ (cal/mole)}$$

Interpolation of entropy.—Self-consistent values of entropy may be obtained with the aid of equation (49), which may be approximated by

$$\delta S_T^\circ = \bar{C}_p^\circ \delta \ln T \quad (60)$$

from which S_T° may be written

$$S_T^\circ = (S_T^\circ)_1 + \bar{C}_p^\circ \delta \ln T \quad (61)$$

where $(S_T^\circ)_1$ is the value listed at T_1 .

Equation (61) yields self-consistent values to within 0.0001 (cal/(mole) $(^\circ K)$) for all substances tabulated at temperatures above $1600^\circ K$ and for all substances except Al_2O_3 (s and g), BF_3 , B_2O_3 (liq and g), CO_2 , and H_2O for temperatures from 1000° to $1600^\circ K$. For these substances, the error due to use of equation (61) does not exceed 0.0003 cal/(mole) $(^\circ K)$, but equation (49) may be used if greater self-consistency is desired.

For example, for H_2O at $1573.4^\circ K$,

$$\begin{aligned} S_T^\circ &= 59.8687 + 11.211 (\ln 1573.4 - \ln 1500) \\ &= 60.4043 \text{ (cal/(mole) } (^\circ K)) \end{aligned}$$

By linear interpolation,

$$S_T^\circ = 60.4010 \text{ (cal/(mole) } (^\circ K))$$

Interpolation of $-\Delta H^\circ/RT$ and $\log K$.—The values of $-\Delta H^\circ/RT$ and $\log K$ for any temperature T are given by

$$\frac{-\Delta H^\circ}{RT} = \left(\frac{-\Delta H^\circ}{RT} \right)_1 - \frac{\delta T}{100} \left(\frac{a}{T} + b \right) \quad (62)$$

and

$$\log K = (\log K)_1 - \frac{\delta T}{100} \left(\frac{c}{T} + d \right) \quad (63)$$

where $(-\Delta H^\circ/RT)_1$ and $(\log K)_1$ are the values corresponding to T_1 and where a , b , c , and d are interpolation coefficients corresponding to T_1 .

For example, for H_2O at $1573.4^\circ K$,

$$\frac{-\Delta H^\circ}{RT} = 76.3615 - \frac{73.4}{100} \left(\frac{7366}{1573.4} + 0.05315 \right) = 72.8862$$

By linear interpolation,

$$\frac{-\Delta H^\circ}{RT} = 72.9433$$

and for $\log K$,

$$\log K = 20.5727 - \frac{73.4}{100} \left(\frac{3276}{1573.4} + 0.02690 \right) = 19.0247$$

By linear interpolation,

$$\log K = 19.0501$$

SOURCES OF DATA

A summary of the heat of formation and spectroscopic constants used in computing the tables and the references from which these data were taken are given in table III together with a summary of the source and the treatment of specific-heat, enthalpy, and entropy data. Additional discussion for a few substances follows.

Al₂O₃(s, liq, g).—The properties of Al₂O₃ in the solid, liquid, and gaseous phases were approximated by starting with data at 298.16° K for the solid phase and computing the properties of each phase from specific-heat data and enthalpy changes associated with phase changes. The specific heat for the solid was computed from a formula for C_p° given in reference 27. The value of $S_{298.16}^\circ$ for the solid was taken from selected values of National Bureau of Standards (issued undated but prior to June 30, 1948). The values of enthalpy and entropy up to 1000° K were then found by integration of the C_p° formula given in reference 27. The heat of fusion ($\Delta H_{\text{fusion}}^\circ = 6000$ cal/mole at 2320° K) was taken from reference 25. The C_p° values for Al₂O₃ (liq) above 2320° K were calculated from a formula based upon data given in reference 25.

Inasmuch as data on gaseous Al₂O₃ are unavailable in the literature, it was assumed that C_p° values for Al₂O₃(g) are the same as those for B₂O₃(g) given in reference 29. The heat of vaporization ($\Delta H_{\text{vaporization}}^\circ = 115.7$ kcal/mole at boiling point of $2980 \pm 60^\circ$ C) was taken from reference 40. The uncertainty in the values given for entropy and enthalpy is estimated to be ± 10 percent.

BF.—Since publication of Technical Note 2161, new thermodynamic data have been computed for BF based upon a $\sum$ ground state and a dissociation energy of 4.3 electron volts as quoted by reference 30.

BF₃.—The thermodynamic functions of BF₃ were computed by the rigid-rotator-harmonic-oscillator approximation with the following spectroscopic data given in references 31 and 41:

Vibrational frequencies			Moment of inertia (g)(cm ²)
	B ¹¹ F ₃ (cm ⁻¹)	B ¹⁰ F ₃ (cm ⁻¹)	
ν_1	888	888	$I_1 = 157.7 \times 10^{-40}$
ν_2	691.3	719.5	$I_2 = 78.84 \times 10^{-40}$
$\nu_2(2)$	1445.9	1497	$I_3 = 78.84 \times 10^{-40}$
$\nu_4(2)$	480.4	482.0	
Relative abundance (percent)	81.17	18.83	

B₂O₃(g).—The value for the heat required to convert solid B₂O₃ at 0° K to gaseous B₂O₃ at 1500° K listed in reference 29 as 106.065 (kcal/mole) was used to compute the value of (H_{1500}°)_{B₂O₃(g)}. The values of (S_{1500}°)_{B₂O₃(g)} and ($H_{1500}^\circ - H_0^\circ$)_{B₂O₃(g)} were taken from reference 29. The remaining values of enthalpy and entropy were computed by integration of the specific-heat data.

Cl₂ and HCl.—The C_p° data for Cl₂ and HCl from 1000° to 6000° K were taken from unpublished data obtained at the Jet Propulsion Laboratory of the California Institute of Technology.

ClF, F, and F⁻.—Recent spectroscopic and thermochemical measurements on compounds of fluorine (reference 42)

have indicated that the values of the heat of formation of ClF, F, and F⁻ are considerably less than the values given in reference 18. In a communication in May 1949, Dr. F. D. Rossini of the National Bureau of Standards listed their best current estimate for the heat of formation of ClF and F as -13.2 and 17.8 (kcal/mole), respectively. In accordance with these new values, the value of the heat of formation of F⁻ has been recalculated from data in reference 18.

HF.—The values of C_p° , $H_T^\circ - H_0^\circ$, and S_T° for HF at 298.16° K, 600° K, and every 1000° from 1000° to 6000° K were computed from spectroscopic data given in reference 36 using the accurate summation process. Intermediate values of C_p° were interpolated. Subsequent to the completion of computations for this substance, new spectroscopic data were made available by Dr. A. H. Nielsen of the University of Tennessee. Values of C_p° , $H_T^\circ - H_0^\circ$, and S_T° at 5000° K computed with these data differ from values herein by 1 percent for C_p° , 0.2 percent for $H_T^\circ - H_0^\circ$, and 0.03 percent for S_T° .

e⁻, F⁻, and Li⁺.—The use of metals with low ionization potentials introduces the possibility of the formation of appreciable quantities of ionized products. Because the partial pressure of ions is expected to be small, the zero-pressure properties of electron gas e⁻ have been tabulated from reference 38. The properties of F⁻ have been computed on the assumption that only the ground electronic state is stable. (See reference 43, p. 218.) The contributions of all energy levels above the ground level to the thermodynamic functions of Li⁺ are negligible. The value of C_p° tabulated for all these substances is $\frac{5}{2} R$.

Li.—In computing the thermodynamic functions of Li, the summation was carried over the first five energy levels.

LiF.—Spectroscopic data for LiF gas were not found in the literature. A vibrational frequency for the ground state of 1343 (cm⁻¹) and a moment of inertia $I_0 = 15.415 \times 10^{-40}$ (gm)(cm²) were graphically estimated from a plot of force constants against difference in atomic number of the two elements composing the substances NaH, C₂, and BeO, each substance of which is isoelectronic with LiF. It is expected that the anharmonicity constant for LiF is sufficiently large to increase materially the computed value of the specific heat. The uncertainty in the value of the enthalpy and the entropy is estimated to be ± 10 percent at 5000° K.

LiH.—Spectroscopic data for Li⁷H¹ given in reference 26 were modified for the normal isotopic mixture LiH using relative abundance percentages of isotopes and atomic weights given in reference 44 (pp. 163 and 188, respectively). The value obtained for the vibrational frequency of the ground state of LiH is 1360.37 (cm⁻¹) and for the moment of inertia is 3.77246×10^{-40} (gm)(cm²).

H₂O(s), N₂(s), and O₂(s).—The heat required to heat solid H₂O, N₂, and O₂ from 0° to 298.16° K in the natural state was taken from reference 37.

New data.—Subsequent to the completion of the computation for H₂, new values for C_p° , internal energy $E^\circ - E_0^\circ$, and S_T° were published in reference 45 and differ from the values in this report at 5000° K by 0.5 percent for C_p° , 0.1 percent for $H_T^\circ - H_0^\circ$, and 0.02 percent for S_T° .

TABLES OF THERMODYNAMIC PROPERTIES

The values of the functions of the 42 substances are given in tables IV to XLV at 298.16° K and every 100° from 300° to 6000° K together with interpolation coefficients for $-\Delta H^\circ/RT$ and $\log K$ at every 100° from 1000° to 6000° K.

LEWIS FLIGHT PROPULSION LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
CLEVELAND, OHIO, January 26, 1950.

APPENDIX—SYMBOLS

The following symbols are used in this report:

A	number of formula weights of reactants
$a, b, \dots$	summation of each atomic type over products of reaction per equivalent formula; with subscript, number of atoms of each element within chemical formula; in thermodynamic tables, interpolation coefficients
C_p°	molar specific heat at constant pressure and standard conditions (cal/(mole) (°K))
C', C''	specific heat coefficient for matrix
C_v°	molar specific heat at constant volume and standard conditions
D	operator $\left(\frac{\partial \log}{\partial \log T}\right)_i$
E_T°	molar internal energy at standard conditions
e	internal energy per equivalent formula
F_T°	molar free energy at standard conditions
g	gas phase of substance
H_0°	chemical energy at 0° K and standard conditions (kcal/mole)
H_T°	sum of sensible enthalpy and chemical energy at temperature T and standard conditions (kcal/mole)
$H_T^\circ - H_0^\circ$	sensible enthalpy at temperature T and standard conditions (kcal/mole)
$\frac{\Delta H^\circ}{RT}$	enthalpy change due to formation of substance from its elements in atomic gas state divided by RT
ΔH_f°	enthalpy change due to formation of substance from its elements in standard state (kcal/mole)
h	enthalpy per equivalent formula
h', h''	enthalpy coefficient for matrix
h^*	sum of heat and kinetic energies per equivalent formula
hc/k	ratio of Planck's constant times velocity of light to Boltzmann's constant, 1.43847 (cm) (°K)
I	moment of inertia (gm) (cm ²)
J	dimensional constant
K	equilibrium constant
liq	liquid phase of substance
M	Mach number
M_r	molecular weight of equivalent formula
m	mass flow per second
N	number of products of reaction

n	number of moles
P	total pressure
p	partial pressure
Q	any function
q, r	any variables; with subscript, matrix symbol for $\frac{\Delta H}{RT}$
R	universal gas constant, 1.98718 (cal/(mole) (°K))
S_T°	molar entropy at standard conditions (cal/(mole) (°K))
s	entropy per equivalent formula; in thermodynamic tables, solid phase of substance
s'	entropy coefficient for matrix
T	temperature (°K)
t	throat area
U_m, U_k	unit matrix
u	velocity of sound
V	volume
v	velocity of flow
$Z, Y, \dots$	elements within representative chemical formula
x	correction variables
$\alpha_1, \alpha_2, \dots$	submatrices
Δ	increment
δ	increment due to a temperature change; with subscript, error parameter
ϵ	total-error parameter
$\nu_1, \nu_2, \nu_3(2)$	spectroscopic constants
$\nu_4(2)$	
$\omega_e, \omega_e x_e$	spectroscopic constants
ρ	density
Subscripts:	
$a, b, \dots$	number of atoms within chemical formula
f	fuel
g	oxidant
l	any point in nozzle
m	number of types of gaseous molecule
o	initial given condition
P	constant pressure
s	constant entropy
T	temperature (°K)
$\dots Y, Z$	product index numbers (i) that designate atomic gases
$1, 2, \dots, i, \dots N$	product index number

REFERENCES

1. Brinkley, Stuart R., Jr.: Calculation of the Equilibrium Composition of Systems of Many Constituents. *Jour. Chem. Phys.*, vol. 15, no. 2, Feb. 1947, pp. 107-110.
2. Krieger, F. J., and White, W. B.: A Simplified Method for Computing the Equilibrium Composition of Gaseous Systems. *Jour. Chem. Phys.*, vol. 16, no. 4, April 1948, pp. 358-360.
3. Huff, Vearl N., and Calvert, Clyde S.: Charts for the Computation of Equilibrium Composition of Chemical Reactions in the Carbon-Hydrogen-Oxygen-Nitrogen System at Temperatures from 2000° to 5000° K. NACA TN 1653, 1948.
4. Winternitz, Paul F.: A Method for Calculating Simultaneous, Homogeneous Gas Equilibria and Flame Temperatures. Third Symposium on Combustion and Flame and Explosion Phenomena. The Williams & Wilkins Co. (Baltimore), 1949, pp. 623-633.

5. Scarborough, James B.: Numerical Mathematical Analysis. Johns Hopkins Press (Baltimore), 1930, pp. 187-190.
6. Crout, Prescott D.: A Short Method for Evaluating Determinants and Solving Systems of Linear Equations with Real or Complex Coefficients. AIEE Trans. (Suppl.), vol. 60, 1941, pp. 1235-1241.
7. Pipes, Louis A.: Applied Mathematics for Engineers and Physicists. McGraw-Hill Book Co., Inc. (New York and London), 1946, pp. 78-79.
8. Woolley, Harold W.: Thermodynamic Functions for Molecular Oxygen in the Ideal Gas State. Nat. Bur. Standards Jour. Res., vol. 40, no. 2, Feb. 1948, pp. 163-168.
9. DuMond, Jesse W. M., and Cohen, E. Richard: Our Knowledge of the Atomic Constants F , N , m , and h in 1947, and of Other Constants Derivable Therefrom. Rev. Modern Phys., vol. 20, no. 1, Jan. 1948, pp. 82-108.
10. Anon.: Tables of Selected Values of Chemical Thermodynamic Properties. Nat. Bur. Standards, Dec. 31, 1947.
11. Anon.: Tables of Selected Values of Chemical Thermodynamic Properties. Nat. Bur. Standards, June 30, 1948.
12. Rossini, Frederick D., Pitzer, Kenneth S., Taylor, William J., Ebert, Joan P., Kilpatrick, John E., Beckett, Charles W., Williams, Mary G., and Werner, Helene G.: Selected Values of Properties of Hydrocarbons. Circular C461, Nat. Bur. Standards, Nov. 1947.
13. Woolley, Harold W.: Dry Air (Ideal Gas State). Table 2.10 of the NBS-NACA Tables of Thermal Properties of Gases. Nat. Bur. Standards, July 1949.
14. Anon.: Handbook of Chemistry and Physics, Charles D. Hodgman, ed. Chem. Rubber Pub. Co. (Cleveland), 29th ed., 1945.
15. Hulme, R. E.: Thermodynamic Properties of Superheated Chlorine. Chem. Eng., vol. 56, no. 11, Nov. 1949, pp. 118-119, 127.
16. Anon.: Tables of Selected Values of Chemical Thermodynamic Properties. Nat. Bur. Standards, Sept. 30, 1949.
17. Beckett, C. W., Clarke, J. T., and Johnston, H. L.: Tentative Thermal Functions for Diborane. Joint Rep. 7 on Proj. 319 and Rep. 4 on Proj. 309, Ohio State Univ. Res. Foundation, Aug. 1, 1948. (USAF Contract W33-038-ac-17721 and ONR Contract N6onr-225, T. O. IX, NR 058 061).
18. Anon.: Tables of Selected Values of Chemical Thermodynamic Properties. Nat. Bur. Standards, June 30, 1947.
19. Woolley, Harold W., Scott, Russell B., and Brickwedde, F. G.: Compilation of Thermal Properties of Hydrogen in Its Various Isotopic and Ortho-Para Modifications. Nat. Bur. Standards Jour. Res., vol. 41, no. 5, Nov. 1948, pp. 379-475.
20. Kelley, K. K.: Contributions to the Data on Theoretical Metallurgy. III. The Free Energies of Vaporization and Vapor Pressures of Inorganic Substances. Bull. 383, Bur. Mines, 1935.
21. Kelley, K. K.: Contributions to the Data on Theoretical Metallurgy. V. Heats of Fusion of Inorganic Substances. Bull. 393, Bur. Mines, 1936.
22. Davis, William D., Mason, Leo S., and Stegeman, Gebhard: Thermal Properties of Some Hydrides. Tech. Rep., Dept. Chem., Univ. Pittsburgh, Dec. 1, 1948. (Contract No. N60RI-43, T. O. No. 1.)
23. Anon.: Tables of Selected Values of Chemical Thermodynamic Properties. Nat. Bur. Standards, March 31, 1948.
24. Bacher, Robert F., and Goudsmit, Samuel: Atomic Energy States. McGraw-Hill Book Co., Inc., 1932.
25. Terebesi, L.: Die Thermodynamischen Funktionen von Aluminium. α -Aluminiumoxyd, β -Graphit, Sauerstoff und Kohlenmonoxyd, Helvetica Chim. Acta, vol. 17, fasc. IV, 1934, pp. 804-819.
26. Herzberg, Gerhard: Molecular Spectra and Molecular Structure. Vol. I. Prentice-Hall, Inc., 1939.
27. Shomate, C. Howard, and Naylor, B. F.: High-Temperature Heat Contents of Aluminum Oxide, Aluminum Sulfate, Potassium Sulfate, Ammonium Sulfate and Ammonium Bisulfate. Jour. Am. Chem. Soc., vol. 67, no. 1, Jan. 1945, pp. 72-75.
28. Anon.: Tables of Selected Values of Chemical Thermodynamic Properties. Nat. Bur. Standards, Sept. 30, 1948.
29. Wacker, Paul F., Woolley, Harold W., and Fair, Myron F.: Thermodynamic Properties and Gaseous Equilibria of Boron, Oxygen and the Oxides of Boron. Tech. Rep. to Bur. Aero., Navy Dept. submitted by Heat and Power Div., Nat. Bur. Standards, Jan. 25, 1945.
30. Herzberg, Gerhard: Molecular Spectra and Molecular Structure. Vol. I. D. Van Nostrand Company, Inc., 1950.
31. Spencer, Hugh M.: Thermodynamic Properties of Gaseous Boron Trifluoride, Boron Trichloride, and Boron Tribromide. Jour. Chem. Phys., vol. 14, no. 12, Dec. 1946, pp. 729-732.
32. Anon.: Tables of Selected Values of Chemical Thermodynamic Properties. Nat. Bur. Standards, March 31, 1947.
33. Anon.: Tables of Selected Values of Chemical Thermodynamic Properties. Nat. Bur. Standards, June 30, 1949.
34. Ward, J. J., and Hussey, M. A.: Estimated Thermodynamic Functions of Free Radicals in Combustion Gases. Third Symposium on Combustion and Flame and Explosion Phenomena, The Williams & Wilkins Co. (Baltimore), 1949, pp. 599-610.
35. Moore, Charlotte E.: Atomic Energy Levels, vol. 1, sec. 1. Circular 467, Nat. Bur. Standards, April 15, 1948.
36. Murphy, George M., and Vance, John E.: Thermodynamic Properties of Hydrogen Fluoride and Fluorine from Spectroscopic Data. Jour. Chem. Phys., vol. 7, no. 9, Sept. 1939, pp. 806-810.
37. Anon.: International Critical Tables. Vol. 5. McGraw-Hill Book Co., Inc., 1929, p. 176.
38. Gordon, A. R.: The Free Energy of Electron Gas. Jour. Chem. Phys., vol. 4, no. 10, Oct. 10, 1936, pp. 678-679.
39. Mulliken, Robert S.: The Low Electronic States of Simple Heteropolar Diatomic Molecules. Phys. Rev., vol. 50, no. 11, 2d ser., Dec. 1, 1936, pp. 1028-1040.
40. Ruff, Otto, und Kenschak, Martin: Arbeiten im Gebiet hoher Temperaturen. Zeitschr. f. Elektrochem., Bd. 32, Nr. 11, Nov. 1926, S. 515-525.
41. Herzberg, Gerhard: Infrared and Raman Spectra of Polyatomic Molecules. D. Van Nostrand Co., Inc., 1945, p. 299.
42. Schmitz, Heinz, und Schumacher, Joachim: Das Bandenspektrum und die Dissoziationsenergie des ClF. Zeitschr. f. Naturforschung, Bd. 2a, Heft 6, Juni 1947, S. 359-363.
43. Herzberg, Gerhard: Atomic Spectra and Atomic Structure. Prentice-Hall, Inc., 1937. (Reprinted, Dover Pub. (New York), 1944, p. 218.)
44. Stranathan, J. D.: The "Particles" of Modern Physics. The Blakiston Co. (Philadelphia), 1942.
45. Johnston, H. L., Savedoff, L. G., and Belzer, J.: The Thermodynamic Properties of Hydrogen Gas from near Zero Degrees to 6000° K. Tech. Rep. No. 2, Proj. RF-316, Ohio State Univ. Res. Foundation, May 1, 1949. (ONR Proj. No. NR 058 005, Navy Contract No. N6onr-225, T. O. XII.)

TABLE I—VALUES OF CONSTANTS FOR REACTION OF DIBORANE WITH OXYGEN BIFLUORIDE ($B_2H_6 + 5F_2O$)

Product	Fixed					Determined at estimated temperature of 4000° K				
	i	b_i	a_i	c_i	d_i	$(H_f^\circ)_i$ kcal/mole	$(\Delta H_f^\circ/RT)_i$	$(S_f^\circ)_i$	$(C_p^\circ)_i$	$\log K_i$
Equivalent formula	0	2	6	10	5					
BF ₃	1	1	0	3	0	72.172	-62.0753	105.951	19.738	5.6953
B ₂ O ₃	2	2	0	0	3	233.435	-80.5932	116.760	25.660	5.1094
BF	3	1	0	1	0	262.961	-17.2884	73.904	8.905	1.6342
BH	4	1	1	0	0	356.994	-8.3004	61.412	8.826	-2.6110
BO	5	1	0	0	1	252.739	-18.1834	69.620	9.065	1.0327
B ₂	6	2	0	0	0	572.053	-7.9892	70.580	8.923	-2.7625
H ₂	7	0	2	0	0	99.593	-13.9385	51.054	9.151	-0.4061
H ₂ O	8	0	2	0	1	57.706	-29.2092	72.458	13.300	-0.3470
OH	9	0	1	0	1	76.560	-13.6031	63.989	9.165	-0.1668
HF	10	0	1	1	0	32.016	-19.6736	61.054	9.045	1.8944
O ₂	11	0	0	0	2	37.310	-15.3125	70.783	9.932	-0.3804
F ₂	12	0	0	2	0	96.012	-8.7047	70.813	9.451	-3.1373
H	13	0	1	0	0	105.192	-----	40.306	4.968	-----
B	14	1	0	0	0	317.778	-----	49.549	4.968	-----
F	15	0	0	1	0	82.601	-----	51.230	4.974	-----
O	16	0	0	0	1	79.493	-----	51.479	5.091	-----

TABLE II—ENTHALPY H_f° ASSIGNED TO SEVERAL SUBSTANCES

Substance	Formula	Phase	Temperature (°K)	Heat of formation, ΔH_f° (kcal/mole)	Enthalpy assigned, H_f° (kcal/mole)	Reference
Acetylene	C ₂ H ₂	Gas	298.16	54.194	307.993	11
		Liquid	191.7	-----	302.75	12
Air ^b		Gas	298.16	0	3.8208	13
Aluminum	Al	Crystal	298.16	0	234.6951	10
Ammonia	NH ₃	Gas	298.16	-11.04	95.01	-----
		Liquid	239.76	-----	88.91	10, 14
Aniline	C ₆ H ₅ NH ₂	Liquid	298.16	8.18	806.19	14
n-Butane	C ₄ H ₁₀	Gas	298.16	-29.812	686.108	12
		Liquid	272.66	-----	680.126	12, 14
Chlorine	Cl ₂	Gas	298.16	0	10.000	-----
		Liquid	239.11	-----	4.61	15
Chlorine trifluoride	ClF ₃	Liquid	298.16	-32.1	67.5	-----
Diborane	B ₂ H ₆	Gas	298.16	7.5	562.6	16
		Liquid	180.63	-----	557.8	16, 17
Ethylene-diamine	C ₂ H ₈ N ₂	Liquid	298.16	-6.36	459.53	11
Fluorine	F ₂	Gas	298.16	0	63.0699	-----
		Liquid	85.24	-----	60.04	18
Fluorine oxide	F ₂ O	Gas	298.16	5.5	70.6	18
		Liquid	128.3	-----	66.4	18
Gasoline ^f	A.N.F-58	Liquid	298.16	-47.50	1346.60	-----
Heptane	C ₇ H ₁₆	Liquid	298.16	-53.63	1147.15	12
Hexane	C ₆ H ₁₄	Liquid	298.16	-47.52	991.64	12
Hydrazine	N ₂ H ₄	Liquid	298.16	12.05	154.70	10
Hydrazine hydrate	N ₂ H ₄ ·H ₂ O	Liquid	298.16	-57.95	156.20	10
Hydrogen	H ₂	Gas	298.16	0	69.4407	-----
		Liquid	20.39	-----	67.546	19
Hydrogen peroxide	H ₂ O ₂	Liquid	298.16	-44.84	28.71	18
Hydroxylamine	NH ₂ OH	Liquid	298.16	-25.5	82.6	10
Lithium	Li	Crystal	298.16	0	132.2250	-----
		Liquid	459.16	-----	134.53	20, 21
Lithium borohydride	LiBH ₄	Crystal	298.16	-44.15	400.34	22
Methane	CH ₄	Gas	298.16	-17.889	213.171	12
		Liquid	111.67	-----	209.71	12
Methanol	CH ₃ OH	Liquid	298.16	-57.036	176.080	23
Nitric acid, white fuming	HNO ₃	Liquid	298.16	-41.404	1.368	10
Nitrogen tetroxide	N ₂ O ₄	Gas	298.16	2.309	14.302	10
		Liquid	294.31	-----	5.120	10
Nitrogen trifluoride	NF ₃	Gas	298.16	-27.2	69.3	10
		Liquid	144.1	-----	64.6	10
Nitromethane	CH ₃ NO ₂	Liquid	298.16	-26.7	175.6	23
n-Octane	C ₈ H ₁₈	Liquid	298.16	-59.74	1302.66	12
Oxygen	O ₂	Gas	298.16	0	4.1109	-----
		Liquid	90.16	-----	1.0314	8, 18
Ozone	O ₃	Gas	298.16	34.0	40.2	18
		Liquid	162.65	-----	36.4	18
Pentaborane	B ₅ H ₉	Liquid	298.16	7.8	1187.2	16
Tetranitromethane	C(NO ₂) ₄	Liquid	298.16	5	121	23
Water	H ₂ O	Gas	298.16	-57.7979	13.6988	18
		Liquid	298.16	-----	3.1788	18

^a For pressure of 900 mm Hg.^b For composition consisting of following mole fractions: N₂, 0.780881; O₂, 0.209495; A, 0.009324; CO₂, 0.006300.^c Energy of gas imperfections included.^d Computed from heat of combustion.^e Estimate based upon unpublished value of -26.4 kcal/mole at 200° C obtained by Dr. Swinehart of Harshaw Chemical Company.^f Based upon representative sample having molecular weight of 122 and hydrogen-carbon atom ratio of 1.942.

TABLE III—THERMOCHEMICAL AND SPECTROSCOPIC DATA AND REFERENCES FOR EACH SUBSTANCE

Substance	Phase	Heat of formation, ΔH_f° (kcal/mole)		Spectroscopic constants		Reference			
		0° K	298.16° K	$\omega_e - 2\omega_e x_e$ (cm ⁻¹)	Moment of inertia (gm)(cm ²) $\times 10^{-40}$	Heat of formation	Spectroscopic constants	Specific heat, enthalpy, and entropy (0° to 1000° K)	Specific heat (1000° to 6000° K)
A	Gas	0	0				24	(*)	(*)
Al	Gas		67.50				24	(*)	(*)
Al	Crystal	0	0			(b)		25	(*)
Al	Liquid							25	c d 25
AlF ₃	Crystal		-329.3			(b)			
AlO	Gas		37.8	963	43.8223	(b)	26	(*)	(f)
Al ₂ O ₃	Gas							27	(f)
Al ₂ O ₃	Crystal, α		-399.09			(b)		27	25
Al ₂ O ₃	Liquid								(*)
B	Gas		97.2			28	29		
B	Crystal	0	0						(*)
B ₂	Gas		124			28		29	c d 29
BF	Gas		-16.9308	1304.84	19.9465	b 30	f 30	(*)	(*)
BF ₃	Gas		-265.2			28	f 31	(*)	(*)
BH	Gas		73.8	2268	2.36939	28	26	(*)	(*)
BO	Gas		-5.2			28		29	c d 29
B ₂ O ₃	Gas					(f)			c d 29
B ₂ O ₃	Crystal		-302.0			28		29	
B ₂ O ₃	Liquid							29	c d 29
C	Gas		171.698			23		11	c d 11
C	Graphite	0	0						c d 11
CO	Gas		-26.4157			23		32	c d 11
CO ₂	Gas		-94.0518			23		32	c d 11
Cl	Gas	28.61				33		11	c d 11
Cl ₂	Gas	0						11	(f)
ClF	Gas		-13.2	773	54.35	(f)	34	(*)	(*)
F	Gas	17.8				(f)	35	(*)	(*)
F ₂	Gas	0	0	856	33.3564		36	(*)	(*)
H	Gas	51.620				18		32	32
H ₂	Gas	0	0					32	c d 19
HCl	Gas		-22.063			11		11	(f)
HF	Gas		-64.2			18	36	(f)	(f)
H ₂ O	Gas		-57.7974			18		11	c d 11
H ₂ O	Crystal	-68.4350				b 37			
H ₂ O	Liquid		-68.3174			18			
e ⁻	Gas	0	0					f 38	38
F ⁻	Gas	-78.5				b 18	(f)	(*)	(*)
Li ⁺	Gas		161.4664			b 24	24	(*)	(*)
Li	Gas		36.150			i 20	24	(*)	(*)
Li	Crystal	0	0						
LiF	Gas		-83.766	1343	15.415	i 20	(f)	(*)	(*)
LiH	Gas		25.4564	1360.37	3.77246	b i 39	f 26	(*)	(*)
N	Gas	85.120				10		32	c d 11
N ₂	Gas	0	0					32	c d 11
N ₂	Crystal	-1.6992				b 37			
NO	Gas	21.477				10		18	c d 11
O	Gas	58.586				32		32	c d 11
O ₂	Gas	0	0					8	c d 8
O ₂	Crystal	-2.0362				b 37			
OH	Gas	10.0				18		32	c d 11

* Specific heat, enthalpy, and entropy at 298.16° and every 100° from 300° to 1000° and specific heat for each 100° from 1000° to 6000° K were computed from spectroscopic data given in reference listed by accurate summation method.

^b Data from selected values of National Bureau of Standards issued undated but prior to June 30, 1948.

^c Graphically smoothed.

^d Extrapolated.

^e Specific heat, enthalpy, and entropy at 298.16° and every 100° from 300° to 1000° and specific heat for each 100° from 1000° to 6000° K were computed from spectroscopic data by rigid rotator-harmonic oscillator approximation.

^f See discussion in text.

^g Interpolated.

^h Computed with aid of data in reference listed.

ⁱ Unpublished data from Battelle Memorial Institute also used.

TABLE IV—THERMODYNAMIC PROPERTIES OF A (ARGON GAS)

[Atomic weight, 39.944]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)
0	-----	0	0	-----
298.16	4.9680	1.4812	1.4812	36.9830
300	4.9680	1.4904	1.4904	37.0135
400	4.9680	1.9872	1.9872	38.4427
500	4.9680	2.4840	2.4840	39.5513
600	4.9680	2.9808	2.9808	40.4570
700	4.9680	3.4776	3.4776	41.2229
800	4.9680	3.9744	3.9744	41.8862
900	4.9680	4.4712	4.4712	42.4714
1,000	4.9680	4.9680	4.9680	42.9948
1,100	4.9680	5.4648	5.4648	43.4683
1,200	4.9680	5.9616	5.9616	43.9006
1,300	4.9680	6.4584	6.4584	44.2982
1,400	4.9680	6.9552	6.9552	44.6664
1,500	4.9680	7.4520	7.4520	45.0091
1,600	4.9680	7.9488	7.9488	45.3298
1,700	4.9680	8.4456	8.4456	45.6310
1,800	4.9680	8.9424	8.9424	45.9149
1,900	4.9680	9.4392	9.4392	46.1835
2,000	4.9680	9.9360	9.9360	46.4383
2,100	4.9680	10.4328	10.4328	46.6807
2,200	4.9680	10.9296	10.9296	46.9118
2,300	4.9680	11.4264	11.4264	47.1327
2,400	4.9680	11.9232	11.9232	47.3441
2,500	4.9680	12.4200	12.4200	47.5469
2,600	4.9680	12.9168	12.9168	47.7418
2,700	4.9680	13.4136	13.4136	47.9293
2,800	4.9680	13.9104	13.9104	48.1099
2,900	4.9680	14.4072	14.4072	48.2843
3,000	4.9680	14.9040	14.9040	48.4527
3,100	4.9680	15.4008	15.4008	48.6156
3,200	4.9680	15.8976	15.8976	48.7733
3,300	4.9680	16.3944	16.3944	48.9262
3,400	4.9680	16.8912	16.8912	49.0745
3,500	4.9680	17.3880	17.3880	49.2185
3,600	4.9680	17.8848	17.8848	49.3584
3,700	4.9680	18.3816	18.3816	49.4946
3,800	4.9680	18.8784	18.8784	49.6270
3,900	4.9680	19.3752	19.3752	49.7561
4,000	4.9680	19.8720	19.8720	49.8819
4,100	4.9680	20.3688	20.3688	50.0045
4,200	4.9680	20.8656	20.8656	50.1243
4,300	4.9680	21.3624	21.3624	50.2412
4,400	4.9680	21.8592	21.8592	50.3554
4,500	4.9680	22.3560	22.3560	50.4670
4,600	4.9680	22.8528	22.8528	50.5763
4,700	4.9680	23.3496	23.3496	50.6831
4,800	4.9680	23.8464	23.8464	50.7876
4,900	4.9680	24.3432	24.3432	50.8901
5,000	4.9680	24.8400	24.8400	50.9905
5,100	4.9680	25.3368	25.3368	51.0888
5,200	4.9680	25.8336	25.8336	51.1853
5,300	4.9680	26.3304	26.3304	51.2799
5,400	4.9680	26.8272	26.8272	51.3728
5,500	4.9680	27.3240	27.3240	51.4639
5,600	4.9680	27.8208	27.8208	51.5535
5,700	4.9680	28.3176	28.3176	51.6414
5,800	4.9680	28.8144	28.8144	51.7278
5,900	4.9680	29.3112	29.3112	51.8127
6,000	4.9680	29.8080	29.8080	51.8960

TABLE V—THERMODYNAMIC PROPERTIES OF Al (GAS)

[Atomic weight, 26.97]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)
0	-----	0	300.5416	-----
298.16	5.1122	1.6535	302.1951	39.3027
300	5.1104	1.6629	302.2045	39.3342
400	5.0469	2.1703	302.7119	40.7943
500	5.0178	2.6734	303.2150	41.9170
600	5.0022	3.1743	303.7159	42.8304
700	4.9929	3.6740	304.2156	43.6007
800	4.9869	4.1730	304.7146	44.2670
900	4.9829	4.6715	305.2131	44.8541
1000	4.9800	5.1696	305.7112	45.3790
1100	4.9778	5.6675	306.2091	45.8535
1200	4.9762	6.1652	306.7068	46.2865
1300	4.9750	6.6628	307.2044	46.6848
1400	4.9740	7.1602	307.7018	47.0534
1500	4.9732	7.6576	308.1992	47.3966
1600	4.9726	8.1549	308.6965	47.7175
1700	4.9720	8.6521	309.1937	48.0190
1800	4.9716	9.1493	309.6909	48.3032
1900	4.9712	9.6464	310.1880	48.5719
2000	4.9709	10.1435	310.6851	48.8269
2100	4.9706	10.6406	311.1822	49.0694
2200	4.9704	11.1376	311.6792	49.3007
2300	4.9702	11.6347	312.1763	49.5216
2400	4.9700	12.1317	312.6733	49.7331
2500	4.9699	12.6287	313.1703	49.9360
2600	4.9698	13.1257	313.6673	50.1309
2700	4.9698	13.6226	314.1642	50.3185
2800	4.9698	14.1196	314.6612	50.4992
2900	4.9699	14.6166	315.1582	50.6736
3000	4.9701	15.1136	315.6552	50.8421
3100	4.9704	15.6106	316.1522	51.0051
3200	4.9709	16.1077	316.6493	51.1629
3300	4.9716	16.6048	317.1464	51.3159
3400	4.9724	17.1020	317.6436	51.4643
3500	4.9736	17.5993	318.1409	51.6085
3600	4.9750	18.0968	318.6384	51.7486
3700	4.9768	18.5943	319.1359	51.8849
3800	4.9791	19.0921	319.6337	52.0177
3900	4.9818	19.5902	320.1318	52.1470
4000	4.9851	20.0885	320.6301	52.2732
4100	4.9890	20.5872	321.1288	52.3964
4200	4.9936	21.0864	321.6280	52.5166
4300	4.9990	21.5860	322.1276	52.6342
4400	5.0052	22.0862	322.6278	52.7492
4500	5.0122	22.5871	323.1287	52.8618
4600	5.0203	23.0887	323.6303	52.9720
4700	5.0294	23.5912	324.1328	53.0801
4800	5.0396	24.0946	324.6362	53.1861
4900	5.0510	24.5992	325.1408	53.2901
5000	5.0637	25.1049	325.6465	53.3923
5100	5.0776	25.6120	326.1536	53.4927
5200	5.0928	26.1205	326.6621	53.5914
5300	5.1094	26.6306	327.1722	53.6886
5400	5.1275	27.1424	327.6840	53.7843
5500	5.1470	27.6562	328.1978	53.8785
5600	5.1680	28.1719	328.7135	53.9714
5700	5.1905	28.6898	329.2314	54.0631
5800	5.2145	29.2101	329.7517	54.1536
5900	5.2401	29.7328	330.2744	54.2430
6000	5.2672	30.2582	330.7998	54.3313

TABLE VI.—THERMODYNAMIC PROPERTIES OF Al
(CRYSTAL)

[Atomic weight, 26.97]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f - H_0$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f ($\frac{\text{kcal}}{\text{mole}}$)	S_f ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\log K$
0	-----	0	233.6251	-----	-----	-----
298.16	5.80	1.0700	234.6951	6.641	113.9245	42.3386
300	5.81	1.0820	234.7071	6.682	113.2214	42.0354
400	6.24	1.6860	235.3111	8.319	84.7945	29.7284
500	6.46	2.3210	235.9461	9.844	67.7029	22.3935
600	6.66	2.9760	236.6011	11.013	56.2898	17.4927
700	6.88	3.6530	237.2781	12.070	48.1210	14.0077
800	7.15	4.3530	237.9781	13.018	41.9794	11.4020
900	7.65	5.0890	238.7141	13.864	37.1822	9.3752
930	7.90	5.3240	238.9491	14.130	35.9365	8.8566

TABLE VII.—THERMODYNAMIC PROPERTIES OF Al (LIQUID)

[Atomic weight, 26.97]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f - H_0$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f ($\frac{\text{kcal}}{\text{mole}}$)	S_f ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
930	6.75	7.8240	241.4491	16.818	34.5838	-----	-----	8.8566	-----	-----
1000	6.84	8.2990	241.9241	17.310	32.0993	3271	0.03246	7.8061	1414	-0.01995
1100	6.966	8.9893	242.6144	17.9678	29.0932	2974	0.03217	6.5406	1285	-0.01953
1200	7.092	9.6922	243.3173	18.5794	26.5827	2726	.03228	5.4893	1178	-0.01955
1300	7.218	10.4077	244.0328	19.1520	24.4535	2517	.03174	4.6027	1087	-0.01923
1400	7.344	11.1358	244.7609	19.6915	22.6239	2337	.03190	3.8455	1009	-0.01887
1500	7.470	11.8765	245.5016	20.2025	21.0340	2181	.03198	3.1917	941	-0.01832
1600	7.596	12.6298	246.2549	20.6886	19.6389	2045	0.03186	2.6219	882	-0.01812
1700	7.722	13.3957	247.0208	21.1529	18.4041	1925	.03165	2.1212	830	-0.01811
1800	7.848	14.1742	247.7993	21.5978	17.3030	1818	.03166	1.6782	784	-0.01803
1900	7.974	14.9653	248.5904	22.0255	16.3145	1722	.03190	1.2836	742	-0.01760
2000	8.100	15.7690	249.3941	22.4377	15.4216	1636	.03175	.9302	705	-0.01751
2100	8.226	16.5853	250.2104	22.8360	14.6108	1558	0.03192	0.6120	671	-0.01740
2200	8.352	17.4142	251.0393	23.2216	13.8707	1487	.03188	.3244	640	-0.01706
2300	8.478	18.1857	251.8808	23.5956	13.1923	1423	.03168	.0632	612	-0.01700
2400	8.604	19.1098	252.7349	23.9591	12.5677	1363	.03190	-.1748	586	-0.01670
2500	8.730	19.9765	253.6016	24.3129	11.9906	1309	.03174	-.3925	562	-0.01655
2600	8.856	20.8558	254.4809	24.6577	11.4554	1259	0.03160	-0.5921	540	-0.01630
2700	8.982	21.7477	255.3728	24.9943	10.9575	1212	.03164	-.7758	520	-0.01631
2800	9.108	22.6522	256.2773	25.3233	10.4930	1169	.03170	-.9452	501	-0.01616
2900	9.234	23.5693	257.1944	25.6451	10.0582	1129	.03147	-1.1018	483	-0.01600
3000	9.360	24.4990	258.1241	25.9602	9.6504	-----	-----	-1.2468	-----	-----

TABLE VIII—THERMODYNAMIC PROPERTIES OF AlO (GAS)

[Molecular weight, 42.97]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ (kcal/mole)	H_T° (kcal/mole)	S_T° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	272.4501	-----	-----	-----	-----	-----	-----	-----
298.16	7.3749	2.1005	274.5506	52.1648	149.9685	-----	-----	59.5342	-----	-----
300	7.3820	2.1140	274.5641	52.2101	148.0580	-----	-----	59.1348	-----	-----
400	7.7510	2.8711	275.3212	54.3856	112.1309	-----	-----	42.9281	-----	-----
500	8.0424	3.6615	276.1116	56.1482	89.9293	-----	-----	33.1770	-----	-----
600	8.2530	4.4769	276.9270	57.6342	75.1020	-----	-----	26.6609	-----	-----
700	8.4035	5.3101	277.7602	58.9183	64.4956	-----	-----	21.9971	-----	-----
800	8.5122	6.1562	278.6063	60.0479	56.5311	-----	-----	18.4930	-----	-----
900	8.5924	7.0116	279.4617	61.0553	50.3305	-----	-----	15.7629	-----	-----
1000	8.6529	7.8740	280.3241	61.9639	45.3658	4456	0.01379	13.5755	1955	0.01513
1100	8.6993	8.7416	281.1917	62.7908	41.3011	4054	0.01097	11.7831	1779	0.01320
1200	8.7356	9.6134	282.0635	63.5493	37.9118	3719	.00853	10.2874	1632	.01202
1300	8.7645	10.4884	282.9385	64.2496	35.0425	3435	.00693	9.0200	1508	.01056
1400	8.7878	11.3660	283.8161	64.9000	32.5820	3191	.00597	7.9323	1402	.00923
1500	8.8069	12.2457	284.6958	65.5070	30.4487	2980	.00470	6.9884	1309	.00888
1600	8.8227	13.1272	285.5773	66.0759	28.5815	2795	0.00388	6.1614	1228	0.00825
1700	8.8359	14.0101	286.4602	66.6111	26.9335	2632	.00308	5.4308	1157	.00752
1800	8.8471	14.8943	287.3444	67.1165	25.4682	2487	.00245	4.7805	1094	.00671
1900	8.8566	15.7795	288.2296	67.5951	24.1568	2357	.00190	4.1980	1037	.00630
2000	8.8647	16.6655	289.1156	68.0496	22.9764	2240	.00163	3.6732	986	.00598
2100	8.8718	17.5524	290.0025	68.4822	21.9081	2134	0.00130	3.1977	940	0.00543
2200	8.8779	18.4399	290.8900	68.8951	20.9368	2037	.00115	2.7650	898	.00517
2300	8.8833	19.3279	291.7780	69.2899	20.0500	1949	.00102	2.3694	859	.00508
2400	8.8880	20.2165	292.6666	69.6680	19.2369	1869	.00050	2.0064	824	.00480
2500	8.8922	21.1055	293.5556	70.0309	18.4888	1795	.00022	1.6720	791	.00477
2600	8.8960	21.9949	294.4450	70.3798	17.7982	1727	-0.00013	1.3630	761	0.00455
2700	8.8993	22.8847	295.3348	70.7156	17.1587	1663	-.00023	1.0766	734	.00416
2800	8.9023	23.7747	296.2248	71.0393	16.5650	1605	-.00065	.8103	708	.00406
2900	8.9050	24.6651	297.1152	71.3517	16.0122	1550	-.00087	.5621	684	.00390
3000	8.9074	25.5557	298.0058	71.6536	15.4964	1500	-.00127	.3302	662	.00375
3100	8.9096	26.4466	298.8967	71.9458	15.0138	1452	-0.00145	0.1129	640	0.00380
3200	8.9116	27.3376	299.7877	72.2287	14.5615	1407	-.00156	-.0909	621	.00362
3300	8.9134	28.2289	300.6790	72.5029	14.1367	1365	-.00177	-.2827	602	.00364
3400	8.9150	29.1203	301.5704	72.7690	13.7370	1326	-.00216	-.4634	584	.00364
3500	8.9166	30.0119	302.4620	73.0275	13.3603	1289	-.00236	-.6339	568	.00352
3600	8.9180	30.9036	303.3537	73.2787	13.0046	1254	-0.00252	-0.7952	552	0.00351
3700	8.9193	31.7955	304.2456	73.5230	12.6682	1221	-.00282	-.9479	538	.00332
3800	8.9205	32.6875	305.1376	73.7609	12.3497	1190	-.00313	-1.0928	524	.00324
3900	8.9216	33.5796	306.0297	73.9927	12.0477	1160	-.00320	-1.2304	511	.00315
4000	8.9226	34.4718	306.9219	74.2186	11.7609	1132	-.00350	-1.3613	498	.00314
4100	8.9235	35.3641	307.8142	74.4389	11.4883	1106	-0.00393	-1.4859	486	0.00319
4200	8.9244	36.2565	308.7066	74.6539	11.2289	1081	-.00420	-1.6048	475	.00303
4300	8.9252	37.1490	309.5991	74.8639	10.9817	1057	-.00443	-1.7183	464	.00295
4400	8.9260	38.0415	310.4916	75.0691	10.7459	1034	-.00468	-1.8267	454	.00291
4500	8.9267	38.9342	311.3843	75.2697	10.5208	1012	-.00490	-1.9305	444	.00288
4600	8.9274	39.8269	312.2770	75.4659	10.3057	991	-0.00515	-2.0299	434	0.00296
4700	8.9280	40.7196	313.1697	75.6579	10.1000	972	-.00560	-2.1252	425	.00286
4800	8.9286	41.6125	314.0626	75.8459	9.9031	953	-.00579	-2.2166	416	.00300
4900	8.9291	42.5054	314.9555	76.0300	9.7144	935	-.00610	-2.3045	407	.00300
5000	8.9296	43.3983	315.8484	76.2104	9.5335	918	-.00650	-2.3889	400	.00287
5100	8.9301	44.2913	316.7414	76.3873	9.3600	901	-0.00667	-2.4702	391	0.00301
5200	8.9306	45.1843	317.6344	76.5607	9.1934	886	-.00707	-2.5484	384	.00295
5300	8.9310	46.0774	318.5275	76.7308	9.0333	871	-.00740	-2.6238	377	.00289
5400	8.9314	46.9705	319.4206	76.8977	8.8794	857	-.00782	-2.6965	369	.00301
5500	8.9318	47.8637	320.3138	77.0616	8.7314	843	-.00814	-2.7666	363	.00298
5600	8.9322	48.7569	321.2070	77.2226	8.5890	830	-0.00851	-2.8344	356	0.00304
5700	8.9326	49.6501	322.1002	77.3807	8.4519	817	-.00876	-2.8999	349	.00313
5800	8.9329	50.5434	322.9935	77.5360	8.3198	805	-.00904	-2.9632	343	.00316
5900	8.9332	51.4367	323.8868	77.6887	8.1924	793	-.00947	-3.0245	337	.00323
6000	8.9335	52.3300	324.7801	77.8389	8.0697	-----	-----	-3.0839	-----	-----

TABLE IX—THERMODYNAMIC PROPERTIES OF Al_2O_3 (CRYSTAL, α)

[Molecular weight, 101.94]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_f^\circ - H_{1000}^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta - \left(\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
298.16	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
300	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
400	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
500	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
600	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
700	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
800	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
900	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
1000	23.360	0	237.0838	91.5490	286.1479	28.430	0.10445	95.1916	12.409	0.01799
1100	23.754	2.3557	239.4395	93.7929	260.1980	25.870	0.08207	83.8927	11.287	0.01187
1200	24.064	4.7466	241.8304	95.8741	238.5576	23.735	.06471	74.4750	10.351	.00779
1300	24.311	7.1654	244.2492	97.8100	220.2352	21.925	.05269	66.5049	9.558	.00516
1400	24.510	9.6064	246.6902	99.6189	204.5218	20.373	.04250	59.6726	8.878	.00293
1500	24.672	12.0655	249.1493	101.3154	190.8973	19.026	.03505	53.7510	8.288	.00170
1600	24.805	14.5394	251.6232	102.9120	178.9710	17.846	0.02935	48.5693	7.771	0.00092
1700	24.916	17.0254	254.1092	104.4191	168.4440	16.805	.02429	43.9972	7.315	.00021
1800	25.010	19.5217	256.6055	105.8459	159.0836	15.878	.02056	39.9331	6.910	-.00054
1900	25.090	22.0267	259.1105	107.2003	150.7062	15.048	.01760	36.2968	6.547	-.00100
2000	25.159	24.5392	261.6230	108.4890	143.1646	14.300	.01538	33.0243	6.220	-.00130
2100	25.221	27.0582	264.1420	109.7180	136.3397	13.623	0.01353	30.0637	5.924	-0.00133
2200	25.277	29.5831	266.6669	110.8925	130.1339	13.008	.01165	27.3723	5.655	-.00160
2300	25.329	32.1134	269.1972	112.0173	124.4666	12.447	.00965	24.9152	5.410	-.00197
2400	25.374	34.6485	271.7323	113.0962	119.2707	11.933	.00780	22.6630	5.185	-.00210
2500	25.408	37.1876	274.2714	114.1327	114.4897	11.460	.00601	20.5911	4.977	-.00193

TABLE X—THERMODYNAMIC PROPERTIES OF Al_2O_3 (GAS)

[Molecular weight, 101.94]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_T^\circ - H_{298.16}^\circ$ (kcal/mole)	H_T° (kcal/mole)	S_T° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta\left(-\frac{\Delta H^\circ}{RT}\right) = \frac{-\delta T}{100}\left(\frac{a}{T} + b\right)$		log K	$\delta \log K = \frac{-\delta T}{100}\left(\frac{c}{T} + d\right)$	
						a	b		c	d
298.16	18.8144	0	76.4666	12.5	1200.9468	-----	-----	481.8955	-----	-----
300	18.9368	0.0347	76.5013	12.6160	1193.6027	-----	-----	478.6965	-----	-----
400	23.3603	2.1790	78.6456	18.7545	895.7357	-----	-----	349.0606	-----	-----
500	25.5720	4.6362	81.1028	24.2285	716.6696	-----	-----	271.2518	-----	-----
600	26.9132	7.2651	83.7317	29.0179	597.1339	-----	-----	219.3803	-----	-----
700	27.8434	10.0054	86.4720	33.2403	511.6641	-----	-----	182.3381	-----	-----
800	28.5546	12.8268	89.2932	37.0064	447.5067	-----	-----	154.5672	-----	-----
900	29.1383	15.7121	92.1787	40.4041	397.5684	-----	-----	132.9781	-----	-----
1000	29.6430	18.6517	95.1183	43.5010	357.5885	35.876	0.11865	115.7171	15.581	-0.05115
1100	30.0960	21.6386	98.1052	46.3475	324.8553	32.625	0.10920	101.6037	14.164	-0.05063
1200	30.5135	24.6691	101.1357	48.9841	297.5586	29.915	0.10186	89.8510	12.984	-0.05099
1300	30.9068	27.7401	104.2067	51.4421	274.4452	27.620	0.09723	79.9143	11.984	-0.05000
1400	31.2819	30.8496	107.3162	53.7462	254.6194	25.653	0.09290	71.4043	11.127	-0.04940
1500	31.6436	33.9958	110.4624	55.9168	237.4245	23.948	0.08950	64.0357	10.385	-0.04913
1600	31.9951	37.1778	113.6444	57.9702	222.3675	22.454	0.08777	57.5942	9.735	-0.04857
1700	32.3389	40.3945	116.8611	59.9202	209.0715	21.137	0.08562	51.9163	9.161	-0.04774
1800	32.6766	43.6452	120.1118	61.7782	197.2431	19.965	0.08421	46.8746	8.652	-0.04769
1900	33.0094	46.9295	123.3961	63.5539	186.6510	18.917	0.08300	42.3686	8.196	-0.04720
2000	33.3383	50.2469	126.7135	65.2554	177.1095	17.973	0.08183	38.3178	7.785	-0.04664
2100	33.6639	53.5970	130.0636	66.8899	168.4691	17.119	0.08114	34.6573	7.412	-0.04549
2200	33.9870	56.9796	133.4462	68.4634	160.6066	16.343	0.07995	31.3337	7.074	-0.04495
2300	34.3079	60.3943	136.8609	69.9812	153.4210	15.634	0.07958	28.3030	6.762	-0.04294
2319.57	34.3704	61.0663	137.5329	70.2722	152.0864	-----	-----	27.7409	-----	-----
2600	25.439	39.7300	276.8138	115.1299	110.0760	11.024	0.00414	18.6788	4.786	-0.00209
2700	25.466	42.2752	279.3590	116.0905	105.9889	10.619	0.00300	16.9083	4.688	-0.00191
2800	25.491	44.8291	281.9069	117.0170	102.1934	10.243	0.00183	15.2645	4.444	-0.00201
2900	25.514	47.3733	284.4571	117.9120	98.6595	9.892	0.00107	13.7341	4.291	-0.00213
3000	25.534	49.9257	287.0095	118.7773	95.3611	9.565	0.00012	12.3059	4.148	-0.00216
3100	25.552	52.4800	289.5638	119.6148	92.2755	9.260	-0.00105	10.9700	4.014	-0.00218
3200	25.570	55.0361	292.1199	120.4263	89.3828	8.974	-0.00209	9.7178	3.889	-0.00218
3300	25.585	57.5939	294.6777	121.2134	86.6655	8.705	-0.00299	8.5415	3.770	-0.00192
3400	25.599	60.1531	297.2369	121.9774	84.1082	8.452	-0.00386	7.4346	3.659	-0.00183
3500	25.613	62.7137	299.7975	122.7197	81.6072	8.214	-0.00487	6.3910	3.555	-0.00190
3600	25.624	65.2755	302.3593	123.4414	79.4804	7.988	-0.00542	5.4054	3.455	-0.00168
3700	25.635	67.8385	304.9223	124.1436	77.2669	7.774	-0.00599	4.4733	3.362	-0.00164
3800	25.645	70.4025	307.4863	124.8273	75.2271	7.572	-0.00664	3.5902	3.273	-0.00163
3900	25.655	72.9675	310.0513	125.4936	73.2922	7.381	-0.00745	2.7526	3.189	-0.00145
4000	25.663	75.5334	312.6172	126.1432	71.4544	7.199	-0.00815	1.9568	3.109	-0.00149
4100	25.671	78.1001	315.1839	126.7770	69.7067	7.027	-0.00899	1.2000	3.033	-0.00134
4200	25.679	80.6676	317.7514	127.3957	68.0426	6.863	-0.00985	0.4792	2.960	-0.00117
4300	25.687	83.2359	320.3197	128.0001	66.4564	6.707	-0.01062	0.2080	2.890	-0.00092
4400	25.693	85.8049	322.8887	128.5907	64.9427	6.557	-0.01111	0.8639	2.824	-0.00076
4500	25.699	88.3745	325.4583	129.1681	63.4967	6.414	-0.01185	-1.4907	2.761	-0.00073
4600	25.705	90.9447	328.0285	129.7330	62.1142	6.278	-0.01244	-2.0902	2.700	-0.00057
4700	25.710	93.5154	330.5992	130.2859	60.7909	6.147	-0.01313	-2.6641	2.642	-0.00042
4800	25.716	96.0867	333.1705	130.8273	59.5234	6.023	-0.01388	-3.2141	2.587	-0.00036
4900	25.720	98.6585	335.7423	131.3575	58.3081	5.903	-0.01440	-3.7417	2.533	-0.00025
5000	25.725	101.2308	338.3146	131.8772	57.1419	5.788	-0.01510	-4.2481	2.482	-0.00007
5100	25.729	103.8035	340.8873	132.3867	56.0221	5.678	-0.01582	-4.7347	2.433	0.00002
5200	25.733	106.3766	343.4604	132.8863	54.9460	5.572	-0.01642	-5.2026	2.385	0.00030
5300	25.737	108.9501	346.0339	133.3765	53.9111	5.471	-0.01715	-5.6529	2.339	0.00045
5400	25.741	111.5240	348.6078	133.8576	52.9151	5.373	-0.01781	-6.0865	2.294	0.00071
5500	25.745	114.0983	351.1821	134.3300	51.9560	5.279	-0.01838	-6.5043	2.252	0.00086
5600	25.748	116.6729	353.7567	134.7939	51.0317	5.188	-0.01908	-6.9073	2.211	0.00091
5700	25.752	119.2479	356.3317	135.2497	50.1406	5.100	-0.01961	-7.2961	2.172	0.01002
5800	25.756	121.8233	358.9071	135.6976	49.2809	5.016	-0.02017	-7.6716	2.133	0.01117
5900	25.760	124.3991	361.4829	136.1379	48.4509	4.935	-0.02100	-8.0343	2.096	0.01137
6000	25.763	126.9753	364.0591	136.5709	47.6494	-----	-----	-8.3850	-----	-----

TABLE XI—THERMODYNAMIC PROPERTIES OF Al_2O_3 (LIQUID)

[Molecular weight, 101.94]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_T^\circ - H_{298.16}^\circ$ (kcal/mole)	H_T° (kcal/mole)	S_T° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta\left(-\frac{\Delta H^\circ}{RT}\right) = \frac{-\delta T}{100}\left(\frac{a}{T} + b\right)$		log K	$\delta \log K = \frac{-\delta T}{100}\left(\frac{c}{T} + d\right)$	
						a	b		c	d
2319.57	35.79	67.0670	143.5336	72.8593	150.7846	15.513	0.04937	27.7409	6.666	-0.05066
2400	35.95	69.9520	146.4186	74.0819	145.5461	14.995	0.04850	25.5477	6.441	-0.05000
2500	36.15	73.5570	150.0236	75.5535	139.4996	14.396	0.04828	23.0213	6.181	-0.04911
2600	36.35	77.1820	153.6486	76.9753	133.9144	13.844	0.04759	20.6931	5.941	-0.04807
2700	36.55	80.8270	157.2936	78.3509	128.7394	13.332	0.04737	18.5408	5.718	-0.04704
2800	36.75	84.4920	160.9586	79.6837	123.9306	12.858	0.04651	16.5457	5.512	-0.04629
2900	36.95	88.1770	164.6436	80.9768	119.4503	12.416	0.04603	14.6913	5.319	-0.04530
3000	37.15	91.8820	168.3486	82.2329	115.2656	12.004	0.04554	12.9636	5.140	-0.04466
3100	37.35	95.6070	172.0736	83.4543	111.3478	11.618	0.04508	11.3502	4.972	-0.04395
3200	37.55	99.3520	175.8186	84.6432	107.6721	11.257	0.04439	9.8404	4.814	-0.04309
3300	37.75	103.1170	179.5836	85.8018	104.2165	10.918	0.04372	8.4247	4.666	-0.04235
3400	37.95	106.9020	183.3686	86.9317	100.9616	10.599	0.04322	7.0947	4.526	-0.04154
3500	38.15	110.7070	187.1736	88.0347	97.8901	10.298	0.04264	5.8431	4.395	-0.04103
3600	38.35	114.5320	190.9986	89.1122	94.9869	10.013	0.04228	4.6633	4.271	-0.04052
3700	38.55	118.3770	194.8436	90.1657	92.2384	9.745	0.04163	3.5495	4.153	-0.03979
3800	38.75	122.2420	198.7086	91.1964	89.6323	9.491	0.04101	2.4964	4.042	-0.03921
3900	38.95	126.1270	202.5936	92.2055	87.1577	9.250	0.04040	1.4992	3.936	-0.03870
4000	39.15	130.0320	206.4986	93.1942	84.8048	-----	-----	.5539	-----	-----

* Enthalpy change in converting Al_2O_3 (crystal, α) at 298.16° to Al_2O_3 (liquid) at temperature indicated.

TABLE XII—THERMODYNAMIC PROPERTIES OF B (GAS)

[Atomic weight, 10.82]

T (°K)	C_p $\left(\frac{\text{cal}}{\text{mole } ^\circ\text{K}}\right)$	$H_T^\circ - H_0^\circ$ $\left(\frac{\text{kcal}}{\text{mole}}\right)$	H_T° $\left(\frac{\text{kcal}}{\text{mole}}\right)$	S_T° $\left(\frac{\text{cal}}{\text{mole } ^\circ\text{K}}\right)$
0	-----	0	269.0696	-----
298.16	4.9704	1.5097	270.5793	36.6493
300	4.9704	1.5188	270.5884	36.6798
400	4.9693	2.0158	271.0854	38.1096
500	4.9688	2.5127	271.5823	39.2183
600	4.9686	3.0096	272.0792	40.1243
700	4.9684	3.5064	272.5760	40.8902
800	4.9683	4.0033	273.0729	41.5536
900	4.9682	4.5001	273.5697	42.1388
1000	4.9682	4.9969	274.0665	42.6625
1100	4.9682	5.4937	274.5633	43.1360
1200	4.9681	5.9905	275.0601	43.5683
1300	4.9681	6.4874	275.5570	43.9659
1400	4.9681	6.9842	276.0538	44.3341
1500	4.9681	7.4810	276.5506	44.6769
1600	4.9681	7.9778	277.0474	44.9975
1700	4.9681	8.4746	277.5442	45.2987
1800	4.9680	8.9714	278.0410	45.5827
1900	4.9680	9.4682	278.5378	45.8513
2000	4.9680	9.9650	279.0346	46.1061
2100	4.9680	10.4618	279.5314	46.3485
2200	4.9680	10.9586	280.0282	46.5796
2300	4.9680	11.4554	280.5250	46.8004
2400	4.9680	11.9522	281.0218	47.0119
2500	4.9680	12.4490	281.5186	47.2147
2600	4.9680	12.9458	282.0154	47.4095
2700	4.9680	13.4426	282.5122	47.5970
2800	4.9680	13.9394	283.0090	47.7777
2900	4.9680	14.4362	283.5058	47.9520
3000	4.9680	14.9330	284.0026	48.1204
3100	4.9680	15.4298	284.4994	48.2833
3200	4.9680	15.9266	284.9962	48.4410
3300	4.9680	16.4234	285.4930	48.5939
3400	4.9680	16.9202	285.9898	48.7422
3500	4.9680	17.4170	286.4866	48.8862
3600	4.9680	17.9138	286.9834	49.0262
3700	4.9680	18.4106	287.4802	49.1623
3800	4.9680	18.9074	287.9770	49.2948
3900	4.9680	19.4042	288.4738	49.4238
4000	4.9681	19.9010	288.9706	49.5496
4100	4.9681	20.3978	289.4674	49.6723
4200	4.9681	20.8946	289.9642	49.7920
4300	4.9682	21.3914	290.4610	49.9089
4400	4.9682	21.8883	290.9579	50.0231
4500	4.9683	22.3851	291.4547	50.1348
4600	4.9685	22.8819	291.9515	50.2440
4700	4.9686	23.3788	292.4484	50.3509
4800	4.9688	23.8756	292.9452	50.4555
4900	4.9690	24.3725	293.4421	50.5579
5000	4.9692	24.8694	293.9390	50.6583
5100	4.9694	25.3664	294.4360	50.7567
5200	4.9697	25.8633	294.9329	50.8532
5300	4.9701	26.3603	295.4299	50.9479
5400	4.9705	26.8574	295.9270	51.0408
5500	4.9710	27.3544	296.4240	51.1320
5600	4.9716	27.8516	296.9212	51.2216
5700	4.9722	28.3487	297.4183	51.3096
5800	4.9728	28.8460	297.9156	51.3961
5900	4.9736	29.3433	298.4129	51.4811
6000	4.9745	29.8407	298.9103	51.5647

TABLE XIII—THERMODYNAMIC PROPERTIES OF B₂ (GAS)

[Molecular weight, 21.64]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta\left(-\frac{\Delta H^\circ}{RT}\right) = \frac{-\delta T}{100}\left(\frac{a}{T} + b\right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100}\left(\frac{c}{T} + d\right)$	
						a	b		c	d
0	-----	0	468.6652	-----	-----	-----	-----	-----	-----	-----
298.16	7.289	2.0934	470.7586	48.652	118.8190	-----	-----	46.2160	-----	-----
300	7.295	2.1068	470.7720	48.697	118.0983	-----	-----	45.8995	-----	-----
400	7.645	2.8540	471.5192	50.844	88.8842	-----	-----	33.0562	-----	-----
500	7.950	3.6338	472.2990	52.584	71.3228	-----	-----	25.3251	-----	-----
600	8.165	4.4398	473.1050	54.052	59.5932	-----	-----	20.1558	-----	-----
700	8.330	5.2646	473.9298	55.323	51.2012	-----	-----	16.4542	-----	-----
800	8.450	6.1042	474.7694	56.444	44.8980	-----	-----	13.6718	-----	-----
900	8.540	6.9537	475.6189	57.445	39.9899	-----	-----	11.5032	-----	-----
1000	8.608	7.8115	476.4767	58.348	36.0593	3526	0.01325	9.7646	1552	0.01399
1100	8.665	8.6751	477.3403	59.1711	32.8406	3209	0.01003	8.3397	1413	0.01210
1200	8.704	9.5436	478.2088	59.9267	30.1564	2944	.00798	7.1501	1296	.01138
1300	8.738	10.4157	479.0809	60.6248	27.8838	2720	.00624	6.1418	1198	.00999
1400	8.764	11.2898	479.9560	61.2732	25.9347	2528	.00467	5.2761	1113	.00940
1500	8.784	12.1682	480.8334	61.8786	24.2447	2359	.00302	4.5247	1040	.00860
1600	8.803	13.0475	481.7127	62.4461	22.7653	2213	0.00404	3.8661	976	0.00788
1700	8.820	13.9287	482.5939	62.9803	21.4595	2084	.00342	3.2841	919	.00764
1800	8.832	14.8113	483.4765	63.4848	20.2983	1969	.00288	2.7659	869	.00693
1900	8.842	15.6950	484.3602	63.9626	19.2591	1866	.00260	2.3016	824	.00630
2000	8.852	16.5797	485.2449	64.4164	18.3235	1774	.00184	1.8831	784	.00597
2100	8.860	17.4653	486.1305	64.8484	17.4769	1690	0.00172	1.5038	747	0.00565
2200	8.868	18.3517	487.0169	65.2608	16.7070	1614	.00136	1.1586	714	.00527
2300	8.874	19.2388	487.9040	65.6551	16.0039	1544	.00127	.8429	683	.00522
2400	8.880	20.1265	488.7917	66.0329	15.3593	1480	.00120	.5531	655	.00500
2500	8.884	21.0147	489.6799	66.3955	14.7661	1421	.00106	.2861	630	.00449
2600	8.888	21.9033	490.5685	66.7440	14.2185	1367	0.00080	0.0393	606	0.00446
2700	8.891	22.7922	491.4574	67.0795	13.7114	1316	.00090	-.1896	584	.00423
2800	8.895	23.6815	492.3467	67.4029	13.2405	1270	.00067	-.4024	564	.00392
2900	8.898	24.5712	493.2364	67.7151	12.8019	1226	.00063	-.6008	545	.00383
3000	8.901	25.4611	494.1263	68.0168	12.3926	1185	.00074	-.7863	527	.00370
3100	8.903	26.3513	495.0165	68.3087	12.0096	1147	0.00066	-0.9600	511	0.00341
3200	8.906	27.2418	495.9070	68.5914	11.6505	1111	.00063	-1.1231	495	.00340
3300	8.908	28.1325	496.7977	68.8655	11.3132	1078	.00054	-1.2765	481	.00323
3400	8.910	29.0234	497.6886	69.1314	10.9956	1046	.00054	-1.4212	466	.00336
3500	8.912	29.9145	498.5797	69.3898	10.6962	1016	.00058	-1.5577	453	.00327
3600	8.915	30.8058	499.4710	69.6409	10.4134	988	0.00057	-1.6868	441	0.00311
3700	8.917	31.6974	500.3626	69.8852	10.1458	961	.00061	-1.8091	429	.00311
3800	8.919	32.5892	501.2544	70.1230	9.8923	936	.00050	-1.9251	419	.00286
3900	8.921	33.4812	502.1464	70.3547	9.6518	912	.00060	-2.0354	408	.00280
4000	8.923	34.3734	503.0386	70.5806	9.4232	889	.00057	-2.1402	398	.00283
4100	8.925	35.2658	503.9310	70.8009	9.2058	867	0.00067	-2.2401	389	0.00268
4200	8.927	36.1584	504.8236	71.0160	8.9987	847	.00052	-2.3354	380	.00253
4300	8.930	37.0513	505.7165	71.2261	8.8012	827	.00055	-2.4263	371	.00258
4400	8.932	37.9444	506.6096	71.4314	8.6127	808	.00054	-2.5132	363	.00253
4500	8.935	38.8377	507.5029	71.6322	8.4326	790	.00066	-2.5964	355	.00253
4600	8.937	39.7313	508.3965	71.8286	8.2602	773	0.00063	-2.6761	348	0.00236
4700	8.940	40.6252	509.2904	72.0208	8.0951	756	.00070	-2.7525	341	.00226
4800	8.944	41.5194	510.1846	72.2091	7.9369	740	.00078	-2.8258	334	.00224
4900	8.947	42.4139	511.0791	72.3935	7.7851	724	.00090	-2.8962	327	.00220
5000	8.951	43.3008	511.9740	72.5743	7.6394	710	.00088	-2.9638	321	.00216
5100	8.955	44.2041	512.8693	72.7516	7.4993	695	0.00105	-3.0289	315	0.00212
5200	8.959	45.0998	513.7650	72.9255	7.3646	682	.00102	-3.0916	309	.00210
5300	8.963	45.9959	514.6611	73.0962	7.2349	669	.00101	-3.1520	303	.00209
5400	8.968	46.8925	515.5577	73.2638	7.1100	656	.00113	-3.2102	298	.00202
5500	8.973	47.7895	516.4547	73.4284	6.9896	644	.00110	-3.2664	293	.00188
5600	8.979	48.6871	517.3523	73.5901	6.8735	632	0.00122	-3.3206	288	0.00187
5700	8.985	49.5853	518.2505	73.7491	6.7614	620	.00130	-3.3730	283	.00191
5800	8.991	50.4841	519.1493	73.9054	6.6532	610	.00131	-3.4237	278	.00188
5900	8.998	51.3836	520.0488	74.0592	6.5455	599	.00137	-3.4727	273	.00190
6000	9.005	52.2837	520.9489	74.2105	6.4473	-----	-----	-3.5201	-----	-----

TABLE XIV—THERMODYNAMIC PROPERTIES OF BF (GAS)

[Molecular weight, 29.82]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	218.1759	-----	-----	-----	-----	-----	-----	-----
298.16	7.0696	2.0789	220.2548	47.9001	169.0481	-----	-----	67.5887	-----	-----
300	7.0729	2.0919	220.2678	47.9437	168.0215	-----	-----	67.1385	-----	-----
400	7.3044	2.8101	220.9860	50.0081	126.4173	-----	-----	48.8690	-----	-----
500	7.5718	3.5539	221.7298	51.6668	101.4208	-----	-----	37.8738	-----	-----
600	7.8149	4.3236	222.4995	53.0694	84.7288	-----	-----	30.5239	-----	-----
700	8.0158	5.1155	223.2914	54.2897	72.7858	-----	-----	25.2615	-----	-----
800	8.1760	5.9254	224.1013	55.3710	63.8146	-----	-----	21.3064	-----	-----
900	8.3024	6.7495	224.9255	56.3416	56.8272	-----	-----	18.2243	-----	-----
1000	8.4025	7.5850	225.7609	57.2217	51.2304	5016	0.02420	15.7542	2207	0.01784
1100	8.4822	8.4293	226.6053	58.0264	46.6462	4565	0.01963	13.7300	2009	0.01523
1200	8.5464	9.2909	227.4568	58.7673	42.8224	4189	.01589	12.0406	1844	.01304
1300	8.5986	10.1382	228.3142	59.4535	39.5842	3870	.01341	10.6091	1704	.01166
1400	8.6415	11.0003	229.1762	60.0924	36.8065	3597	.01100	9.3803	1584	.01040
1500	8.6770	11.8663	230.0422	60.6898	34.3975	3360	.00900	8.3139	1480	.00920
1600	8.7068	12.7355	230.9115	61.2508	32.2885	3152	0.00778	7.3797	1388	0.00883
1700	8.7319	13.6075	231.7834	61.7794	30.4266	2969	.00635	6.5544	1308	.00793
1800	8.7533	14.4818	232.6577	62.2791	28.7708	2805	.00588	5.8198	1236	.00747
1900	8.7717	15.3580	233.5340	62.7529	27.2886	2659	.00500	5.1618	1172	.00690
2000	8.7875	16.2360	234.4120	63.2032	25.9541	2528	.00399	4.5689	1115	.00615
2100	8.8013	17.1155	235.2914	63.6323	24.7463	2408	0.00376	4.0318	1062	0.00597
2200	8.8133	17.9962	236.1722	64.0420	23.6480	2300	.00310	3.5431	1015	.00550
2300	8.8239	18.8781	237.0540	64.4340	22.6449	2201	.00282	3.0963	971	.00542
2400	8.8332	19.7610	237.9369	64.8098	21.7250	2110	.00240	2.6863	931	.00520
2500	8.8415	20.6447	238.8206	65.1705	20.8786	2026	.00227	2.3087	894	.00505
2600	8.8489	21.5292	239.7052	65.5175	20.0971	1949	0.00195	1.9598	860	0.00498
2700	8.8555	22.4145	240.5904	65.8516	19.3733	1877	.00184	1.6363	829	.00453
2800	8.8614	23.3003	241.4763	66.1737	18.7011	1811	.00152	1.3357	801	.00399
2900	8.8668	24.1867	242.3626	66.4848	18.0751	1749	.00130	1.0555	774	.00380
3000	8.8716	25.0736	243.2496	66.7855	17.4908	1691	.00132	.7937	749	.00359
3100	8.8760	25.9610	244.1370	67.0764	16.9440	1637	0.00104	0.5485	725	0.00354
3200	8.8800	26.8488	245.0248	67.3583	16.4314	1586	.00099	.3184	703	.00327
3300	8.8837	27.7370	245.9130	67.6316	15.9498	1538	.00105	.1021	682	.00321
3400	8.8870	28.6256	246.8015	67.8969	15.4964	1493	.00093	-.1017	662	.00326
3500	8.8901	29.5144	247.6904	68.1545	15.0689	1451	.00084	-.2941	644	.00301
3600	8.8929	30.4036	248.5795	68.4050	14.6650	1411	0.00065	-.4760	626	0.00301
3700	8.8955	31.2930	249.4690	68.6487	14.2830	1373	.00068	-.6482	610	.00287
3800	8.8979	32.1827	250.3586	68.8860	13.9210	1337	.00058	-.8116	593	.00295
3900	8.9001	33.0726	251.2485	69.1171	13.5776	1303	.00055	-.9666	579	.00275
4000	8.9022	33.9627	252.1386	69.3425	13.2513	1270	.00074	-1.1141	565	.00270
4100	8.9041	34.8530	253.0290	69.5623	12.9408	1239	0.00060	-1.2546	551	0.00261
4200	8.9059	35.7435	253.9195	69.7769	12.6452	1210	.00050	-1.3884	538	.00268
4300	8.9075	36.6342	254.8101	69.9865	12.3633	1182	.00056	-1.5162	526	.00255
4400	8.9091	37.5250	255.7009	70.1913	12.0941	1155	.00053	-1.6383	514	.00258
4500	8.9105	38.4160	256.5920	70.3915	11.8369	1129	.00057	-1.7551	503	.00245
4600	8.9119	39.3071	257.4830	70.5874	11.5909	1105	0.00049	-1.8669	492	0.00252
4700	8.9132	40.1983	258.3743	70.7790	11.3553	1082	.00038	-1.9741	482	.00238
4800	8.9144	41.0897	259.2657	70.9667	11.1295	1060	.00017	-2.0769	472	.00227
4900	8.9155	41.9812	260.1571	71.1505	10.9130	1039	.00020	-2.1755	463	.00230
5000	8.9165	42.8728	261.0487	71.3306	10.7050	1018	.00019	-2.2704	454	.00218
5100	8.9175	43.7645	261.9405	71.5072	10.5052	998	0.00018	-2.3616	445	0.00222
5200	8.9184	44.6563	262.8322	71.6804	10.3131	979	.00008	-2.4494	437	.00205
5300	8.9193	45.5483	263.7242	71.8503	10.1283	960	.00022	-2.5339	429	.00206
5400	8.9202	46.4402	264.6161	72.0170	9.9503	943	.00015	-2.6154	421	.00205
5500	8.9210	47.3323	265.5082	72.1807	9.7787	926	.00004	-2.6940	414	.00197
5600	8.9247	48.2244	266.4003	72.3414	9.6133	910	0.00005	-2.7699	407	0.00190
5700	8.9224	49.1166	267.2926	72.4994	9.4536	894	-.00004	-2.8432	400	.00183
5800	8.9231	50.0089	268.1848	72.6546	9.2995	879	-.00008	-2.9140	393	.00179
5900	8.9237	50.9012	269.0771	72.8071	9.1506	864	-.00010	-2.9824	387	.00180
6000	8.9244	51.7936	269.9696	72.9571	9.0067	-----	-----	-3.0487	-----	-----

TABLE XV—THERMODYNAMIC PROPERTIES OF BF_3 (GAS)

[Molecular weight, 67.82]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f - H_o$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0		0	0							
298.18	12.0621	2.7842	2.7842	60.6958	704.3124			286.2741		
300	12.0981	2.8066	2.8066	60.7712	700.0206			284.3981		
400	13.7643	4.1030	4.1030	64.4894	526.0482			208.3238		
500	15.0594	5.5471	5.5471	67.7061	421.4916			162.5976		
600	16.0460	7.1046	7.1046	70.5433	351.6741			132.0708		
700	16.7919	8.7482	8.7482	73.0755	301.7303			110.2416		
800	17.3580	10.4570	10.4570	75.3564	264.2235			93.8552		
900	17.7915	12.2154	12.2154	77.4270	235.0185			81.1010		
1000	18.1277	14.0120	14.0120	79.3197	211.6314	20.979	0.07908	70.8915	9168	0.02195
1100	18.3922	15.8380	15.8380	81.0598	192.4805	19.089	0.06350	62.5341	8339	0.01823
1200	18.6030	17.6878	17.6878	82.6692	176.5095	17.513	.05116	55.5667	7648	.01462
1300	18.7731	19.5566	19.5566	84.1650	162.9868	16.178	.04179	49.6690	7063	.01180
1400	18.9122	21.4409	21.4409	85.5613	151.3893	15.033	.03430	44.6122	6561	.00970
1500	19.0270	23.3378	23.3378	86.8700	141.3330	14.039	.02882	40.2285	6126	.00795
1600	19.1228	25.3453	25.2453	88.1010	132.5298	13.169	0.02423	36.3918	5745	0.00699
1700	19.2035	27.1616	27.1616	89.2628	124.7591	12.400	.02091	33.0057	5409	.00540
1800	19.2720	29.0854	29.0854	90.3623	117.8493	11.717	.01746	29.9953	5110	.00443
1900	19.3306	31.0155	31.0155	91.4059	111.6650	11.105	.01520	27.3014	4842	.00400
2000	19.3813	32.9511	32.9511	92.3987	106.0973	10.554	.01309	24.8764	4601	.00335
2100	19.4252	34.8915	34.8915	93.3454	101.0585	10.055	0.01135	22.6821	4383	0.00283
2200	19.4635	36.8359	36.8359	94.2499	96.4767	9.601	.00995	20.6870	4184	.00267
2300	19.4971	38.7839	38.7839	95.1159	92.2924	8.186	.00890	18.8652	4003	.00228
2400	19.5268	40.7351	40.7351	95.9463	88.4560	8.806	.00780	17.1950	3837	.00190
2500	19.5532	42.6891	42.6891	96.7440	84.9258	8.456	.00689	15.6583	3684	.00168
2600	19.5766	44.6456	44.6456	97.5113	81.6666	8.133	0.00608	14.2397	3543	0.00148
2700	19.5976	46.6043	46.6043	98.2505	78.0483	7.833	.00560	12.9260	3412	.00133
2800	19.6165	48.5650	48.5650	98.9636	75.8452	7.555	.00503	11.7061	3291	.00107
2900	19.6332	50.5275	50.5275	99.6522	73.2350	7.296	.00450	10.5702	3178	.00087
3000	19.6486	52.4916	52.4916	100.3181	70.7985	7.054	.00412	9.5100	3072	.00083
3100	19.6626	54.4572	54.4572	100.9626	68.5189	6.827	0.00396	8.5182	2973	0.00094
3200	19.6753	56.4241	56.4241	101.5871	66.3815	6.615	.00346	7.5882	2880	.00087
3300	19.6867	58.3922	58.3922	102.1927	64.3735	6.416	.00304	6.7146	2793	.00083
3400	19.6972	60.3614	60.3614	102.7805	62.4834	6.228	.00287	5.8923	2711	.00083
3500	19.7069	62.3316	62.3316	103.3516	60.7011	6.051	.00257	5.1169	2634	.00063
3600	19.7158	64.3027	64.3027	103.9069	59.0177	5.884	0.00233	4.3846	2561	0.00054
3700	19.7240	66.2747	66.2747	104.4472	57.4251	5.725	.00232	3.6919	2492	.00051
3800	19.7316	68.2475	68.2475	104.9733	55.9162	5.575	.00211	3.0356	2427	.00039
3900	19.7386	70.2210	70.2210	105.4860	54.4846	5.433	.00185	2.4129	2365	.00035
4000	19.7451	72.1952	72.1952	105.9858	53.1245	5.298	.00171	1.8213	2306	.00026
4100	19.7511	74.1700	74.1700	106.4734	51.8306	5.169	0.00169	1.2586	2250	0.00029
4200	19.7567	76.1454	76.1454	106.9494	50.5982	5.046	.00161	.7226	2196	.00030
4300	19.7619	78.1213	78.1213	107.4144	49.4231	4.929	.00157	.2116	2145	.00030
4400	19.7668	80.0977	80.0977	107.8688	48.3013	4.817	.00146	-.2762	2097	.00020
4500	19.7714	82.0746	82.0746	108.3130	47.2294	4.711	.00127	-.7424	2050	.00025
4600	19.7737	84.0519	84.0519	108.7476	46.2040	4.609	.00116	-1.1883	2006	0.00019
4700	19.7777	86.0295	86.0295	109.1729	45.2222	4.512	.00100	-1.6153	1963	.00014
4800	19.7814	88.0074	88.0074	109.5893	44.2812	4.418	.00097	-2.0244	1922	.00026
4900	19.7849	89.9857	89.9857	109.9972	43.3786	4.328	.00090	-2.4169	1883	.00020
5000	19.7882	91.9644	91.9644	110.3970	42.5121	4.242	.00084	-2.7937	1845	.00024
5100	19.7913	93.9434	93.9434	110.7889	41.6795	4.159	0.00079	-3.1557	1809	0.00022
5200	19.7942	95.9226	95.9226	111.1732	40.8789	4.079	.00078	-3.5038	1775	.00009
5300	19.7969	97.9022	97.9022	111.5503	40.1085	4.002	.00079	-3.8388	1741	.00019
5400	19.7995	99.8820	99.8820	111.9204	39.3666	3.929	.00064	-4.1614	1708	.00025
5500	19.8020	101.8621	101.8621	112.2837	38.6516	3.857	.00065	-4.4722	1678	.00016
5600	19.8044	103.8424	103.8424	112.6405	37.9622	3.789	0.00056	-4.7720	1648	0.00008
5700	19.8067	105.8230	105.8230	112.9910	37.2969	3.722	.00058	-5.0612	1619	.00016
5800	19.8089	107.8037	107.8037	113.3355	36.6546	3.659	.00043	-5.3405	1591	.00014
5900	19.8110	109.7847	109.7847	113.6742	36.0340	3.597	.00040	-5.6103	1564	.00013
6000	19.8131	111.7659	111.7659	114.0072	35.4341			-5.8711		

TABLE XVI—THERMODYNAMIC PROPERTIES OF BH (GAS)

[Molecular weight, 11.828]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	6.9593	0	279.8258	41.0362	127.4086	-----	-----	50.3050	-----	-----
298.16	6.9596	2.0739	281.9125	41.0790	126.6364	-----	-----	49.9656	-----	-----
300	6.9931	2.7839	282.6097	43.0847	95.3504	-----	-----	36.1918	-----	-----
400	7.0796	3.4871	283.3129	44.6534	76.5727	-----	-----	27.8950	-----	-----
500	7.2130	4.2014	284.0272	45.9554	64.0449	-----	-----	22.3429	-----	-----
600	7.3713	4.9305	284.7563	47.0790	55.0858	-----	-----	18.3628	-----	-----
700	7.5344	5.6758	285.5016	48.0740	48.3563	-----	-----	15.3677	-----	-----
800	7.6897	6.4371	286.2629	48.9705	43.1133	-----	-----	13.0308	-----	-----
900	7.8309	7.2133	287.0391	49.7881	38.9114	3754	0.03127	11.1558	1667	0.02285
1000	7.9561	8.0026	287.8284	50.5404	35.4674	3417	0.02740	9.6175	1519	0.01927
1100	8.0656	8.8037	288.6295	51.2373	32.5925	3136	0.02419	8.3324	1395	0.01702
1200	8.1607	9.6150	289.4408	51.8867	30.1560	2899	0.02089	7.2423	1290	0.01507
1300	8.2431	10.4352	290.2610	52.4945	28.0644	2696	0.01807	6.3058	1200	0.01340
1400	8.3144	11.2631	291.0889	53.0656	26.2490	2520	0.01550	5.4924	1122	0.01195
1500	8.3764	12.0976	291.9234	53.6042	24.6585	2366	0.01344	4.7792	1054	0.01060
1600	8.4302	12.9350	292.7638	54.1136	23.2533	2229	0.01217	4.1486	993	0.00983
1700	8.4772	13.7833	293.6091	54.5968	22.0028	2108	0.01033	3.5871	939	0.00919
1800	8.5184	14.6331	294.4589	55.0562	20.8830	2000	0.00900	3.0837	891	0.00840
1900	8.5546	15.4868	295.3126	55.4941	19.8740	1902	0.00789	2.6298	848	0.00749
2000	8.5866	16.3438	296.1696	55.9123	18.9604	1813	0.00721	2.2185	809	0.00687
2100	8.6150	17.2039	297.0297	56.3124	18.1291	1733	0.00612	1.8439	773	0.00651
2200	8.6401	18.0667	297.8925	56.6959	17.3695	1659	0.00555	1.5013	740	0.00627
2300	8.6626	18.9318	298.7576	57.0641	16.6727	1591	0.00510	1.1867	710	0.00580
2400	8.6828	19.7991	299.6249	57.4181	16.0312	1529	0.00442	0.8969	682	0.00569
2500	8.7009	20.6683	300.4941	57.7590	15.4387	1471	0.00409	0.6289	657	0.00527
2600	8.7172	21.5392	301.3650	58.0877	14.8898	1418	0.00357	0.3803	633	0.00503
2700	8.7320	22.4116	302.2374	58.4050	14.3798	1368	0.00338	0.1492	611	0.00481
2800	8.7453	23.2855	303.1113	58.7116	13.9047	1322	0.00293	0.0663	590	0.00473
2900	8.7575	24.1606	303.9864	59.0083	13.4611	1279	0.00262	0.2677	571	0.00451
3000	8.7686	25.0369	304.8627	59.2956	13.0459	1238	0.00252	0.4564	554	0.00408
3100	8.7788	25.9143	305.7401	59.5742	12.6565	1200	0.00236	0.6336	537	0.00397
3200	8.7881	26.7926	306.6184	59.8445	12.2905	1164	0.00225	0.8003	521	0.00386
3300	8.7966	27.6719	307.4977	60.1070	11.9459	1130	0.00204	0.9574	506	0.00373
3400	8.8045	28.5519	308.3777	60.3621	11.6210	1099	0.00182	1.1057	492	0.00363
3500	8.8117	29.4327	309.2585	60.6102	11.3139	1068	0.00185	1.2460	479	0.00344
3600	8.8184	30.3143	310.1401	60.8517	11.0234	1040	0.00162	1.3789	466	0.00347
3700	8.8246	31.1964	311.0222	61.0870	10.7481	1013	0.00156	1.5050	454	0.00339
3800	8.8304	32.0792	311.9050	61.3163	10.4868	988	0.00130	1.6248	443	0.00315
3900	8.8357	32.9625	312.7883	61.5399	10.2385	964	0.00118	1.7387	432	0.00313
4000	8.8407	33.8463	313.6721	61.7581	10.0022	940	0.00119	1.8472	422	0.00302
4100	8.8453	34.7306	314.5564	61.9712	9.7772	918	0.00111	1.9507	412	0.00299
4200	8.8497	35.6153	315.4411	62.1794	9.5626	897	0.00114	2.0495	403	0.00291
4300	8.8537	36.5005	316.3263	62.3829	9.3576	877	0.00091	2.1440	394	0.00274
4400	8.8575	37.3861	317.2119	62.5819	9.1618	858	0.00088	2.2343	386	0.00269
4500	8.8611	38.2720	318.0978	62.7766	8.9744	840	0.00078	2.3209	378	0.00257
4600	8.8644	39.1583	318.9841	62.9672	8.7949	822	0.00075	2.4039	370	0.00252
4700	8.8676	40.0449	319.8707	63.1539	8.6229	805	0.00072	2.4835	363	0.00242
4800	8.8705	40.9318	320.7576	63.3368	8.4579	789	0.00060	2.5600	356	0.00230
4900	8.8733	41.8190	321.6448	63.5160	8.2995	774	0.00054	2.6335	349	0.00227
5000	8.8759	42.7064	322.5322	63.6918	8.1472	758	0.00063	2.7042	342	0.00233
5100	8.8784	43.5941	323.4199	63.8641	8.0008	744	0.00052	2.7723	336	0.00220
5200	8.8807	44.4821	324.3079	64.0333	7.8599	730	0.00052	2.8379	330	0.00209
5300	8.8830	45.3703	325.1961	64.1993	7.7242	717	0.00044	2.9011	325	0.00201
5400	8.8850	46.2587	326.0845	64.3623	7.5934	704	0.00039	2.9622	319	0.00194
5500	8.8870	47.1473	326.9731	64.5224	7.4673	692	0.00040	3.0211	313	0.00199
5600	8.8889	48.0361	327.8619	64.6797	7.3455	680	0.00026	3.0780	308	0.00200
5700	8.8907	48.9251	328.7509	64.8343	7.2280	668	0.00028	3.1331	303	0.00184
5800	8.8924	49.8142	329.6400	64.9863	7.1145	657	0.00030	3.1863	298	0.00183
5900	8.8941	50.7035	330.5293	65.1358	7.0047	-----	-----	3.2378	-----	-----

TABLE XVII—THERMODYNAMIC PROPERTIES OF BO (GAS)

[Molecular weight, 26.82]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = -\frac{\delta T}{100} \left(\frac{a}{T} + b \right)$		$\log K$	$\delta \log K = -\frac{\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	168.1616	-----	-----	-----	-----	-----	-----	-----
298.16	6.976	2.0731	170.2347	48.605	272.6696	-----	-----	112.6245	-----	-----
300	6.977	2.0859	170.2475	48.647	271.0072	-----	-----	111.8980	-----	-----
400	7.062	2.7872	170.9488	50.664	203.6499	-----	-----	82.4477	-----	-----
500	7.230	3.5018	171.6634	52.259	163.2146	-----	-----	64.7442	-----	-----
600	7.427	4.2347	172.3963	53.594	136.2389	-----	-----	52.9207	-----	-----
700	7.635	4.9880	173.1496	54.755	116.9540	-----	-----	44.4620	-----	-----
800	7.810	5.7600	173.9216	55.785	102.4775	-----	-----	38.1086	-----	-----
900	7.970	6.5490	174.7106	56.714	91.2079	-----	-----	33.1604	-----	-----
1000	8.109	7.3530	175.5146	57.563	82.1842	8094	0.03092	29.1974	3547	0.02195
1100	8.225	8.1697	176.3313	58.3413	74.7951	7363	0.02647	25.9509	3229	0.01747
1200	8.325	8.9972	177.1588	59.0613	68.6328	6754	.02252	23.2426	2963	.01467
1300	8.411	9.8340	177.9956	59.7310	63.4149	6239	.01907	20.9487	2737	.01310
1400	8.485	10.6788	178.8404	60.3570	58.9394	5797	.01653	18.9806	2544	.01120
1500	8.5485	11.5305	179.6921	60.9446	55.0582	5414	.01425	17.2734	2376	.01010
1600	8.6025	12.3881	180.5497	61.4980	51.6602	5079	0.01195	15.7783	2229	0.00912
1700	8.6483	13.2506	181.4122	62.0209	48.6606	4783	.01048	14.4580	2099	.00839
1800	8.6883	14.1174	182.2790	62.5164	45.9929	4520	.00885	13.2835	1984	.00749
1900	8.7235	14.9886	183.1496	62.9871	43.6051	4284	.00790	12.2318	1880	.00720
2000	8.7550	15.8620	184.0236	63.4354	41.4552	4071	.00733	11.2846	1787	.00665
2100	8.7835	16.7389	184.9005	63.8632	39.5093	3879	0.00632	10.4270	1703	0.00611
2200	8.8095	17.6185	185.7801	64.2724	37.7398	3704	.00587	9.6468	1627	.00561
2300	8.8333	18.5007	186.6623	64.6645	36.1235	3545	.00492	8.9339	1556	.00547
2400	8.8549	19.3851	187.5467	65.0409	34.6415	3398	.00460	8.2801	1493	.00540
2500	8.8749	20.2716	188.4332	65.4028	33.2777	3264	.00382	7.6781	1434	.00456
2600	8.8934	21.1600	189.3216	65.7513	32.0185	3140	0.00324	7.1220	1379	0.00436
2700	8.9106	22.0502	190.2118	66.0872	30.8523	3024	.00320	6.6069	1329	.00406
2800	8.9268	22.9421	191.1037	66.4116	29.7691	2918	.00249	6.1282	1282	.00393
2900	8.9421	23.8355	191.9971	66.7251	28.7604	2818	.00217	5.6822	1238	.00383
3000	8.9565	24.7304	192.8920	67.0285	27.8189	2725	.00197	5.2657	1197	.00367
3100	8.9702	25.6268	193.7884	67.3224	26.9379	2638	0.00162	4.8759	1159	0.00351
3200	8.9833	26.5244	194.6860	67.6074	26.1119	2557	.00115	4.5102	1124	.00319
3300	8.9959	27.4234	195.5850	67.8840	25.3359	2480	.00109	4.1664	1090	.00311
3400	9.0081	28.3236	196.4852	68.1527	24.6054	2408	.00070	3.8427	1058	.00311
3500	9.0200	29.2250	197.3866	68.4140	23.9167	2340	.00060	3.5373	1028	.00304
3600	9.0316	30.1276	198.2892	68.6683	23.2661	2275	0.00054	3.2487	1000	0.00293
3700	9.0430	31.0313	199.1929	68.9159	22.6507	2214	.00037	2.9755	973	.00285
3800	9.0543	31.9362	200.0978	69.1572	22.0677	2157	.00012	2.7166	948	.00272
3900	9.0654	32.8422	201.0038	69.3925	21.5145	2102	.00000	2.4708	924	.00270
4000	9.0763	33.7492	201.9108	69.6222	20.9890	2050	-.00020	2.2371	901	.00264
4100	9.0870	34.6574	202.8190	69.8464	20.4892	2000	-0.00009	2.0147	879	0.00261
4200	9.0976	35.5666	203.7282	70.0655	20.0131	1953	-.00029	1.8028	859	.00243
4300	9.1080	36.4769	204.6385	70.2797	19.5592	1908	-.00044	1.6006	839	.00242
4400	9.1183	37.3882	205.5498	70.4892	19.1260	1865	-.00044	1.4075	820	.00238
4500	9.1284	38.3006	206.4622	70.6943	18.7120	1824	-.00062	1.2229	802	.00235
4600	9.1384	39.2139	207.3755	70.8950	18.3161	1785	-0.00069	1.0462	785	0.00238
4700	9.1482	40.1282	208.2898	71.0916	17.9370	1747	-.00076	.8768	768	.00230
4800	9.1579	41.0325	209.2051	71.2843	17.5738	1710	-.00058	.7145	753	.00213
4900	9.1675	41.9598	210.1214	71.4733	17.2254	1676	-.00070	.5587	738	.00210
5000	9.1769	42.8770	211.0386	71.6586	16.8909	1642	-.00066	.4090	722	.00223
5100	9.1862	43.7952	211.9568	71.8404	16.5696	1610	-0.00072	0.2652	709	0.00215
5200	9.1954	44.7143	212.8759	72.0189	16.2607	1580	-.00091	.1267	695	.00217
5300	9.2045	45.6343	213.7959	72.1941	15.9635	1550	-.00084	-.0066	682	.00210
5400	9.2135	46.5552	214.7168	72.3662	15.6773	1521	-.00085	-.1350	669	.00216
5500	9.2224	47.4770	215.6386	72.5354	15.4016	1494	-.00099	-.2588	657	.00218
5600	9.2312	48.3996	216.5612	72.7016	15.1358	1467	-0.00077	-0.3783	645	0.00214
5700	9.2399	49.3232	217.4848	72.8651	14.8792	1441	-.00085	-.4936	634	.00219
5800	9.2485	50.2476	218.4092	73.0259	14.6316	1416	-.00080	-.6051	623	.00211
5900	9.2570	51.1729	219.3345	73.1840	14.3924	1393	-.00097	-.7128	612	.00220
6000	9.2654	52.0990	220.2606	73.3397	14.1612	-----	-----	-.8170	-----	-----

TABLE XVIII—THERMODYNAMIC PROPERTIES OF B₂O₃ (GAS)

[Molecular weight, 69.64]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta\left(-\frac{\Delta H^\circ}{RT}\right) = \frac{-\delta T}{100}\left(\frac{a}{T} + b\right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100}\left(\frac{c}{T} + d\right)$	
						a	b		c	d
0	-----	0	126.0889	-----	-----	-----	-----	-----	-----	-----
298.16	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
300	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
400	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
500	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
600	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
700	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
800	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
900	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
1000	23.360	16.5945	142.6834	82.1636	301.8037	29.998	0.10329	101.1271	13.090	0.01700
1100	23.754	18.9502	145.0391	84.4085	274.4295	27.295	0.08117	89.2101	11.905	0.01197
1200	24.064	21.3411	147.4300	86.4886	251.6025	25.041	0.06409	79.2773	10.917	0.00821
1300	24.311	23.7599	149.8488	88.4246	232.2761	23.131	0.05176	70.8714	10.081	0.00499
1400	24.510	26.2009	152.2898	90.2335	215.7022	21.492	0.04230	63.6657	9.363	0.00340
1500	24.672	28.6600	154.7489	91.9300	201.3319	20.071	0.03442	57.4203	8.741	0.00178
1600	24.805	31.1339	157.2228	93.5266	188.7531	18.825	0.02917	51.9554	8.196	0.00092
1700	24.916	33.6199	159.7088	95.0337	177.6504	17.726	0.02422	47.1333	7.715	0.00009
1800	25.010	36.1162	162.2051	96.4605	167.7784	16.748	0.02046	42.8471	7.287	0.00026
1900	25.090	38.6212	164.7101	97.8148	158.9432	15.872	0.01760	39.0121	6.904	0.00070
2000	25.159	41.1337	167.2226	99.1036	150.9896	15.083	0.01532	35.5608	6.560	0.00131
2100	25.221	43.6527	169.7416	100.3326	143.7919	14.368	0.01369	32.4383	6.248	0.00160
2200	25.277	46.1776	172.2665	101.5071	137.2473	13.718	0.01235	29.5999	5.965	0.00198
2300	25.329	48.7079	174.7968	102.6319	131.2706	13.127	0.00992	27.0084	5.705	0.00278
2400	25.374	51.2430	177.3319	103.7108	125.7911	12.587	0.00710	24.6331	5.468	0.00210
2500	25.408	53.7821	179.8710	104.7473	120.7492	12.087	0.00565	22.4480	5.249	0.00195
2600	25.439	56.3245	182.4134	105.7445	116.0947	11.626	0.00418	20.4311	5.047	0.00196
2700	25.466	58.8697	184.9586	106.7050	111.7846	11.197	0.00367	18.5638	4.861	0.00227
2800	25.491	61.4176	187.5065	107.6316	107.7820	10.802	0.00177	16.8300	4.687	0.00221
2900	25.514	63.9678	190.0567	108.5266	104.0554	10.432	0.00097	15.2160	4.525	0.00203
3000	25.534	66.5202	192.6091	109.3919	100.5771	10.087	0.00003	13.7097	4.374	0.00207
3100	25.562	69.0745	195.1634	110.2294	97.3232	9.764	0.00065	12.3008	4.233	0.00201
3200	25.570	71.6306	197.7195	111.0409	94.2726	9.463	0.00197	10.9800	4.100	0.00182
3300	25.585	74.1884	200.2773	111.8280	91.4070	9.178	0.00261	9.7394	3.976	0.00191
3400	25.599	76.7476	202.8365	112.5920	88.7102	8.910	0.00301	8.5719	3.858	0.00159
3500	25.613	79.3082	205.3971	113.3342	86.1675	8.659	0.00418	7.4712	3.748	0.00161
3600	25.624	81.8700	207.9589	114.0560	83.7664	8.420	0.00447	6.4317	3.644	0.00166
3700	25.635	84.4330	210.5219	114.7582	81.4952	8.195	0.00528	5.4485	3.545	0.00159
3800	25.645	86.9970	213.0859	115.4419	79.3439	7.980	0.00535	4.5172	3.452	0.00153
3900	25.655	89.5620	215.6509	116.1082	77.3031	7.779	0.00635	3.6336	3.362	0.00130
4000	25.663	92.1279	218.2168	116.7578	75.3647	7.585	0.00640	2.7944	3.278	0.00121
4100	25.671	94.6946	220.7835	117.3916	73.5211	7.401	0.00664	1.9961	3.198	0.00123
4200	25.679	97.2621	223.3510	118.0103	71.7656	7.226	0.00696	1.2359	3.122	0.00125
4300	25.687	99.8304	225.9193	118.6147	70.0921	7.060	0.00755	0.5111	3.049	0.00115
4400	25.693	102.3994	228.4883	119.2053	68.4951	6.900	0.00753	0.1807	2.979	0.00090
4500	25.699	104.9690	231.0579	119.7827	66.9693	6.748	0.00776	0.8418	2.912	0.00084
4600	25.705	107.5392	233.6281	120.3476	65.5101	6.602	0.00808	0.14740	2.849	0.00087
4700	25.710	110.1099	236.1988	120.9005	64.1135	6.461	0.00784	0.20793	2.788	0.00073
4800	25.716	112.6812	238.7701	121.4418	62.7753	6.328	0.00823	0.26594	2.730	0.00074
4900	25.720	115.2530	241.3419	121.9721	61.4921	6.198	0.00800	0.32158	2.673	0.00050
5000	25.725	117.8253	243.9142	122.4918	60.2605	6.075	0.00818	0.37499	2.619	0.00043
5100	25.729	120.3980	246.4869	123.0013	59.0775	5.956	0.00828	0.42630	2.568	0.00045
5200	25.733	122.9711	249.0600	123.5009	57.9404	5.842	0.00836	0.47564	2.518	0.00029
5300	25.737	125.5446	251.6335	123.9911	56.8465	5.731	0.00820	0.52312	2.470	0.00021
5400	25.741	128.1185	254.2074	124.4722	55.7934	5.625	0.00823	0.56884	2.424	0.00013
5500	25.745	130.6928	256.7817	124.9446	54.7789	5.524	0.00843	0.61290	2.379	0.00002
5600	25.748	133.2674	259.3563	125.4085	53.8009	5.423	0.00800	0.65538	2.337	0.00010
5700	25.752	135.8424	261.9313	125.8643	52.8575	5.327	0.00785	0.69637	2.295	0.00011
5800	25.756	138.4178	264.5067	126.3122	51.9469	5.235	0.00789	0.73595	2.255	0.00020
5900	25.760	140.9936	267.0825	126.7525	51.0675	5.148	0.00810	0.77419	2.216	0.00027
6000	25.763	143.5698	269.6587	127.1855	50.2176	-----	-----	0.8115	-----	-----

TABLE XIX—THERMODYNAMIC PROPERTIES OF B₂O₃ (CRYSTAL)

[Molecular weight, 69.64]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\log K$
0	-----	0	48.6839	-----	-----	-----
298.16	14.73	2.2410	50.9249	13.07	1137.336	455.554
300	14.79	2.2680	50.9519	13.16	1130.390	452.523
400	18.40	3.9240	52.6079	17.90	848.915	329.717
500	21.12	5.9080	54.5919	22.31	679.678	255.953
600	23.26	8.1300	56.8139	26.36	566.642	206.747
700	25.15	10.5520	59.2359	30.09	485.753	171.589
723.16	25.57	11.1400	59.8239	30.91	470.191	164.833

TABLE XX—THERMODYNAMIC PROPERTIES OF B₂O₃ (LIQUID)

[Molecular weight, 69.64]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_o^\circ$ ($\frac{\text{kcal}}{\text{mole}}$) (s)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
723.16	31.75	16.5600	65.2439	38.41	466.420	-----	-----	164.833	-----	-----
800	31.660	18.9962	67.6801	41.6117	421.294	-----	-----	145.384	-----	-----
900	31.242	22.1413	70.8252	45.3166	374.121	-----	-----	125.064	-----	-----
1000	30.835	25.2452	73.9291	48.5872	336.4027	34,014	-0.07512	108.8151	14,670	-0.06036
1100	30.580	28.3159	76.9998	51.5141	305.5560	30.893	-0.04637	95.5391	13,333	-0.05693
1200	30.442	31.3670	80.0509	54.1690	279.8582	28.291	-0.02131	84.4852	12,216	-0.05142
1300	30.392	34.4087	83.0926	56.6037	258.1172	26.095	-0.00489	75.1397	11,271	-0.04681
1400	30.377	37.4472	86.1311	58.8554	239.4828	24.226	-0.00097	67.1358	10,462	-0.04377
1500	30.372	40.4846	89.1685	60.9511	223.3331	22.610	-0.00015	60.2049	9,760	-0.04040
1600	30.370	43.5217	92.2056	62.9112	209.2020	21.197	-0.00022	54.1453	9,146	-0.03770
1700	30.370	46.5587	95.2426	64.7523	196.7334	19.951	-0.00049	48.8030	8,605	-0.03586
1800	30.370	49.5957	98.2796	66.4882	185.6500	18.842	-0.00024	44.0583	8,123	-0.03346
1900	30.370	52.6327	101.3166	68.1303	175.7334	17.851	-0.00030	39.8165	7,692	-0.03150
2000	30.370	55.6697	104.3536	69.6880	166.8082	16.958	-0.00024	36.0020	7,304	-0.02990
2100	30.370	58.7067	107.3906	71.1698	158.7332	16.151	-0.00036	32.5538	6,954	-0.02861
2200	30.370	61.7437	110.4276	72.5826	151.3922	15.416	-0.00011	29.4215	6,635	-0.02738
2300	30.370	64.7807	113.4646	73.9326	144.6897	14.746	-0.00027	26.5641	6,344	-0.02613
2400	30.370	67.8177	116.5016	75.2251	138.5458	14.131	-0.00010	23.9469	6,077	-0.02500
2500	30.370	70.8547	119.5386	76.4649	132.8935	-----	-----	21.5411	-----	-----

* Enthalpy change in converting B₂O₃ (crystal) at 0° K to B₂O₃ (liquid) at temperature indicated.

TABLE XXI—THERMODYNAMIC PROPERTIES OF C (GAS)

[Atomic weight, 12.010]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_o^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)
0	-----	0	262.3181	-----
298.16	4.9803	1.5589	263.8770	37.7611
300	4.9801	1.5681	263.8862	37.7917
400	4.9747	2.0658	264.3839	39.2235
500	4.9723	2.5631	264.8812	40.3333
600	4.9709	3.0603	265.3784	41.2398
700	4.9701	3.5573	265.8754	42.0060
800	4.9697	4.0543	266.3724	42.6696
900	4.9693	4.5513	266.8694	43.2550
1000	4.9691	5.0482	267.3663	43.7785
1100	4.9691	5.5451	267.8632	44.2521
1200	4.9697	6.0421	268.3602	44.6845
1300	4.9705	6.5391	268.8572	45.0823
1400	4.9725	7.0362	269.3543	45.4507
1500	4.9747	7.5336	269.8517	45.7939
1600	4.9783	8.0312	270.3493	46.1150
1700	4.9835	8.5293	270.8474	46.4170
1800	4.9899	9.0280	271.3461	46.7020
1900	4.9980	9.5274	271.8455	46.9720
2000	5.0075	10.0277	272.3458	47.2287
2100	5.0189	10.5290	272.8471	47.4732
2200	5.0316	11.0315	273.3496	47.7070
2300	5.0455	11.5354	273.8535	47.9310
2400	5.0607	12.0407	274.3588	48.1460
2500	5.0769	12.5476	274.8657	48.3530
2600	5.0941	13.0561	275.3742	48.5524
2700	5.1118	13.5664	275.8845	48.7450
2800	5.1299	14.0785	276.3966	48.9312
2900	5.1486	14.5924	276.9105	49.1116
3000	5.1677	15.1082	277.4263	49.2864
3100	5.1866	15.6259	277.9440	49.4562
3200	5.2055	16.1455	278.4636	49.6212
3300	5.2243	16.6670	278.9851	49.7816
3400	5.2428	17.1904	279.5085	49.9379
3500	5.2610	17.7156	280.0337	50.0901
3600	5.2786	18.2426	280.5607	50.2386
3700	5.2959	18.7713	281.0894	50.3834
3800	5.3126	19.3017	281.6198	50.5249
3900	5.3286	19.8338	282.1519	50.6631
4000	5.3442	20.3674	282.6855	50.7982
4100	5.3590	20.9026	283.2207	50.9303
4200	5.3732	21.4392	283.7573	51.0596
4300	5.3866	21.9772	284.2953	51.1862
4400	5.3994	22.5165	284.8347	51.3102
4500	5.4115	23.0570	285.3751	51.4317
4600	5.4227	23.5987	285.9168	51.5508
4700	5.4331	24.1415	286.4596	51.6675
4800	5.4427	24.6853	287.0034	51.7820
4900	5.4514	25.2300	287.5481	51.8943
5000	5.4592	25.7755	288.0936	52.0045
5100	5.4661	26.3218	288.6399	52.1127
5200	5.4720	26.8687	289.1868	52.2189
5300	5.4770	27.4162	289.7343	52.3231
5400	5.4810	27.9641	290.2822	52.4256
5500	5.4841	28.5123	290.8304	52.5262
5600	5.4865	29.0608	291.3789	52.6250
5700	5.4882	29.6096	291.9277	52.7221
5800	5.4893	30.1585	292.4769	52.8176
5900	5.4898	30.7074	293.0255	52.9114
6000	5.4899	31.2564	293.5745	53.0037

TABLE XXII—THERMODYNAMIC PROPERTIES OF CO (GAS)

[Molecular weight, 28.010]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f^\circ - H_o^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	65.7461	-----	-----	-----	-----	-----	-----	-----
298.16	6.965	2.0727	67.8188	47.301	434.2122	-----	-----	182.2536	-----	-----
300	6.965	2.0855	67.8316	47.342	431.5591	-----	-----	181.0967	-----	-----
400	7.013	2.7838	68.5299	49.352	324.0685	-----	-----	134.2147	-----	-----
500	7.120	3.4900	69.2361	50.927	259.5584	-----	-----	106.0510	-----	-----
600	7.276	4.2095	69.9556	52.238	216.5368	-----	-----	87.2536	-----	-----
700	7.451	4.9458	70.6919	53.373	185.7931	-----	-----	73.8126	-----	-----
800	7.624	5.6998	71.4459	54.379	162.7232	-----	-----	63.7218	-----	-----
900	7.787	6.4706	72.2167	55.287	144.7699	-----	-----	55.8663	-----	-----
1000	7.932	7.2565	73.0026	56.1160	130.3992	12.904	0.03349	49.5767	5640	0.02293
1100	8.058	8.0560	73.8021	56.8779	118.6348	11.736	0.02870	44.4265	5131	0.01927
1200	8.167	8.8673	74.6134	57.5837	108.8261	10.762	0.02534	40.1314	4706	0.01680
1300	8.265	9.6889	75.4350	58.2413	100.5223	9.939	0.02161	36.4946	4346	0.01491
1400	8.349	10.5196	76.2657	58.8569	93.4014	9.235	0.01733	33.3754	4038	0.01310
1500	8.419	11.3580	77.1041	59.4353	87.2274	8.623	0.01492	30.6703	3771	0.01142
1600	8.481	12.2030	77.9491	59.9806	81.8231	8.087	0.01304	28.3020	3537	0.01031
1700	8.536	13.0538	78.7999	60.4964	77.0530	7.615	0.01094	26.2111	3320	0.00960
1800	8.585	13.9099	79.6560	60.9857	72.8115	7.196	0.00863	24.3515	3146	0.00891
1900	8.627	14.7705	80.5166	61.4510	69.0155	6.820	0.00720	22.6868	2982	0.00800
2000	8.665	15.6351	81.3812	61.8945	65.5983	6.482	0.00563	21.1878	2834	0.00748
2100	8.699	16.5033	82.2494	62.3181	62.5060	6.176	0.00443	19.8308	2700	0.00693
2200	8.730	17.3747	83.1208	62.7234	59.6943	5.898	0.00325	18.5866	2578	0.00653
2300	8.758	18.2491	83.9952	63.1121	57.1267	5.644	0.00213	17.4692	2467	0.00608
2400	8.784	19.1262	84.8723	63.4854	54.7729	5.412	0.00100	16.4352	2365	0.00580
2500	8.806	20.0057	85.7518	63.8444	52.6073	5.197	0.00025	15.4834	2271	0.00544
2600	8.827	20.8874	86.6335	64.1902	50.6082	4.999	-0.00048	14.6045	2184	0.00531
2700	8.847	21.7711	87.5172	64.5238	48.7572	4.815	-0.00084	13.7903	2104	0.00507
2800	8.865	22.6567	88.4028	64.8458	47.0384	4.645	-0.00163	13.0338	2029	0.00494
2900	8.882	23.5440	89.2901	65.1572	45.4383	4.486	-0.00203	12.3292	1959	0.00490
3000	8.898	24.4330	90.1791	65.4586	43.9450	4.338	-0.00236	11.6713	1894	0.00483
3100	8.913	25.3246	91.0707	65.7506	42.5480	4.199	-0.00289	11.0555	1834	0.00448
3200	8.927	26.2166	91.9627	66.0338	41.2387	4.070	-0.00353	10.4779	1777	0.00432
3300	8.939	27.1099	92.8560	66.3087	40.0089	3.946	-0.00339	9.9351	1723	0.00444
3400	8.952	28.0044	93.7505	66.5757	38.8517	3.832	-0.00386	9.4239	1673	0.00420
3500	8.963	28.9002	94.6463	66.8354	37.7607	3.723	-0.00417	8.9417	1626	0.00403
3600	8.974	29.7970	95.5431	67.0880	36.7307	3.619	-0.00391	8.4860	1580	0.00417
3700	8.985	30.6950	96.4411	67.3340	35.7565	3.522	-0.00414	8.0548	1538	0.00406
3800	8.995	31.5940	97.3401	67.5738	34.8338	3.429	-0.00403	7.6460	1498	0.00390
3900	9.005	32.4940	98.2401	67.8076	33.9586	3.341	-0.00405	7.2580	1460	0.00380
4000	9.015	33.3950	99.1411	68.0357	33.1274	3.258	-0.00423	6.8892	1423	0.00393
4100	9.024	34.2969	100.0430	68.2584	32.3370	3.177	-0.00383	6.5382	1389	0.00379
4200	9.034	35.1998	100.9459	68.4760	31.5844	3.103	-0.00423	6.2037	1356	0.00365
4300	9.042	36.1036	101.8497	68.6887	30.8670	3.030	-0.00394	5.8847	1325	0.00366
4400	9.051	37.0083	102.7544	68.8966	30.1823	2.961	-0.00400	5.5799	1295	0.00362
4500	9.059	37.9138	103.6599	69.1001	29.5283	2.895	-0.00395	5.2885	1266	0.00358
4600	9.067	38.8201	104.5662	69.2993	28.9029	2.832	-0.00395	5.0097	1239	0.00358
4700	9.074	39.7271	105.4732	69.4944	28.3043	2.770	-0.00358	4.7425	1213	0.00349
4800	9.082	40.6349	106.3810	69.6855	27.7308	2.712	-0.00347	4.4863	1187	0.00356
4900	9.089	41.5435	107.2896	69.8728	27.1808	2.655	-0.00310	4.2405	1164	0.00340
5000	9.096	42.4527	108.1988	70.0565	26.6529	2.601	-0.00300	4.0043	1140	0.00347
5100	9.103	43.3627	109.1088	70.2367	26.1459	2.549	-0.00279	3.7773	1118	0.00340
5200	9.110	44.2733	110.0194	70.4136	25.6585	2.499	-0.00261	3.5589	1097	0.00332
5300	9.117	45.1847	110.9308	70.5872	25.1896	2.451	-0.00239	3.3486	1077	0.00326
5400	9.123	46.0967	111.8428	70.7576	24.7391	2.403	-0.00201	3.1459	1057	0.00322
5500	9.130	47.0093	112.7554	70.9251	24.3032	2.358	-0.00167	2.9505	1037	0.00332
5600	9.137	47.9227	113.6688	71.0897	23.8838	2.316	-0.00172	2.7620	1019	0.00333
5700	9.143	48.8367	114.5828	71.2514	23.4792	2.273	-0.00140	2.5799	1001	0.00321
5800	9.150	49.7513	115.4974	71.4105	23.0887	2.234	-0.00135	2.4041	984	0.00322
5900	9.156	50.6666	116.4127	71.5670	22.7114	2.196	-0.00130	2.2341	968	0.00317
6000	9.162	51.5825	117.3286	71.7209	22.3467	-----	-----	2.0696	-----	-----

TABLE XXIII—THERMODYNAMIC PROPERTIES OF CO₂ (GAS)

[Molecular weight, 44.010]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f^\circ - H^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(\frac{-\Delta^\circ H}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	0	-----	-----	-----	-----	-----	-----	-----
298.16	8.874	2.2381	2.2381	51.061	648.2084	-----	-----	267.6053	-----	-----
300	8.894	2.2546	2.2546	51.116	644.2527	-----	-----	265.8787	-----	-----
400	9.871	3.1948	3.1948	53.815	483.9359	-----	-----	195.8795	-----	-----
500	10.662	4.2228	4.2228	56.113	287.6422	-----	-----	153.8214	-----	-----
600	11.311	5.3224	5.3224	58.109	323.3792	-----	-----	125.7469	-----	-----
700	11.849	6.4813	6.4813	59.895	277.4307	-----	-----	105.6751	-----	-----
800	12.300	7.6894	7.6894	61.507	242.9362	-----	-----	90.6085	-----	-----
900	12.678	8.9399	8.9399	62.980	216.0824	-----	-----	78.8821	-----	-----
1000	12.995	10.2220	10.2220	64.3310	194.5822	19.288	0.07095	69.4953	8428	0.02238
1100	13.265	11.5350	11.5350	65.5822	176.9768	17.548	0.05867	61.8111	7666	0.01797
1200	13.490	12.8728	12.8728	66.7461	162.2948	16.097	.04919	55.4048	7031	.01434
1300	13.680	14.2312	14.2312	67.8334	149.8633	14.868	.04210	49.9820	6494	.01103
1400	13.844	15.6074	15.6074	68.8532	139.2012	13.814	.03647	45.3324	6033	.00880
1500	13.988	16.9990	16.9990	69.8132	129.9554	12.900	.03180	41.3016	5633	.00718
1600	14.116	18.4042	18.4042	70.7200	121.8611	12.100	0.02796	37.7738	5283	0.00585
1700	14.230	19.8216	19.8216	71.5792	114.7155	11.395	.02394	34.6603	4974	.00467
1800	14.331	21.2496	21.2496	72.3955	108.3610	10.767	.02116	31.8923	4699	.00394
1900	14.421	22.6872	22.6872	73.1727	102.6730	10.206	.01820	29.4152	4453	.00320
2000	14.502	24.1334	24.1334	73.9145	97.5518	9.701	.01548	27.1855	4231	.00274
2100	14.576	25.5872	25.5872	74.6238	92.9168	9.244	0.01318	25.1680	4030	0.00248
2200	14.643	27.0482	27.0482	75.3034	88.7018	8.828	.01124	23.3337	3848	.00196
2300	14.705	28.5156	28.5156	75.9557	84.8523	8.447	.01002	21.6587	3681	.00175
2400	14.763	29.9890	29.9890	76.5828	81.3227	8.099	.00850	20.1232	3529	.00120
2500	14.817	31.4680	31.4680	77.1865	78.0746	7.778	.00716	18.7104	3388	.00112
2600	14.868	32.9522	32.9522	77.7687	75.0739	7.482	0.00599	17.4062	3258	0.00093
2700	14.916	34.4414	34.4414	78.3307	72.2988	7.208	.00491	16.1986	3138	.00069
2800	14.961	35.9353	35.9353	78.8740	69.7196	6.955	.00322	15.0772	3026	.00065
2900	15.003	37.4335	37.4335	79.3997	67.3181	6.717	.00260	14.0331	2922	.00060
3000	15.043	38.9358	38.9358	79.9090	65.0765	6.496	.00172	13.0585	2825	.00041
3100	15.081	40.4420	40.4420	80.4029	62.9793	6.289	0.00079	12.1468	2734	0.00032
3200	15.117	41.9519	41.9519	80.8822	61.0132	6.094	.00043	11.2921	2649	.00027
3300	15.152	43.4654	43.4654	81.3480	59.1661	5.912	.00052	10.4891	2568	.00051
3400	15.185	44.9822	44.9822	81.8008	57.4278	5.740	.00100	9.7333	2493	.00031
3500	15.216	46.5022	46.5022	82.2414	55.7888	5.578	.00154	9.0207	2421	.00050
3600	15.246	48.0254	48.0254	82.6705	54.2409	5.424	-0.00184	8.3477	2354	0.00048
3700	15.275	49.5514	49.5514	83.0886	52.7768	5.280	.00257	7.7110	2290	.00047
3800	15.302	51.0802	51.0802	83.4963	51.3899	5.141	.00250	7.1079	2230	.00051
3900	15.329	52.6118	52.6118	83.8941	50.0742	5.011	.00295	6.5356	2173	.00045
4000	15.355	54.1460	54.1460	84.2826	48.8244	4.886	.00311	5.9919	2119	.00037
4100	15.380	55.6828	55.6828	84.6620	47.6338	4.767	-0.00310	5.4747	2067	0.00046
4200	15.405	57.2220	57.2220	85.0329	46.5039	4.654	.00322	4.9821	2018	.00040
4300	15.429	58.7637	58.7637	85.3957	45.4248	4.546	.00318	4.5124	1970	.00057
4400	15.452	60.3078	60.3078	85.7507	44.3948	4.443	.00333	4.0641	1926	.00050
4500	15.475	61.8541	61.8541	86.0982	43.4108	4.343	.00313	3.6356	1883	.00055
4600	15.498	63.4028	63.4028	86.4386	42.4698	4.249	-0.00314	3.2257	1842	0.00049
4700	15.520	64.9536	64.9536	86.7721	41.5689	4.158	.00305	2.8333	1803	.00048
4800	15.542	66.5068	66.5068	87.0991	40.7057	4.070	.00271	2.4572	1765	.00060
4900	15.564	68.0620	68.0620	87.4198	39.8778	3.986	.00250	2.0964	1729	.00060
5000	15.586	69.6196	69.6196	87.7344	39.0831	3.905	.00239	1.7500	1694	.00064
5100	15.608	71.1792	71.1792	88.0433	38.3198	3.827	-0.00196	1.4172	1661	0.00068
5200	15.630	72.7412	72.7412	88.3466	37.5858	3.752	.00172	1.0971	1629	.00064
5300	15.652	74.3052	74.3052	88.6445	36.8796	3.680	.00158	.7891	1598	.00067
5400	15.674	75.8716	75.8716	88.9373	36.1997	3.610	.00116	.4925	1568	.00071
5500	15.696	77.4400	77.4400	89.2251	35.5445	3.543	.00088	.2067	1540	.00070
5600	15.718	79.0108	79.0108	89.5081	34.9127	3.478	-0.00068	-0.0690	1512	0.00074
5700	15.740	80.5836	80.5836	89.7865	34.3032	3.416	.00037	-.3350	1486	.00069
5800	15.762	82.1588	82.1588	90.0604	33.7146	3.356	.00031	-.5919	1460	.00074
5900	15.784	83.7360	83.7360	90.3301	33.1461	3.298	.00007	-.8401	1435	.00073
6000	15.806	85.3156	85.3156	90.5955	32.5965	-----	-----	-1.0900	-----	-----

TABLE XXIV—THERMODYNAMIC PROPERTIES OF
Cl (GAS)

[Atomic weight, 33.457]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)
0	-----	0	32.5131	-----
298.16	5.2203	1.4991	34.0122	39.4569
300	5.2237	1.5087	34.0218	39.4890
400	5.3705	2.0391	34.5522	41.0138
500	5.4363	2.5801	35.0932	42.2206
600	5.4448	3.1244	35.6375	43.2132
700	5.4232	3.6880	36.1811	44.0511
800	5.3887	4.2086	36.7217	44.7731
900	5.3506	4.7456	37.2587	45.4056
1000	5.3133	5.2788	37.7919	45.9674
1100	5.2788	5.8084	38.3215	46.4722
1200	5.2477	6.3347	38.8478	46.9302
1300	5.2201	6.8581	39.3712	47.3491
1400	5.1958	7.3789	39.8920	47.7351
1500	5.1745	7.8974	40.4105	48.0928
1600	5.1557	8.4139	40.9270	48.4262
1700	5.1392	8.9286	41.4417	48.7383
1800	5.1246	9.4418	41.9549	49.0316
1900	5.1117	9.9536	42.4667	49.3083
2000	5.1002	10.4642	42.9773	49.5702
2100	5.0900	10.9737	43.4868	49.8188
2200	5.0809	11.4823	43.9954	50.0554
2300	5.0727	11.9900	44.5031	50.2811
2400	5.0654	12.4969	45.0100	50.4968
2500	5.0588	13.0031	45.5162	50.7034
2600	5.0528	13.5087	46.0218	50.9017
2700	5.0474	14.0137	46.5268	51.0923
2800	5.0425	14.5182	47.0313	51.2758
2900	5.0380	15.0222	47.5353	51.4527
3000	5.0339	15.5258	48.0389	51.6234
3100	5.0301	16.0290	48.5421	51.7884
3200	5.0267	16.5318	49.0449	51.9480
3300	5.0235	17.0343	49.5474	52.1027
3400	5.0206	17.5365	50.0496	52.2526
3500	5.0179	18.0385	50.5516	52.3981
3600	5.0154	18.5401	51.0532	52.5394
3700	5.0131	19.0416	51.5547	52.6768
3800	5.0109	19.5428	52.0559	52.8105
3900	5.0089	20.0437	52.5568	52.9406
4000	5.0070	20.5445	53.0576	53.0674
4100	5.0052	21.0452	53.5583	53.1910
4200	5.0035	21.5456	54.0587	53.3116
4300	5.0020	22.0459	54.5590	53.4293
4400	5.0006	22.5460	55.0591	53.5443
4500	4.9993	23.0460	55.5591	53.6566
4600	4.9981	23.5459	56.0590	53.7665
4700	4.9970	24.0456	56.5587	53.8740
4800	4.9960	24.5453	57.0584	53.9792
4900	4.9950	25.0448	57.5579	54.0822
5000	4.9941	25.5443	58.0574	54.1831
5100	4.9932	26.0436	58.5567	54.2820
5200	4.9924	26.5429	59.0560	54.3789
5300	4.9916	27.0421	59.5552	54.4740
5400	4.9908	27.5412	60.0543	54.5673
5500	4.9901	28.0403	60.5534	54.6589
5600	4.9894	28.5393	61.0524	54.7488
5700	4.9887	29.0382	61.5513	54.8371
5800	4.9880	29.5370	62.0501	54.9239
5900	4.9873	30.0358	62.5489	55.0091
6000	4.9866	30.5345	63.0476	55.0929

TABLE XXV—THERMODYNAMIC PROPERTIES OF Cl₂ (GAS)

[Molecular weight, 70.914]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta\left(-\frac{\Delta H^\circ}{RT}\right) = \frac{-\delta T}{100}\left(\frac{a}{T} + b\right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100}\left(\frac{c}{T} + d\right)$	
						a	b		c	d
0	-----	0	7.8061	-----	-----	-----	-----	-----	-----	-----
298.16	8.11	2.1939	10.0000	53.286	97.9319	-----	-----	36.9304	-----	-----
300	8.12	2.2089	10.0150	53.336	97.3383	-----	-----	36.6695	-----	-----
400	8.44	3.0384	10.8445	55.720	73.2947	-----	-----	26.0820	-----	-----
500	8.62	3.8920	11.6981	57.625	58.8656	-----	-----	19.7044	-----	-----
600	8.74	4.7610	12.5671	59.207	49.2389	-----	-----	15.4354	-----	-----
700	8.82	5.6399	13.4460	60.562	42.3545	-----	-----	12.3755	-----	-----
800	8.88	6.5248	14.3309	61.744	37.1837	-----	-----	10.0725	-----	-----
900	8.92	7.4142	15.2203	62.792	33.1554	-----	-----	8.2757	-----	-----
1000	8.96	8.3090	16.1151	63.7350	29.9262	2884	0.02478	6.8337	1282	0.01765
1100	8.99	9.2065	17.0126	64.5904	27.2796	2624	0.02283	5.6506	1168	0.01517
1200	9.02	10.1070	17.9131	65.3739	25.0701	2410	.01895	4.6621	1073	.01312
1300	9.04	11.0100	18.8161	66.0967	23.1973	2227	.01719	3.8236	992	.01183
1400	9.06	11.9150	19.7211	66.7674	21.5894	2070	.01570	3.1032	923	.01047
1500	9.08	12.8220	20.6281	67.3931	20.1937	1930	.01695	2.4774	863	.00923
1600	9.109	13.7315	21.5376	67.9800	18.9705	1817	0.01228	1.9288	810	0.00863
1700	9.124	14.6431	22.4492	68.5327	17.8894	1712	.01109	1.4437	764	.00766
1800	9.139	15.5563	23.3624	69.0547	16.9272	1618	.01042	1.0116	722	.00720
1900	9.155	16.4709	24.2770	69.5492	16.0652	1534	.00990	.6244	686	.00620
2000	9.171	17.3873	25.1934	70.0192	15.2883	1460	.00856	.2752	653	.00545
2100	9.185	18.3051	26.1112	70.4670	14.5845	1391	0.00823	-.0412	623	0.00492
2200	9.200	19.2243	27.0304	70.8946	13.9440	1329	.00767	-.3293	595	.00480
2300	9.215	20.1451	27.9512	71.3039	13.3585	1272	.00740	-.5928	570	.00440
2400	9.230	21.0673	28.8734	71.6964	12.8211	1220	.00700	-.8347	547	.00410
2500	9.244	21.9910	29.7971	72.0735	12.3261	1172	.00663	-1.0576	526	.00379
2600	9.259	22.9161	30.7222	72.4364	11.8687	1128	0.00632	-1.2637	506	0.00359
2700	9.273	23.8427	31.6488	72.7860	11.4446	1087	.00599	-1.4547	488	.00331
2800	9.287	24.7707	32.5768	73.1235	11.0504	1049	.00568	-1.6323	471	.00319
2900	9.300	25.7001	33.5062	73.4497	10.6830	1012	.00597	-1.7979	456	.00280
3000	9.315	26.6309	34.4370	73.7652	10.3397	981	.00495	-1.9527	441	.00264
3100	9.327	27.5629	35.3690	74.0708	10.0183	949	0.00514	-2.0976	427	0.00256
3200	9.341	28.4963	36.3024	74.3672	9.7166	919	.00532	-2.2336	415	.00224
3300	9.355	29.4311	37.2372	74.6548	9.4328	893	.00465	-2.3616	402	.00226
3400	9.368	30.3673	38.1734	74.9343	9.1655	866	.00487	-2.4821	391	.00209
3500	9.382	31.3048	39.1109	75.2060	8.9132	843	.00443	-2.5959	381	.00177
3600	9.395	32.2437	40.0498	75.4705	8.6746	819	0.00455	-2.7035	371	0.00163
3700	9.409	33.1839	40.9900	75.7281	8.4487	798	.00430	-2.8054	361	.00160
3800	9.422	34.1254	41.9315	75.9792	8.2344	776	.00458	-2.9020	352	.00144
3900	9.436	35.0683	42.8744	76.2241	8.0309	758	.00410	-2.9937	343	.00155
4000	9.448	36.0125	43.8186	76.4632	7.8373	739	.00406	-3.0810	334	.00154
4100	9.461	36.9579	44.7640	76.6966	7.6530	721	0.00403	-3.1640	327	0.00134
4200	9.474	37.9047	45.7108	76.9248	7.4773	703	.00431	-3.2432	319	.00131
4300	9.488	38.8528	46.6589	77.1479	7.3095	688	.00394	-3.3187	312	.00129
4400	9.501	39.8023	47.6084	77.3662	7.1492	672	.00397	-3.3909	305	.00122
4500	9.514	40.7530	48.5591	77.5798	6.9959	658	.00386	-3.4599	299	.00110
4600	9.527	41.7051	49.5112	77.7891	6.8490	644	0.00378	-3.5260	292	0.00117
4700	9.540	42.6584	50.4645	77.9941	6.7082	630	.00385	-3.5893	286	.00112
4800	9.553	43.6131	51.4192	78.1951	6.5731	617	.00378	-3.6500	281	.00095
4900	9.566	44.5690	52.3751	78.3922	6.4434	605	.00370	-3.7083	275	.00100
5000	9.579	45.5263	53.3324	78.5856	6.3187	593	.00363	-3.7643	270	.00086
5100	9.592	46.4868	54.2909	78.7754	6.1988	581	0.00377	-3.8181	265	0.00084
5200	9.605	47.4447	55.2508	78.9618	6.0833	569	.00384	-3.8699	260	.00084
5300	9.619	48.4059	56.2120	79.1449	5.9721	559	.00378	-3.9198	255	.00078
5400	9.632	49.3684	57.1745	79.3248	5.8648	549	.00368	-3.9678	250	.00085
5500	9.645	50.3323	58.1384	79.5016	5.7613	539	.00365	-4.0141	246	.00077
5600	9.658	51.2974	59.1035	79.6755	5.6614	530	0.00362	-4.0588	242	0.00074
5700	9.671	52.2639	60.0700	79.8466	5.5648	520	.00364	-4.1020	238	.00067
5800	9.684	53.2316	61.0377	80.0149	5.4715	511	.00369	-4.1437	233	.00071
5900	9.697	54.2007	62.0068	80.1806	5.3812	503	.00357	-4.1839	230	.00067
6000	9.710	55.1660	62.9771	80.3436	5.2938	-----	-----	-4.2229	-----	-----

TABLE XXVI—THERMODYNAMIC PROPERTIES OF ClF (GAS)

[Molecular weight, 54.457]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	21.2069	-----	-----	-----	-----	-----	-----	-----
298.16	7.6517	2.1281	23.3350	52.0433	102.1326	-----	-----	38.8196	-----	-----
300	7.6599	2.1422	23.3491	52.0904	101.5154	-----	-----	38.5476	-----	-----
400	8.0382	2.9283	24.1351	54.3491	76.4945	-----	-----	27.5018	-----	-----
500	8.2920	3.7456	24.9525	56.1720	61.4528	-----	-----	20.8444	-----	-----
600	8.4594	4.5838	25.7907	57.6997	51.4044	-----	-----	16.3882	-----	-----
700	8.5723	5.4357	26.6426	59.0127	44.2125	-----	-----	13.1937	-----	-----
800	8.6511	6.2971	27.5040	60.1628	38.8081	-----	-----	10.7898	-----	-----
900	8.7077	7.1652	28.3721	61.1852	34.5972	-----	-----	8.9145	-----	-----
1000	8.7496	8.0381	29.2450	62.1049	31.2228	3018	0.02146	7.4100	1337	0.01885
1100	8.7814	8.9147	30.1216	62.9404	28.4577	2748	0.01730	6.1757	1218	0.01620
1200	8.8060	9.7940	31.0009	63.7055	26.1504	2522	0.01490	5.1445	1119	0.01383
1300	8.8255	10.6756	31.8825	64.4111	24.1955	2331	0.01250	4.2699	1035	0.01211
1400	8.8411	11.5589	32.7658	65.0657	22.5180	2167	0.01073	3.5185	962	0.01127
1500	8.8538	12.4437	33.6506	65.6761	21.0626	2025	0.00908	2.8659	900	0.00980
1600	8.8644	13.3296	34.5365	66.2479	19.7879	1901	0.00747	2.2936	845	0.00904
1700	8.8732	14.2165	35.4234	66.7855	18.6622	1790	0.00696	1.7875	796	0.00848
1800	8.8808	15.1042	36.3111	67.2929	17.6608	1692	0.00617	1.3368	753	0.00778
1900	8.8875	15.9926	37.1995	67.7733	16.7641	1605	0.00500	0.9327	714	0.00730
2000	8.8937	16.8817	38.0886	68.2293	15.9566	1525	0.00491	0.5684	680	0.00649
2100	8.8997	17.7713	38.9782	68.6634	15.2255	1453	0.00465	0.2381	648	0.00635
2200	8.9059	18.6616	39.8685	69.0775	14.5604	1388	0.00412	0.0628	619	0.00607
2300	8.9126	19.5525	40.7594	69.4736	13.9528	1327	0.00448	0.3380	593	0.00562
2400	8.9202	20.4442	41.6511	69.8530	13.3954	1272	0.00430	0.5907	569	0.00530
2500	8.9291	21.3366	42.5435	70.2174	12.8823	1220	0.00477	0.8236	547	0.00492
2600	8.9398	22.2301	43.4370	70.5678	12.4083	1173	0.00476	-1.0389	527	0.00461
2700	8.9525	23.1247	44.3316	70.9054	11.9691	1128	0.00534	-1.2387	508	0.00437
2800	8.9678	24.0207	45.2276	71.2313	11.5609	1086	0.00602	-1.4245	490	0.00433
2900	8.9858	24.9184	46.1253	71.5463	11.1804	1047	0.00650	-1.5978	474	0.00390
3000	9.0072	25.8180	47.0249	71.8513	10.8249	1009	0.00762	-1.7597	459	0.00374
3100	9.0320	26.7200	47.9269	72.1470	10.4918	974	0.00832	-1.9115	445	0.00344
3200	9.0606	27.6246	48.8315	72.4342	10.1791	941	0.00915	-2.0540	432	0.00319
3300	9.0932	28.5323	49.7392	72.7135	9.8848	909	0.01015	-2.1881	419	0.00306
3400	9.1299	29.4435	50.6504	72.9855	9.6073	879	0.01116	-2.3144	408	0.00273
3500	9.1709	30.3585	51.5654	73.2508	9.3450	850	0.01229	-2.4337	397	0.00252
3600	9.2162	31.2779	52.4848	73.5098	9.0966	823	0.01317	-2.5465	387	0.00221
3700	9.2658	32.2020	53.4089	73.7630	8.8610	797	0.01416	-2.6533	377	0.00219
3800	9.3197	33.1312	54.3381	74.0108	8.6371	772	0.01535	-2.7547	368	0.00184
3900	9.3778	34.0661	55.2730	74.2536	8.4238	749	0.01605	-2.8509	359	0.00175
4000	9.4399	35.0070	56.2139	74.4918	8.2205	726	0.01713	-2.9424	351	0.00149
4100	9.5059	35.9543	57.1612	74.7257	8.0263	705	0.01804	-3.0295	343	0.00133
4200	9.5754	36.9084	58.1153	74.9556	7.8404	684	0.01893	-3.1125	336	0.00106
4300	9.6485	37.8696	59.0765	75.1818	7.6624	666	0.01954	-3.1917	329	0.00093
4400	9.7246	38.8382	60.0451	75.4045	7.4915	647	0.02032	-3.2674	322	0.00074
4500	9.8035	39.8146	61.0215	75.6239	7.3274	630	0.02084	-3.3397	316	0.00050
4600	9.8849	40.7990	62.0059	75.8403	7.1696	614	0.02146	-3.4089	310	0.00024
4700	9.9686	41.7917	62.9986	76.0538	7.0175	599	0.02181	-3.4751	305	0.00004
4800	10.0539	42.7928	63.9997	76.2645	6.8709	585	0.02221	-3.5386	299	0.00012
4900	10.1407	43.8026	65.0095	76.4727	6.7293	572	0.02230	-3.5995	294	0.00030
5000	10.2287	44.8210	66.0279	76.6785	6.5926	560	0.02250	-3.6580	289	0.00057
5100	10.3173	45.8483	67.0552	76.8819	6.4603	548	0.02272	-3.7141	284	0.00072
5200	10.4064	46.8845	68.0914	77.0831	6.3322	538	0.02259	-3.7680	280	0.00093
5300	10.4955	47.9296	69.1365	77.2822	6.2081	528	0.02252	-3.8199	275	0.00103
5400	10.5844	48.9836	70.1905	77.4792	6.0878	519	0.02244	-3.8698	270	0.00109
5500	10.6725	50.0465	71.2534	77.6742	5.9710	511	0.02215	-3.9178	267	0.00138
5600	10.7597	51.1181	72.3250	77.8673	5.8576	504	0.02178	-3.9641	263	0.00154
5700	10.8457	52.1983	73.4052	78.0585	5.7474	497	0.02141	-4.0087	259	0.00156
5800	10.9302	53.2871	74.4940	78.2478	5.6403	491	0.02098	-4.0518	255	0.00182
5900	11.0129	54.3843	75.5912	78.4354	5.5361	485	0.02067	-4.0932	252	0.00190
6000	11.0937	55.4896	76.6965	78.6212	5.4346	-----	-----	-4.1333	-----	-----

TABLE XXVII—THERMODYNAMIC PROPERTIES OF F (GAS)

[Molecular weight, 19.00]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ (kcal/mole)	H_f° (kcal/mole)	S_f° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)
0	-----	0	48.2781	-----
298.16	5.4364	1.5580	49.8361	37.9173
300	5.4355	1.5680	49.8461	37.9507
400	5.3612	2.1081	50.3862	39.5050
500	5.2819	2.6401	50.9182	40.6926
600	5.2179	3.1650	51.4431	41.6497
700	5.1692	3.6842	51.9623	42.4502
800	5.1324	4.1992	52.4773	43.1379
900	5.1043	4.7110	52.9891	43.7407
1000	5.0826	5.2203	53.4984	44.2774
1100	5.0655	5.7277	54.0058	44.7610
1200	5.0519	6.2336	54.5117	45.2012
1300	5.0409	6.7382	55.0163	45.6051
1400	5.0318	7.2419	55.5200	45.9783
1500	5.0244	7.7447	56.0228	46.3252
1600	5.0181	8.2468	56.5249	46.6493
1700	5.0129	8.7483	57.0264	46.9534
1800	5.0084	9.2494	57.5275	47.2398
1900	5.0045	9.7501	58.0282	47.5105
2000	5.0012	10.2503	58.5284	47.7671
2100	4.9983	10.7503	59.0284	48.0110
2200	4.9957	11.2500	59.5281	48.2435
2300	4.9935	11.7495	60.0276	48.4655
2400	4.9915	12.2487	60.5268	48.6780
2500	4.9898	12.7478	61.0259	48.8817
2600	4.9882	13.2467	61.5248	49.0774
2700	4.9868	13.7454	62.0235	49.2656
2800	4.9855	14.2441	62.5222	49.4469
2900	4.9844	14.7425	63.0206	49.6219
3000	4.9834	15.2409	63.5190	49.7908
3100	4.9824	15.7392	64.0173	49.9542
3200	4.9816	16.2374	64.5155	50.1124
3300	4.9808	16.7355	65.0136	50.2657
3400	4.9801	17.2336	65.5117	50.4143
3500	4.9794	17.7316	66.0097	50.5587
3600	4.9788	18.2295	66.5076	50.6990
3700	4.9782	18.7273	67.0054	50.8354
3800	4.9777	19.2251	67.5032	50.9681
3900	4.9772	19.7229	68.0010	51.0974
4000	4.9768	20.2206	68.4987	51.2234
4100	4.9764	20.7182	68.9963	51.3463
4200	4.9760	21.2158	69.4939	51.4662
4300	4.9756	21.7134	69.9915	51.5833
4400	4.9753	22.2110	70.4891	51.6977
4500	4.9750	22.7085	70.9866	51.8095
4600	4.9747	23.2060	71.4841	51.9188
4700	4.9744	23.7034	71.9815	52.0258
4800	4.9741	24.2009	72.4790	52.1305
4900	4.9739	24.6983	72.9764	52.2331
5000	4.9737	25.1956	73.4737	52.3336
5100	4.9735	25.6930	73.9711	52.4320
5200	4.9732	26.1903	74.4684	52.5286
5300	4.9731	26.6876	74.9657	52.6234
5400	4.9729	27.1849	75.4630	52.7163
5500	4.9727	27.6822	75.9603	52.8076
5600	4.9725	28.1795	76.4576	52.8972
5700	4.9724	28.6767	76.9548	52.9852
5800	4.9723	29.1740	77.4521	53.0716
5900	4.9721	29.6712	77.9493	53.1566
6000	4.9720	30.1684	78.4465	53.2402

TABLE XXVIII—THERMODYNAMIC PROPERTIES OF F₂ (GAS)

[Molecular weight, 38.00]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	60.9562	-----	-----	-----	-----	-----	-----	-----
298.16	7.5183	2.1137	63.0699	48.5590	61.7763	-----	-----	20.8681	-----	-----
300	7.5262	2.1276	63.0838	48.6053	61.4076	-----	-----	20.7035	-----	-----
400	7.9077	2.9001	63.8563	50.8251	46.4428	-----	-----	14.0101	-----	-----
500	8.1822	3.7055	64.6617	52.6211	37.4145	-----	-----	9.9626	-----	-----
600	8.3704	4.5337	65.4899	54.1306	31.3646	-----	-----	7.2467	-----	-----
700	8.5004	5.3776	66.3338	55.4313	27.0238	-----	-----	5.2959	-----	-----
800	8.5924	6.2325	67.1887	56.5727	23.7560	-----	-----	3.8255	-----	-----
900	8.6593	7.0953	68.0515	57.5888	21.2063	-----	-----	2.6768	-----	-----
1000	8.7092	7.9643	68.9205	58.5042	19.1610	1825	0.01791	1.7540	817	0.01517
1100	8.7472	8.8371	69.7933	59.3360	17.4840	1663	0.01427	0.9961	745	0.01297
1200	8.7768	9.7133	70.6695	60.0984	16.0839	1528	0.01142	0.3623	684	0.01205
1300	8.8002	10.5921	71.5483	60.8019	14.8971	1413	0.00931	-1.759	633	0.01076
1400	8.8190	11.4731	72.4293	61.4547	13.8785	1315	0.00733	-3.0388	589	0.00983
1500	8.8326	12.3557	73.3119	62.0636	12.9945	1229	0.00618	-1.0413	551	0.00883
1600	8.8471	13.2397	74.1959	62.6341	12.2202	1154	0.00518	-1.3945	518	0.00799
1700	8.8577	14.1249	75.0811	63.1708	11.5362	1087	0.00461	-1.7072	488	0.00759
1800	8.8666	15.0111	75.9673	63.6774	10.9277	1028	0.00375	-1.9859	462	0.00694
1900	8.8742	15.8982	76.8544	64.1570	10.3829	975	0.00330	-2.2360	438	0.00680
2000	8.8807	16.7859	77.7421	64.6123	9.8921	927	0.00287	-2.4618	417	0.00633
2100	8.8863	17.6742	78.6304	65.0457	9.4478	884	0.00238	-2.6667	398	0.00579
2200	8.8912	18.5631	79.5193	65.4592	9.0436	844	0.00224	-2.8534	381	0.00535
2300	8.8955	19.4525	80.4087	65.8546	8.6744	808	0.00193	-3.0244	365	0.00512
2400	8.8993	20.3422	81.2984	66.2332	8.3358	775	0.00170	-3.1816	350	0.00500
2500	8.9026	21.2323	82.1885	66.5966	8.0241	744	0.00165	-3.3266	336	0.00497
2600	8.9056	22.1227	83.0789	66.9458	7.7363	716	0.00152	-3.4608	324	0.00460
2700	8.9082	23.0134	83.9696	67.2819	7.4696	690	0.00117	-3.5854	313	0.00431
2800	8.9106	23.9043	84.0605	67.6060	7.2220	666	0.00104	-3.7015	302	0.00416
2900	8.9127	24.7955	85.1517	67.9187	6.9913	643	0.00107	-3.8098	292	0.00397
3000	8.9146	25.6869	86.6431	68.2209	6.7759	622	0.00085	-3.9111	283	0.00371
3100	8.9164	26.5784	87.5346	68.5132	6.5744	602	0.00088	-4.0061	275	0.00346
3200	8.9180	27.4701	88.4263	68.7963	6.3854	584	0.00063	-4.0955	266	0.00349
3300	8.9194	28.3620	89.3182	69.0708	6.2078	566	0.00063	-4.1796	258	0.00352
3400	8.9209	29.2540	90.2102	69.3370	6.0407	550	0.00056	-4.2590	251	0.00339
3500	8.9220	30.1462	91.1024	69.5956	5.8830	534	0.00057	-4.3341	244	0.00322
3600	8.9231	31.0384	91.9946	69.8470	5.7341	520	0.00046	-4.4051	238	0.00308
3700	8.9241	31.9308	92.8870	70.0915	5.5931	506	0.00034	-4.4725	232	0.00295
3800	8.9250	32.8232	93.7794	70.3295	5.4596	492	0.00055	-4.5365	226	0.00295
3900	8.9259	33.7158	94.6720	70.5614	5.3329	480	0.00040	-4.5974	221	0.00275
4000	8.9267	34.6084	95.5646	70.7873	5.2125	468	0.00035	-4.6554	215	0.00276
4100	8.9274	35.5011	96.4573	71.0078	5.0980	457	0.00039	-4.7106	210	0.00280
4200	8.9281	36.3939	97.3501	71.2229	4.9888	446	0.00028	-4.7634	206	0.00259
4300	8.9288	37.2867	98.2429	71.4330	4.8848	436	0.00021	-4.8139	201	0.00262
4400	8.9294	38.1796	99.1358	71.6383	4.7855	426	0.00023	-4.8622	196	0.00264
4500	8.9300	39.0726	100.0288	71.8390	4.6906	417	0.00025	-4.9084	192	0.00256
4600	8.9305	39.9656	100.9218	72.0352	4.5997	407	0.00030	-4.9527	188	0.00260
4700	8.9310	40.8587	101.8149	72.2273	4.5128	399	0.00028	-4.9953	184	0.00247
4800	8.9315	41.7518	102.7080	72.4153	4.4294	391	0.00010	-5.0361	181	0.00246
4900	8.9319	42.6450	103.6012	72.5995	4.3495	383	0.00020	-5.0755	177	0.00240
5000	8.9323	43.5382	104.4944	72.7800	4.2727	375	0.00027	-5.1133	174	0.00228
5100	8.9327	44.4315	105.3877	72.9568	4.1989	368	0.00013	-5.1497	171	0.00222
5200	8.9331	45.3248	106.2810	73.1303	4.1280	361	0.00019	-5.1848	168	0.00220
5300	8.9334	46.2181	107.1743	73.3005	4.0597	354	0.00014	-5.2187	165	0.00214
5400	8.9337	47.1114	108.0676	73.4674	3.9940	348	0.00013	-5.2514	161	0.00223
5500	8.9340	48.0048	108.9610	73.6314	3.9306	341	0.00021	-5.2829	159	0.00221
5600	8.9343	48.8982	109.8544	73.7924	3.8695	335	0.00023	-5.3135	156	0.00213
5700	8.9346	49.7917	110.7479	73.9505	3.8105	329	0.00018	-5.3430	153	0.00222
5800	8.9349	50.6852	111.6414	74.1059	3.7536	324	0.00008	-5.3716	150	0.00218
5900	8.9351	51.5787	112.5349	74.2586	3.6986	318	0.00020	-5.3992	148	0.00213
6000	8.9353	52.4722	113.4284	74.4088	3.6454	-----	-----	-5.4260	-----	-----

TABLE XXIX—THERMODYNAMIC PROPERTIES OF H (GAS)

[Atomic weight, 1.008]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_T^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_T° ($\frac{\text{kcal}}{\text{mole}}$)	S_T° ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)
0	-----	0	85.3285	-----
298.16	4.9680	1.4812	86.8097	27.3927
300	4.9680	1.4904	86.8189	27.4232
400	4.9680	1.9872	87.3157	28.8524
500	4.9680	2.4840	87.8125	29.9610
600	4.9680	2.9808	88.3093	30.8667
700	4.9680	3.4776	88.8061	31.6326
800	4.9680	3.9744	89.3029	32.2959
900	4.9680	4.4712	89.7997	32.8811
1000	4.9680	4.9680	90.2965	33.4045
1100	4.9680	5.4648	90.7933	33.8780
1200	4.9680	5.9616	91.2901	34.3103
1300	4.9680	6.4584	91.7869	34.7079
1400	4.9680	6.9552	92.2837	35.0761
1500	4.9680	7.4520	92.7805	35.4188
1600	4.9680	7.9488	93.2773	35.7395
1700	4.9680	8.4456	93.7741	36.0407
1800	4.9680	8.9424	94.2709	36.3246
1900	4.9680	9.4392	94.7677	36.5932
2000	4.9680	9.9360	95.2645	36.8480
2100	4.9680	10.4328	95.7613	37.0904
2200	4.9680	10.9296	96.2581	37.3215
2300	4.9680	11.4264	96.7549	37.5424
2400	4.9680	11.9232	97.2517	37.7538
2500	4.9680	12.4200	97.7485	37.9566
2600	4.9680	12.9168	98.2453	38.1515
2700	4.9680	13.4136	98.7421	38.3390
2800	4.9680	13.9104	99.2389	38.5196
2900	4.9680	14.4072	99.7357	38.6940
3000	4.9680	14.9040	100.2325	38.8624
3100	4.9680	15.4008	100.7293	39.0253
3200	4.9680	15.8976	101.2261	39.1830
3300	4.9680	16.3944	101.7229	39.3359
3400	4.9680	16.8912	102.2197	39.4842
3500	4.9680	17.3880	102.7165	39.6282
3600	4.9680	17.8848	103.2133	39.7681
3700	4.9680	18.3816	103.7101	39.9043
3800	4.9680	18.8784	104.2069	40.0368
3900	4.9680	19.3752	104.7037	40.1658
4000	4.9680	19.8720	105.2005	40.2916
4100	4.9680	20.3688	105.6973	40.4142
4200	4.9680	20.8656	106.1941	40.5340
4300	4.9680	21.3624	106.6909	40.6509
4400	4.9680	21.8592	107.1877	40.7651
4500	4.9680	22.3560	107.6845	40.8767
4600	4.9680	22.8528	108.1813	40.9859
4700	4.9680	23.3496	108.6781	41.0928
4800	4.9680	23.8464	109.1749	41.1973
4900	4.9680	24.3432	109.6717	41.2998
5000	4.9680	24.8400	110.1685	41.4002
5100	4.9680	25.3368	110.6653	41.4985
5200	4.9680	25.8336	111.1621	41.5950
5300	4.9680	26.3304	111.6589	41.6896
5400	4.9680	26.8272	112.1557	41.7825
5500	4.9680	27.3240	112.6525	41.8736
5600	4.9680	27.8208	113.1493	41.9632
5700	4.9680	28.3176	113.6461	42.0511
5800	4.9680	28.8144	114.1429	42.1375
5900	4.9680	29.3112	114.6397	42.2224
6000	4.9680	29.8080	115.1365	42.3059

TABLE XXX—THERMODYNAMIC PROPERTIES OF H₂ (GAS)

[Molecular weight, 2.016]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	67.4169	-----	-----	-----	-----	-----	-----	-----
298.16	6.892	2.0238	69.4407	31.211	175.8297	-----	-----	71.2098	-----	-----
300	6.895	2.0365	69.4534	31.253	174.7610	-----	-----	70.7414	-----	-----
400	6.974	2.7310	70.1479	33.250	131.4465	-----	-----	51.7421	-----	-----
500	6.993	3.4295	70.8464	34.809	105.4544	-----	-----	40.3099	-----	-----
600	7.008	4.1286	71.5455	36.084	88.1261	-----	-----	32.6669	-----	-----
700	7.035	4.8315	72.2484	37.167	75.7454	-----	-----	27.1921	-----	-----
800	7.078	5.5374	72.9543	38.108	66.4582	-----	-----	23.0744	-----	-----
900	7.139	6.2480	73.6649	38.946	59.2322	-----	-----	19.8636	-----	-----
1000	7.219	6.9658	74.3827	39.7040	53.4478	5185	0.02306	17.2883	2291	0.03007
1100	7.310	7.6923	75.1092	40.3963	48.7111	4712	0.02453	15.1755	2087	0.02583
1200	7.407	8.4281	75.8450	41.0365	44.7599	4318	.02556	13.4105	1916	.02315
1300	7.509	9.1739	76.5908	41.6334	41.4128	3985	.02637	11.9135	1772	.02029
1400	7.615	9.9301	77.3470	42.1938	38.5400	3700	.02653	10.6275	1648	.01833
1500	7.7202	10.6969	78.1138	42.7227	36.0468	3454	.02605	9.5105	1541	.01628
1600	7.8232	11.4740	78.8909	43.2243	33.8620	3240	0.02502	8.5311	1447	0.01472
1700	7.9229	12.2613	79.6782	43.7016	31.9311	3051	.02400	7.6652	1364	.01332
1800	8.0185	13.0584	80.4753	44.1571	30.2121	2884	.02261	6.8941	1291	.01173
1900	8.1093	13.8648	81.2817	44.5931	28.6716	2734	.02170	6.2029	1225	.01060
2000	8.1949	14.6800	82.0969	45.0112	27.2829	2600	.02031	5.5798	1166	.00946
2100	8.2762	15.5036	82.9205	45.4130	26.0245	2478	0.01954	5.0151	1112	0.00865
2200	8.3537	16.3351	83.7520	45.7998	24.8786	2367	.01867	4.5010	1063	.00793
2300	8.4274	17.1741	84.5910	46.1728	23.8308	2267	.01752	4.0309	1018	.00733
2400	8.4977	18.0204	85.4373	46.5329	22.8687	2174	.01690	3.5994	978	.00680
2500	8.5647	18.8735	86.2904	46.8812	21.9822	2089	.01604	3.2018	940	.00636
2600	8.6286	19.7331	87.1500	47.2183	21.1627	2011	0.01519	2.8344	905	0.00541
2700	8.6896	20.5991	88.0160	47.5451	20.4027	1938	.01466	2.4938	873	.00481
2800	8.7479	21.4709	88.8878	47.8622	19.6959	1870	.01417	2.1772	843	.00441
2900	8.8042	22.3485	89.7654	48.1702	19.0369	1807	.01377	1.8821	815	.00403
3000	8.8587	23.2317	90.6486	48.4696	18.4208	1748	.01323	1.6064	789	.00368
3100	8.9118	24.1202	91.5371	48.7609	17.8437	1693	0.01294	1.3482	764	0.00335
3200	8.9636	25.0140	92.4309	49.0447	17.3017	1640	.01283	1.1059	741	.00325
3300	9.0143	25.9129	93.3298	49.3213	16.7919	1591	.01266	.8781	719	.00313
3400	9.0639	26.8168	94.2337	49.5911	16.3113	1546	.01219	.6635	699	.00279
3500	9.1125	27.7256	95.1425	49.8545	15.8574	1502	.01208	.4610	680	.00241
3600	9.1602	28.6392	96.0561	50.1119	15.4281	1461	0.01184	0.2697	662	0.00228
3700	9.2070	29.5576	96.9745	50.3635	15.0214	1423	.01153	.0885	644	.00223
3800	9.2529	30.4806	97.8975	50.6097	14.6354	1386	.01132	-.0832	628	.00197
3900	9.2979	31.4081	98.8250	50.8506	14.2687	1351	.01125	-.2462	613	.00175
4000	9.3421	32.3401	99.7570	51.0866	13.9197	1318	.01104	-.4012	598	.00165
4100	9.3856	33.2765	100.6934	51.3178	13.5872	1287	0.01067	-.5487	584	0.00145
4200	9.4283	34.2172	101.6341	51.5445	13.2701	1257	.01057	-.6892	571	.00131
4300	9.4704	35.1621	102.5790	51.7668	12.9672	1229	.01038	-.8233	558	.00118
4400	9.5118	36.1113	103.5282	51.9850	12.6775	1201	.01031	-.9513	546	.00097
4500	9.5526	37.0645	104.4814	52.1992	12.4003	1175	.01017	-1.0736	535	.00080
4600	9.5928	38.0217	105.4386	52.4096	12.1347	1150	0.01002	-1.1907	524	0.00071
4700	9.6324	38.9830	106.3999	52.6164	11.8800	1127	.00971	-1.3029	514	.00042
4800	9.6714	39.9482	107.3651	52.8196	11.6355	1104	.00969	-1.4104	503	.00045
4900	9.7099	40.9173	108.3342	53.0194	11.4005	1082	.00950	-1.5135	494	.00030
5000	9.7479	41.8901	109.3070	53.2159	11.1746	1061	.00936	-1.6126	484	.00020
5100	9.7853	42.8668	110.2837	53.4093	10.9572	1041	0.00931	-1.7077	475	0.00015
5200	9.8222	43.8472	111.2641	53.5997	10.7477	1021	.00916	-1.7992	466	.00018
5300	9.8586	44.8312	112.2481	53.7871	10.5459	1003	.00906	-1.8873	458	-.00001
5400	9.8945	45.8189	113.2358	53.9717	10.3511	985	.00891	-1.9721	450	-.00002
5500	9.9299	46.8101	114.2270	54.1536	10.1631	967	.00882	-2.0539	442	-.00013
5600	9.9649	47.8048	115.2217	54.3328	9.9816	951	0.00876	-2.1327	434	-0.00014
5700	9.9994	48.8031	116.2200	54.5095	9.8060	934	.00867	-2.2087	427	-.00012
5800	10.0334	49.8047	117.2216	54.6837	9.6363	919	.00854	-2.2822	420	-.00029
5900	10.0670	50.8097	118.2266	54.8555	9.4720	904	.00843	-2.3531	413	-.00033
6000	10.1001	51.8181	119.2350	55.0250	9.3129	-----	-----	-2.4216	-----	-----

TABLE XXXI—THERMODYNAMIC PROPERTIES OF HCl (GAS)

[Molecular weight, 36.465]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(\frac{-\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	15.5926	-----	-----	-----	-----	-----	-----	-----
298.16	6.96	2.0648	17.6574	44.617	174.1180	-----	-----	70.7596	-----	-----
300	6.96	2.0778	17.6704	44.661	173.0598	-----	-----	70.2960	-----	-----
400	6.97	2.7740	18.3666	46.656	130.2113	-----	-----	51.4775	-----	-----
500	7.00	3.4730	19.0656	48.224	104.5100	-----	-----	40.1522	-----	-----
600	7.07	4.1766	19.7692	49.506	87.3747	-----	-----	32.5758	-----	-----
700	7.17	4.8881	20.4807	50.603	75.1291	-----	-----	27.1468	-----	-----
800	7.29	5.6112	21.2038	51.568	65.9356	-----	-----	23.0623	-----	-----
900	7.42	6.3468	21.9394	52.434	58.7762	-----	-----	19.8761	-----	-----
1000	7.554	7.0950	22.6876	53.2220	53.0404	5124	0.04282	17.3201	2274	0.02943
1100	7.690	7.8572	23.4498	53.9484	48.3394	4661	0.04023	15.2234	2071	0.02567
1200	7.819	8.6326	24.2252	54.6230	44.4150	4277	.03660	13.4719	1903	.02145
1300	7.938	9.4205	25.0131	55.2536	41.0884	3952	.03354	11.9866	1700	.01876
1400	8.046	10.2197	25.8123	55.8458	38.2320	3676	.02903	10.7107	1537	.01667
1500	8.140	11.0290	26.6216	56.4041	35.7573	3437	.02498	9.6027	1350	.01505
1600	8.221	11.8470	27.4396	56.9320	33.5792	3227	0.02197	8.6314	1437	0.01331
1700	8.292	12.6727	28.2653	57.4325	31.6590	3040	.02031	7.7728	1355	.01172
1800	8.358	13.5052	29.0978	57.9084	29.9498	2871	.02045	7.0083	1282	.01036
1900	8.426	14.3444	29.9370	58.3621	28.4183	2723	.01870	6.3232	1216	.00960
2000	8.485	15.1901	30.7827	58.7958	27.0381	2590	.01717	5.7056	1157	.00855
2100	8.545	16.0417	31.6343	59.2114	25.7876	2471	0.01502	5.1461	1103	0.00804
2200	8.595	16.8987	32.4913	59.6100	24.6494	2361	.01398	4.6367	1055	.00710
2300	8.643	17.7606	33.3532	59.9932	23.6089	2262	.01240	4.1709	1010	.00667
2400	8.685	18.6270	34.2196	60.3619	22.6540	2169	.01190	3.7434	969	.00620
2500	8.726	19.4976	35.0902	60.7172	21.7745	2086	.01039	3.3496	931	.00582
2600	8.762	20.3720	35.9646	61.0602	20.9618	2007	0.00997	2.9857	896	0.00545
2700	8.796	21.2499	36.8425	61.3915	20.2085	1934	.00939	2.6484	864	.00513
2800	8.829	22.1311	37.7237	61.7120	19.5084	1868	.00846	2.3347	834	.00471
2900	8.858	23.0155	38.6081	62.0223	18.8558	1805	.00783	2.0424	806	.00433
3000	8.885	23.9026	39.4952	62.3231	18.2463	1745	.00700	1.7693	780	.00409
3100	8.912	24.7925	40.3851	62.6148	17.6756	1691	0.00706	1.5136	756	0.00375
3200	8.937	25.6849	41.2775	62.8982	17.1401	1639	.00683	1.2736	733	.00358
3300	8.961	26.5798	42.1724	63.1736	16.6366	1591	.00626	1.0479	711	.00348
3400	8.983	27.4770	43.0696	63.4414	16.1624	1545	.00607	.8353	691	.00327
3500	9.004	28.3764	43.9690	63.7021	15.7149	1502	.00578	.6346	671	.00321
3600	9.024	29.2778	44.8704	63.9560	15.2919	1462	0.00527	0.4450	654	0.00284
3700	9.043	30.1811	45.7737	64.2035	14.8915	1421	.00565	.2654	636	.00283
3800	9.063	31.0864	46.6790	64.4450	14.5119	1386	.00502	.0952	620	.00273
3900	9.081	31.9936	47.5862	64.6806	14.1515	1352	.00460	-.0665	604	.00270
4000	9.098	32.8026	48.4952	64.9108	13.8089	1318	.00474	-.2202	590	.00250
4100	9.115	33.8132	49.4058	65.1356	13.4827	1287	0.00437	-0.3666	576	0.00226
4200	9.131	34.7255	50.3181	65.3554	13.1719	1256	.00451	-.5060	563	.00217
4300	9.147	35.6394	51.2320	65.5705	12.8753	1228	.00421	-.6391	551	.00197
4400	9.162	36.5549	52.1475	65.7810	12.5920	1202	.00379	-.7693	538	.00194
4500	9.176	37.4718	53.0644	65.9870	12.3211	1174	.00408	-.8878	527	.00183
4600	9.191	38.3901	53.9827	66.1888	12.0618	1150	0.00372	-1.0042	516	0.00171
4700	9.205	39.3099	54.9025	66.3867	11.8134	1127	.00341	-1.1157	505	.00169
4800	9.218	40.2311	55.8237	66.5806	11.5752	1102	.00380	-1.2226	495	.00158
4900	9.232	41.1536	56.7462	66.7708	11.3465	1081	.00340	-1.3252	485	.00160
5000	9.245	42.0774	57.6700	66.9574	11.1269	1060	.00336	-1.4238	476	.00147
5100	9.257	43.0025	58.5951	67.1406	10.9157	1038	0.00348	-1.5186	467	0.00139
5200	9.270	43.9289	59.5215	67.3205	10.7126	1020	.00325	-1.6098	458	.00138
5300	9.282	44.8565	60.4491	67.4972	10.5169	1001	.00313	-1.6976	449	.00145
5400	9.294	45.7853	61.3779	67.6708	10.3284	982	.00325	-1.7822	441	.00142
5500	9.306	46.7153	62.3079	67.8415	10.1466	964	.00316	-1.8638	433	.00138
5600	9.318	47.6465	63.2391	68.0093	9.9713	947	0.00326	-1.9425	426	0.00126
5700	9.330	48.5789	64.1715	68.1743	9.8019	930	.00326	-2.0185	419	.00116
5800	9.342	49.5125	65.1051	68.3367	9.6383	914	.00328	-2.0919	412	.00117
5900	9.354	50.4473	66.0399	68.4965	9.4801	900	.00300	-2.1629	405	.00110
6000	9.365	51.3832	66.9758	68.6538	9.3271	-----	-----	-2.2315	-----	-----

TABLE XXXII—THERMODYNAMIC PROPERTIES OF HF (GAS)

[Molecular weight, 20.008]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	0	0	-----	-----	-----	-----	-----	-----
298.16	6.9615	2.0553	2.0553	41.5114	227.1579	-----	-----	93.4523	-----	-----
300	6.9615	2.0681	2.0681	41.5542	225.7754	-----	-----	92.8472	-----	-----
400	6.9652	2.7645	2.7645	43.5575	169.7599	-----	-----	68.3058	-----	-----
500	6.9715	3.4613	3.4613	45.1124	136.1421	-----	-----	53.5438	-----	-----
600	6.9858	4.1592	4.1592	46.3848	113.7233	-----	-----	43.6784	-----	-----
700	7.0150	4.8592	4.8592	47.4639	97.7043	-----	-----	36.6149	-----	-----
800	7.0627	5.5631	5.5631	48.4038	85.6849	-----	-----	31.3051	-----	-----
900	7.1290	6.2727	6.2727	49.2396	76.3316	-----	-----	27.1660	-----	-----
1000	7.2108	6.9897	6.9897	49.9950	68.8439	6714	0.02746	23.8476	2959	0.03070
1100	7.3038	7.7154	7.7154	50.6866	62.7128	6103	0.02817	21.1269	2694	0.02690
1200	7.4035	8.4508	8.4508	51.3264	57.5988	5594	0.02852	18.8550	2473	0.02359
1300	7.5059	9.1962	9.1962	51.9231	53.2672	5164	0.02823	16.9291	2287	0.02033
1400	7.6084	9.9520	9.9520	52.4831	49.5504	4797	0.02700	15.2752	2126	0.01847
1500	7.7084	10.7178	10.7178	53.0114	46.3254	4479	0.02573	13.8394	1987	0.01653
1600	7.8048	11.4935	11.4935	53.5120	43.5003	4201	0.02452	12.5810	1866	0.01455
1700	7.8967	12.2785	12.2785	53.9879	41.0046	3956	0.02332	11.4688	1758	0.01258
1800	7.9836	13.0726	13.0726	54.4417	38.7835	3739	0.02161	10.4788	1662	0.01063
1900	8.0657	13.8750	13.8750	54.8756	36.7940	3545	0.02030	9.5917	1577	0.00870
2000	8.1427	14.6854	14.6854	55.2912	35.0012	3370	0.01904	8.7922	1500	0.00710
2100	8.2150	15.5033	15.5033	55.6903	33.3774	3212	0.01790	8.0678	1430	0.005920
2200	8.2828	16.3282	16.3282	56.0740	31.8995	3069	0.01655	7.4086	1367	0.00505
2300	8.3464	17.1597	17.1597	56.4436	30.5486	2938	0.01543	6.8059	1309	0.00438
2400	8.4061	17.9973	17.9973	56.8001	29.3090	2818	0.01450	6.2528	1256	0.00380
2500	8.4623	18.8407	18.8407	57.1444	28.1673	2707	0.01385	5.7433	1206	0.00333
2600	8.5152	19.6896	19.6896	57.4773	27.1123	2605	0.01289	5.2726	1161	0.00300
2700	8.5650	20.5436	20.5436	57.7996	26.1346	2510	0.01247	4.8363	1119	0.00275
2800	8.6124	21.4025	21.4025	58.1119	25.2257	2423	0.01148	4.4307	1080	0.00254
2900	8.6572	22.2660	22.2660	58.4149	24.3787	2341	0.01087	4.0528	1044	0.00236
3000	8.7000	23.1338	23.1338	58.7091	23.5875	2264	0.01058	3.6997	1010	0.00219
3100	8.7408	24.0059	24.0059	59.0051	22.8466	2192	0.00992	3.3691	979	0.00202
3200	8.7797	24.8819	24.8819	59.2732	22.1513	2126	0.00945	3.0589	949	0.00186
3300	8.8169	25.7617	25.7617	59.5439	21.4976	2062	0.00933	2.7673	921	0.00172
3400	8.8527	26.6452	26.6452	59.8077	20.8818	2003	0.00881	2.4926	895	0.00159
3500	8.8872	27.5322	27.5322	60.0648	20.3007	1947	0.00847	2.2334	870	0.00147
3600	8.9205	28.4226	28.4226	60.3156	19.7514	1894	0.00821	1.9884	847	0.00136
3700	8.9528	29.3162	29.3162	60.5605	19.2313	1843	0.00820	1.7564	824	0.00126
3800	8.9842	30.2131	30.2131	60.7996	18.7381	1795	0.00794	1.5366	804	0.00117
3900	9.0151	31.1130	31.1130	61.0334	18.2699	1750	0.00780	1.3278	784	0.00109
4000	9.0453	32.0161	32.0161	61.2620	17.8246	1707	0.00756	1.1294	765	0.00102
4100	9.0750	32.9221	32.9221	61.4858	17.4007	1666	0.00733	0.9405	746	0.00096
4200	9.1043	33.8310	33.8310	61.7048	16.9967	1626	0.00756	0.7606	730	0.00091
4300	9.1336	34.7429	34.7429	61.9194	16.6111	1588	0.00759	0.5888	713	0.00086
4400	9.1631	35.6578	35.6578	62.1297	16.2426	1551	0.00773	0.4248	697	0.00081
4500	9.1936	36.5756	36.5756	62.3359	15.8902	1513	0.00849	0.2680	682	0.00077
4600	9.2271	37.4966	37.4966	62.5384	15.5528	1479	0.00872	0.1180	668	0.00072
4700	9.2623	38.4211	38.4211	62.7372	15.2294	1446	0.00915	0.0258	654	0.00067
4800	9.2978	39.3491	39.3491	62.9326	14.9190	1416	0.00902	0.0000	641	0.00063
4900	9.3335	40.2807	40.2807	63.1246	14.6210	1387	0.00910	0.0000	629	0.00059
5000	9.3690	41.2158	41.2158	63.3136	14.3345	1360	0.00893	0.0000	616	0.00055
5100	9.4040	42.1545	42.1545	63.4994	14.0589	1334	0.00876	-0.5450	604	0.00051
5200	9.4386	43.0966	43.0966	63.6824	13.7936	1309	0.00872	-0.6624	592	0.00047
5300	9.4726	44.0422	44.0422	63.8625	13.5379	1285	0.00844	-0.7754	582	0.00043
5400	9.5061	44.9911	44.9911	64.0399	13.2915	1262	0.00845	-0.8843	571	0.00039
5500	9.5391	45.9434	45.9434	64.2146	13.0536	1240	0.00817	-0.9893	561	0.00035
5600	9.5718	46.8989	46.8989	64.3868	12.8240	1218	0.00822	-1.0906	551	0.00031
5700	9.6040	47.8577	47.8577	64.5565	12.6021	1198	0.00795	-1.1883	541	0.00027
5800	9.6358	48.8197	48.8197	64.7238	12.3876	1177	0.00801	-1.2826	532	0.00023
5900	9.6673	49.7848	49.7848	64.8888	12.1801	1158	0.00780	-1.3738	524	0.00019
6000	9.6986	50.7531	50.7531	65.0515	11.9793	-----	-----	-1.4621	-----	-----

TABLE XXXIII—THERMODYNAMIC PROPERTIES OF H₂O (GAS)

[Molecular weight, 18.016]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = -\frac{\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	11.3311	-----	-----	-----	-----	-----	-----	-----
298.16	8.025	2.3677	13.6988	45.106	373.2200	-----	-----	151.5648	-----	-----
300	8.026	2.3820	13.7131	45.154	370.9541	-----	-----	150.5708	-----	-----
400	8.185	3.1940	14.5251	47.490	279.0956	-----	-----	110.2372	-----	-----
500	8.415	4.0255	15.3566	49.344	223.9534	-----	-----	85.9609	-----	-----
600	8.677	4.8822	16.2133	50.903	187.1673	-----	-----	69.7280	-----	-----
700	8.959	5.7715	17.1026	52.269	160.8663	-----	-----	58.0996	-----	-----
800	9.254	6.6896	18.0207	53.490	141.1213	-----	-----	49.3548	-----	-----
900	9.559	7.6347	18.9658	54.599	125.7485	-----	-----	42.5361	-----	-----
1000	9.861	8.6080	19.9391	55.6180	113.4355	11.015	0.07316	37.0674	4875	0.05158
1100	10.145	9.6083	20.9394	56.5712	103.3487	10.019	0.06833	32.5840	4439	0.04493
1200	10.413	10.6362	21.9673	57.4654	94.9312	9.188	.06491	28.8399	4076	.03902
1300	10.668	11.6902	23.0213	58.3090	87.7986	8.486	.06137	25.6655	3768	.03457
1400	10.909	12.7691	24.1002	59.1084	81.6758	7.886	.05696	22.9395	3505	.03013
1500	11.134	13.8712	25.2023	59.8687	76.3615	7.366	.05315	20.5727	3276	.02590
1600	11.343	14.9951	26.3262	60.5939	71.7046	6.913	0.04853	18.4983	3075	0.02428
1700	11.534	16.1389	27.4700	61.2873	67.5896	6.514	.04401	16.6652	2898	.02200
1800	11.708	17.3010	28.6321	61.9515	63.9267	6.160	.03959	15.0332	2741	.01957
1900	11.865	18.4797	29.8108	62.5887	60.6450	5.843	.03590	13.5710	2600	.01770
2000	12.008	19.6733	31.0044	63.2010	57.6876	5.557	.03271	12.2533	2474	.01570
2100	12.138	20.8806	32.2117	63.7900	55.0087	5.299	0.02966	11.0595	2359	0.01423
2200	12.256	22.1003	33.4314	64.3574	52.5704	5.064	.02686	9.9730	2255	.01277
2300	12.364	23.3313	34.6624	64.9045	50.3418	4.849	.02468	8.9798	2160	.01150
2400	12.463	24.5727	35.9038	65.4328	48.2967	4.653	.02220	8.0683	2072	.01060
2500	12.554	25.8235	37.1546	65.9434	46.4133	4.471	.02049	7.2289	1991	.00993
2600	12.638	27.0831	38.4142	66.4374	44.6732	4.304	0.01863	6.4532	1916	0.00927
2700	12.715	28.3508	39.6819	66.9159	43.0605	4.149	.01691	5.7343	1846	.00896
2800	12.786	29.6258	40.9569	67.3796	41.5618	4.005	.01547	5.0661	1781	.00856
2900	12.852	30.9077	42.2388	67.8294	40.1653	3.871	.01407	4.4434	1721	.00803
3000	12.913	32.1960	43.5271	68.2661	38.8609	3.747	.01239	3.8617	1666	.00728
3100	12.968	33.4900	44.8211	68.6904	37.6398	3.630	0.01113	3.3170	1613	0.00694
3200	13.018	34.7893	46.1204	69.1029	36.4943	3.520	.00993	2.8060	1564	.00656
3300	13.064	36.0934	47.4245	69.5042	35.4177	3.417	.00890	2.3255	1517	.00642
3400	13.107	37.4020	48.7331	69.8949	34.4038	3.320	.00793	1.8729	1473	.00624
3500	13.147	38.7147	50.0458	70.2754	33.4473	3.228	.00703	1.4458	1431	.00610
3600	13.184	40.0312	51.3623	70.6463	32.5436	3.142	0.00601	1.0422	1392	0.00588
3700	13.218	41.3513	52.6824	71.0080	31.6884	3.059	.00550	.6601	1355	.00572
3800	13.250	42.6747	54.0058	71.3609	30.8779	2.981	.00484	.2978	1320	.00554
3900	13.280	44.0012	55.3323	71.7054	30.1087	2.907	.00425	-.0462	1287	.00525
4000	13.308	45.3306	56.6617	72.0420	29.3777	2.836	.00379	-.3732	1255	.00520
4100	13.334	46.6627	57.9938	72.3710	28.6822	2.769	0.00321	-0.6845	1266	0.00480
4200	13.358	47.9973	59.3284	72.6926	28.0197	2.705	.00283	-.9812	1198	.00450
4300	13.381	49.3343	60.6654	73.0071	27.3878	2.643	.00262	-1.2643	1170	.00449
4400	13.403	50.6735	62.0046	73.3150	26.7845	2.584	.00238	-1.5347	1144	.00438
4500	13.424	52.0148	63.3459	73.6164	26.2079	2.528	.00204	-1.7933	1119	.00434
4600	13.444	53.3582	64.6893	73.9117	25.6563	2.473	0.00203	-2.0409	1095	0.00412
4700	13.464	54.7036	66.0347	74.2011	25.1281	2.421	.00193	-2.2780	1073	.00396
4800	13.483	56.0510	67.3821	74.4847	24.6218	2.371	.00182	-2.5055	1051	.00381
4900	13.502	57.4002	68.7313	74.7629	24.1361	2.323	.00170	-2.7238	1030	.00370
5000	13.521	58.7414	70.0825	75.0359	23.6698	2.276	.00183	-2.9335	1010	.00356
5100	13.540	60.1044	71.4355	75.3038	23.2217	2.232	0.00177	-3.1351	991	0.00342
5200	13.559	61.4594	72.7905	75.5669	22.7907	2.189	.00168	-3.3291	973	.00322
5300	13.577	62.8162	74.1473	75.8254	22.3760	2.148	.00172	-3.5159	955	.00315
5400	13.596	64.1748	75.5059	76.0794	21.9765	2.108	.00163	-3.6959	938	.00305
5500	13.614	65.5353	76.8664	76.3290	21.5916	2.070	.00166	-3.8695	921	.00304
5600	13.633	66.8977	78.2288	76.5745	21.2203	2.033	0.00163	-4.0370	905	0.00293
5700	13.651	68.2619	79.5930	76.8159	20.8620	1.998	.00162	-4.1987	890	.00285
5800	13.669	69.6279	80.9590	77.0535	20.5159	1.963	.00159	-4.3550	875	.00279
5900	13.687	70.9957	82.3268	77.2873	20.1816	1.930	.00163	-4.5061	860	.00277
6000	13.705	72.3653	83.6964	77.5175	19.8583	-----	-----	-4.6522	-----	-----

TABLE XXXIV—THERMODYNAMIC PROPERTIES OF
 e^- (ELECTRON GAS)[Atomic weight, 5.4847×10^{-4}]

T (°K)	C_p $\left(\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}\right)$	$H_T - H_0$ (kcal/mole)	H_T (kcal/mole)	S_T $\left(\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}\right)$
0	---	0	60.0000	---
298.16	4.9680	1.4812	61.4812	4.9882
300	4.9680	1.4904	61.4904	5.0188
400	4.9680	1.9872	61.9872	6.4480
500	4.9680	2.4840	62.4840	7.5565
600	4.9680	2.9808	62.9808	8.4623
700	4.9680	3.4776	63.4776	9.2281
800	4.9680	3.9744	63.9744	9.8915
900	4.9680	4.4712	64.4712	10.4766
1000	4.9680	4.9680	64.9680	11.0001
1100	4.9680	5.4648	65.4648	11.4736
1200	4.9680	5.9616	65.9616	11.9058
1300	4.9680	6.4584	66.4584	12.3035
1400	4.9680	6.9552	66.9552	12.6717
1500	4.9680	7.4520	67.4520	13.0144
1600	4.9680	7.9488	67.9488	13.3350
1700	4.9680	8.4456	68.4456	13.6362
1800	4.9680	8.9424	68.9424	13.9202
1900	4.9680	9.4392	69.4392	14.1888
2000	4.9680	9.9360	69.9360	14.4436
2100	4.9680	10.4328	70.4328	14.6860
2200	4.9680	10.9296	70.9296	14.9171
2300	4.9680	11.4264	71.4264	15.1379
2400	4.9680	11.9232	71.9232	15.3494
2500	4.9680	12.4200	72.4200	15.5522
2600	4.9680	12.9168	72.9168	15.7470
2700	4.9680	13.4136	73.4136	15.9345
2800	4.9680	13.9104	73.9104	16.1152
2900	4.9680	14.4072	74.4072	16.2895
3000	4.9680	14.9040	74.9040	16.4579
3100	4.9680	15.4008	75.4008	16.6208
3200	4.9680	15.8976	75.8976	16.7786
3300	4.9680	16.3944	76.3944	16.9314
3400	4.9680	16.8912	76.8912	17.0797
3500	4.9680	17.3880	77.3880	17.2237
3600	4.9680	17.8848	77.8848	17.3637
3700	4.9680	18.3816	78.3816	17.4998
3800	4.9680	18.8784	78.8784	17.6323
3900	4.9680	19.3752	79.3752	17.7614
4000	4.9680	19.8720	79.8720	17.8871
4100	4.9680	20.3688	80.3688	18.0098
4200	4.9680	20.8656	80.8656	18.1295
4300	4.9680	21.3624	81.3624	18.2464
4400	4.9680	21.8592	81.8592	18.3606
4500	4.9680	22.3560	82.3560	18.4723
4600	4.9680	22.8528	82.8528	18.5815
4700	4.9680	23.3496	83.3496	18.6883
4800	4.9680	23.8464	83.8464	18.7929
4900	4.9680	24.3432	84.3432	18.8953
5000	4.9680	24.8400	84.8400	18.9957
5100	4.9680	25.3368	85.3368	19.0941
5200	4.9680	25.8336	85.8336	19.1906
5300	4.9680	26.3304	86.3304	19.2852
5400	4.9680	26.8272	86.8272	19.3781
5500	4.9680	27.3240	87.3240	19.4692
5600	4.9680	27.8208	87.8208	19.5587
5700	4.9680	28.3176	88.3176	19.6467
5800	4.9680	28.8144	88.8144	19.7331
5900	4.9680	29.3112	89.3112	19.8180
6000	4.9680	29.8080	89.8080	19.9015

TABLE XXXV—THERMODYNAMIC PROPERTIES OF F- (GAS)

[Atomic weight, 19.00]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = -\frac{\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	11.9781	-----	-----	-----	-----	-----	-----	-----
298.16	4.9680	1.4812	13.4593	34.7682	165.1619	-----	-----	69.9505	-----	-----
300	4.9680	1.4904	13.4685	34.7988	164.1656	-----	-----	69.5105	-----	-----
400	4.9680	1.9872	13.9653	36.2280	123.8037	-----	-----	51.6419	-----	-----
500	4.9680	2.4840	14.4621	37.3366	99.5784	-----	-----	40.8614	-----	-----
600	4.9680	2.9808	14.9589	38.2423	83.4222	-----	-----	33.6357	-----	-----
700	4.9680	3.4776	15.4557	39.0081	71.8780	-----	-----	28.4472	-----	-----
800	4.9680	3.9744	15.9525	39.6715	63.2172	-----	-----	24.5355	-----	-----
900	4.9680	4.4712	16.4493	40.2566	56.4793	-----	-----	21.4775	-----	-----
1000	4.9680	4.9679	16.9460	40.7801	51.0877	4849	0.00402	19.0187	2164	0.05463
1100	4.9680	5.4647	17.4428	41.2536	46.6755	4409	0.00333	16.9968	1972	0.04997
1200	4.9680	5.9615	17.9396	41.6858	42.9980	4042	.00297	15.3035	1812	.04605
1300	4.9680	6.4583	18.4364	42.0835	39.8858	3732	.00219	13.8636	1677	.04234
1400	4.9680	6.9551	18.9332	42.4517	37.2179	3466	.00193	12.6234	1561	.03953
1500	4.9680	7.4519	19.4300	42.7944	34.9053	3235	.00172	11.5432	1461	.03648
1600	4.9680	7.9487	19.9268	43.1150	32.8817	3034	0.00110	10.5936	1373	0.03435
1700	4.9680	8.4455	20.4236	43.4162	31.0959	2855	.00129	9.7516	1296	.03210
1800	4.9680	8.9423	20.9204	43.7002	29.5085	2697	.00103	8.9995	1227	.03021
1900	4.9680	9.4391	21.4172	43.9688	28.0880	2555	.00110	8.3235	1165	.02890
2000	4.9680	9.9359	21.9140	44.2236	26.8094	2428	.00061	7.7121	1110	.02713
2100	4.9680	10.4327	22.4108	44.4660	25.6526	2312	0.00079	7.1564	1059	0.02614
2200	4.9680	10.9295	22.9076	44.6971	24.6009	2207	.00073	6.6489	1014	.02473
2300	4.9680	11.4263	23.4044	44.9179	23.6406	2112	.00040	6.1833	972	.02380
2400	4.9680	11.9231	23.9012	45.1294	22.7602	2024	.00030	5.7545	934	.02270
2500	4.9680	12.4199	24.3980	45.3322	21.9503	1943	.00039	5.3682	899	.02173
2600	4.9680	12.9167	24.8948	45.5270	21.2026	1868	0.00045	4.9907	866	0.02106
2700	4.9680	13.4135	25.3916	45.7145	20.5103	1799	.00040	4.6489	836	.02023
2800	4.9680	13.9103	25.8884	45.8952	19.8674	1735	.00032	4.3301	808	.01958
2900	4.9680	14.4071	26.3852	46.0695	19.2688	1675	.00037	4.0319	782	.01893
3000	4.9680	14.9039	26.8820	46.2379	18.7101	1619	.00044	3.7523	758	.01818
3100	4.9680	15.4007	27.3788	46.4008	18.1874	1567	0.00031	3.4896	735	0.01761
3200	4.9680	15.8975	27.8756	46.5586	17.6974	1518	.00030	3.2423	714	.01704
3300	4.9680	16.3943	28.3724	46.7114	17.2371	1472	.00036	3.0089	694	.01658
3400	4.9680	16.8911	28.8692	46.8598	16.8038	1429	.00021	2.7882	675	.01614
3500	4.9680	17.3879	29.3660	47.0038	16.3953	1388	.00024	2.5792	658	.01542
3600	4.9680	17.8847	29.8628	47.1437	16.0095	1350	0.00014	2.3810	641	0.01506
3700	4.9680	18.3815	30.3596	47.2798	15.6445	1313	.00027	2.1927	625	.01473
3800	4.9680	18.8783	30.8564	47.4123	15.2987	1279	.00005	2.0135	610	.01429
3900	4.9680	19.3751	31.3532	47.5414	14.9707	1246	.00010	1.8428	595	.01415
4000	4.9680	19.8719	31.8500	47.6672	14.6591	1215	.00016	1.6799	582	.01365
4100	4.9680	20.3687	32.3468	47.7898	14.3626	1185	0.00026	1.5243	569	0.01332
4200	4.9680	20.8655	32.8436	47.9095	14.0802	1157	.00013	1.3755	557	.01297
4300	4.9680	21.3623	33.3404	48.0264	13.8110	1130	.00018	1.2330	545	.01274
4400	4.9680	21.8591	33.8372	48.1406	13.5540	1104	.00017	1.0964	534	.01243
4500	4.9680	22.3559	34.3340	48.2523	13.3085	1080	.00012	.9653	524	.01199
4600	4.9680	22.8527	34.8308	48.3615	13.0736	1057	0.00001	0.8394	514	0.01174
4700	4.9680	23.3495	35.3276	48.4683	12.8487	1034	.00008	.7183	504	.01150
4800	4.9680	23.8463	35.8244	48.5729	12.6332	1013	.00007	.6018	494	.01138
4900	4.9680	24.3431	36.3212	48.6754	12.4264	992	.00000	.4896	485	.01110
5000	4.9680	24.8399	36.8180	48.7757	12.2280	972	.00011	.3815	477	.01087
5100	4.9680	25.3367	37.3148	48.8741	12.0373	953	0.00013	0.2771	468	0.01070
5200	4.9680	25.8335	37.8116	48.9706	11.8539	934	.00017	.1764	460	.01051
5300	4.9680	26.3303	38.3084	49.0652	11.6775	917	.00009	.0791	453	.01031
5400	4.9680	26.8271	38.8052	49.1581	11.5076	900	.00016	-.0151	445	.01009
5500	4.9680	27.3239	39.3020	49.2492	11.3438	883	.00012	-.1061	439	.00981
5600	4.9680	27.8207	39.7988	49.3387	11.1860	868	0.00012	-.01943	432	0.00961
5700	4.9680	28.3175	40.2956	49.4267	11.0336	853	.00003	-.2797	425	.00942
5800	4.9680	28.8143	40.7924	49.5131	10.8865	838	.00007	-.3624	419	.00928
5900	4.9680	29.3111	41.2892	49.5980	10.7444	824	.00007	-.4427	413	.00917
6000	4.9680	29.8079	41.7860	49.6815	10.6070	-----	-----	-.5207	-----	-----

TABLE XXXVI—THERMODYNAMIC PROPERTIES OF Li+ (GAS)

[Atomic weight, 6.940]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(\frac{-\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	230.7290	-----	-----	-----	-----	-----	-----	-----
298.16	4.9680	1.4812	232.2102	31.7662	-211.5052	-----	-----	-91.0660	-----	-----
300	4.9680	1.4903	232.2193	31.7967	-210.2229	-----	-----	-90.5026	-----	-----
400	4.9680	1.9871	232.7161	33.2259	-158.2922	-----	-----	-67.6371	-----	-----
500	4.9680	2.4839	233.2129	34.3345	-127.1337	-----	-----	-53.8629	-----	-----
600	4.9680	2.9807	233.7097	35.2403	-106.3614	-----	-----	-44.6436	-----	-----
700	4.9680	3.4775	234.2065	36.0061	-91.5241	-----	-----	-38.0325	-----	-----
800	4.9680	3.9743	234.7033	36.6695	-80.3961	-----	-----	-33.0546	-----	-----
900	4.9680	4.4711	235.2001	37.2546	-71.7410	-----	-----	-29.1679	-----	-----
1000	4.9680	4.9679	235.6969	37.7780	-64.8169	-6232	0.00035	-26.0464	-2761	-0.05380
1100	4.9680	5.4647	236.1937	38.2515	-59.1518	-5665	-0.00017	-23.4826	-2515	-0.04897
1200	4.9680	5.9615	236.6905	38.6838	-54.4308	-5193	-0.00008	-21.3378	-2310	-0.04488
1300	4.9680	6.4583	237.1873	39.0814	-50.4361	-4793	-0.00043	-19.5160	-2136	-0.04179
1400	4.9680	6.9551	237.6841	39.4496	-47.0121	-4451	-0.00017	-17.9485	-1987	-0.03893
1500	4.9680	7.4519	238.1809	39.7924	-44.0446	-4154	-0.00035	-16.5849	-1858	-0.03655
1600	4.9680	7.9487	238.6777	40.1130	-41.4480	-3894	-0.00051	-15.3871	-1745	-0.03433
1700	4.9680	8.4455	239.1745	40.4142	-39.1569	-3664	-0.00104	-14.3263	-1646	-0.03206
1800	4.9680	8.9423	239.6713	40.6981	-37.1203	-3460	-0.00125	-13.3798	-1558	-0.03000
1900	4.9680	9.4391	240.1681	40.9668	-35.2980	-3277	-0.00180	-12.5298	-1479	-0.02840
2000	4.9680	9.9359	240.6649	41.2216	-33.6577	-3112	-0.00220	-11.7619	-1408	-0.02682
2100	4.9680	10.4327	241.1617	41.4640	-32.1736	-2963	-0.00258	-11.0646	-1343	-0.02585
2200	4.9680	10.9295	241.6585	41.6951	-30.8242	-2827	-0.00327	-10.4283	-1285	-0.02440
2300	4.9680	11.4263	242.1553	41.9159	-29.5918	-2702	-0.00407	-9.8452	-1231	-0.02348
2400	4.9680	11.9231	242.6521	42.1273	-28.4619	-2588	-0.00480	-9.3088	-1183	-0.02220
2500	4.9680	12.4199	243.1489	42.3301	-27.4219	-2482	-0.00569	-8.8134	-1138	-0.02111
2600	4.9680	12.9167	243.6457	42.5250	-26.4616	-2384	-0.00674	-8.3546	-1097	-0.02010
2700	4.9680	13.4135	244.1425	42.7125	-25.5719	-2293	-0.00777	-7.9282	-1058	-0.01944
2800	4.9680	13.9103	244.6393	42.8932	-24.7452	-2208	-0.00892	-7.5309	-1023	-0.01844
2900	4.9680	14.4071	245.1361	43.0675	-23.9749	-2129	-0.00983	-7.1597	-990	-0.01770
3000	4.9680	14.9039	245.6329	43.2359	-23.2554	-2055	-0.01090	-6.8120	-959	-0.01695
3100	4.9680	15.4007	246.1297	43.3966	-22.5816	-1986	-0.01173	-6.4857	-930	-0.01639
3200	4.9680	15.8975	246.6265	43.5505	-21.9492	-1921	-0.01268	-6.1787	-903	-0.01566
3300	4.9680	16.3943	247.1233	43.7094	-21.3544	-1860	-0.01354	-5.8894	-878	-0.01506
3400	4.9680	16.8911	247.6201	43.8577	-20.7938	-1802	-0.01444	-5.6161	-854	-0.01440
3500	4.9680	17.3879	248.1169	44.0017	-20.2645	-1748	-0.01524	-5.3577	-831	-0.01397
3600	4.9680	17.8847	248.6137	44.1417	-19.7637	-1698	-0.01568	-5.1129	-810	-0.01348
3700	4.9680	18.3815	249.1105	44.2778	-19.2891	-1649	-0.01645	-4.8805	-790	-0.01291
3800	4.9680	18.8783	249.6073	44.4103	-18.8387	-1603	-0.01707	-4.6597	-771	-0.01231
3900	4.9680	19.3751	250.1041	44.5393	-18.4106	-1561	-0.01745	-4.4497	-753	-0.01195
4000	4.9680	19.8719	250.6009	44.6651	-18.0029	-1520	-0.01787	-4.2495	-736	-0.01149
4100	4.9680	20.3687	251.0977	44.7878	-17.6143	-1481	-0.01838	-4.0585	-720	-0.01097
4200	4.9680	20.8655	251.5945	44.9075	-17.2433	-1444	-0.01869	-3.8761	-705	-0.01045
4300	4.9680	21.3623	252.0913	45.0244	-16.8888	-1410	-0.01895	-3.7017	-690	-0.01008
4400	4.9680	21.8591	252.5881	45.1386	-16.5494	-1376	-0.01922	-3.5348	-676	-0.00968
4500	4.9680	22.3559	253.0849	45.2502	-16.2244	-1345	-0.01941	-3.3749	-663	-0.00927
4600	4.9680	22.8527	253.5817	45.3594	-15.9126	-1315	-0.01961	-3.2215	-650	-0.00890
4700	4.9680	23.3495	254.0785	45.4663	-15.6132	-1286	-0.01978	-3.0743	-639	-0.00838
4800	4.9680	23.8463	254.5753	45.5709	-15.3255	-1259	-0.01986	-2.9328	-626	-0.00814
4900	4.9680	24.3431	255.0721	45.6733	-15.0487	-1233	-0.01990	-2.7969	-616	-0.00770
5000	4.9680	24.8399	255.5689	45.7737	-14.7822	-1208	-0.01994	-2.6660	-605	-0.00737
5100	4.9680	25.3367	256.0657	45.8721	-14.5254	-1185	-0.01982	-2.5400	-595	-0.00708
5200	4.9680	25.8335	256.5625	45.9685	-14.2777	-1163	-0.01967	-2.4185	-585	-0.00672
5300	4.9680	26.3303	257.0593	46.0632	-14.0386	-1142	-0.01942	-2.3014	-576	-0.00633
5400	4.9680	26.8271	257.5561	46.1560	-13.8077	-1122	-0.01930	-2.1884	-568	-0.00593
5500	4.9680	27.3239	258.0529	46.2472	-13.5844	-1103	-0.01904	-2.0792	-558	-0.00576
5600	4.9680	27.8207	258.5497	46.3367	-13.3684	-1084	-0.01882	-1.9738	-550	-0.00551
5700	4.9680	28.3175	259.0465	46.4246	-13.1594	-1067	-0.01853	-1.8718	-541	-0.00532
5800	4.9680	28.8143	259.5433	46.5110	-12.9569	-1050	-0.01833	-1.7732	-534	-0.00499
5900	4.9680	29.3111	260.0401	46.5960	-12.7606	-1033	-0.01813	-1.6777	-526	-0.00473
6000	4.9680	29.8079	260.5369	46.6795	-12.5703	-----	-----	-1.5853	-----	-----

TABLE XXXVII—THERMODYNAMIC PROPERTIES OF
Li (GAS)

[Atomic weight, 6.940]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)	$H_g - H_0$ (kcal/mole)	H_g (kcal/mole)	S_g ($\frac{\text{cal}}{\text{mole } ^\circ\text{K}}$)
0	-----	0	166.8941	-----
298.16	4.9680	1.4809	168.3750	33.1418
300	4.9680	1.4904	168.3845	33.1734
300	4.9680	1.9872	168.8813	34.6026
500	4.9680	2.4840	169.3781	35.7112
600	4.9680	2.9808	169.8749	36.6169
700	4.9680	3.4776	170.3717	37.3828
800	4.9680	3.9743	170.8684	38.0461
900	4.9680	4.4711	171.3652	38.6312
1000	4.9680	4.9679	171.8620	39.1547
1100	4.9680	5.4647	172.3588	39.6282
1200	4.9680	5.9615	172.8556	40.0605
1300	4.9681	6.4583	173.3524	40.4581
1400	4.9683	6.9551	173.8492	40.8263
1500	4.9687	7.4520	174.3461	41.1691
1600	4.9696	7.9489	174.8430	41.4898
1700	4.9711	8.4459	175.3400	41.7911
1800	4.9736	8.9432	175.8373	42.0753
1900	4.9775	9.4407	176.3348	42.3443
2000	4.9828	9.9388	176.8329	42.5998
2100	4.9908	10.4374	177.3315	42.8431
2200	5.0011	10.9370	177.8311	43.0755
2300	5.0142	11.4378	178.3319	43.2981
2400	5.0304	11.9400	178.8341	43.5119
2500	5.0506	12.4441	179.3382	43.7176
2600	5.0742	12.9503	179.8444	43.9162
2700	5.1017	13.4591	180.3532	44.1082
2800	5.1332	13.9709	180.8650	44.2943
2900	5.1687	14.4859	181.3800	44.4750
3000	5.2083	15.0048	181.8989	44.6509
3100	5.2520	15.5278	182.4219	44.8224
3200	5.2997	16.0554	182.9495	44.9899
3300	5.3497	16.5879	183.4820	45.1538
3400	5.4034	17.1255	184.0196	45.3143
3500	5.4619	17.6688	184.5629	45.4718
3600	5.5223	18.2180	185.1121	45.6265
3700	5.5841	18.7733	185.6674	45.7786
3800	5.6495	19.3350	186.2291	45.9284
3900	5.7182	19.9034	186.7975	46.0760
4000	5.7870	20.4786	187.3727	46.2217
4100	5.8586	21.0609	187.9550	46.3655
4200	5.9316	21.6504	188.5445	46.5075
4300	6.0053	22.2473	189.1414	46.6480
4400	6.0813	22.8516	189.7457	46.7869
4500	6.1576	23.4636	190.3577	46.9244
4600	6.2356	24.0832	190.9773	47.0606
4700	6.3129	24.7106	191.6047	47.1955
4800	6.3919	25.3459	192.2400	47.3293
4900	6.4702	25.9890	192.8831	47.4619
5000	6.5495	26.6400	193.5341	47.5934
5100	6.6275	27.2988	194.1929	47.7238
5200	6.7059	27.9655	194.8596	47.8533
5300	6.7833	28.6399	195.5340	47.9818
5400	6.8608	29.3222	196.2163	48.1093
5500	6.9373	30.0121	196.9062	48.2359
5600	7.0130	30.7096	197.6037	48.3616
5700	7.0880	31.4146	198.3087	48.4863
5800	7.1617	32.1271	199.0212	48.6102
5900	7.2348	32.8469	199.7410	48.7333
6000	7.3063	33.5740	200.4681	48.8555

TABLE XXXVIII—THERMODYNAMIC PROPERTIES OF LiF (GAS)

[Molecular weight, 25.940]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(\frac{-\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0		0	77.9204							
298.16	7.0836	2.0796	80.0000	47.1209	233.2688			96.0757		
300	7.0872	2.0926	80.0130	47.1645	231.8488			95.4543		
400	7.3314	2.8129	80.7333	49.2349	174.2849			70.2551		
500	7.6048	3.5598	81.4802	50.9005	139.7116			55.1023		
600	7.8484	4.3328	82.2532	52.3092	116.6350			44.9810		
700	8.0471	5.1280	83.0484	53.5345	100.1316			37.7391		
800	8.2042	5.9409	83.8613	54.6197	87.7402			32.2995		
900	8.3275	6.7677	84.6881	55.5935	78.0930			28.0630		
1000	8.4245	7.6055	85.5259	56.4761	70.3683	6931	0.02399	24.6694	3039	0.01707
1100	8.5016	8.4518	86.3722	57.2826	64.0434	6306	0.01920	21.8896	2765	0.01483
1200	8.5635	9.3050	87.2254	58.0250	58.7692	5785	0.01560	19.5706	2536	0.01353
1300	8.6137	10.1639	88.0843	58.7124	54.3036	5344	0.01236	17.6063	2343	0.01173
1400	8.6549	11.0273	88.9477	59.3523	50.4741	4965	0.01050	15.9210	2177	0.01077
1500	8.6890	11.8945	89.8149	59.9506	47.1536	4637	0.00858	14.4589	2033	0.00988
1600	8.7175	12.7649	90.6853	60.5123	44.2469	4350	0.00668	13.1784	1907	0.00904
1700	8.7416	13.6378	91.5582	61.0415	41.6814	4096	0.00564	12.0476	1796	0.00842
1800	8.7621	14.5130	92.4334	61.5417	39.4002	3871	0.00433	11.0414	1697	0.00774
1900	8.7796	15.3901	93.3105	62.0159	37.3585	3669	0.00330	10.1405	1609	0.00710
2000	8.7948	16.2688	94.1892	62.4666	35.5207	3488	0.00215	9.3289	1529	0.00680
2100	8.8079	17.1489	95.0693	62.8961	33.8576	3324	0.00109	8.5940	1457	0.00633
2200	8.8194	18.0303	95.9507	63.3061	32.3456	3176	0.00037	7.9254	1392	0.00578
2300	8.8295	18.9128	96.8332	63.6983	30.9651	3040	0.00117	7.3144	1332	0.00560
2400	8.8384	19.7961	97.7165	64.0743	29.6996	2917	0.00280	6.7538	1277	0.00540
2500	8.8463	20.6804	98.6008	64.4353	28.5356	2803	0.00375	6.2376	1226	0.00526
2600	8.8534	21.5654	99.4858	64.7824	27.4613	2698	0.00486	5.7608	1179	0.00523
2700	8.8597	22.4510	100.3714	65.1166	26.4669	2602	0.00639	5.3189	1136	0.00499
2800	8.8653	23.3373	101.2577	65.4389	25.5440	2512	0.00731	4.9082	1095	0.00511
2900	8.8704	24.2241	102.1445	65.7501	24.6851	2429	0.00867	4.5255	1058	0.00483
3000	8.8750	25.1113	103.0317	66.0509	23.8841	2351	0.00969	4.1680	1023	0.00480
3100	8.8792	25.9990	103.9194	66.3420	23.1354	2279	0.01079	3.8332	990	0.00473
3200	8.8830	26.8871	104.8075	66.6239	22.4340	2210	0.01160	3.5191	959	0.00479
3300	8.8865	27.7756	105.6960	66.8973	21.7759	2146	0.01248	3.2237	930	0.00477
3400	8.8897	28.6644	106.5848	67.1627	21.1572	2088	0.01397	2.9454	903	0.00470
3500	8.8926	29.5535	107.4739	67.4204	20.5746	2030	0.01439	2.6827	877	0.00469
3600	8.8953	30.4429	108.3633	67.6710	20.0251	1975	0.01473	2.4344	852	0.00493
3700	8.8978	31.3326	109.2530	67.9147	19.5061	1925	0.01568	2.1992	829	0.00484
3800	8.9001	32.2225	110.1429	68.1520	19.0152	1878	0.01674	1.9762	807	0.00498
3900	8.9022	33.1126	111.0330	68.3832	18.5504	1830	0.01670	1.7643	786	0.00510
4000	8.9041	34.0029	111.9233	68.6087	18.1096	1787	0.01735	1.5627	766	0.00507
4100	8.9060	34.8934	112.8138	68.8286	17.6911	1745	0.01788	1.3708	748	0.00500
4200	8.9076	35.7841	113.7045	69.0432	17.2935	1704	0.01788	1.1877	729	0.00517
4300	8.9092	36.6749	114.5953	69.2528	16.9151	1668	0.01879	1.0130	712	0.00528
4400	8.9107	37.5659	115.4863	69.4576	16.5548	1630	0.01872	0.8459	696	0.00523
4500	8.9121	38.4571	116.3775	69.6579	16.2113	1596	0.01926	0.6860	680	0.00537
4600	8.9134	39.3484	117.2688	69.8538	15.8836	1561	0.01923	0.5328	665	0.00541
4700	8.9146	40.2398	118.1602	70.0445	15.5707	1529	0.01944	0.3859	651	0.00538
4800	8.9157	41.1313	119.0517	70.2332	15.2716	1497	0.01941	0.2449	636	0.00560
4900	8.9168	42.0229	119.9433	70.4170	14.9855	1468	0.01970	0.1095	623	0.00570
5000	8.9178	42.9146	120.8350	70.5972	14.7116	1437	0.01936	0.0208	611	0.00560
5100	8.9187	43.8064	121.7268	70.7738	14.4492	1409	0.01946	0.01462	598	0.00580
5200	8.9196	44.6984	122.6188	70.9470	14.1977	1381	0.01917	0.02670	586	0.00583
5300	8.9204	45.5904	123.5108	71.1169	13.9563	1355	0.01933	0.03834	575	0.00592
5400	8.9212	46.4824	124.4028	71.2836	13.7247	1328	0.01885	0.04958	564	0.00595
5500	8.9220	47.3746	125.2950	71.4474	13.5021	1303	0.01878	0.06043	553	0.00605
5600	8.9227	48.2668	126.1872	71.6081	13.2882	1279	0.01869	0.07091	543	0.00604
5700	8.9234	49.1591	127.0795	71.7661	13.0825	1255	0.01828	0.08104	534	0.00603
5800	8.9240	50.0515	127.9719	71.9213	12.8844	1232	0.01811	0.09085	524	0.00609
5900	8.9246	50.9439	128.8643	72.0738	12.6937	1209	0.01780	0.10034	515	0.00617
6000	8.9252	51.8364	129.7568	72.2238	12.5100			0.10954		

TABLE XXXIX—THERMODYNAMIC PROPERTIES OF LiH (GAS)

[Molecular weight, 7.948]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_g^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = -\frac{\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = -\frac{\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0		0	190.3225							
298.16	7.0763	2.0793	192.4018	40.7963	105.9633			41.7055		
300	7.0797	2.0923	192.4148	40.8398	105.3228			41.4233		
400	7.3175	2.8115	193.1340	42.9072	79.3373			29.9651		
500	7.5879	3.5568	193.8793	44.5692	63.7197			23.0611		
600	7.8313	4.3281	194.6506	45.9748	53.2862			18.4412		
700	8.0312	5.1216	195.4441	47.1975	45.8178			15.1301		
800	8.1899	5.9330	196.2555	48.2807	40.2052			12.6394		
900	8.3148	6.7585	197.0810	49.2529	35.8319			10.6968		
1000	8.4134	7.5951	197.9176	50.1342	32.3277	3136	0.02009	9.1388	1388	0.01598
1100	8.4918	8.4403	198.7628	50.9398	29.4567	2856	0.01550	7.8610	1264	0.01387
1200	8.5548	9.2927	199.6152	51.6814	27.0612	2621	.01305	6.7938	1160	.01269
1300	8.6060	10.1507	200.4732	52.3681	25.0320	2423	.01019	5.8888	1072	.01149
1400	8.6481	11.0134	201.3359	53.0074	23.2911	2252	.00877	5.1116	997	.01033
1500	8.6829	11.8899	202.2024	53.6053	21.7810	2104	.00730	4.4366	932	.00930
1600	8.7121	12.7497	203.0722	54.1666	20.4587	1975	0.00574	3.8448	875	0.00839
1700	8.7367	13.6221	203.9446	54.6955	19.2912	1861	.00451	3.3217	824	.00812
1800	8.7576	14.4969	204.8194	55.1955	18.2528	1759	.00371	2.8558	780	.00707
1900	8.7756	15.3735	205.6960	55.6694	17.3232	1669	.00230	2.4382	739	.00700
2000	8.7911	16.2518	206.5743	56.1200	16.4565	1588	.00121	2.0617	703	.00644
2100	8.8046	17.1316	207.4541	56.5492	15.7291	1514	0.00042	1.7205	670	0.00625
2200	8.8162	18.0127	208.3352	56.9591	15.0405	1448	-.00087	1.4097	640	.00804
2300	8.8266	18.8948	209.2173	57.3512	14.4118	1387	-.00172	1.1254	613	.00558
2400	8.8358	19.7779	210.1004	57.7270	13.8356	1332	-.00290	.8644	588	.00530
2500	8.8439	20.6619	210.9844	58.0879	13.3057	1282	-.00428	.6239	565	.00519
2600	8.8511	21.5467	211.8692	58.4349	12.8169	1235	-.00511	0.4014	543	0.00519
2700	8.8576	22.4321	212.7546	58.7691	12.3646	1193	-.00647	.1951	524	.00486
2800	8.8634	23.3182	213.6407	59.0913	11.9450	1154	-.00773	.0031	505	.00486
2900	8.8686	24.2048	214.5273	59.4024	11.5548	1117	-.00883	-.1759	486	.00483
3000	8.8733	25.0918	215.4143	59.7032	11.1913	1083	-.00975	-.3434	472	.00464
3100	8.8776	25.9794	216.3019	59.9942	10.8517	1052	-.01115	-.5001	457	0.00459
3200	8.8815	26.8673	217.1898	60.2761	10.5341	1021	-.01159	-.6477	443	.00456
3300	8.8851	27.7557	218.0782	60.5495	10.2363	993	-.01256	-.7865	429	.00472
3400	8.8884	28.6444	218.9669	60.8148	9.9568	969	-.01416	-.9174	417	.00456
3500	8.8913	29.5333	219.8558	61.0724	9.6941	943	-.01454	-.1.0411	404	.00478
3600	8.8941	30.4226	220.7451	61.3230	9.4467	918	-.01491	-.1.1581	394	0.00461
3700	8.8966	31.3121	221.6346	61.5667	9.2135	897	-.01595	-.1.2692	383	.00461
3800	8.8990	32.2019	222.5244	61.8040	8.9934	877	-.01687	-.1.3746	373	.00466
3900	8.9011	33.0919	223.4144	62.0352	8.7854	854	-.01670	-.1.4749	363	.00475
4000	8.9032	33.9821	224.3046	62.2605	8.5886	836	-.01760	-.1.5704	354	.00476
4100	8.9050	34.8726	225.1951	62.4804	8.4023	817	-.01792	-.1.6615	345	0.00486
4200	8.9068	35.7631	226.0856	62.6950	8.2257	798	-.01808	-.1.7485	336	.00506
4300	8.9084	36.6539	226.9764	62.9046	8.0582	783	-.01885	-.1.8317	328	.00505
4400	8.9099	37.5448	227.8673	63.1094	7.8991	765	-.01880	-.1.9113	321	.00507
4500	8.9113	38.4395	228.7584	63.3097	7.7479	750	-.01924	-.1.9877	313	.00526
4600	8.9126	39.3271	229.6496	63.5056	7.6041	733	-.01906	-.2.0610	305	0.00541
4700	8.9138	40.2184	230.5409	63.6972	7.4672	720	-.01970	-.2.1313	299	.00541
4800	8.9150	41.1098	231.4323	63.8849	7.3369	704	-.01947	-.2.1990	292	.00561
4900	8.9161	42.0014	232.3239	64.0688	7.2127	691	-.01970	-.2.2642	285	.00570
5000	8.9171	42.8933	233.2156	64.2489	7.0942	675	-.01925	-.2.3269	279	.00579
5100	8.9181	43.7848	234.1073	64.4255	6.9811	663	-.01950	-.2.3874	274	0.00581
5200	8.9190	44.6767	234.9992	64.5987	6.8731	649	-.01935	-.2.4459	267	.00602
5300	8.9199	45.5686	235.8911	64.7686	6.7700	637	-.01936	-.2.5023	262	.00598
5400	8.9207	46.4606	236.7831	64.9353	6.6714	624	-.01905	-.2.5568	257	.00607
5500	8.9215	47.3527	237.6752	65.0990	6.5770	611	-.01881	-.2.6096	251	.00628
5600	8.9222	48.2449	238.5674	65.2598	6.4867	599	-.01859	-.2.6607	247	0.00627
5700	8.9229	49.1372	239.4597	65.4177	6.4002	587	-.01841	-.2.7103	241	.00645
5800	8.9236	50.0295	240.3520	65.5729	6.3174	576	-.01823	-.2.7583	236	.00660
5900	8.9242	50.9219	241.2444	65.7254	6.2380	564	-.01780	-.2.8049	232	.00663
6000	8.9248	51.8144	242.1369	65.8754	6.1618			-.2.8502		

TABLE XL—THERMODYNAMIC PROPERTIES OF N (GAS)

[Atomic weight, 14.008]

T (°K)	C_p $\left(\frac{\text{cal}}{\text{mole}^\circ\text{K}}\right)$	$H_T^0 - H_0^0$ $\left(\frac{\text{kcal}}{\text{mole}}\right)$	H_T^0 $\left(\frac{\text{kcal}}{\text{mole}}\right)$	S_T^0 $\left(\frac{\text{cal}}{\text{mole}^\circ\text{K}}\right)$
0		0	85.9696	
298.16	4.9680	1.4812	87.4508	36.6145
300	4.9680	1.4904	87.4600	36.6450
400	4.9680	1.9872	87.9568	38.0742
500	4.9680	2.4840	88.4536	39.1828
600	4.9680	2.9808	88.9504	40.0885
700	4.9680	3.4776	89.4472	40.8544
800	4.9680	3.9744	89.9440	41.5177
900	4.9680	4.4712	90.4408	42.1029
1000	4.9680	4.9680	90.9376	42.6263
1100	4.9680	5.4648	91.4344	43.0998
1200	4.9680	5.9616	91.9312	43.5321
1300	4.9680	6.4584	92.4280	43.9297
1400	4.9680	6.9552	92.9248	44.2979
1500	4.9680	7.4520	93.4216	44.6406
1600	4.9680	7.9488	93.9184	44.9613
1700	4.9681	8.4456	94.4152	45.2625
1800	4.9683	8.9424	94.9120	45.5464
1900	4.9685	9.4393	95.4089	45.8151
2000	4.9690	9.9362	95.9058	46.0699
2100	4.9697	10.4331	96.4027	46.3124
2200	4.9708	10.9301	96.8997	46.5436
2300	4.9724	11.4273	97.3969	46.7646
2400	4.9746	11.9246	97.8942	46.9763
2500	4.9777	12.4222	98.3918	47.1794
2600	4.9816	12.9202	98.8898	47.3747
2700	4.9869	13.4186	99.3882	47.5628
2800	4.9935	13.9177	99.8873	47.7443
2900	5.0015	14.4174	100.3870	47.9197
3000	5.0108	14.9180	100.8876	48.0894
3100	5.0222	15.4197	101.3893	48.2539
3200	5.0354	15.9226	101.8922	48.4135
3300	5.0504	16.4268	102.3964	48.5687
3400	5.0675	16.9327	102.9023	48.7197
3500	5.0866	17.4404	103.4100	48.8669
3600	5.1079	17.9502	103.9198	49.0105
3700	5.1312	18.4621	104.4317	49.1508
3800	5.1567	18.9765	104.9461	49.2880
3900	5.1844	19.4936	105.4632	49.4223
4000	5.2143	20.0135	105.9831	49.5539
4100	5.2461	20.5365	106.5061	49.6830
4200	5.2800	21.0628	107.0324	49.8099
4300	5.3158	21.5926	107.5622	49.9345
4400	5.3533	22.1261	108.0957	50.0572
4500	5.3927	22.6634	108.6330	50.1779
4600	5.4335	23.2047	109.1743	50.2969
4700	5.4759	23.7502	109.7198	50.4142
4800	5.5197	24.2999	110.2695	50.5299
4900	5.5646	24.8542	110.8238	50.6442
5000	5.6109	25.4129	111.3825	50.7571
5100	5.6581	25.9764	111.9460	50.8687
5200	5.7063	26.5446	112.5142	50.9790
5300	5.7553	27.1177	113.0873	51.0882
5400	5.8052	27.6957	113.6653	51.1962
5500	5.8558	28.2788	114.2484	51.3032
5600	5.9070	28.8669	114.8365	51.4092
5700	5.9588	29.4602	115.4298	51.5142
5800	6.0114	30.0587	116.0283	51.6183
5900	6.0644	30.6625	116.6321	51.7215
6000	6.1179	31.2716	117.2412	51.8238

TABLE XLI—THERMODYNAMIC PROPERTIES OF N₂ (GAS)

[Molecular weight, 28.016]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta\left(-\frac{\Delta H^\circ}{RT}\right) = \frac{-\delta T}{100}\left(\frac{a}{T} + b\right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100}\left(\frac{c}{T} + d\right)$	
						a	b		c	d
0	-----	0	1.6992	-----	-----	-----	-----	-----	-----	-----
298.16	6.980	2.0723	3.7715	45.767	288.8283	-----	-----	119.4348	-----	-----
300	6.961	2.0851	3.7843	45.809	287.0664	-----	-----	118.6656	-----	-----
400	6.991	2.7824	4.4816	47.818	215.6725	-----	-----	87.4738	-----	-----
500	7.070	3.4850	5.1842	49.385	172.8306	-----	-----	68.7259	-----	-----
600	7.197	4.1980	5.8972	50.685	144.2610	-----	-----	56.2064	-----	-----
700	7.351	4.9253	6.6245	51.805	123.8438	-----	-----	47.2492	-----	-----
800	7.512	5.6686	7.3678	52.797	108.5207	-----	-----	40.5214	-----	-----
900	7.671	6.4280	8.1272	53.692	96.5937	-----	-----	35.2815	-----	-----
1000	7.816	7.2025	8.9017	54.5090	87.0447	8565	0.03284	31.0841	3756	0.02405
1100	7.947	7.9907	9.6899	55.2601	79.2255	7790	0.02943	27.6455	3418	0.02057
1200	8.063	8.7912	10.4904	55.9565	72.7044	7145	.02594	24.7766	3136	.01779
1300	8.165	9.6026	11.3018	56.6060	67.1823	6600	.02241	22.3465	2898	.01510
1400	8.253	10.4235	12.1227	57.2143	62.4456	6133	.01923	20.2614	2693	.01347
1500	8.330	11.2526	12.9518	57.7863	58.3377	5727	.01732	18.4526	2515	.01233
1600	8.399	12.0891	13.7883	58.3261	54.7410	5373	0.01481	16.8684	2360	0.01076
1700	8.459	12.9320	14.6312	58.8371	51.5656	5060	.01309	15.4694	2223	.00970
1800	8.512	13.7805	15.4797	59.3221	48.7414	4781	.01188	14.2247	2101	.00881
1900	8.560	14.6341	16.3333	59.7836	46.2132	4532	.01050	13.1101	1991	.00800
2000	8.602	15.4922	17.1914	60.2237	43.9367	4308	.00927	12.1063	1893	.00757
2100	8.640	16.3543	18.0535	60.6443	41.8760	4105	0.00819	11.1973	1804	0.00700
2200	8.674	17.2200	18.9192	61.0471	40.0019	3921	.00702	10.3703	1723	.00647
2300	8.705	18.0890	19.7882	61.4333	38.2901	3753	.00595	9.6147	1649	.00602
2400	8.733	18.9609	20.6601	61.8044	36.7204	3599	.00490	8.9216	1581	.00570
2500	8.759	19.8355	21.5347	62.1614	35.2759	3457	.00419	8.2835	1519	.00527
2600	8.783	20.7126	22.4118	62.5054	33.9421	3327	0.00298	7.6940	1461	0.00499
2700	8.805	21.5920	23.2912	62.8373	32.7069	3207	.00174	7.1479	1407	.00500
2800	8.8253	22.4735	24.1727	63.1579	31.5598	3096	.00061	6.6404	1357	.00477
2900	8.8440	23.3570	25.0562	63.4679	30.4916	2992	-.00043	6.1677	1311	.00460
3000	8.8610	24.2422	25.9414	63.7680	29.4947	2896	-.00169	5.7261	1268	.00427
3100	8.8774	25.1291	26.8283	64.0588	28.5622	2806	-0.00268	5.3128	1227	0.00436
3200	8.8928	26.0177	27.7169	64.3409	27.6880	2722	-.00395	4.9250	1189	.00420
3300	8.9073	26.9077	28.6069	64.6148	26.8671	2644	-.00525	4.5605	1153	.00418
3400	8.9210	27.7991	29.4983	64.8809	26.0947	2570	-.00628	4.2172	1120	.00390
3500	8.9340	28.6918	30.3910	65.1397	25.3667	2501	-.00772	3.8933	1088	.00388
3600	8.9462	29.5858	31.2850	65.3915	24.6797	2436	-0.00878	3.5872	1058	0.00385
3700	8.9577	30.4810	32.1802	65.6368	24.0301	2375	-.01020	3.2974	1029	.00391
3800	8.9686	31.3773	33.0765	65.8758	23.4153	2317	-.01140	3.0227	1002	.00398
3900	8.9790	32.2747	33.9739	66.1089	22.8326	2262	-.01240	2.7618	976	.00400
4000	8.9890	33.1731	34.8723	66.3364	22.2795	2210	-.01362	2.5138	952	.00390
4100	8.9987	34.0725	35.7717	66.5585	21.7541	2160	-0.01459	2.2777	928	0.00405
4200	9.0082	34.9729	36.6721	66.7754	21.2544	2113	-.01570	2.0527	906	.00410
4300	9.0174	35.8741	37.5733	66.9875	20.7787	2068	-.01660	1.8379	885	.00406
4400	9.0263	36.7763	38.4755	67.1949	20.3253	2026	-.01772	1.6327	864	.00420
4500	9.0350	37.6794	39.3786	67.3979	19.8928	1984	-.01840	1.4365	845	.00420
4600	9.0435	38.5833	40.2825	67.5965	19.4799	1945	-0.01923	1.2486	827	0.00414
4700	9.0518	39.4881	41.1873	67.7911	19.0853	1907	-.02009	1.0685	809	.00426
4800	9.0600	40.3937	42.0929	67.9818	18.7081	1870	-.02053	.8957	792	.00427
4900	9.0681	41.3001	42.9993	68.1687	18.3470	1835	-.02120	.7298	775	.00450
5000	9.0760	42.2073	43.9065	68.3520	18.0012	1801	-.02184	.5703	759	.00448
5100	9.0838	43.1153	44.8145	68.5318	17.6699	1768	-0.02220	0.4170	744	0.00462
5200	9.0915	44.0240	45.7232	68.7082	17.3521	1737	-.02284	.2693	729	.00475
5300	9.0991	44.9336	46.6328	68.8815	17.0472	1706	-.02313	.1270	714	.00498
5400	9.1066	45.8438	47.5430	69.0516	16.7544	1677	-.02361	-.0102	700	.00513
5500	9.1140	46.7549	48.4541	69.2188	16.4731	1648	-.02389	-.1426	686	.00530
5600	9.1214	47.6666	49.3658	69.3831	16.2027	1620	-0.02421	-0.2704	673	0.00543
5700	9.1287	48.5791	50.2783	69.5446	15.9427	1594	-.02453	-.3939	661	.00553
5800	9.1359	49.4924	51.1916	69.7034	15.6924	1568	-.02486	-.5134	648	.00577
5900	9.1431	50.4063	52.1055	69.8596	15.4515	1543	-.02507	-.6290	637	.00583
6000	9.1502	51.3210	53.0202	70.0134	15.2194	-----	-----	-.7410	-----	-----

TABLE XLII—THERMODYNAMIC PROPERTIES OF NO (GAS)

[Molecular weight, 30.008]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta\left(\frac{-\Delta H^\circ}{RT}\right) = \frac{-\delta T}{100}\left(\frac{a}{T} + b\right)$		$\log K$	$\delta \log K = \frac{-\delta T}{100}\left(\frac{c}{T} + d\right)$	
						a	b		c	d
0	-----	0	23.3447	-----	-----	-----	-----	-----	-----	-----
298.16	7.137	2.1942	25.5389	50.339	207.8039	-----	-----	84.8403	-----	-----
300	7.134	2.2068	25.5515	50.384	206.5399	-----	-----	84.2876	-----	-----
400	7.162	2.9208	26.2655	52.436	155.2832	-----	-----	61.8373	-----	-----
500	7.289	3.4440	26.7887	54.048	124.5125	-----	-----	48.3348	-----	-----
600	7.468	4.3812	27.7259	55.392	103.9835	-----	-----	39.3132	-----	-----
700	7.657	5.1366	28.4813	56.556	89.3049	-----	-----	32.8556	-----	-----
800	7.833	5.9096	29.2543	57.589	78.2839	-----	-----	28.0035	-----	-----
900	7.990	6.7005	30.0452	58.520	69.7015	-----	-----	24.2228	-----	-----
1000	8.126	7.5060	30.8507	59.3700	62.8276	6160	0.03020	21.1937	2707	0.02129
1100	8.243	8.3245	31.6692	60.1500	57.1974	5606	0.02567	18.7115	2465	0.01723
1200	8.342	9.1537	32.4984	60.8715	52.5009	5143	.02155	16.6401	2262	.01500
1300	8.426	9.9921	33.3368	61.5425	48.5232	4751	.01873	14.8851	2090	.01334
1400	8.498	10.8383	34.1830	62.1696	45.1109	4415	.01637	13.3789	1943	.01147
1500	8.560	11.6912	35.0359	62.7580	42.1512	4124	.01410	12.0721	1815	.01033
1600	8.614	12.5499	35.8946	63.3122	39.5596	3869	0.01232	10.9274	1703	0.00944
1700	8.660	13.4136	36.7583	63.8358	37.2714	3644	.01086	9.9162	1604	.00859
1800	8.702	14.2817	37.6264	64.3319	35.2361	3444	.00937	9.0165	1516	.00791
1900	8.738	15.1537	38.4984	64.8034	33.4141	3265	.00830	8.2107	1437	.00740
2000	8.771	16.0292	39.3739	65.2524	31.7733	3104	.00711	7.4848	1367	.00645
2100	8.801	16.9078	40.2525	65.6811	30.2881	2958	0.00616	6.8274	1302	0.00628
2200	8.828	17.7892	41.1339	66.0912	28.9374	2826	.00520	6.2293	1244	.00583
2300	8.852	18.6732	42.0179	66.4841	27.7035	2704	.00473	5.6826	1191	.00525
2400	8.874	19.5595	42.9042	66.8613	26.5721	2593	.00410	5.1811	1142	.00500
2500	8.895	20.4480	43.7927	67.2240	25.5308	2491	.00332	4.7193	1097	.00468
2600	8.914	21.3384	44.6831	67.5732	24.5694	2398	0.00235	4.2927	1056	0.00429
2700	8.932	22.2307	45.5754	67.9100	23.6789	2311	.00154	3.8973	1017	.00419
2800	8.949	23.1248	46.4695	68.2351	22.8520	2230	.00103	3.5299	981	.00410
2900	8.966	24.0205	47.3652	68.5494	22.0820	2156	.00013	3.1875	948	.00380
3000	8.981	24.9179	48.2626	68.8537	21.3632	2086	-.00060	2.8677	916	.00392
3100	8.996	25.8167	49.1614	69.1484	20.6909	2021	-0.00136	2.5683	887	0.00371
3200	9.010	26.7170	50.0617	69.4342	20.0607	1951	-.00224	2.2874	860	.00360
3300	9.024	27.6187	50.9634	69.7117	19.4687	1904	-.00310	2.0233	834	.00351
3400	9.037	28.5218	51.8665	69.9813	18.9118	1851	-.00386	1.7745	810	.00337
3500	9.049	29.4261	52.7708	70.2434	18.3868	1800	-.00450	1.5397	787	.00329
3600	9.061	30.3316	53.6763	70.4985	17.8913	1753	-.00528	1.3178	765	0.00334
3700	9.073	31.2383	54.5830	70.7469	17.4228	1708	-.00597	1.1077	745	.00315
3800	9.085	32.1462	55.4909	70.9891	16.9793	1666	-.00668	.9085	725	.00320
3900	9.096	33.0552	56.3999	71.2252	16.5588	1626	-.00740	.7194	707	.00315
4000	9.107	33.9654	57.3101	71.4556	16.1597	1588	-.00812	.5395	689	.00325
4100	9.118	34.8766	58.2213	71.6806	15.7805	1552	-0.00872	0.3682	672	0.00320
4200	9.128	35.7889	59.1336	71.9005	15.4197	1518	-.00952	.2050	656	.00324
4300	9.138	36.7022	60.0469	72.1154	15.0762	1485	-.01000	.0492	641	.00322
4400	9.148	37.6165	60.9612	72.3256	14.7487	1453	-.01039	-.0997	627	.00317
4500	9.158	38.5318	61.8765	72.5312	14.4362	1422	-.01063	-.2422	612	.00326
4600	9.168	39.4481	62.7928	72.7326	14.1377	1393	-0.01108	-0.3785	599	0.00325
4700	9.178	40.3654	63.7101	72.9299	13.8524	1365	-.01148	-.5092	586	.00332
4800	9.188	41.2837	64.6284	73.1232	13.5795	1338	-.01176	-.6346	574	.00336
4900	9.198	42.2030	65.5477	73.3128	13.3182	1313	-.01220	-.7551	561	.00350
5000	9.208	43.1233	66.4680	73.4957	13.0678	1288	-.01255	-.8708	550	.00356
5100	9.218	44.0446	67.3893	73.6812	12.8278	1264	-0.01268	-0.9822	539	0.00355
5200	9.227	44.9669	68.3116	73.8602	12.5974	1240	-.01276	-1.0894	528	.00368
5300	9.237	45.8901	69.2348	74.0351	12.3762	1218	-.01306	-1.1927	517	.00386
5400	9.246	46.8142	70.1589	74.2088	12.1637	1196	-.01315	-1.2923	507	.00392
5500	9.256	47.7393	71.0840	74.3786	11.9594	1175	-.01332	-1.3884	497	.00405
5600	9.266	48.6654	72.0101	74.5454	11.7629	1155	-0.01353	-1.4812	488	0.00409
5700	9.275	49.5925	72.9372	74.7095	11.5738	1136	-.01366	-1.5709	478	.00429
5800	9.285	50.5205	73.8652	74.8709	11.3916	1117	-.01382	-1.6576	469	.00441
5900	9.294	51.4494	74.7941	75.0297	11.2161	1099	-.01397	-1.7415	461	.00447
6000	9.304	52.3793	75.7240	75.1860	11.0469	-----	-----	-1.8228	-----	-----

TABLE XLIII—THERMODYNAMIC PROPERTIES OF O (GAS)

[Atomic weight, 16.0000]

T (°K)	C_p° $\left(\frac{\text{cal}}{\text{mole}^\circ\text{K}}\right)$	$H_f^\circ - H_g^\circ$ $\left(\frac{\text{kcal}}{\text{mole}}\right)$	H_f° $\left(\frac{\text{kcal}}{\text{mole}}\right)$	S_f° $\left(\frac{\text{cal}}{\text{mole}^\circ\text{K}}\right)$
0	-----	0	59.6041	-----
298.16	5.2364	1.6074	61.2115	38.4689
300	5.2338	1.6170	61.2211	38.5010
400	5.1341	2.1349	61.7390	39.9915
500	5.0802	2.6454	62.2495	41.1308
600	5.0486	3.1517	62.7558	42.0540
700	5.0284	3.6555	63.2596	42.8307
800	5.0150	4.1576	63.7617	43.5011
900	5.0055	4.6587	64.2628	44.0914
1000	4.9988	5.1588	64.7629	44.6183
1100	4.9936	5.6584	65.2625	45.0945
1200	4.9894	6.1576	65.7617	45.5288
1300	4.9864	6.6564	66.2605	45.9281
1400	4.9838	7.1549	66.7590	46.2975
1500	4.9819	7.6532	67.2573	46.6413
1600	4.9805	8.1513	67.7554	46.9628
1700	4.9792	8.6493	68.2534	47.2646
1800	4.9784	9.1471	68.7512	47.5492
1900	4.9778	9.6450	69.2491	47.8184
2000	4.9776	10.1427	69.7468	48.0737
2100	4.9778	10.6405	70.2446	48.3166
2200	4.9784	11.1383	70.7424	48.5481
2300	4.9796	11.6362	71.2403	48.7695
2400	4.9812	12.1343	71.7384	48.9814
2500	4.9834	12.6325	72.2366	49.1848
2600	4.9862	13.1310	72.7351	49.3803
2700	4.9897	13.6298	73.2339	49.5686
2800	4.9935	14.1289	73.7330	49.7501
2900	4.9986	14.6285	74.2326	49.9254
3000	5.0041	15.1287	74.7328	50.0950
3100	5.0102	15.6294	75.2335	50.2592
3200	5.0170	16.1307	75.7348	50.4183
3300	5.0245	16.6328	76.2369	50.5728
3400	5.0325	17.1357	76.7398	50.7229
3500	5.0411	17.6393	77.2434	50.8689
3600	5.0502	18.1439	77.7480	51.0111
3700	5.0599	18.6494	78.2535	51.1496
3800	5.0700	19.1559	78.7600	51.2846
3900	5.0805	19.6634	79.2675	51.4165
4000	5.0914	20.1720	79.7761	51.5452
4100	5.1026	20.6817	80.2858	51.6711
4200	5.1140	21.1925	80.7966	51.7942
4300	5.1257	21.7045	81.3086	51.9147
4400	5.1375	22.2177	81.8218	52.0326
4500	5.1495	22.7320	82.3361	52.1482
4600	5.1616	23.2476	82.8517	52.2615
4700	5.1738	23.7644	83.3685	52.3727
4800	5.1860	24.2824	83.8865	52.4817
4900	5.1981	24.8016	84.4057	52.5888
5000	5.2102	25.3220	84.9261	52.6939
5100	5.2223	25.8436	85.4477	52.7972
5200	5.2344	26.3664	85.9705	52.8988
5300	5.2464	26.8905	86.4946	52.9986
5400	5.2583	27.4157	87.0198	53.0968
5500	5.2701	27.9421	87.5462	53.1933
5600	5.2818	28.4697	88.0738	53.2884
5700	5.2933	28.9985	88.6026	53.3820
5800	5.3047	29.5284	89.1325	53.4742
5900	5.3159	30.0594	89.6635	53.5649
6000	5.3270	30.5916	90.1957	53.6544

TABLE XLIV—THERMODYNAMIC PROPERTIES OF O₂ (GAS)

[Molecular weight, 32.0000]

T (°K)	C_p ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$H_f^\circ - H_0^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole} \cdot ^\circ\text{K}}$)	$-\frac{\Delta H^\circ}{RT}$	$\delta \left(\frac{-\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0		0	2.0362							
298.16	7.021	2.0747	4.1109	49.011	199.6835			80.6182		
300	7.023	2.0876	4.1238	49.056	198.4695			80.0867		
400	7.196	2.7977	4.8339	51.098	149.2619			58.5109		
500	7.431	3.5288	5.5650	52.728	119.7013			45.5311		
600	7.670	4.2841	6.3203	54.105	99.9669			36.8580		
700	7.883	5.0620	7.0982	55.303	85.8510			30.6499		
800	8.063	5.8596	7.8958	56.368	75.2496			25.9854		
900	8.212	6.6737	8.7099	57.327	66.9937			22.3515		
1000	8.336	7.5012	9.5374	58.1990	60.3812	5926	0.02853	19.4400	2604	0.01823
1100	8.439	8.3400	10.3762	58.9983	54.9654	5392	0.02417	17.0545	2370	0.01550
1200	8.527	9.1883	11.2245	59.7364	50.4479	4947	.02061	15.0640	2173	.01322
1300	8.604	10.0448	12.0810	60.4220	46.6219	4569	.01873	13.3777	2009	.01200
1400	8.674	10.9087	12.9449	61.0622	43.3396	4245	.01700	11.9307	1867	.01083
1500	8.738	11.7793	13.8155	61.6628	40.4926	3963	.01642	10.6752	1744	.00960
1600	8.800	12.6562	14.6924	62.2287	37.9993	3717	0.01523	9.5756	1637	0.00826
1700	8.858	13.5391	15.5753	62.7640	35.7976	3499	.01501	8.6044	1542	.00743
1800	8.916	14.4278	16.4640	63.2719	33.8787	3305	.01473	7.7403	1458	.00643
1900	8.973	15.3223	17.3585	63.7555	32.0845	3132	.01420	6.9665	1383	.00550
2000	9.029	16.2224	18.2586	64.2172	30.5043	2976	.01386	6.2695	1315	.00491
2100	9.084	17.1280	19.1642	64.6590	29.0733	2835	0.01356	5.6384	1253	0.00455
2200	9.139	18.0392	20.0754	65.0829	27.7711	2707	.01324	5.0643	1197	.00407
2300	9.194	18.9558	20.9920	65.4904	26.5809	2590	.01283	4.5398	1146	.00370
2400	9.248	19.8779	21.9141	65.8828	25.4889	2484	.01200	4.0586	1099	.00330
2500	9.301	20.8054	22.8416	66.2614	24.4833	2386	.01161	3.6157	1056	.00295
2600	9.354	21.7381	23.7743	66.6272	23.5540	2295	0.01120	3.2066	1016	0.00260
2700	9.405	22.6761	24.7123	66.9812	22.6928	2212	.01060	2.8277	980	.00210
2800	9.455	23.6191	25.6553	67.3241	21.8922	2135	.00979	2.4756	945	.00204
2900	9.503	24.5670	26.6032	67.6568	21.1462	2063	.00913	2.1477	913	.00187
3000	9.551	25.5197	27.5559	67.9797	20.4494	1997	.00831	1.8415	883	.00166
3100	9.596	26.4770	28.5132	68.2936	19.7969	1935	0.00761	1.5550	855	0.00161
3200	9.640	27.4388	29.4750	68.5990	19.1846	1877	.00671	1.2862	829	.00129
3300	9.682	28.4049	30.4411	68.8963	18.6091	1822	.00622	1.0337	804	.00123
3400	9.723	29.3752	31.4114	69.1859	18.0670	1771	.00540	.7960	781	.00106
3500	9.762	30.3494	32.3856	69.4683	17.5556	1723	.00469	.5718	759	.00097
3600	9.799	31.3275	33.3637	69.7439	17.0723	1677	0.00426	0.3600	739	0.00077
3700	9.835	32.3092	34.3454	70.0128	16.6148	1634	.00350	.1595	719	.00069
3800	9.869	33.2944	35.3306	70.2756	16.1813	1594	.00278	-.0304	700	.00071
3900	9.901	34.2829	36.3191	70.5323	15.7698	1555	.00225	-.2106	682	.00070
4000	9.932	35.2745	37.3107	70.7834	15.3788	1519	.00161	-.3818	665	.00070
4100	9.960	36.2691	38.3053	71.0290	15.0067	1484	0.00107	-0.5447	649	0.00068
4200	9.987	37.2665	39.3027	71.2693	14.6523	1450	.00069	-.6999	634	.00056
4300	10.013	38.2665	40.3027	71.5046	14.3144	1419	.00010	-.8479	619	.00062
4400	10.037	39.2690	41.3052	71.7351	13.9918	1388	-.00014	-.9892	606	.00043
4500	10.060	40.2738	42.3100	71.9609	13.6835	1359	-.00063	-1.1243	592	.00050
4600	10.081	41.2809	43.3171	72.1822	13.3887	1331	-0.00089	-1.2535	579	0.00051
4700	10.103	42.2901	44.3263	72.3993	13.1064	1304	-.00127	-1.3772	567	.00048
4800	10.121	43.3013	45.3375	72.6122	12.8360	1279	-.00162	-1.4958	555	.00053
4900	10.139	44.3143	46.3505	72.8210	12.5766	1254	-.00190	-1.6096	543	.00060
5000	10.156	45.3290	47.3652	73.0261	12.3277	1230	-.00208	-1.7188	533	.00049
5100	10.172	46.3454	48.3816	73.2273	12.0886	1207	-0.00232	-1.8238	522	0.00062
5200	10.187	47.3634	49.3996	73.4250	11.8588	1184	-.00240	-1.9248	512	.00060
5300	10.201	48.3828	50.4190	73.6192	11.6378	1162	-.00248	-2.0220	502	.00064
5400	10.215	49.4036	51.4398	73.8100	11.4251	1143	-.00282	-2.1156	493	.00056
5500	10.228	50.4257	52.4619	73.9975	11.2201	1122	-.00286	-2.2058	484	.00057
5600	10.239	51.4496	53.4858	74.1819	11.0226	1102	-0.00293	-2.2928	475	0.00067
5700	10.250	52.4735	54.5097	74.3632	10.8322	1084	-.00310	-2.3768	466	.00076
5800	10.261	53.4991	55.5353	74.5416	10.6483	1067	-.00335	-2.4579	458	.00077
5900	10.270	54.5256	56.5618	74.7171	10.4709	1049	-.00343	-2.5363	450	.00080
6000	10.279	55.5531	57.5893	74.8898	10.2995			-2.6121		

TABLE XLV—THERMODYNAMIC PROPERTIES OF OH (GAS)

[Molecular weight, 17.008]

T (°K)	C_p° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$H_f^\circ - H_f^\circ$ ($\frac{\text{kcal}}{\text{mole}}$)	H_f° ($\frac{\text{kcal}}{\text{mole}}$)	S_f° ($\frac{\text{cal}}{\text{mole}^\circ\text{K}}$)	$\frac{\Delta H^\circ}{RT}$	$\delta \left(-\frac{\Delta H^\circ}{RT} \right) = \frac{-\delta T}{100} \left(\frac{a}{T} + b \right)$		log K'	$\delta \log K = \frac{-\delta T}{100} \left(\frac{c}{T} + d \right)$	
						a	b		c	d
0	-----	0	44.7266	-----	-----	-----	-----	-----	-----	-----
298.16	7.141	2.1062	46.8328	43.888	170.7827	-----	-----	69.3677	-----	-----
300	7.139	2.1225	46.8491	43.934	169.7395	-----	-----	68.9110	-----	-----
400	7.074	2.8296	47.5562	45.978	127.6916	-----	-----	50.4585	-----	-----
500	7.048	3.5350	48.2616	47.553	102.4572	-----	-----	39.3522	-----	-----
600	7.053	4.2408	48.9674	48.840	85.6303	-----	-----	31.9260	-----	-----
700	7.087	4.9469	49.6735	49.927	73.6091	-----	-----	26.6057	-----	-----
800	7.150	5.6584	50.3850	50.877	64.5888	-----	-----	22.6043	-----	-----
900	7.234	6.3774	51.1040	51.723	57.5682	-----	-----	19.4833	-----	-----
1000	7.333	7.1060	51.8326	52.4910	51.9464	5034	0.02814	16.9801	2227	0.02885
1100	7.440	7.8446	52.5712	53.1949	47.3419	4575	0.02930	14.9267	2029	0.02457
1200	7.551	8.5942	53.3208	53.8470	43.5001	4194	0.02915	13.2113	1863	.02172
1300	7.663	9.3549	54.0815	54.4559	40.2448	3873	.02777	11.7565	1723	.01909
1400	7.772	10.1266	54.8532	55.0278	37.4506	3599	.02607	10.5067	1603	.01673
1500	7.875	10.9090	55.6356	55.5675	35.0252	3361	.02478	9.4213	1498	.01535
1600	7.973	11.7014	56.4280	56.0788	32.8998	3153	0.02340	8.4697	1407	0.01355
1700	8.066	12.5033	57.2299	56.5650	31.0217	2970	.02210	7.6285	1327	.01198
1800	8.152	13.3142	58.0408	57.0285	29.3496	2808	.02041	6.8793	1255	.01087
1900	8.233	14.1335	58.8601	57.4714	27.8513	2663	.01890	6.2079	1190	.01020
2000	8.308	14.9605	59.6871	57.8956	26.5009	2533	.01731	5.6027	1133	.00898
2100	8.378	15.7948	60.5214	58.3027	25.2774	2415	0.01617	5.0542	1080	0.00839
2200	8.443	16.6359	61.3625	58.6939	24.1635	2308	.01482	4.5549	1032	.00790
2300	8.504	17.4832	62.2098	59.0705	23.1452	2210	.01387	4.0983	989	.00702
2400	8.561	18.3365	63.0631	59.4337	22.2105	2121	.01260	3.6792	949	.00650
2500	8.614	19.1952	63.9218	59.7842	21.3495	2038	.01185	3.2931	912	.00603
2600	8.663	20.0591	64.7857	60.1230	20.5538	1962	0.01093	2.9363	878	0.00561
2700	8.710	20.9277	65.6543	60.4508	19.8162	1892	.00999	2.6055	846	.00546
2800	8.755	21.8010	66.5276	60.7684	19.1305	1826	.00934	2.2979	817	.00488
2900	8.798	22.6786	67.4052	61.0764	18.4915	1765	.00877	2.0113	790	.00457
3000	8.838	23.5604	68.2870	61.3753	17.8944	1708	.00813	1.7434	764	.00435
3100	8.877	24.4462	69.1728	61.6658	17.3353	1655	0.00741	1.4926	740	0.00415
3200	8.913	25.3357	70.0623	61.9482	16.8107	1605	.00694	1.2572	718	.00372
3300	8.949	26.2288	70.9554	62.2230	16.3174	1558	.00646	1.0359	697	.00360
3400	8.982	27.1253	71.8519	62.4906	15.8527	1513	.00621	.8273	677	.00327
3500	9.015	28.0252	72.7518	62.7515	15.4142	1471	.00579	.6306	659	.00304
3600	9.047	28.9283	73.6549	63.0059	14.9998	1432	0.00527	0.4445	641	0.00286
3700	9.077	29.8345	74.5611	63.2542	14.6075	1395	.00489	.2684	624	.00279
3800	9.107	30.7437	75.4703	63.4966	14.2355	1359	.00464	.1014	608	.00270
3900	9.135	31.6558	76.3824	63.7336	13.8824	1325	.00435	-.0572	593	.00255
4000	9.162	32.5706	77.2972	63.9652	13.5468	1293	.00413	-.2080	579	.00238
4100	9.189	33.4882	78.2148	64.1917	13.2273	1263	0.00379	-.3516	565	0.00228
4200	9.215	34.4084	79.1350	64.4135	12.9228	1234	.00352	-.4884	552	.00223
4300	9.241	35.3312	80.0578	64.6306	12.6323	1206	.00331	-.6190	539	.00220
4400	9.266	36.2565	80.9831	64.8434	12.3549	1179	.00320	-.7437	528	.00197
4500	9.290	37.1843	81.9109	65.0518	12.0897	1154	.00303	-.8630	517	.00181
4600	9.314	38.1145	82.8411	65.2563	11.8358	1129	0.00289	-.9772	506	0.00174
4700	9.338	39.0471	83.7737	65.4569	11.5927	1106	.00268	-.1.0866	495	.00178
4800	9.362	39.9821	84.7087	65.6537	11.3596	1084	.00258	-.1.1915	485	.00172
4900	9.384	40.9194	85.6460	65.8470	11.1358	1062	.00250	-.1.2922	476	.00160
5000	9.406	41.8589	86.5855	66.0368	10.9209	1041	.00248	-.1.3890	466	.00163
5100	9.427	42.8006	87.5272	66.2233	10.7143	1021	0.00235	-.1.4820	458	0.00152
5200	9.448	43.7443	88.4709	66.4065	10.5156	1002	.00214	-.1.5716	449	.00148
5300	9.469	44.6902	89.4168	66.5867	10.3244	984	.00208	-.1.6578	440	.00152
5400	9.489	45.6381	90.3647	66.7639	10.1401	966	.00206	-.1.7408	433	.00137
5500	9.509	46.5880	91.3146	66.9382	9.9624	949	.00194	-.1.8209	425	.00141
5600	9.529	47.5399	92.2665	67.1097	9.7910	932	0.00189	-.1.8982	418	0.00127
5700	9.548	48.4937	93.2203	67.2785	9.6256	916	.00187	-.1.9728	411	.00124
5800	9.567	49.4495	94.1761	67.4447	9.4658	901	.00179	-.2.0449	404	.00123
5900	9.585	50.4071	95.1337	67.6084	9.3113	886	.00163	-.2.1146	397	.00123
6000	9.603	51.3665	96.0931	67.7697	9.1620	-----	-----	-.2.1820	-----	-----

REPORT 1038

WIND-TUNNEL INVESTIGATION OF AIR INLET AND OUTLET OPENINGS ON A STREAMLINE BODY¹

By JOHN V. BECKER

SUMMARY

In connection with the general problem of providing air flow to an aircraft power plant located within a fuselage, an investigation was conducted in the Langley 8-foot high-speed tunnel to determine the effect on external drag and pressure distribution of air inlet openings located at the nose of a streamline body. Air outlet openings located at the tail and at the 21-percent and 63-percent stations of the body were also investigated. Boundary-layer transition measurements were made and correlated with the force and the pressure data. Individual openings were investigated with the aid of a blower and then practicable combinations of inlet and outlet openings were tested. Various modifications to the internal duct shape near the inlet opening and the aerodynamic effects of a simulated gun in the duct were also studied.

The results showed that the external drag (measured drag less computed drag due to internal duct losses) of the body with suitably designed nose-inlet and tail-outlet openings was no higher than the drag of the streamline body over a wide range of rates of internal air flow. The static-pressure distribution with the best inlet profiles developed during the investigation was almost identical with that of the corresponding portion of the streamline body. As a consequence, the same favorable laminar-boundary-layer flow as on the streamline body was obtained. The local velocity increments over the nose profiles were so low that the critical speed of a fuselage employing these shapes would depend on the peak-velocity increments occurring elsewhere than on the nose.

The results of the tests suggested that outlet openings should be designed so that the static pressure of the internal flow at the outlet would be the same as the static pressure of the external flow in the vicinity of the opening. Radical changes in the internal-duct arrangement near the inlet openings had little effect on the external drag or pressure distribution.

INTRODUCTION

Various wind-tunnel tests of full-scale airplanes and of airplane models have shown large external losses associated with the power-plant installations. The external drag of clean NACA cowlings with no protruding scoops or surface irregularities was shown in reference 1 to be considerable. At high speeds, prohibitive increases in the

external drag may occur as a result of the formation of compression shocks on cowlings (reference 2) or protruding scoops. The excessive cowlings drag costs at high speeds can be reduced to some extent by reducing the bluntness of the nose profile (reference 2); however, the improvement which can be made in this direction is limited for conventional installations in which the engine is located at the nose of the fuselage. If it is assumed that the engine is located near the center of the fuselage or nacelle, then radical changes in the shape of the nose are possible. The present investigation was designed to explore the possibilities of high-speed drag reduction by use of nose inlets proportioned solely from aerodynamic requirements without any restrictions arising from engine dimensions, location, or air-flow requirements.

At the outset of the present investigation little information was available in regard to the characteristics of inlet openings at the nose of a streamline body. Previous tests had been made without air flow into the openings, a condition seldom occurring in practice, and the results were therefore inconclusive. Little pressure-distribution or critical-speed data were available, and it was not known whether any appreciable laminar boundary layer could exist behind an inlet opening.

The principal purpose of this investigation was to develop nose-inlet openings of various relative sizes which would have the lowest possible external drag and the highest possible critical compressibility speed. The air-inlet flow rates used were not restricted to the relatively small requirements of radial engines but were varied from zero flow to inlet-velocity ratios in excess of unity. Pressure-distribution and boundary-layer data were obtained to aid in interpreting the drag results and to permit the estimation of the critical speeds. Typical annular and tail-outlet openings were similarly investigated. In order to avoid possible confusing interference effects, the inlet and the outlet openings were tested separately with the internal air flow being supplied through wing ducts from a blower located outside of the wind tunnel. Representative combinations of the inlet and outlet openings were then tested without the use of the blower. The effects on external drag of a protruding simulated gun in the inlet opening and of various internal-duct arrangements near the nose were also included in the investigation of the individual inlet openings.

¹ Supersedes NACA ACR, "Wind-Tunnel Tests of Air Inlet and Outlet Openings on a Streamline Body" by John V. Becker, November 1940.

The external drag cost of an inlet opening at the nose of a smooth streamline body is generally greater at low Reynolds numbers, when the opening may disturb extensive low-drag laminar boundary layers, than at high-speed flight Reynolds numbers, where the boundary-layer flow may be almost wholly turbulent. Although it was impossible to attain full-scale conditions in this investigation, the boundary-layer-flow condition corresponding to high Reynolds numbers was simulated by artificially forcing transition to take place near the nose of the models. The tests were made both with the natural-transition and the fixed-transition boundary-layer conditions. The results thus show the effect of the openings at conditions corresponding to extremes of the Reynolds number range.

SYMBOLS

V	free-stream velocity
p_0	free-stream static pressure
ρ_0	free-stream density
q_0	free-stream dynamic pressure $\left(\frac{1}{2} \rho_0 V^2\right)$
v	mean velocity in duct
p	local static pressure
ρ	density in duct
V_i	initial velocity of air passing through duct
V_w	hypothetical final velocity of air passing through duct based on total pressure at discharge
A	cross-sectional area of inlet or outlet opening
d	diameter of inlet or outlet opening
D	maximum diameter of streamline body
F	maximum cross-sectional area of streamline body
L	length of streamline body
l	distance between end of streamline body and end of nose
R	maximum radius of streamline body
R	fuselage Reynolds number (VL/ν)
ν	kinematic viscosity
P	pressure coefficient $((p - p_0)/q_0)$
a	velocity of sound in air
M	Mach number (V/a)
Q	volume of flow through duct, cubic feet per second
α	angle of attack referred to center line of streamline body, degrees
C_{D_F}	external-drag coefficient
	$\left(\frac{[(\text{Measured drag of model}) - (\text{Drag of wing alone}) - (\text{Drag due to internal flow})]}{q_0 F} \right)$
$C_{D_{Fi}}$	calculated drag coefficient due to internal air flow
U	velocity just outside the boundary layer
u	velocity in the boundary layer
x	distance from nose of streamline body, along major axis
x'	distance from nose of inlet openings, along major axis
X	length of nose measured from $L/4$ station
y	ordinate measured from center line of streamline body
y'	nose-profile ordinate measured from inlet-opening radius
Y	value of y' at $L/4$ station

APPARATUS AND METHODS

The tests were made in the Langley 8-foot high-speed tunnel in its original form incorporating an 8,000-horsepower drive motor. The tunnel was of the closed-throat, circular-section, single-return type and was capable of air speeds of about 500 miles per hour at the time of these tests. This tunnel was chosen for the investigation principally because of the low turbulence of the air stream, which permitted the boundary-layer-flow conditions more nearly to approach those obtained in free air than in streams of high turbulence. Most of the tests were run at low speed (140 miles per hour).

Streamline body.—The streamline body (fig. 1 and table I) is a slightly modified version of fuselage form No. 111 of reference 3. The thickness distribution was modified slightly to eliminate the unfavorable pressure gradient occurring ahead of the 50-percent station of the original 111 form. This modification was made to encourage a more extensive laminar boundary layer. The fineness ratio of 5 is representative of several current pursuit-type fuselages.

The streamline body was mounted in the wind tunnel on a 24-inch-chord airfoil of NACA 27-212 section, which completely spanned the jet (fig. 2). The wing contained two large ducts to permit air to be supplied to or drawn from the openings on the body. The ratio of wing chord at the body to the length of the body is within the range of current practice.

Inlet openings.—Nose-inlet openings of three sizes were tested (figs. 1 and 3; ordinates in table I). The largest opening, nose A, was approximately the size, relative to the maximum cross section of the body, of average NACA cowling

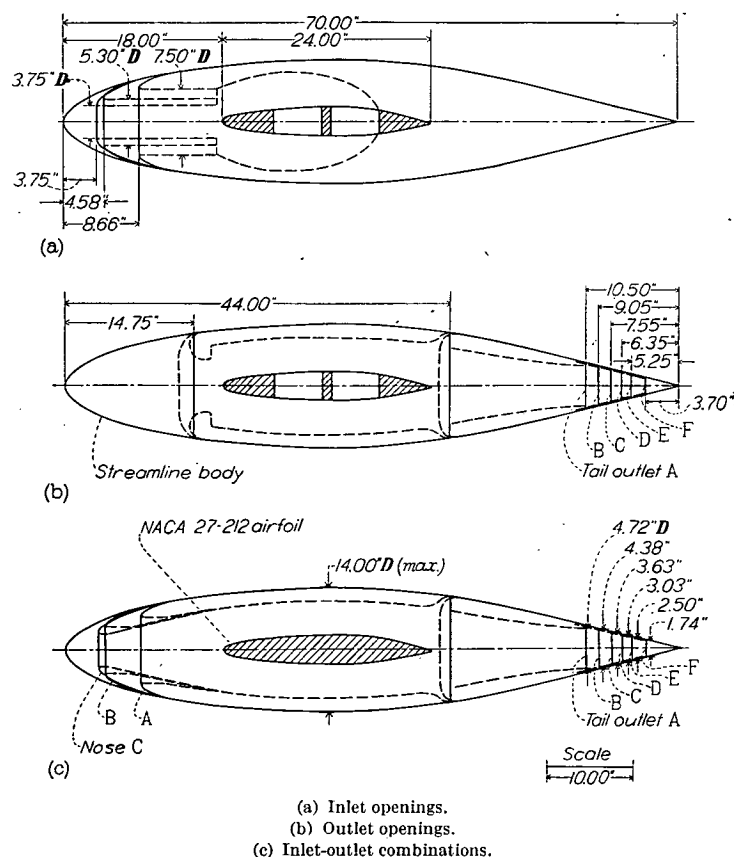


FIGURE 1.—Streamline body with general arrangement of inlet and outlet openings

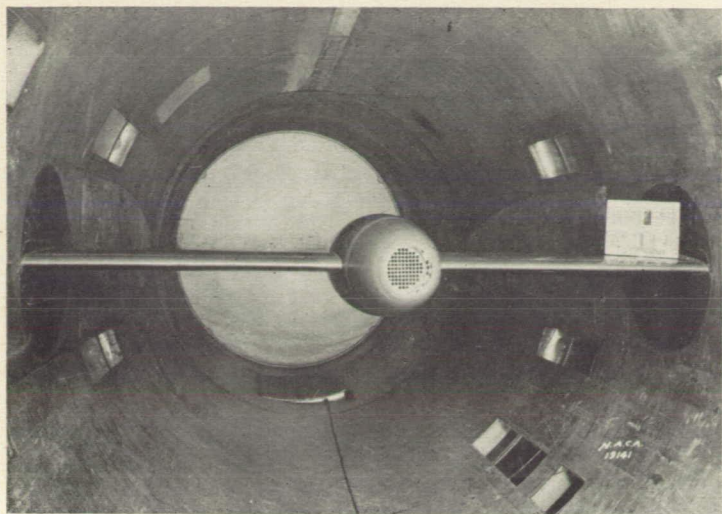


FIGURE 2.—Model installed in Langley 8-foot high-speed tunnel. Nose B.

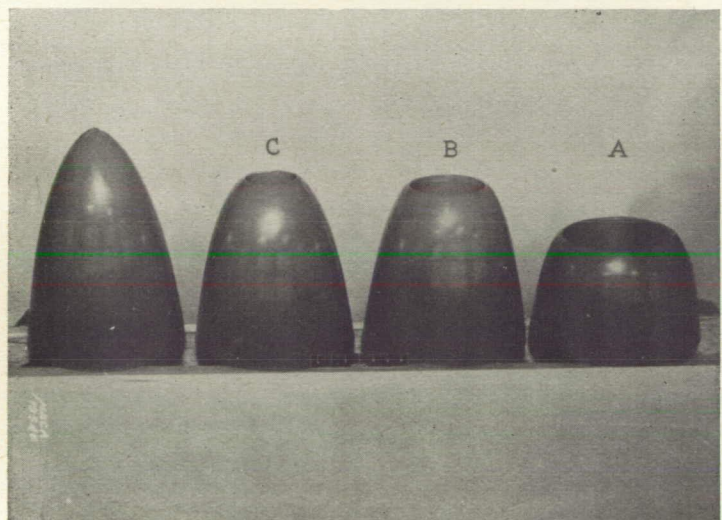


FIGURE 3.—Nose inlet openings compared with streamline nose.

inlet openings. Nose B had one-half the area of nose A, and nose C, one-quarter the area of nose A. The profile shapes of the noses were developed in a series of tests (not discussed in detail in this report) in which the nose lengths and profiles were progressively modified until the most satisfactory pressure-distribution characteristics were obtained. The profiles all fall within the profile of the streamline body. It will be noticed that the nose ordinates (table I) are given only to the quarter-length station. Beyond this point, streamline-body ordinates apply. Several modifications of the straight duct (fig. 1) that was used in most of the nose-inlet tests will be described later in the discussion of the results.

Outlet openings.—Outlet openings at the tail and annular outlets located ahead of and behind the wing were investigated (figs. 1, 4, and 5). The tail-outlet profiles coincide with the streamline-body lines. Various tail-outlet areas were obtained by successively decreasing the length of the body. The internal duct was of converging section to represent typical practice in the design of outlet openings. The annular outlet openings were designed primarily to exhaust the air as nearly as possible in the stream direction. The

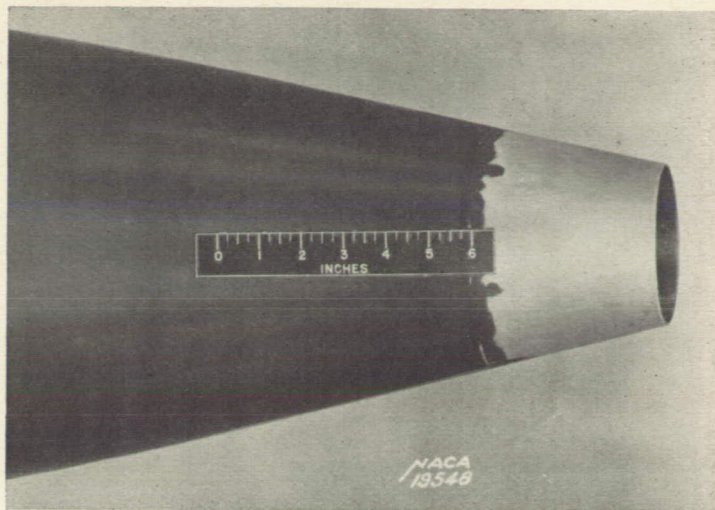
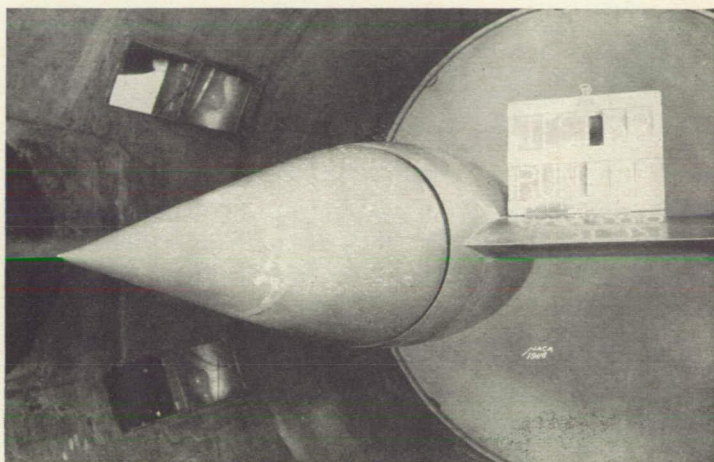


FIGURE 4.—Typical tail outlet opening. Tail C.

FIGURE 5.—Annular outlet opening at $x=0.63L$.

areas were selected from consideration of the quantity of air required by a radial engine large enough to occupy the maximum section of a fuselage. The outlet openings are not in any sense optimum shapes arrived at on the basis of experiment as in the case of the inlet openings; they merely represent typical design practice.

Blower setup.—Air flow in the tests of the individual openings was supplied by a 50-horsepower centrifugal blower mounted outside the wind tunnel on the floor of the test chamber (figs. 6 and 7). Freedom of the floating balance structure was maintained by a mercury seal that connected the blower duct to the wing duct leading to the model. The air flow through the mercury seal was at right angles to the longitudinal (drag) axis of the wind tunnel so that the flow had no momentum in the drag direction. Preliminary tests were made throughout the range of blower speeds at zero air speed in the tunnel, with and without air inlet, to insure that the pressures and flow at the mercury seal had no effect on the drag scale readings.

The flow was metered by a venturi installed on the balance ring between the mercury seal and the model. Several calibrations were made with the venturi in its operating position by surveying the flow in the duct with a rake of 25 total-pressure and 7 static tubes.

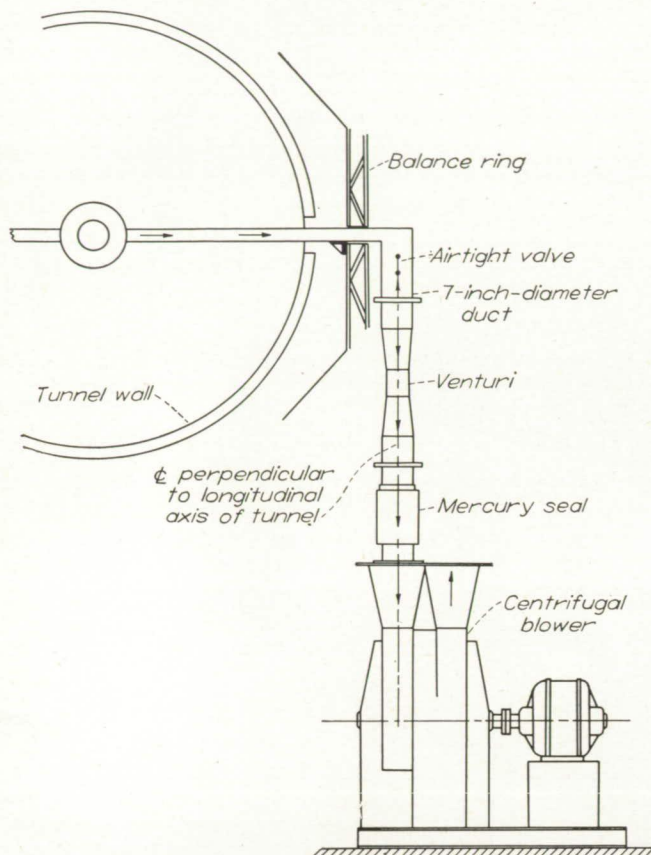


FIGURE 6.—Schematic diagram of setup for tests with blower.

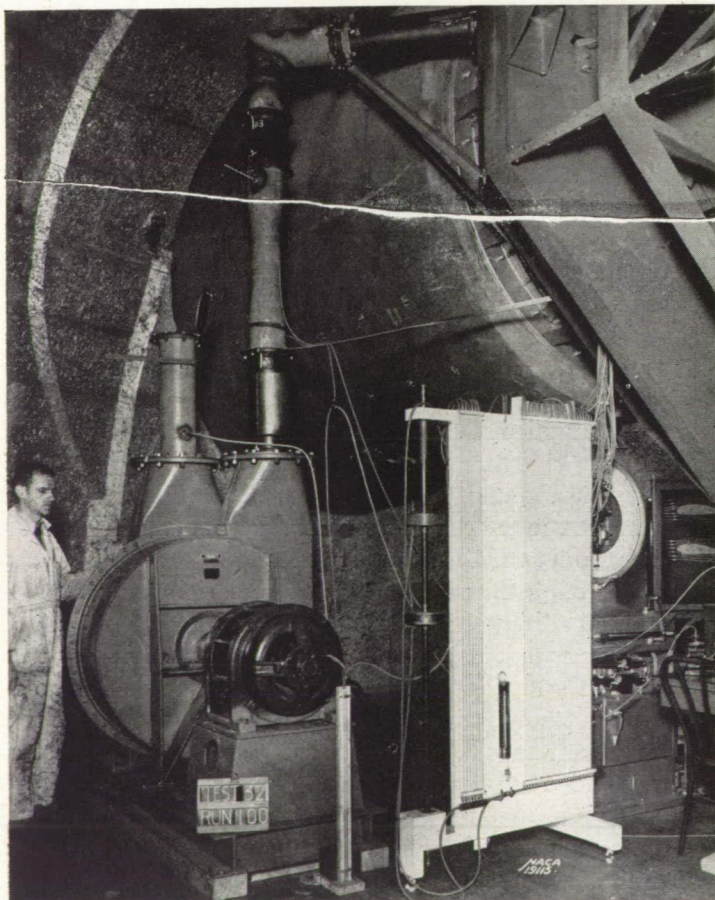


FIGURE 7.—General view of blower setup in the test chamber of the Langley 8-foot high-speed tunnel.

The flow in the system was controlled either by regulating the blower speed or by adjusting a butterfly valve. The zero flow condition was obtained by closing a special airtight valve located near the end of the wing duct. Utmost care was taken to prevent leakage in the system.

Inlet-outlet combinations.—The combinations tested and the internal-duct arrangement are shown in figure 1. The wing-duct openings within the body were faired over and sealed to prevent leakage. Flow regulation was accomplished by means of perforated plates of various conductance installed as shown in figure 2. Static-pressure orifices were installed at two stations in the converging section of the duct ahead of the outlet openings. The flow quantity was determined from the magnitude of the pressure drop between these stations according to a calibration obtained during the tests of the individual outlet openings with the blower-venturi setup. The total pressure and the static pressure at the outlet were determined from this same calibration. In several cases, as a check on the calibration, the quantities were measured directly by means of a small pitot-static tube mounted in the outlet opening.

No particular attempt was made to design an efficient internal-duct system because interest was centered on the external drag and because the blower was more than adequate to overcome large internal losses. However, in the combination tests with the duct open—that is, with no resistance plates inserted to restrict the flow—the internal losses were practically negligible owing to the low duct velocities.

Method of fixing boundary-layer transition.—Transition in the tests designated “with fixed transition” was fixed artificially by means of a $\frac{1}{4}$ -inch wide ring of No. 180 carborundum grains glued to the surface at the desired station. It was found necessary to fix transition on the wing at the 10-percent station by the same method in order to make the drag of the wing constant so that the effective drag of the body could be obtained accurately.

Except for the strips of carborundum, the surfaces of the model were made aerodynamically smooth—that is, further conditioning would result in no decrease in drag.

Static-pressure measurement.—Flush orifices, closely spaced near the nose and in the vicinity of the openings, were installed along the top of the body. Additional static pressures on the bottom and on the side of the streamline body were obtained by means of a small movable static tube. The pressure tubing was led through a channel in the wing to a multiple-tube alcohol manometer in the test chamber.

Boundary-layer measurements.—The measurement of the boundary-layer profiles used in determining the transition point and in showing the effect on skin friction of air inlet was made with small survey units comprised of a single static and four total-pressure tubes. A discussion of the details of the method of determining the location of transition and a description of the small survey unit are given in reference 4.

Wake surveys.—In order to ascertain whether the drag-force measurements were affected by possible variations in the wind-tunnel pressure gradient due to air inlet at the nose, momentum-loss measurements were made in the wake behind the model with nose B. Vertical total-pressure-loss

profiles were obtained at 23 spanwise stations behind the wing and body at several rates of air inlet. The effective drag of the body was obtained by subtracting from the total drag of the section surveyed the drag of a corresponding section of the wing. The wake surveys were made only with nose B.

TESTS

The drag and the pressure-distribution measurements were made simultaneously. The transition determinations required a separate series of runs for each configuration.

Tests of the wing alone and of the wing with the streamline body were carried to 450 miles per hour. The tests of the openings were made at one speed only: 140 miles per hour. This speed was selected from considerations of the available blower performance and of the magnitude of the drag forces required for adequate precision.

The tests were made at an angle of attack of 0° (referred to the axis of the streamline body) with the exception of the runs with the gun in the inlet opening of nose B, which were carried to 3.5° .

RESULTS

The method of computing the velocity, the Mach number, and the Reynolds number in the Langley 8-foot high-speed tunnel is described in reference 5.

The drag data are presented in terms of the external-drag coefficient C_{D_F} plotted as a function of the internal-flow quantity coefficient $\rho Q/\rho_0 FV$. The external-drag coefficient represents the effective external drag of the body in the presence of the wing; the drag due to the internal flow was deducted from the measured effective body drag in all the tests.

The drag due to the internal flow arises from the change in the momentum of the flow in the drag direction. From the momentum equation,

$$\text{Drag force} = \text{Mass flow} \times (V_i - V_w)$$

where V_i and V_w are taken at the same static pressure and in the same direction as the air stream.

For the inlet-opening tests,

$$V_i = V$$

and

$$V_w = 0$$

because the air was brought to rest in the drag direction. The drag-coefficient increment due to the internal flow therefore is

$$C_{D_{F_{i_{inlet}}}} = \frac{\rho Q V}{q_0 F} = 2 \left(\frac{\rho Q}{\rho_0 F V} \right)$$

For the outlet-opening tests, the air exhausted through the outlets had no initial velocity in the drag direction—that is, $V_i = 0$. The velocity at the exit opening v_e was necessarily measured where the static pressure p_e was generally different from the stream static pressure. Therefore the final outlet velocity attained at some distance behind the model where the pressure had returned to the free-stream static pressure

p_0 was computed by Bernoulli's theorem

$$V_w = \left[v_e^2 + \frac{2(p_e - p_0)}{\rho_0} \right]^{1/2}$$

and

$$C_{D_{F_{i_{outlet}}}} = -\frac{\rho Q}{q_0 F} \left[v_e^2 + \frac{2(p_e - p_0)}{\rho_0} \right]^{1/2}$$

or

$$C_{D_{F_{i_{outlet}}}} = -2 \left(\frac{\rho Q}{\rho_0 F V} \right) \left[\left(\frac{v_e}{V} \right)^2 + \frac{(p_e - p_0)}{q_0} \right]^{1/2}$$

For the tests in which inlet-outlet combinations were investigated,

$$C_{D_{F_{i_{combination}}}} = C_{D_{F_{i_{inlet}}}} + C_{D_{F_{i_{outlet}}}}$$

whence

$$C_{D_{F_{i_{combination}}}} = 2 \left(\frac{\rho Q}{\rho_0 F V} \right) \left\{ 1 - \left[\left(\frac{v_e}{V} \right)^2 + \frac{(p_e - p_0)}{q_0} \right]^{1/2} \right\}$$

or, in terms of the mean total-pressure loss in the duct ΔH ,

$$C_{D_{F_{i_{combination}}}} = 2 \left(\frac{\rho Q}{\rho_0 F V} \right) \left[1 - \left(1 - \frac{\Delta H}{q_0} \right)^{1/2} \right]$$

It will be observed from these equations that the large internal drag in the inlet tests and the thrust in the outlet tests are balanced in the combination tests, so that only a relatively small internal drag due to total-pressure losses in the duct occurs.

The internal flow quantity coefficient $\rho Q/\rho_0 FV$ appearing in the equations is the ratio of the mass flow through the internal ducts to the mass flow at stream velocity through the area F , the maximum cross-sectional area of the body.

The pressure-distribution results obtained in the tests of the individual inlet and outlet openings are presented for a number of values of the ratio of mean velocity in the opening to stream velocity v/V . This parameter determines the local angle of attack at the inlet nose lip and hence governs the pressure distribution over a given nose shape.

The characteristics of the streamline body are shown in figures 8 to 11. Low-speed pressure-distribution data are given in figure 8, and the variation with Mach number of the peak pressure coefficients P_{max} on top of the body for the wing-body combination is presented in figure 9 with extrapolations to show the critical Mach number of the combination and of the body alone. The variation with Mach number of the critical pressure coefficient P_{cr} , the pressure coefficient corresponding to the local attainment of sonic velocity, is also shown in figure 9. Figure 10 shows the results of the transition measurements, and in figure 11 the force-test results, for Reynolds numbers ranging from 4,000,000 to 20,000,000 and the corresponding M values of 0.10 to 0.60 are given.

The results of the tests of the inlet openings with the inlet air exhausting through the external-blower system are presented in figures 12 to 21. Figure 12 shows the pressure distributions about the three inlet openings for various values of v/V (ratio of mean inlet velocity to stream velocity) compared with the streamline-body distribution. Only the

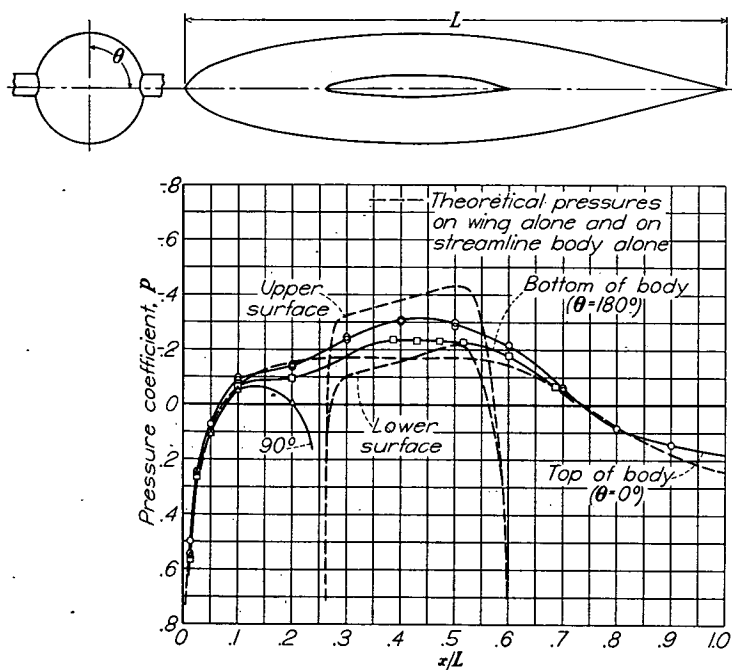


FIGURE 8.—Measured static-pressure distribution about streamline body mounted on NACA 27-212 airfoil compared with theoretical distribution on wing alone and body alone. $M=0.18$; $R=7.6 \times 10^6$.

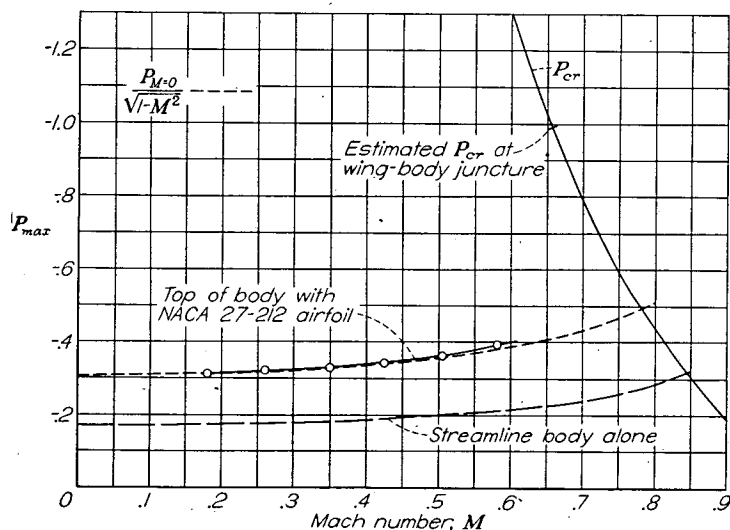


FIGURE 9.—Variation with Mach number of peak pressure coefficient on top of streamline body. Comparison with theory.

forward quarter of the body is represented because the pressures over the remainder of the body were essentially unaffected by the inlet openings. In figure 13 the pressure distributions on the bodies with noses A, B, and C and on the streamline body are compared at the condition of zero inlet flow and at a flow coefficient of 0.057, a practicable high-speed value. The inlet-velocity ratios corresponding to this flow coefficient are approximately 0.20, 0.40, and 0.80, respectively, for noses A, B, and C.

The drag and the transition results obtained are correlated in figure 14. Figure 15 shows the drag-force data obtained with transition artificially fixed near the leading edge of the noses as compared with that of the streamline body with transition fixed at corresponding locations. The drag obtained from the wake surveys is also plotted in figure 15.

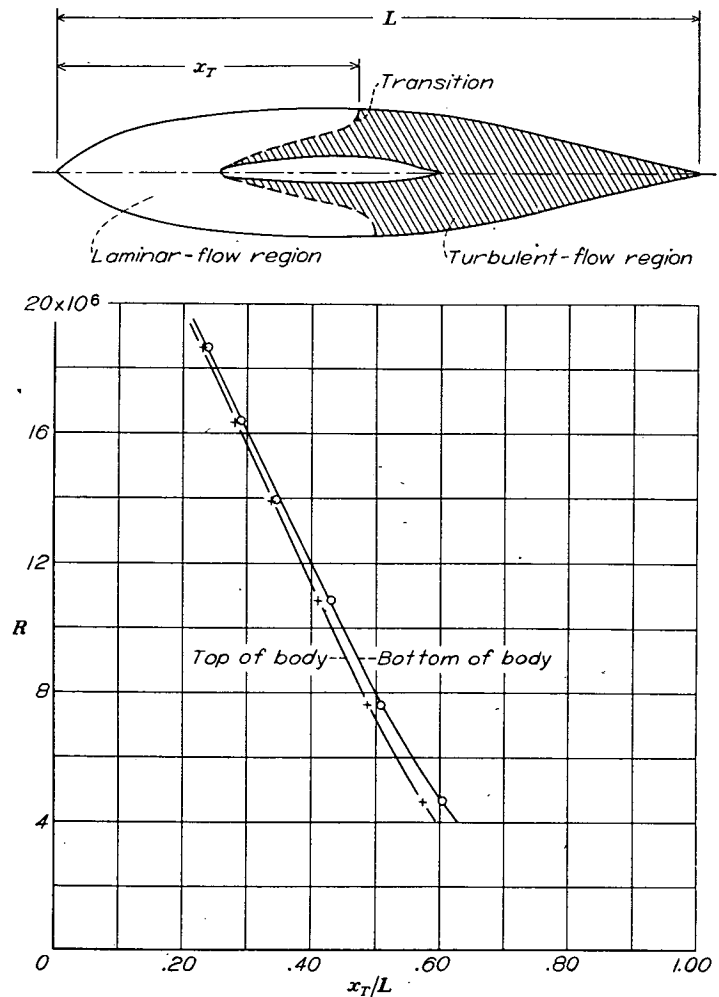


FIGURE 10.—Variation with Reynolds number of the location of the transition point on top and bottom of streamline body.

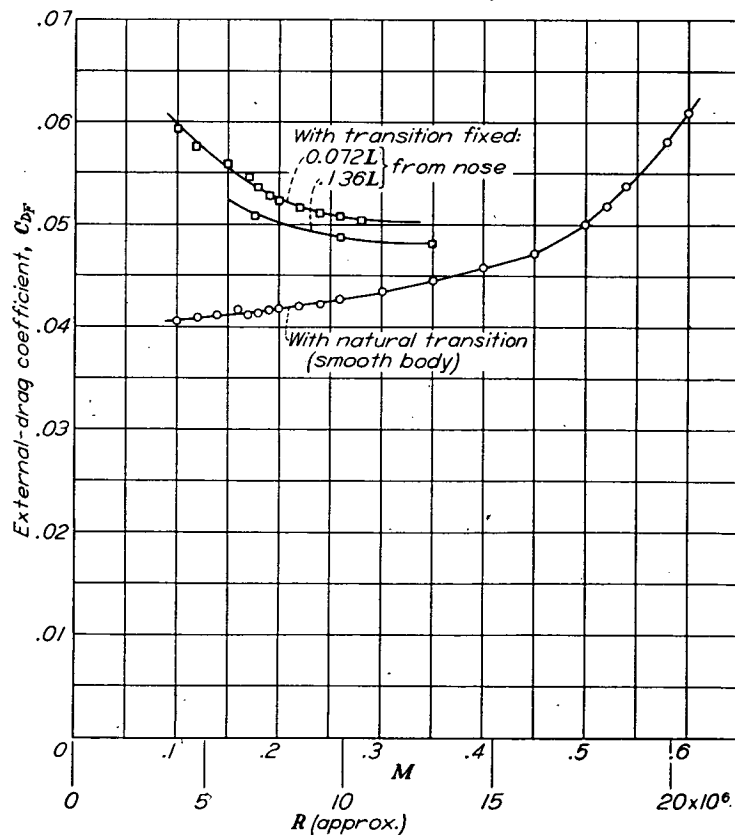


FIGURE 11.—The external-drag coefficient of the streamline body with fixed and with natural transition.

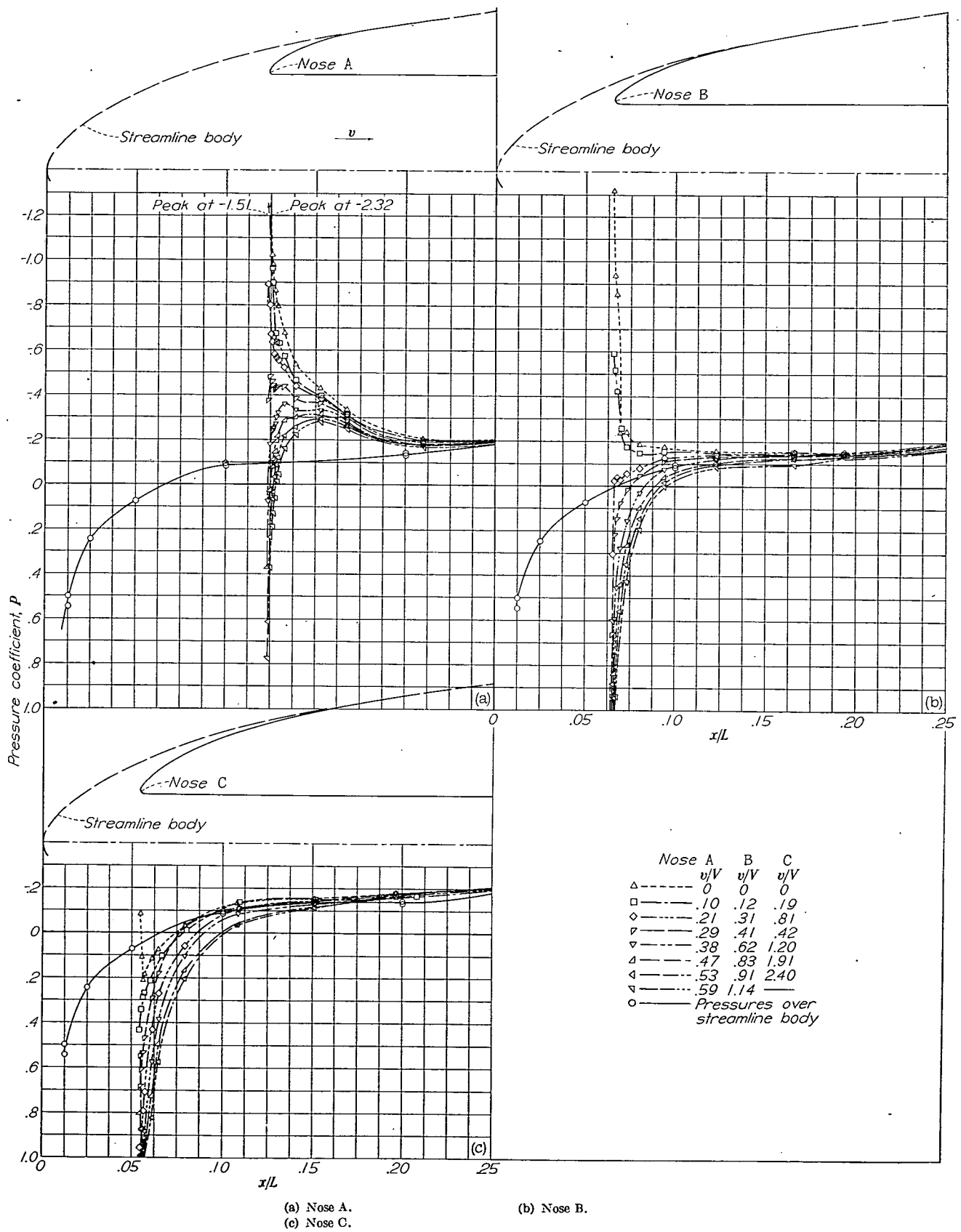


FIGURE 12.—The static-pressure distribution over the nose inlet openings.

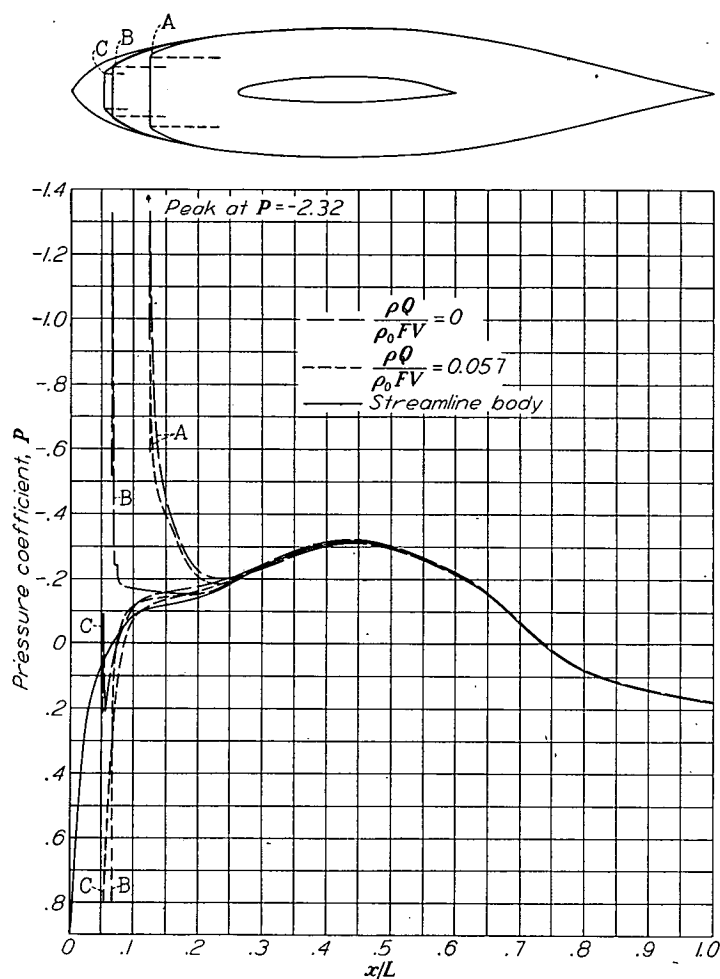


FIGURE 13.—Comparison of the static-pressure distribution over the three nose inlet openings at two rates of air inlet.

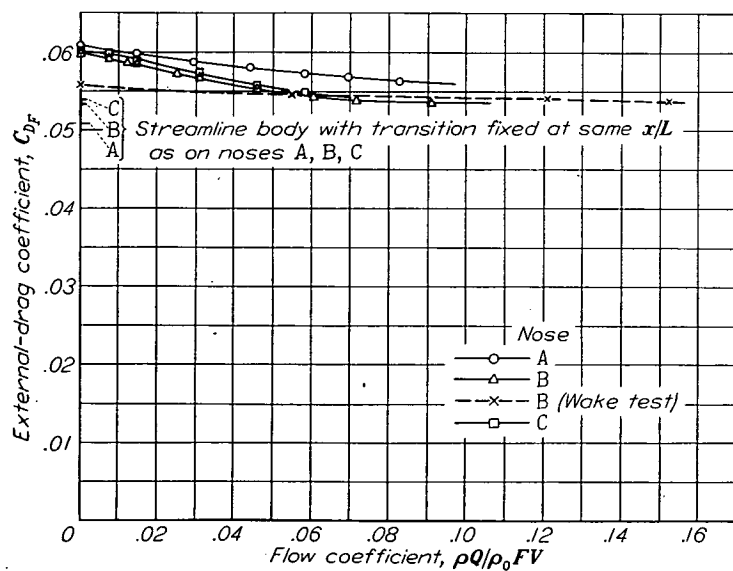


FIGURE 15.—The external-drag coefficient determined from both force and wake-survey tests with transition fixed at the nose of each inlet opening.

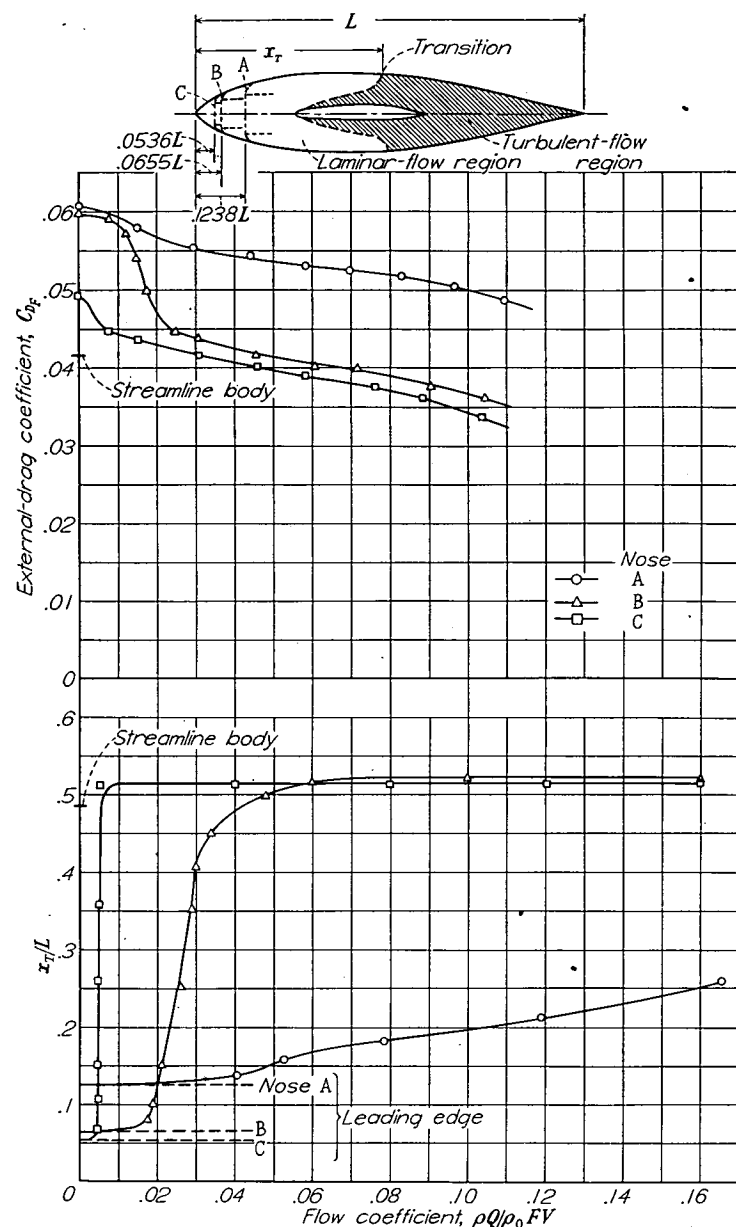


FIGURE 14.—Correlation of external-drag coefficient and transition-point location for the three nose inlet openings.

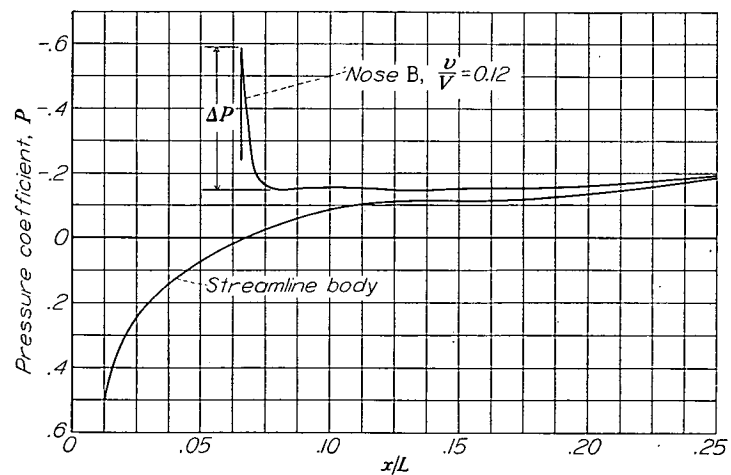


FIGURE 16.—Typical static-pressure distribution diagram for nose-inlet openings B and C at low rates of air inlet.

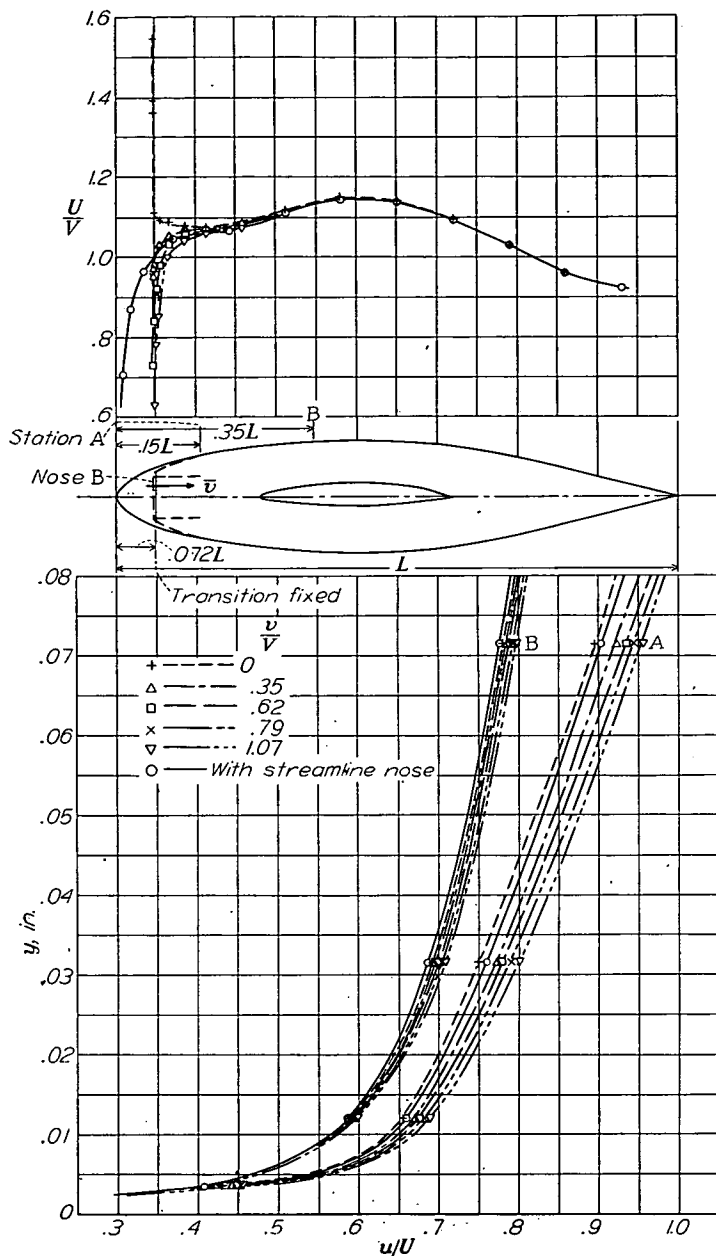


FIGURE 17.—Boundary-layer velocity profiles at two stations for various air-inlet velocity ratios, and the corresponding diagrams of local velocity distribution over the top of the body. Nose B; fixed transition.

In the correlation of the pressure distribution and the transition data, it was found that for values of ΔP (fig. 16) greater than approximately 0.2, transition (fig. 14) occurred at the location of the pressure peak. For lower values of ΔP (higher rates of air inlet), extensive laminar boundary layers existed in spite of the large adverse pressure gradient that followed the peak.

The effect of air inflow (nose B) on the boundary-layer velocity profiles at two stations on the body is shown in figure 17.

In figure 18 the changes in pressure distribution resulting from modifications of the lip shape of nose inlet B-4 (one of the intermediate shapes tested in developing nose B) are given. The force-test results obtained with these modifications showed that shapes B-4a and B-4b caused very slight

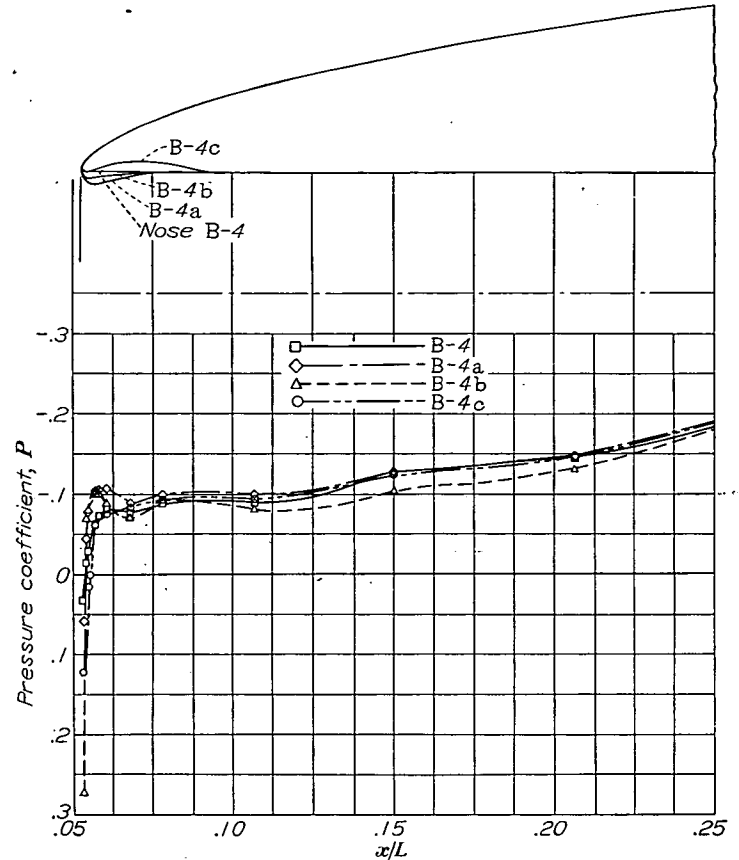


FIGURE 18.—The effect on static-pressure distribution of variations in nose shape at the duct entry. $\frac{v}{V} = 0.30$.

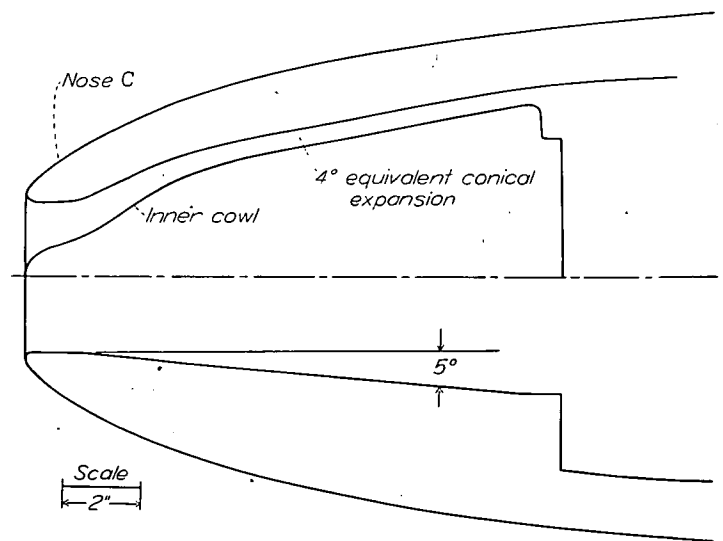


FIGURE 19.—Nose C with conical expanding duct and with inner cowl.

increases in external drag; the cut-out, B-4c, had no effect on the drag. Major changes in the internal duct employed with nose C (fig. 19) had no measurable effects on either the external pressure distribution or the external drag.

Optimum nose shapes for arbitrary inlet-duct sizes.—In order to make possible the derivation of optimum nose profiles for inlet-opening sizes other than those investigated, the three nose profiles tested were reduced to the same length

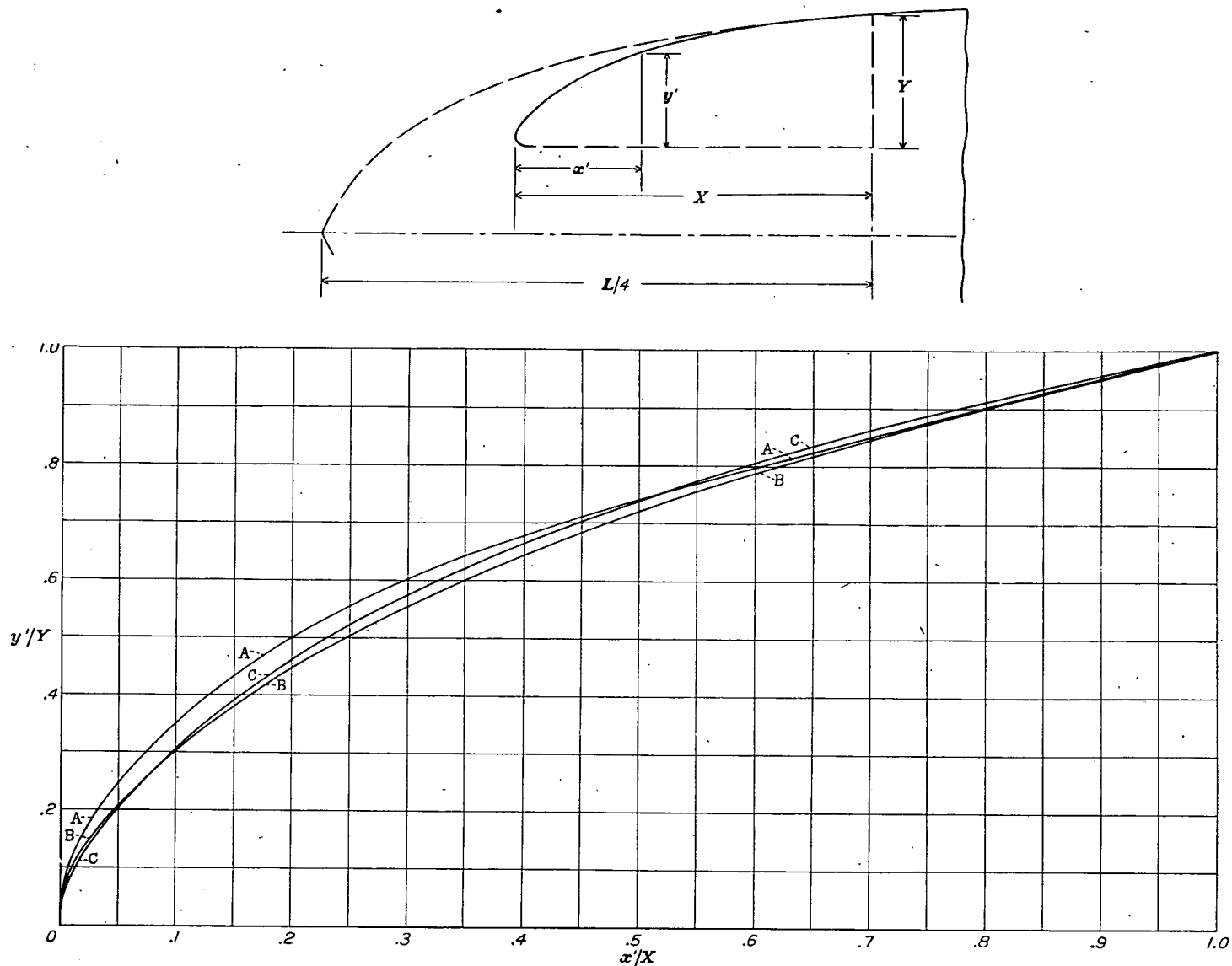


FIGURE 20.—Comparison of the three inlet-opening nose profiles reduced to the same length and depth.

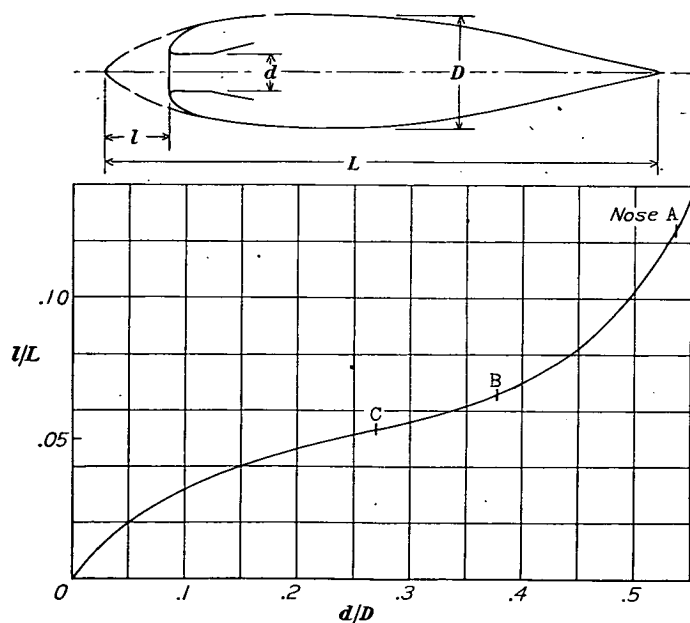


FIGURE 21.—Variation with duct diameter of the distance between the end of the nose and the tip of the streamline body.

(measured from the $L/4$ station of the streamline body) and the same depth. The ordinates thus obtained are given in table II and plotted in figure 20. The marked similarity of the profiles plotted in this way suggested that optimum nose shapes for intermediate inlet-opening sizes on the streamline body could be obtained either by interpolation or by the use of the mean of the three profiles of figure 20. The optimum nose length as a function of the inlet-opening diameter is given in figure 21. The actual nose-profile ordinates for a given inlet diameter are related to the nondimensional ordinates of figure 20 and table II as follows:

$$x' = \left(\frac{x'}{X} \right) X$$

or

$$x' = \left(\frac{x'}{X} \right) \left(\frac{L}{4} - l \right)$$

where l is obtained from figure 21. Similarly,

$$y' = \left(\frac{y'}{Y} \right) \left(y_{L/4} - \frac{d}{2} \right)$$

where y_{LA} is the ordinate of the streamline body at the quarter-length station. If desired, the nose ordinates referred to the end and center line of the streamline body (as in table I) may be obtained from the relations

$$\frac{x}{L} = \frac{x'}{L} + \frac{l}{L}$$

and

$$\frac{y}{R} = \frac{y'}{R} + \frac{d}{D}$$

or

$$\frac{x}{L} = \left(\frac{x'}{X}\right) \left(0.25 - \frac{l}{L}\right) + \frac{l}{L}$$

and

$$\frac{y}{R} = \left(\frac{y'}{Y}\right) \left(0.90 - \frac{d}{D}\right) + \frac{d}{D}$$

The results obtained in the tests of the outlet openings with air supplied by the blower are shown in figures 22 to 26. Figures 22 and 23 show the pressure and force-test results for outlets at the tail. Transition measurements with the largest tail outlet showed that transition occurred at the same station as on the streamline body (fig. 10). The pressure distribution obtained with the two annular outlets is shown in figure 24. Force-test results for the 63-percent annular outlet are given in figure 25. Transition measurements with the 21-percent annular outlet showed that transition occurred at the outlet for all rates of flow. The 63-percent

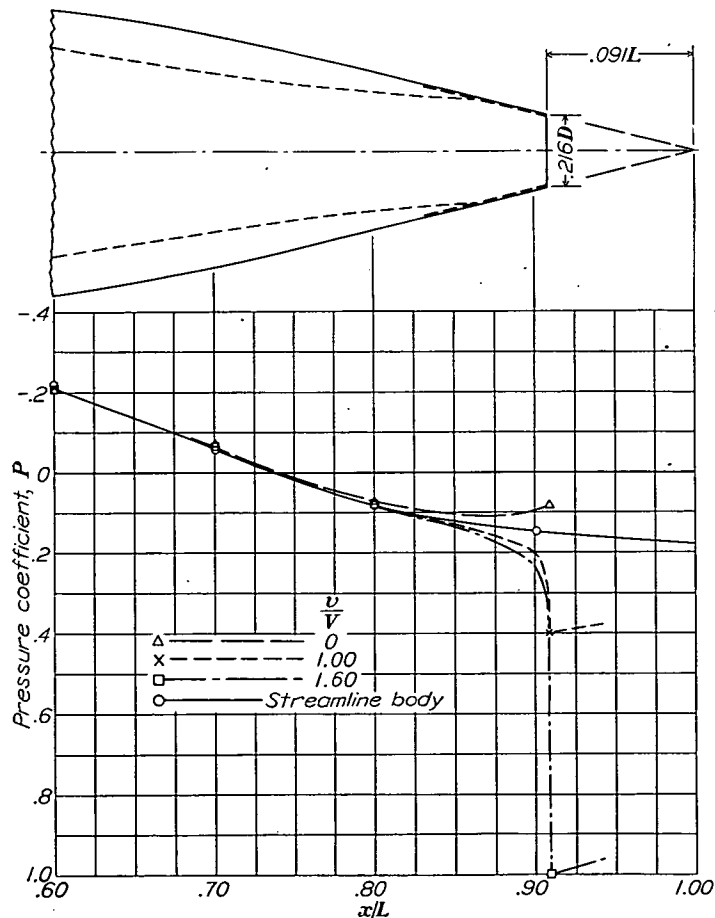


FIGURE 22.—The static-pressure distribution on the afterbody with a typical tail outlet. Tail D.

annular outlet was about $0.14L$ behind the most rearward position of the transition point but appeared to have a slight influence on the transition location, such that the location was displaced somewhat toward the tail as the flow rate was increased.

Figure 26 shows a sketch of the probable outlet flow conditions with tail outlet D and with a suggested improved form of tail outlet.

Before the results of the inlet-outlet combination tests are presented, figure 27 is given in order to show the relatively small internal drag occurring in the combination tests. At high flow rates, where no internal resistance plate was required, this internal drag approached zero; whereas in the individual opening tests the internal drag was several times the external drag of the body. The external-drag determinations in the combination tests were consequently more reliable than in the tests of the single openings.

Figures 28 to 32 show the drag results obtained for the combinations of inlets with three tail outlets. Figure 33 compares the drag of the 63-percent annular outlet with that of tail outlet C when tested in combination with nose B.

The pressure-distribution results obtained with the combinations are not shown because no consistent measurable interference effects occurred—that is, the outlets had no appreciable effects on the pressures at the inlets and vice versa. Similarly, the transition locations on the combinations were the same as in the tests of the inlet openings alone.

In figure 34 the drag of the nose B and tail C combination is compared with an estimate of the drag based on the tests of the single openings. The drag increments (above the streamline-body drag) due to nose B and tail C were added to the streamline-body drag in making the estimate.

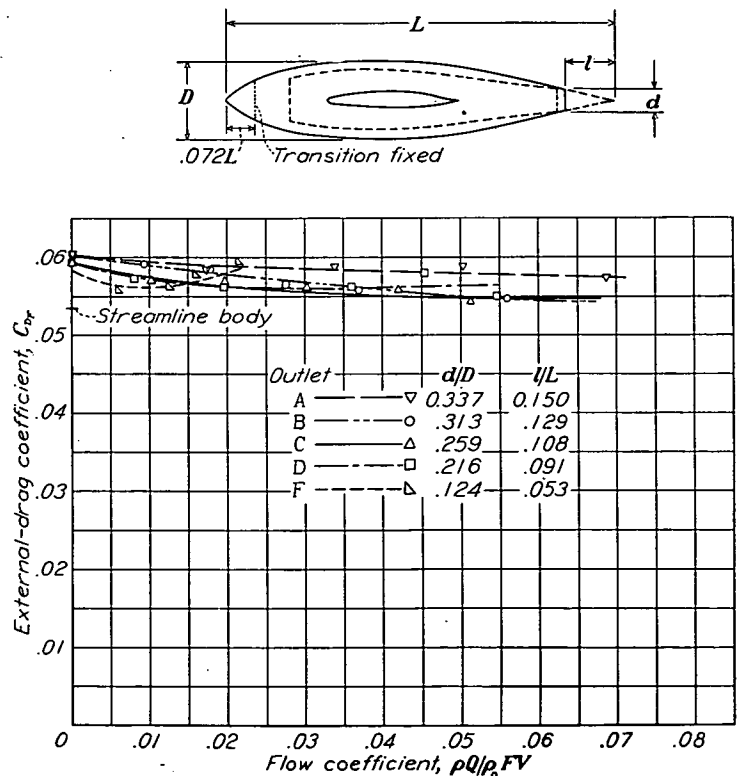


FIGURE 23.—Variation with flow coefficient of the external-drag coefficient obtained with the tail outlet openings. Fixed transition on nose.

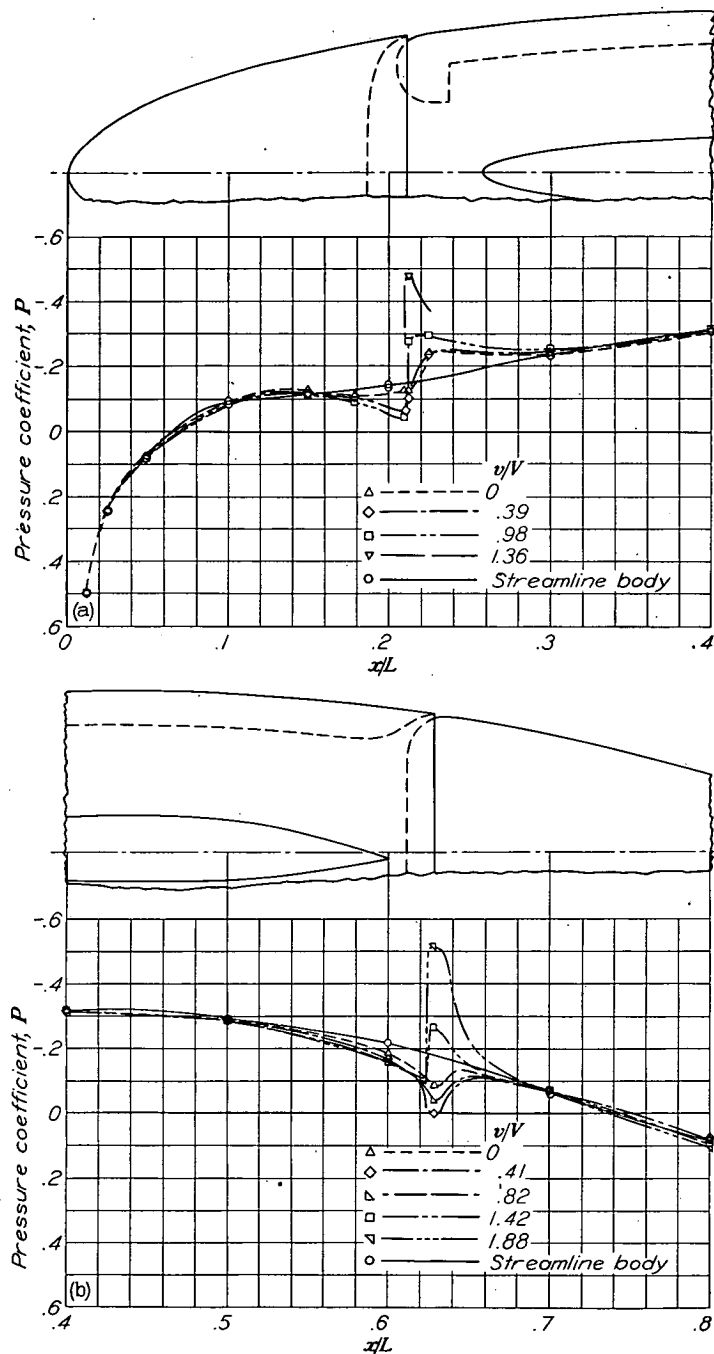


FIGURE 24.—The static-pressure distribution with annular outlet openings.

The main increase in drag due to the guns (table III) occurred at angles of attack other than zero as a result of partial separation of the external flow at the top of the nose as evidenced by the pressure-distribution plots (fig. 35). Increasing the rate of air inlet had a beneficial effect in reducing or preventing this separation. The smooth-barrel cannon had considerably less drag than the machine gun (sketched in fig. 35). Decreases in the length of the barrel extending beyond the nose resulted in appreciable drag reductions. It has been found that the drag of a smooth-barrel gun was considerably reduced by replacing the sharp edge at the muzzle

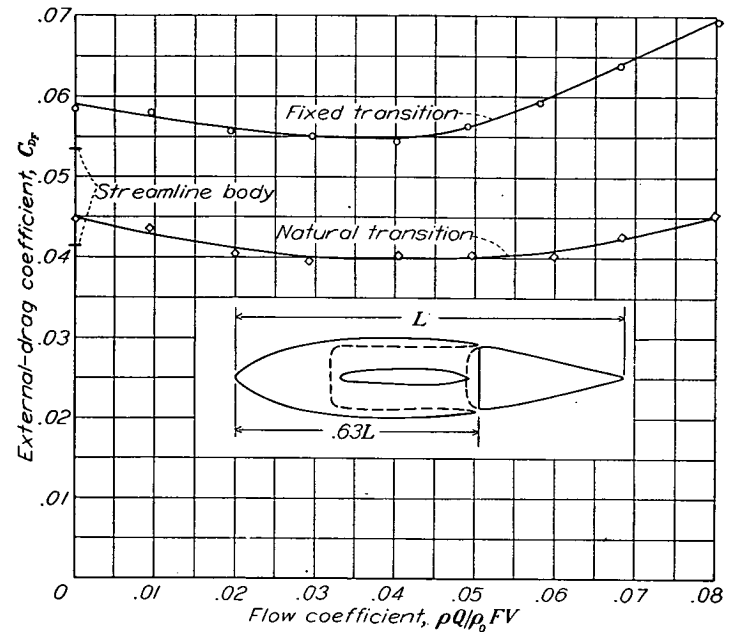


FIGURE 25.—Variation with flow coefficient of the external-drag coefficient obtained with the annular outlet opening at the 63-percent station.

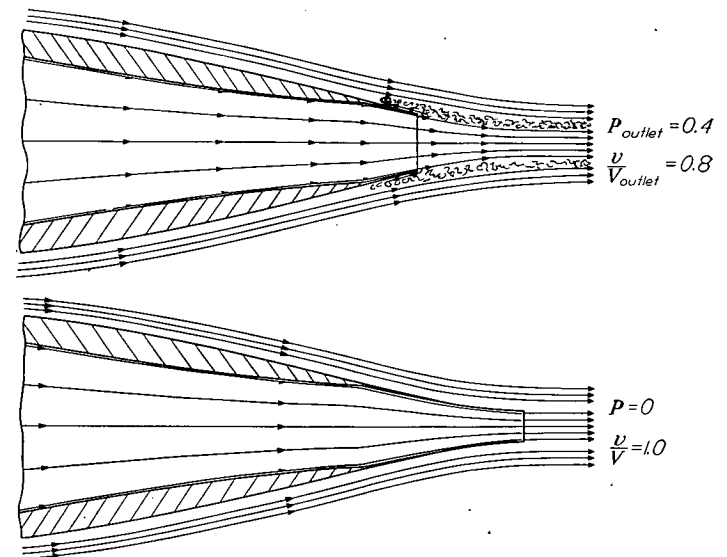


FIGURE 26.—Sketch of flow conditions (no internal losses) for a tail outlet opening, similar to those tested, and for a suggested improved type.

of the gun with a rounded edge of small radius. It is considered likely that the unfavorable effects of the guns would be somewhat less in the high Reynolds number (fixed transition) condition than shown in table III, because no drag would result from disturbance of the laminar flow.

PRECISION

The accuracy of the body-drag determinations was somewhat impaired by the high drag of the wing with fixed transition relative to the body drag, the effective body drag varying from about 0.5 to 0.3 of the wing drag. In the tests of the individual openings, additional sources of error were the leakage of air in the external ducts and possible changes in the tunnel-pressure gradient due to the removal or the

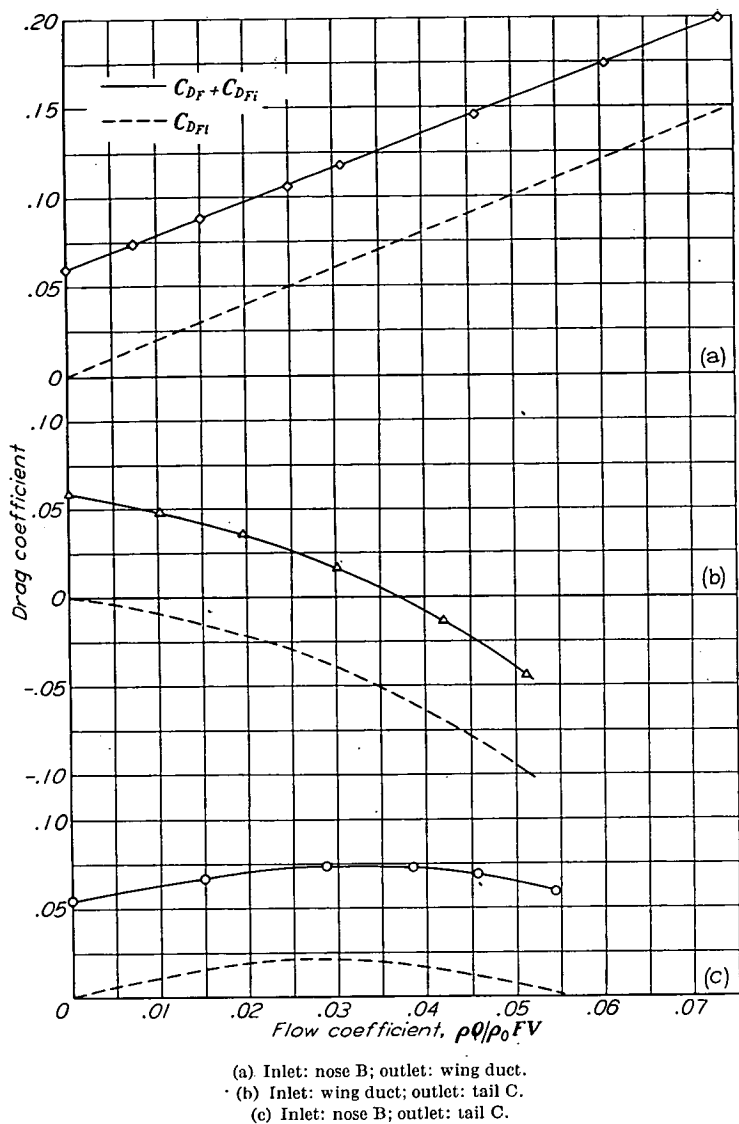


FIGURE 27.—Comparison of magnitude of total body drag with internal drag in typical tests of individual openings and of a combination of an inlet and an outlet opening.

addition of air to the tunnel stream. The results obtained with the inlet-outlet combinations, however, are believed free of these two sources of inaccuracy because no air was added or removed from the tunnel and no leakage was likely due to the absence of all external ducts. A buoyancy correction of about 10 percent of the effective body drag was applied to all of the force-test results.

The wake measurements are more nearly free of these sources of error that affect the force tests. Evaluation of the possible magnitude of the drag-test errors will be made in the discussion of the results.

The precision of measurement of the rate of internal air flow is considered to be of a high enough order so that the external-drag determinations are practically unaffected by the small error in obtaining the internal drag, except possibly in the case of the individual opening tests at the highest rates of air flow. Repeated calibrations of the venturi during the tests showed excellent agreement.

The only significant sources of error in the pressure data are due to the inaccuracy of flow measurement and the

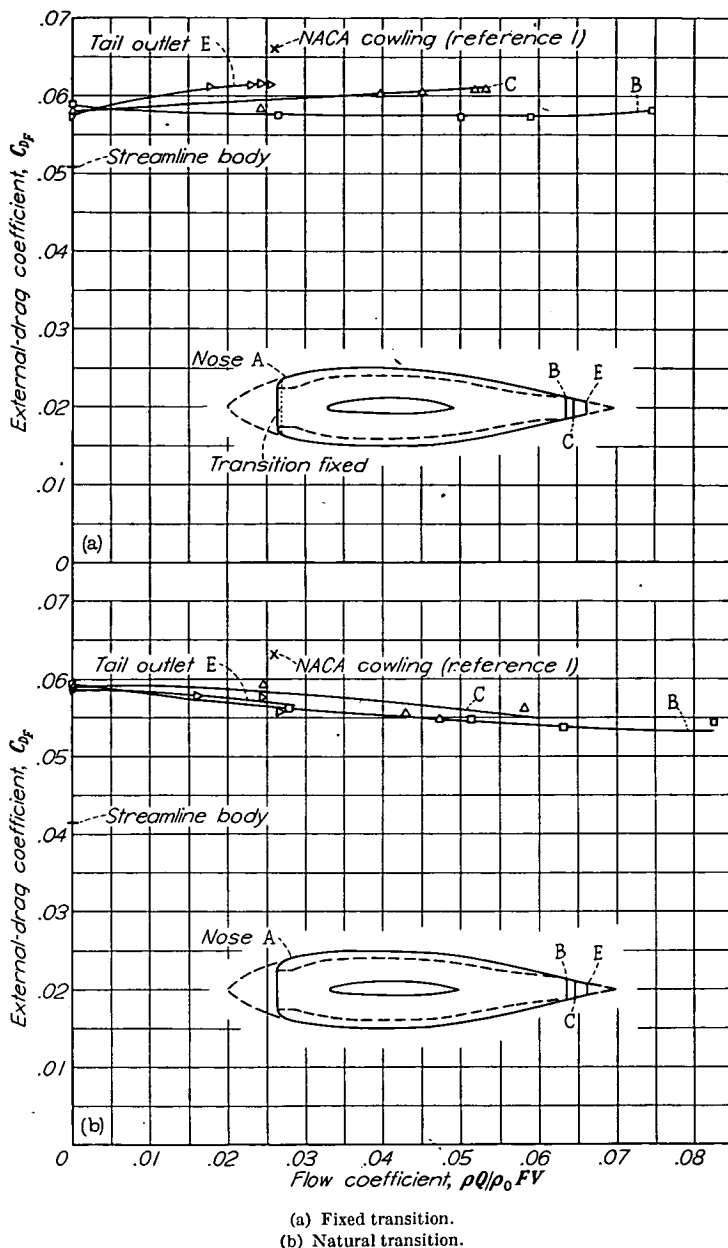


FIGURE 28.—Variation with flow coefficient of the external-drag coefficients for nose inlet opening A combined with various tail outlet openings.

tunnel-wall effects. The maximum possible change in the pressure coefficients due to the tunnel-wall effects was computed to be only about 3 percent. Possible errors in flow measurement could cause measurable changes in pressure coefficients only at the lowest inlet-velocity ratios.

DISCUSSION

STREAMLINE BODY

Pressure distribution and transition.—The presence of the wing had a pronounced effect on the pressure distribution over the body (fig. 8). The local velocities over the central portion were increased and the peak-pressure point was moved forward. At low Reynolds numbers the disturbances due to the wing controlled the location of transition on the body. (See sketch in fig. 10.) There was a rapid forward movement of the transition point with Reynolds number so

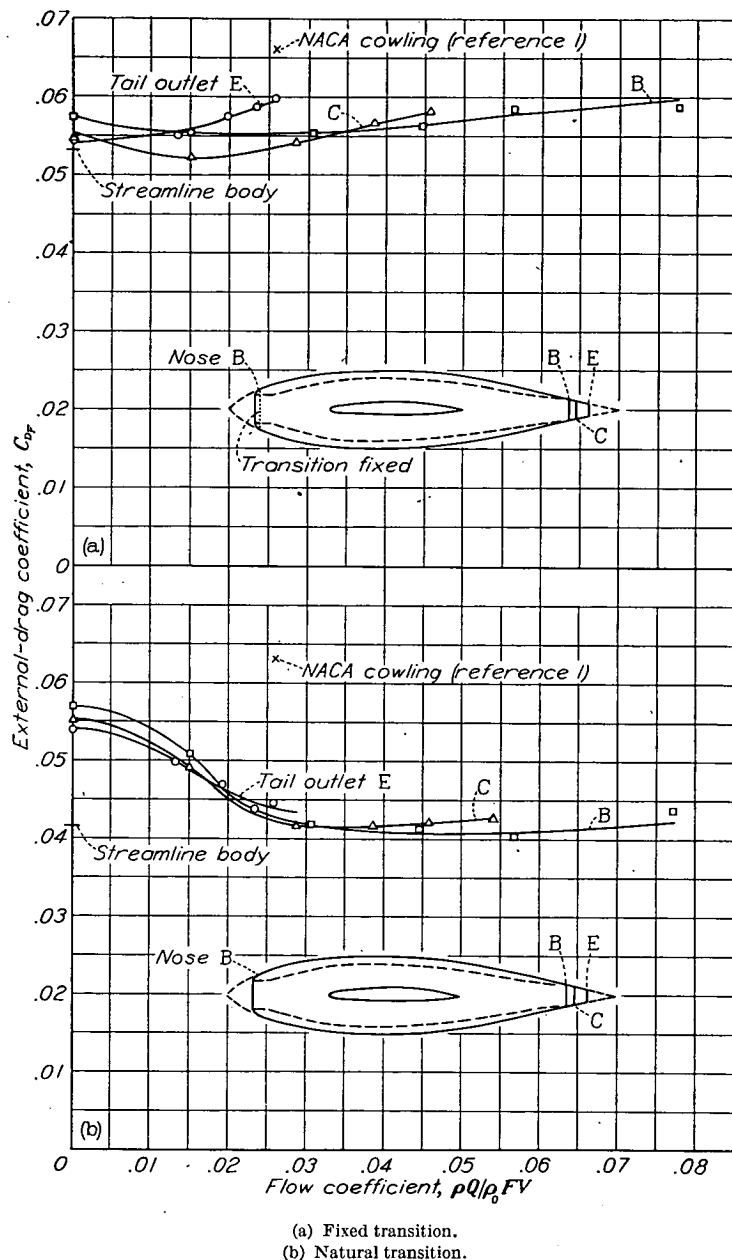


FIGURE 29.—Variation with flow coefficient of the external-drag coefficients for nose inlet opening B combined with various tail outlet openings.

that at the highest test Reynolds number transition occurred considerably ahead of the leading edge of the wing (fig. 10). If a similar forward movement of transition with Reynolds number should occur under flight conditions, the extent of laminar flow obtainable at full-scale Reynolds number would be slight.

Critical speed.—The variation with the Mach number of the peak pressure coefficient on top of the body (fig. 9) was found to agree well with the theoretical variation (obtained from reference 6). Extrapolations of the low-speed, peak negative pressure coefficients to the critical pressure coefficient (at which the speed of sound is attained locally) were made according to the theory. The critical Mach number

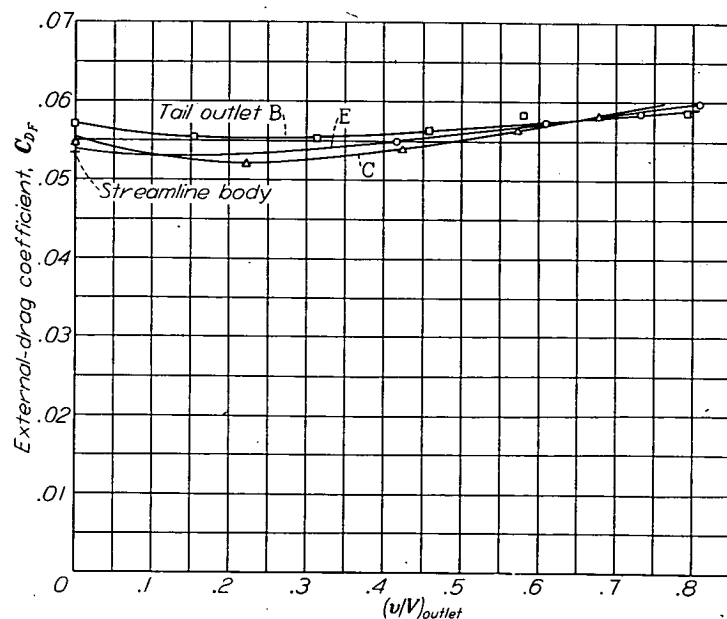


FIGURE 30.—Variation with outlet velocity ratio of the external-drag coefficients for nose B combined with various tail outlet openings. Fixed transition.

of the streamline body alone was thus found to be 0.84 (fig. 9), which corresponds to 600 miles per hour at 20,000 feet (-12° F) in standard air. The critical speed of a wing-body combination is considerably less than that of either component, owing to the increase in peak negative pressures on the wing due to the presence of the body. (See reference 6.)

Effective body drag.—Figure 11 shows the large differences in drag at low Reynolds numbers between the fixed and the natural transition conditions. Calculations based on flat-plate skin-friction coefficients showed that these differences are wholly accounted for by the changes in skin friction on the body. The difference decreases with increasing Reynolds number due to the forward movement of the transition point (fig. 10). The rise in the drag coefficient at the high Mach numbers is indicative of the approaching critical speed of the wing-body combination (estimated $M_{cr}=0.66$). Comparison of the magnitude of the low-speed drag coefficients with the results obtained in reference 3 for the NACA 111 form indicated that the flow over the body was satisfactory. Tuft surveys corroborated this conclusion. It was found, however, that the addition of the body to the wing caused a local separation of the flow at the trailing edge of the wing. The effective drag of the body was therefore somewhat higher than it would have been had a more efficient wing-body juncture been employed.

NOSE-INLET OPENINGS

Pressure distribution.—The nose-inlet shapes employed in this investigation were developed in a series of tests in which the nose shape and the length for a given inlet size were progressively modified to obtain the most satisfactory drag and pressure-distribution characteristics. It was found that by taking air into the body at sufficiently high velocities the

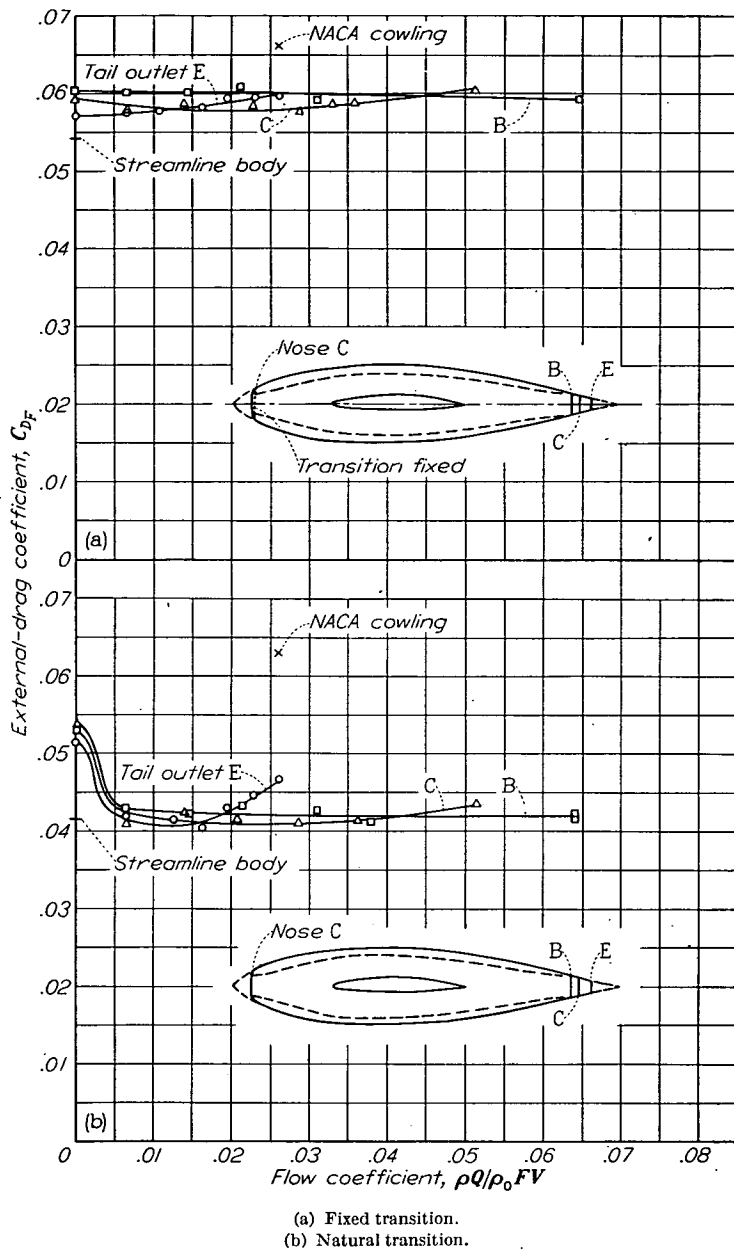


FIGURE 31.—Variation with flow coefficient of the external-drag coefficients for nose inlet opening C combined with various tail outlet openings.

high negative-pressure peak which occurred over the noses at low flows could be greatly reduced in magnitude; for the smaller inlet sizes, the peak could be entirely eliminated. This result has the obvious beneficial effect of greatly increasing the critical compressibility speed which, as in the case of NACA cowling installations (reference 2), is generally fixed by the magnitude of the peak-negative pressures at the nose. In addition, it was found that laminar boundary layers as extensive as the ones with the streamline nose could be obtained. The design objectives then aimed at in developing noses B and C were to eliminate the pressure peak at as low an inlet-velocity ratio as possible and to obtain a uniform favorable pressure gradient similar to that of the streamline

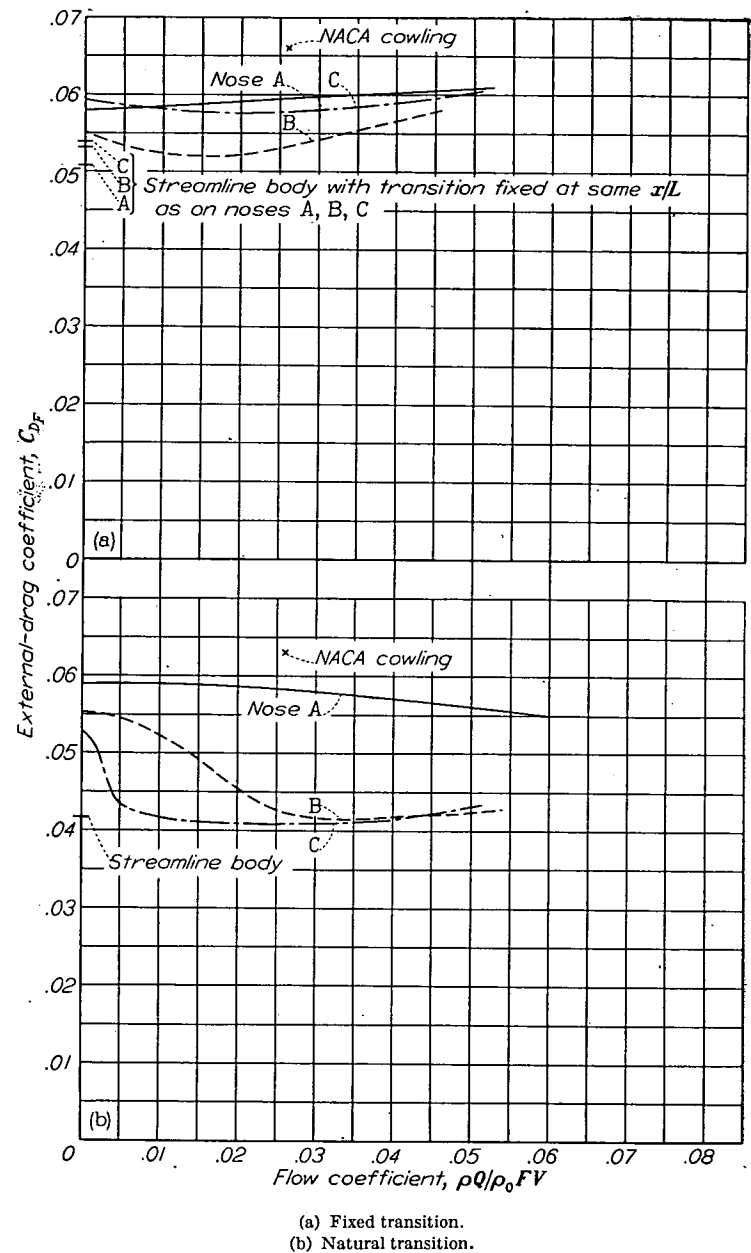


FIGURE 32.—Comparison of the external-drag coefficients of the three nose-inlet openings tested with tail outlet opening C.

body. Figures 12 (b) and 12 (c) show that the desired results were achieved when the inlet-velocity ratios reached or exceeded 0.3 or 0.2 for noses B and C, respectively. Extensive laminar boundary layers (fig. 14) were formed even before the peak was fully eliminated with values of ΔP (fig. 16) as high as 0.2.

For the largest inlet opening, nose A, it was impossible entirely to eliminate the pressure peak, even with impractically high rates of air inlet (fig. 12 (a)). The peak was greatly reduced at practical inlet velocities but little advantage due to laminar flow was attainable (fig. 14).

Comparison of the pressure distribution of the streamline body with those for the three noses is made in figure 13 at a

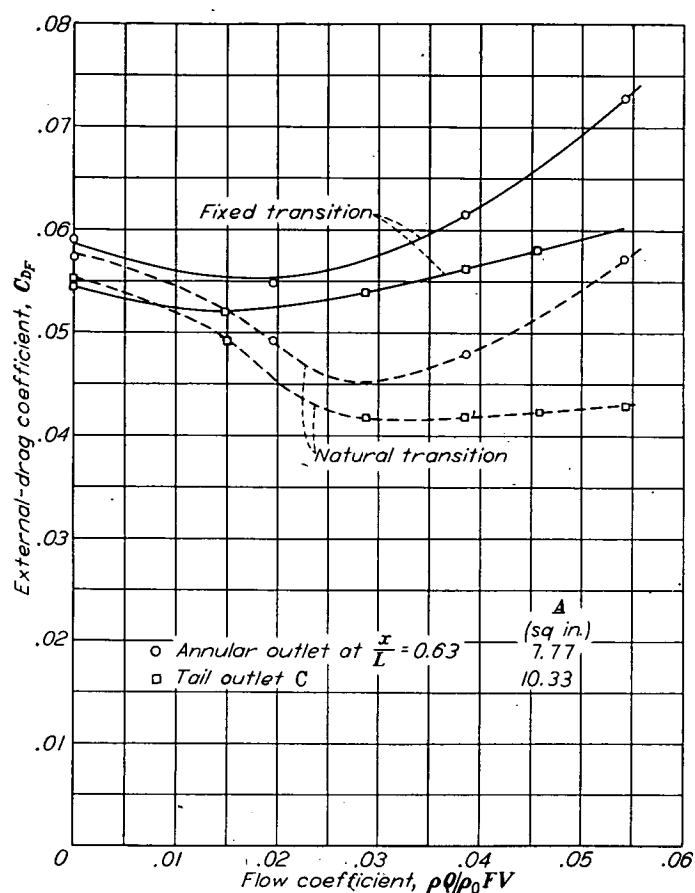


FIGURE 33.—Comparison of the external-drag coefficients of tail outlet C and the annular outlet at the 63-percent station of the fuselage. Nose B.

value of the flow coefficient corresponding to high-speed flight conditions and at zero flow.

Critical speeds.—The critical Mach number corresponding to the pressure peak on the largest inlet opening, nose A, at a practical rate of air inlet for a large radial engine (fig. 13), is 0.64. With the smaller inlets, noses B and C, no pressure peak occurred, and the indicated local velocity increments were so small that the critical speed of a fuselage employing these shapes would be determined by the cockpit enclosure or the wing-fuselage juncture—that is, the highest local velocity would occur at some point other than on the nose.

External drag.—Figure 14 shows that the abrupt decreases in external-drag coefficient of noses B and C at low rates of flow occurred as a consequence of the formation of extensive low-drag laminar boundary layers. This phenomenon did not occur with nose A because, as previously discussed, the unfavorable pressure distribution near the nose precluded the possibility of appreciable laminar flow. It will be noticed, however, that the drag of nose A showed a general decrease with increasing flow coefficient as did the drags of noses B and C after the laminar boundary layers had been formed. Similar decreases occurred with transition fixed (fig. 15).

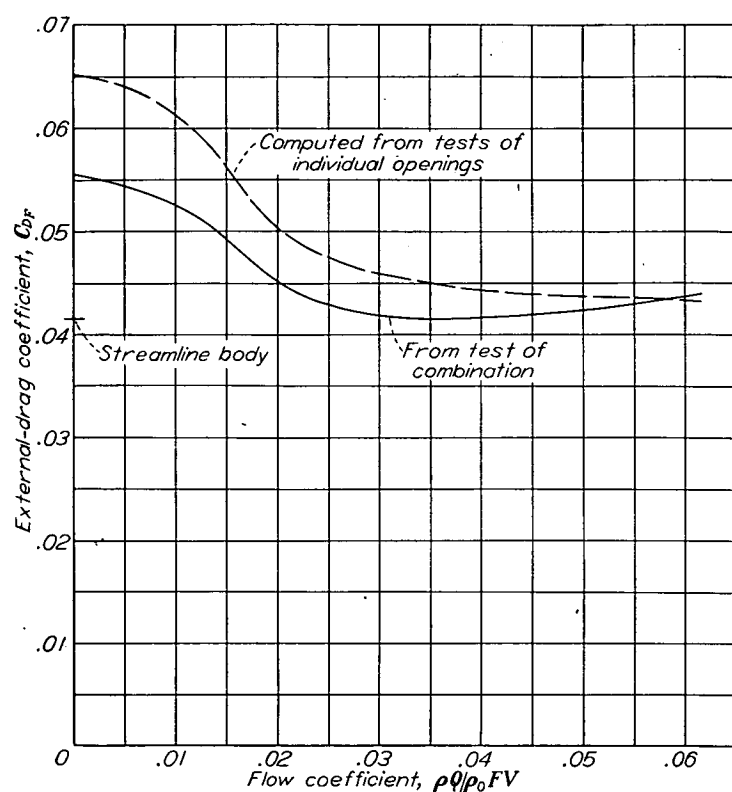


FIGURE 34.—The external-drag coefficient of the nose B and tail C combination compared with an estimate based on the drag increments obtained in the tests of the individual openings. Natural transition.

In order to aid in finding the cause of the decrease in drag with increasing air-inlet velocity, partial boundary-layer velocity profiles were measured at two stations, $0.15L$ and $0.35L$, behind nose B with fixed transition for a wide range of inlet-flow ratios. The results (fig. 17) showed a decrease in the thickness of the turbulent boundary layer as the rate of air-inlet velocity was increased in spite of slight decreases in the velocity outside of the boundary layer. Two conclusions may be drawn from this result:

- (1) The losses over the forward part of the nose are decreased as air inlet is increased.
- (2) The skin friction over the main part of the body (to the rear of the $0.15L$ station) should increase slightly with air inlet.

From the drag results (fig. 15), it is evident that the decrease in losses at the nose more than compensates for the slight increases in skin friction behind the nose because an over-all decrease in external drag with air inlet occurs.

In regard to the magnitude of the external drag with air inlet, figure 14 shows that the external drag with noses B and C was reduced to less than that of the streamline body. For the fixed transition condition, the drag of these noses was approximately the same as for the streamline body. With nose A, in both cases, the drag was considerably higher. Tests of the three noses in combination with tail outlet C

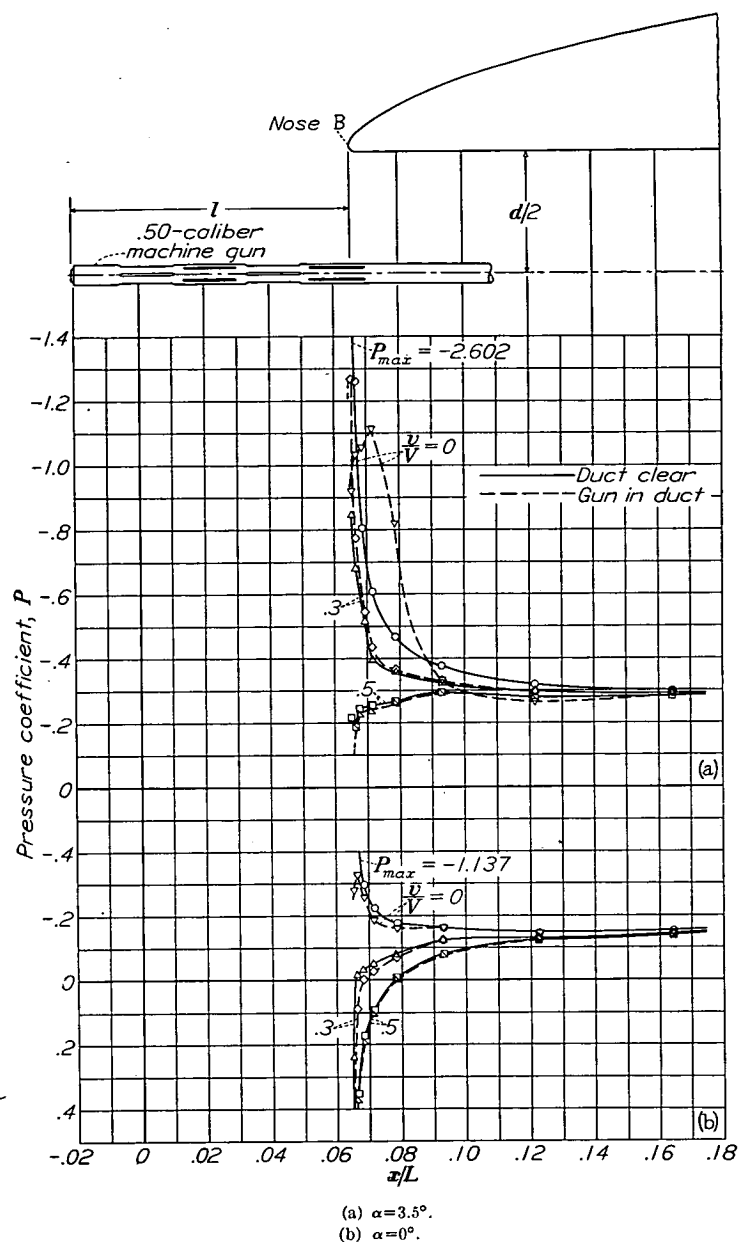


FIGURE 35.—The effect of the presence of a .50-caliber machine-gun model in the inlet opening on static-pressure distribution on top of the body. Nose B; natural transition.

(fig. 32) showed about the same relative drag characteristics as the tests of the single openings. The fact that the external drag with the openings decreased to that of the streamline body may be accounted for by the fact that the wetted area with the openings is somewhat less than that for the streamline body. In addition, the passage of air through the internal system has an effect on the external flow similar to a decrease in the effective thickness of the body.

The wake-survey results (fig. 15) show that the rate of drag decrease with air-inlet-flow coefficient was actually somewhat less than that indicated by the force-test results.

The exaggerated effect shown by the force data is believed due to leakage and possible changes in the tunnel-pressure gradient as air was removed at the nose of the body. Maximum leakage would occur where the pressure in the duct system was the greatest and may account for part of the maximum discrepancy (7 percent) between force and wake drags occurring at zero flow, where stagnation pressure existed in the ducts. At a flow coefficient of about 0.11, the mean duct pressures, and hence leakage, reached a minimum; at this point the force and wake data agree closely. At least for the range of flow covered in figure 15, leakage effects apparently predominated over possible changes in the buoyancy effect in exaggerating the rate of drag decrease with air inlet.

Inlet-opening size.—The size of the inlet opening in an actual installation should be governed by considerations of both the external and internal flow. In a consideration of the external drag, it has been shown that nose B, although twice as large in area as nose C, was equally effective, so that either nose might be employed, the choice depending on the quantity of air flow required. It has also been shown that the opening must be designed for an inlet-velocity ratio of at least 0.3 in order to permit the nose-pressure peak to be eliminated. Higher inlet velocities would be of some benefit externally.

High inlet-velocity ratios are detrimental to the internal-duct efficiency because they necessitate large expansions and make the friction and bend losses high. It is suggested in reference 7 that low inlet velocities may have an additional advantage to the internal flow in that comparatively large expansions can be made efficiently near the inlet owing to the natural spreading of the streamlines at this point.

The final compromise between the conflicting requirements of the internal and the external flows will depend on the internal arrangement and the space available for the ducts. In general, it is believed that efficient installations incorporating nose B or nose C should have inlet-velocity ratios in the range of 0.3 to 0.6.

Derivation of optimum nose profiles for arbitrary inlet-opening sizes.—The method, described in section entitled "Results," for obtaining suitable nose-inlet shapes for inlet sizes other than those tested (see figs. 20 and 21) is obviously strictly applicable only to openings on the modified 111 body form. It was thought possible, however, that the shapes obtained by this method could be applied with good results where only the basic forebody profile was similar to the 111 body form. In a subsequent investigation (reference 8) it was found that considerable stretching of the nose B profile was permissible with the stretched profile still retaining the desirable flat pressure contour and low values of the pressure peak. It was thus indicated that the profile ordinates of the present tests could be directly used in the design of nose inlets having proportions greatly different from the shapes

developed in these tests. In order to simplify the procedure of designing inlets of other proportions, it was found desirable to base the nondimensional ordinates on the distance X from the maximum diameter station ($0.4L$) to the nose rather than on the distance from the $0.25L$ station to the nose, as suggested under "Results." Reference 9, which was an outgrowth of the present work, presents high-speed-test results for related inlets derived by the foregoing method and covers a wide range of proportions.

Internal-duct shape near inlet opening.—The modifications of figure 19 consisted of a conical expansion with a 10° included angle, a large irregular expansion formed by the cut-out for the inner cowl, and a gradual (4° equivalent cone) annular expansion obtained with the inner cowl. None of these changes had a measurable effect on either the external drag or the pressure distribution. Modification B-4c of figure 18 likewise had no effects. Modifications B-4a and B-4b of figure 18, however, caused slight drag increases and disturbed the external pressures at the nose. These latter modifications are equivalent to inferior nose shapes corresponding to smaller inlet sizes than the basic nose B inlet. It will be observed that the internal-duct shapes included both satisfactory and very inefficient designs and that neither had any external effects, provided that the size of the inlet was not altered.

Angle of attack.—The effect of increase in angle of attack from 0° to 3.5° on the pressure distribution over the top of nose B can be seen in figure 35. A considerably higher air-inlet-velocity ratio is required to reduce the pressure peak at 3.5° angle of attack than at 0° angle of attack. In flight, the inlet-velocity ratio would automatically increase with angle-of-attack increases owing to decreases in the flight speed, if the engine power were assumed constant. Force-test data obtained with fixed transition on a fuselage model employing nose C (reference 8) showed that the external drag, at an inlet-velocity ratio of 0.56, was practically constant over the angle-of-attack range of 0° to 3.5° .

OUTLET OPENINGS

The outlet openings tested were not optimum shapes arrived at by a series of tests, as were the inlet openings. As previously stated, they merely represented typical practice in the design and the construction of outlets. It became apparent during the course of the tests that the openings had several undesirable characteristics, but it was not feasible at the time to extend the investigation to include modifications. Further outlet research embracing the improvements that suggested themselves in the course of this investigation are included in references 8 and 10.

Pressure distribution.—The effect on the pressure distribution of air flow from the outlets was generally unfavorable. In the case of the annular outlets (fig. 24) a negative-pressure peak occurred at the higher flow rates, owing to an effective

thickening of the body due to the flow of exhaust air in the rear of the openings. In some cases, the peak was sufficiently high to fix the critical speed of the body. The pressure disturbance at the 21-percent outlet precipitated boundary-layer transition at all outlet velocities.

The static pressure at the tail outlets (fig. 22) became more positive as the flow was increased. This effect was due to the fact that the streamlines of both the internal and the external flows were converging at the opening, so that considerable contraction of the flow in the rear of the outlet resulted. Thus, about one-third of the total pressure (measured from p_0) at the tail outlets was in the form of static pressure which, of course, increased as the flow ratio was advanced. The static pressure in the internal flow at the outlet tended to be considerably more positive than that of the external flow near the tail outlet. The high outlet pressures are believed to have caused local separation of the external flow near the tail outlets.

External drag.—The external drag with the 63-percent annular outlet (fig. 25) at first decreased as the flow rate was advanced probably because of the elimination of locally separated flow in the wake of the opening, and then it increased rapidly, probably because of the increasing skin friction over the part of the body in the wake of the outlet.

Similar drag characteristics were exhibited by the tail outlets wherever velocity ratios v/V , up to 0.5 or greater, could be attained, as in the case of tails D and F tested singly (fig. 23) and tails B, C, and E tested in combination with the nose inlets (figs. 28 to 31). The rise in drag at the higher flow rates in the combination tests is shown conclusively in figure 30 to be due to the tail outlets. When compared on the basis of tail-outlet-velocity ratio (fig. 30) instead of flow coefficient (fig. 29 (a)), the drag obtained with three outlets of widely different size shows close agreement. The drag increase at the higher tail-outlet-velocity ratios is believed to be due to local separation of the external flow as a result of the high outlet pressures.

The tail outlets were superior to the annular outlets. A comparison of tail C with the 63-percent annular outlet in combination with nose B (fig. 33) shows that in spite of a somewhat larger area the tail outlet had the lower drag throughout the range, particularly at the higher outlet velocities. As would be expected, the comparison was independent of the location of boundary-layer transition because neither opening had any appreciable effect on the transition location.

Outlet-opening design.—The outlet velocity is not arbitrary as is the inlet velocity but is fixed by the internal total-pressure losses and the pressure drop across the system. From the standpoint of the internal drag, it is desirable to have the outlet total pressure as nearly equal to the free-stream total pressure as possible so that a minimum amount

of energy will be left in the wake. In well-designed cooling systems, the internal total-pressure losses are only a few percent of the free-stream total pressure at high speeds. Under these conditions, the ideal outlet total pressure is approached and the internal drag is small. The relation between internal total-pressure loss and the internal drag was shown in the section entitled "Results." The outlet velocity at a given flight speed is readily calculable from estimates of the total-pressure losses and the pressure drop across the system. A contraction or an orifice coefficient (dependent on the outlet shape) should be applied to the velocity as computed from the pressure characteristics. With the tail outlets tested, for example, the velocity at the outlet was about 0.8 of the final velocity. For the annular outlets, the coefficient was roughly 0.9. The velocity having thus been obtained at the outlet, the size of the opening will depend on the required quantity of air flow.

The economy of passing exactly the required amount of cooling air through the internal system at all flight speeds is generally appreciated. Variation in the size of the exit opening is the most efficient method of controlling the rate of flow.

The shape of the opening is not critical as far as the internal flow is concerned, provided there are no expansions. But the present tests have indicated that the external flow may be adversely affected if the static pressures are different from those of the main stream near the outlet. The shape of the opening, therefore, should permit the internal air to exhaust at the same static pressure as exists in the external flow near the opening. A suggested optimum tail-outlet shape is sketched in figure 26, and the flow characteristics are compared with those existing at one of the outlets tested for the ideal outlet condition of free-stream total pressure in the opening. The desired conditions at the outlet are obtained in the proposed opening by eliminating the contraction of the outlet flow. The desired outlet conditions can be attained at any outlet location by making the streamlines of both internal and external flows parallel.

The optimum shape for an annular-outlet opening is not as obvious as in the case of the tail outlets. It is evident from figure 24, however, that the body fairing immediately behind the outlet should be altered to reduce the thickening of the body and thus to relieve the thickening effect of the outlet flow. Further research is recommended to determine in detail the shapes required to give the minimum disturbance to the static-pressure distribution.

In regard to the relative merits of the annular and the tail outlets for efficient internal systems, it is probable that the optimum tail outlet will be superior to the best possible annular outlet because the high-velocity flow from the annular openings will generally increase the skin friction of the portion of the body in the wake of the outlet.

INLET-OUTLET COMBINATIONS

The combination tests (figs. 28 and 34) are of principal interest in showing that the over-all external drag of the body with suitable inlet and outlet openings of practicable size was no higher than that of the basic streamline form. This result was obtained at rates of internal air flow sufficient for cooling a radial engine located at the maximum fuselage section at moderate to high-speed flight conditions.

The variation of the rate of internal flow in the combination tests was accomplished by means of varying the internal resistance. At the condition of maximum flow attainable with a given outlet size, the internal losses were very small and consequently the outlet conditions closely approached the ideal. The outlet velocities over approximately the higher 25 percent of the flow range covered with each outlet correspond to high-speed-flight outlet conditions for typical heat-exchanger installations; at lower flow rates the internal-resistance losses were considerably higher than would be encountered in present practice. The actual magnitude of the internal drag throughout the flow range covered with tail C is shown in figure 27.

The rise in drag at the higher flow rates has been shown to be due to the unfavorable outlet conditions at the higher outlet velocities (fig. 30). It is believed that by improving the outlet design as suggested in figure 26 the rise in drag at the high outlet velocities would be eliminated.

It will be observed that the drag obtained for the best combinations with fixed transition was, in general, slightly greater than for the streamline body with transition fixed at the same station. The difference may be entirely accounted for by the higher drag of the carborundum strip itself when located at the nose of the inlet openings than when located in the thicker boundary layer on the streamline body. In addition, it should be remembered that the stations selected for fixing the transition on the streamline body are entirely arbitrary. Under actual flight conditions, transition on the streamline body might occur somewhat ahead of the corresponding station on the noses owing to the greater length of the streamline body. In this case, the drag of the streamline body would be relatively higher than in the present comparisons.

The drag of the inlet openings in the presence of the outlets, and vice versa, was considerably less than it was when the openings were tested individually. (See fig. 34.) A part of this effect, particularly at low rates of internal flow, may be due to leakage in the tests of the individual openings, as has previously been pointed out. Another contributing factor of secondary importance may be the difference in the methods of restricting the internal flow—that is, the resistance plates inserted near the inlet opening in the combination tests (fig. 2) may have had some small tendency to affect the external flow. In general, however, it is reasonable to expect

that the openings, in combination, would contribute less drag than when tested individually.

Comparison with NACA cowling.—The results of reference 1 provide a comparison of the inlet-outlet combinations with the NACA cowling. In the investigation cited, the best NACA cowling shape of reference 2 was adapted in a typical fuselage installation to the NACA 111 fuselage form. This basic streamline shape was almost identical with the body employed in the present tests, and the effective body-drag coefficients with natural and fixed transition, 0.040 and 0.055, respectively, were practically equal to the corresponding drag coefficients, 0.042 and 0.054, obtained in this investigation. The flow and the boundary-layer conditions on the basic shapes employed were evidently quite similar. The drags of the cowling with fixed and natural transition as given in reference 1, with cooling air flow, were reduced about 5 percent to obtain the external drag necessary to the comparison. The results, taken at the same Mach number and at very nearly the same Reynolds number as in the present tests, are shown on each of the figures along with the results of the combination tests (figs. 28, 29, and 31) with the tail outlets.

The combinations tested were aerodynamically superior to the NACA cowling, particularly in the natural transition condition where the inlets B and C permitted extensive laminar flow and caused no increase in drag. The NACA cowling produced a drag increase of some 56 percent in this case.

The NACA cowling shape employed in the tests of reference 1 was developed (reference 2) to have the highest critical speed, $M_{cr}=0.63$, of eight typical cowling shapes of the same over-all dimensions. The critical speed of the body alone with the largest of the present inlets, nose A, was $M_{cr}=0.64$ at a practical rate of air inlet (fig. 13). With the smaller inlets the critical speed was advanced to $M_{cr}=0.84$, the critical speed of the basic 111 fuselage shape.

CONCLUSIONS

The results of this investigation of inlet and outlet openings led to the following conclusions:

1. Modification of a streamline body permitting air inlet at the nose and outlet at the tail can be accomplished without increasing the external drag.
2. Inlet profiles were developed which, with practicable rates of air flow, produced velocity distributions approaching

closely that of the basic streamline body. Consequently, the critical speed was as high as that of the streamline body and the same favorable laminar-boundary-layer-flow conditions were realized.

3. The test results indicated that outlet openings should be designed so that the static pressure of the internal flow at the outlet would be the same as the static pressure of the external flow in the vicinity of the opening.

4. The internal-duct shape near an inlet of given size had no appreciable effect on the external drag or pressure distribution.

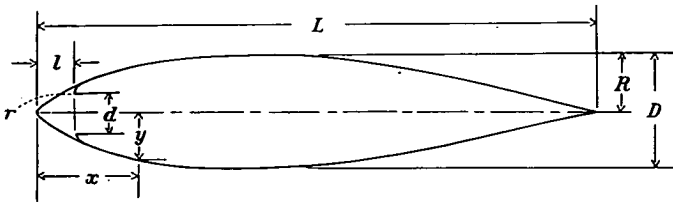
5. The location of a simulated smooth-barrel gun in the nose-inlet opening caused no appreciable increase in drag at low angles of attack. The muzzle of the gun should be slightly rounded, and the length of barrel extending beyond the inlet should be as small as possible.

LANGLEY MEMORIAL AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *September 11, 1940.*

REFERENCES

1. Draley, Eugene C.: High-Speed Drag Tests of Several Fuselage Shapes in Combination with a Wing. NACA ACR, Aug. 1940.
2. Robinson, Russell G., and Becker, John V.: High-Speed Tests of Conventional Radial-Engine Cowlings. NACA Rep. 745, 1942.
3. Abbott, Ira H.: Fuselage-Drag Tests in the Variable-Density Wind Tunnel: Streamline Bodies of Revolution, Fineness Ratio of 5. NACA TN 614, 1937.
4. Becker, John V.: Boundary-Layer Transition on the N.A.C.A. 0012 and 23012 Airfoils in the 8-Foot High-Speed Wind Tunnel. NACA ACR, Jan. 1940.
5. Hood, Manley J.: The Effects of Some Common Surface Irregularities on Wing Drag. NACA TN 695, 1939.
6. Robinson, Russell G., and Wright, Ray H.: Estimation of Critical Speeds of Airfoils and Streamline Bodies. NACA ACR, March 1940.
7. Patterson, G. N.: The Design of Aeroplane Ducts. Rules To Be Followed for the Reduction of Internal and External Drag. Aircraft Engineering, vol. XI, no. 125, July 1939, pp. 263-268.
8. Becker, John V., and Baals, Donald D.: Wind-Tunnel Tests of a Submerged-Engine Fuselage Design. NACA ACR, Oct. 1940.
9. Baals, Donald D., Smith, Norman F., and Wright, John B.: The Development and Application of High-Critical-Speed Nose Inlets. NACA Rep. 920, 1948.
10. Becker, John V., and Baals, Donald D.: High-Speed Tests of a Ducted Body with Various Air-Outlet Openings. NACA ACR, May 1942.

TABLE I
ORDINATES OF STREAMLINE BODY AND NOSE INLETS



Streamline body			
x/L	y/R	x/L	y/R
0	0	0.5000	0.986
.0125	.190	.5500	.952
.0250	.287	.6000	.898
.0500	.431	.6500	.820
.0750	.536	.7000	.724
.1000	.620	.7500	.612
.1500	.742	.8000	.491
.2000	.825	.8500	.371
.2500	.898	.9000	.249
.3000	.948	.9500	.124
.3500	.982	.9750	.0624
.4000	1.000	1.0000	0
.4500	1.000		

$\frac{D}{L} = 0.20$

Nose A		Nose B		Nose C	
x/L	y/R	x/L	y/R	x/L	y/R
0.1238	0.550	0.0654	0.396	0.0536	0.286
.1244	.568	.0657	.408	.0540	.297
.1250	.578	.0660	.416	.0543	.304
.1258	.587	.0664	.420	.0546	.310
.1265	.595	.0668	.425	.0550	.317
.1272	.602	.0674	.432	.0557	.326
.1286	.614	.0682	.439	.0564	.334
.1300	.626	.0696	.451	.0578	.349
.1322	.640	.0724	.472	.0593	.361
.1358	.661	.0796	.516	.0607	.375
.1394	.680	.0868	.550	.0643	.404
.1465	.708	.0939	.580	.0678	.429
.1536	.732	.1082	.630	.0750	.473
.1608	.752	.1224	.670	.0822	.510
.1679	.768	.1368	.706	.0964	.574
.1822	.797	.1654	.768	.1107	.626
.1965	.821	.1939	.817	.1393	.707
.2108	.842	.2064	.837	.1678	.770
.2322	.875	.2321	.875	.1966	.821
.2500	.898	.2500	.898	.2322	.875
				.2500	.898

$\frac{r}{R} = 0.0143$ $\frac{r}{R} = 0.0143$ $\frac{r}{R} = 0.0143$

$\frac{l}{L} = 0.1238$ $\frac{l}{L} = 0.0653$ $\frac{l}{L} = 0.0536$

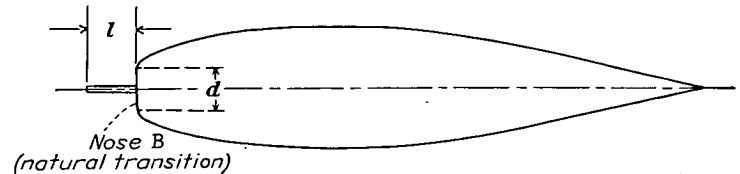
$\frac{d}{D} = 0.5360$ $\frac{d}{D} = 0.3785$ $\frac{d}{D} = 0.2680$

TABLE II
ORDINATES FOR DERIVING OPTIMUM NOSE SHAPES FOR INLET DUCT SIZES OTHER THAN THOSE TESTED *

x'/X	y'/Y			
	$\frac{d}{D} = 0.536$	$\frac{d}{D} = 0.379$	$\frac{d}{D} = 0.268$	Mean
0	0.039	0.036	0.027	0.034
.005	.091	.079	.069	.080
.010	.119	.101	.089	.103
.020	.160	.135	.123	.139
.030	.192	.163	.152	.169
.050	.247	.211	.204	.221
.075	.303	.259	.258	.273
.100	.352	.304	.308	.321
.150	.435	.382	.392	.403
.200	.501	.448	.462	.470
.300	.603	.555	.574	.577
.400	.679	.644	.664	.662
.500	.740	.721	.740	.734
.600	.798	.788	.804	.797
.800	.902	.899	.910	.904
1.000	1.000	1.000	1.000	1.000

* See figure 20.

TABLE III
INCREASE IN DRAG WITH SIMULATED GUN IN NOSE-INLET OPENING



α (deg)	l/d	v/V	Increase in body drag due to gun, percent
.50-caliber machine gun			
0	1.13	0	12.8
0	1.13	.25	12.5
0	1.13	.50	10.6
0	.57	0	9.6
0	0	0	2.6
1.5	1.13	0	21.5
1.5	1.13	.25	16.1
1.5	1.13	.50	10.2
3.5	1.13	0	38.6
3.5	1.13	.25	20.4
3.5	1.13	.50	7.2
3.5	.57	0	18.4
3.5	0	0	4.8
37-millimeter cannon (smooth barrel)			
0	1.13	0	6.7
0	1.13	.25	4.3
0	1.13	.50	0
1.5	1.13	0	11.5
1.5	1.13	.25	9.0
1.5	1.13	.50	0
3.5	1.13	0	19.2
3.5	1.13	.25	9.1
3.5	1.13	.50	0

REPORT 1039

ON THE PARTICULAR INTEGRALS OF THE PRANDTL-BUSEMANN ITERATION EQUATIONS FOR THE FLOW OF A COMPRESSIBLE FLUID¹

By CARL KAPLAN

SUMMARY

The particular integrals of the second-order and third-order Prandtl-Busemann iteration equations for the flow of a compressible fluid are obtained by means of the method in which the complex conjugate variables z and $\bar{z}$ are utilized as the independent variables of the analysis. The assumption is made that the Prandtl-Glauert solution of the linearized or first-order iteration equation for the two-dimensional flow of a compressible fluid is known. The forms of the particular integrals, derived for subsonic flow, are readily adapted to supersonic flows with only a change in sign of one of the parameters of the problem.

INTRODUCTION

For the past several years iteration methods have been increasingly applied to the solution of compressible-flow problems. The most useful method from the point of view of aeronautical applications and the one discussed in this report is based on small perturbations with respect to the undisturbed flow. The Prandtl-Glauert and Ackeret solutions in two-dimensional subsonic and supersonic flow, respectively, obtained by means of the linearization of the fundamental nonlinear differential equation for compressible flow, are presumed to be known and are taken as the initial steps in this iteration process. Higher-order solutions are then obtained by retaining appropriate powers and products of the perturbation quantities. This method of iteration has been variously labeled the Ackeret iteration process and the Prandtl-Busemann small perturbation method when limited to two-dimensional subsonic flow. The procedure has been extended in recent years to both two-dimensional and axisymmetrical supersonic-flow problems.

In a recent publication (reference 1), Van Dyke succeeded in obtaining by trial the particular integral of the nonhomogeneous second-order iteration equation for the velocity potential in supersonic flow. The general solution is then easily obtained by adding solutions of the homogeneous equation with proper regard to the boundary conditions at the surface of the solid and at infinity.

The purpose of the present report is to show a procedure by means of which the particular integrals of the higher-than-first-order iteration equations can be derived in a

systematic manner. The explicit expressions obtained for the particular integrals of the second- and third-order iteration equations are believed to yield essentially the solution of the problem of high subsonic flow past an arbitrary two-dimensional profile, since it is never a difficult problem to supply the solutions of the homogeneous equation necessary for the fulfillment of the boundary conditions. It is noteworthy that the particular integrals, derived for subsonic flow, can be adapted to supersonic flow with simply a change in sign of one of the parameters.

FUNDAMENTAL EQUATIONS

The fundamental nonlinear differential equation governing the flow of a compressible fluid is

$$(c^2 - u^2) \frac{\partial u}{\partial X} + (c^2 - v^2) \frac{\partial v}{\partial Y} - uv \left(\frac{\partial v}{\partial X} + \frac{\partial u}{\partial Y} \right) = 0 \quad (1)$$

where

X, Y rectangular Cartesian coordinates in flow plane

u, v fluid velocity components along X - and Y -axis, respectively

c local speed of sound

The condition for irrotational motion is that

$$\frac{\partial u}{\partial Y} = \frac{\partial v}{\partial X}$$

and leads to a velocity potential Φ defined by

$$\left. \begin{aligned} u &= \frac{\partial \Phi}{\partial X} \\ v &= \frac{\partial \Phi}{\partial Y} \end{aligned} \right\} \quad (2)$$

If the body is held fixed in a uniform stream of velocity U , the relation between the local speed of sound c and the speed of the fluid $\sqrt{u^2 + v^2}$ is given for adiabatic processes by

$$\frac{c^2}{c_\infty^2} = 1 + \frac{\gamma - 1}{2} M_\infty^2 \left(1 - \frac{u^2 + v^2}{U^2} \right) \quad (3)$$

¹ Supersedes NACA TN 2159, "On the Particular Integrals of the Prandtl-Busemann Iteration Equations for the Flow of a Compressible Fluid" by Carl Kaplan, 1950.

where

c_∞ sound speed in undisturbed fluid

γ ratio of specific heats at constant pressure and constant volume

M_∞ Mach number of undisturbed stream (U/c_∞)

With the introduction of a characteristic length l as unit of length and the undisturbed stream velocity U as unit of velocity, the quantities X , Y , u , v , and Φ for the remainder of the analysis denote, respectively, the nondimensional quantities X/l , Y/l , u/U , v/U , and Φ/Ul , while c and c_∞ retain their original meanings. By means of equations (2), equations (1) and (3) then become, respectively,

$$\left(\frac{c^2}{c_\infty^2} - M_\infty^2 u^2\right) \Phi_{XX} + \left(\frac{c^2}{c_\infty^2} - M_\infty^2 v^2\right) \Phi_{YY} - 2M_\infty^2 \Phi_X \Phi_Y \Phi_{XY} = 0 \quad (4)$$

and

$$\frac{c^2}{c_\infty^2} = 1 + \frac{\gamma-1}{2} M_\infty^2 [1 - (u^2 + v^2)] \quad (5)$$

where the subscripts X and Y denote partial differentiations with respect to the designated variables.

In order to obtain the iteration equations based on small perturbations of the undisturbed stream, the assumption is made that the velocity potential Φ can be expanded in the form

$$\Phi = X + \Phi_1 + \Phi_2 + \Phi_3 + \dots \quad (6)$$

For the purpose of defining and controlling the iteration procedure, the function Φ_{n+1} and its derivatives are then regarded as small compared with the preceding approximation Φ_n and its derivatives.

From equations (2) and (6),

$$u = \Phi_X = 1 + \Phi_{1X} + \Phi_{2X} + \Phi_{3X} + \dots$$

and

$$v = \Phi_Y = \Phi_{1Y} + \Phi_{2Y} + \Phi_{3Y} + \dots$$

When these expressions for u , v are introduced into equation (4), together with the expression for c^2/c_∞^2 given by equation (5), and the powers and products of Φ_n and their derivatives are grouped according to the assumptions of the small perturbation method, the following iteration equations for the first three approximations Φ_1 , Φ_2 , and Φ_3 result:

$$(1 - M_\infty^2) \Phi_{1XX} + \Phi_{1YY} = 0 \quad (7)$$

$$(1 - M_\infty^2) \Phi_{2XX} + \Phi_{2YY} = M_\infty^2 [(\gamma+1) \Phi_{1X} \Phi_{1XX} + (\gamma-1) \Phi_{1X} \Phi_{1YY} + 2 \Phi_{1Y} \Phi_{1XY}] \quad (8)$$

$$(1 - M_\infty^2) \Phi_{3XX} + \Phi_{3YY} = M_\infty^2 \left\{ \left(\frac{1}{2} \Phi_{1X}^2 + \Phi_{2X} \right) [(\gamma+1) \Phi_{1XX} + (\gamma-1) \Phi_{1YY}] + \frac{1}{2} \Phi_{1Y}^2 [(\gamma-1) \Phi_{1XX} + (\gamma+1) \Phi_{1YY}] + \Phi_{1X} [(\gamma+1) \Phi_{2XX} + (\gamma-1) \Phi_{2YY}] + 2(\Phi_{1X} \Phi_{1Y} \Phi_{1XY} + \Phi_{2Y} \Phi_{1XY} + \Phi_{1Y} \Phi_{2XY}) \right\} \quad (9)$$

For slender bodies, the first few steps of this iteration process may be expected to yield an accurate result with the exception of a small region in the neighborhood of a stagnation point. Even at stagnation points, the iteration method has been shown to represent correctly the effect of compressibility (reference 2). The accuracy of the calculations obviously depends upon the number of terms determined, each additional term reducing the region of inaccuracy in the neighborhood of a stagnation point.

The iteration equations (7), (8), and (9) may be put into more familiar forms by the introduction of a new set of independent variables x and y , where

$$\left. \begin{aligned} x &= X \\ y &= Y \sqrt{1 - M_\infty^2} \end{aligned} \right\} \quad (10)$$

Thus, for $M_\infty < 1$, equation (7) is transformed into a Laplace equation; whereas equations (8) and (9) are transformed into Poisson equations with the right-hand sides composed of, respectively, double products and triple products of previously determined perturbation quantities. It is further assumed that the solution of equation (7) is available. This initial step in the approximation to the exact nonlinear solution is usually easily obtained, as it represents the Prandtl-Glauert approximation (reference 3, appendix B). The purpose of the present report is then to derive explicit expressions for the particular solutions of the second- and third-order iteration equations (8) and (9).

CALCULATION OF THE PARTICULAR INTEGRAL OF THE SECOND-ORDER ITERATION EQUATION

By introducing the independent variables x and y defined by equation (10), the second-order iteration equation (8) becomes

$$\Phi_{2xx} + \Phi_{2yy} = 2M_\infty^2 [(1 + \sigma) \Phi_{1x} \Phi_{1xx} + \Phi_{1y} \Phi_{1xy}] \quad (11)$$

where

$$\sigma = \frac{\gamma+1}{2} \frac{M_\infty^2}{\beta^2}$$

$$\beta^2 = 1 - M_\infty^2$$

and where use has been made of Laplace's equation

$$\Phi_{1xx} + \Phi_{1yy} = 0$$

The procedure for obtaining the particular integrals of the higher-order iteration equations is based on the use of the complex conjugate variables z and $\bar{z}$ as independent variables. Thus,

$$z = x + iy$$

$$\bar{z} = x - iy$$

and the equivalence of operators

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}$$

$$\frac{\partial}{\partial y} = i \left(\frac{\partial}{\partial z} - \frac{\partial}{\partial \bar{z}} \right)$$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial z^2} + 2 \frac{\partial^2}{\partial z \partial \bar{z}} + \frac{\partial^2}{\partial \bar{z}^2}$$

$$\frac{\partial^2}{\partial y^2} = -\frac{\partial^2}{\partial z^2} + 2 \frac{\partial^2}{\partial z \partial \bar{z}} - \frac{\partial^2}{\partial \bar{z}^2}$$

Then equation (7) for Φ_1 becomes

$$4 \frac{\partial^2 \Phi_1}{\partial z \partial \bar{z}} = 0 \quad (12)$$

The most general real solution of this equation is

$$\Phi_1 = \frac{1}{2} [w_1(z) + \bar{w}_1(\bar{z})] \quad (13)$$

or

$$\Phi_1 = \text{R.P. } w_1(z) = \text{R.P. } \bar{w}_1(\bar{z})$$

where $w_1(z)$ is an arbitrary analytic function of z , $\bar{w}_1(\bar{z})$ is its conjugate complex, and where the symbol R.P. stands for "real part of." The imaginary part of $w_1(z)$ is a function ψ_1 , say, related to Φ_1 by means of the Cauchy-Riemann equations and hence also satisfies Laplace's equation. The function ψ_1 does not represent the stream function of the actual compressible flow and does not appear in the final expressions of the particular integrals. The following relations will be found useful and are easily verified:

$$\Phi_1 = \frac{1}{2} (w_1 + \bar{w}_1)$$

$$\Phi_{1x} = \frac{1}{2} (w_{1z} + \bar{w}_{1\bar{z}})$$

$$\Phi_{1y} = \frac{i}{2} (w_{1z} - \bar{w}_{1\bar{z}})$$

$$\Phi_{1xz} = \frac{i}{2} (w_{1zz} - \bar{w}_{1\bar{z}\bar{z}})$$

$$\Phi_{1zz} = \frac{1}{2} (w_{1zz} + \bar{w}_{1\bar{z}\bar{z}})$$

Then

$$2\Phi_{1x}\Phi_{1xz} = \text{R.P. } (w_{1z} + \bar{w}_{1\bar{z}})w_{1zz}$$

$$2\Phi_{1y}\Phi_{1zy} = -\text{R.P. } (w_{1z} - \bar{w}_{1\bar{z}})w_{1zz}$$

and equation (11) for the second approximation Φ_2 becomes

$$\begin{aligned} \Phi_{2z\bar{z}} &= \frac{1}{4} M_\infty^2 \text{R.P. } [\sigma w_{1z}w_{1zz} + (2 + \sigma) \bar{w}_{1\bar{z}}w_{1zz}] \\ &= \frac{1}{4} M_\infty^2 \text{R.P. } \left[\frac{\sigma}{2} (w_{1z}^2)_z + (2 + \sigma) (w_{1z}\bar{w}_{1\bar{z}})_z \right] \end{aligned}$$

If Φ_2 is defined to be the real part of a nonanalytic function $w_2(z, \bar{z})$, then

$$w_{2z\bar{z}} = \frac{1}{4} M_\infty^2 \left[\frac{\sigma}{2} (w_{1z}^2)_z + (2 + \sigma) (w_{1z}\bar{w}_{1\bar{z}})_z \right] \quad (14)$$

This equation can be integrated immediately by inspection and yields the general solution

$$w_2 = \frac{1}{4} M_\infty^2 \left[\frac{\sigma}{2} \bar{z} w_{1z}^2 + (2 + \sigma) \bar{w}_1 w_{1z} + F(z) \right] \quad (15)$$

where, because only the real part Φ_2 of w_2 is of interest, only one arbitrary analytic function $F(z)$ need be included. The function $F(z)$ satisfies Laplace's equation and is so chosen as to satisfy the required boundary conditions at the surface of the body and at infinity. The part of the expression on the right-hand side of equation (15), excluding the arbitrary function $F(z)$, is the particular integral of equation (14) and may be expressed in real form in the following manner:

Suppose

$$F(z) = -\frac{\sigma}{2} z w_{1z}^2 + (2 + \sigma) w_1 w_{1z} + f(z) \quad (16)$$

where $f(z)$ is again an arbitrary analytic function of z .

Then with the aid of the relation

$$w_{1z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right) (\Phi_1 + i\psi_1) = \Phi_{1x} - i\Phi_{1y}$$

where use has been made of the Cauchy-Riemann conditions

$$\Phi_{1x} = \psi_{1y}$$

$$\Phi_{1y} = -\psi_{1x}$$

the expression for Φ_2 , obtained from equation (15), becomes

$$\Phi_2 = \frac{1}{4} M_\infty^2 [-2\sigma y \Phi_{1x} \Phi_{1y} + 2(2 + \sigma) \Phi_1 \Phi_{1x} + \text{R.P. } f(z)] \quad (17)$$

The expression on the right-hand side of this equation, excluding R.P. $f(z)$, namely,

$$\Phi_2 = M_\infty^2 \left[\left(1 + \frac{\sigma}{2} \right) \Phi_1 - \frac{1}{2} \sigma y \Phi_{1y} \right] \Phi_{1x} \quad (18)$$

corresponds precisely to the particular integral obtained by Van Dyke (reference 1) for two-dimensional supersonic flow with σ replaced by $-\sigma$, where for supersonic flow the definition of σ is

$$\sigma = \frac{\gamma+1}{2} \frac{M_\infty^2}{M_\infty^2-1}$$

It is rather noteworthy that the particular integral of the second-order iteration equation (8) can be obtained for both subsonic and supersonic flows by simply interchanging the sign of the parameter σ .

CALCULATION OF THE PARTICULAR INTEGRAL OF THE THIRD-ORDER ITERATION EQUATION

In this section, the particular integral of equation (9) involving only Φ_1 , Φ_2 , and their derivatives is derived. For this purpose, the variables x, y and the parameter σ are introduced. Equation (9) then takes the following form:

$$\begin{aligned} \Phi_{3xx} + \Phi_{3yy} = 2M_\infty^2 \left\{ (1+\sigma)(\Phi_{1xx}\Phi_{2x} + \Phi_{1x}\Phi_{2xx}) + [2\beta^2(1+\sigma)-1]\Phi_{1x}\Phi_{1y}\Phi_{1xy} + (1+\sigma) \left[2\beta^2(1+\sigma) - \frac{3}{2} \right] \Phi_{1xx}\Phi_{1x}^2 + \right. \\ \left. \frac{1}{2}(\sigma\beta^2-1)\Phi_{1xx}\Phi_{1y}^2 + \Phi_{1xy}\Phi_{2y} + \Phi_{1y}\Phi_{2xy} \right\} \end{aligned} \quad (19)$$

Use is again made of the complex conjugate variables z and $\bar{z}$ as independent variables. Thus,

$$\begin{aligned} \Phi_2 &= \text{R. P. } w_2(z, \bar{z}) \\ \Phi_{2x} &= \text{R. P. } (w_{2z} + w_{2\bar{z}}) \\ \Phi_{2y} &= \text{R. P. } i(w_{2z} - w_{2\bar{z}}) \\ \Phi_{2xx} &= \text{R. P. } (w_{2zz} + 2w_{2z\bar{z}} + w_{2\bar{z}\bar{z}}) \\ \Phi_{2xy} &= \text{R. P. } i(w_{2zz} - w_{2\bar{z}\bar{z}}) \\ \Phi_{1xx}\Phi_{1x}^2 &= \text{R. P. } \frac{1}{4}(w_{1z} + \bar{w}_{1\bar{z}})^2 w_{1zz} \\ \Phi_{1xx}\Phi_{1y}^2 &= -\text{R. P. } \frac{1}{4}(w_{1z} - \bar{w}_{1\bar{z}})^2 w_{1zz} \\ \Phi_{1xx}\Phi_{2x} + \Phi_{1x}\Phi_{2xx} &= \text{R. P. } \frac{1}{2}(w_{1zz} + \bar{w}_{1\bar{z}\bar{z}})(w_{2z} + w_{2\bar{z}}) + \text{R. P. } \frac{1}{2}(w_{1z} + \bar{w}_{1\bar{z}})(w_{2zz} + 2w_{2z\bar{z}} + w_{2\bar{z}\bar{z}}) \\ \Phi_{1xy}\Phi_{2y} &= -\text{R. P. } \frac{1}{2}(w_{1zz} - \bar{w}_{1\bar{z}\bar{z}})(w_{2z} - w_{2\bar{z}}) \\ \Phi_{1y}\Phi_{2xy} &= -\text{R. P. } \frac{1}{2}(w_{1z} - \bar{w}_{1\bar{z}})(w_{2zz} - w_{2\bar{z}\bar{z}}) \\ \Phi_{1x}\Phi_{1y}\Phi_{1xy} &= -\text{R. P. } \frac{1}{4}(w_{1z}^2 - \bar{w}_{1\bar{z}}^2)w_{1zz} \end{aligned}$$

Then equation (19), with $\Phi_3 = \text{R. P. } w_3(z, \bar{z})$, can be written as follows:

$$\begin{aligned} w_{3z\bar{z}} &= \frac{1}{4}(1-\beta^2)\sigma(w_{2z}w_{1zz} + w_{2zz}w_{1z} + w_{2\bar{z}}\bar{w}_{1\bar{z}\bar{z}} + w_{2\bar{z}\bar{z}}\bar{w}_{1\bar{z}}) + \frac{1}{4}(1-\beta^2)(2+\sigma)(w_{2z}\bar{w}_{1\bar{z}\bar{z}} + w_{2zz}\bar{w}_{1\bar{z}} + w_{2\bar{z}}w_{1zz} + w_{2\bar{z}\bar{z}}w_{1z}) + \\ &\frac{1}{16}(1-\beta^2)\sigma[\beta^2(1+2\sigma)-(1-2\sigma)]w_{1z}^2w_{1zz} + \frac{1}{16}(1-\beta^2)[\beta^2(4+5\sigma+2\sigma^2)+\sigma(3+2\sigma)]\bar{w}_{1\bar{z}}^2w_{1zz} + \\ &\frac{1}{8}(1-\beta^2)[\beta^2(2+5\sigma+2\sigma^2)+(-2+\sigma+2\sigma^2)]w_{1z}\bar{w}_{1\bar{z}}w_{1zz} \end{aligned}$$

Introducing the expressions for $w_2(z, \bar{z})$ and its derivatives with respect to z and $\bar{z}$ from equation (15) yields for $w_{3z\bar{z}}$ the following equation:

$$\begin{aligned} w_{3z\bar{z}} &= \frac{1}{16}(1-\beta^2)^2\sigma^2\bar{z}(w_{1z}^2w_{1zz})_z + \frac{1}{16}(1-\beta^2)^2\sigma(2+\sigma)(\bar{z}\bar{w}_{1\bar{z}})_z(w_{1z}w_{1zz})_z + \frac{1}{16}(1-\beta^2)^2\sigma(w_{1z}F_z)_z + \frac{1}{16}(1-\beta^2)\sigma[\beta^2(2+\sigma) + \\ &(4+3\sigma)]\bar{w}_{1\bar{z}}(w_{1z}^2)_z + \frac{1}{32}(1-\beta^2)^2\sigma(2+\sigma)(w_{1z}^2)_z(\bar{z}\bar{w}_{1\bar{z}})_z + \frac{1}{16}(1-\beta^2)^2(2+\sigma)^2w_{1zz}(\bar{w}_{1\bar{z}}\bar{w}_{1\bar{z}})_z + \frac{1}{16}(1-\beta^2)^2(2+\sigma)(\bar{w}_{1\bar{z}}F_z)_{z\bar{z}} + \\ &\frac{1}{32}(1-\beta^2)^2(2+\sigma)^2(\bar{w}_{1\bar{z}}^2)_z w_{1zz} + \frac{1}{96}(1-\beta^2)\sigma^2(5+3\beta^2)(w_{1z}^3)_z + \frac{1}{32}(1-\beta^2)[\beta^2(8+10\sigma+3\sigma^2)+\sigma(6+5\sigma)]\bar{w}_{1\bar{z}}w_{1z}^2 \end{aligned} \quad (20)$$

In the derivation of this expression for $w_{3,\bar{z}}$, free use was made of the fact that insofar as the real part Φ_3 of w_3 is concerned, terms of the nature $g(z)\bar{h}(\bar{z})$ and $\bar{g}(\bar{z})h(z)$ are equivalent. It is important to note that this type of operation leaves unaltered the real part Φ_3 of w_3 . Since Φ_3 is the quantity sought in the calculations, changes in the imaginary part of w_3 are of no consequence in the final results.

Equation (20) can be integrated immediately by inspection and yields the following result:

$$\begin{aligned} w_3 = & \frac{1}{32}(1-\beta^2)^2\sigma^2\bar{z}^2w_{1z}^2w_{1zz} + \frac{1}{16}(1-\beta^2)^2\sigma\bar{z}w_{1z}F_z + \frac{1}{16}(1-\beta^2)^2\sigma(2+\sigma)\bar{z}\bar{w}_1w_{1z}w_{1zz} + \frac{1}{16}(1-\beta^2)\sigma[\beta^2(2+\sigma)+(4+3\sigma)]\bar{w}_1w_{1z}^2 + \\ & \frac{1}{32}(1-\beta^2)^2\sigma(2+\sigma)\bar{z}\bar{w}_{1z}w_{1z}^2 + \frac{1}{16}(1-\beta^2)^2(2+\sigma)\bar{w}_1\bar{w}_{1z}w_{1z} + \frac{1}{16}(1-\beta^2)^2(2+\sigma)(F\bar{w}_{1z} + F_z\bar{w}_1) + \frac{1}{32}(1-\beta^2)^2(2+\sigma)^2\bar{w}_1^2w_{1zz} + \\ & \frac{1}{96}(1-\beta^2)\sigma^2(3\beta^2+5)\bar{z}w_{1z}^3 + \frac{1}{32}(1-\beta^2)[\beta^2(8+10\sigma+3\sigma^2)+\sigma(6+5\sigma)]\bar{w}_{1z}\int w_{1z}^2dz \end{aligned} \quad (21)$$

Equation (21) is the particular integral of equation (20). The most general solution is obtained by adding an arbitrary analytic function $G(z)$, satisfying the homogeneous or Laplace equation $G_{zz}=0$. An arbitrary function of $\bar{z}$, customarily included in the general solution, need not be considered here because only the real part Φ_3 of w_3 is of interest. In fact, the omitted arbitrary function is the complex conjugate $\bar{G}(\bar{z})$.

In order to obtain the desired form of Φ_3 (the real part of w_3) from equation (21), $F(z)$ is replaced by its expression given in equation (16), and the real part of $f(z)$ in equation (17) is replaced by

$$\text{R.P. } f(z) = \frac{4}{M_\infty^2}\Phi_2 + 2\sigma y\Phi_{1x}\Phi_{1y} - 2(2+\sigma)\Phi_1\Phi_{1x}$$

The final form of the particular integral of the third-order iteration equation (9) then becomes

$$\begin{aligned} \Phi_3 = & -\frac{1}{2}(1-\beta^2)\sigma y(\Phi_{1x}\Phi_{2y} + \Phi_{1y}\Phi_{2x}) + \frac{1}{2}(1-\beta^2)(2+\sigma)(\Phi_{1x}\Phi_2 + \Phi_1\Phi_{2x}) + \\ & \frac{1}{8}(1-\beta^2)^2\sigma^2y^2\Phi_{1xx}(\Phi_{1x}^2 - \Phi_{1y}^2) - \frac{1}{4}(1-\beta^2)^2\sigma^2y^2\Phi_{1x}\Phi_{1y}\Phi_{1xy} + \frac{1}{48}(1-\beta^2)\sigma[(6+5\sigma)+3(-2+\sigma)\beta^2]y\Phi_{1y}^3 + \\ & \frac{1}{16}(1-\beta^2)\sigma[(10-\sigma)-(10+7\sigma)\beta^2]y\Phi_{1x}^2\Phi_{1y} + \frac{1}{4}(1-\beta^2)^2\sigma(2+\sigma)y\Phi_1(\Phi_{1x}\Phi_{1xy} + \Phi_{1y}\Phi_{1xz}) + \\ & \frac{1}{16}(1-\beta^2)[(-16-10\sigma+\sigma^2)+(16+22\sigma+7\sigma^2)\beta^2]\Phi_1\Phi_{1x}^2 - \frac{1}{16}(1-\beta^2)\sigma[(6+5\sigma)+3(2+\sigma)\beta^2]\Phi_1\Phi_{1y}^2 - \\ & \frac{1}{8}(1-\beta^2)^2(2+\sigma)^2\Phi_1^2\Phi_{1xx} + \frac{1}{16}(1-\beta^2)[\sigma(6+5\sigma)+(8+10\sigma+3\sigma^2)\beta^2]\Phi_{1x}\int[(\Phi_{1x}^2 - \Phi_{1y}^2)dx + 2\Phi_{1x}\Phi_{1y}dy] \end{aligned} \quad (22)$$

The corresponding expression for Φ_3 for supersonic flow is obtained by simply replacing σ by $-\sigma$ with

$$\sigma = \frac{\gamma+1}{2} \frac{M_\infty^2}{M_\infty^2-1}$$

and β^2 by $-\beta^2$ with $\beta^2 = M_\infty^2 - 1$. The physical plane variables X and Y are easily inserted into both equations (18) and (22) by means of the transformation equations (10),

$$x = X$$

$$y = \beta Y$$

It is pointed out that the forms of the two particular integrals, equations (18) and (22), derived in this report are identical for both subsonic and supersonic flow. The apparent differences are caused by a change in sign of the parameter β^2 . Thus, β^2 and σ are positive for both subsonic and supersonic flow. Actually, of course, the functions represented by Φ_1 , Φ_2 , Φ_3 , . . . are different for the two types of flow. For subsonic flow, these functions are derived from analytic and nonanalytic functions of z and $\bar{z}$; whereas for supersonic flow, they involve the real "characteristics" variables $x \pm \beta y$.

Note that the last term of the expression on the right-hand side of equation (22) contains the indefinite integral

$$I = \int [(\Phi_{1x}^2 - \Phi_{1y}^2)dx + 2\Phi_{1x}\Phi_{1y}dy] \quad (23)$$

It is obvious from the corresponding complex integral in equation (21) that the integrand of equation (23) is an exact differential. This fact can also be easily verified with the help of Laplace's equation

$$\Phi_{1xx} + \Phi_{1yy} = 0$$

Thus,

$$\frac{\partial}{\partial y} (\Phi_{1x}^2 - \Phi_{1y}^2) = \frac{\partial}{\partial x} (2\Phi_{1x}\Phi_{1y}) \quad (24)$$

Equation (24) represents the necessary and sufficient condition that the integrand of equation (23) be an exact differential. Further, according to the theory of exact differential equations, the integral I may be expressed as follows:

$$I = \int^x M dx + \int \left(N - \frac{\partial}{\partial y} \int^x M dx \right) dy \quad (25)$$

where

$$M = \Phi_{1x}^2 - \Phi_{1y}^2$$

$$N = 2\Phi_{1x}\Phi_{1y}$$

and where by $\int^x M dx$ is meant the result of integrating $M dx$ with y considered constant. The expression within the parentheses, namely

$$N - \frac{\partial}{\partial y} \int^x M dx$$

is a function of y only. This statement can be verified as

follows: Thus,

$$\frac{\partial}{\partial x} \left(N - \frac{\partial}{\partial y} \int^x M dx \right) = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y} \frac{\partial}{\partial x} \int^x M dx \quad (26)$$

and because y is considered constant in the process of integrating $M dx$, it is clear that

$$\frac{\partial}{\partial x} \int^x M dx = M$$

Hence the right-hand side of equation (26) is $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$, which vanishes because of the condition for the existence of an exact differential.

Note that in general it is simpler to perform the complex integration $\int w_{1z}^2 dz$ rather than transform to a real integral and then perform the integration.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., May 29, 1950.

REFERENCES

1. Van Dyke, Milton D.: A Study of Second Order Supersonic Flow. GALCIT Thesis, 1949.
2. Kaplan, Carl: Effect of Compressibility at High Subsonic Velocities on the Lifting Force Acting on an Elliptic Cylinder. NACA Rep. 834, 1946. (Supersedes NACA TN 1118.)
3. Kaplan, Carl: The Flow of a Compressible Fluid past a Curved Surface. NACA Rep. 768, 1943. (Supersedes NACA ARR 3K02.)

REPORT 1040

SPECTRA AND DIFFUSION IN A ROUND TURBULENT JET¹

By STANLEY CORRSIN and MAHINDER S. UBEROI

SUMMARY

In a round turbulent jet at room temperature, measurement of the shear correlation coefficient as a function of frequency (through band-pass filters) has given a rather direct verification of Kolmogoroff's local-isotropy hypothesis.

One-dimensional power spectra of velocity and temperature fluctuations, measured in unheated and heated jets, respectively, have been contrasted. Under the same conditions, the two corresponding transverse correlation functions have been measured and compared.

Finally, measurements have been made of the mean thermal wakes behind local (line) heat sources in the unheated turbulent jet, and the order of magnitude of the temperature fluctuations has been determined.

INTRODUCTION

At the present time there apparently exists no statistical theory of turbulent shear flow. One significant theoretical consideration has been proposed: The hypothesis of local isotropy, originated by Kolmogoroff (references 1 and 2). Kolmogoroff has suggested that the fine structure in turbulent shear flow may be isotropic; recent experiments of Townsend (references 3 and 4) in a turbulent wake seem to verify this hypothesis.

In view of this situation, the various experimental researches in the field follow two courses: First, they attempt to verify or disprove local isotropy; second, they try in all conceivable ways to make measurements that may shed light upon the basic nature of the turbulent shear flow, so that the foundations for a successful theory can be laid.

During recent years it has become evident that measurements of the intensity of turbulence alone cannot provide sufficient information about the statistical and dynamical properties of the flow fields. Such quantities as correlations, spectra, probability densities, the various terms in the turbulent kinetic-energy balance, and so forth may be expected to reveal many essential features of the problem.

The types of turbulent shear flow that have come under close experimental scrutiny are the boundary layer (references 5 and 6), the plane channel (reference 7), the plane wake (references 3 and 4), the plane single free-mixing region (reference 8), and the round jet (references 9 and 10). The present work is a continuation of that reported in references 9 and 10.

The general objectives of this investigation have been to learn something more about the flow in a fully developed round turbulent jet and about the heat transfer in such a flow. The work reported here has fallen into three phases:

(a) An attempt to establish the presence or absence of local isotropy, (b) a comparison of velocity- and temperature-fluctuation fields when the over-all boundary conditions on mean velocity and temperature are effectively the same, and (c) a study of the diffusion of heat from a local (line) source in the turbulent flow.

The only specific experimental verification of local isotropy in a turbulent shear flow to date was by Townsend in the plane wake behind a circular rod (references 3 and 4). He found that the skewness and flattening factors of the probability density of $\partial u/\partial t$ in the shear flow are very closely equal to those in the (effectively isotropic) turbulence far behind a grid. Since differentiation emphasizes the higher frequencies, his measurement shows, in essence, that the values of certain statistical quantities related to the smaller eddies in a shear flow are the same as the values for the smaller eddies in isotropic turbulence. He also found the microscale of u in the stream direction to be nearly $\sqrt{2}$ times the microscale of v in that direction, a relation which is exactly true for isotropic turbulence.

Until recently, only mean-velocity and mean-temperature distributions were measured to provide a comparison of the transfer rates of momentum and of heat in turbulent shear flows with over-all heat transfer. The previous report in this round-jet investigation (reference 10) included a beginning on the problem of direct comparison of the velocity and temperature fluctuations as well as some measurements of velocity-temperature correlations. The fluctuations in a warm turbulent wake have been studied by Townsend (reference 11).

Up to the present time, however, there appears still to be no successful hypothesis to account for the well-known fact that heat (and other scalar quantities, like material) is diffused more rapidly than momentum in a turbulent flow. Thus, more detailed study of the fluctuations seemed in order.

The first real analysis on the diffusive property (for scalar quantities) of a homogeneous turbulent field was Taylor's well-known work "Diffusion by Continuous Movements" (reference 12).

The mean thermal wake behind a line source of heat in a flowing isotropic turbulence has been carefully measured by Schubauer (reference 13) in the region close to the source, and by Simmons (measurements reported in reference 14) over an extended range. Taylor's theory of diffusion by continuous movements is directly applicable to the diffusion from a line source of heat in a homogeneous turbulent field, and he has

¹ Supersedes NACA TN 2124, "Spectrums and Diffusion in a Round Turbulent Jet" by Stanley Corrsin and Mahinder S. Uberoi, 1950.

made a generalization to permit application of the method in a decaying isotropic turbulence (reference 14).

In a turbulent shear flow, the only published measurements of the thermal wake of a local source seem to be those of Skramstad and Schubauer in a turbulent boundary layer.² These were reported by the experimenters (reference 15) and by Dryden (reference 16). The temperature distribution across the wake in shear flow is decidedly skew. On the other hand, in the isotropic turbulence, it is to all intents and purposes a Gaussian curve. Of course, in a homogeneous field this curve (measured close enough to the source so that the Lagrangian correlation coefficient is still effectively unity) is simply the probability density of the lateral velocity fluctuations. This relation may be roughly true for shear flow as well, in which case the contrasting results mentioned above would mean that the probability density of $v(t)$ is skew in shear flow, but Gaussian in decaying isotropic turbulence.

No measurements had been made of the fluctuations in these thermal wakes, and it was felt that some information on the nature of these might help further a general understanding of the diffusive process.

This investigation was conducted at the Aeronautics Department of The Johns Hopkins University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics. The authors would like to acknowledge many stimulating conversations with Dr. L. S. G. Kovátszay, and to thank Dr. F. H. Clauser for his helpful criticism. Donation of the hot-jet unit by Dr. C. B. Millikan, Director of the Guggenheim Aeronautical Laboratory at the California Institute of Technology, is greatly appreciated. Mr. Philip Lebowitz helped to set up much of the laboratory equipment.

SYMBOLS

d	diameter of orifice (1 in.)
x	axial distance from orifice
r	radial distance from jet axis
$\bar{U}$	axial component of mean velocity
$\bar{V}$	radial component of mean velocity
$\bar{W}$	tangential component of mean velocity
$\bar{U}_{max}$	maximum $\bar{U}$ at a cross section (on axis)
u	axial component of instantaneous velocity fluctuation
v	radial component of instantaneous velocity fluctuation
w	tangential component of instantaneous velocity fluctuation
$u' \equiv \sqrt{u^2}$	
$v' \equiv \sqrt{v^2}$	
θ	instantaneous temperature difference (measured above room temperature as reference)
$\bar{\theta}$	mean temperature difference (measured above room temperature as reference)

² Wieghardt has recently measured the diffusion of heat from a local source in a turbulent boundary layer (reference 17), but his source was flush with the solid surface and thus he was studying a different problem, that is, one more directly related to micrometeorological conditions.

$\bar{\theta}_{max}$	maximum mean temperature difference at a cross section
$\bar{\theta}_o$	maximum mean temperature in jet at orifice
ϑ	instantaneous temperature fluctuation ($\vartheta \equiv \theta - \bar{\theta}$)
$\vartheta' \equiv \sqrt{\vartheta^2}$	
t	time
e	voltage fluctuation
α, β	sensitivity of a diagonal hot-wire to u and v , respectively
R_{uv}	shear correlation coefficient ($\overline{uv}/u'v'$)
u_n, v_n	instantaneous contributions of $u(t)$ and $v(t)$, respectively, in a narrow frequency band of nominal frequency n ; in a Fourier series discussion, the n th harmonic of periodic $u(t)$ and $v(t)$, respectively
$u_n' \equiv \sqrt{u_n^2}$	
$v_n' \equiv \sqrt{v_n^2}$	
${}_nR_{uv}$	shear correlation coefficient for a narrow band of frequencies ($\overline{u_n v_n}/u_n' v_n'$); this function is referred to as the "shear-correlation spectrum" or, briefly, the "shear spectrum"
ϕ, ψ	phase angles
${}_nF_u, {}_nF_v$	in Fourier series analysis, ${}_nF_u \equiv a_n^2 / \sum_1^\infty a_n^2$, ${}_nF_v \equiv b_n^2 / \sum_1^\infty b_n^2$, where a_n and b_n are Fourier series coefficients
n	cyclic frequency in general; in particular, magnitude of radial coordinate in three-dimensional frequency space
n_1	cyclic frequency of one-dimensional spectra of $u(t)$ and $\vartheta(t)$
k	wave-number magnitude for three-dimensional spectra ($2\pi n/\bar{U}$)
k_1	wave-number magnitude for one-dimensional spectra ($2\pi n_1/\bar{U}$)
k_o	a reference constant with dimensions of wave numbers
$F_1(k_1)$	one-dimensional power spectrum of $u(t)$ in terms of wave number
$F_1^*(n_1)$	one-dimensional power spectrum of $u(t)$ in terms of frequency
$F(k)$	three-dimensional power spectrum of velocity fluctuation
$G_1(k_1)$	one-dimensional power spectrum of $\vartheta(t)$ in terms of wave number
$G_1^*(n_1)$	one-dimensional power spectrum of $\vartheta(t)$ in terms of frequency
$G(k)$	three-dimensional power spectrum of temperature fluctuation
R_y	transverse correlation function of u measured symmetrically about jet axis ($\overline{u_1 u_2}/u'^2$)
R_x	longitudinal correlation function of u
S_y	transverse correlation function of ϑ measured symmetrically about jet axis ($\overline{\vartheta_1 \vartheta_2}/\vartheta'^2$)
L_x	longitudinal scale of u -fluctuations
Λ_x	longitudinal scale of ϑ -fluctuations

L_y	lateral scale of u -fluctuations
Δ_y	lateral scale of ϑ -fluctuations
λ_x	longitudinal microscale of u -fluctuations
λ	lateral microscale of u -fluctuations
l	lateral microscale of ϑ -fluctuations
l_x	longitudinal microscale of ϑ -fluctuations
ξ	distance downstream from local heat source in x-direction
ζ	lateral distance, perpendicular to source line, from local heat source
Φ	dimensionless temperature ratio $(\bar{\theta}/\bar{\theta}_{max})$
$\eta = \zeta/\xi$	
Δ	standard deviation of mean-temperature distribution in wake behind local heat source
τ	pulse spacing
h	pulse height
j	pulse width

EQUIPMENT

AERODYNAMIC EQUIPMENT

The 1-inch hot-jet unit is shown schematically in figure 1. The centrifugal blower is driven by a $\frac{1}{2}$ -horsepower direct-current motor. Heat is added through two double banks of coils of No. 16 Nichrome wire. As can be seen in the sketch, a good part of the heated air is directed around the outside of the jet-air pipe in order to maintain a flat initial temperature distribution in the jet. A vacuum-cleaner blower is used to help the air through this secondary heating annulus, and this warm air is fed back into the intake of the main blower.

The section of relatively high velocity between heaters and final pressure box permits adequate mixing behind the grid, to insure thermally homogeneous initial jet air.

Figure 2 is a photograph of the unit as set up previously (reference 10); the present arrangement is essentially the same.

All turbulence measurements were made with an initial

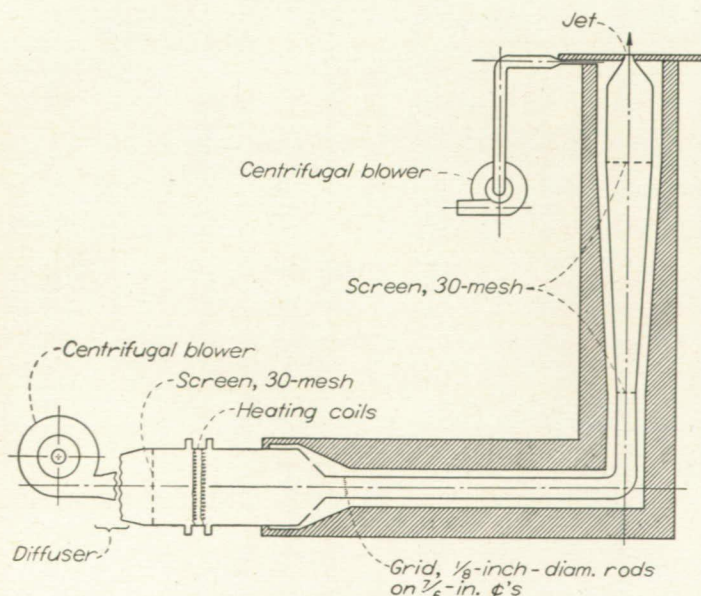


FIGURE 1.—Schematic diagram of 1-inch hot-jet unit.

jet total head in the range from 3.5 to 5.0 inches of water. In free turbulent flows there is no detectable effect of jet Reynolds number over a much wider range of Reynolds numbers than this.

When the jet was run unheated, there was a slight temperature rise through the blower and duct. In the measurements of thermal wake behind a local heat source, correction for this ambient-temperature field was necessary. For all hot runs, the orifice air temperature was very close to 200° C, about 175° above room temperature.

Three different "local heat sources" were used:

(a) A straight diametrically strung wire of 0.008-inch Nichrome

(b) A 2-inch-diameter Nichrome ring

(c) A 4-inch-diameter Nichrome ring

Because of the extremely high turbulence levels encountered in a free jet, a measurable thermal wake could only be obtained by using source temperatures in the range from 300° to 700° C. This undoubtedly led to some local buoyancy effects, but, even with this order of temperatures, the thermal wake was barely detectable 1 inch downstream.

The Reynolds numbers of these heat-source wires were about as follows:

For straight wire on axis,

150, based on air temperature

22, based on wire temperature

For 2-inch-diameter ring,

110, based on air temperature

16, based on wire temperature

For 4-inch-diameter ring,

59, based on air temperature

9, based on wire temperature

No noticeable additional turbulence was generated by these wires, and no average momentum defect could be detected with a flattened total-head tube, even as close as $\frac{1}{4}$ inch downstream.

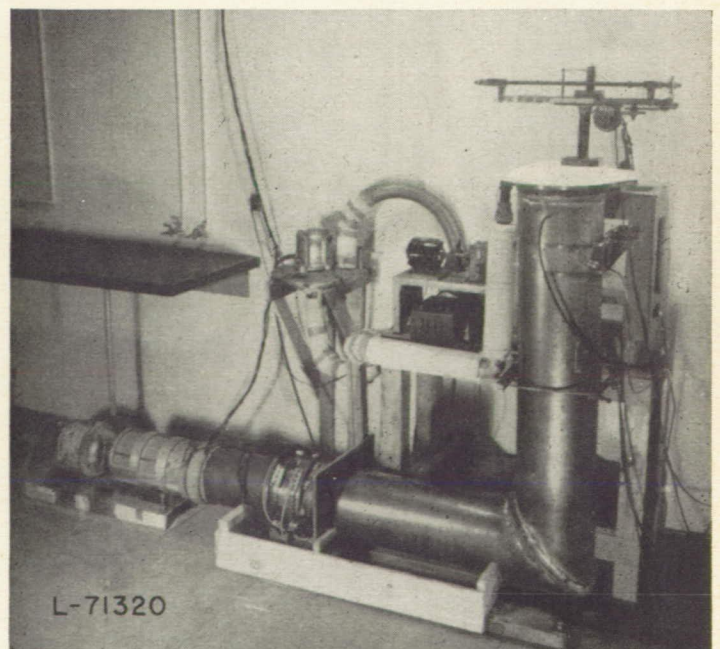


FIGURE 2.—The jet unit.

MEASURING EQUIPMENT

The measuring instruments used were: Total-head tube, Chromel-Alumel thermocouple, and hot-wire anemometer (also used as resistance thermometer).

The hot-wires were nominally 0.000635-centimeter platinum, about 1.5 millimeters in length, etched from Wollaston wire. The etched platinum was soft-soldered to the tips of small steel needle supports. A discussion of heat loss from a wire at various ambient temperatures is given in appendix A.

The basic hot-wire-anemometry equipment was purchased from Mr. Carl L. Thiele of Altadena, California. One of the two identical heating circuits is shown in figure 3.

The amplifier, with resistance-capacitance compensation network, is given in figure 4. The uncompensated gain is constant to within ± 2 percent over a frequency range from 3 to 12,000 cycles per second (fig. 5). With the wires and operating conditions used (time constants on the order of 1 millisecond), the over-all compensated response was good over the same range. Correct setting of the compensation network was determined by superimposing a square wave upon the hot-wire bridge (reference 18). Unfortunately, in a free turbulent shear flow the ambient disturbance is so great (because of the extremely high turbulence levels) that calibration cannot be made in the flow to be studied.

The vacuum-thermocouple signal output was measured

either with a millivoltmeter or by the average deflection rate of a fluxmeter.

The various spectra reported here were measured with a modified General Radio Type 760-A Sound Analyzer (reference 19). The changes in output stage (fig. 6) were made to eliminate the direct-current component and to obtain linear instead of logarithmic response. As modified, the sound analyzer had rather undesirable frequency-response characteristics, particularly a day-to-day shift in relative amplification of the higher-frequency ranges. The frequency-response calibration in figure 7 is plotted in terms of voltage squared, since this was the quantity ultimately measured.

The frequency pass band for this analyzer is far from the optimum rectangular shape. However, the slopes of the two sides are sufficiently steep that no appreciable error is attributable to noninfinite slopes with the spectra measured in this investigation. Figure 8 is an experimentally determined band shape. There was fair similarity of band shape over the entire frequency range. For computational purposes, an equivalent rectangular pass band was defined as indicated in the figure.

The instrument is a type recording constant-percent band width, measuring the product of power spectrum times frequency. This has obvious advantages in the high-frequency range where there is so little turbulent energy.

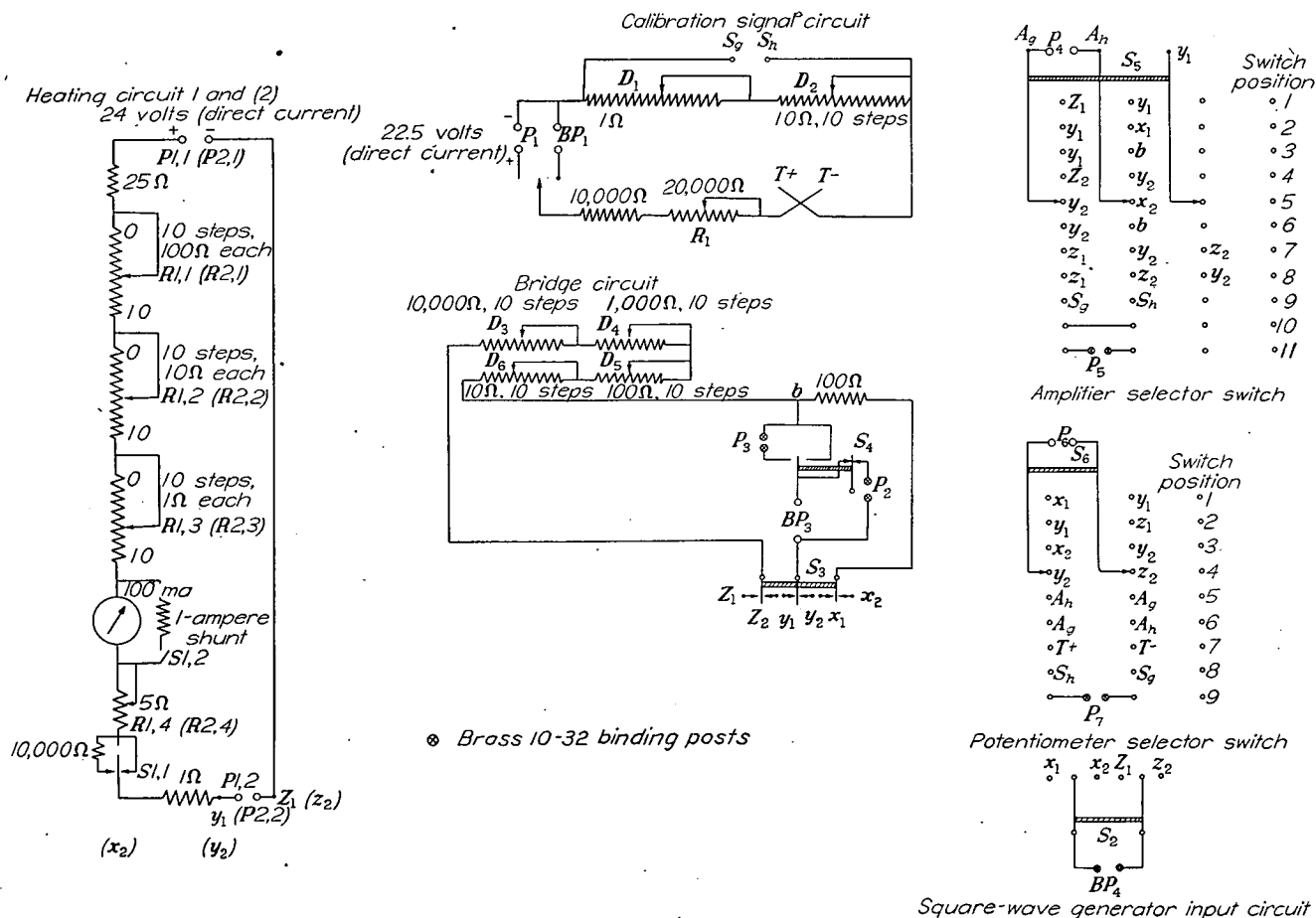


FIGURE 3.—Control circuits.

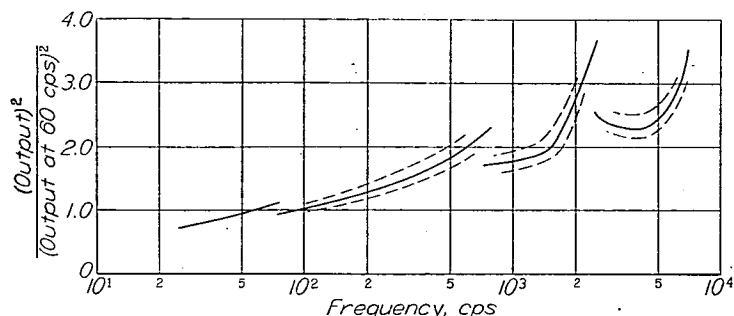


FIGURE 7.—Response of sound analyzer with modified output circuit.

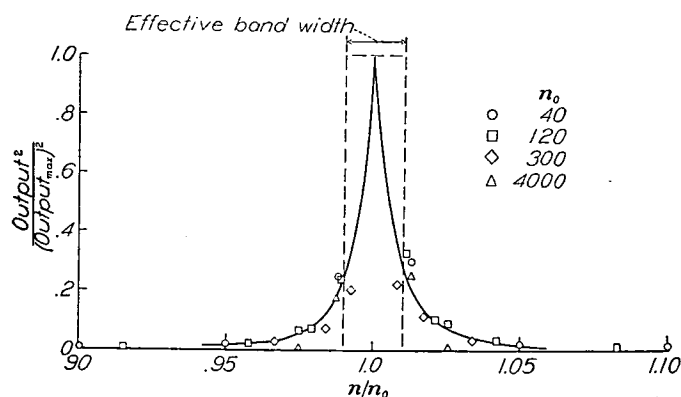


FIGURE 8.—Filtration characteristics of sound analyzer with modified output circuit. Area of rectangle equals area under curve.

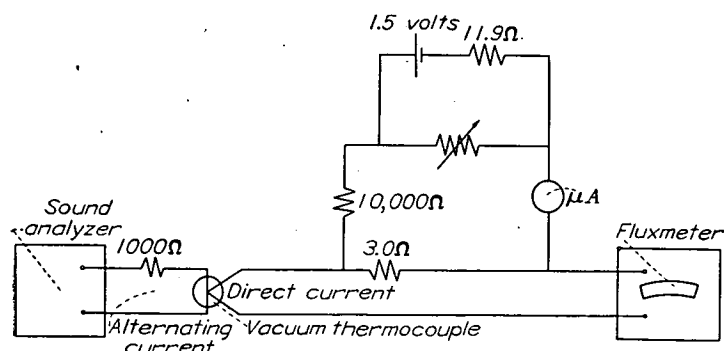


FIGURE 9.—Integrator and bucking circuit.

deflection point during the time of integration, and its reading at the end of this time ordinarily gave a small correction on the result.

In the highest-frequency range, overload considerations on the sound analyzer limited the signal drastically, and only a part of the thermocouple direct current was bucked.

For the determination of average wake temperatures behind the local heat sources, the thermocouple voltage was measured with a Leeds & Northrup type K-2 potentiometer.

Oscillograms were taken from a blue oscilloscope tube by means of a General Radio Type 651-AE camera, using fast film.

PROCEDURES

VELOCITY SPECTRUM

The power (or energy) spectrum of the longitudinal velocity fluctuations at a point in the unheated jet was measured by conventional hot-wire-anemometry technique,

with a continuously adjustable band-pass filter, as described under "Equipment."

TEMPERATURE SPECTRUM

The power spectrum of the temperature fluctuations in the heated jet was measured by using the hot-wire effectively as a simple resistance thermometer (reference 20). The amplified voltage signal was analyzed exactly as in the measurement of velocity spectra.

SHEAR-CORRELATION SPECTRUM

For the shear-correlation spectrum, the quantity to be measured is the correlation coefficient between a narrow frequency band of u -fluctuations and the same narrow frequency band of v -fluctuations, at the same point in the flow field.

The method, for any particular nominal frequency, was to pass the various voltage signals (e_1 , e_2 , $e_1 + e_2$, $e_1 - e_2$) from an $\times$ -type shear- (or v' -) meter through the band-pass filter after amplification. By appropriate combination of the mean-square values of these four signals (identical with total-shear measurement), there results

$${}_n R_{uv} \equiv \overline{u_n v_n} / \overline{u_n'^2 v_n'^2}$$

where the subscript n indicates the narrow band of nominal frequency n cycles per second. A justification for the validity of this procedure is obtainable by considering the two velocity-fluctuation components as periodic functions. Of course, this is not a real proof.

If a symmetrical $\times$ -meter is assumed, the two instantaneous voltage signals are

$$\begin{aligned} e_1 &= \alpha u + \beta v \\ e_2 &= \alpha u - \beta v \end{aligned} \quad (1)$$

For periodic fluctuations,

$$\left. \begin{aligned} u &= \sum_{n=1}^{\infty} a_n \cos(2\pi n t + \phi_n) \\ v &= \sum_{n=1}^{\infty} b_n \cos(2\pi n t + \psi_n) \end{aligned} \right\} \quad (2)$$

In this simple case, the correlation coefficient for any spectral line is merely

$${}_n R_{uv} = \cos(\phi_n - \psi_n) \quad (3)$$

If equations (2) are substituted into equations (1), it is easily shown that

$$\cos(\phi_n - \psi_n) = \frac{\overline{a_1^2} - \overline{a_2^2}}{\sqrt{(\overline{a_1^2} + \overline{a_2^2})(\overline{b_1^2} + \overline{b_2^2})}} \quad (4)$$

in complete analogy to the conventional method for measuring R_{uv} with an $\times$ -meter. The algebraic details are given in appendix B.

When the meter is not perfectly symmetrical ($\alpha_1 \neq \alpha_2$; $\beta_1 \neq \beta_2$), the formal processes, both algebraic and experimental, become excessively involved. Consequently, in

actual practice the effect of unavoidable unsymmetry in the $\times$ -meter was essentially nullified by taking double sets of readings at each frequency, rotating the instrument 180° about the axis of flow direction between sets.

VELOCITY CORRELATION FUNCTION

The double correlation R_v between longitudinal velocity fluctuation at pairs of points on opposite sides of the jet axis was measured only in the unheated jet. Hence the standard hot-wire-anemometry technique was used. The two hot-wires were always equidistant from the axis (on a diameter), so that they were under identical operating conditions.

TEMPERATURE CORRELATION FUNCTION

In the hot jet, the wires traversing symmetrically as for R_v were operated as simple resistance thermometers, so that the double temperature correlation S_v could be determined directly.

MEAN TEMPERATURES BEHIND LOCAL SOURCE

To determine mean temperatures behind a local source, traverses were made with a Chromel-Alumel thermocouple, whose voltage was measured with a Leeds & Northrup type K-2 potentiometer.

TEMPERATURE FLUCTUATIONS BEHIND LOCAL SOURCES

To determine temperature fluctuations behind local sources, the fine platinum wire was operated at small currents, so that it worked essentially as a resistance thermometer.

EXPERIMENTAL RESULTS

Mean-velocity and mean-temperature distributions for various orifice temperatures are presented in reference 10.

SHEAR-CORRELATION SPECTRUM

The spectrum of shear correlation coefficients ${}_n R_{uv} = \overline{u_n v_n} / \overline{u_n' v_n'}$ was measured in the unheated jet at $x/d=20$ at a radial station corresponding to maximum shear at this cross section. Figure 10 shows quite definitely that ${}_n R_{uv}(n)$ is a function decreasing monotonically to zero. Thus, the

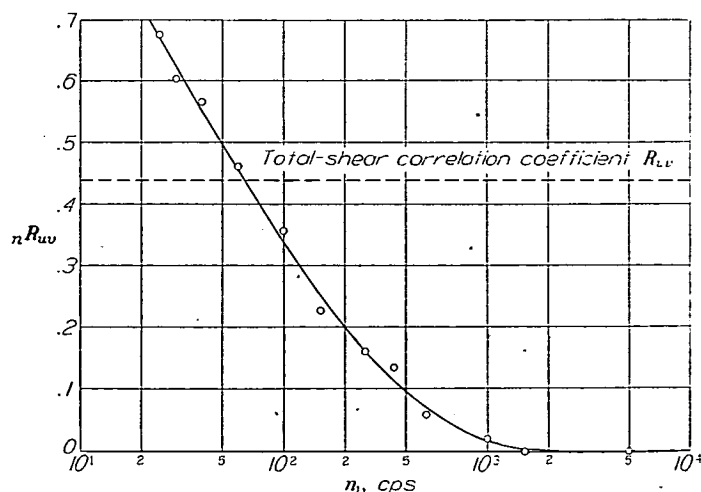


FIGURE 10.—Variation of shear correlation coefficient ${}_n R_{uv}$ with frequency. Round turbulent jet. $x/d=20$; $\bar{U}=270$ centimeters per second.

hypothesis of local isotropy is seen to be verified in a very direct way. The value of the directly measured total-shear correlation coefficient $R_{uv} = \overline{uv} / \overline{u'v'}$ is indicated in the figure.

VELOCITY AND TEMPERATURE SPECTRA

The one-dimensional power spectrum $F_1(k_1)$ of the longitudinal velocity fluctuations $u(t)$ was measured at two radial positions in the unheated jet at $x/d=20$. Figure 11 gives the two spectra, one measured on the axis and one measured at about the maximum-shear location. Plotted against wave number ($k_1 = 2\pi n_1 / \bar{U}$), the two spectra are identical within the experimental scatter. The solid line drawn as approximation to the points is made up as follows:

- (a) For $0 < k_1 < 1.25$, it is Von Kármán's semiempirical formula (reference 21):

$$F_1\left(\frac{k_1}{k_0}\right) = \frac{\text{Constant}}{\left[1 + \left(\frac{k_1}{k_0}\right)^2\right]^{5/6}} \quad (5)$$

- (b) For $k_1 > 1.25$, a nonanalytical curve has been faired in

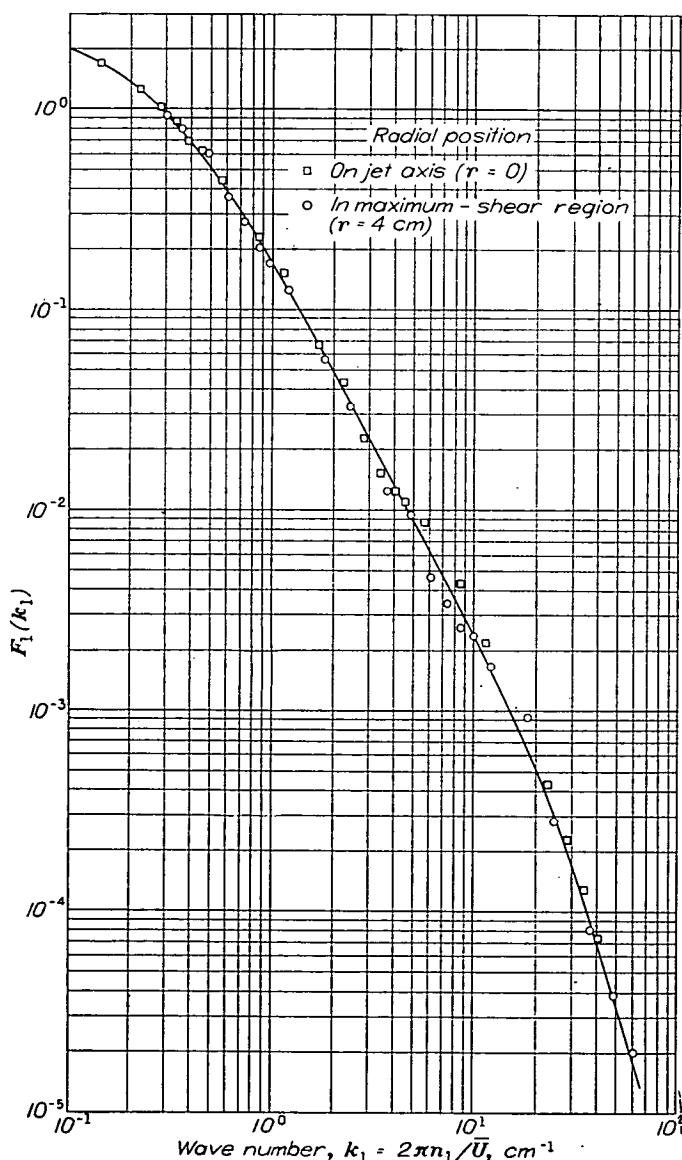


FIGURE 11.—One-dimensional power spectra of $u(t)$ measured in 1-inch unheated jet at $x/d=20$. Computed scales: $L_z=3.3$ centimeters for both stations, $\lambda_z=0.31$ centimeter for both stations.

The Von Kármán expression was used primarily to simplify the problem of extrapolation to $k_1=0$ and to shorten the work of transformation to three-dimensional spectra. (See "Velocity and Temperature Spectra" under "Analysis of Results.")

The corresponding one-dimensional power spectra of temperature fluctuations $G_1(k_1)$ were measured in the heated jet ($\bar{\theta}_0=170^\circ\text{C}$) at $x/d=20$. One spectrum is on the jet axis; one is at about the radial station of maximum heat transfer (and shear). From figure 12, it can be seen that they differ noticeably in the high-frequency range, but are essentially identical in the low- and moderate-frequency ranges. The curves used to approximate the experimental points are as follows: On the axis—the Von Kármán formula is used over the entire range. At the maximum-heat-transfer point—

- (a) For $0 < k_1 < 1.0$, the Von Kármán formula is used
- (b) For $k_1 > 1.0$, a nonanalytical curve has been faired in

Figures 13 and 14 contrast velocity spectra with temperature spectra at corresponding radial stations.

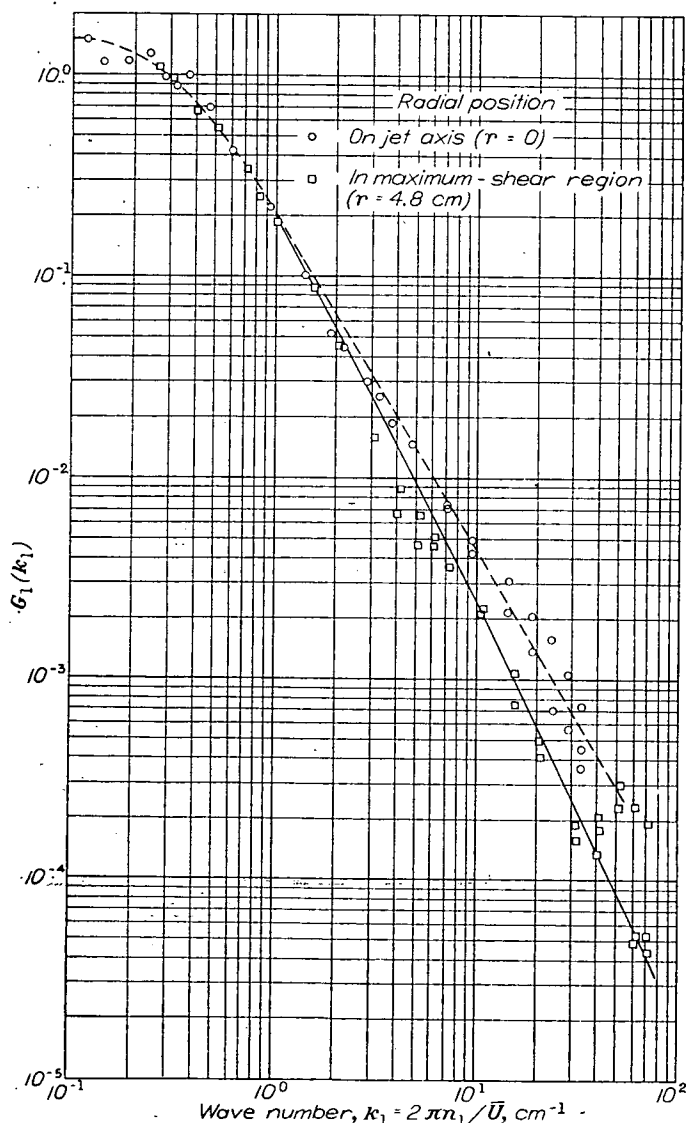


FIGURE 12.—One-dimensional power spectra of $\vartheta(t)$ measured in 1-inch heated jet at $x/d=20$.
Computed scales: $\Lambda_z = \frac{\pi}{2} G_1(0) = 2.18$ centimeters for both stations.

TRANSVERSE CORRELATION FUNCTIONS

The double correlation function $R_v = \overline{u_1 u_2} / u_1' u_2'$, measured symmetrically about the axis at $x/d=20$ in the unheated jet, is plotted in figure 15. Of course, since the wires are in identical flow conditions, $u_1' = u_2' = u'$ (say), and $R_v = \overline{u_1 u_2} / u_2^2$.

The double correlation function $S_v = \overline{\vartheta_1 \vartheta_2} / \vartheta_1' \vartheta_2'$ in the heated jet at $x/d=20$ is plotted in figure 16. Since this was also measured symmetrically, $S_v = \overline{\vartheta_1 \vartheta_2} / \vartheta^2$, where $\vartheta_1' = \vartheta_2' = \vartheta'$ (say).

Clearly, the range of measurable temperature correlation exceeds the range of measurable velocity correlation by an amount greater than can be attributed simply to the fact that the hot jet is wider than the unheated jet (reference 10).

MEAN THERMAL WAKES BEHIND LOCAL HEAT SOURCES

Typical radial distributions of average temperature behind a straight (diametrical) wire, a 2-inch-diameter ring, and a 4-inch-diameter ring at $x/d=20$ in the unheated jet are shown in figures 17, 18, and 19, respectively. The points in these figures are not direct experimental points, but merely

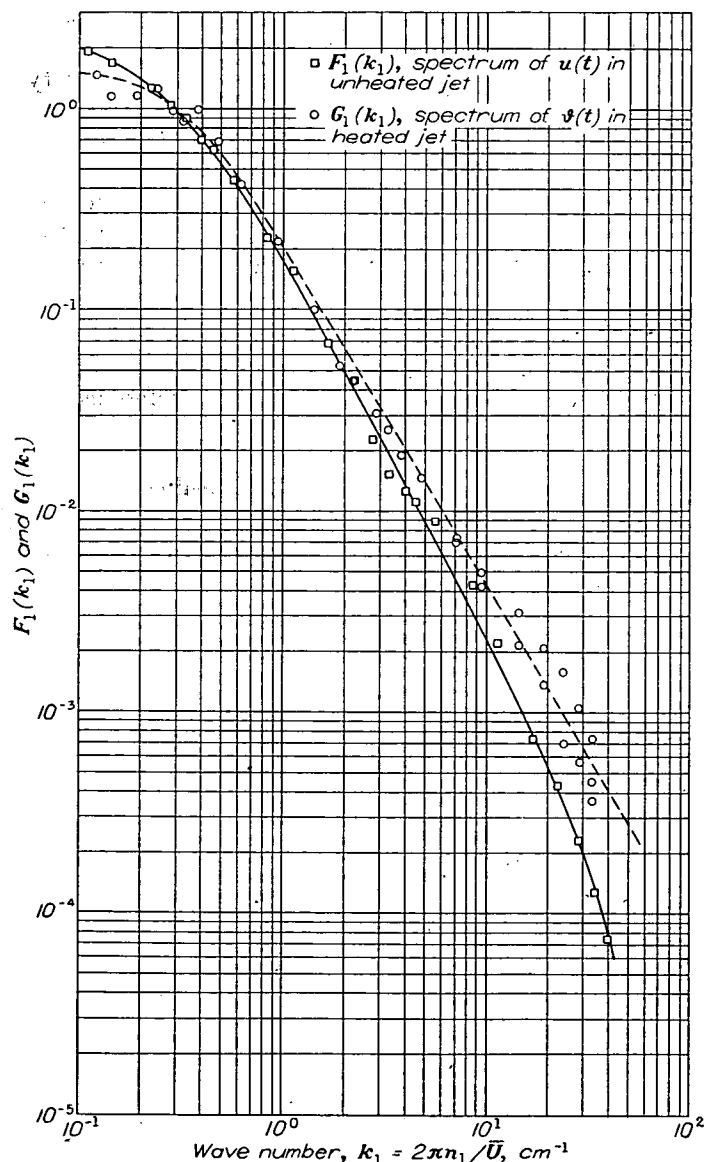


FIGURE 13.—One-dimensional spectra of $u(t)$ and $\vartheta(t)$ on axis of 1-inch jet at $x/d=20$.

serve to indicate the faired results for different distances downstream. All these have been corrected for the previously mentioned small ambient-temperature field in the unheated jet. There was rather large scatter (illustrated only in fig. 20) due to the small temperature differences measured and to the extremely large degree of fluctuation present. The 4-inch ring is slightly outside of the fully turbulent jet core (reference 9); consequently the results for this case are not of direct interest in a study of fully developed turbulence. The figures show that each of the thermal wakes possesses similarity well within the accuracy of measurement.

All of the thermal wakes spread linearly in the measured range (figs. 21, 22, and 23). From Taylor's theory of diffusion by continuous movements, this simply indicates that, for the maximum downstream station studied, the Lagrangian correlation coefficient of the v -fluctuations has still not departed appreciably from unity.

For a straight-line source at the jet axis, because of con-

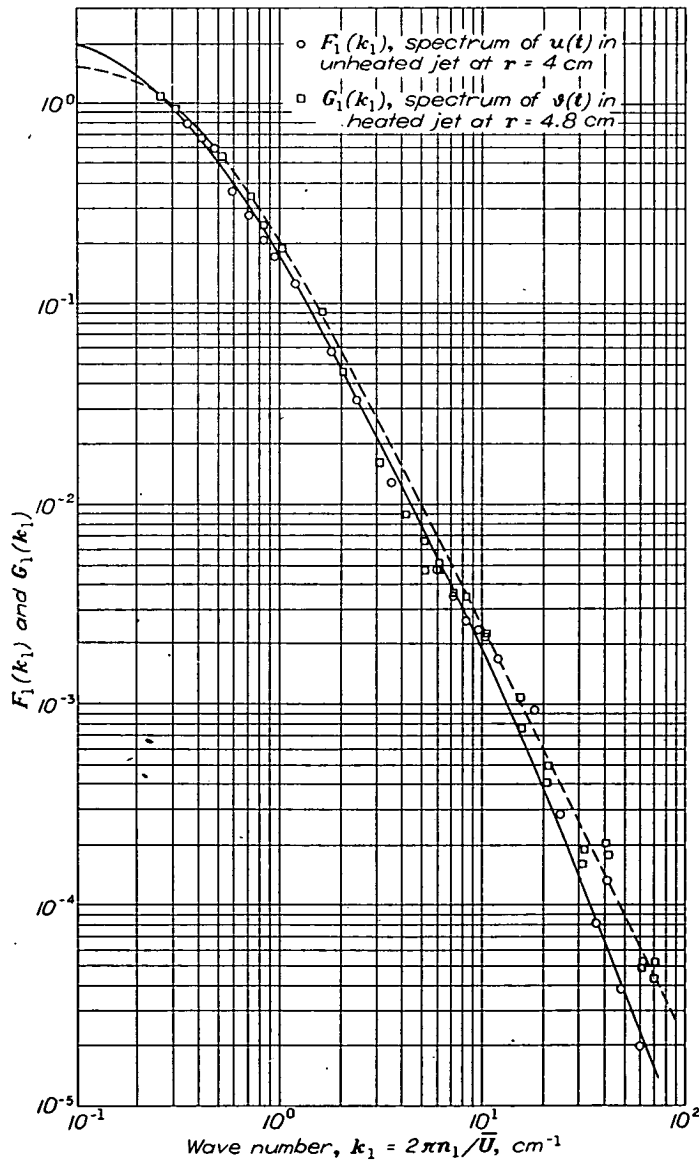


FIGURE 14.—One-dimensional power spectra of $u(t)$ and $v(t)$ in maximum-shear region of 1-inch jet at $x/d=20$.

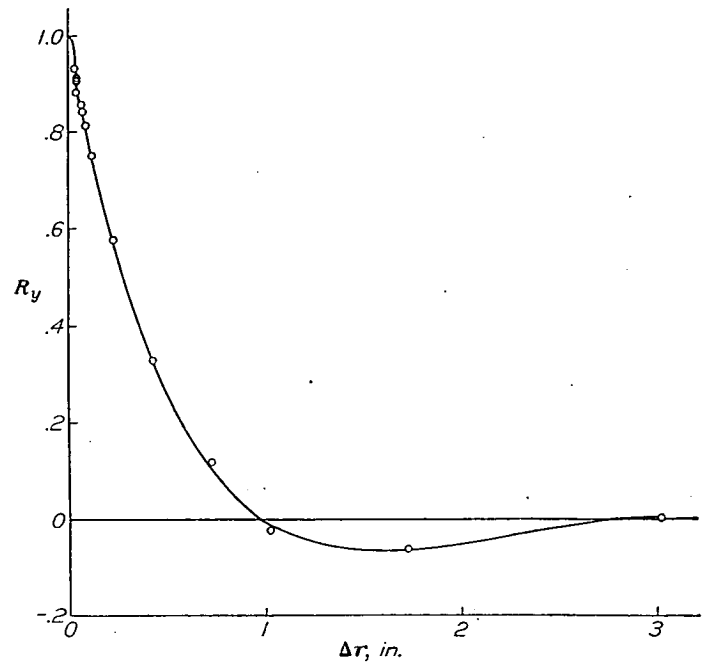


FIGURE 15.—Symmetric transverse correlation of u , measured about axis at $x/d=20$ in 1-inch unheated jet. $R_y = \overline{u_1 u_2} / \bar{u}^2$.

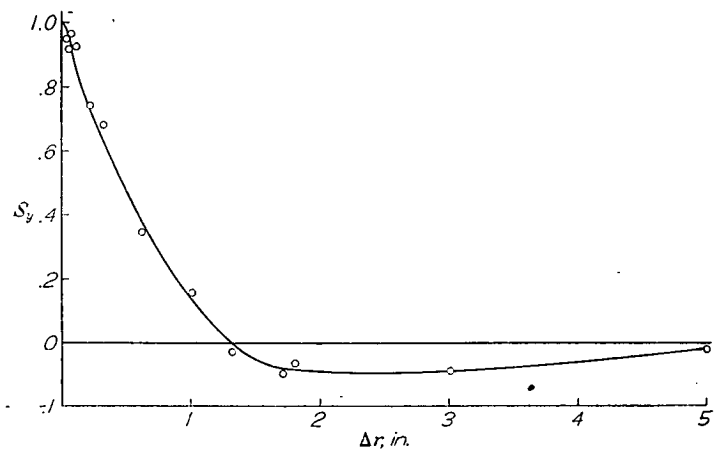


FIGURE 16.—Symmetric transverse correlation of ϑ , measured at $x/d=20$ in 1-inch heated jet. $S_y = \overline{\vartheta_1 \vartheta_2} / \bar{\vartheta}^2$.

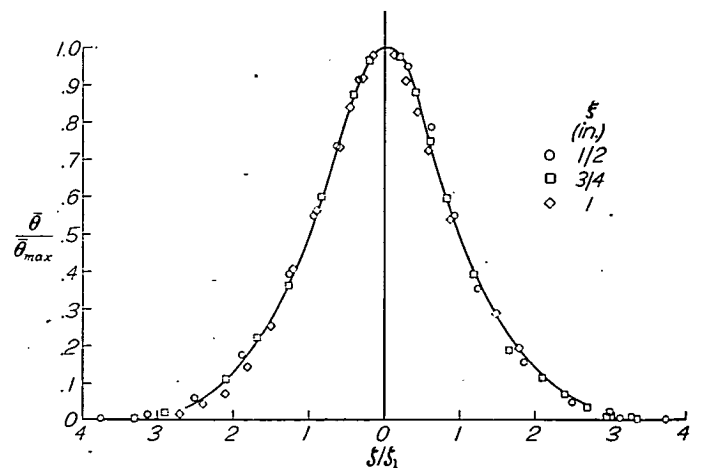


FIGURE 17.—Similarity of temperature behind line source of heat on jet axis at $x/d=20$ in unheated jet. $\xi_1 = \xi$ at which $\bar{\vartheta} = \frac{1}{2} \bar{\vartheta}_{max}$.

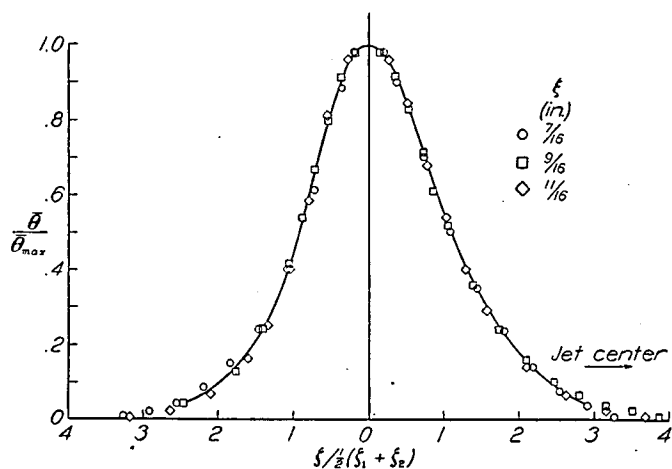


FIGURE 18.—Similarity of temperature distribution behind 2-inch-diameter-ring source of heat in unheated jet at $x/d=20$. $\xi_1, \xi_2=\xi$ at which $\bar{\theta}=\frac{1}{2}\bar{\theta}_{max}$.

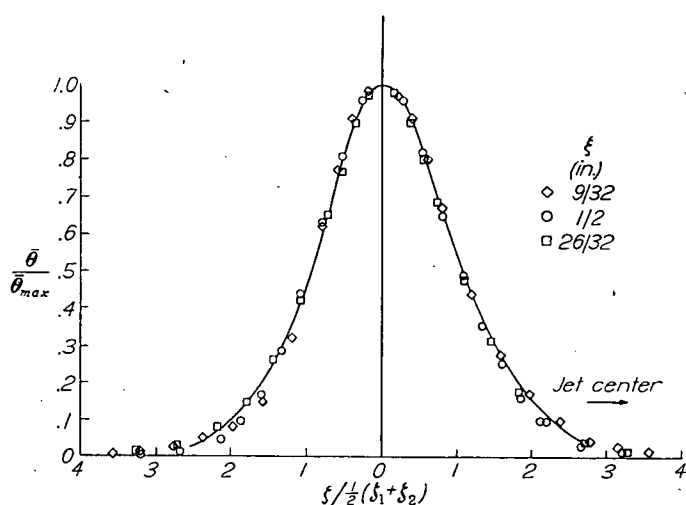


FIGURE 19.—Similarity of temperature behind 4-inch-diameter-ring source of heat in unheated jet at $x/d=20$. $\xi_1, \xi_2=\xi$ at which $\bar{\theta}=\frac{1}{2}\bar{\theta}_{max}$.

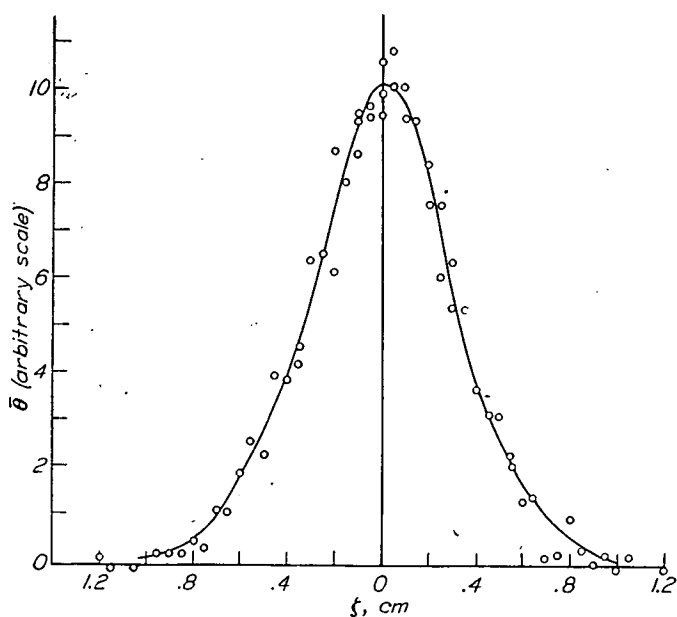


FIGURE 20.—Experimental scatter; temperature behind line source of heat on jet axis. $\xi=1.27$ centimeters; $x/d=20$.

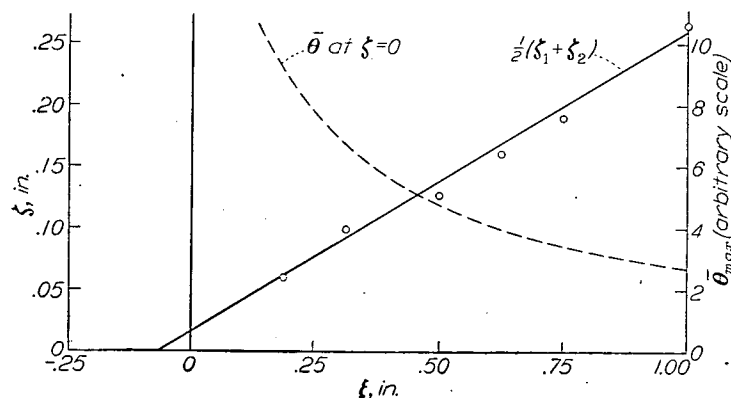


FIGURE 21.—Spread of heat from a line source in center of unheated jet. $x/d=20$. $\xi_1=\xi_2$ at which $\bar{\theta}=\frac{1}{2}\bar{\theta}_{max}$.

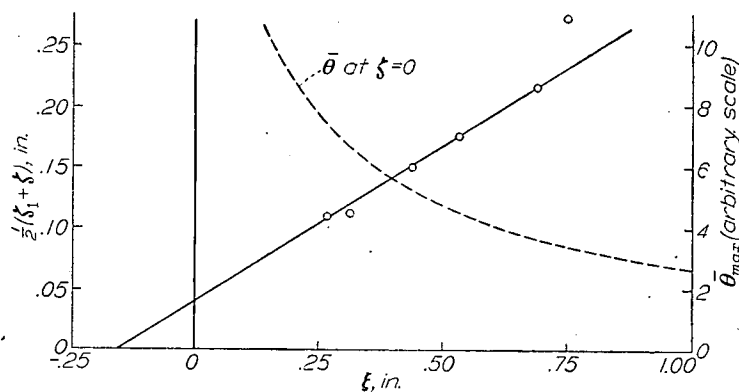


FIGURE 22.—Spread of heat from 2-inch-diameter-ring source in unheated jet. $x/d=20$. $\xi_1, \xi_2=\xi$ at which $\bar{\theta}=\frac{1}{2}\bar{\theta}_{max}$.

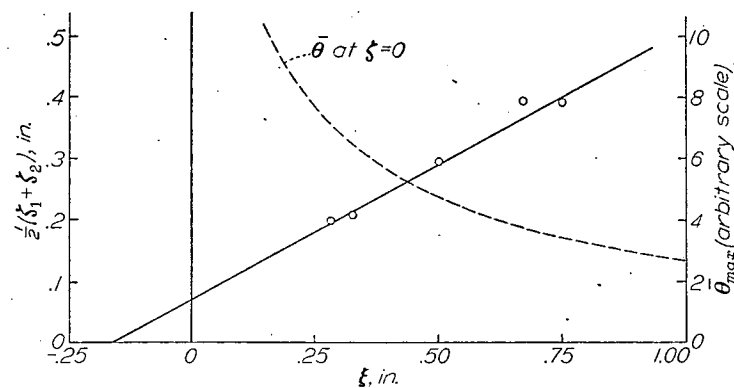


FIGURE 23.—Spread of heat from 4-inch-diameter-ring source in unheated jet. $x/d=20$. $\xi_1, \xi_2=\xi$ at which $\bar{\theta}=\frac{1}{2}\bar{\theta}_{max}$.

servation of heat, it follows immediately from similarity and linear spread that the maximum temperature at a cross section in the wake $\bar{\theta}_{max}$ must decrease hyperbolically with increasing downstream distance. Let $\bar{\theta}/\bar{\theta}_{max}=\Phi(\eta)$, where $\eta=\xi/\delta$ and δ is some characteristic width of the wake, for example, the ξ at which $\Phi=\frac{1}{2}$; conservation of heat gives

$$\int_0^{\infty} \bar{\theta} d\xi = \text{Constant} \quad (6)$$

where the mean-velocity changes are neglected. Then,

$$\bar{\theta}_{max}\delta = \text{Constant}/I_1 \quad (7)$$

where

$$I_1 = \int_0^\infty \Phi(\eta) d\eta$$

The same is true of the annular wake in the range of ξ so small that $\delta \ll r$.

Single traverses were also made behind a straight-line heat source for two other cases:

(a) With the line source set perpendicular to r at a radius of 1 inch, a temperature traverse was made in the r -direction, at $\xi = \frac{1}{2}$ inch (fig. 24)

(b) With the line source set on a diametral line, a temperature traverse was made perpendicular to r at a radius of 1 inch; $\xi = \frac{1}{2}$ inch (fig. 25)

TEMPERATURE FLUCTUATIONS BEHIND LOCAL HEAT SOURCE

As can be anticipated, the temperature fluctuations close behind a local heat source are quite different in nature from the velocity fluctuations at the same point or from the temperature and velocity fluctuations in a turbulent flow with over-all heat transfer. Since a suitable source produces no additional turbulence,³ the velocity fluctuations should be

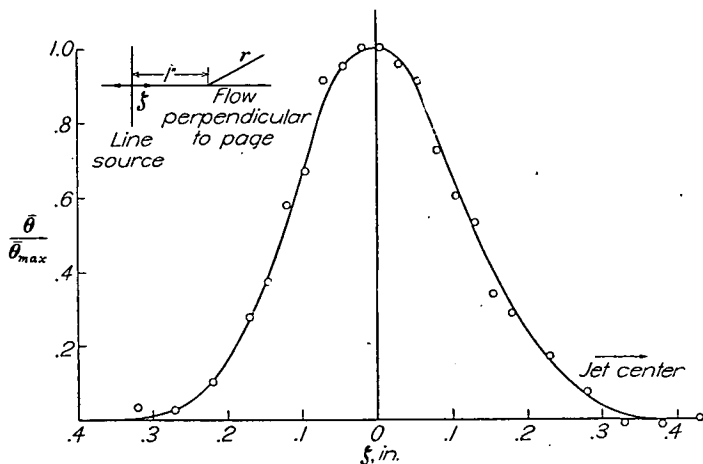


FIGURE 24.—Temperature behind line source of heat in unheated jet. $x/d=20$; $\xi=1/2$ inch. Line source set perpendicular to r at a radius of 1 inch. Traverse made in r -direction.

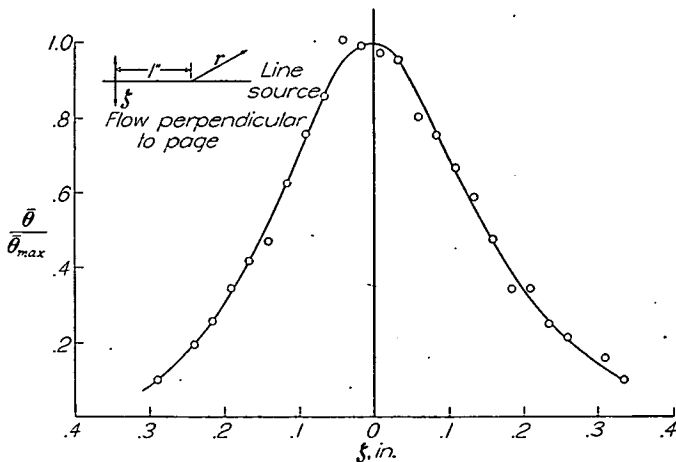


FIGURE 25.—Temperature behind line source of heat in unheated jet. $x/d=20$; $\xi=1/2$ inch. Line source set on a diametral line. Traverse made perpendicular to r at a radius of 1 inch.

³ An ideal source would also produce no average momentum defect. However, as mentioned previously, the momentum wakes of the local sources were relatively so small as to be completely undetectable as close as $\frac{1}{4}$ inch downstream.

the same as in undisturbed flow. On the other hand, the "turbulent" thermal wake close to the source must be simply a very narrow laminar thermal wake which is fluctuating in direction as $v(t)$ fluctuates. Since all of the fluid outside of this unsteady laminar wake is of constant temperature and the temperature fluctuations can only be positive, the general character of the oscillogram of $\vartheta(t)$ in figure 26 is understandable. These records were taken about $\frac{3}{8}$ inch downstream from the straight-line heat source, and about $\frac{1}{16}$ inch off the wake axis. All of the oscillograms were made with *insufficient compensation* for the hot-wire thermal lag, in order to suppress the (high frequency) noise and thus permit the basic form of $\vartheta(t)$ to stand out. From these two oscillograms of $u(t)$ and check measurements of the turbulence levels for the two cases, it appears that the source wire has made no appreciable change in the turbulence.

Measurements of the intensity of the temperature fluctuations across a section at $\xi=0.4$ inch are given in figure 27.

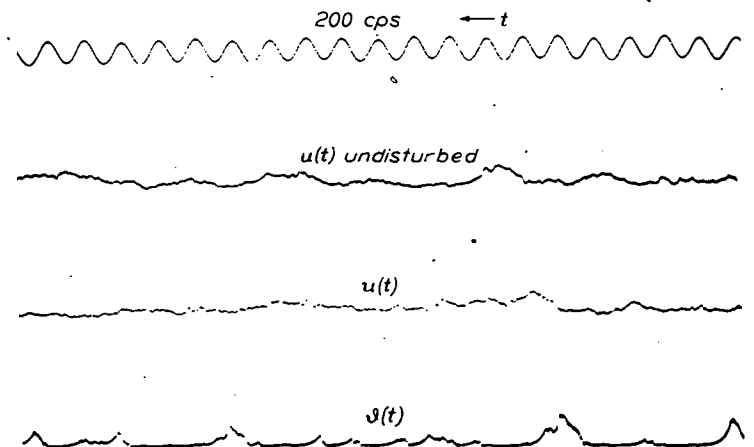


FIGURE 26.—Oscillograms of velocity and temperature fluctuations.

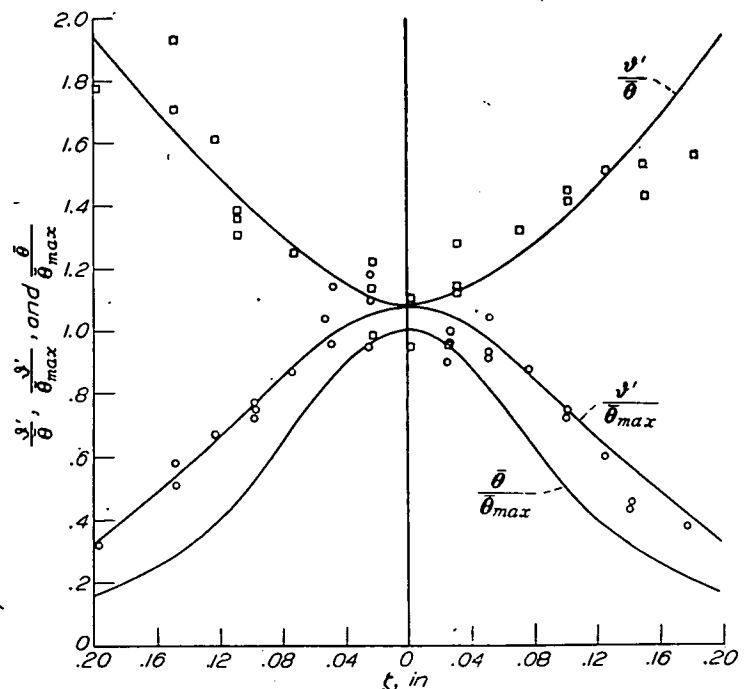


FIGURE 27.—Temperature fluctuations behind line source of heat in center of unheated jet. $x/d=20$; $\xi=0.4$ inch.

In the same vicinity the values of $u'/\bar{U}$ and $v'/\bar{U}$ are on the order of 20 percent. The extremely high values of $\vartheta'/\bar{\theta}$ are not surprising since the mean temperature difference is due only to the presence of the fluctuation. Since the simple-resistance-thermometer theory is based upon the assumption of small fluctuations compared with absolute temperature (reference 20), it is expected that these measurements are about as accurate as the measurements of much lower $\vartheta'/\bar{\theta}$ in the hot jet (reference 10).

ANALYSIS OF RESULTS

SHEAR-CORRELATION SPECTRUM

A very convenient check upon the shear-spectrum measurements can be gotten by the simple expedient of computing the total (or "net") turbulent shear correlation coefficient (which was also directly measured) from this spectrum and the turbulent-energy spectrum. Again, an elementary Fourier series treatment serves to justify (not prove) the intuitive idea that the total correlation coefficient R_{uv} is simply a weighted average of the ${}_n R_{uv}$ (i. e., the cosines of the phase angles), weighted simply by the product of the square roots of the energy spectra of u and v . The calculation is given in appendix B, and yields the relation

$$R_{uv} = \sum_{n=1}^{\infty} ({}_n F_u {}_n F_v)^{1/2} {}_n R_{uv} \quad (8)$$

Unfortunately, 25 cycles per second is the lower limit of the measured frequency range, so that some extrapolation must be made to lower frequencies which contain much of the turbulent energy. Since no theoretical basis yet exists to guide this extrapolation (like the Von Kármán formula in the case of energy spectrum), guesses had to be made as to the maximum and minimum of reasonable-looking extrapolation curves. These are plotted in figure 28, along with the energy spectrum of the u -fluctuations. Since no spectrum of the v -fluctuations was measured, the expression actually used for computing R_{uv} is

$$R_{uv} = \int_0^{\infty} \bar{F}_1(k_1) {}_n R_{uv} dk_1 \quad (9)$$

The best of the three extrapolations tried (extrapolation ③) gives $R_{uv}=0.46$, which is satisfactorily close to the directly

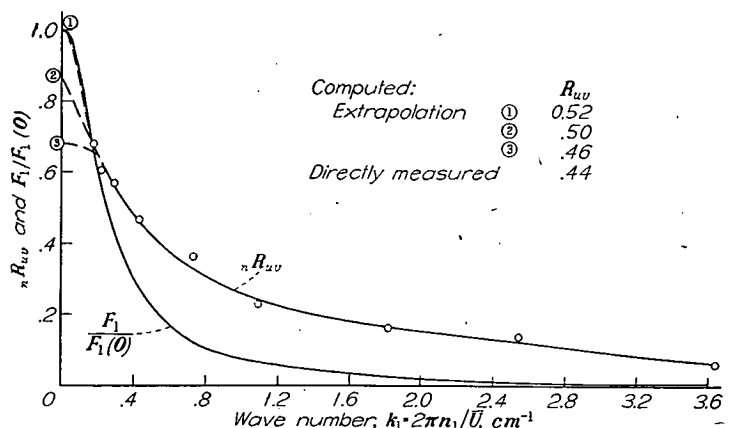


FIGURE 28.—Shear spectrum and power spectrum in 1-inch unheated jet at $x/d=20$ in region of maximum shear.

measured value of 0.44, especially since there is no reason to suppose that the spectrum of the v -fluctuations is identical with the spectrum of the u -fluctuations, except in the range of local isotropy.

VELOCITY AND TEMPERATURE SPECTRA

The one-dimensional spectra of velocity and temperature fluctuations, as plotted in figures 11 and 12, respectively, are area-normalized; that is, they are defined such that

$$\int_0^{\infty} \bar{F}_1(k_1) dk_1 = 1.0 \quad (10)$$

$$\int_0^{\infty} \bar{G}_1(k_1) dk_1 = 1.0 \quad (11)$$

However, the original measurements were made on an *absolute-value* basis, so that the total fluctuation levels $u'/\bar{U}$ and $\vartheta'/\bar{\theta}$ could be used as checks on the spectra. The spectra as measured were $F_1^*(n_1)$ and $G_1^*(n_1)$, defined such that, ideally,

$$\int_0^{\infty} F_1^*(n_1) dn_1 = \overline{u^2} \quad (12)$$

$$\int_0^{\infty} G_1^*(n_1) dn_1 = \overline{\vartheta^2} \quad (13)$$

Integration of F_1^* as indicated in equation (12) yielded the following:

On the axis ($r=0$),

$$\frac{1}{\bar{U}} \left(\int_0^{\infty} F_1^* dn_1 \right)^{1/2} = 0.28$$

($u'/\bar{U}=0.22$, directly measured)

In the maximum-shear region ($r=4.0$ cm),

$$\frac{1}{\bar{U}} \left(\int_0^{\infty} F_1^* dn_1 \right)^{1/2} = 0.52$$

($u'/\bar{U}=0.40$, directly measured)

Similar integration of the measured temperature spectrum in the heated jet yielded the following:

On the axis ($r=0$),

$$\frac{1}{\bar{\theta}} \left(\int_0^{\infty} G_1^* dn_1 \right)^{1/2} = 0.21$$

($\vartheta'/\bar{\theta}=0.18$, directly measured)

In the maximum-shear region ($r=4.8$ cm),

$$\frac{1}{\bar{\theta}} \left(\int_0^{\infty} G_1^* dn_1 \right)^{1/2} = 0.41$$

($\vartheta'/\bar{\theta}=0.36$, directly measured)

On the jet axis, where conventional small-perturbation hot-wire theory may still be moderately accurate, the agreement is satisfactory.

It should be noted in passing that these directly measured values of $\vartheta'/\bar{\theta}$ are appreciably higher than those reported in

reference 10. No explanation for this difference is apparent.

The longitudinal scale of u -fluctuations is obtainable approximately from the power spectrum:

$$L_x = \frac{\pi}{2} F_1(0) \quad (14)$$

which follows from the fundamental Fourier transformation,

$$F_1(k_1) = \frac{2}{\pi} \int_0^\infty R_x \cos(k_1 x) dx \quad (15)$$

in the limit $k_1 \rightarrow 0$.

An analogous treatment of a temperature-fluctuation field leads to an identical expression for the longitudinal scale of ϑ -fluctuations:

$$\Lambda_x = \frac{\pi}{2} G_1(0) \quad (16)$$

Within the accuracy of measurement, the longitudinal scale of $u(t)$ was found to be the same on the jet axis and in the maximum-shear region:

$$L_x = 3.6 \text{ centimeters}$$

The same was true of the longitudinal temperature scales in the hot jet:

$$\Lambda_x = 2.4 \text{ centimeters}$$

In all these computations the measured spectra were extrapolated to zero wave number with a parabola, as first suggested by Dryden (reference 22). Approximately the same numerical values are obtained with the Von Kármán formula illustrated in the plotted curves.

In order to compare L_x in the unheated jet with Λ_x in the heated jet, L_x may be multiplied by 1.15, the jet-width ratio at $x/d=20$ for these two initial temperatures (reference 10). The "corrected" L_x is then 4.15 centimeters.

It must be recalled that the Fourier transformation relation between time spectrum and space correlation would be exactly true only if the turbulent fluctuations at a point were due to pure rectilinear translation (by $\bar{U}$) of a fixed fluctuation pattern. For the free jet flow, the extremely high turbulence levels make such a transformation very uncertain. Therefore, equations (14) and (16) (and the resulting scales) can only be considered as crude approximations.

The longitudinal microscale,

$$\lambda_x = \left[-\frac{2}{R_x''(0)} \right]^{1/2} \quad (17)$$

may also be computed approximately from the one-dimensional power spectrum, by

$$\frac{1}{\lambda_x^2} = \frac{1}{2} \int_0^\infty k_1^2 F_1(k_1) dk_1 \quad (18)$$

(Primes signify differentiation when applied to correlations and spectra.)

Again, an analogous approach to temperature fluctuations gives

$$\frac{1}{\Lambda_x^2} = \frac{1}{2} \int_0^\infty k_1^2 G_1(k_1) dk_1 \quad (19)$$

Unfortunately, the high-frequency ranges of the G_1 's are too uncertain to permit reasonable extrapolation and the use of equation (19), although it can be seen from figure 13 that $\lambda_x > l_x$. However, equation (18) has been used to compute the longitudinal velocity microscale. Figure 29 is a plot of the integrand of equation (18). The integration and appropriate computation give

$$\lambda_x = 0.44 \text{ centimeter}$$

and the same value for both radial positions.

Since the turbulence on the axis of such a jet seems to be rather isotropic (the experimental evidence is that $\overline{uv}=0$ and $u' \approx v'$), the lateral microscale

$$\lambda = \left[-\frac{2}{R_y''(0)} \right]^{1/2} \quad (20)$$

is of the order of $\lambda_x/\sqrt{2}$; that is,

$$\lambda \approx 0.31 \text{ centimeter}$$

With the assumption of isotropic turbulence on the jet axis, it is possible to compute the three-dimensional power spectrum $F(k)$ from the one-dimensional spectrum $F_1(k_1)$. Heisenberg (reference 23) has given the inverse transformation

$$F_1(k_1) = \frac{1}{4} \int_{k_1}^\infty \frac{F(k)}{k^2} (k^2 - k_1^2) dk \quad (21)$$

and the desired $F(F_1)$ is readily found to be

$$F(k_1) = 2k_1 [k_1 F_1''(k_1) - F_1'(k_1)] \quad (22)$$

A three-dimensional spectrum computed in this way is given in figure 30.

The corresponding spectral transformations for the three-dimensional isotropic fluctuation field of a scalar quantity

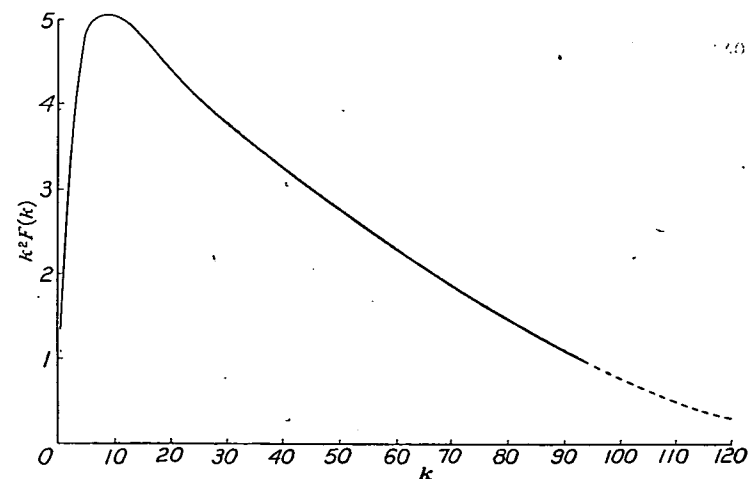
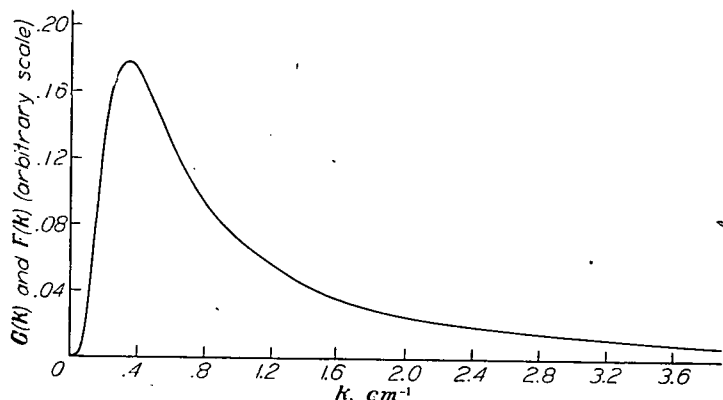


FIGURE 29.—Viscous dissipation and microscale λ on axis of 1-inch jet. $x/d=20$;

$$\frac{1}{\lambda^2} = \frac{1}{2} \int_0^\infty k_1^2 F_1(k_1) dk_1; \lambda = 0.31 \text{ centimeter.}$$

FIGURE 30.—General nature of three-dimensional spectra on jet axis. $x/d=20$

(temperature, for example) are simply (reference 24)

$$G_1(k_1) = \frac{1}{2} \int_{k_1}^{\infty} \frac{G(k)}{k} dk \quad (23)$$

and

$$G(k_1) = -2k_1 G_1'(k_1) \quad (24)$$

With the assumption of isotropic temperature fluctuations on the jet axis, equation (24) has been used to compute $G(k)$.

The three-dimensional velocity and temperature spectra on the jet axis are very nearly the same—provided that the isotropy assumption is reasonably good. The curve in figure 30, which shows the general nature of both F and G , is simply the transformation of the Von Kármán approximations to F_1 and G_1 .

It must be recalled, however, that F_1 is in the unheated jet, while G_1 is in the heated jet. Although the results of references 9 and 10 indicate no essential change in the detailed dynamics of jet turbulence as a result of moderate increase in jet temperature, there is an appreciable increase in jet width at a given x/d . As mentioned earlier, the width ratio between $\bar{\theta}_0=175^\circ$ and $\bar{\theta}_0=0^\circ$ is about 1.15 at $x/d=20$.

TRANSVERSE CORRELATION FUNCTIONS

The integral area under the velocity correlation function (fig. 15) may be considered to give a sort of lateral scale of turbulence in the jet, although the result is not associated with any particular region in the jet. The conventional expression

$$L_v = \int_0^{\infty} R_v(r) dr \quad (25)$$

gives a scale, $L_v=0.67$ centimeter.

If a lateral temperature scale is defined in similar fashion,

$$\Lambda_v = \int_0^{\infty} S_v(r) dr \quad (26)$$

then for the heated jet at $x/d=20$ it turns out that $\Lambda_v=0.77$ centimeter from the function as given in figure 16.

Appropriate comparison of these transverse scales may be had if L_v is multiplied by the jet-width ratio: $1.15L_v=0.77$ centimeter, the same value as Λ_v . However, the two correlation functions that yield these net areas are still quite

different in shape. The contrast is shown in figure 31. Clearly, even though the net areas are identical, there is nonzero temperature correlation over appreciably greater distances.

A rough approximation to the microscale of turbulence can be gotten by guessing at the osculating parabola for $\Delta r=0$. In this particular case "guessing" is more appropriate than "fitting," since the job is entirely extrapolatory in nature. Figure 32 (a) shows the vertex region of R_v with the parabola that corresponds to

$$\lambda = \left[-\frac{2}{R_v''(0)} \right]^{1/2} = 0.28 \text{ centimeter} \quad (27)$$

This value is in surprisingly good agreement with the 0.31 centimeter obtained from the power spectrum on the axis. In fact, the agreement must be regarded as fortuitous, since the difference is appreciably less than the experimental uncertainty.

If the temperature-fluctuation field is again considered analogously, the transverse microscale of temperature fluctuations (fig. 32 (b)) is

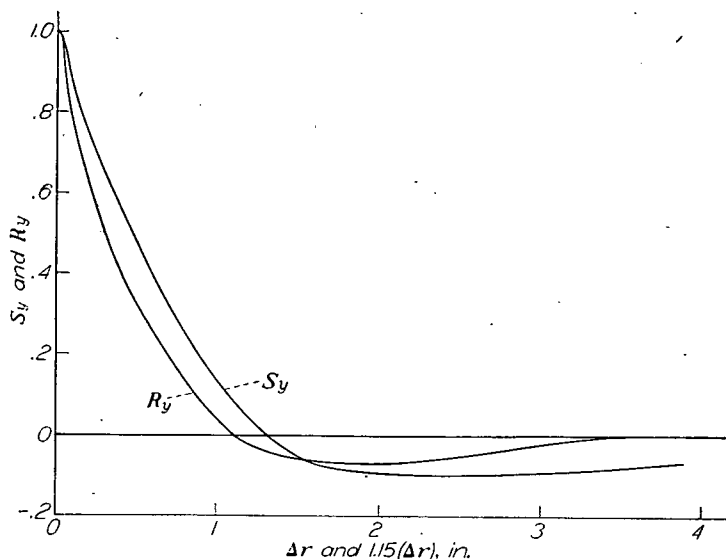
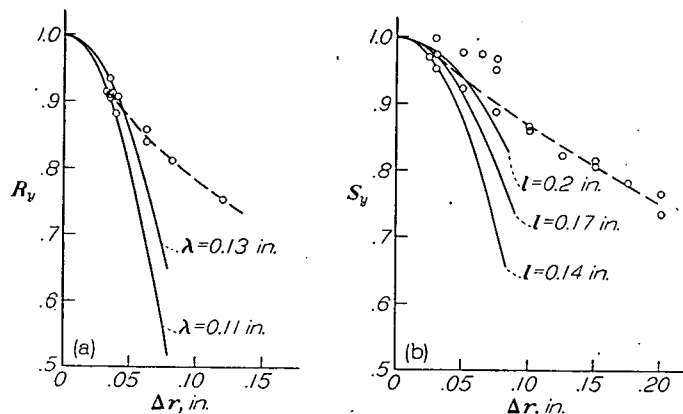


FIGURE 31.—Comparison of transverse correlation functions. 1-inch heated jet.

(a) Lateral microscale of u -fluctuations.(b) Lateral microscale of θ -fluctuations.FIGURE 32.—Estimates of microscales. 1-inch jet. $x/d=20$.

$$l = \left[-\frac{2}{S_v''(0)} \right]^{1/2} = 0.43 \text{ centimeter} \quad (28)$$

For comparison, $1.15\lambda = 0.32$.

MEAN THERMAL WAKES BEHIND LOCAL HEAT SOURCES

The rate of spread of the thermal wake close behind a local source of heat was first used by Schubauer (reference 13) as a means of measuring the intensity of lateral velocity fluctuations $v'/\bar{U}$. A detailed discussion of this technique has been given by Taylor (reference 14) and need not be repeated here. The results of such a computation, compared with direct $\times$ -meter measurements of $v'/\bar{U}$, are as follows:

r (in.)	$\left(\frac{v'}{\bar{U}}\right)_{\text{wake}}$	$\left(\frac{v'}{\bar{U}}\right)_{\times\text{-meter}}$
0	0.18	0.185
1	0.22	0.20
2	0.40	0.30

The $\times$ -meter measurements were corrected for the effects of both u' and $\bar{u}v$ upon the slightly unsymmetrical meter.

It is possible to get some additional information about the fluctuation field by computation from the turbulent-heat-transfer equation. In particular, an estimate of the distribution $\bar{\theta}v$ across a section of the thermal wake behind the line source may be made as follows.

The steady turbulent-heat-transfer equation for low velocity (negligible viscous dissipation to heat), negligible molecular heat conduction, and constant density is, in Cartesian tensor notation,

$$\bar{U}_i \frac{\partial \bar{\theta}}{\partial x_i} = -\frac{\partial}{\partial x_k} (\bar{\theta} u_k) \quad (29)$$

For the region in the immediate vicinity of the jet axis, assume that conditions approximate those in a homogeneous field of turbulence; that is, $\bar{V} = \bar{W} = 0$ and $\bar{U} = \text{Constant} = \bar{U}_{\max}$.

Then equation (29) becomes simply

$$\bar{U}_{\max} \frac{\partial \bar{\theta}}{\partial \xi} \approx -\frac{\partial}{\partial \xi} (\bar{\theta} u) - \frac{\partial}{\partial \xi} (\bar{\theta} v) \quad (30)$$

The assumption of small turbulence level implies $\bar{\theta} u \ll \bar{\theta} \bar{U}_{\max}$ and hence

$$\bar{U}_{\max} \frac{\partial \bar{\theta}}{\partial \xi} \approx -\frac{\partial}{\partial \xi} (\bar{\theta} v) \quad (31)$$

This would be a good approximation in turbulence far behind a grid placed in a uniform stream, but is certainly rather crude here.

The final assumption, that of similarity in the thermal wake, is well-supported by the experimental results. Then let

$$\frac{\bar{\theta}}{\bar{\theta}_{\max}} = f\left(\frac{\xi}{\Delta}\right) \equiv f(\eta) \quad (32)$$

where Δ is the standard deviation of the mean-temperature distribution, and, according to the theory of diffusion by continuous movements, is therefore proportional to the standard deviation of the probability density of $v(t)$ as well. Specifically, for a small ξ in a homogeneous turbulent flow,

$$\Delta = \frac{v'}{\bar{U}} \xi \quad (33)$$

Similarity also implies that

$$\bar{\theta}v = \bar{\theta}_{\max} \bar{U}_{\max} \omega(\eta) \quad (\text{say}) \quad (34)$$

Equation (7) may be written

$$\bar{\theta}_{\max} \Delta = \text{Constant} \quad (7a)$$

Then, with equations (32), (34), and (7a), equation (31) may be transformed to

$$\frac{d\omega}{d\eta} = \frac{d\Delta}{d\xi} \frac{d}{d\eta} (\eta f) \quad (35)$$

or, with equation (33),

$$\frac{d\omega}{d\eta} = \frac{v'}{\bar{U}_{\max}} \frac{d}{d\eta} (\eta f) \quad (36)$$

Since $v'/\bar{U}_{\max}$ is constant in this approximation,

$$\omega(\eta) = \frac{v'}{\bar{U}_{\max}} \eta f(\eta) + \text{Constant}$$

and the boundary condition, $\omega = 0$ at $\eta = 0$, gives finally

$$\frac{\bar{\theta}v}{\bar{\theta}_{\max} \bar{U}_{\max}} = \frac{v'}{\bar{U}_{\max}} \frac{\xi}{\Delta} \frac{\bar{\theta}}{\bar{\theta}_{\max}} \quad (37)$$

which is conveniently written in the form

$$\frac{\bar{\theta}v}{\bar{\theta}_{\max} \bar{U}_{\max}} = \frac{\theta}{\bar{\theta}_{\max}} \frac{\xi}{\xi} \quad (38)$$

In figure 33 this function is plotted against ξ/ξ for the traverse $\frac{1}{2}$ inch downstream from the straight-wire heat source across the jet center.

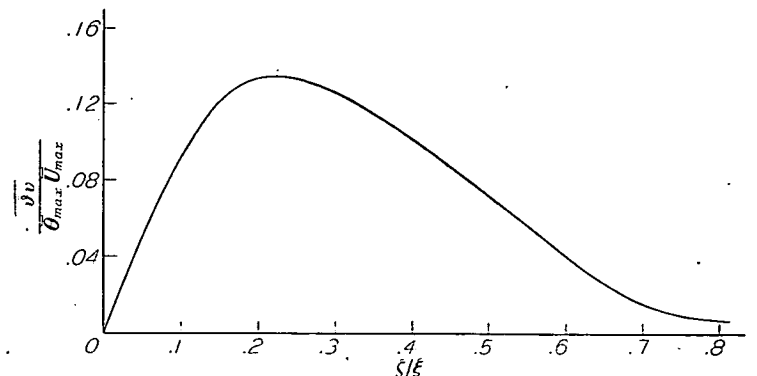


FIGURE 33.—Thermal wake behind straight-line heat source. Temperature-velocity correlation computed from $\bar{\theta}/\bar{\theta}_{\max}$. $\xi = 0.5$ inch.

DISCUSSION

LOCAL ISOTROPY

The monotonic decrease to zero in shear correlation coefficient with increasing frequency seems to be decisive evidence for the existence of local isotropy at sufficiently high Reynolds numbers. It is interesting to note that the spectral region of negligible shear ($n_1 > 1000$ cps in the present particular determination, for example) contains only about 1.5 percent of the turbulent kinetic energy in $\bar{u}^2$. Of course, this is by no means an indication of the importance of the existence of local isotropy in a turbulent shear flow. A more pertinent comparison would be with figure 29, which shows in effect dissipation as a function of frequency. From this it appears that about 90 percent of the dissipation of turbulent kinetic energy to heat takes place in essentially isotropic turbulence. This permits the use of the Taylor expression for dissipation in isotropic turbulence (reference 25). Of course it also implies that the isotropic relation between longitudinal and lateral microscales, $\lambda_x = \sqrt{2} \lambda$, will be fairly accurate even in the region of high turbulent shear. Furthermore, it implies a universal dimensionless spectral function for all turbulent flows, in the high-frequency region. For turbulence at this Reynolds number, it appears that a universal part of the spectrum exists only for $k_1 > 7.9$, which is well beyond the point of slope $-\frac{5}{3}$. In fact, the " $-\frac{5}{3}$ point" in this spectrum is just at $k_1 = 1.0$.

At this point a few remarks on the appropriate type of measurement for verification of local isotropy may be in order. In particular, a careful distinction must be made between the shear spectrum ${}_n R_{uv}$ as presented in this report and the power spectrum of the randomly fluctuating quantity uv which might be measured with a multiplying circuit followed by a frequency analyzer.

Local isotropy specifies that restriction to a sufficiently small domain in a turbulent shear flow shows up isotropy in the various statistical properties that are studied within that domain. It implies that the frequency (or wave-number) vector must be large in magnitude. Clearly then a good indication of isotropy is zero correlation between orthogonal velocity-fluctuation components; this means that the highest-frequency parts of u , v , and w are uncorrelated with each other. Hence it is clear that if $\overline{u_n v_n}$ decreases to zero with increasing frequency faster than the product $u_n' v_n'$ decreases to zero, local isotropy exists. In terms of coefficient, this merely requires that ${}_n R_{uv}$ decrease to zero eventually.

Now consider the fluctuating quantity uv . In a turbulent shear flow $\overline{uv} \neq 0$, so that uv consists of a direct-current component with superimposed random fluctuations. Since the conventional electronic techniques eliminate the direct current, the quantity to be analyzed would be $uv - \overline{uv}$ as a function of time. If local isotropy were present, the lower frequencies of u and v would be rectified in the multiplying process, and therefore the oscillogram and power spectrum of $uv - \overline{uv}$ would have relatively great emphasis on the high-frequencies. In other words, if the naively measured power spectrum of uv were used as an indication of local isotropy, it would show a trend opposite to that of $\overline{u_n v_n}$; that is, it would

decrease more slowly than the product $u_n' v_n'$. In general, the measurement of ${}_n R_{uv}$ seems like a much more specific and direct approach than the measurement of the power spectrum of uv . Presumably, a (somewhat more complicated) Fourier series discussion like that in appendix B could also be carried out for the power spectrum of uv .

VELOCITY AND TEMPERATURE SPECTRA

The apparent identity of the velocity power spectra $F_1(k_1)$ on the jet axis and in the region of maximum shear is only approximate and has been determined only down to $k_1 = 0.1$. There still exists the possibility of measurable divergence in the lowest wave-number range. The good degree of agreement indicates that, in diffusing from the region of maximum production (near the maximum-shear region) to the region of maximum dissipation (on the jet axis), the turbulent kinetic energy has not done any gross migrating in the wave-number space.

On the other hand, the apparent decided difference between temperature power spectra measured on the axis and in the maximum-heat-transfer region seems to indicate such a migration. However, the considerable scatter at the highest measured frequencies renders definite conclusions impossible.

Somewhat more specific conclusions can be drawn from the comparison between one-dimensional velocity and temperature spectra. On the jet axis, for example, in spite of distinct differences between these two spectra, it turns out that within the experimental scatter (which is considerable) the three-dimensional power spectra may be much more nearly identical. The fact that they did in fact come out to be identical over a wide range of wave number when computed from the empirically fitted Von Kármán formula must certainly be regarded as pure chance. This is true not only because of the experimental uncertainty, but also because these spectra were measured in two similar but different flows, whose characteristic lengths probably differed by 15 percent.

KINEMATIC AND THERMAL SCALES

From the extrapolated zero-wave-number intercepts of the one-dimensional spectra, the following longitudinal scales were obtained at $x/d = 20$:

$$L_x = 4.15 \text{ centimeters}$$

$$\Lambda_x = 2.4 \text{ centimeters}$$

This L_x is 15 percent greater than the unheated-jet value, to allow for the greater width of the heated jet. Thus, $L_x/\Lambda_x = 1.7$. In a homogeneous, isotropic field of velocity and temperature fluctuations, it turns out (reference 24) that, if the three-dimensional power spectra of velocity and temperature are proportional, $L_x/\Lambda_x = 1.50$. It may also be noted that, if the measured ratio were in an isotropic field, $L_y = \frac{1}{2} L_x$ and $\Lambda_y = \Lambda_x$, so that $L_y/\Lambda_y = 0.85$. The ideal value would be 0.75. Actually, the integrals of the transverse correlation functions R_y and S_y are considerably less than the scales that would be expected, according to these relations, in a homogeneous isotropic turbulence.

On the other hand, the relative values of longitudinal and lateral kinematic microscales follow the isotropic relation,

$\lambda_x = \sqrt{2}\lambda$, at least within the experimental uncertainty. This is on the order of ± 25 percent in the case of the parabola "fitted" at the vertex of R_y .

Unfortunately, the temperature spectrum on the jet axis is not extended sufficiently far to permit computation of longitudinal microscale l_x there. The spectra in figure 13 show only that l_x is considerably less than λ_x ; that is, l_x is considerably less than 0.44 centimeter. It may be remarked in passing that isotropy for a scalar quantity means equality of longitudinal and lateral correlation functions. The lateral microscale, $l = 0.43$ centimeter, obtained by "fitting" a parabola at the vertex of S_y seems of reasonable magnitude relative to $1.15\lambda = 0.32$ centimeter.

TRANSVERSE CORRELATION FUNCTIONS

Of course, the reason L_y and Λ_y as determined by integration of functions R_y and S_y are not related isotropically to L_x and Λ_x is that over most of the range of Δr the probes are in decidedly nonisotropic turbulence. Thus, there is no reason to expect $L_y = \frac{1}{2}L_x$ or $\Lambda_y = \Lambda_x$, when L_x and Λ_x are computed from the spectra.

An examination of the behavior of these two symmetrically measured correlation functions shows that there is nonzero correlation over a considerable part of the jet, but that the relatively small scales result from the rather extensive regions of negative correlation. This behavior is emphasized by a comparison of R_y with the corresponding function in some typical isotropic turbulence downstream of 1-inch-mesh grid (reference 26). Figure 34 shows the contrast clearly.

It is conceivable that such an extended region of negative correlation is characteristic of turbulent shear flow. However, until someone establishes this in a shear flow whose transverse extent is very large compared with the maximum correlation distance, it may be safer to guess that the "excess" amount of negative correlation is simply due to a slight irregular waving of the jet as a whole. In reference 9 it was assumed that, since R_y actually goes to zero at large values of Δr , there is no over-all "whipping" of the jet. However, such a conclusion does not appear to be completely warranted.

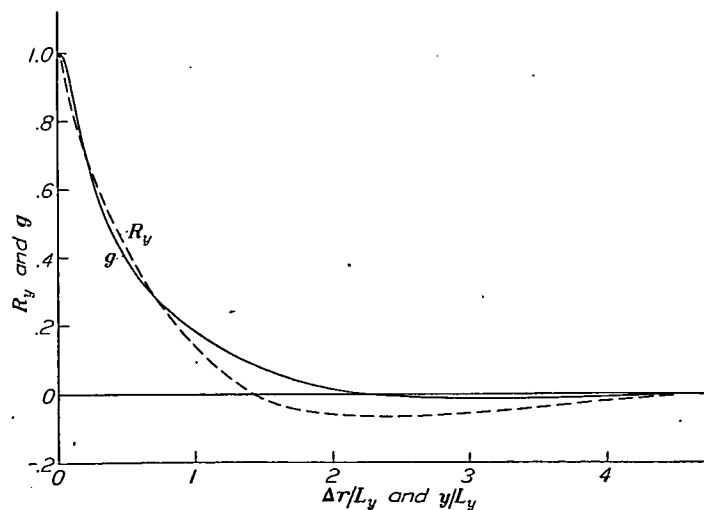


FIGURE 34.—Transverse velocity correlation functions. $R_y = \overline{u_1 u_2} / \overline{u^2}$ in round jet; $q = \overline{u_1 u_2} / \overline{u^2}$ in isotropic turbulence.

In the section entitled "Transverse Correlation Functions" under "Analysis of Results," it was found that in the heated jet $L_y \approx \Lambda_y$. On the other hand, it is well-known that the lateral rate of transfer of heat is appreciably greater than the lateral rate of transfer of momentum, as was first found by Ruden (reference 27) from mean-velocity and mean-temperature measurements. Since diffusion is essentially Lagrangian in nature, while L and Λ are Eulerian scales, the above results are not necessarily in contradiction. The appreciably greater distance over which $S_y \neq 0$ (as contrasted with R_y) may, however, be related to the fact that the mean thermal jet diameter is appreciably greater than the mean momentum jet diameter.

PROBABILITY DENSITY OF $v(t)$ AND $w(t)$

The mean-temperature distribution close behind the straight-line heat source on the jet axis is effectively symmetrical, and closely resembles a Gaussian curve in shape (fig. 20); this shows that the probability density of $v(t)$ on the axis is more or less Gaussian, as in isotropic turbulence.

The mean-temperature distributions close behind the two ring heat sources are decidedly skew. However, some of this skewness seems to be due simply to the curvature of the line sources. Therefore, the temperature distribution across the wake of a straight wire set tangent to the circle $r=1$ inch was measured. Neglecting the effects of mean-velocity gradient, this curve (fig. 24) is proportional to the probability density of the radial velocity fluctuation $v(t)$ in the shear region. It is seen to be slightly skew; the skewness factor

$$S = \frac{\overline{v^3}}{(\overline{v^2})^{3/2}} \approx -0.1$$

is computed directly from this curve. The thermal wake measurements of Skramstad and Schubauer behind a line source in a turbulent boundary layer (reported in reference 16) show a skewness of 0.38. The differences in sign and magnitude of these two skewness factors suggest lateral turbulence-level gradient as the cause. The gradients in $v'/\overline{U}$ are of opposite sign in these two flows.

Calculation from figure 25 shows that the probability density of the tangential fluctuation $w(t)$ is symmetrical. It may be noted that on the axis of such an axially symmetric flow there is no distinction between radial and tangential velocity fluctuation; hence figure 20 also applies to $w(t)$ on the axis.

TEMPERATURE FLUCTUATIONS BEHIND LOCAL HEAT SOURCE

The extremely high temperature-fluctuation levels ($\sigma'/\bar{\theta} > 1.0$) encountered in the wake of the line heat source are easily understood from a brief consideration of the nature of the temperature field. Close behind the source, there is just a single narrow laminar thermal wake which is being blown in random deviations from the ξ -direction by the turbulent fluctuations. The gross turbulent thermal wake is simply the wedge-shaped region over which this relatively narrow wake wanders. Hence the total thermal signal at any fixed point in the gross wake consists simply

of a series of pulses, where each pulse corresponds to an occasion upon which the laminar wake swept over the point. Obviously, the frequency of occurrence of pulses will decrease monotonically with increasing transverse distance from the center of the gross wake.

If this type of temperature signal is represented schematically by periodic square pulses of height h , width j , and fundamental wave length τ (fig. 35), then it can be easily deduced that the fluctuation level is

$$\frac{\vartheta'}{\bar{\vartheta}} = \sqrt{\frac{\tau}{j} - 1}$$

where it is recalled that $\vartheta = \theta - \bar{\theta}$ by definition.

Two pulse spacings of interest are

$$(1) \tau = 2j; \text{ then } \vartheta'/\bar{\vartheta} = 1.0$$

$$(2) \tau \rightarrow \infty; \text{ then } \vartheta'/\bar{\vartheta} \rightarrow \infty$$

Hence, the measured results for $\vartheta'/\bar{\vartheta}$ seem quite reasonable in both order of magnitude and in qualitative behavior across the gross thermal wake of the local heat source.

The distribution of $\bar{\vartheta}v/\bar{\theta}_{max}\bar{U}_{max}$ computed from the thermal wake, plus the availability of the measurements of $\vartheta'/\bar{\theta}_{max}$ and $v'/\bar{U}_{max}$, suggests the computation of the heat-transfer correlation coefficient $\bar{\vartheta}v/\vartheta'v'$. Unfortunately, when the results of figures 33 and 27 are used, a part of the correlation-coefficient distribution reaches impossible values (slightly above unity). It must be concluded that the absolute values of $\bar{\vartheta}v/\bar{\theta}_{max}\bar{U}_{max}$ are too inaccurate for such a computation.

SOURCES OF ERROR

Aside from the specific instances mentioned earlier in this section, the sources of experimental error are much the same as outlined on pages 27 and 28 of reference 10. Additional uncertainties arise in the spectrum measurements, especially in the higher-frequency range, because of (a) rapid changes in the calibration of the sound analyzer (band peak response against frequency, fig. 7) and (b) slight static friction of fluxmeter bearings.

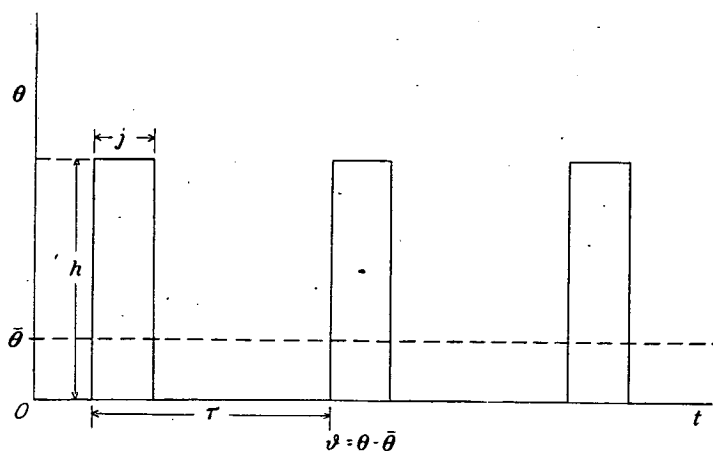


FIGURE 35.—Simulation of temperature signal close behind local heat source.

In general, it should be emphasized that measurements by conventional (small-perturbation) hot-wire anemometry in a flow of this high level of turbulence cannot be considered as accurate absolute-value measurements. Even on the jet axis, where the level is a minimum and conditions are relatively steady, there is no reason to believe that absolute values are better than within, say, ± 10 percent of the "correct" values. However, relative behaviors are undoubtedly determined, and dimensionless measures of the type of correlation coefficients are more accurate than absolute values.

None of the measurements reported here have been corrected for finite length of hot-wires.

SUMMARY OF RESULTS

From measurements in a round turbulent jet at room temperature of the shear correlation coefficient as a function of frequency, of velocity and temperature fluctuations with and without jet heating, and of the mean thermal wakes behind local heat sources, the following statements may be made:

1. The Kolmogoroff hypothesis of local isotropy is verified for the shear flow in a round, turbulent jet. This is concluded from the monotonic decrease to zero of the shear-correlation spectrum $(\overline{u_n v_n} / \overline{u_n' v_n'})$ with increasing frequency n .

2. The one-dimensional power spectra of longitudinal velocity fluctuations and of temperature fluctuations appear to be basically different.

3. The three-dimensional power spectra of velocity and temperature fluctuations on the jet axis seem to be roughly alike—if the assumption of isotropy in this region be true. It may then follow that the difference in the one-dimensional power spectra is a direct manifestation of the fact that velocity and heat are vector and scalar quantities, respectively.

4. The ratio of longitudinal to lateral scale (for both velocity and temperature fluctuations) is considerably larger than would follow from isotropy. Longitudinal scales are measured on the jet axis, while lateral scales involve a traverse of most of the fully turbulent core of the jet.

5. The ratio of longitudinal to lateral kinematic microscale on the jet axis is about equal to the isotropic value.

6. The longitudinal thermal microscale (from one-dimensional power spectra) is less than the longitudinal kinematic microscale, but the lateral microscales (from correlation measurements) have the opposite relation; that is, the thermal is greater than the kinematic.

7. The probability density of the radial fluctuation $v(t)$ on the jet axis is effectively Gaussian. The probability density in the shear region is slightly skew.

8. The temperature-fluctuation field in the wake behind a local heat source consists of a randomly waving narrow laminar thermal wake. Hence the temperature signal at a fixed point is a random-pulse type of function. Its fluctuation intensity is on the order of 100 percent on the center line, and increases toward the edges.

The JOHNS HOPKINS UNIVERSITY,
BALTIMORE, MD., August 17, 1949.

APPENDIX A

HEAT LOSS FROM A WIRE AT VARIOUS AMBIENT TEMPERATURES

In figure 3 of reference 20 a rough check was made on the temperature-variation first term in King's (reference 28) equation for the steady static heat loss from a cylinder perpendicular to a fluid stream, at low Reynolds numbers. The conventional form is

$$\frac{i^2 R}{R - R_a} = A + B\sqrt{U} \quad (\text{A1})$$

where

$$A = c_1 \frac{l^* k^*}{R_0 \alpha^*}$$

$$B = c_2 \frac{l^*}{R_0 \alpha^*} \sqrt{d^* c_p \rho k^*}$$

and

R	wire resistance
R_a	wire resistance at ambient fluid temperature
R_0	wire resistance at 0° C
l^*	wire length
d^*	wire diameter
α^*	temperature coefficient of change of resistivity of wire material
k^*	thermal conductivity of fluid at ambient temperature
c_p	specific heat of fluid at ambient temperature
ρ	density of fluid at ambient temperature
i	current
c_1, c_2	empirical constants

In reference 20 the check on A as a function of temperature was made by assuming the second term in equation (A1) to be exact in its temperature variation. Then each measured calibration point at any velocity and temperature led to a value for A .

The present check was carried out more completely; a full calibration curve was run for each ambient temperature. From this, both A and B were determined. Figure 36 gives the results compared with King's predicted variation, using physical constants from reference 29. Each point corresponds to a calibration. The vertical line through a

point obviously does not represent the over-all uncertainty; it simply shows the range of values that could be gotten by drawing *different* reasonable-looking straight lines through the same set of original calibration points. From the figure it can be seen that King's equation predicts the temperature variation of A quite well. The changes in B (the slope of the calibration line in the plot of $i^2 R / (R - R_a)$ against $\sqrt{U}$) are so small that the experimental scatter is as great as the changes predicted for these temperature differences.

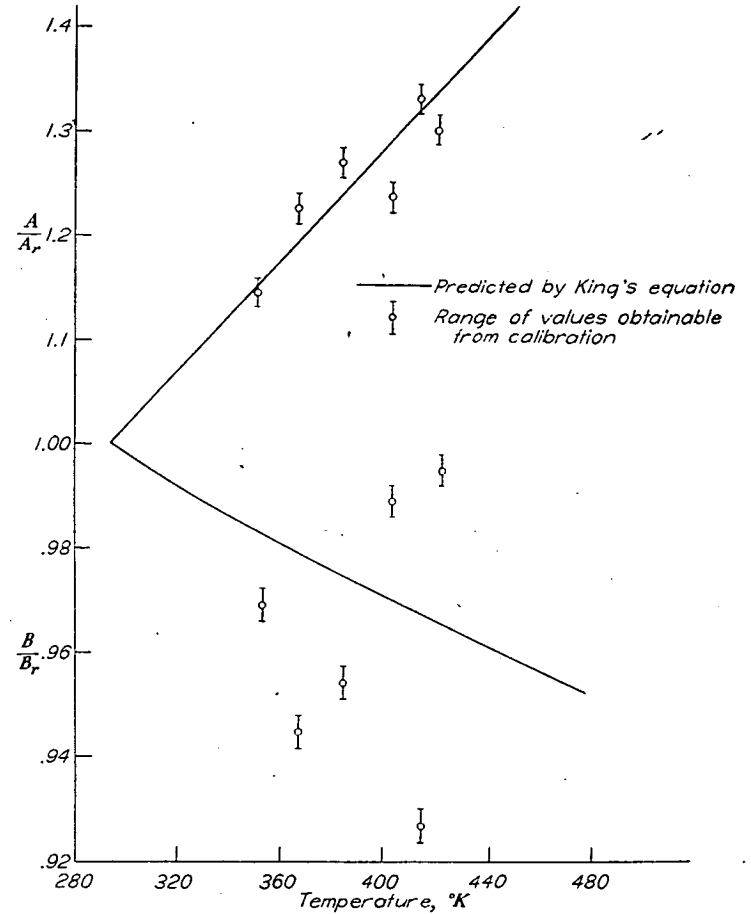


FIGURE 36.—Variation of hot-wire constants with air temperature. ()_r, room temperature,

APPENDIX B

MEASUREMENT OF SHEAR-CORRELATION SPECTRUM

The two voltage signals from an ideal symmetrical X-meter are

$$\left. \begin{aligned} e_1 &= \alpha u + \beta v \\ e_2 &= \alpha u - \beta v \end{aligned} \right\} \quad (\text{B1})$$

Suppose that the velocity fluctuations are periodic:

$$\left. \begin{aligned} u &= \sum_{n=1}^{\infty} a_n \cos(2\pi n t + \phi_n) \\ v &= \sum_{n=1}^{\infty} b_n \cos(2\pi n t + \psi_n) \end{aligned} \right\} \quad (\text{B2})$$

Of course, there would be no loss in generality if ϕ_n or ψ_n were taken as zero.

The quantity to be measured is

$${}_n R_{uv} = \frac{\overline{u_n v_n}}{u_n' v_n'} \quad (\text{B3})$$

For two simple harmonic functions the correlation coefficient is simply the cosine of the phase angle. Thus,

$${}_n R_{uv} = \cos(\phi_n - \psi_n) \quad (\text{B4})$$

Substitution of equations (B2) into equations (B1), followed

by trigonometric transformation, gives

$$e_1 = \sum_1^{\infty} [(\alpha a_n \cos \phi_n + \beta b_n \cos \psi_n) \cos 2\pi n t - (\alpha a_n \sin \phi_n + \beta b_n \sin \psi_n) \sin 2\pi n t] \quad (\text{B5a})$$

$$e_2 = \sum_1^{\infty} [(\alpha a_n \cos \phi_n - \beta b_n \cos \psi_n) \cos 2\pi n t - (\alpha a_n \sin \phi_n - \beta b_n \sin \psi_n) \sin 2\pi n t] \quad (\text{B5b})$$

When these two signals are put separately through a narrow band-pass filter that passes only the n th harmonic, the two output voltages may be represented as

$${}_n e_1 = K [(\alpha a_n \cos \phi_n + \beta b_n \cos \psi_n) \cos 2\pi n t - (\alpha a_n \sin \phi_n + \beta b_n \sin \psi_n) \sin 2\pi n t] \quad (\text{B6a})$$

$${}_n e_2 = K [(\alpha a_n \cos \phi_n - \beta b_n \cos \psi_n) \cos 2\pi n t - (\alpha a_n \sin \phi_n - \beta b_n \sin \psi_n) \sin 2\pi n t] \quad (\text{B6b})$$

where K is an attenuation factor.

For brevity, write

$${}_n e_1 = K (A_n \cos 2\pi n t - B_n \sin 2\pi n t) \quad (\text{B6c})$$

$${}_n e_2 = K (C_n \cos 2\pi n t - D_n \sin 2\pi n t) \quad (\text{B6d})$$

These filtered signals go next into the vacuum-thermocouple unit, which puts out the mean-square values,

$$\begin{aligned} \frac{1}{K'} \overline{{}_n e_1^2} &= A_n^2 \overline{\cos^2(2\pi n t)} - 2 A_n B_n \overline{\cos(2\pi n t) \sin(2\pi n t)} + B_n^2 \overline{\sin^2(2\pi n t)} \\ \frac{1}{K'} \overline{{}_n e_2^2} &= C_n^2 \overline{\cos^2(2\pi n t)} - 2 C_n D_n \overline{\cos(2\pi n t) \sin(2\pi n t)} + D_n^2 \overline{\sin^2(2\pi n t)} \end{aligned}$$

where K' is an over-all attenuation factor.

But, $\overline{\cos^2} \approx \overline{\sin^2} \approx \frac{1}{2}$, and $\overline{\cos \sin} \approx 0$, over a large number of wavelengths. Thus,

$$\left. \begin{aligned} \overline{{}_n e_1^2} &\approx \frac{K'}{2} (A_n^2 + B_n^2) \\ \overline{{}_n e_2^2} &\approx \frac{K'}{2} (C_n^2 + D_n^2) \end{aligned} \right\} \quad (\text{B7})$$

Then, within the approximation,

$$\overline{{}_n e_1^2} - \overline{{}_n e_2^2} = \frac{K'}{2} (A_n^2 - C_n^2 + B_n^2 - D_n^2) \quad (\text{B8})$$

and when the expressions for A , B , C , and D are substituted, it turns out that

$$\overline{{}_n e_1^2} - \overline{{}_n e_2^2} = 2K' \alpha \beta a_n b_n \cos(\phi_n - \psi_n) \quad (\text{B9})$$

The necessity of determining α and β is ordinarily avoided with a symmetrical meter, if only the correlation coefficient is required.

The sum and difference of the two wire voltages are

$$\left. \begin{aligned} e_1 + e_2 &= 2\alpha u = 2\alpha \sum_1^{\infty} a_n \cos(2\pi n t + \phi_n) \\ e_1 - e_2 &= 2\beta v = 2\beta \sum_1^{\infty} b_n \cos(2\pi n t + \psi_n) \end{aligned} \right\} \quad (\text{B10})$$

Filtering gives

$${}_n(e_1 + e_2) = 2K\alpha a_n \cos(2\pi n t + \phi_n)$$

$${}_n(e_1 - e_2) = 2K\beta b_n \cos(2\pi n t + \psi_n)$$

Passage through the vacuum thermocouple gives

$$\left. \begin{aligned} \overline{{}_n(e_1 + e_2)^2} &= 2K' \alpha^2 a_n^2 \\ \overline{{}_n(e_1 - e_2)^2} &= 2K' \beta^2 b_n^2 \end{aligned} \right\} \quad (\text{B11})$$

Combination of equations (B11), (B9), and (B4) gives the final result:

$${}_n R_{uv} = \frac{\overline{{}_n e_1^2} - \overline{{}_n e_2^2}}{[\overline{{}_n(e_1 + e_2)^2} \overline{{}_n(e_1 - e_2)^2}]^{1/2}} \quad (\text{B12})$$

The computation of total-shear correlation coefficient from shear-coefficient spectrum suggests itself as a useful check possibility:

$$R_{uv} = \overline{uv} / u'v' \quad (\text{B13})$$

with the Fourier series for u and v ,

$$\left. \begin{aligned} u'^2 &\equiv \overline{u^2} = \frac{1}{2} \sum_1^{\infty} a_n^2 \\ v'^2 &\equiv \overline{v^2} = \frac{1}{2} \sum_1^{\infty} b_n^2 \end{aligned} \right\} \quad (\text{B14})$$

and the instantaneous cross product can be transformed to

$$uv = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} a_n b_m (\cos \phi_n \cos 2\pi n t - \sin \phi_n \sin 2\pi n t) \times (\cos \psi_m \cos 2\pi m t - \sin \psi_m \sin 2\pi m t)$$

The time average of this expression is

$$\overline{uv} = \frac{1}{2} \sum_{n=1}^{\infty} a_n b_n (\cos \phi_n \cos \psi_n + \sin \phi_n \sin \psi_n)$$

or

$$\overline{uv} = \frac{1}{2} \sum_1^{\infty} a_n b_n \cos(\phi_n - \psi_n) \quad (\text{B15})$$

Thus,

$$u'v'R_{uv} = \frac{1}{2} \sum_1^{\infty} a_n b_n R_{uv}$$

In terms of the Fourier coefficients,

$$R_{uv} = \frac{\sum_1^{\infty} a_n b_n R_{uv}}{\left(\sum_1^{\infty} a_n^2 \sum_1^{\infty} b_n^2 \right)^{1/2}} \quad (\text{B16})$$

But $nF_u \equiv a_n^2 / \sum_1^{\infty} a_n^2$ and $nF_v \equiv b_n^2 / \sum_1^{\infty} b_n^2$ are simply the normalized one-dimensional energy spectra of u and v , respectively. Therefore,

$$R_{uv} = \sum_{n=1}^{\infty} (nF_u nF_v^{1/2})_n R_{nn} \quad (\text{B17})$$

REFERENCES

1. Kolmogoroff, A.: The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds' Numbers. *Comp. rend., acad. sci. URSS*, vol. 30, no. 4, Feb. 10, 1941, pp. 301-305.
2. Kolmogoroff, A. N.: Dissipation of Energy in the Locally Isotropic Turbulence. *Comp. rend., acad. sci. URSS*, vol. 32, no. 1, July 10, 1941, pp. 16-18.
3. Townsend, A. A.: Measurements in the Turbulent Wake of a Cylinder. *Proc. Roy. Soc. (London)*, ser. A, vol. 190, no. 1023, Sept. 9, 1947, pp. 551-561.
4. Townsend, A. A.: Local Isotropy in the Turbulent Wake of a Cylinder. *Australian Jour. Sci. Res.*, ser. A, vol. 1, no. 2, June 1948, pp. 161-174.
5. Schubauer, G. B., and Klebanoff, P. S.: Theory and Application of Hot-Wire Instruments in the Investigation of Turbulent Boundary Layers. *NACA ACR 5K27*, 1946.
6. Schubauer, G. B., and Klebanoff, P. S.: Investigation of Separation of the Turbulent Boundary Layer. *NACA Rep. 1030*, 1951. (Formerly *NACA TN 2133*.)
7. Laufer, John: Investigation of Turbulent Flow in a Two-Dimensional Channel. *NACA Rep. 1053*, 1951. (Formerly *NACA TN 2123*.)
8. Liepmann, Hans Wolfgang, and Laufer, John: Investigations of Free Turbulent Mixing. *NACA TN 1257*, 1947.
9. Corrsin, Stanley: Investigation of Flow in an Axially Symmetrical Heated Jet of Air. *NACA ACR 3L23*, 1943.
10. Corrsin, Stanley, and Uberoi, Mahinder S.: Further Experiments on the Flow and Heat Transfer in a Heated Turbulent Air Jet. *NACA Rep. 998*, 1950. (Formerly *NACA TN 1865*.)
11. Townsend, A. A.: The Fully Developed Turbulent Wake of a Circular Cylinder. *Australian Jour. Sci. Res.*, ser. A, vol. 2, no. 4, Dec. 1949, pp. 451-468.
12. Taylor, G. I.: Diffusion by Continuous Movements. *Proc. London Math. Soc.*, ser. A, vol. 20, 1922, pp. 196-212.
13. Schubauer, G. B.: A Turbulence Indicator Utilizing the Diffusion of Heat. *NACA Rep. 524*, 1935.
14. Taylor, G. I.: Statistical Theory of Turbulence. IV—Diffusion in a Turbulent Air Stream. *Proc. Roy. Soc. (London)*, ser. A, vol. 151, no. 873, Sept. 2, 1935, pp. 465-478.
15. Skramstad, H. K., and Schubauer, G. B.: The Application of Thermal Diffusion to the Study of Turbulent Air Flow. *Phys. Rev.*, vol. 53, no. 11, June 1, 1938, p. 927.
16. Dryden, Hugh L.: Turbulence and Diffusion. *Ind. and Eng. Chem.*, vol. 31, no. 4, April 1939, pp. 416-425.
17. Wiegardt, K.: Über Ausbreitungsvorgänge in turbulenten Reibungsschichten. *Z. f. a. M. M.*, Bd. 28, Heft 11/12, Nov./Dec. 1948, pp. 346-355.
18. Kovasznay, L.: Calibration and Measurement in Turbulence Research by the Hot-Wire Method. *NACA TM 1130*, 1947.
19. Scott, H. H.: The Degenerative Sound Analyzer. *Jour. Acous. Soc. Am.*, vol. 11, no. 2, Oct. 1939, pp. 225-232.
20. Corrsin, Stanley: Extended Applications of the Hot-Wire Anemometer. *NACA TN 1864*, 1949; abstract in *Rev. Sci. Instr.*, vol. 18, no. 7, July 1947, pp. 469-471.
21. Von Kármán, Th.: Progress in the Statistical Theory of Turbulence. *Proc. Nat. Acad. Sci.*, vol. 34, no. 11, Nov. 1948, pp. 530-539.
22. Dryden, Hugh L.: A Review of the Statistical Theory of Turbulence. *Quart. Appl. Math.*, vol. I, no. 1, April 1943, pp. 7-42.
23. Heisenberg, W.: Zur statistischen Theorie der Turbulenz. *Zeitschr. Phys.*, Bd. 124, Heft 7/12, 1948, pp. 628-657.
24. Kovasznay, L. S. G., Uberoi, M. S., and Corrsin, S.: The Transformation between One- and Three-Dimensional Power Spectra for an Isotropic Scalar Fluctuation Field. *Phys. Rev.*, vol. 76, no. 8, Oct. 15, 1949, pp. 1263-1264.
25. Taylor, G. I.: Statistical Theory of Turbulence. Part I. *Proc. Roy. Soc. (London)*, ser. A, vol. 151, no. 873, Sept. 2, 1935, pp. 423-444.
26. Corrsin, S.: Decay of Turbulence behind Three Similar Grids. *A. E. Thesis*, C. I. T., 1942.
27. Ruden, P.: Turbulente Ausbreitungsvorgänge im Freistrahle. *Die Naturwissenschaften*, Jahrg. 21, Heft 21/23, May 26, 1933, pp. 375-378.
28. King, Louis Vessot: On the Convection of Heat from Small Cylinders in a Stream of Fluid: Determination of the Convection Constants of Small Platinum Wires with Applications to Hot-Wire Anemometry. *Phil. Trans. Roy. Soc. (London)*, ser. A, vol. 214, Nov. 12, 1914, pp. 373-432.
29. National Research Council (Edward W. Washburn, ed.): *International Critical Tables*. Vol. V. First ed., McGraw-Hill Book Co., Inc., 1929.

REPORT 1041

EQUATIONS AND CHARTS FOR THE RAPID ESTIMATION OF HINGE-MOMENT AND EFFECTIVENESS PARAMETERS FOR TRAILING-EDGE CONTROLS HAVING LEADING AND TRAILING EDGES SWEEPED AHEAD OF THE MACH LINES¹

By KENNETH L. GOIN

SUMMARY

Existing conical-flow solutions have been used to calculate the hinge-moment and effectiveness parameters of trailing-edge controls having leading and trailing edges swept ahead of the Mach lines and having streamwise root and tip chords. Equations and detailed charts are presented for the rapid estimation of these parameters. Also included is an approximate method by which these parameters may be corrected for airfoil-section thickness.

Deflected controls are assumed to be located either at the wing tip or far enough inboard to prevent the outermost Mach lines from the controls from crossing the wing tip. For either of these locations, the innermost Mach lines are assumed not to cross the wing root chord. The method for determining control hinge moment resulting from wing angle-of-attack loading is valid for wing plan forms having the leading edges swept ahead of the Mach lines and having streamwise tips. The only additional restrictions are that the controls must not be influenced by the tip conical flow from the opposite wing panel or by the interaction of the wing-root Mach cone with the wing tip.

INTRODUCTION

Linearized theory, though neglecting viscosity and second-order effects existing in practice, is the most practical method now available for estimating the characteristics of control surfaces at supersonic speeds. A general application of this theory to control surfaces having edges swept either ahead of or behind the Mach lines is presented in reference 1. (Edges swept ahead of or behind the Mach lines are subsequently referred to as supersonic or subsonic edges.) Conical-flow solutions for various deflected control configurations are presented in reference 2. Such solutions were used in reference 3 to evaluate the characteristics of a restricted family of trailing-edge control surfaces.

In the present report a general analysis based on existing conical-flow solutions has been made which will apply to a broad range of trailing-edge control configurations having supersonic edges and will provide for a comprehensive coverage of control location, aspect ratio, taper ratio, and sweep. Equations and detailed charts are presented from which lift, pitching-moment, rolling-moment, and hinge-moment coefficients due to control deflection and hinge-moment

coefficient due to wing angle of attack, as predicted by linearized theory, may be determined in an estimated 5 percent of the time required without the use of such equations and charts. Also included is an approximate method by which these hinge-moment and effectiveness parameters may be corrected for airfoil-section thickness.

The equations and charts presented are applicable to control-surface plan forms that vary throughout the range in which the leading and trailing edges are supersonic and the root and tip chords are in a streamwise direction. Deflected controls are assumed to be located either at the wing tip or far enough inboard to prevent the outermost Mach lines from the controls from crossing the wing tip. For either of these locations, the innermost Mach lines are assumed not to cross the wing root chord. The method for calculating the hinge-moment coefficient due to wing angle of attack is valid for wing plan forms having straight supersonic edges and streamwise tips. This method is restricted only in that the controls must not lie in a region influenced by the tip conical flow from the opposite wing panel or by the interaction of the wing-root Mach cone with the wing tip.

SYMBOLS

M	free-stream Mach number
$\beta = \sqrt{M^2 - 1}$	
C_1, C_2	functions of Mach number used in calculating two-dimensional-flow characteristics
Λ	angle of sweep of wing leading edge, positive when swept back
Λ_{HL}	angle of sweep of control hinge line, positive when swept back
Λ_{TE}	angle of sweep of wing trailing edge, positive when swept back
b_f	span of control surface
c_f	root chord of control surface
c_{ft}	tip chord of control surface
λ_f	control-surface taper ratio (c_{ft}/c_f)
S_f	area of control surface
A_f	aspect ratio of control surface (b_f^2/S_f)
$A_f' = \beta A_f$	
M_a	area moment of control surface about hinge axis

¹ Supersedes NACA TN 2221, "Equations and Charts for the Rapid Estimation of Hinge-Moment and Effectiveness Parameters for Trailing-Edge Controls Having Leading and Trailing Edges Swept Ahead of the Mach Lines" by Kenneth L. Goin, 1950.

S_L	area of a loaded region	$C_l = \frac{\text{Rolling moment about wing root chord}}{2qbS}$	
S_{L_0}	area of part of deflected control surface lying in two-dimensional-flow region less area lying in region of overlap of conical-flow fields	$C_m = \frac{\text{Pitching moment about wing axis of pitch}}{qS\bar{c}}$	
m_0	moment of S_{L_0} about hinge axis	F_1	thickness correction factor for C_{L_i}' and C_{l_i}'
l_0	moment of S_{L_0} about control root chord	F_2	thickness correction factor for C_{h_i} and C_{m_i}'
$\bar{x}$	distance of center of loading from control hinge axis measured normal to hinge axis	F_3	thickness correction factor for C_{h_a}
$\bar{y}$	spanwise distance of center of loading from control root chord	Δp	difference between local pressure and stream static pressure
θ	slope of airfoil-section contour	C_p	pressure coefficient ($\Delta p/q$)
$t/2c$	one-half airfoil-thickness ratio measured in plane normal to control hinge axis	C_{p_0}	two-dimensional pressure coefficient $\left(\frac{2\delta}{57.3\beta\sqrt{1-a^2}}\right)$ or $\left(\frac{2\alpha}{57.3\beta\sqrt{1-q^2}}\right)$
$(t/c)_{max}$	maximum airfoil-thickness ratio measured in plane normal to control hinge axis	P'	local pressure ratio (C_p/C_{p_0})
x/c	chordwise position measured in plane normal to control hinge axis	P	average value of pressure ratio P' over conical-flow region $\left(\frac{\int P' dS_L}{S_L}\right)$
x_h/c	chordwise location of control hinge axis measured in plane normal to control hinge axis	τ	angle denoting arbitrary position of ray in conical-flow field
$(t/2c)', (x/c)'$	dimensions measured in plane normal to wing leading edge	$\tau' = \tau + \Delta_{TE}$	
x_f	distance of leading edge of control root chord behind wing axis of pitch	$t = \beta \tan \tau$	
y_f	distance of root chord of control from root chord of wing	$t' = \beta \tan \tau'$	
b	wing span	$r = \frac{1}{t}$	
c_r	wing root chord	n, r'	nondimensional coordinates used in integration of wing root and tip conical pressures
c_t	wing tip chord	η	angle of sweep of line intersecting conical-flow regions of wing at angle of attack
$\bar{c}$	mean aerodynamic chord of wing	Subscripts:	
S	area of semispan wing	δ, α	denote partial derivative of force and moment coefficients with respect to δ or α
$g = \frac{\tan \Lambda}{\beta}$		cp	denotes center-of-pressure ray location
$a = \frac{\tan \Lambda_{HL}}{\beta}$		Superscript:	
$d = \frac{\tan \Lambda_{TE}}{\beta}$		*	indicates that parameters $P, PS_L, PS_L\bar{x}, PS_L\bar{y}, t_{cp}'$, and r_{cp} refer to loss of loading from two-dimensional value rather than to actual loading
α	wing angle of attack, degrees		
δ	angle of control-surface deflection measured in streamwise direction, degrees		
q	free-stream dynamic pressure		
$C_L' = \frac{\text{Lift induced by deflected control}}{qS_f}$			
	Moment about control root chord induced by deflected control		
$C_l' = \frac{\text{Moment about control root chord induced by deflected control}}{qb_f S_f}$			
$C_m' = \frac{\text{Moment about hinge axis induced by deflected control}}{2qM_a}$			
$C_h = \frac{\text{Hinge moment}}{2qM_a}$			
$C_L = \frac{\text{Lift induced by deflected control}}{qS}$			

ANALYSIS

CHARACTERISTICS DUE TO DEFLECTION OF CONTROL SURFACES

Scope.—Existing solutions of the linearized equations of fluid motion have been used as a basis for calculating the characteristics due to deflection of trailing-edge control surfaces on wings in steady flight at supersonic speeds. These solutions, as presented in reference 2, are applicable to configurations for which the leading and trailing edges of the control are supersonic and the root and tip chords are streamwise. Two control-surface locations are considered.

The control is assumed to be located either at the wing tip or far enough inboard to prevent the outermost Mach line from the control from crossing the wing tip. For either of these locations, the innermost Mach lines are assumed not to cross the wing root chord. For these locations, deflected control-surface characteristics are functions only of Mach number and control-surface plan form. (The parameter C_h depends on control-surface location only when the control is located inboard from the wing tip and lies in a region influenced either by the interaction of the control-tip Mach cone with the wing tip or by the reflection from the wing root chord of the innermost control Mach line.) If the limitations previously mentioned are considered, the analysis is valid for all controls except those located at the wing tip and having the inboard conical-flow regions intersecting the tip. In such cases, the conical pressures on the control, as given in reference 2, are not applicable in the region influenced by the interaction of the Mach cone with the wing tip. Necessary corrections for this region can be determined by the method described in reference 4. Such corrections are not considered in the present report because of the prohibitive amount of computation involved. Results not including these corrections are presented, however, because they should be very useful as an indication of trends and should in many cases closely approximate the corrected result.

Method.—In order to determine control-surface characteristics, the two-dimensional region and the triangular segments of the conical-flow regions (fig. 1) are considered independently. The characteristics are obtained by summing the products of pressure ratio and nondimensional-area and moment-arm parameters for all parts (table I). The nature of conical flow is such that the pressure is constant along any ray from the origin of the flow field. Any infinitesimal triangle having the origin of the flow field as an apex, therefore, has its center of pressure located at two-thirds of the distance from the apex to the base. It follows that the summation of the loading of such infinitesimal triangles results in a finite triangle having its center of pressure lying on a line parallel to the base and located at two-thirds of the distance from the apex to the base. The center-of-pressure location and, consequently, the desired moment arms can therefore be determined from the location of the ray on which the center of pressure lies. General equations for the average pressure ratio and center-of-pressure ray location for each conical segment (tables II (a) and II (b)) were obtained by integrating the pressure equations of reference 2. (See appendix A.) Table II(c) presents equations for the nondimensional-area and moment-arm parameters (in terms of center-of-pressure ray location) for each conical segment. Equations pertaining to the two-dimensional region were obtained by treating

this region as a simple geometric area and are also included in table II(c). Results obtained by evaluating the general equations of table II when they become indeterminate at taper ratios of 1.0 are presented in table III.

For regions in which the two conical-flow fields overlap, the method of superposition must be used wherein the losses in pressure ratio from the two-dimensional value ($P'=1.0$) in the two conical-flow regions are additive; that is,

$$\begin{aligned} P' &= 1.0 - (1.0 - P_{mc_1}') - (1.0 - P_{mc_2}') \\ &= -1.0 + P_{mc_1}' + P_{mc_2}' \end{aligned}$$

(Subscripts mc_1 and mc_2 refer to inboard and outboard conical-flow regions, respectively.) The net effects of the pressure distribution in this region are obtained by adding the effects of the two conical-flow regions as though the flow regions did not overlap and by subtracting the effects of a two-dimensional pressure distribution. This subtraction is accomplished by use of the equations for the two-dimensional region (tables II (c) and III (b)). In calculating control hinge moments it was convenient to calculate the effects of regions I_c and II_c or III (fig. 1) and then to subtract the effects of the parts of these regions lying off the control. For controls located at the wing tip and having the inboard Mach cone intersecting the tip, a similar procedure was also used to reduce to zero the lift, pitching moment, and rolling moment contributed by the triangular part of the inboard conical-flow region lying beyond the tip. As previously mentioned for this case, a rigid application of linearized theory would require a correction, as described in reference 4, to the loading assumed in the region influenced by the interaction of the root Mach cone with the free edge. It should be pointed out that the areas influenced by such interactions become appreciable for extreme conditions and approximate results for such configurations should be used with caution.

HINGE MOMENT DUE TO WING ANGLE-OF-ATTACK CHANGE

Scope.—Conical-flow solutions for swept wings at supersonic speeds, as presented in reference 5, are used as a basis for the analysis. These solutions are applicable to wing plan forms having straight supersonic edges and streamwise tips.

As in the analysis for deflected control surfaces, only control surfaces having supersonic edges and streamwise root and tip chords are considered. The only restrictions regarding control location are that the control must not lie in a region influenced by the tip conical flow from the opposite wing panel or by the interaction of the wing-root Mach cone with the wing tip.

Method.—The method consists essentially of determining the hinge-moment parameter $PS_L\bar{x}$ for the flap by assuming two-dimensional loading and then subtracting the losses

resulting from the wing-root and wing-tip conical flows. The conical-flow losses are obtained by dividing the conical regions into a series of triangular segments, each having its apex at the origin of the Mach cone, and by summing the hinge-moment parameters $(PS_L\bar{x})^*$ for these segments as illustrated in figure 2. In determining $(PS_L\bar{x})^*$ for the triangular segments, integrations of the loading are necessary for obtaining P^* and $\bar{x}$. As has been previously explained for this type of conical-flow segment, it is sufficient to determine P^* and t_{cp} because the moment arm $\bar{x}$ can be determined from t_{cp} . The method for obtaining P^* and t_{cp} is illustrated in figure 3 and involves integrating the pressure losses along the bases of the segments. From integrations of the pressure losses between 0 and n_1 (or 0 and r_1'), values of P^* and n_{cp} (or r_{cp}') are obtained. Values of P^* and values of t_{cp} , corresponding to n_{cp} (or r_{cp}'), obtained in this manner are applicable to the triangular segment bounded by the Mach line, the ray $\tau=\tau_1$, and the section intersecting the Mach cone. Results have been obtained by numerical integration using Simpson's rule (reference 6) except in regions where the slopes of the pressure curves become infinite (fig. 3). In these regions, integrating coefficients, as presented in reference 7, have been used. Forms by which the integrations were made are presented in tables IV to VII. The upper parts of these forms are used for computing the pressure distributions $(1-P')$ along the sections intersecting the Mach cones (fig. 3). In the lower part of the form, the areas and area moments about n (or r') = 0 of the curves of $1-P'$ plotted against n (or r') are determined and are used to obtain P^* and t_{cp} for the corresponding triangular segments. Tables IV to VII can be used directly for calculating the loading distribution for intermediate cases or cases not included in the present report.

METHOD FOR APPROXIMATELY CORRECTING RESULTS OBTAINED FROM USE OF LINEARIZED THEORY FOR AIRFOIL-SECTION THICKNESS

Scope.—The method for approximately correcting the theoretical results for airfoil-section thickness is based on the assumption that, at any chordwise position on an airfoil having finite thickness, the ratio of conical to two-dimensional pressure is the same as that predicted by linearized theory for an infinitely thin flat plate. (This method is a variation of the method presented in reference 8.) The method can be logically applied only to configurations having similar sections at all spanwise positions affected. The method is expected to give most accurate results at moderate and high Mach numbers for thin controls located inboard from the wing tip and having relatively large areas over which the flow is two-dimensional.

Method.—On the basis of the preceding assumption, the method requires the determination of the following three factors:

$$F_1 = \frac{C_{L'}(\text{Two-dimensional with thickness})}{C_{L'}(\text{Two-dimensional flat plate})} = \frac{C_{l'}(\text{Two-dimensional with thickness})}{C_{l'}(\text{Two-dimensional flat plate})} \quad (1)$$

$$F_2 = \frac{C_{m'}(\text{Two-dimensional with thickness})}{C_{m'}(\text{Two-dimensional flat plate})} = \frac{C_h(\text{Two-dimensional with thickness})}{C_h(\text{Two-dimensional flat plate})} \quad (2)$$

$$F_3 = \frac{C_h(\text{Two-dimensional with thickness})}{C_h(\text{Two-dimensional flat plate})} \quad (3)$$

(The coefficients in equations (1) and (2) are for deflected controls, and the coefficients in equation (3) are those resulting from wing angle-of-attack loading.) Corrected values of $C_{L_i'}$, $C_{l_i'}$, $C_{m_i'}$, C_{h_i} , and C_{h_a} are obtained by multiplying the results obtained by use of the linearized theory for three-dimensional flat plates by the appropriate factors.

The factors are determined, as described in appendix B, by using the Busemann second-order approximation to determine the coefficients for sections having thickness. This approximation gives results which are generally in good agreement with results obtained by use of the more involved exact theories. The theory is not considered accurate, however, at Mach numbers for which the shocks become detached or at Mach numbers below about 1.3 (reference 9). For the general group of airfoil sections that are symmetrical about the chord plane, equations for the correction factors as derived in appendix B are:

$$F_1 = \frac{1}{\left(1 - \frac{x_h}{c}\right)} \int_{x_h/c}^{1.0} \left(1 + 2 \frac{C_2}{C_1} \frac{d \frac{t}{2c}}{d \frac{x}{c}}\right) d \frac{x}{c} \quad (4)$$

$$F_2 = \frac{2}{\left(1 - \frac{x_h}{c}\right)^2} \int_{x_h/c}^{1.0} \left(\frac{x}{c} - \frac{x_h}{c}\right) \left(1 + 2 \frac{C_2}{C_1} \frac{d \frac{t}{2c}}{d \frac{x}{c}}\right) d \frac{x}{c} \quad (5)$$

$$F_3 = \frac{2}{\left(1 - \frac{x_h}{c}\right)^2} \int_{x_h/c}^{1.0} \left(\frac{x}{c} - \frac{x_h}{c}\right) \left[1 + 2 \frac{C_2}{C_1} \frac{d \left(\frac{t}{2c}\right)'}{d \left(\frac{x}{c}\right)'}\right] d \frac{x}{c} \quad (6)$$

CHARTS PRESENTATION

Aside from the restrictions regarding location, the characteristics of deflected control surfaces are functions only of control plan form and Mach number. The effects of plan form and Mach number are determined from solutions to equations (tables I to III) involving the variables $\frac{\tan \Lambda_{HL}}{\beta}$, $\frac{\tan \Lambda_{TE}}{\beta}$, and λ_r . (For untapered controls the variables are $\frac{\tan \Lambda_{HL}}{\beta}$ and βA_r .) Figure 4 presents $\beta C_{L_i'}$, $\beta C_{l_i'}$, $\beta C_{m_i'}$, and βC_{h_i} as functions of these variables for controls located at the wing tip. Each chart of figure 4 presents the characteristics of a series of plan forms having a fixed hinge-line sweep angle (if the Mach number is considered to be fixed)

and varying trailing-edge sweep angles and taper ratios. The solid-line curves present the effects of varying taper ratio for plan forms having fixed hinge line and trailing-edge sweep angles. The characteristics of controls having constant aspect ratios are indicated in the charts for βC_{h_s} by dashed lines. Constant-aspect-ratio curves are not included in the charts for the other characteristics because, in many cases, they would be quite confusing. If desired, such curves can be drawn by simply determining the taper ratio at which the curve will intersect each of the curves of constant d from the following relation:

$$\lambda_f = \frac{2 - A_f'(a-d)}{2 + A_f'(a-d)}$$

For inversely tapered controls, the parameter $1/\lambda_f$ is used as a coordinate to avoid elongation of the curves. Calculations were made at values of λ_f and $\frac{1}{\lambda_f} = 0, 0.20, 0.40, 0.60, 0.80$, and 0.95 and at values of $A_f' = 0.8, 2.0, 4.0, 6.0, 8.0$, and 10.0 for untapered controls. Calculated results not included in the charts are presented in table VIII. The results not included in the charts are mainly for configurations having values of $\frac{\tan \Lambda_{TE}}{\beta}$ near $|1.0|$ and, consequently, having extremely large areas of induced loading on the wing. Results for such configurations are of little practical value because if these large areas are to lie entirely on the wing, as has been assumed, the wing must have a very large span or the control must have a very small chord.

Charts presenting the characteristics of deflected controls located inboard from the wing tip are presented in figures 5 and 6. These charts vary somewhat from those for controls located at the wing tip. Equations for $\beta C_{L_s}'$ and $\beta C_{m_s}'$ were simplified and found to be dependent only on $\frac{\tan \Lambda_{HL}}{\beta}$ and $\frac{\tan \Lambda_{TE}}{\beta}$. These equations, with results in chart form, are presented in figure 5. Charts for βC_{h_s} and $\beta C_{L_s}'$ (fig. 6) are presented only for normal taper ratios because the characteristics of inversely tapered controls can be obtained by entering the charts at $\frac{-\tan \Lambda_{HL}}{\beta}$, $\frac{-\tan \Lambda_{TE}}{\beta}$, and $1/\lambda_f$.

The computing form for C_{h_s} is presented in table IX and is self-explanatory. Supplementary charts for determining the loading distribution (P^* and t_{cp}) for the various triangular segments of the conical-flow regions are presented in figures 7 to 10. It should be pointed out that figures 8 and 10 can easily be used for determining the spanwise and chordwise loading of the wings considered in this report and will therefore be of value in making loads analyses.

USE

In order to use the charts for determining the characteristics of deflected controls, values of $\frac{\tan \Lambda_{HL}}{\beta}$, $\frac{\tan \Lambda_{TE}}{\beta}$, and λ_f for the configuration being considered must be determined.

These values are then used for entry into the charts, figures 4 or 5 and 6, depending on control location. The coefficients obtained from the charts have been made nondimensional by use of control geometric parameters. For determining the coefficients based on the usual wing parameters, the following equations are given (approximate thickness correction factors are included but can be neglected by letting the factors equal 1.0):

$$(C_{L_s})_c = F_1 C_{L_s}' \frac{S_f}{S} \quad (7)$$

$$(C_{L_s})_c = \frac{(C_{L_s})_c}{2b} \left(y_f + b_f \frac{C_{L_s}'}{C_{L_s}'} \right) \quad (8)$$

$$(C_{m_s})_c = \frac{(C_{L_s})_c}{\bar{c}} \left(\frac{F_2 C_{m_s}'}{F_1 C_{L_s}'} \frac{2M_a}{S_f} \sqrt{1 + \beta^2 a^2} - \beta a b_f \frac{C_{L_s}'}{C_{L_s}'} - x_f \right) \quad (9)$$

$$(C_{h_s})_c = F_2 C_{h_s} \quad (10)$$

(The subscript c indicates that the approximate thickness correction factors have been included.)

For determining the control hinge moment due to wing angle of attack, preliminary calculations are first made on the computing form of table IX. Results of these computations indicate positions in the charts (figs. 7 to 10) from which P^* and t_{cp} are to be obtained. Values from the charts are then inserted in table IX and the operations indicated in the computing form are completed. The approximate thickness correction factor can be applied by use of the following equation:

$$(C_{h_s})_c = F_3 C_{h_s} \quad (11)$$

ILLUSTRATIVE EXAMPLE

As an example of the use of the charts, the control-surface characteristics are determined for the configuration shown in figure 11. The wing is assumed to have 5-percent-thick symmetrical parabolic sections in planes normal to the control hinge line.

Lift and pitching-moment coefficients are obtained by entering the charts of figure 5 at values of $\frac{\tan \Lambda_{HL}}{\beta} = 0.40$ and $\frac{\tan \Lambda_{TE}}{\beta} = 0.35$. Hinge-moment and rolling-moment coefficients are obtained by entering the charts of figure 6 (g) at values of $\frac{\tan \Lambda_{TE}}{\beta} = 0.35$ and $\lambda_f = 0.713$. Coefficients obtained from the charts are $\beta C_{L_s}' = 0.0748$, $\beta C_{m_s}' = -0.0365$, $\beta C_{L_s}' = 0.0372$, and $\beta C_{h_s} = -0.0345$. The calculation of C_{h_s} for the example is presented in table IX. Preliminary calculations are made in table IX (a) and in column (1) of table IX (b). Values of n and r' calculated in column (1) are used to enter the charts (figs. 7 to 10). Values of P^* and t_{cp} obtained from the charts are inserted in columns (2) and (3) of table IX (b) and the computations are completed. The theoretical value of C_{h_s} is -0.0194 .

The equation for the section contour in a plane normal to the control hinge axis is

$$\frac{t}{2c} = 2 \left(\frac{t}{c} \right)_{\max} \left[\frac{x}{c} - \left(\frac{x}{c} \right)^2 \right] \quad (12)$$

The slope in this plane at any point along the airfoil is

$$\frac{d \frac{t}{2c}}{d \frac{x}{c}} = 2 \left(\frac{t}{c} \right)_{\max} \left(1 - 2 \frac{x}{c} \right) \quad (13)$$

Substitution of equation (13) in equations (4) and (5) yields the following equations for F_1 and F_2 :

$$F_1 = 1 - 4 \frac{C_2}{C_1} \left(\frac{t}{c} \right)_{\max} \frac{x_h}{c} \quad (14)$$

$$F_2 = 1 - \frac{4}{3} \frac{C_2}{C_1} \left(\frac{t}{c} \right)_{\max} \left(1 + 2 \frac{x_h}{c} \right) \quad (15)$$

For determining F_3 , the equation for the section contour in a plane normal to the wing leading edge is written as

$$\left(\frac{t}{2c} \right)' = \frac{2 \left(\frac{t}{c} \right)_{\max}}{\cos (\Lambda - \Lambda_{HL})} \left[\frac{\left(\frac{x}{c} \right)' - \left(\frac{x}{c} \right)^{1/2}}{1 + K \left(\frac{x}{c} \right)'} \right] \quad (16)$$

where

$$K = \tan (\Lambda - \Lambda_{HL}) \tan (\Lambda - \Lambda_{TE})$$

The slope of the airfoil contour in this plane is

$$\frac{d \left(\frac{t}{2c} \right)'}{d \left(\frac{x}{c} \right)'} = \frac{2 \left(\frac{t}{c} \right)_{\max}}{\cos (\Lambda - \Lambda_{HL})} \left\{ \frac{1 - 2 \left(\frac{x}{c} \right)' - K \left(\frac{x}{c} \right)^{1/2}}{\left[1 + K \left(\frac{x}{c} \right)' \right]^2} \right\} \quad (17a)$$

or in terms of x/c

$$\frac{d \left(\frac{t}{2c} \right)'}{d \left(\frac{x}{c} \right)'} = \frac{2 \left(\frac{t}{c} \right)_{\max}}{\cos (\Lambda - \Lambda_{HL})} \left[1 - 2 \frac{x}{c} + \frac{K}{1 + K} \left(\frac{x}{c} \right)^2 \right] \quad (17b)$$

Substitution of equation (17b) in equation (6) yields the following equation for F_3 :

$$F_3 = 1 - \frac{2 C_2 \left(\frac{t}{c} \right)_{\max}}{3 C_1 (1 + K) \cos (\Lambda - \Lambda_{HL})} \left[2 \left(1 + 2 \frac{x_h}{c} \right) - K \left(1 - \frac{x_h}{c} \right)^2 \right] \quad (18)$$

From equations (14), (15), and (18), the following correction factors are obtained for the sample configuration: $F_1 = 0.8077$, $F_2 = 0.7889$, and $F_3 = 0.7355$. It is of interest to note that these values indicate appreciable losses in loading due to airfoil-section thickness, and it might be pointed out that greater losses would be obtained for thicker airfoil sections.

The coefficients obtained from the charts and the preceding correction factors are then substituted in equations (4) to (8). The results obtained are

$$(C_{L_s})_c = 0.00411$$

$$(C_{m_s})_c = -0.00318$$

$$(C_{i_s})_c = 0.000619$$

$$(C_{h_s})_c = -0.0182$$

$$(C_{h_a})_c = -0.0143$$

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., September 8, 1950.

APPENDIX A

METHOD OF INTEGRATING PRESSURES OVER CONICAL REGIONS OF DEFLECTED CONTROLS

The pressure distributions in the conical-flow regions shown in figure 12 are given in reference 2. With suitable changes in notation these are:

For region I,

$$P' = \frac{1}{\pi} \cos^{-1} \frac{a-t}{1-at} \quad (A1)$$

For region III,

$$P' = \frac{1}{\pi} \cos^{-1} \frac{1-(2+a)t}{1+at} \quad (A2)$$

Because the flow is conical in regions I and III, integrations of the pressures along the trailing edge within these regions are representative of integrations over corresponding triangular segments having the Mach cone origin as apexes. For such integrations, a coordinate for distance along the trailing edge must be introduced. The nondimensional coordinate chosen was $t' = \beta \tan \tau'$ (fig. 12 and reference 2). The integrations required for determining average pressure ratio and center-of-pressure ray location for any segment are

$$P = \frac{\int_{t_1'}^{t_2'} P' dt'}{\int_{t_1'}^{t_2'} dt'} \quad (A3)$$

and

$$t_{cp}' = \frac{\int_{t_1'}^{t_2'} t' P' dt'}{\int_{t_1'}^{t_2'} P' dt'} \quad (A4)$$

(Subscripts 1 and 2 indicate values of t' corresponding to the end points of the part of t' over which integrations were made.)

A Mach number of $\sqrt{2}$ was assumed for convenience ($\beta=1$) in making the integrations of equations (A3) and (A4). This assumption is valid because any case of Mach number greater than 1 can readily be reduced to an equivalent case at $M=\sqrt{2}$ by an affine transformation corresponding to the Prandtl-Glauert transformation for the subsonic case (reference 5). An example of this transformation is shown in figure 13. The equivalent plan form is obtained by dividing all streamwise dimensions by β and leaving lateral dimensions unchanged; consequently, values of a , d , and t (for equivalent points) are the same. From equations (A1) and (A2), it can readily be seen that values of P' for equivalent points are the same. It follows that summation of P' over equivalent regions results in equal values of P and t_{cp} . It is apparent from figure 13, however, that values of t_{cp}' are different. This difference is of no consequence because values of t_{cp} for the equivalent wing (obtained from t_{cp}' and geometric relations) are the same as values of t_{cp} for the initial wing.

The procedures followed in the integrations of equations (A3) and (A4) are the same for regions I and III and are only shown for region I. If the Mach number is assumed to equal $\sqrt{2}$, where $\beta=1$, equation (A1) may be written in terms of t' as follows:

$$P' = \frac{1}{\pi} \cos^{-1} \frac{(a+d)-(1-ad)t'}{(1+ad)-(a-d)t'}$$

If y is substituted for $\cos \pi P'$, equations (A3) and (A4) become

$$P = \frac{\frac{-(1-a^2)(1+d^2)}{\pi} \int_{y_1}^{y_2} \frac{\cos^{-1} y}{[(1-ad)-(a-d)y]^2} dy}{\frac{-(1-a^2)(1+d^2)}{\pi} \int_{y_1}^{y_2} \frac{dy}{[(1-ad)-(a-d)y]^2}} \quad (A5)$$

$$t_{cp}' = \frac{\frac{-(1-a^2)(1+d^2)}{\pi} \int_{y_1}^{y_2} \frac{(a+d)-(1+ad)y}{[(1-ad)-(a-d)y]^3} \cos^{-1} y dy}{\frac{-(1-a^2)(1+d^2)}{\pi} \int_{y_1}^{y_2} \frac{\cos^{-1} y}{[(1-ad)-(a-d)y]^2} dy} \quad (A6)$$

Integration by parts was then employed in the solutions of equations (A5) and (A6).

For cases in which the conical-flow region overlaps the opposite parting line, the average pressure loss and center-of-pressure ray location are required for regions I_a and I_b (fig. 1). Equations (A3) and (A4) may be used in obtaining the solutions for region I_a by a slight modification requiring no additional integration. Thus,

$$P^* = \frac{\int_{t_1'}^{t_2'} (1-P') dt'}{\int_{t_1'}^{t_2'} dt'} = 1 - \frac{\int_{t_1'}^{t_2'} P' dt'}{\int_{t_1'}^{t_2'} dt'} \quad (A7)$$

$$t_{cp}' = \frac{\int_{t_1'}^{t_2'} t' (1-P') dt'}{\int_{t_1'}^{t_2'} (1-P') dt'} = \frac{\left[\frac{t'^2}{2} \right]_{t_1'}^{t_2'} - \int_{t_1'}^{t_2'} t' P' dt'}{\left[t' \right]_{t_1'}^{t_2'} - \int_{t_1'}^{t_2'} P' dt'} \quad (A8)$$

In obtaining the solutions for region I_b (fig. 1), essentially the same procedure as previously outlined was used. The parameter $r = \frac{1}{t}$ was used to represent distance along the parting line nondimensionally. Values of P^* and r_{cp} were obtained by making integrations similar to those in equations (A7) and (A8) (before simplifications).

Results of integrations over all regions shown in figure 1 are presented in tables II (a) and II (b). Results of evaluating these equations at taper ratios of 1.0, where they become indeterminate, are presented in table III (a).

APPENDIX B

METHOD FOR DETERMINING THICKNESS CORRECTION FACTORS

The pressure coefficient at any point on a two-dimensional surface as given by the Busemann second-order approximation (reference 8, with suitable changes in notation) is

$$C_p = C_1(\delta + \theta) + C_2(\delta + \theta)^2 \quad (\text{B1})$$

(The angle δ is considered positive when calculating C_p for lower surface and negative when calculating C_p for upper surface. Throughout appendix B, δ is considered to be in radians.) The constants C_1 and C_2 are functions only of Mach number. Equations for these constants and tabulated values are presented in reference 10.

The lifting pressure coefficient at any chordwise position is simply the difference between the pressure coefficients on the lower and upper surfaces. The net lift coefficient is obtained by integrating the local lifting pressure coefficients between the hinge line $\left(\frac{x}{c} = \frac{x_h}{c}\right)$ and the trailing edge $\left(\frac{x}{c} = 1.0\right)$. (See fig. 14.) Thus,

$$C_{L' \text{ thickness } (\delta)} = \frac{1}{1 - \frac{x_h}{c}} \int_{x_h/c}^{1.0} [(C_p)_L - (C_p)_U] d \frac{x}{c} \quad (\text{B2})$$

(The subscripts L and U denote lower and upper surfaces.) Similarly, the hinge-moment coefficient is obtained by integrating the products of local lifting pressure coefficient and moment arm between the hinge line and the trailing edge. Thus,

$$C_{h \text{ thickness } (\delta)} = \frac{-1}{\left(1 - \frac{x_h}{c}\right)^2} \int_{x_h/c}^{1.0} \left(\frac{x}{c} - \frac{x_h}{c}\right) [(C_p)_L - (C_p)_U] d \frac{x}{c} \quad (\text{B3})$$

An application of sweepback theory, as explained in reference 10, must be used for determining $C_{pL} - C_{pU}$. It is important to note that, for deflected controls, this theory requires the use of the Mach number component and the airfoil section in a plane normal to the control hinge axis. Values of C_L' and C_h thus obtained are based on the dynamic-pressure component normal to the hinge line and the deflection angle measured in a plane normal to the hinge line. Values of C_L' and C_h for a two-dimensional flat-plate control, based on the same q and δ , are obtained by considering the Mach number normal to the hinge line in determining values of C_1 . Equations for these coefficients are

$$C_{L' \text{ flat plate}} = 2C_1\delta \quad (\text{B4})$$

$$C_{h \text{ flat plate}} = -C_1\delta \quad (\text{B5})$$

The following correction factors are then determined by dividing equations (B2) and (B3) by equations (B4) and (B5), respectively:

$$F_1 = \frac{1}{2C_1\delta \left(1 - \frac{x_h}{c}\right)} \int_{x_h/c}^{1.0} [(C_p)_L - (C_p)_U] d \frac{x}{c} \quad (\text{B6})$$

$$F_2 = \frac{1}{C_1\delta \left(1 - \frac{x_h}{c}\right)^2} \int_{x_h/c}^{1.0} \left(\frac{x}{c} - \frac{x_h}{c}\right) [(C_p)_L - (C_p)_U] d \frac{x}{c} \quad (\text{B7})$$

If the sections are assumed to be symmetrical about the chord plane, equations (B6) and (B7) can be simplified because

$$(C_p)_L - (C_p)_U = 2\delta \left(C_1 + 2C_2 \frac{d \frac{t}{2c}}{d \frac{x}{c}} \right) \quad (\text{B8})$$

Equations (B6) and (B7) then become

$$F_1 = \frac{1}{1 - \frac{x_h}{c}} \int_{x_h/c}^{1.0} \left(1 + 2 \frac{C_2}{C_1} \frac{d \frac{t}{2c}}{d \frac{x}{c}} \right) d \frac{x}{c} \quad (\text{B9})$$

$$F_2 = \frac{2}{\left(1 - \frac{x_h}{c}\right)^2} \int_{x_h/c}^{1.0} \left(\frac{x}{c} - \frac{x_h}{c}\right) \left(1 + 2 \frac{C_2}{C_1} \frac{d \frac{t}{2c}}{d \frac{x}{c}} \right) d \frac{x}{c} \quad (\text{B10})$$

The equation for F_3 , may be written as equation (B7) for F_2 (substituting α for δ)

$$F_3 = \frac{1}{C_1\alpha \left(1 - \frac{x_h}{c}\right)^2} \int_{x_h/c}^{1.0} \left(\frac{x}{c} - \frac{x_h}{c}\right) [(C_p)_L - (C_p)_U] d \frac{x}{c} \quad (\text{B11})$$

In this case, however, the airfoil section and Mach number component in a plane normal to the wing leading edge must be used in determining values of C_1 and $(C_p)_L - (C_p)_U$.

For symmetrical sections, the equation for F_3 may be simplified in the same manner as the equivalent equation for F_2 . Thus,

$$F_3 = \frac{2}{\left(1 - \frac{x_h}{c}\right)^2} \int_{x_h/c}^{1.0} \left(\frac{x}{c} - \frac{x_h}{c}\right) \left[1 + 2 \frac{C_2}{C_1} \frac{d \left(\frac{t}{2c}\right)'}{d \left(\frac{x}{c}\right)'} \right] d \frac{x}{c} \quad (\text{B12})$$

Equation (B12) will in some cases become somewhat involved because $\frac{d(t/c)'}{d(x/c)'}$ must be determined from the equation for the airfoil section in a plane normal to the wing leading edge and must then be written in terms of x/c (unless the surfaces are plane). It should be pointed out that suitable approximations for most symmetrical biconvex airfoils (which in general require involved expressions for defining the contour) may be obtained by assuming the sections to have parabolic contours. General equations for the thickness correction factors for symmetrical sections having parabolic contours have been derived in the illustrative example of the present report.

REFERENCES

1. Frick, Charles W., Jr.: Application of the Linearized Theory of Supersonic Flow to the Estimation of Control-Surface Characteristics. NACA TN 1554, 1948.
2. Lagerstrom, P. A., and Graham, Martha E.: Linearized Theory of Supersonic Control Surfaces. Rep. No. SM-13060, Douglas Aircraft Co., Inc., July 24, 1947.
3. Kainer, Julian H., and Marte, Jack E.: Theoretical Supersonic Characteristics of Inboard Trailing-Edge Flaps Having Arbitrary Sweep and Taper. Mach Lines behind Flap Leading and Trailing Edges. NACA TN 2205, 1950.
4. Cohen, Doris: The Theoretical Lift of Flat Swept-Back Wings at Supersonic Speeds. NACA TN 1555, 1948.
5. Lagerstrom, P. A., Wall, D., and Graham, M. E.: Formulas in Three-Dimensional Wing Theory. Rep. No. SM-11901, Douglas Aircraft Co., Inc., July 8, 1946.
6. Sokolnikoff, Ivan S., and Sokolnikoff, Elizabeth S.: Higher Mathematics for Engineers and Physicists. Second ed., McGraw-Hill Book Co., Inc., 1941.
7. Lock C. N. H., and Knowler, A. E.: Integrating Coefficients for Airscrew Analysis. R. & M. No. 2043, British A.R.C., 1941.
8. Bonney, E. Arthur: Aerodynamic Characteristics of Rectangular Wings at Supersonic Speeds. Jour. Aero. Sci., vol. 14, no. 2, Feb. 1947, pp. 110-116.
9. Laitone, Edmund V.: Exact and Approximate Solutions of Two-Dimensional Oblique Shock Flow. Jour. Aero. Sci., vol. 14, no. 1, Jan. 1947, pp. 25-41.
10. The Staff of the Ames 1- by 3-Foot Supersonic Wind-Tunnel Section: Notes and Tables for Use in the Analysis of Supersonic Flow. NACA TN 1428, 1947.

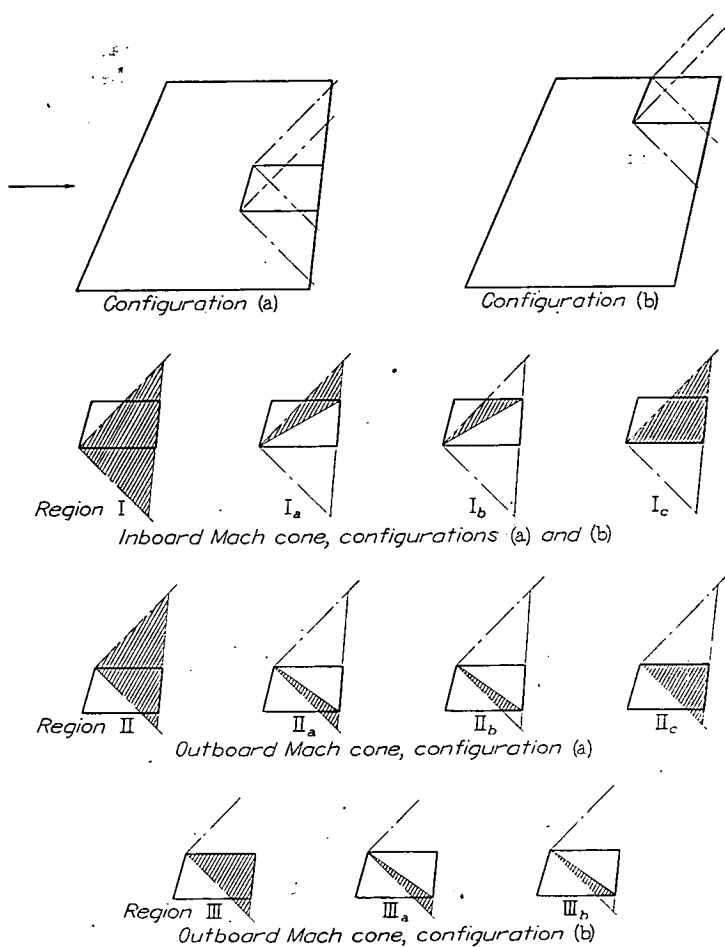


FIGURE 1.—Conical-flow regions for which solutions were obtained in the calculation of deflected control characteristics.

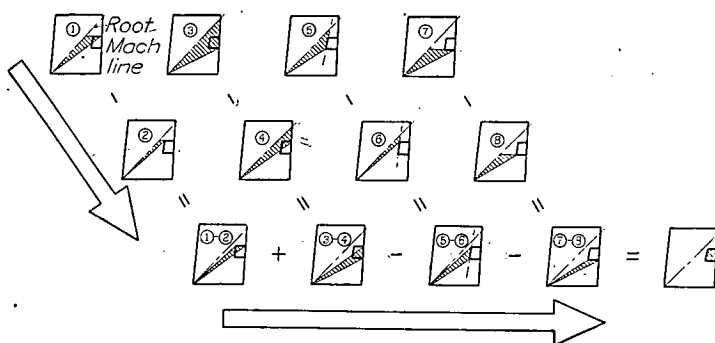


FIGURE 2.—Procedure followed in summing $(PSL\bar{x})^*$ of conical-flow regions for calculation of C_{ha} . (Encircled numbers correspond to regions as designated in computing form for C_{ha} .)

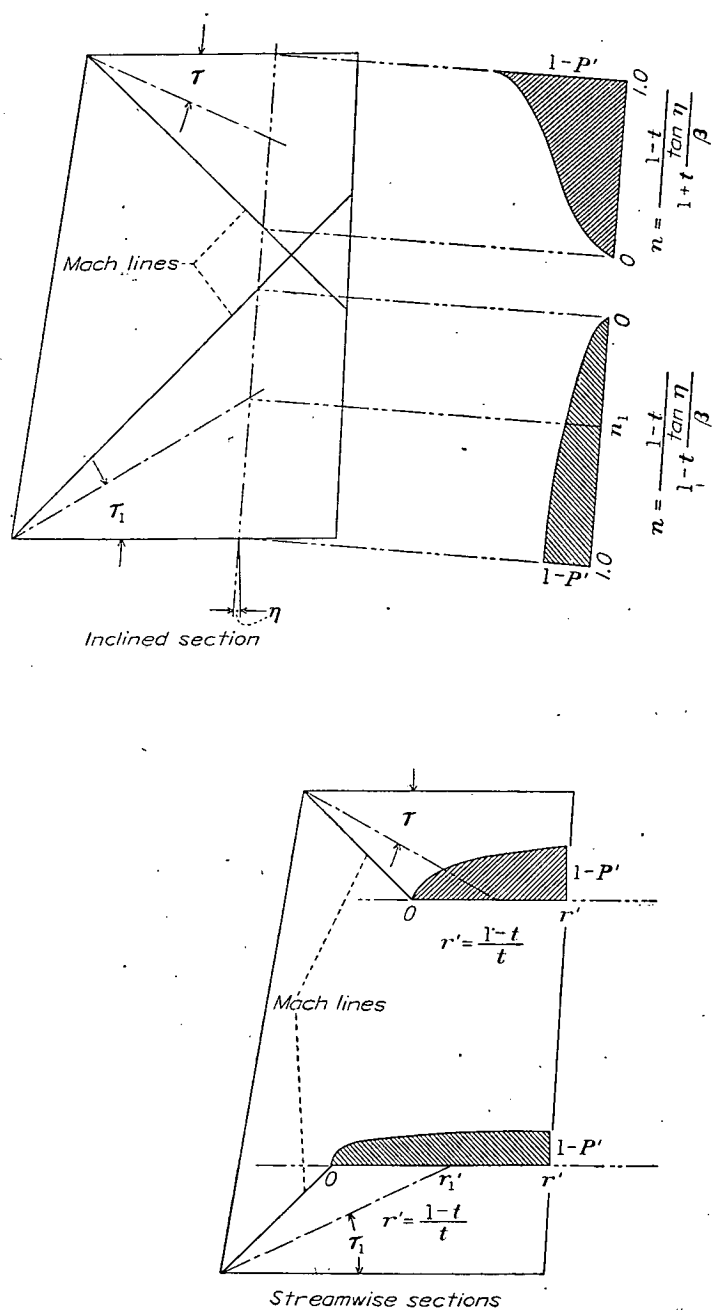


FIGURE 3.—Illustration of method by which P^* and $t_{a,p}$ for triangular segments of the wing-root and wing-tip Mach onese are obtained for use in determining C_{ha} .

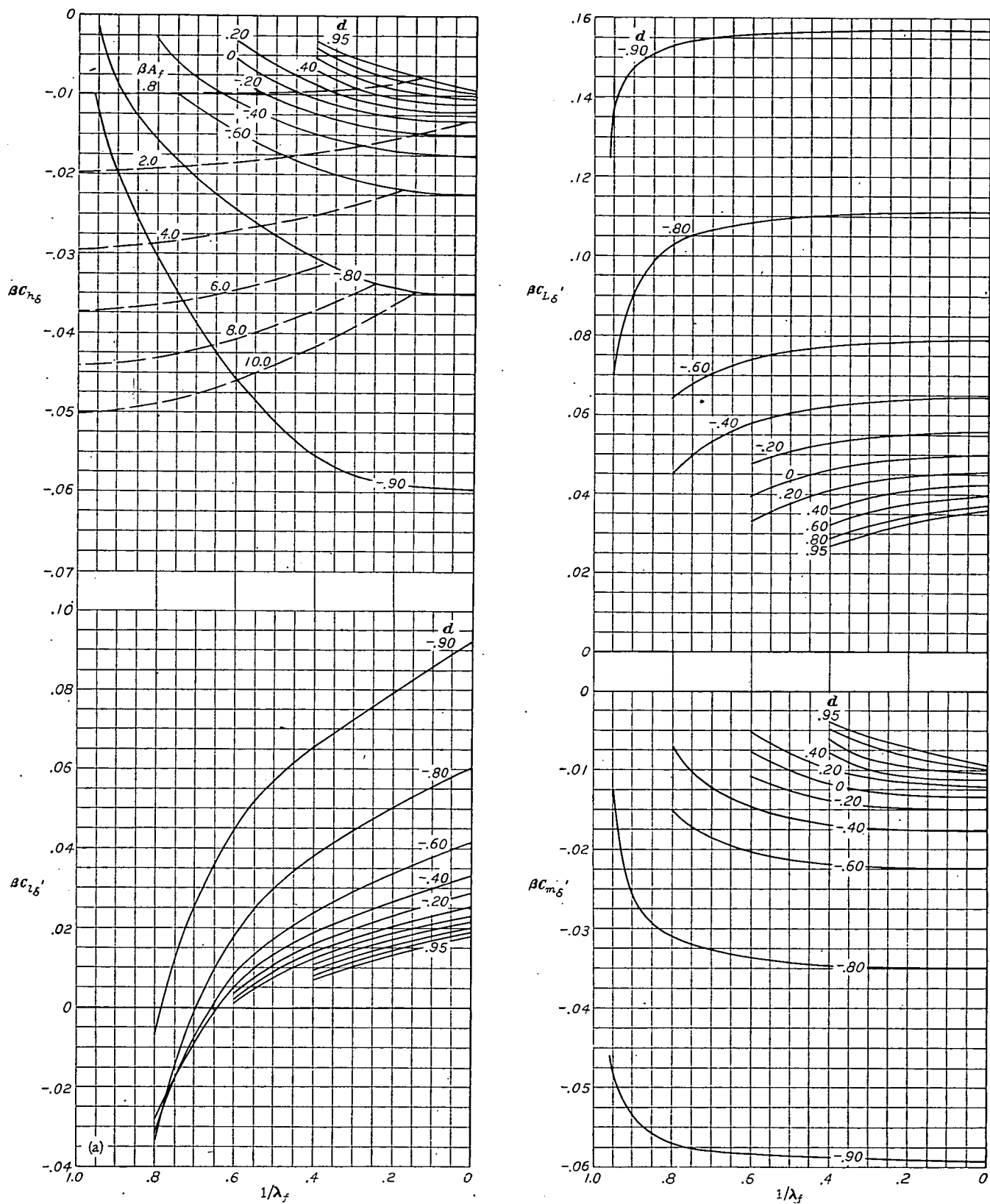
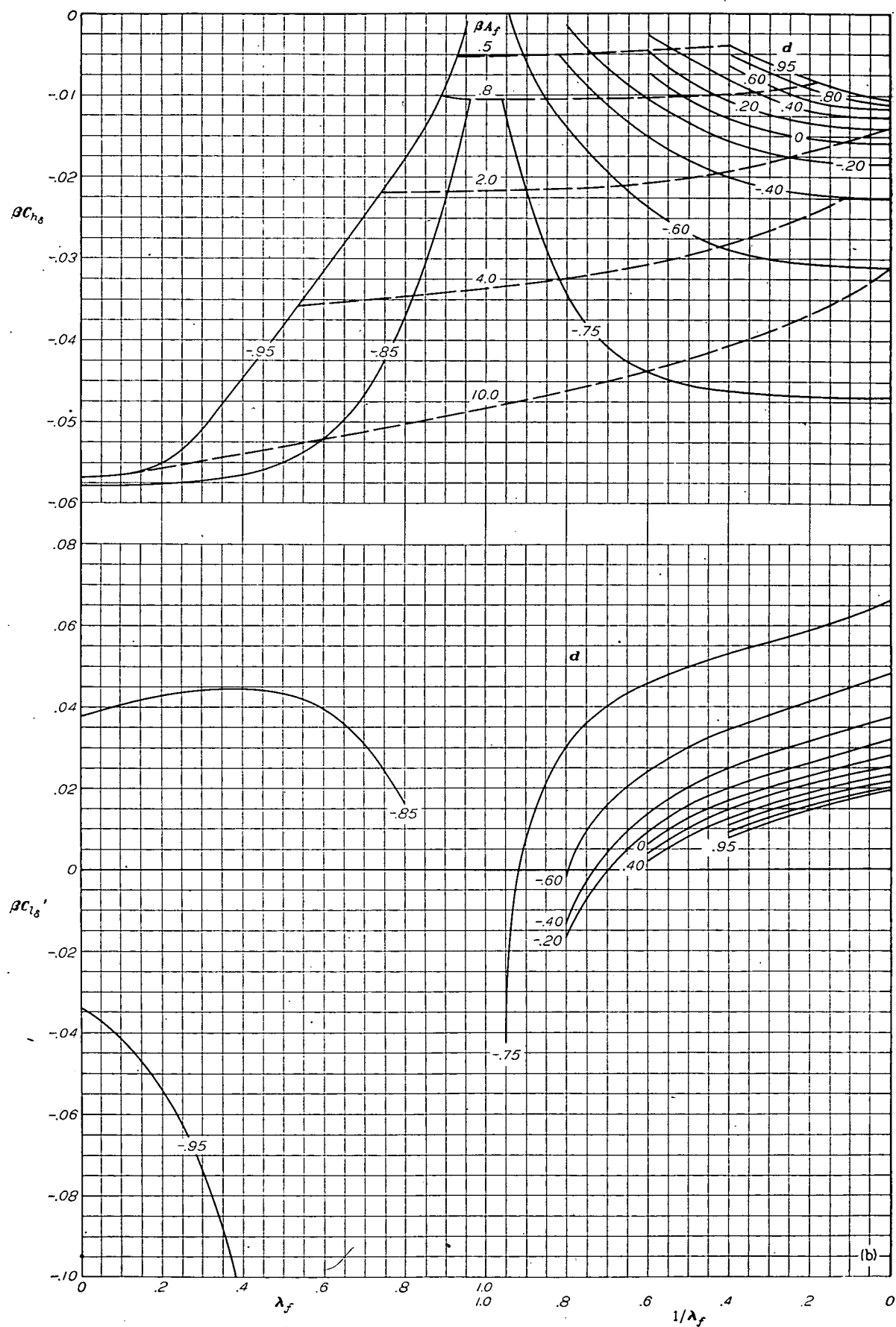
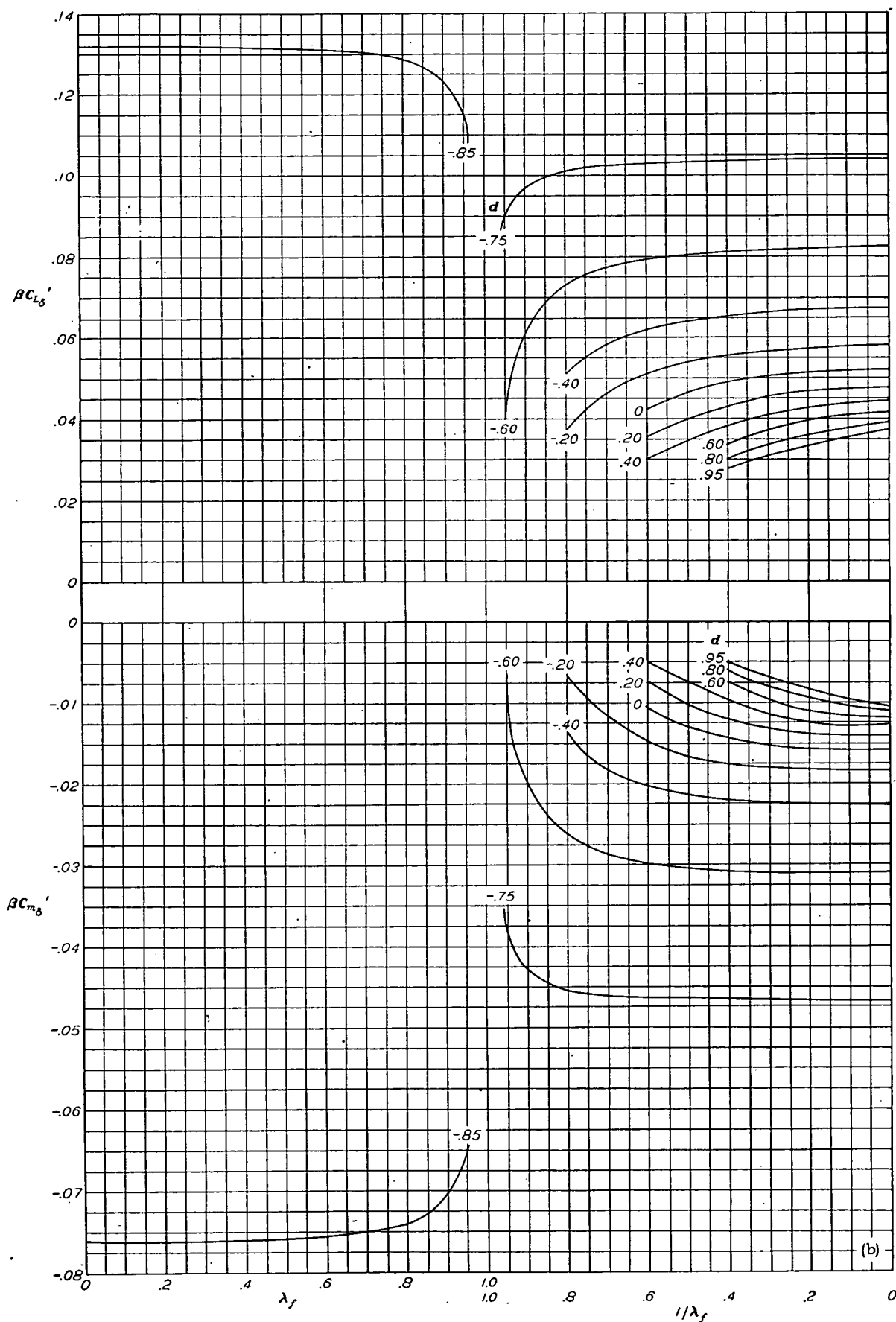


FIGURE 4.—Characteristics of deflected trailing-edge controls located at the wing tip. Results for values of $\lambda_f \geq \frac{1-a}{1-d}$ have been obtained by use of an approximation and should be used with caution.

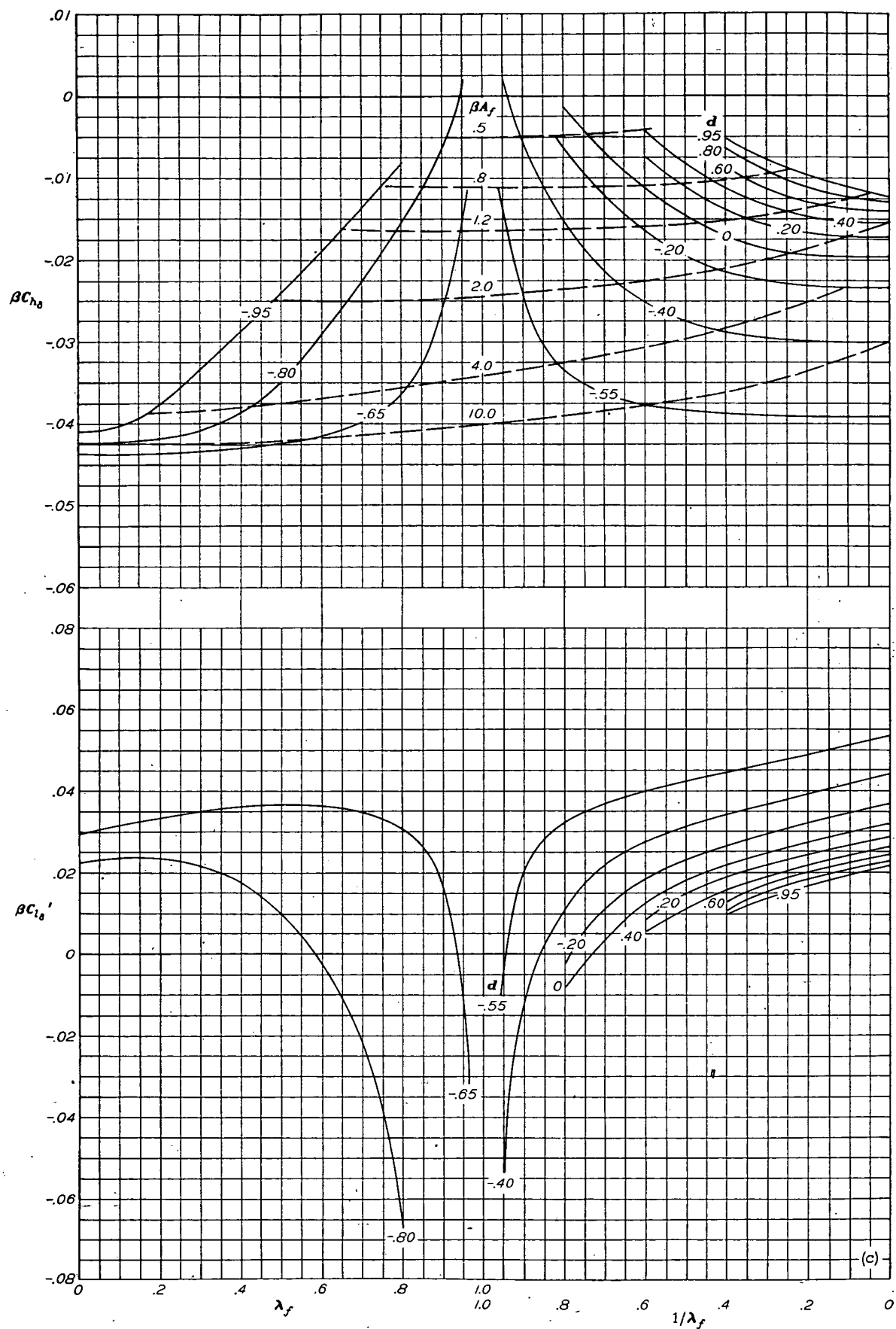


b) $\frac{\tan \Lambda_{HL}}{\beta} = -0.80$.

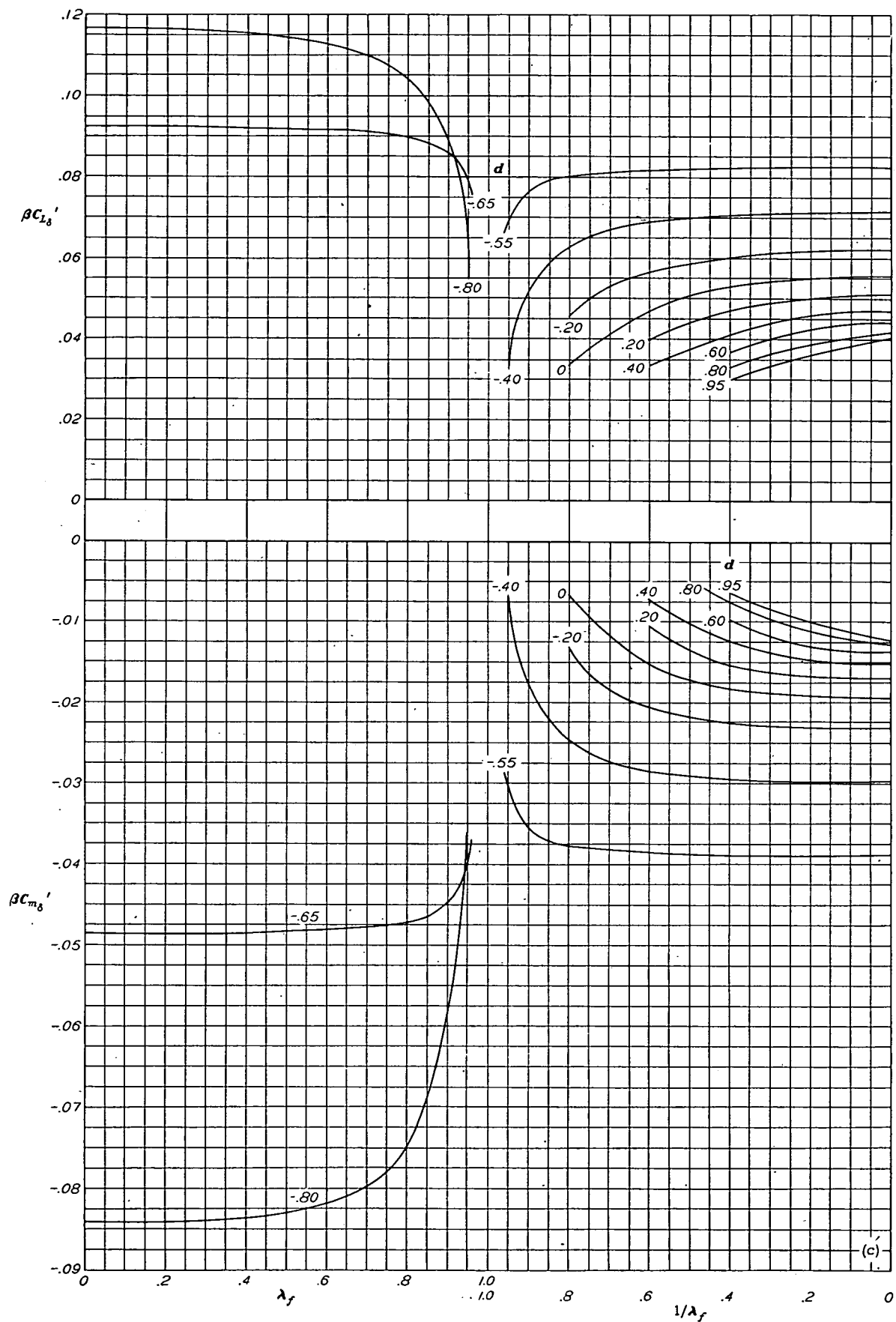
FIGURE 4.—Continued.



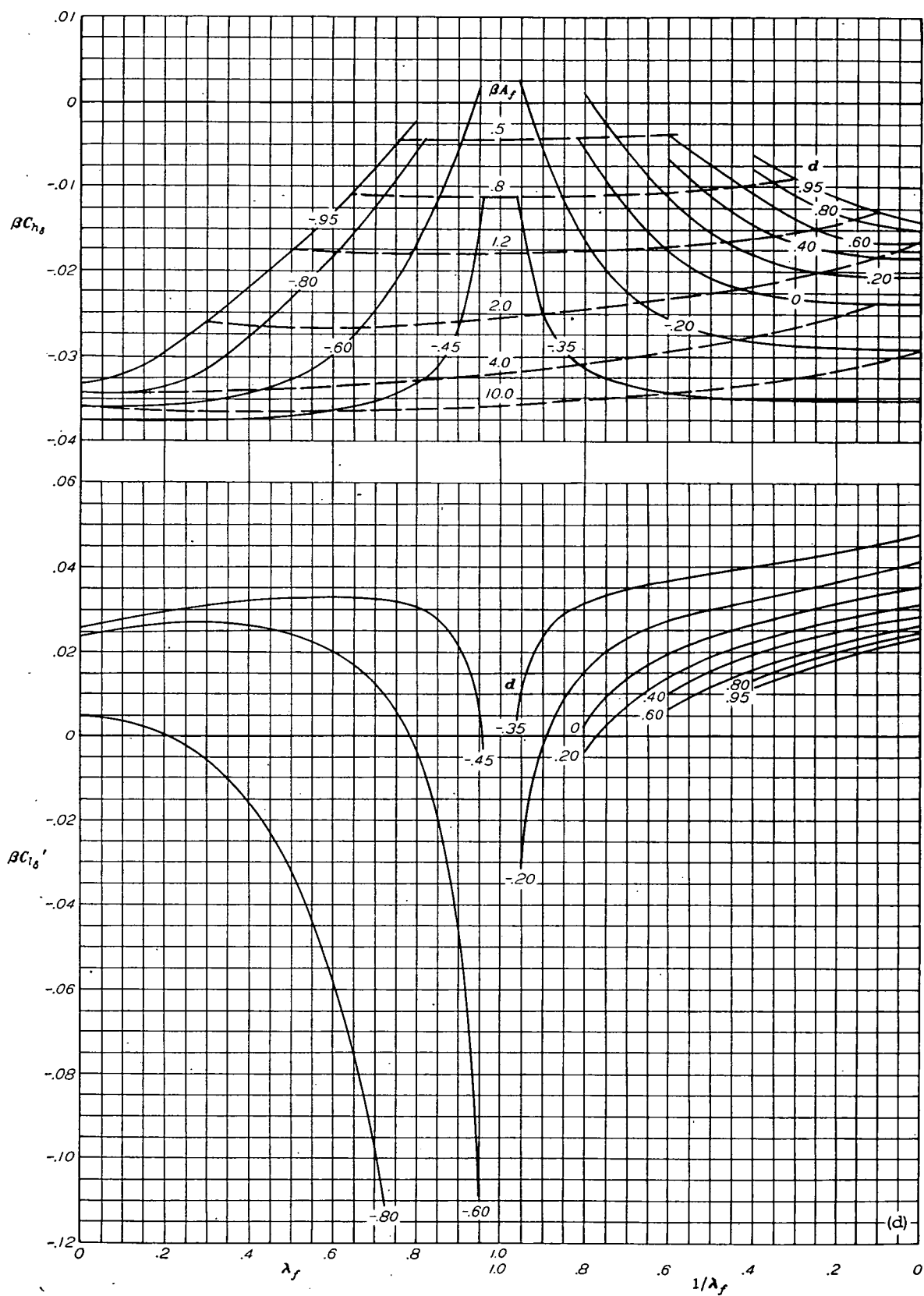
(b) Concluded.
FIGURE 4.—Continued.



(c) $\frac{\tan \Lambda_{BL}}{\beta} = -0.60$.
 FIGURE 4.—Continued.

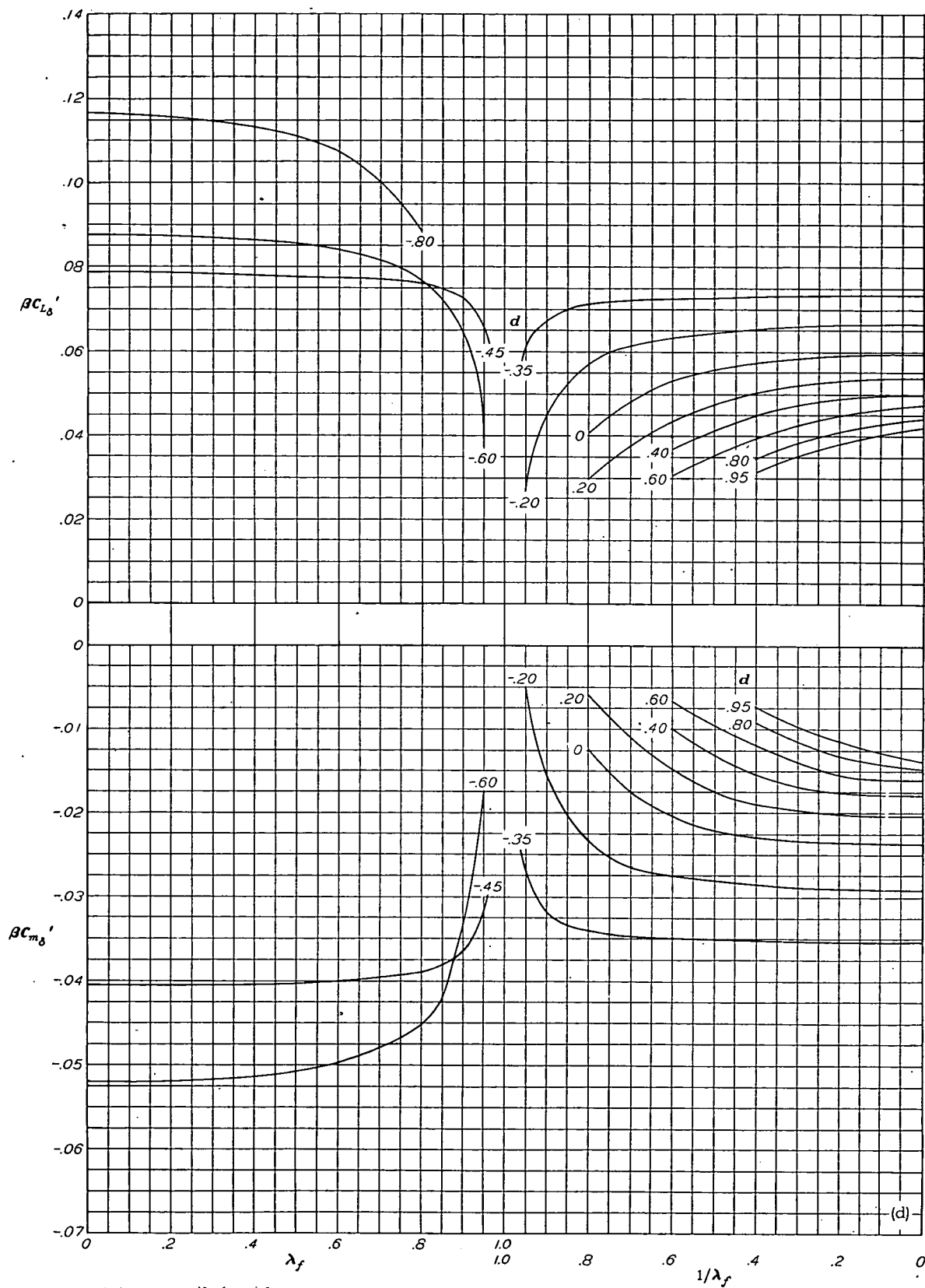


(c) Concluded.
FIGURE 4.—Continued.

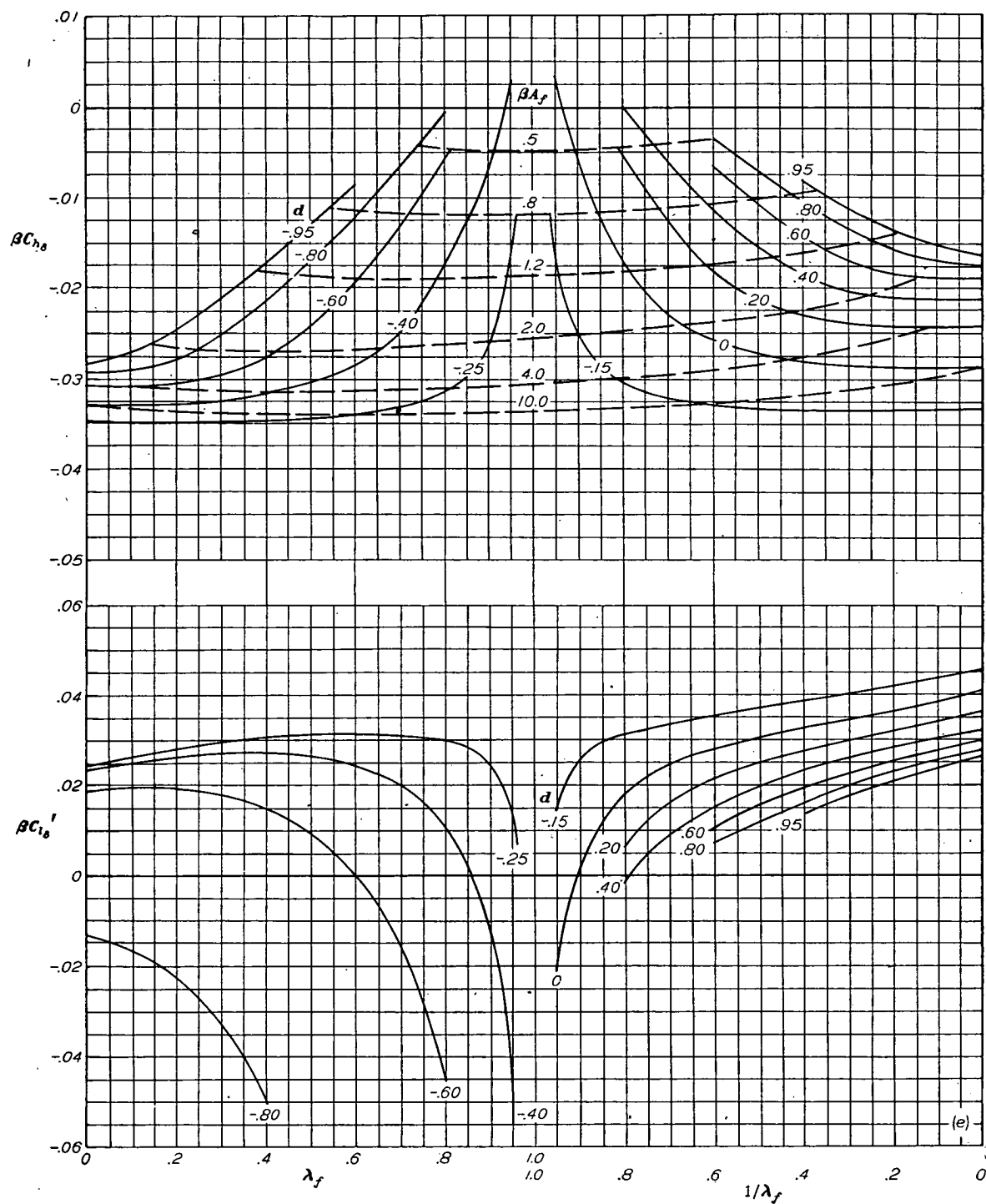


(d) $\frac{\tan \Lambda_{HL}}{\beta} = -0.40$.

FIGURE 4.—Continued.

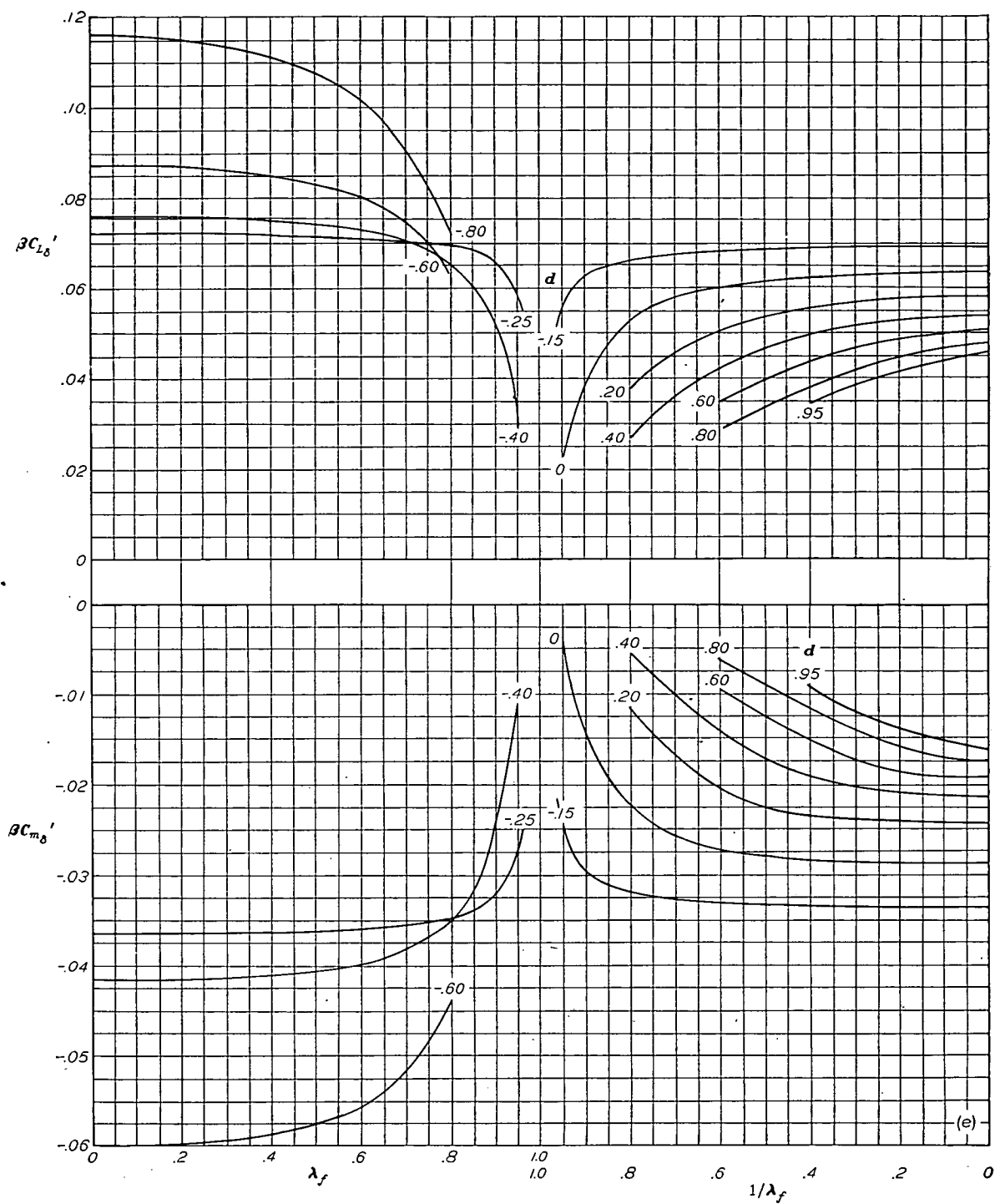


(d) Concluded.
FIGURE 4.—Continued.

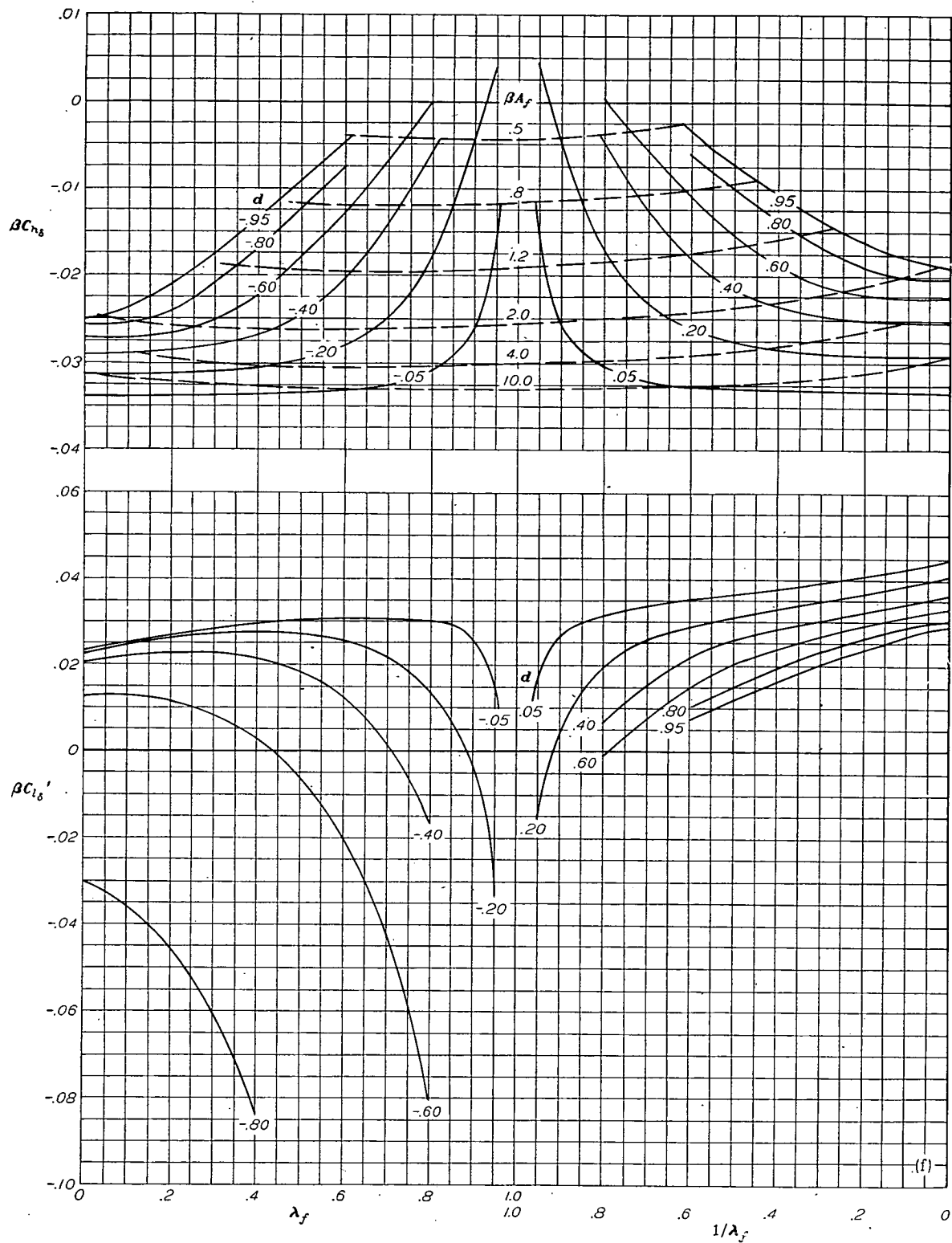


(e). $\frac{\tan \Delta_{HL}}{\beta} = -0.20$.

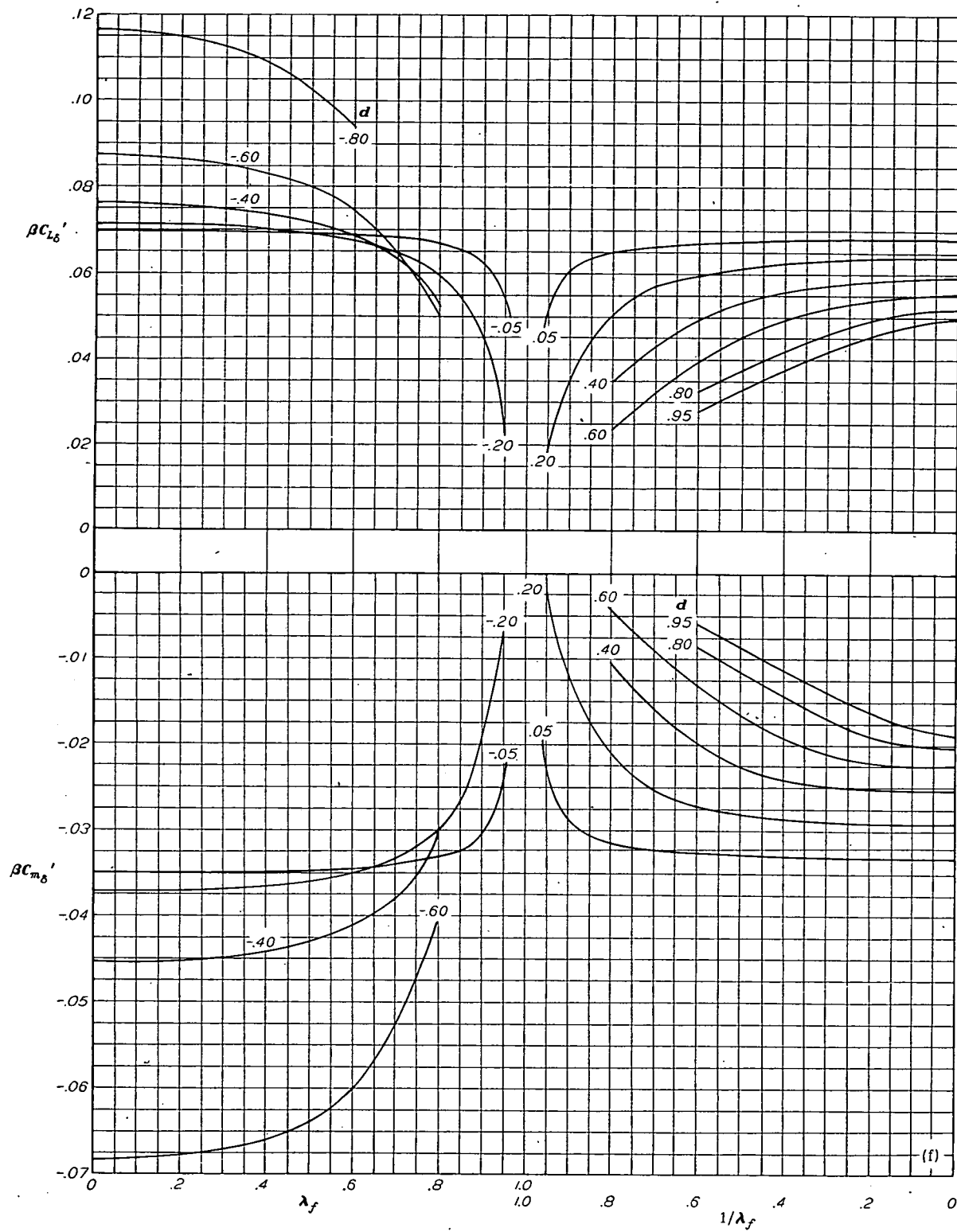
FIGURE 4.—Continued.



(e) Concluded.
FIGURE 4.—Continued.



(f) $\frac{\tan \Lambda_{HL}}{\beta} = 0$.
FIGURE 4.—Continued.



(f) Concluded.
FIGURE 4.—Continued.

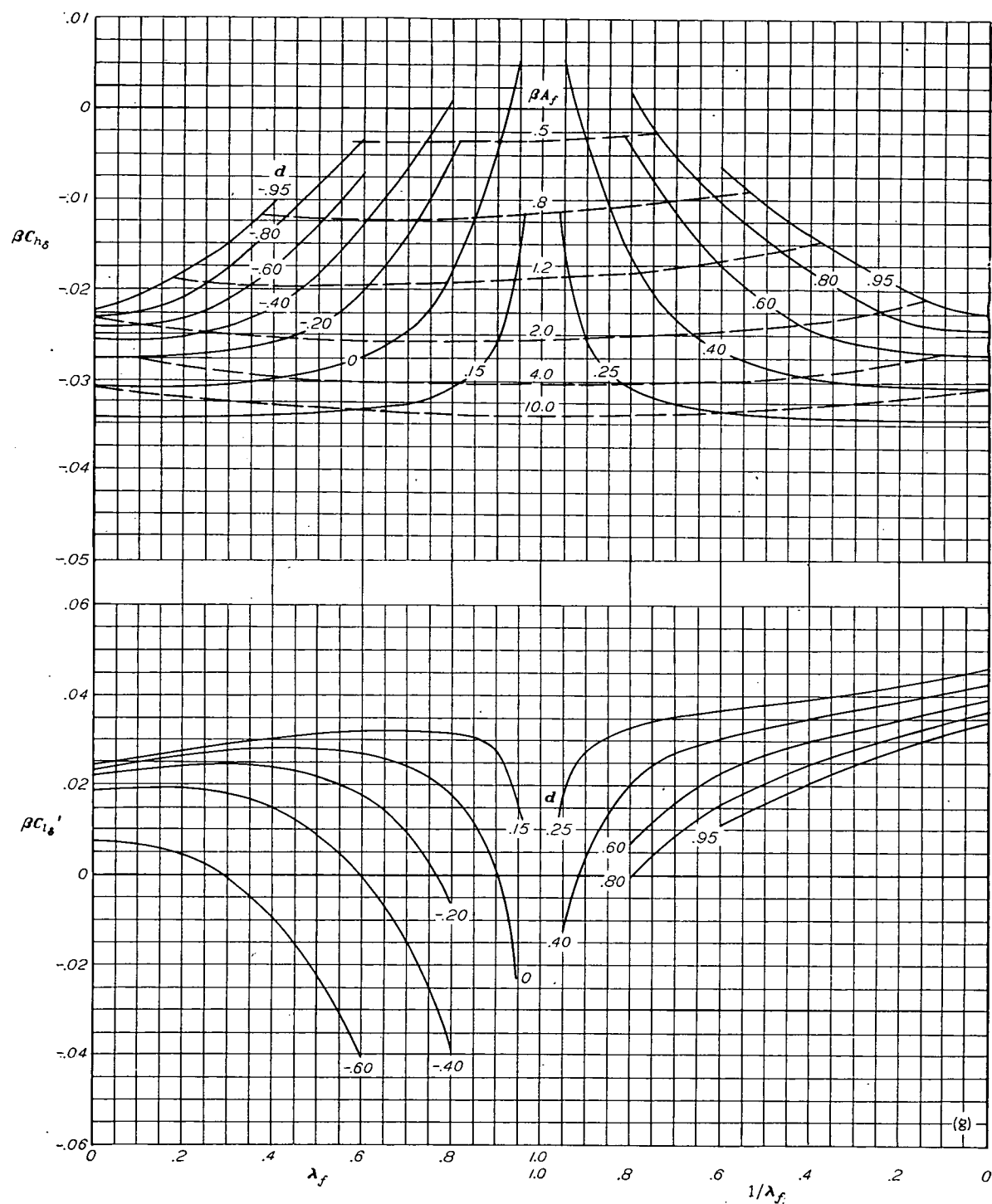
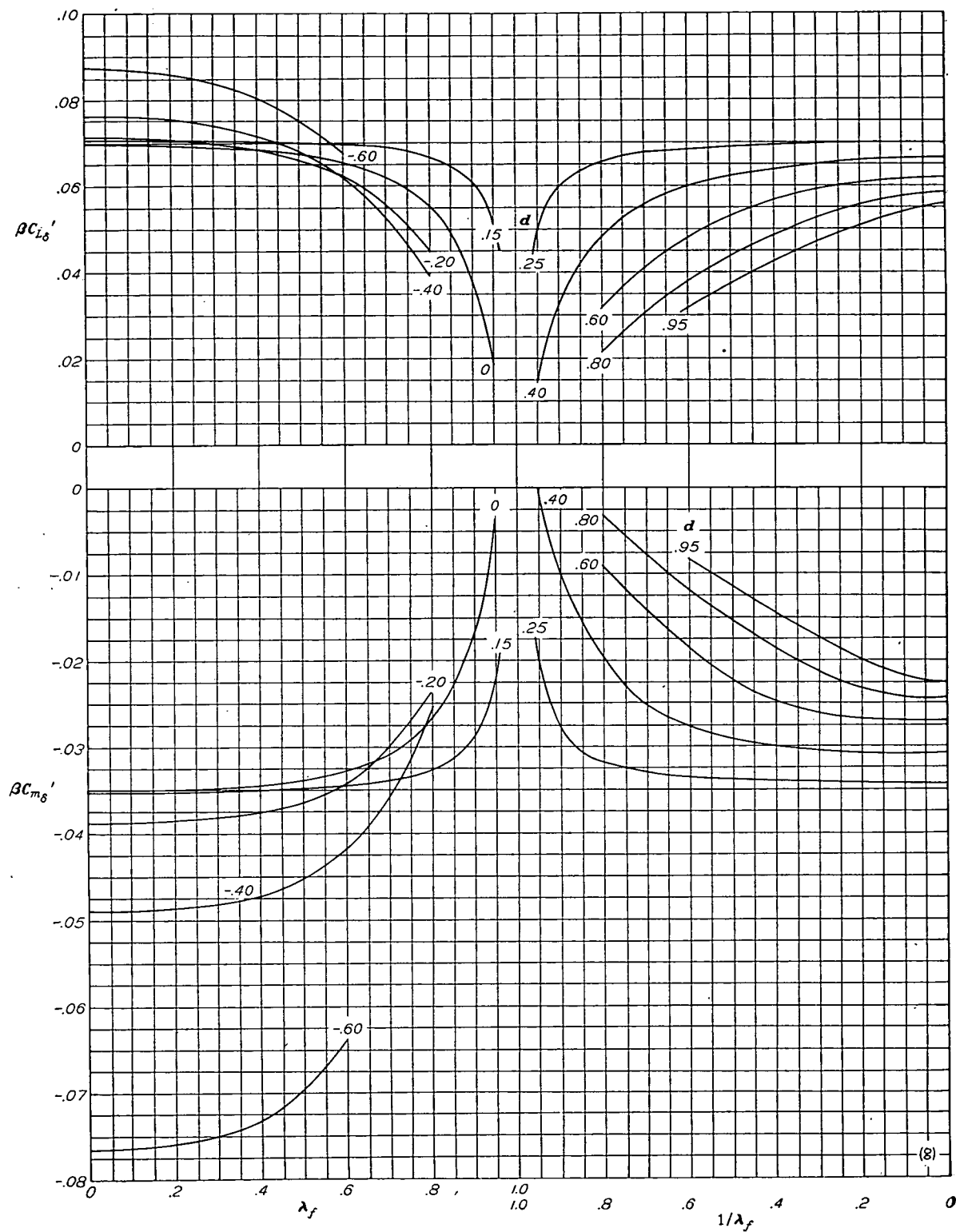
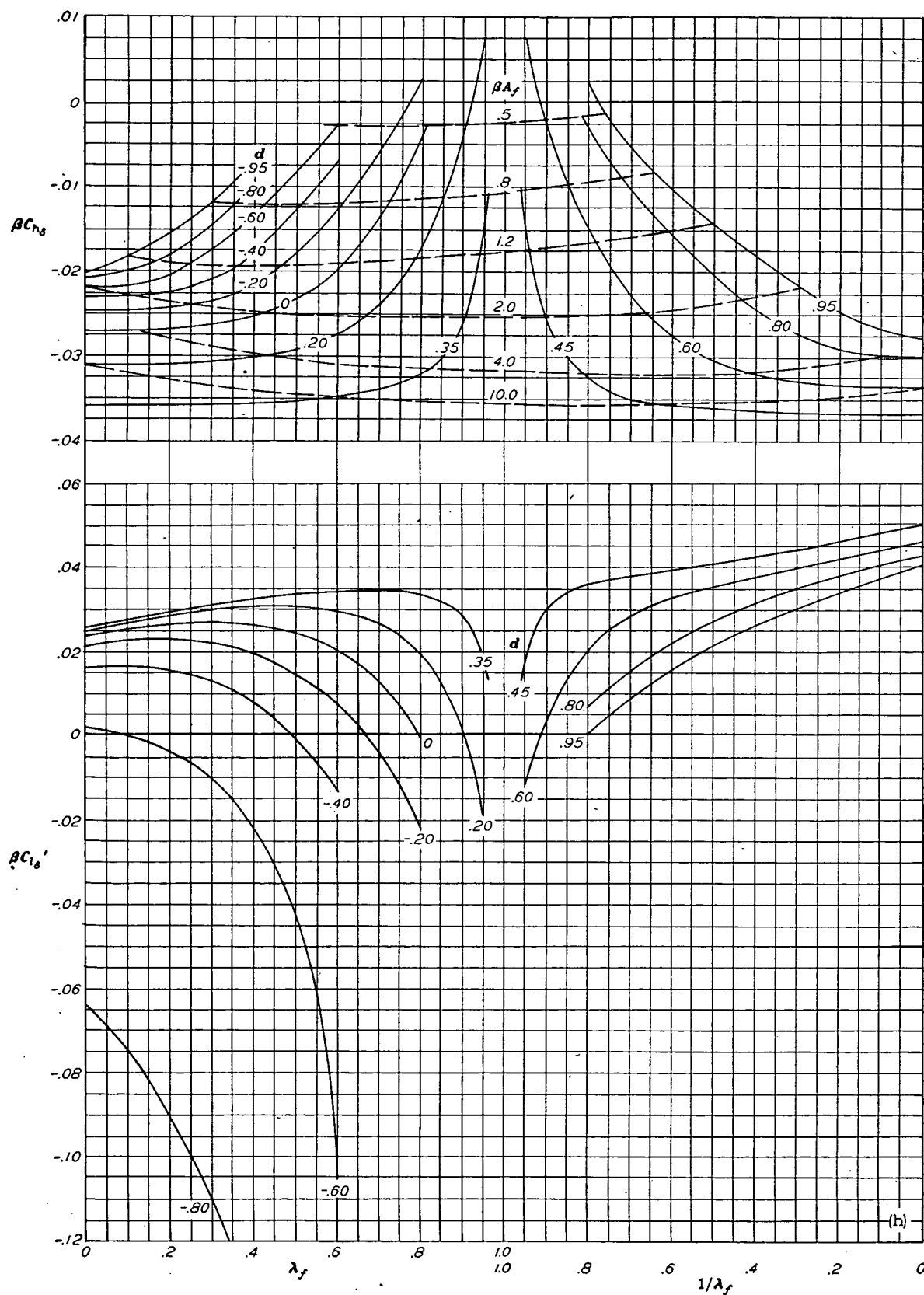


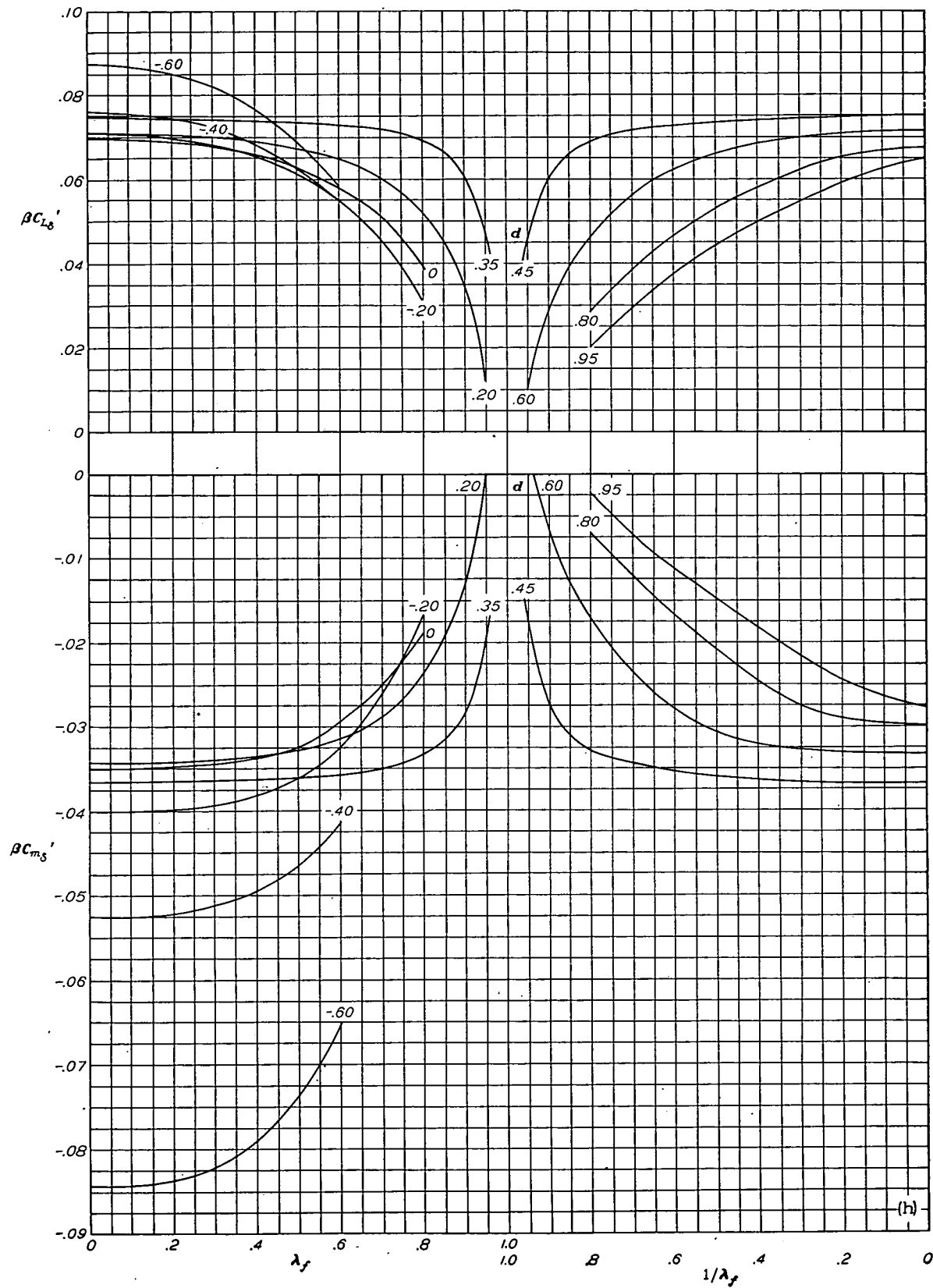
FIGURE 4.—Continued.



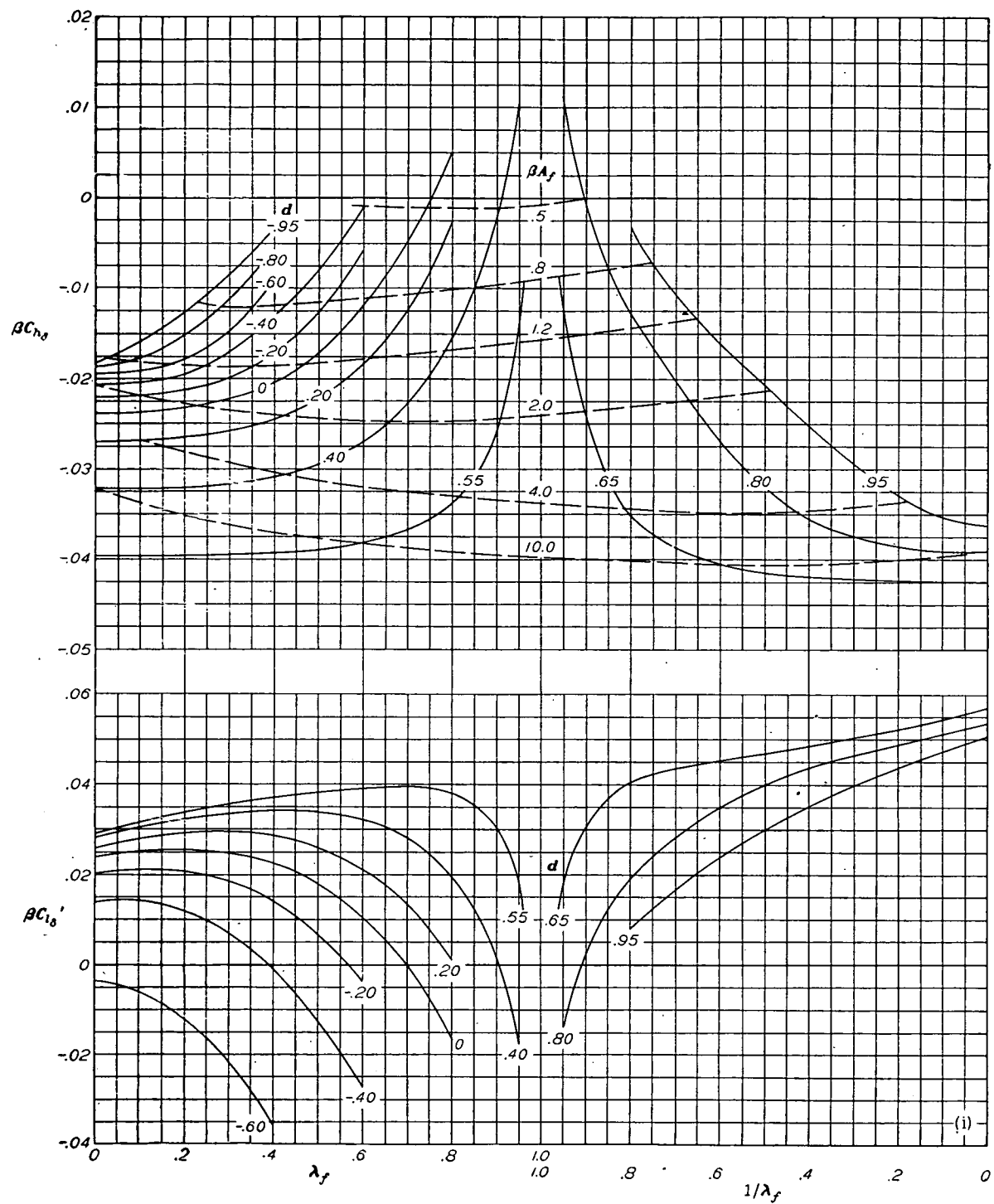
(g) Concluded.
FIGURE 4.—Continued.



(h) $\frac{\tan \Lambda_{BL}}{\beta} = 0.40$.
 FIGURE 4.—Continued.



(h) Concluded.
FIGURE 4.—Continued.



(i) $\frac{\tan \Delta_{HL}}{\beta} = 0.60$.
 FIGURE 4.—Continued.

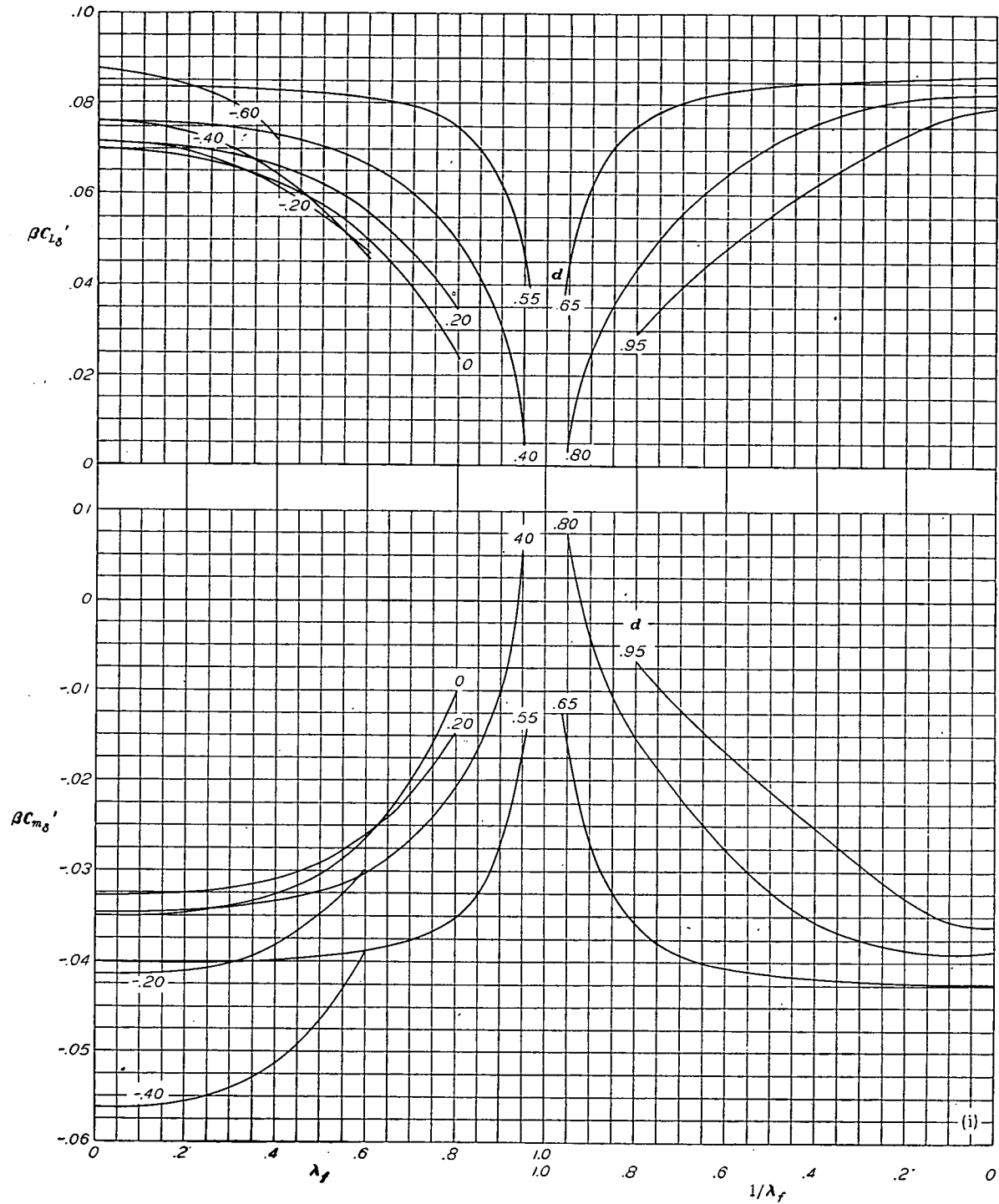
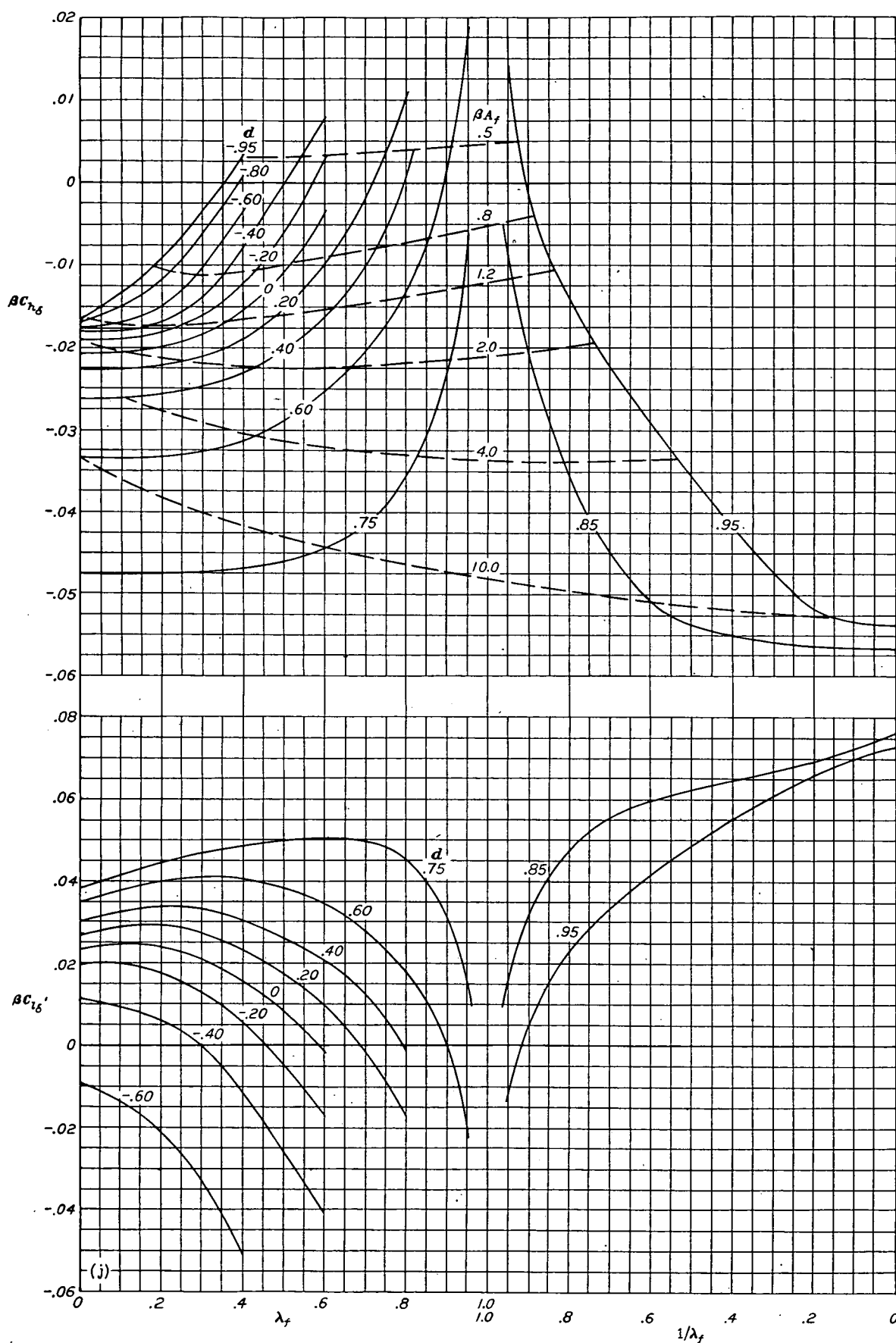
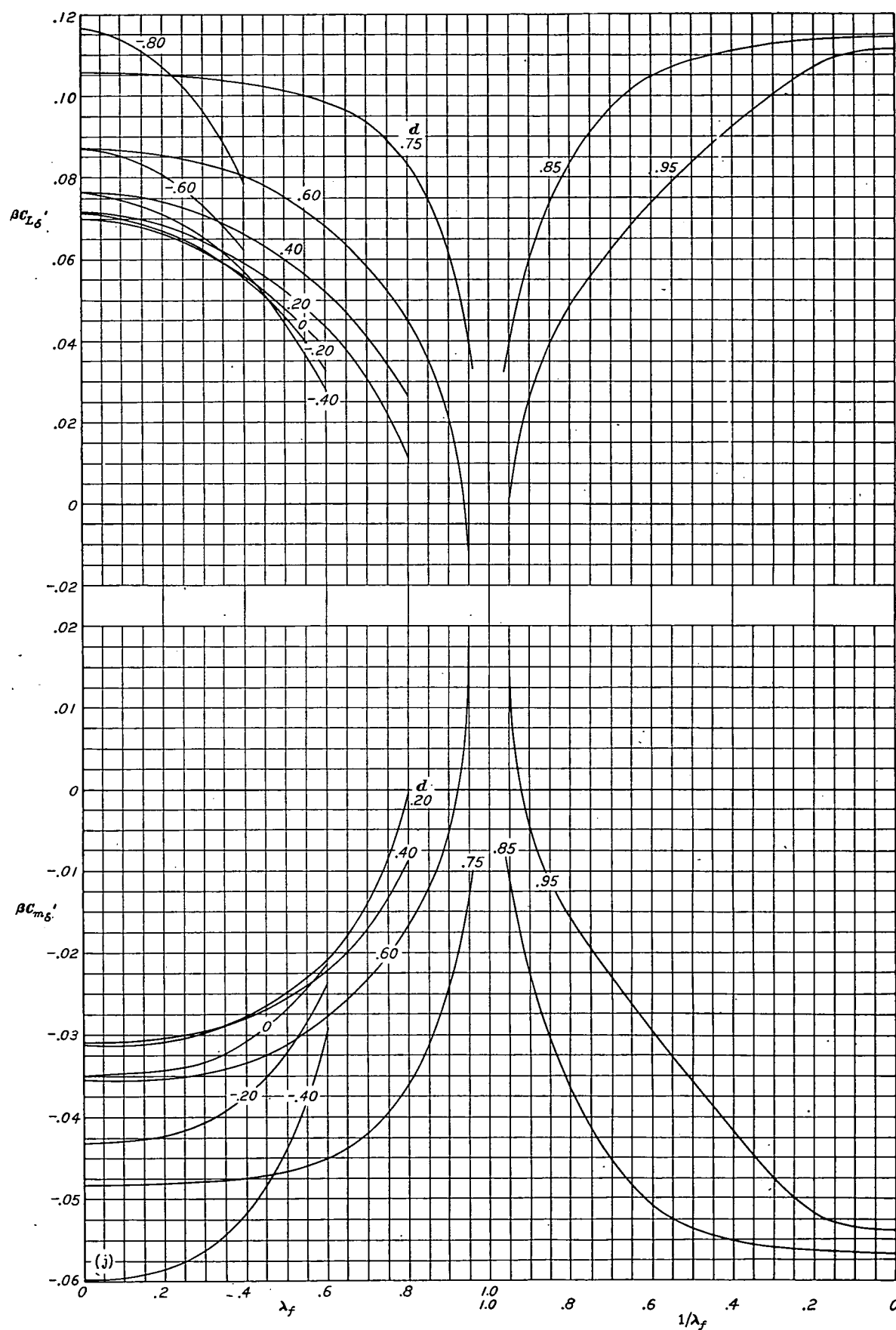


FIGURE 4.—Continued.

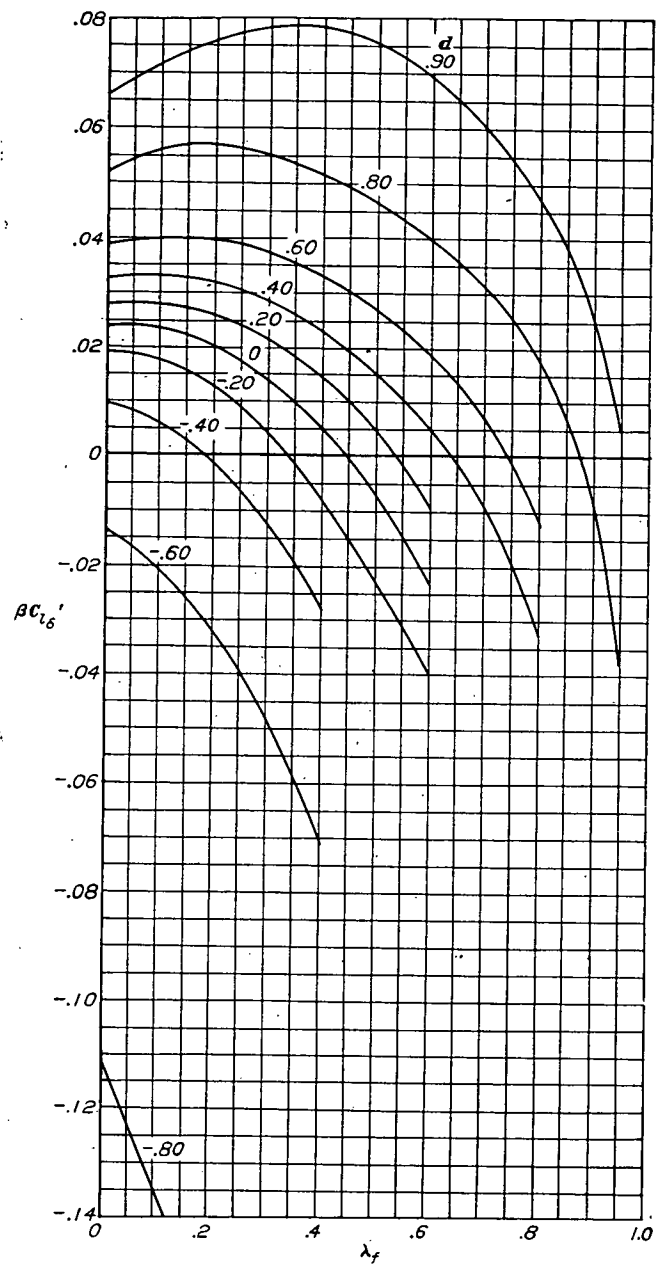
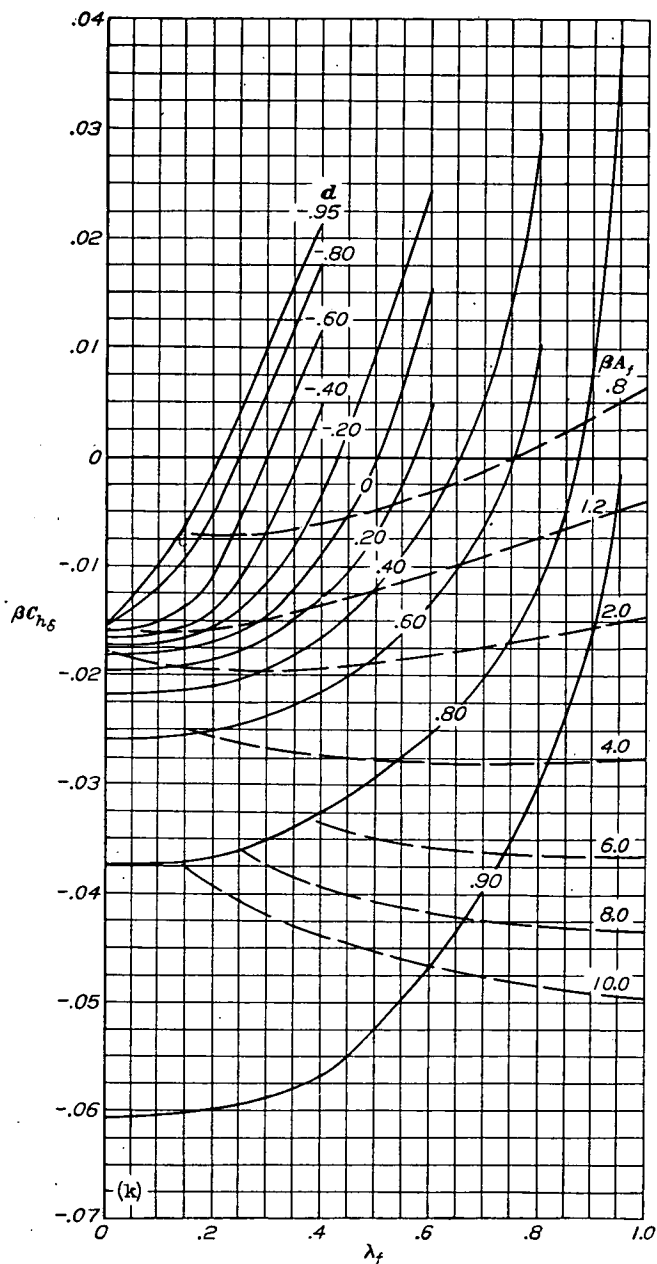


(j) $\frac{\tan \Lambda_{HL}}{\beta} = 0.80$.

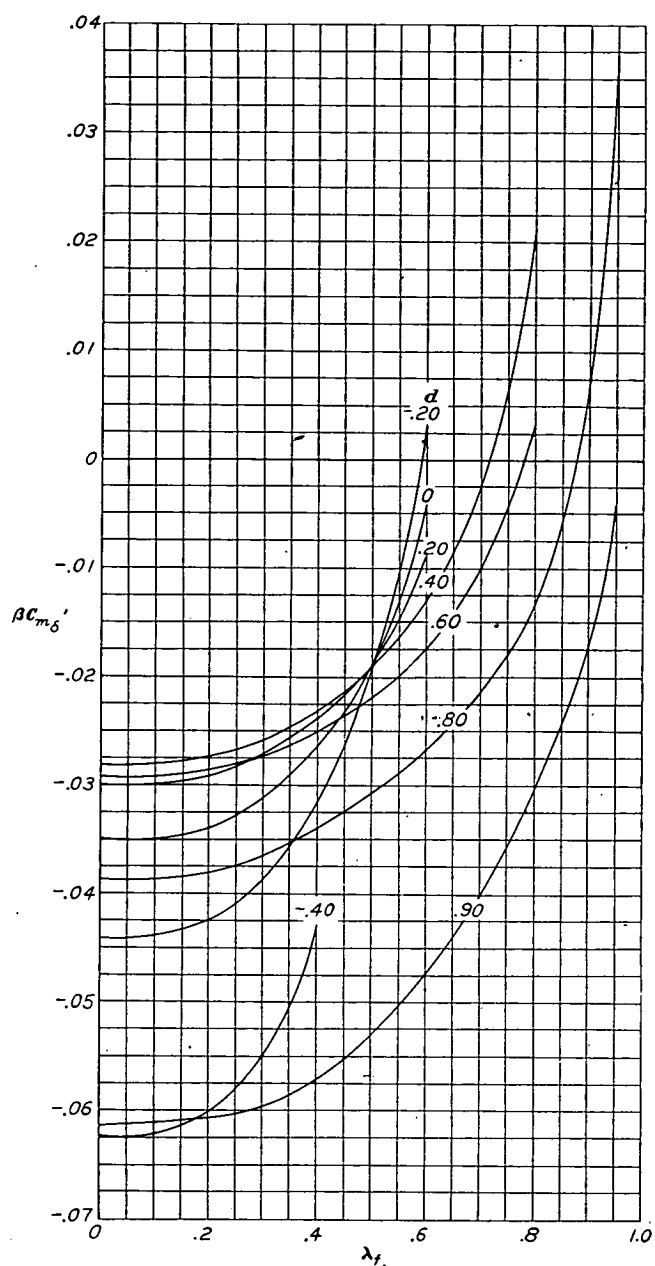
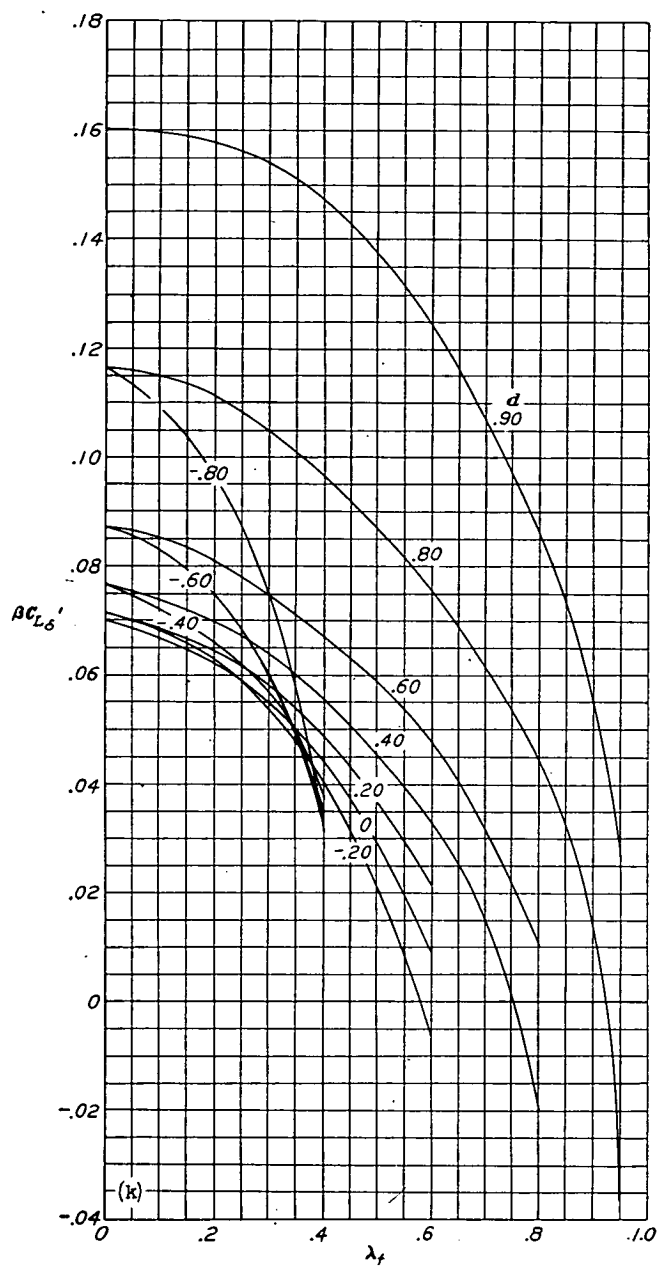
FIGURE 4.—Continued.



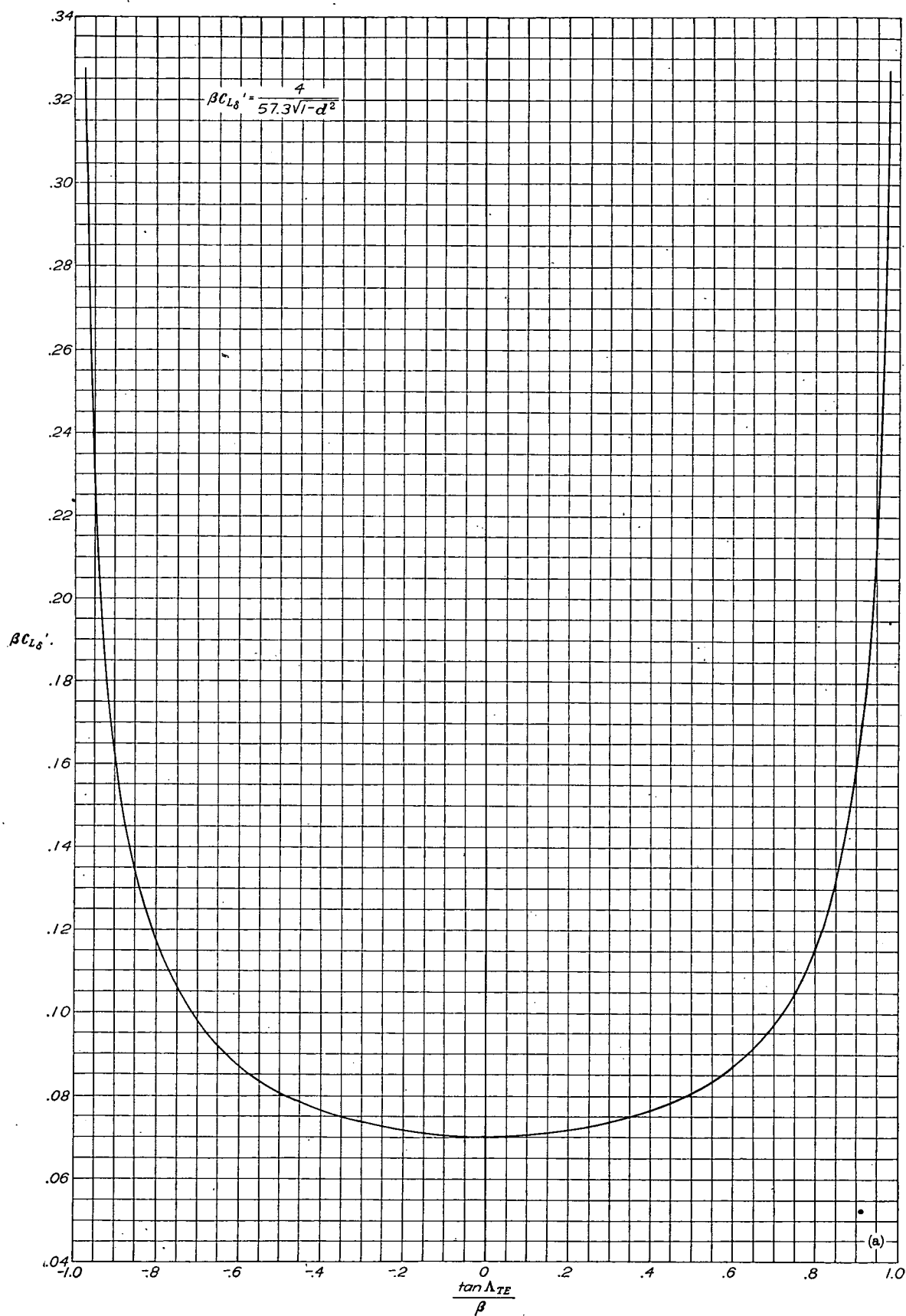
(j) Concluded.
FIGURE 4.—Continued.



(k) $\frac{\tan \Lambda_{HL}}{\beta} = 0.95$.
FIGURE 4.—Continued.

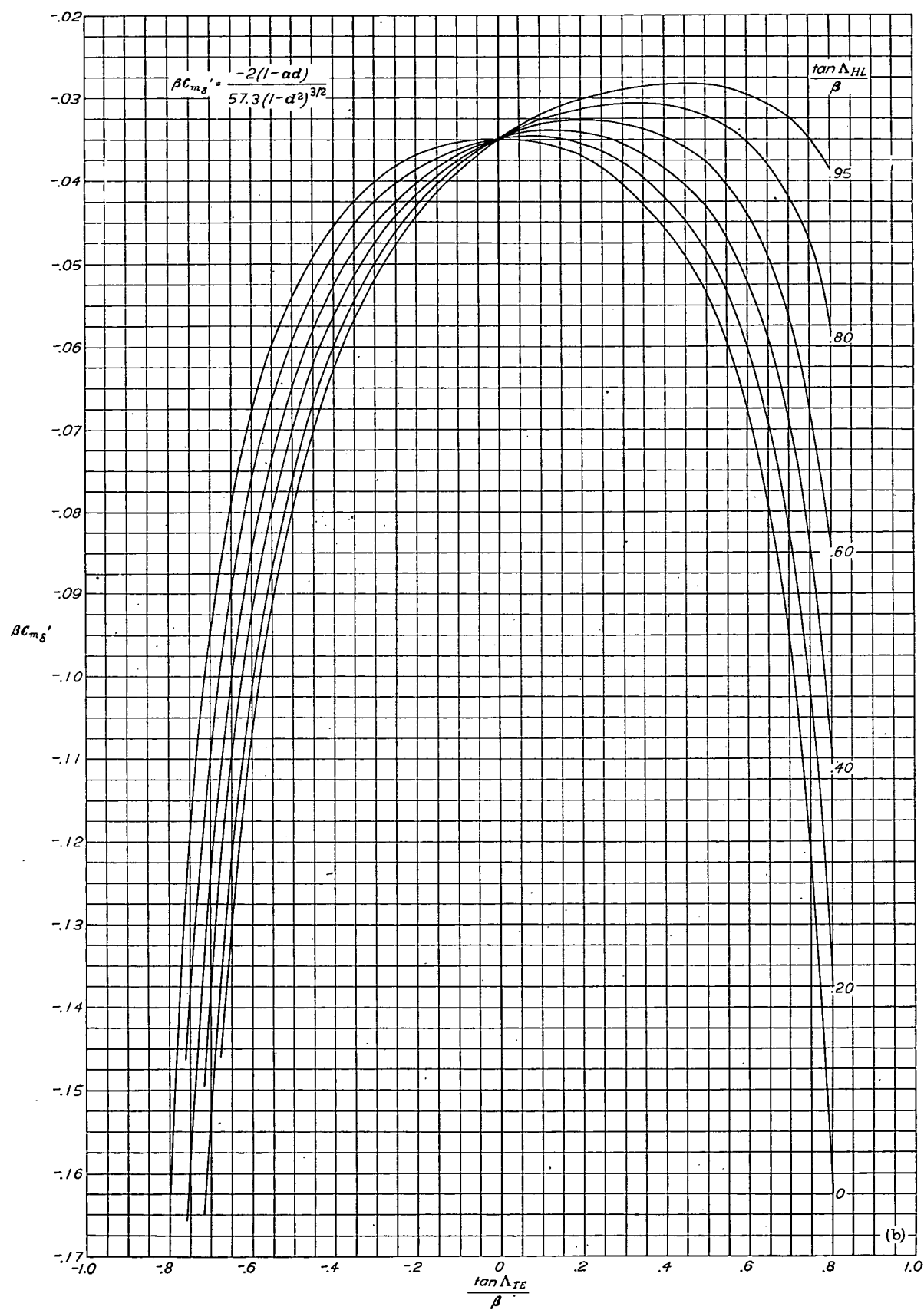


(k) Concluded.
FIGURE 4.—Concluded.



(a) Lift coefficient.

FIGURE 5.—Lift and pitching-moment parameters for deflected trailing-edge flaps located inboard from wing tip.



(b) Pitching-moment coefficient.
FIGURE 5.—Concluded.

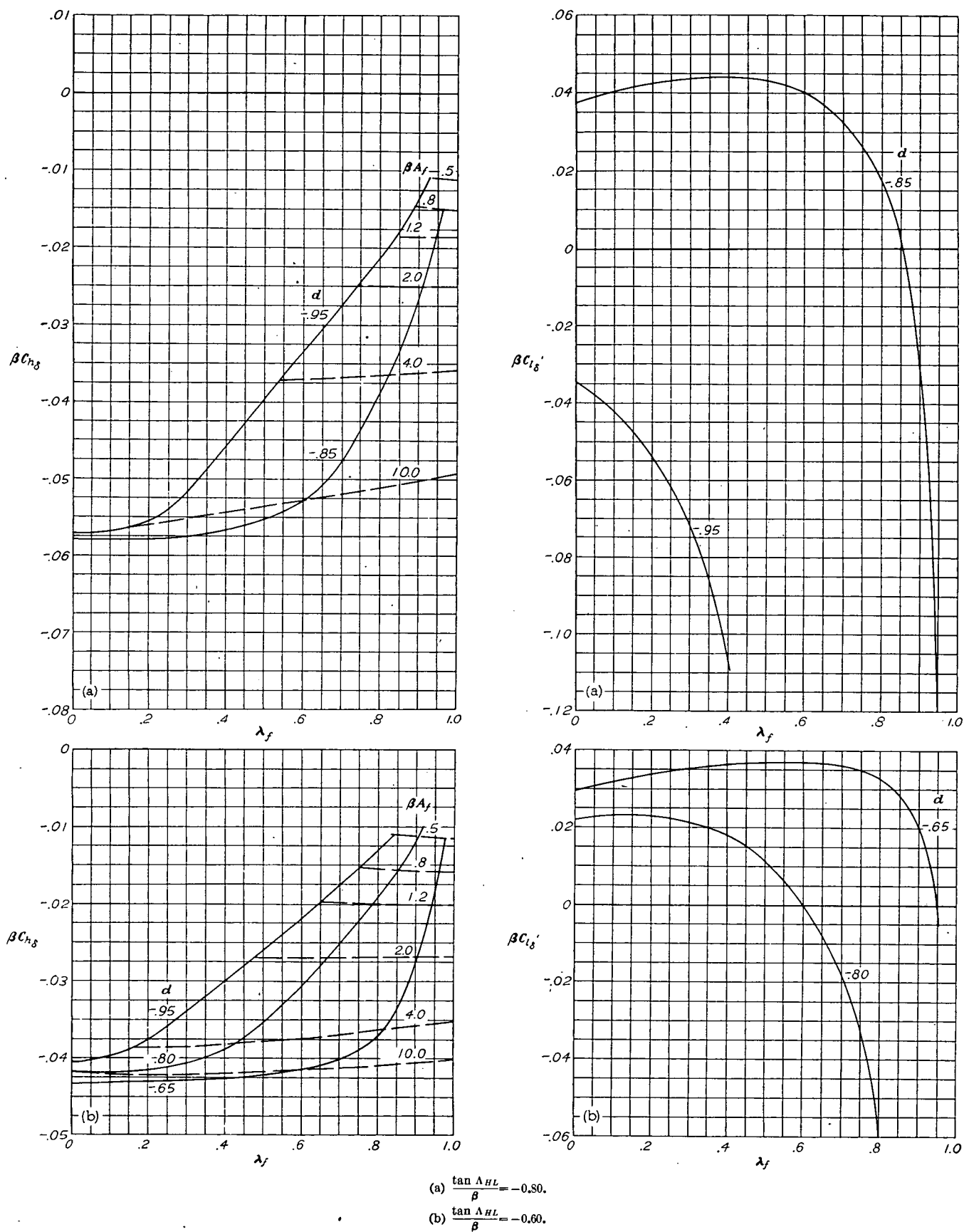
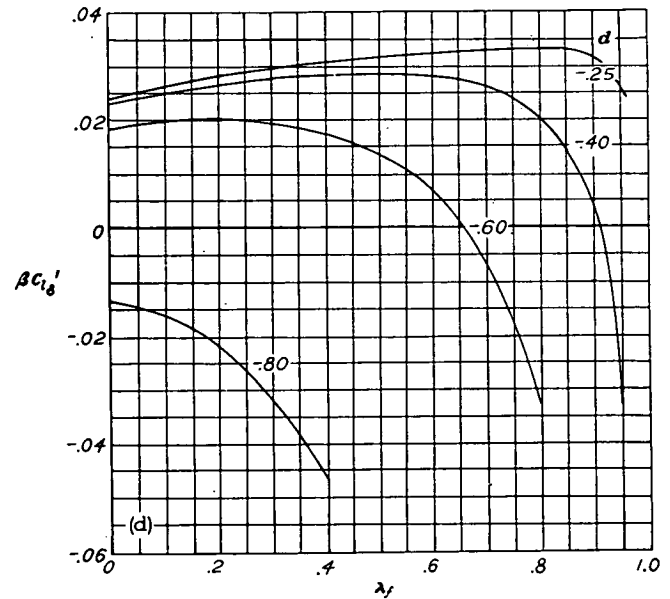
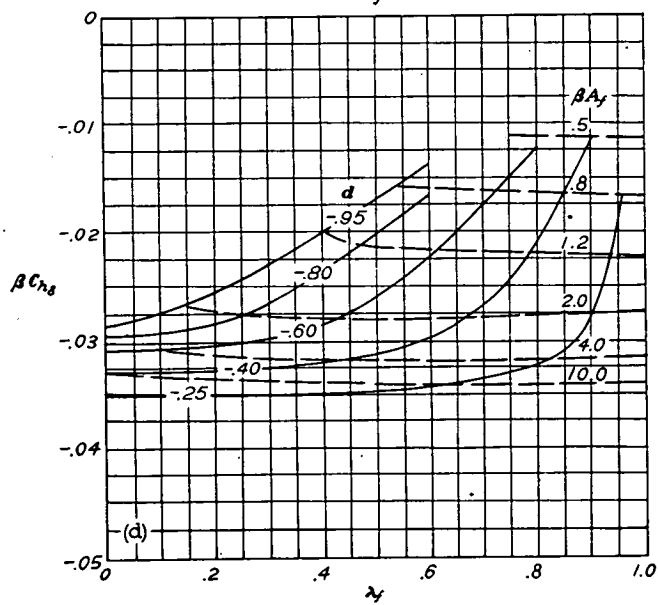
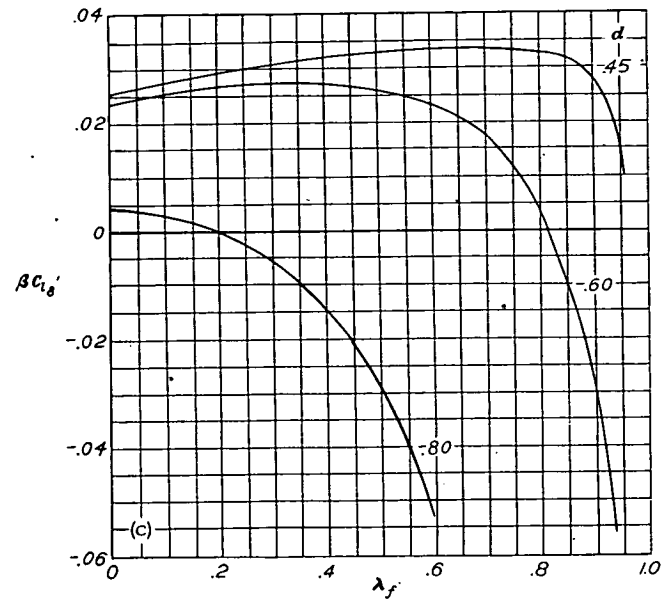
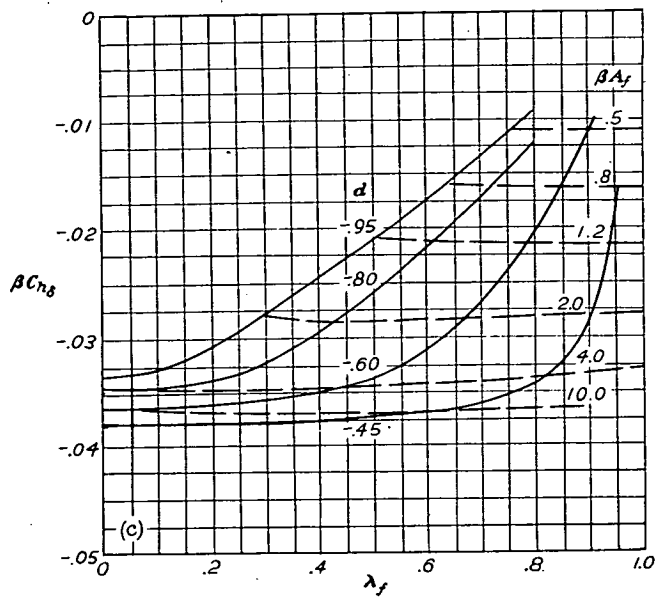


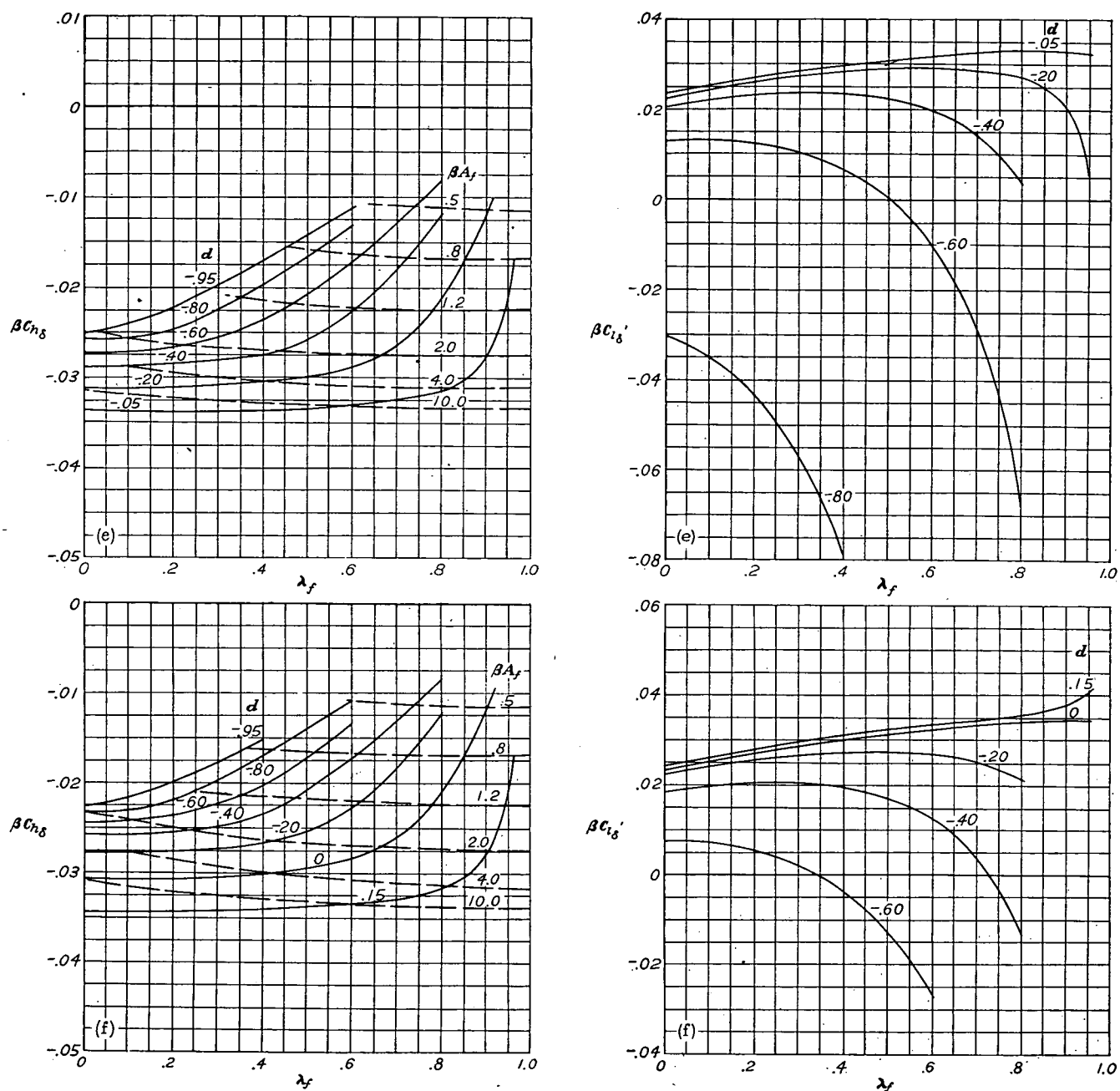
FIGURE 6.—Hinge-moment and rolling-moment parameters for control surfaces located inboard from wing tip.



(c) $\frac{\tan \Delta_{HL}}{\beta} = -0.40.$

(d) $\frac{\tan \Delta_{HL}}{\beta} = -0.20.$

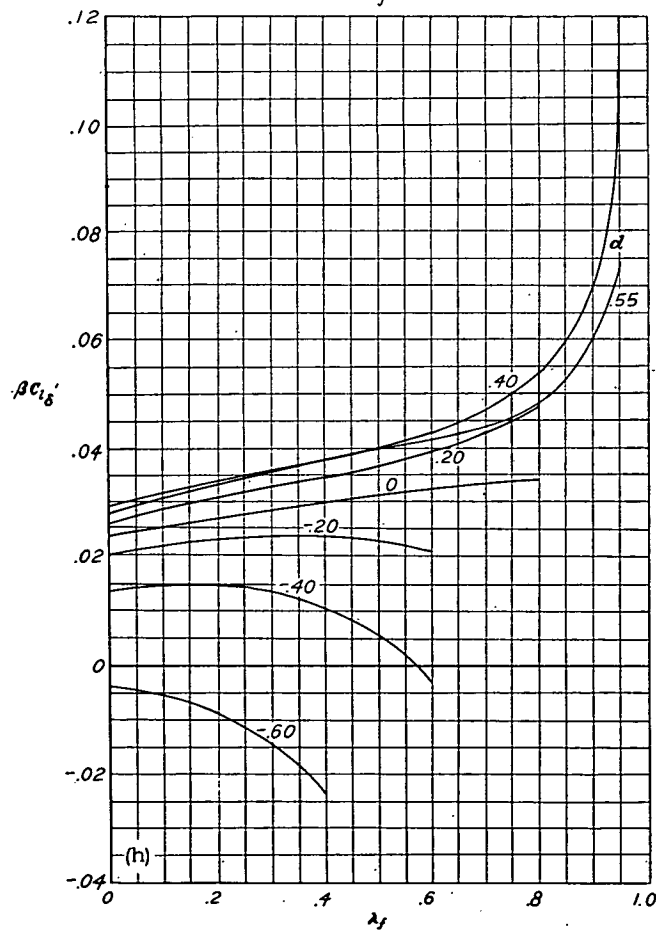
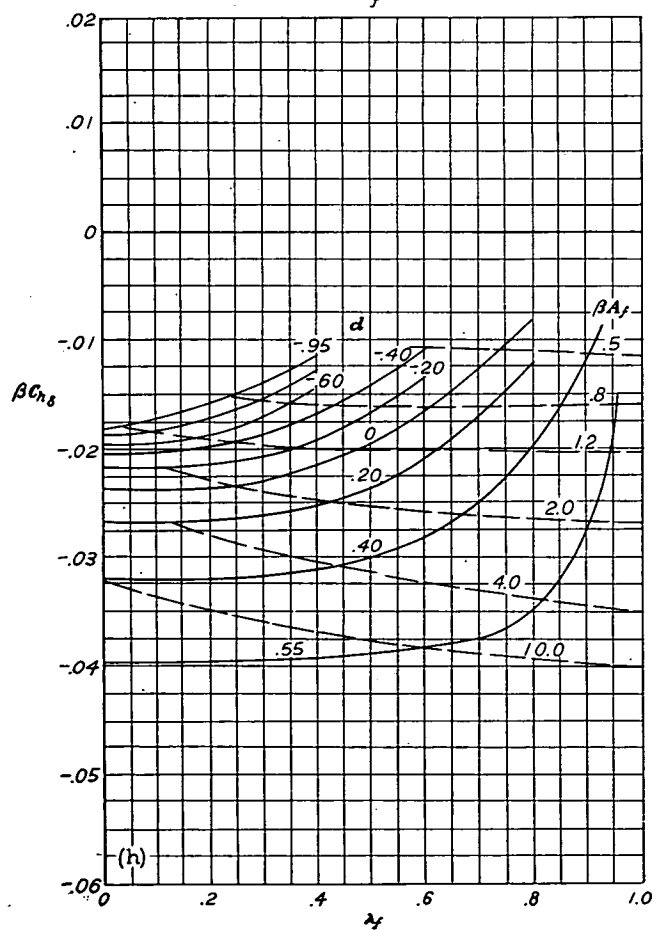
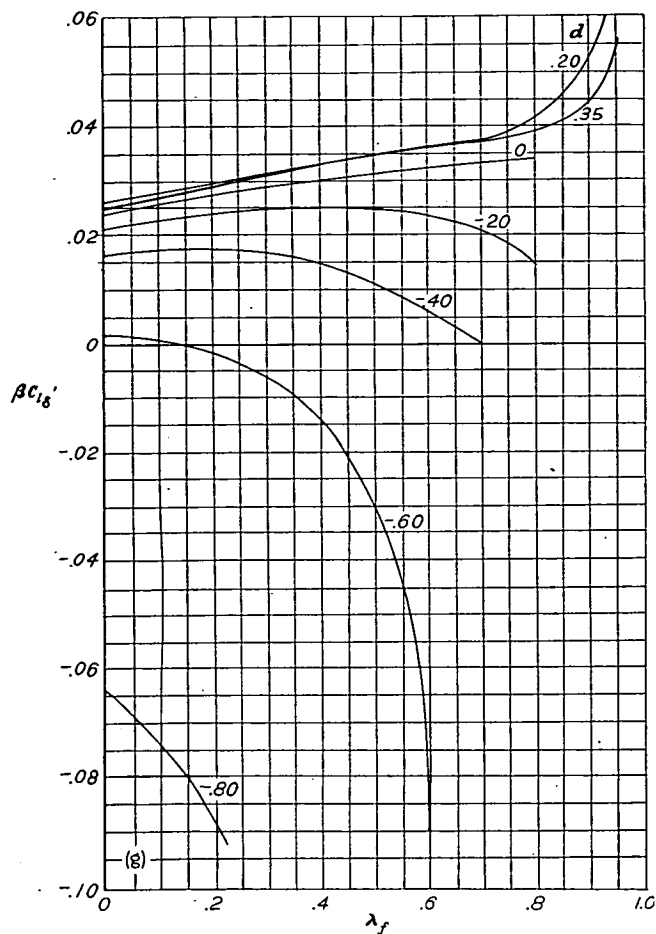
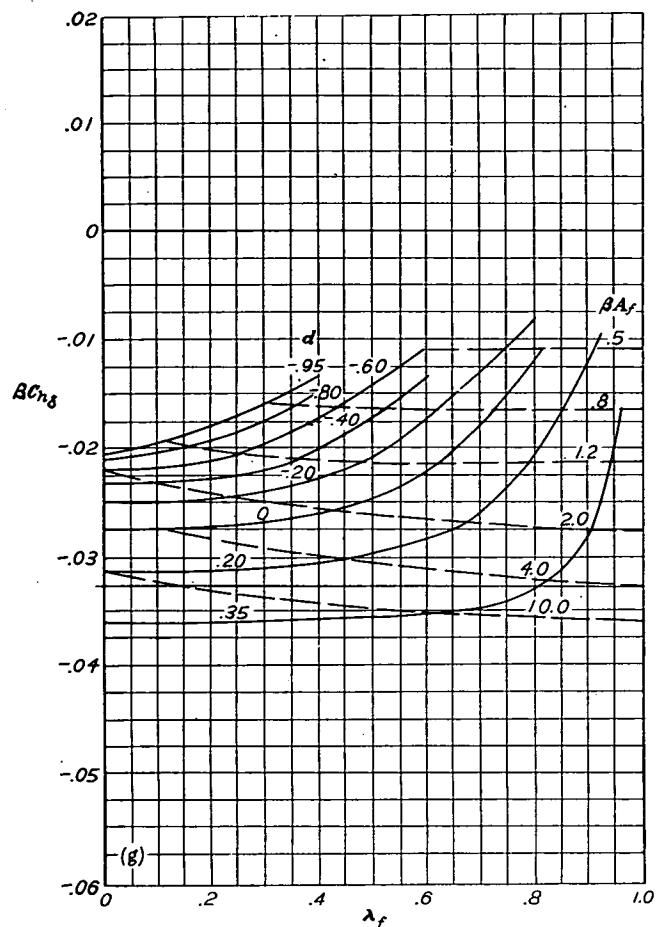
FIGURE 6.—Continued.



(e) $\frac{\tan \Delta_{HL}}{\beta} = 0.$

(f) $\frac{\tan \Delta_{HL}}{\beta} = 0.20.$

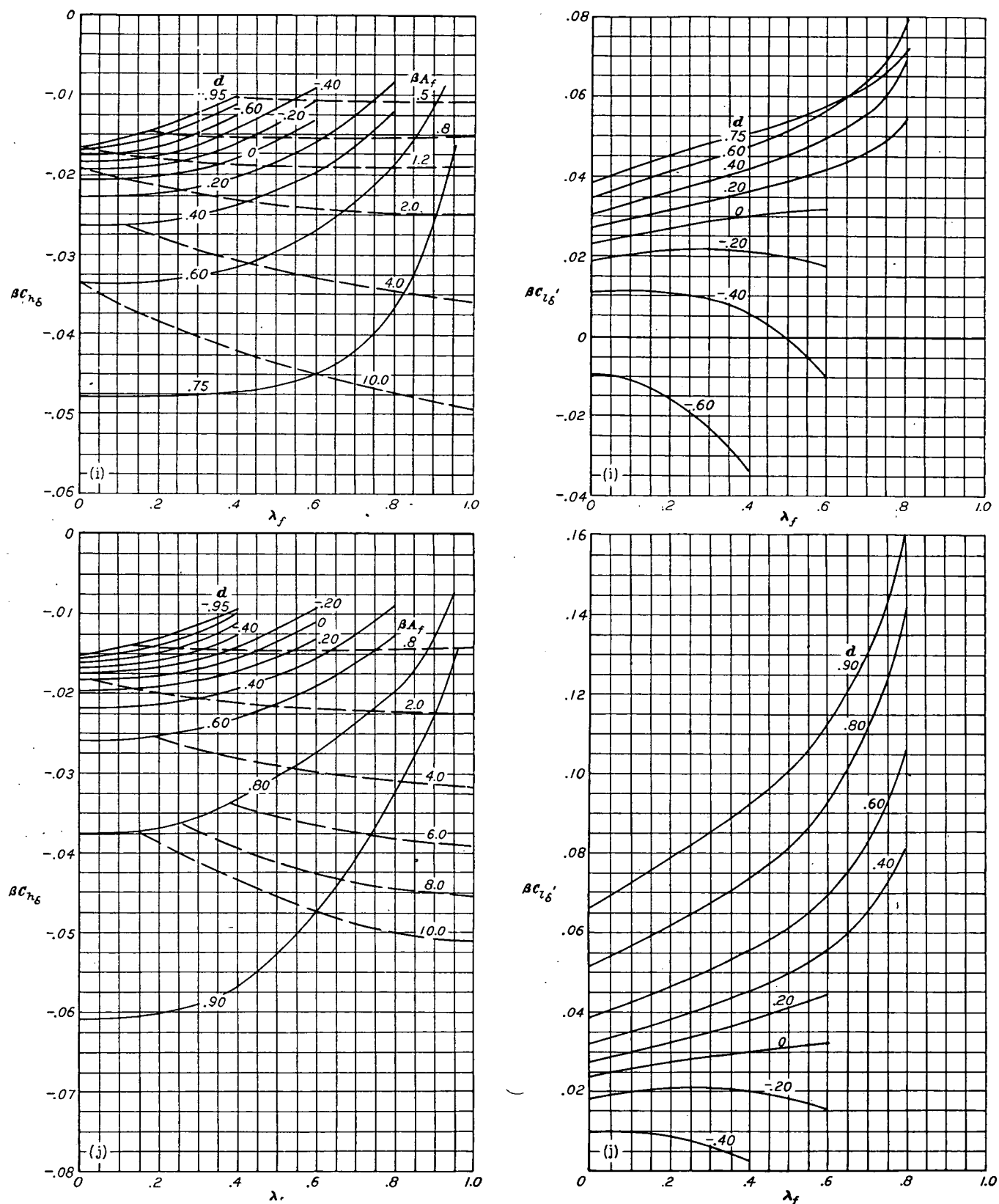
FIGURE 6.—Continued.



(g) $\frac{\tan \Delta_{BL}}{\beta} = 0.40$.

(h) $\frac{\tan \Delta_{BL}}{\beta} = 0.60$.

FIGURE 6.—Continued.



$$(i) \frac{\tan \Lambda_{HL}}{\beta} = 0.80.$$

$$(j) \frac{\tan \Lambda_{HL}}{\beta} = 0.95.$$

FIGURE 6.—Concluded.

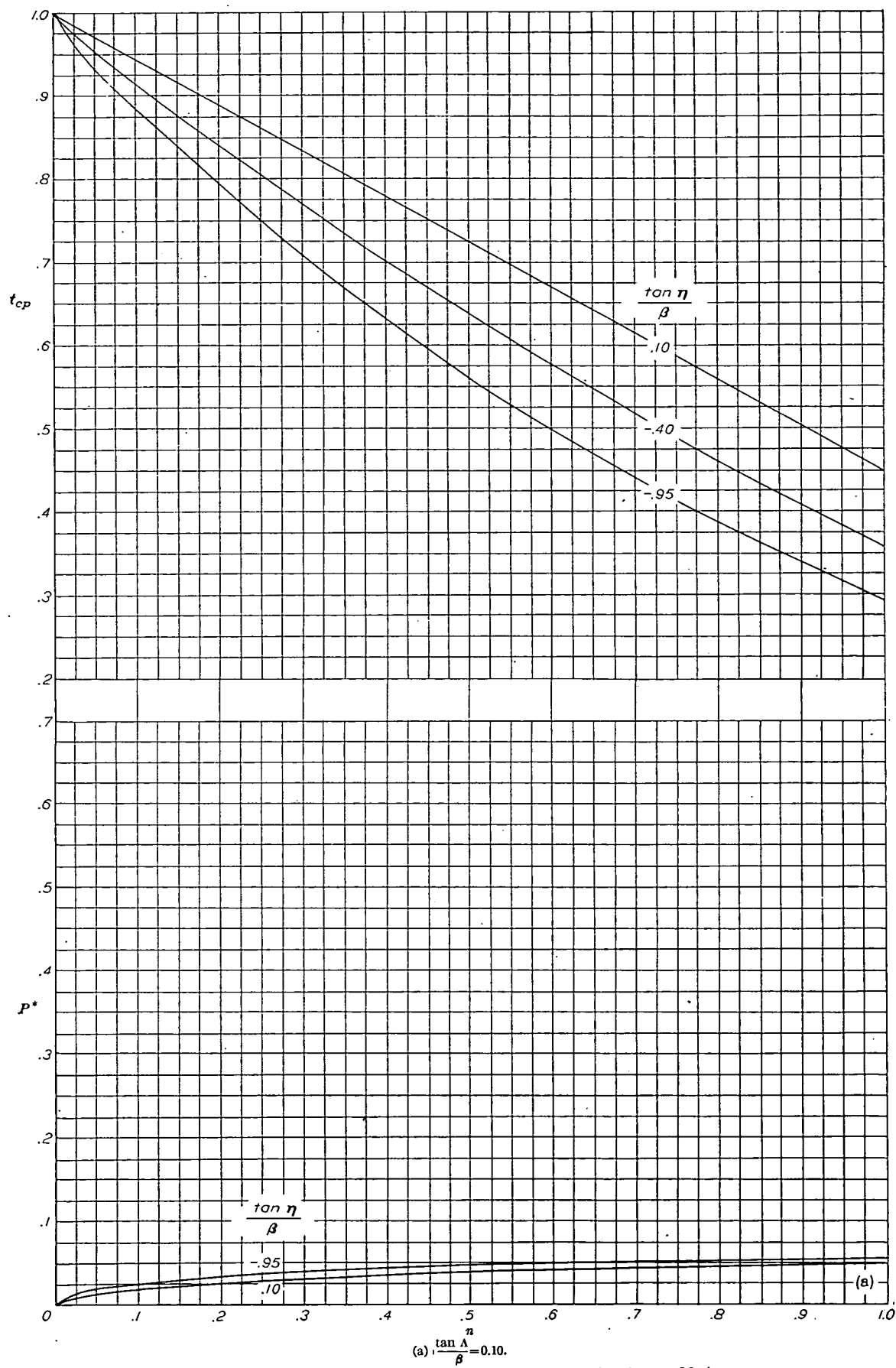
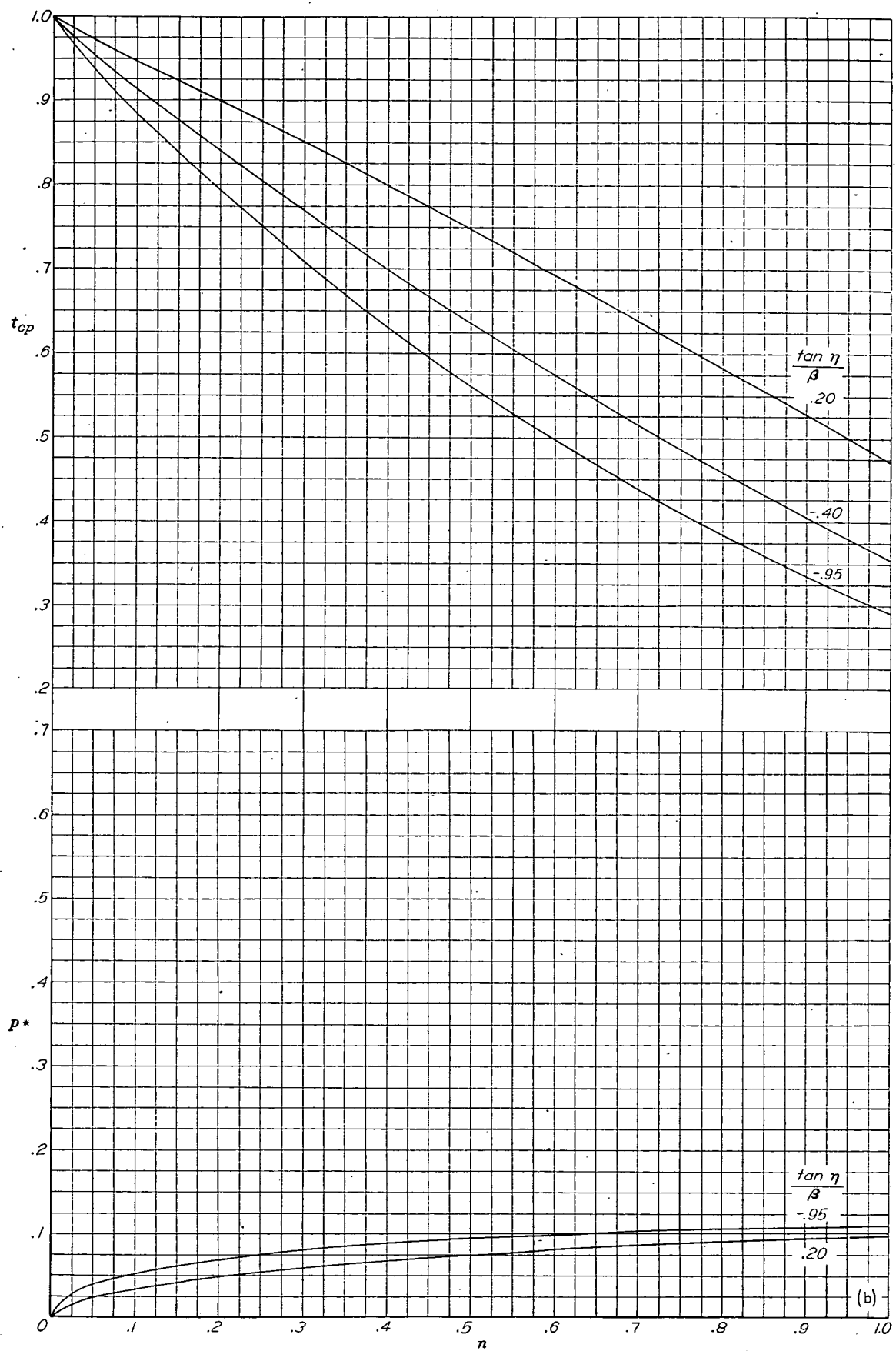
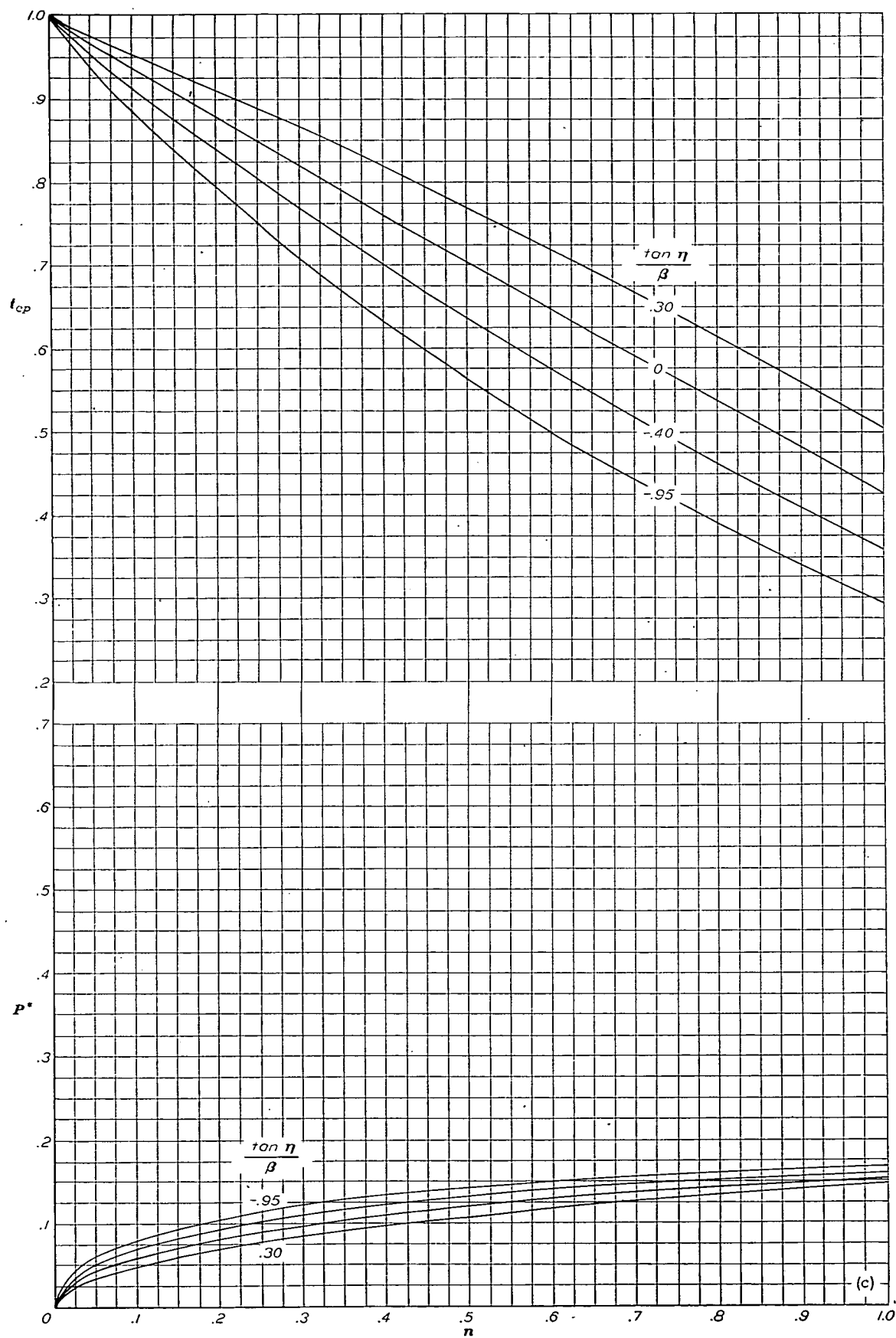


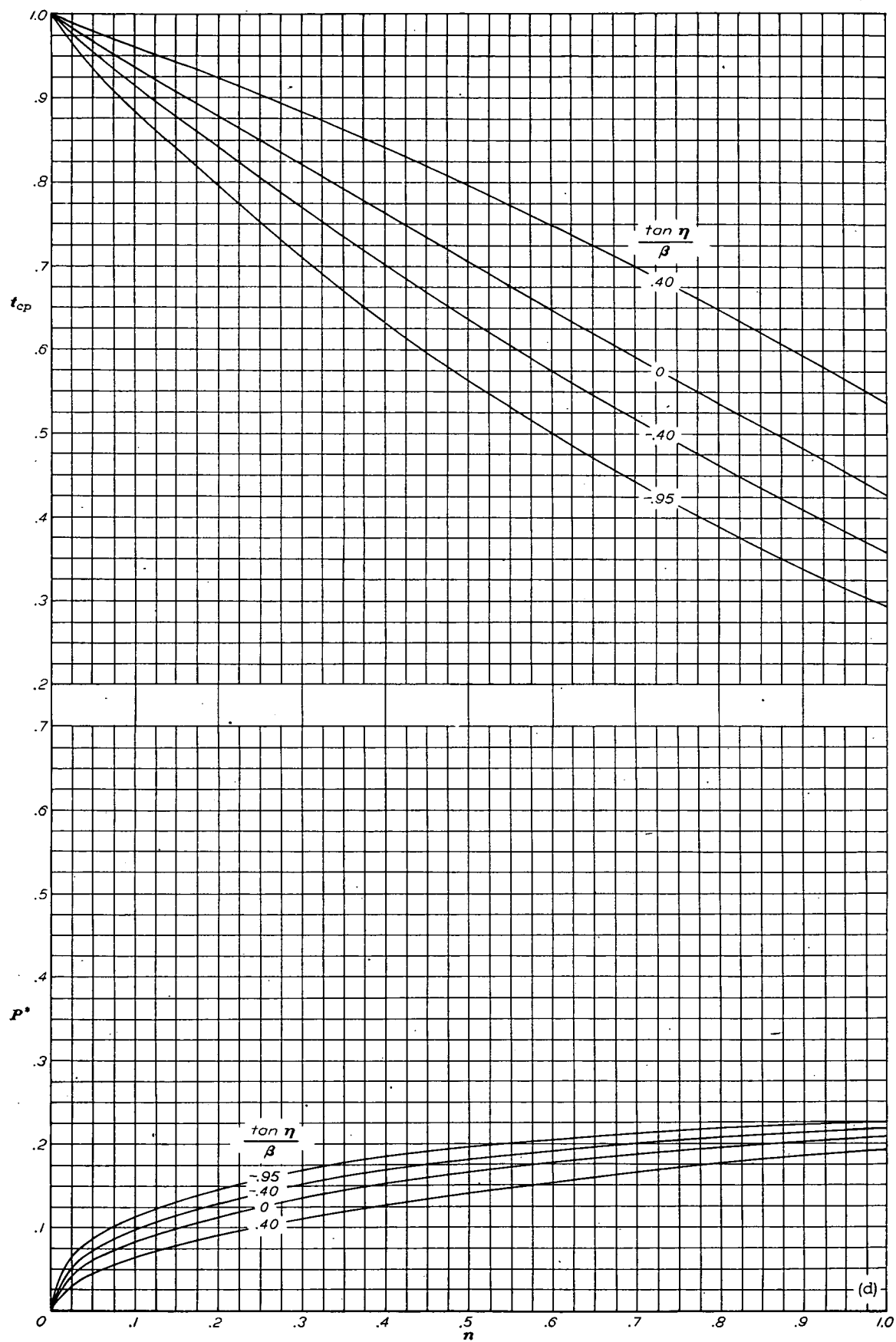
FIGURE 7.—Loading distribution along inclined sections intersecting the wing-root Mach cone.



(b) $\frac{\tan \Lambda}{\beta} = 0.20$.
FIGURE 7.—Continued.

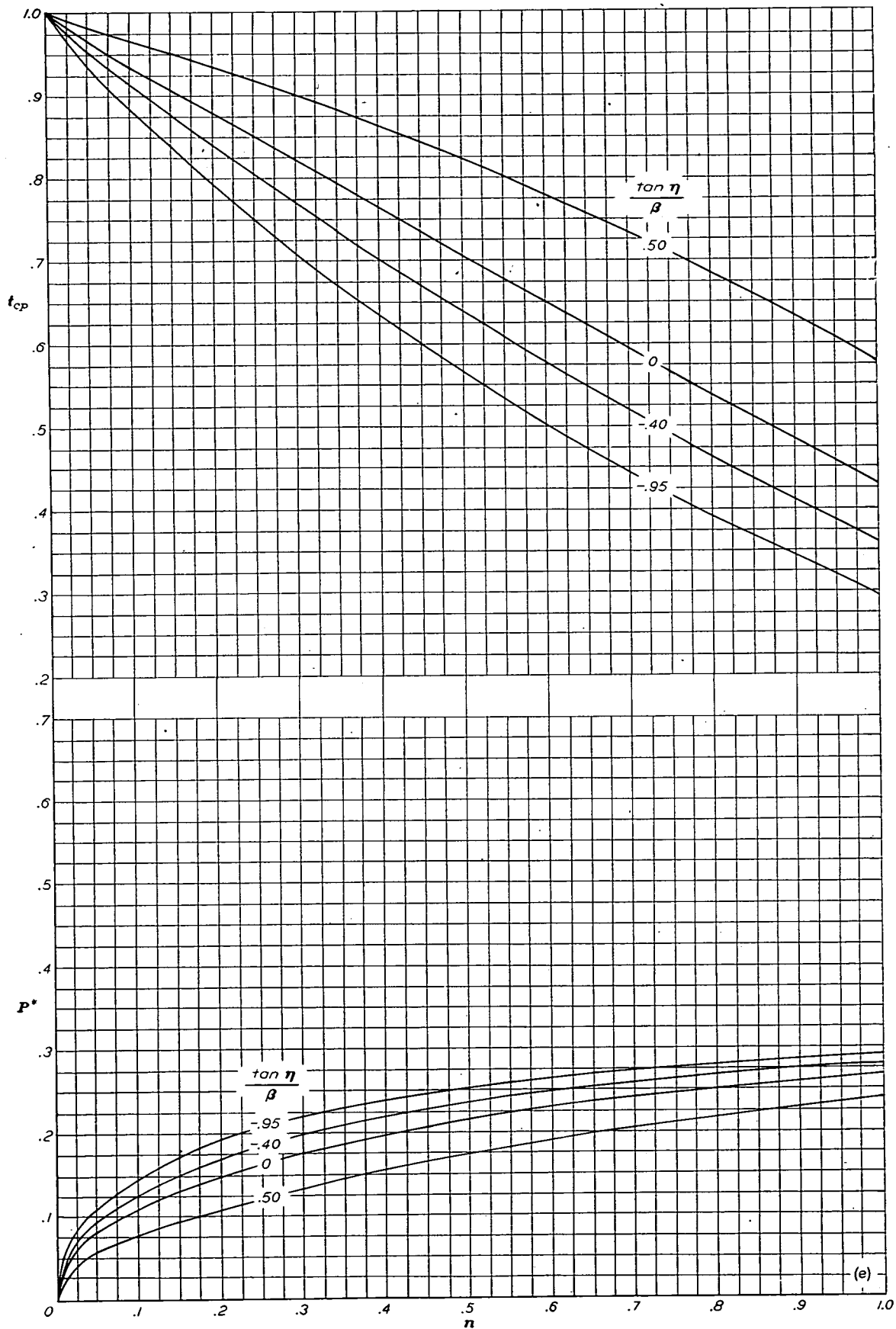


(c) $\frac{\tan \Lambda}{\beta} = 0.30$.
FIGURE 7.—Continued.



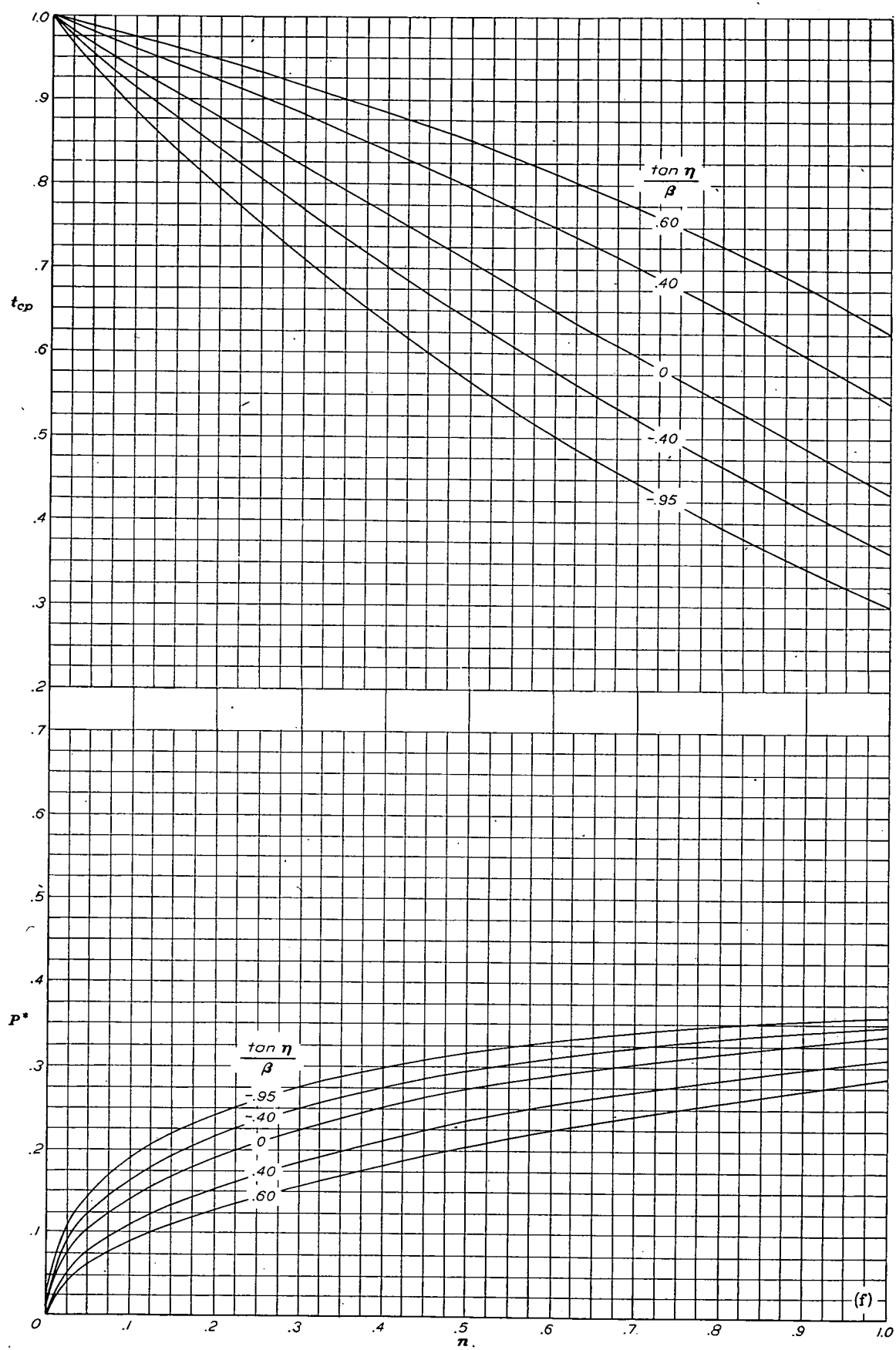
(d) $\frac{\tan \Lambda}{\beta} = 0.40.$

FIGURE 7.—Continued.

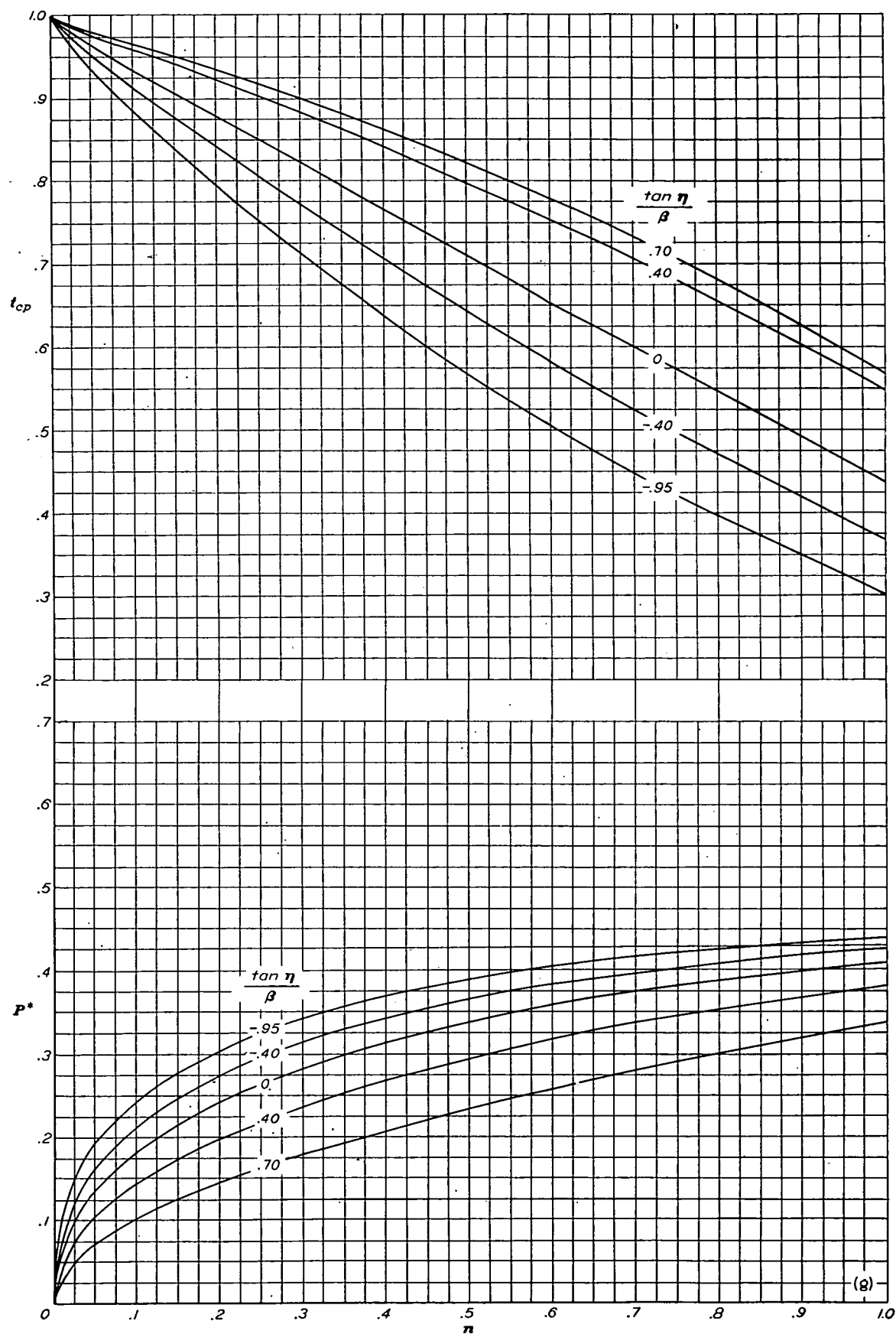


(e) $\frac{\tan \Lambda}{\beta} = 0.50.$

FIGURE 7.—Continued.

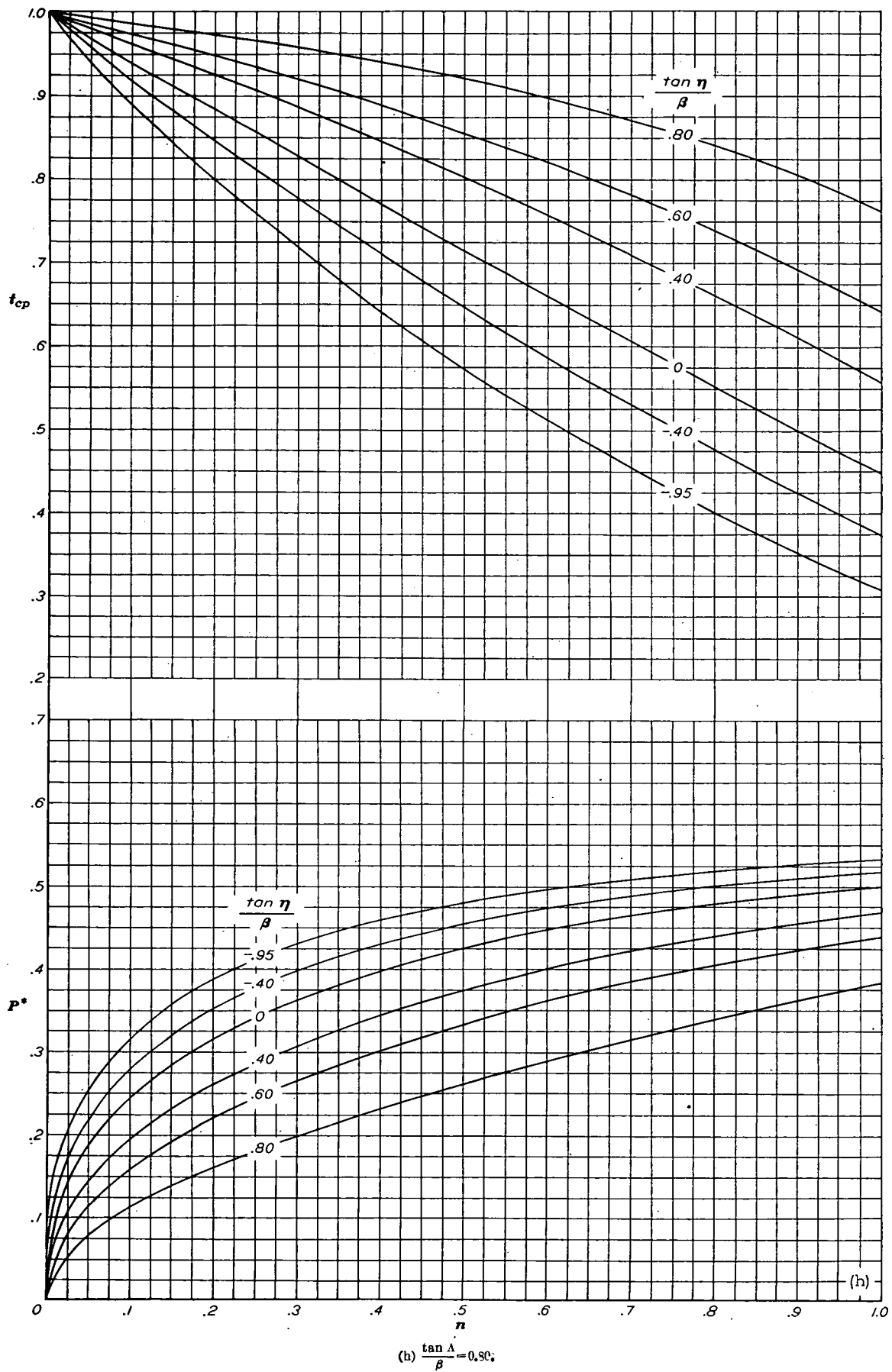


(f) $\frac{\tan A}{\beta} = 0.60.$
 FIGURE 7—Continued



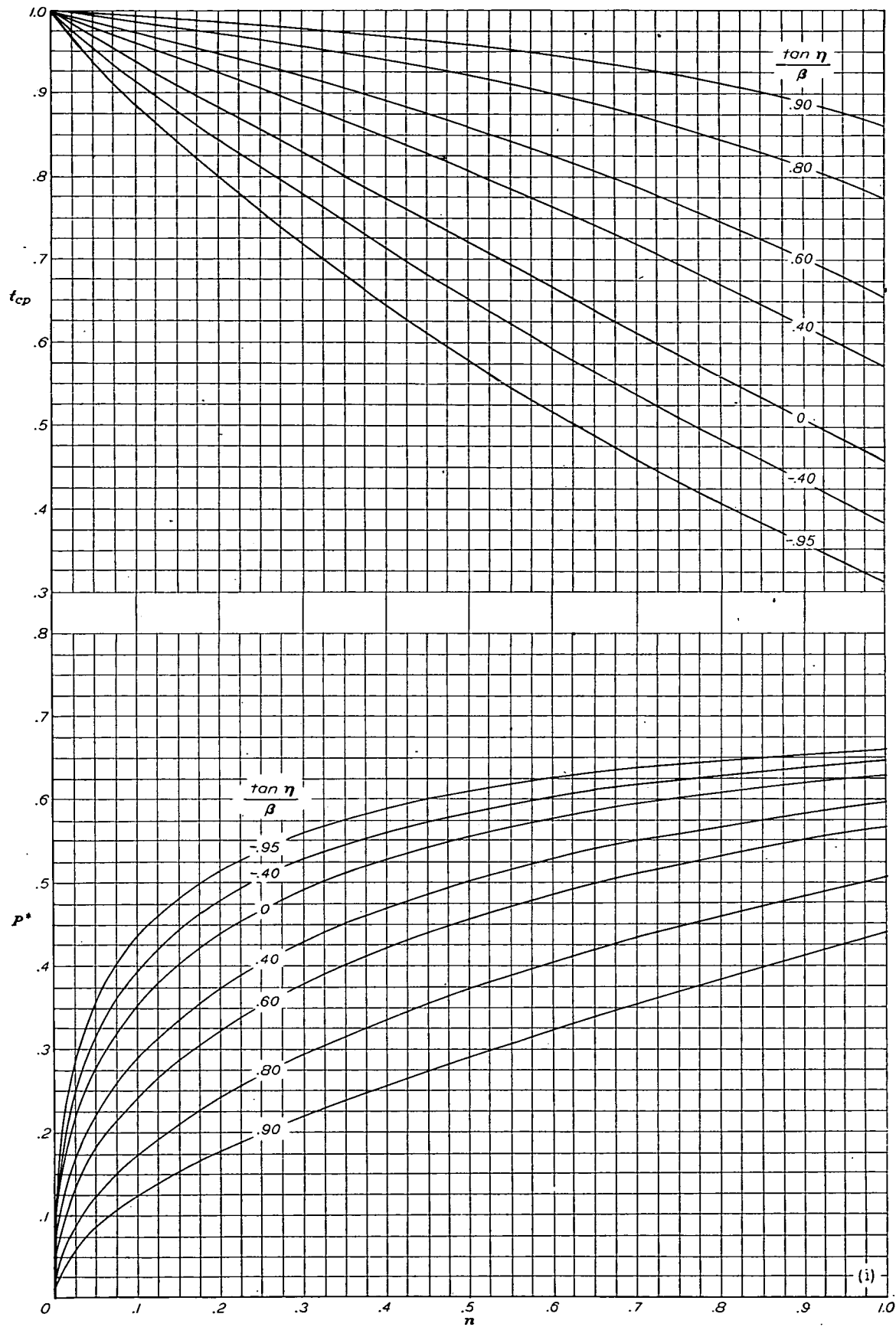
(g) $\frac{\tan \Lambda}{\beta} = 0.70$.

FIGURE 7.—Continued.



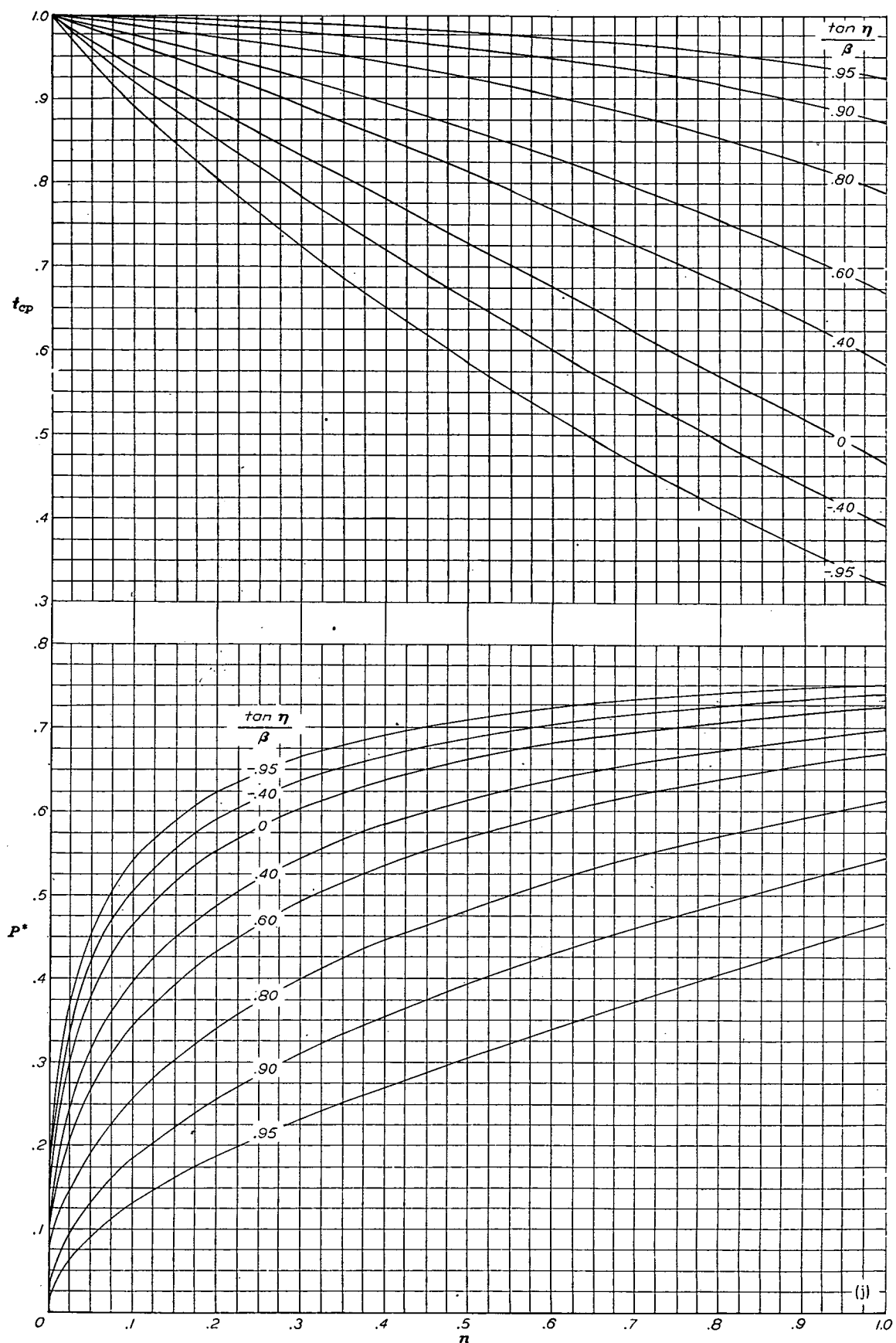
(h) $\frac{\tan \Lambda}{\beta} = 0.90$

FIGURE 7.—Continued.



(i) $\frac{\tan \Lambda}{\beta} = 0.90$.

FIGURE 7.—Continued.



(j) $\frac{\tan \Lambda}{\beta} = 0.95.$

FIGURE 7.—Concluded.

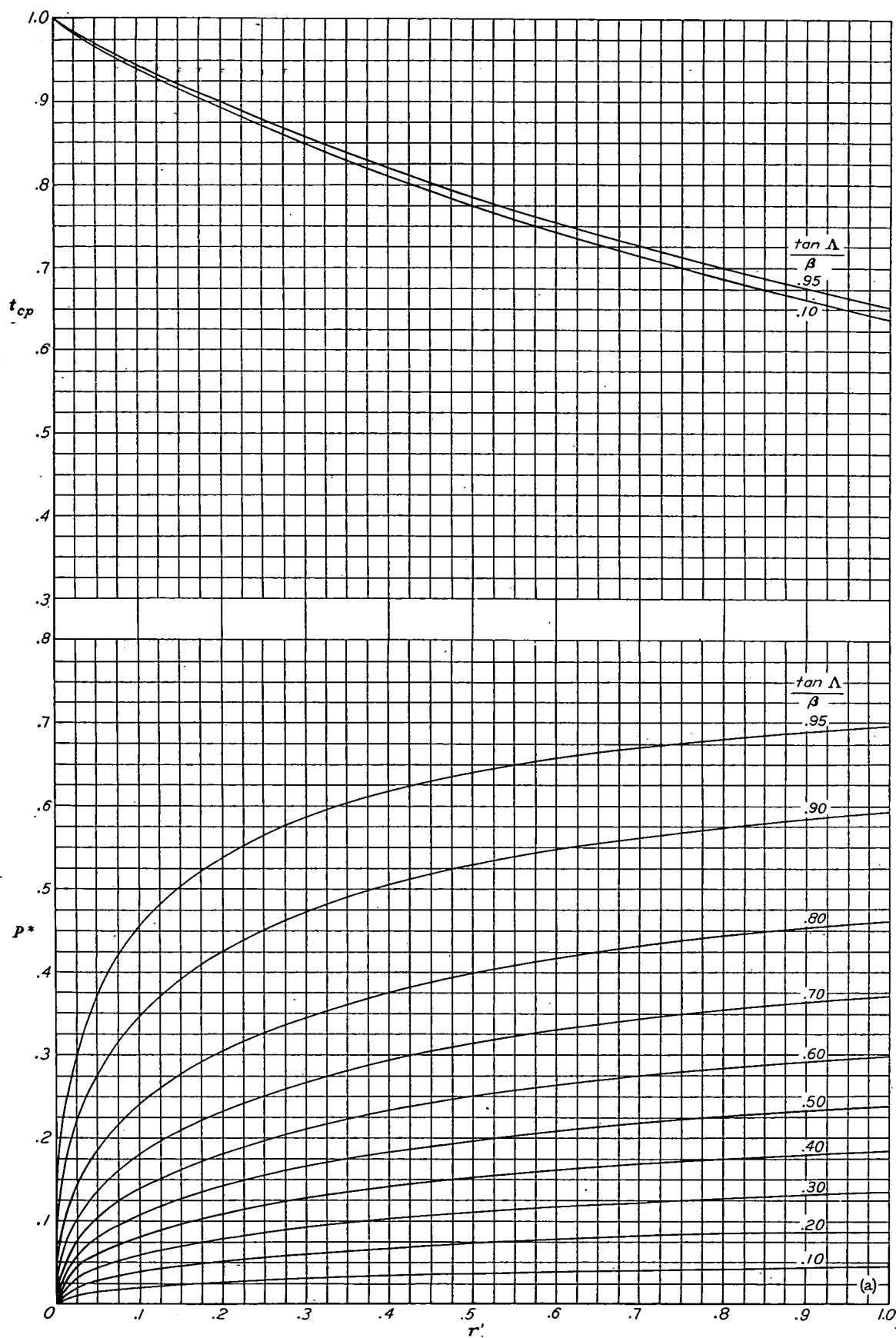
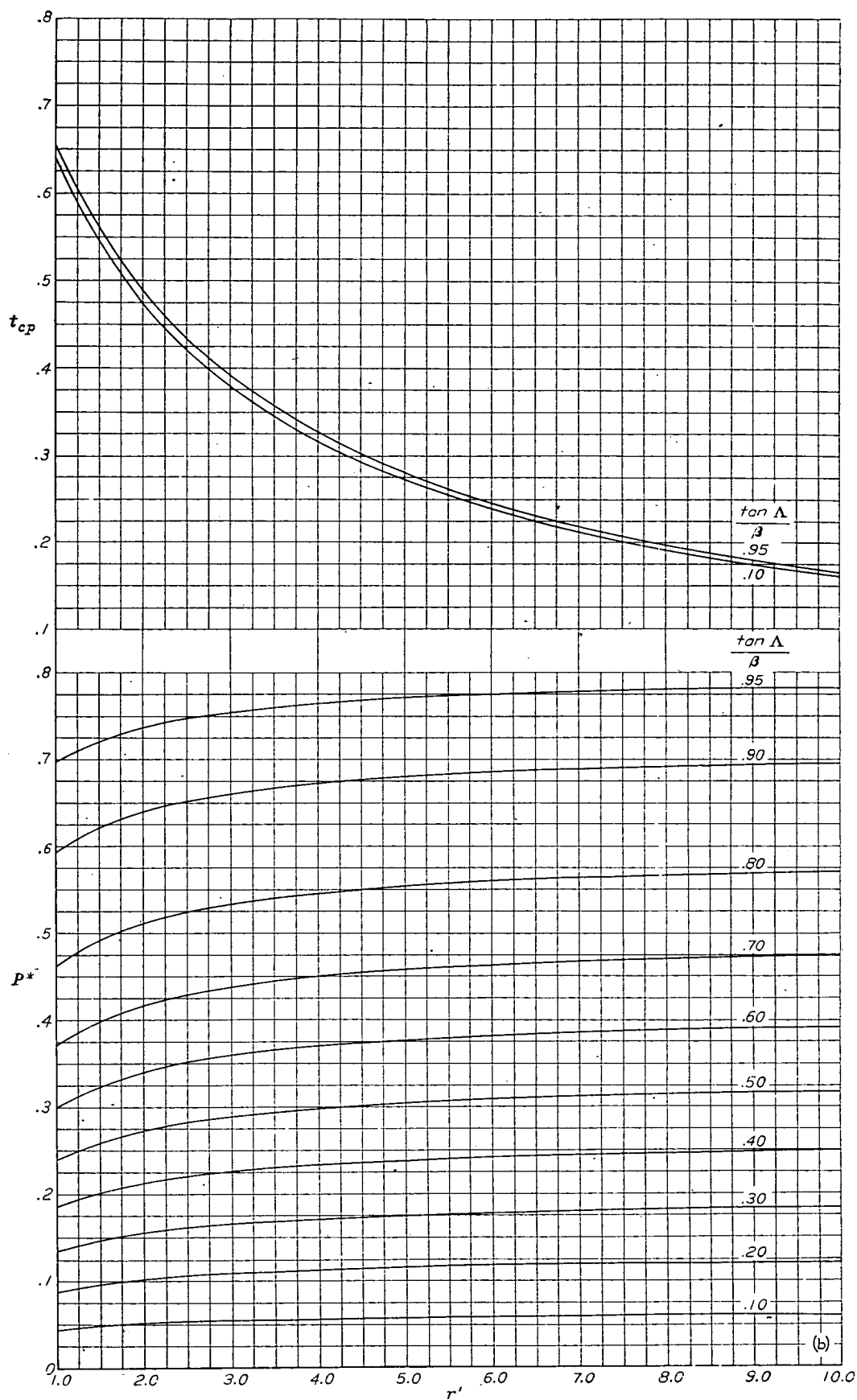
(a) $r' = 0$ to 1.0.

FIGURE 8.—Loading distribution along streamwise sections intersecting wing-root Mach cone.



(b) $r' = 1.0$ to 10.0 .
FIGURE 8.—Concluded.

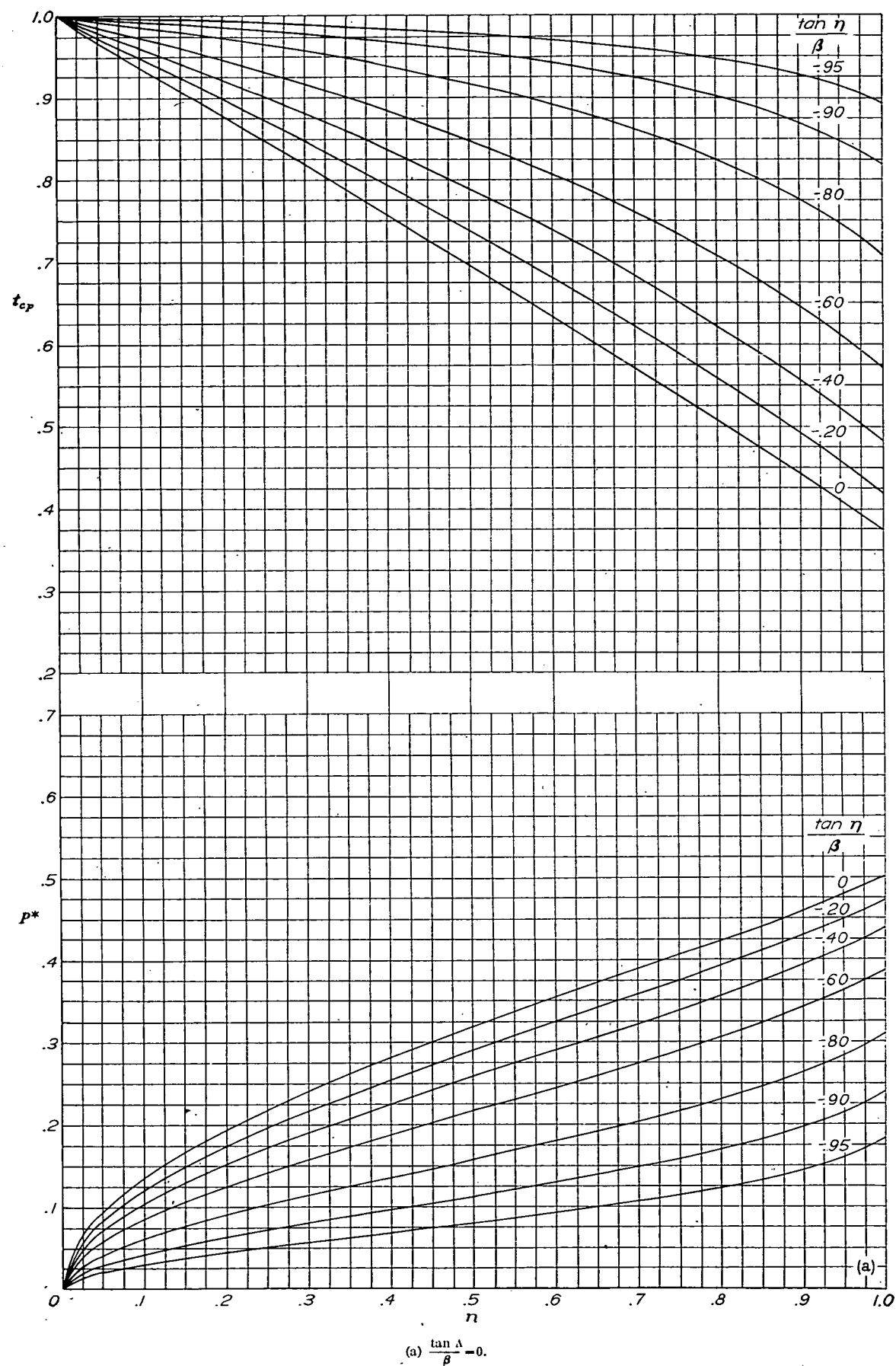
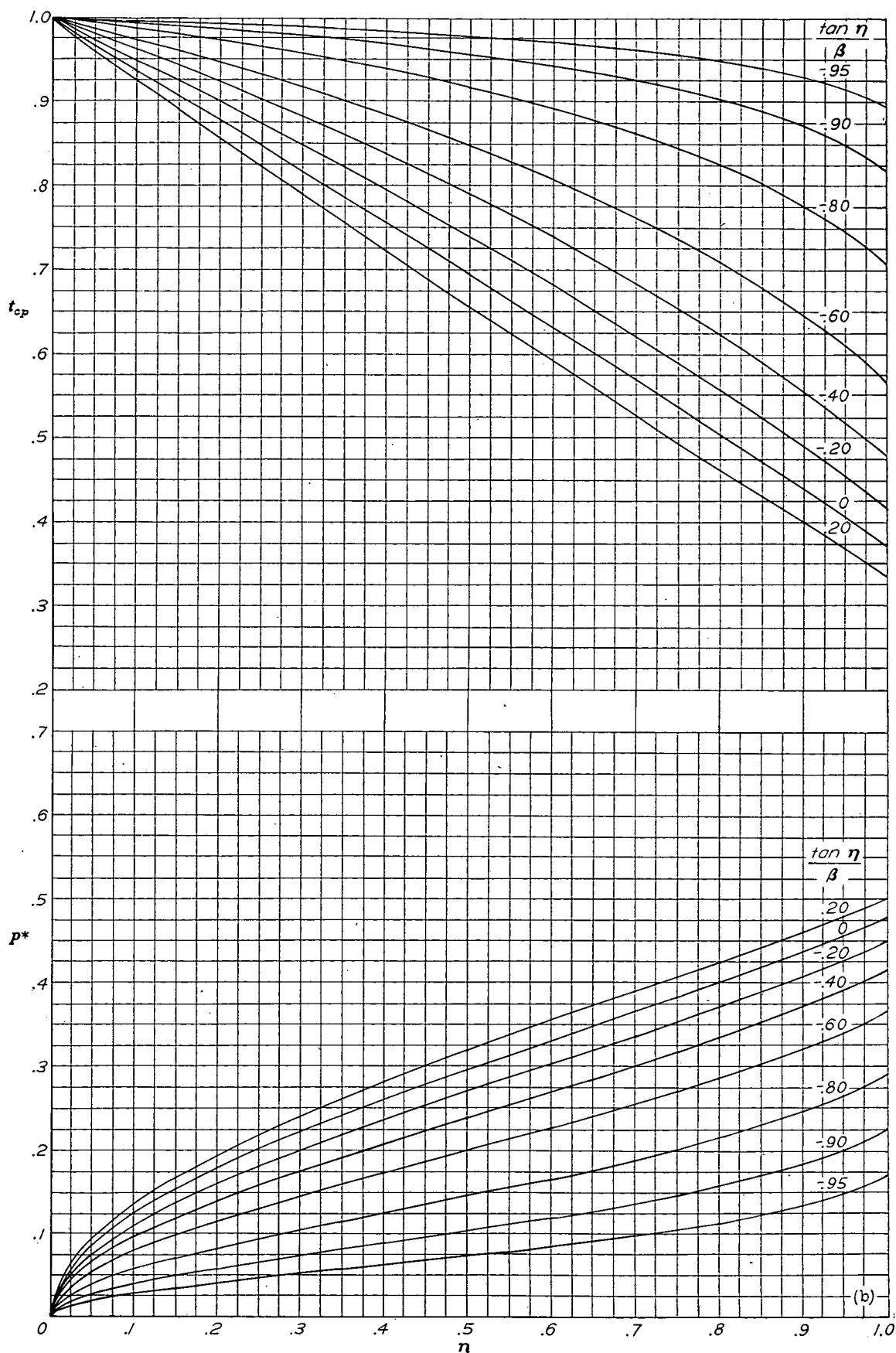
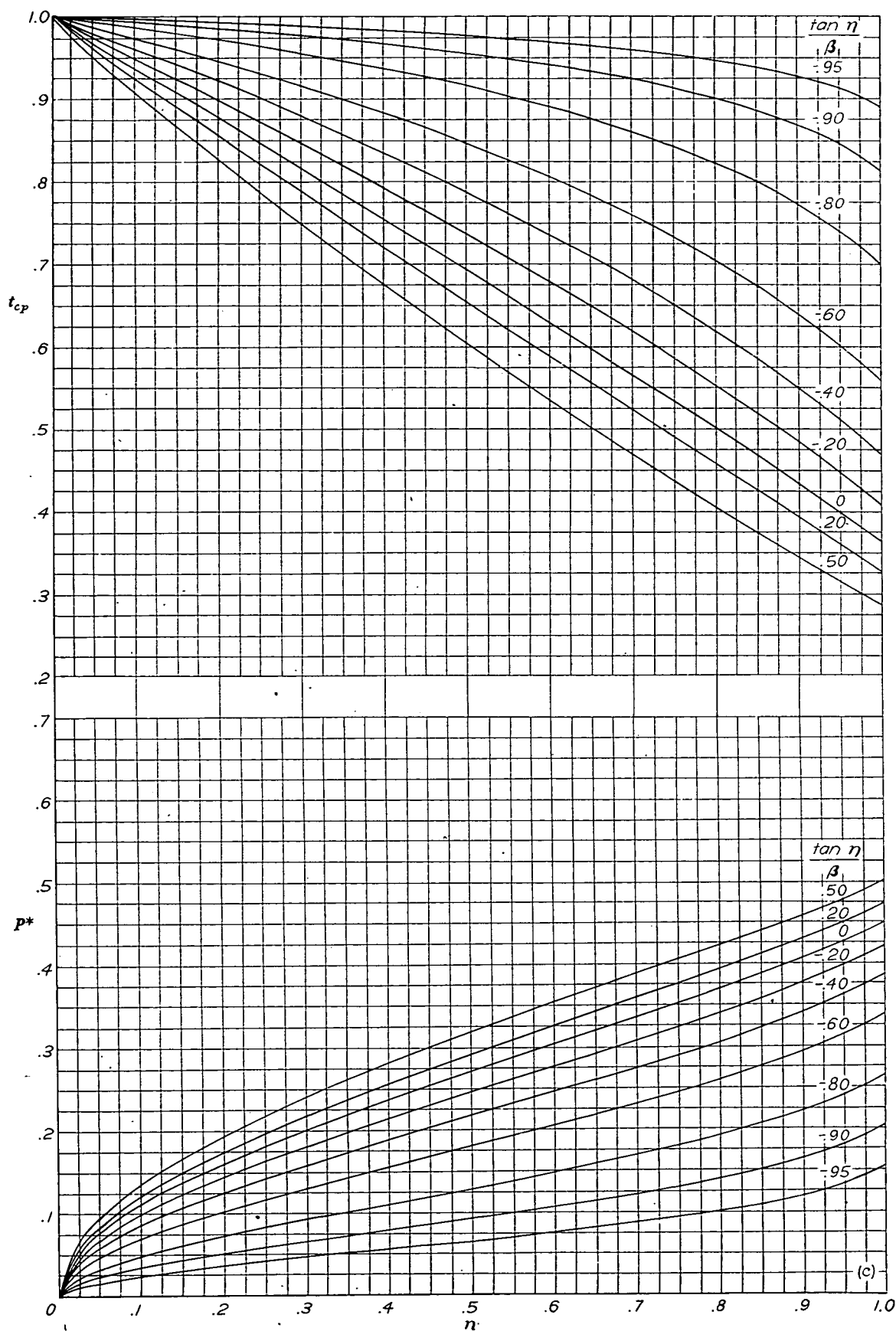


FIGURE 9.—Loading distribution along inclined sections intersecting wing-tip Mach cone.



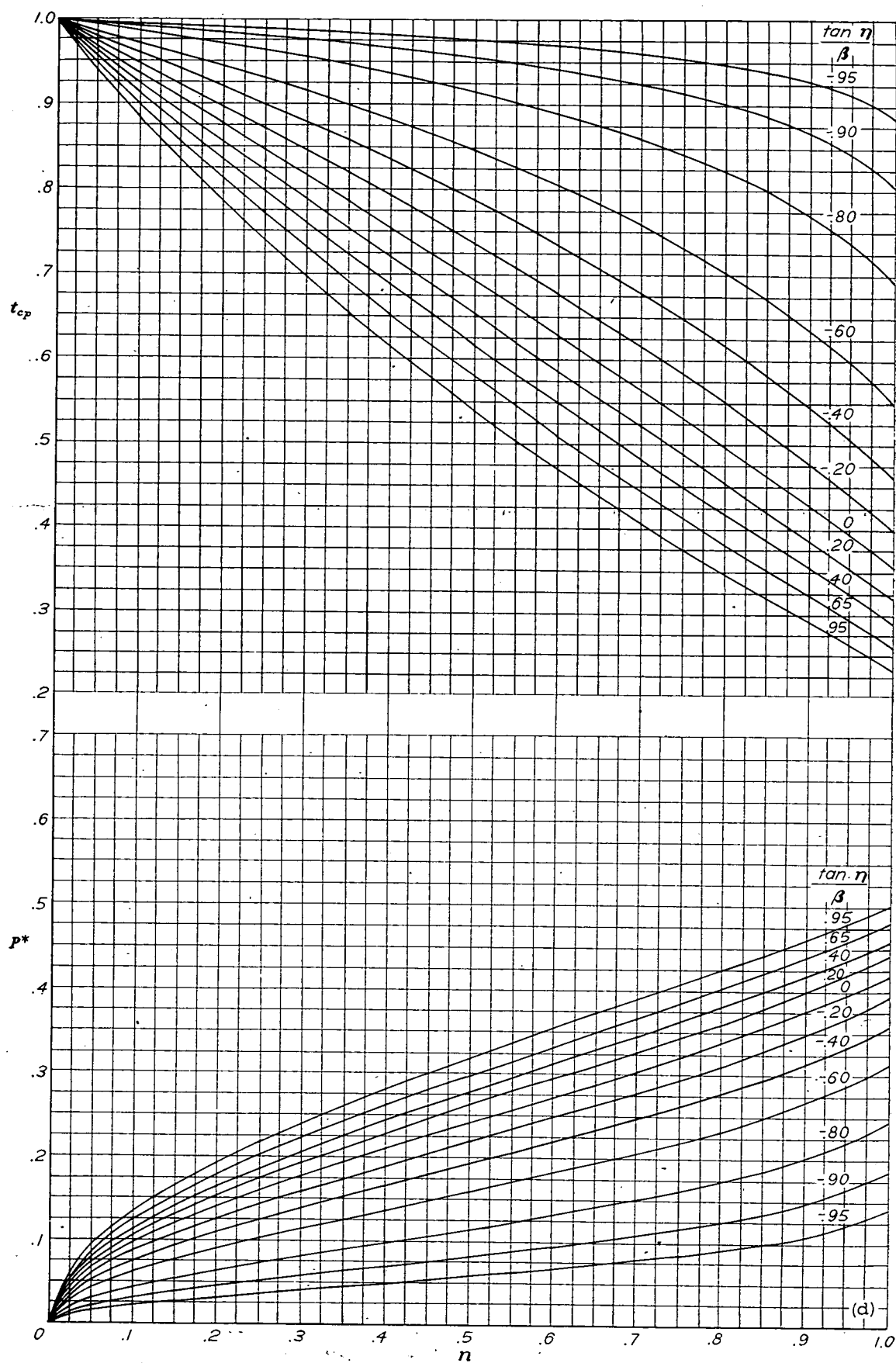
(b) $\frac{\tan \Delta}{\beta} = 0.20.$

FIGURE 9.—Continued.



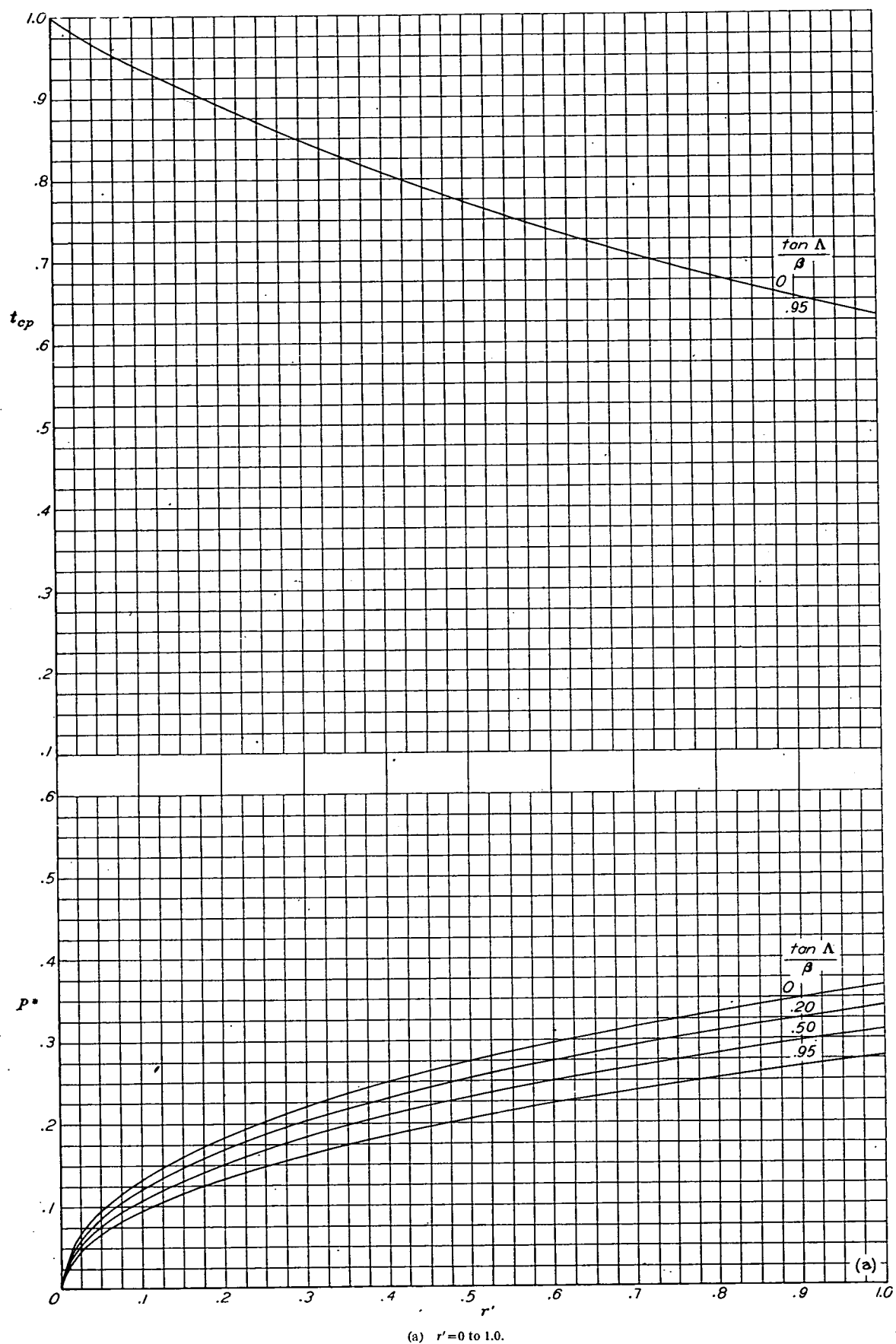
(c) $\frac{\tan \Lambda}{\beta} = 0.50$.

FIGURE 9.—Continued.

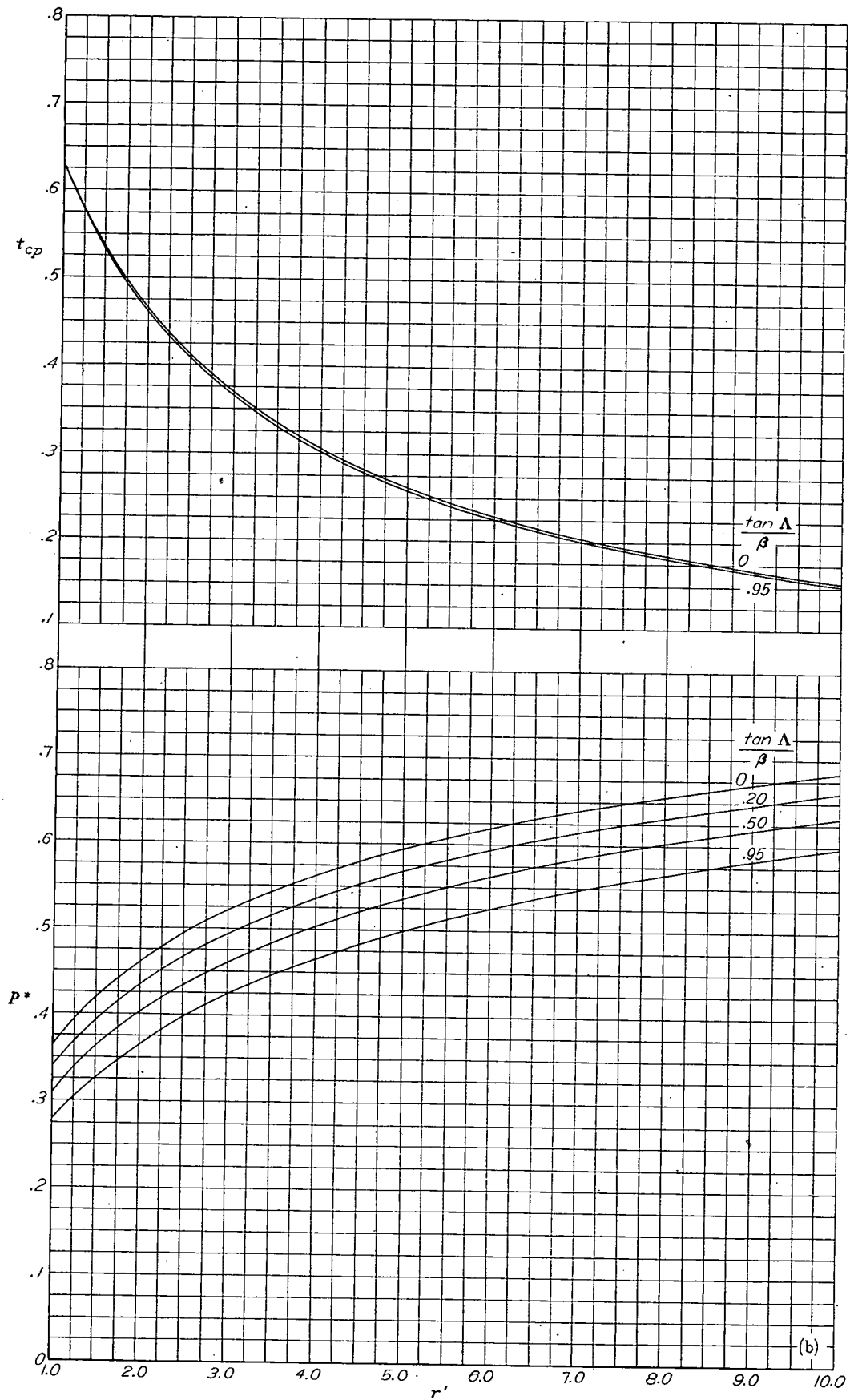


(d) $\frac{\tan \Lambda}{\beta} = 0.95.$

FIGURE 9.—Concluded



(a) $r' = 0$ to 1.0.
 FIGURE 10.—Loading distribution along streamwise sections intersecting wing-tip Mach cone.



(b) $r' = 1.0$ to 10.0 .
FIGURE 10.—Concluded.

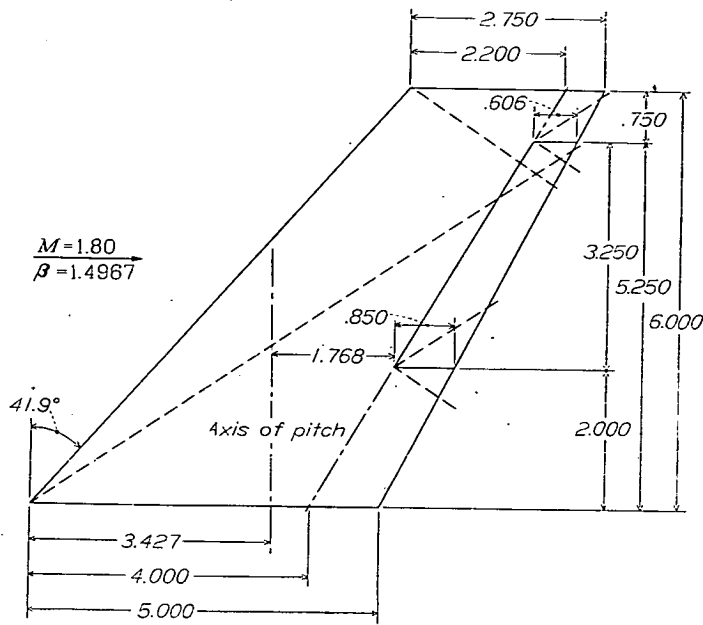


FIGURE 11.—Configuration used in sample calculation. $\left(\frac{\tan \Lambda}{\beta} = 0.5095; \frac{\tan \Lambda_{HL}}{\beta} = 0.3990; \frac{\tan \Lambda_{TE}}{\beta} = 0.3489; S = 23.250; \bar{c} = 3.984; \lambda_T = 0.713; S_T = 2.366; 2M_\infty = 1.490.\right)$

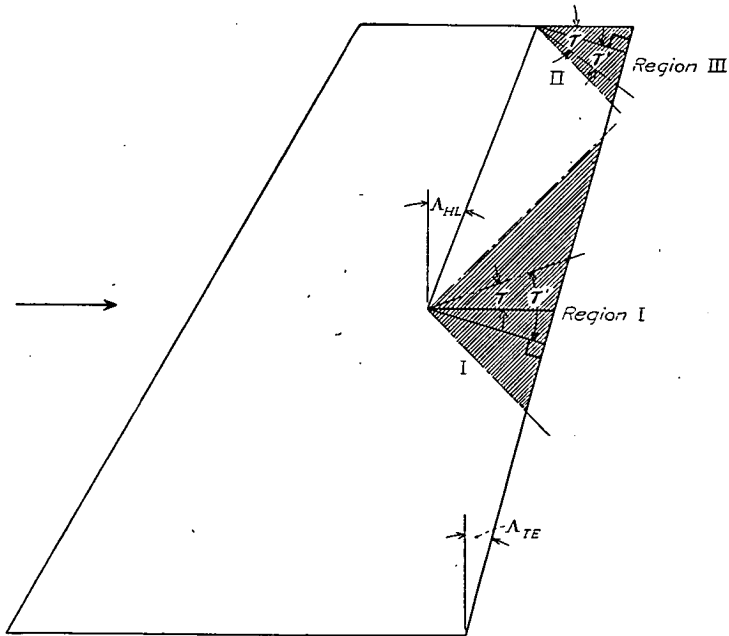


FIGURE 12.—Illustration of ordinates used in integrating pressures over conical regions of deflected controls.

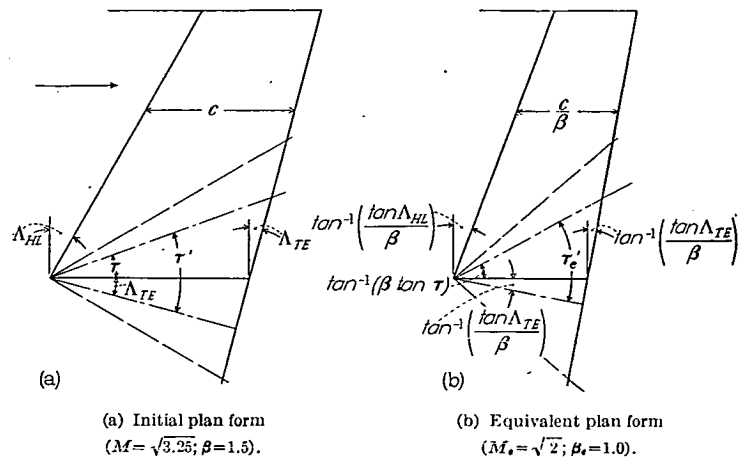


FIGURE 13.—Example transformation from one plan-form Mach number configuration to an equivalent plan form at a Mach number of $\sqrt{2}$.

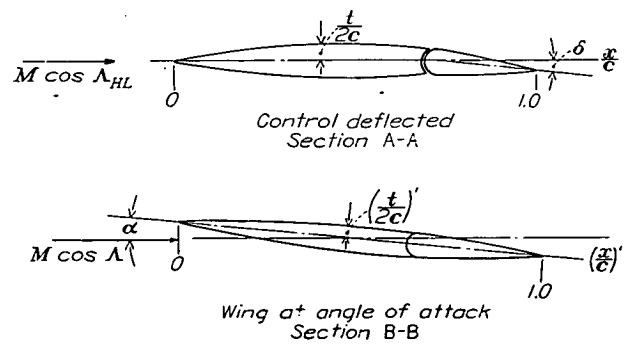
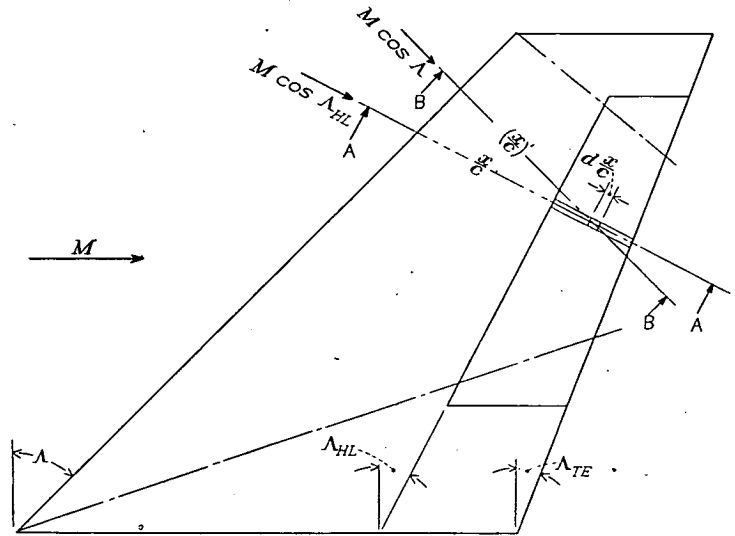


FIGURE 14.—Illustration of parameters used in determining the two-dimensional characteristics of trailing-edge controls having thickness.

TABLE I.—GENERAL EQUATIONS USED FOR DETERMINING CHARACTERISTICS OF DEFLECTED CONTROLS

[Subscripts I, Ia, Ib, Ic, II, IIa, IIb, IIc, III, IIIa, and IIIb refer to regions defined in fig. 1]

(a) Configuration Having Control Located Inboard from the Wing Tip

Parameter	Formula
$C_{L\delta}'$	$\frac{2C_{p_0}}{\delta} \left[\frac{S_{L_0}}{S_f} + \left(P \frac{S_L}{S_f} \right)_I + \left(P \frac{S_L}{S_f} \right)_{II} \right]$
$C_{m\delta}'$	$-\frac{2C_{p_0}}{\delta} \left[\frac{m_0}{2M_a} + \left(P \frac{S_L \bar{x}}{2M_a} \right)_I + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{II} \right]$
$C_{l\delta}'$	$\frac{2C_{p_0}}{\delta} \left[\frac{l_0}{b_f S_f} + \left(P \frac{S_L \bar{y}}{b_f S_f} \right)_I + \left(P \frac{S_L \bar{y}}{b_f S_f} \right)_{II} \right]$
$C_{h\delta}$	$-\frac{2C_{p_0}}{\delta} \left[\frac{m_0}{2M_a} + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{Ic} + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{Ia}^* - \left(P \frac{S_L \bar{x}}{2M_a} \right)_{Ib}^* + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{IIc} + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{IIa}^* - \left(P \frac{S_L \bar{x}}{2M_a} \right)_{IIb}^* \right]$

(b) Configuration Having Control Located at the Wing Tip

Parameter	Formula
$C_{L\delta}'$	$\frac{2C_{p_0}}{\delta} \left[\frac{S_{L_0}}{S_f} + \left(P \frac{S_L}{S_f} \right)_I + \left(P \frac{S_L}{S_f} \right)_{Ia}^* - \left(P \frac{S_L}{S_f} \right)_{Ib}^* + \left(P \frac{S_L}{S_f} \right)_{III} \right]$
$C_{m\delta}'$	$-\frac{2C_{p_0}}{\delta} \left[\frac{m_0}{2M_a} + \left(P \frac{S_L \bar{x}}{2M_a} \right)_I + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{Ia}^* - \left(P \frac{S_L \bar{x}}{2M_a} \right)_{Ib}^* + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{III} \right]$
$C_{l\delta}'$	$\frac{2C_{p_0}}{\delta} \left[\frac{l_0}{b_f S_f} + \left(P \frac{S_L \bar{y}}{b_f S_f} \right)_I + \left(P \frac{S_L \bar{y}}{b_f S_f} \right)_{Ia}^* - \left(P \frac{S_L \bar{y}}{b_f S_f} \right)_{Ib}^* + \left(P \frac{S_L \bar{y}}{b_f S_f} \right)_{III} \right]$
$C_{h\delta}$	$-\frac{2C_{p_0}}{\delta} \left[\frac{m_0}{2M_a} + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{Ic} + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{Ia}^* - \left(P \frac{S_L \bar{x}}{2M_a} \right)_{Ib}^* + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{III} + \left(P \frac{S_L \bar{x}}{2M_a} \right)_{IIIa}^* - \left(P \frac{S_L \bar{x}}{2M_a} \right)_{IIIb}^* \right]$

TABLE II.—COMPONENT PARTS OF EQUATIONS USED IN CALCULATING CHARACTERISTICS OF DEFLECTED CONTROLS HAVING TAPERED PLAN FORMS

(a) Average Pressure Ratio

Values of P for regions II, II_a, II_b, and II_c are obtained by substituting $-a$, $-d$, and $1/\lambda_f$ for a , d , and λ_f in equations for regions I, I_a, I_b, and I_c, respectively. In cases where plus and minus signs are together ($\pm$), the upper sign must be used when values of a and d substituted are such that $a-d$ is negative and the lower sign must be used when values of a and d substituted are such that $a-d$ is positive.]

Region (fig. 1)	Average pressure ratio
I	$P = \frac{\sqrt{(1-a^2)(1-d^2)} - (1-a)(1+d)}{2(a-d)}$
I _a	$P^* = \frac{1-d}{\lambda_f(1-d) - (1-a)} \left\{ \frac{\lambda_f}{\pi} \cos^{-1} \left[\frac{(1-a^2) - \lambda_f(1-ad)}{\lambda_f(a-d)} \right] - \right.$ $\left. \frac{1}{\pi} \sqrt{\frac{1-a^2}{1-d^2}} \cos^{-1} \left[\frac{(1-ad) - \lambda_f(1-d^2)}{(a-d)} \right] \right\}$
I _b	$P^* = \frac{1}{\lambda_f(1-d) - (1-a)} \left\{ \frac{\lambda_f}{\pi} (a-d) \cos^{-1} \left[\frac{(1-a^2) - \lambda_f(1-ad)}{\lambda_f(a-d)} \right] + \right.$ $\left. \frac{(1-\lambda_f)\sqrt{1-a^2}}{\pi} \log_e \left \frac{(a-\lambda_f d) \pm \sqrt{2\lambda_f(1-ad) - \lambda_f^2(1-d^2) - (1-a^2)}}{1-\lambda_f} \right \right\}$
I _c	$P = \frac{1-d}{a-d} \left[\sqrt{\frac{1-a^2}{1-d^2}} \left(1 - \frac{1}{\pi} \cos^{-1} d \right) + \frac{1}{\pi} \cos^{-1} a - \frac{1-a}{1-d} \right]$
III	$P = \frac{(1+a) - \sqrt{(1+a)(1+d)}}{a-d}$
III _a	$P^* = \frac{1}{(1+d) - \lambda_f(1+a)} \left\{ \frac{1+d}{\pi} \cos^{-1} \left[\frac{(2+a+d) - 2\lambda_f(1+a)}{(a-d)} \right] - \right.$ $\left. \frac{\lambda_f \sqrt{(1+a)(1+d)}}{\pi} \cos^{-1} \left[\frac{2(1+d) - \lambda_f(2+a+d)}{\lambda_f(a-d)} \right] \right\}$
III _b	$P^* = \frac{1}{\lambda_f(1+a) - (1+d)} \left\{ \frac{a-d}{\pi} \cos^{-1} \left[\frac{(2+a+d) - 2\lambda_f(1+a)}{(a-d)} \right] \mp \right.$ $\left. \frac{2}{\pi} \sqrt{(1+a)(1-\lambda_f)[\lambda_f(1+a) - (1+d)]} \right\}$

TABLE II.—COMPONENT PARTS OF EQUATIONS USED IN CALCULATING CHARACTERISTICS OF DEFLECTED CONTROLS HAVING TAPERED PLAN FORMS—Continued

(b) Center-of-Pressure Ray Location

[Values of t_{cp}' (or r_{cp}) for regions II, II_a, II_b, and II_c are obtained by substituting $-a$, $-d$, and $1/\lambda_f$ for a , d , and λ_f in equations for regions I, I_a, I_b, and I_c, respectively. In cases where plus and minus signs are together ($\pm$), the upper sign must be used when values of a and d substituted are such that $a-d$ is negative and the lower sign must be used when values of a and d substituted are such that $a-d$ is positive.]

Region (fig. 1)	Center-of-pressure ray location (t_{cp}' or r_{cp})
I	$t_{cp}' = \frac{1}{4P(a-d)^2} \left\{ \sqrt{\frac{1-a^2}{1-d^2}} (1+3ad-3d^2-ad^3) - \frac{1+d}{1-d} [(1-a^2)(1-d^2)-2d(1-a)^2] \right\}$
II _a	$t_{cp}'^* = \frac{1}{2P^*(a-d)(1+d)[\lambda_f(1-d)-(1-a)]} \left\{ \left(\frac{1-d^2}{\pi} \right) [2\lambda_f(1+ad)-\lambda_f^2(1+d^2)] \cos^{-1} \left[\frac{(1-a^2)-\lambda_f(1-ad)}{\lambda_f(a-d)} \right] - \right.$ $\left. \frac{1}{\pi} \sqrt{\frac{1-a^2}{1-d^2}} (1+3ad-3d^2-ad^3) \cos^{-1} \left[\frac{(1-ad)-\lambda_f(1-d^2)}{a-d} \right] \pm \frac{(1+d^2)}{\pi} \sqrt{(1-a^2)[2\lambda_f(1-ad)-\lambda_f^2(1-d^2)-(1-a^2)]} \right\}$
II _b	$r_{cp}^* = \frac{1}{2P^*[\lambda_f(1-d)-(1-a)]} \left\{ \frac{[\lambda_f(a-d)[2a-\lambda_f(a+d)]]}{\pi(1-\lambda_f)} \cos^{-1} \left[\frac{(1-a^2)-\lambda_f(1-ad)}{\lambda_f(a-d)} \right] \pm \right.$ $\left. \frac{\sqrt{(1-a^2)[2\lambda_f(1-ad)-\lambda_f^2(1-d^2)-(1-a^2)]}}{\pi} + \frac{a(1-\lambda_f)\sqrt{1-a^2}}{\pi} \log_e \left \frac{(a-\lambda_f d) \pm \sqrt{2\lambda_f(1-ad)-\lambda_f^2(1-d^2)-(1-a^2)}}{1-\lambda_f} \right \right\}$
II _c	$t_{cp}' = \frac{1}{2P(1+d)(a-d)^2} \left\{ \sqrt{\frac{1-a^2}{1-d^2}} (1+3ad-3d^2-ad^3) \left(1 - \frac{1}{\pi} \cos^{-1} d \right) + \frac{(a-d)(1+d^2)\sqrt{1-a^2}}{\pi} - \right.$ $\left. \frac{1+d}{1-d} [(1-a^2)(1-d^2)-2d(1-a)^2] + \frac{(1-d^2)(1+2ad-d^2)}{\pi} \cos^{-1} a \right\}$
III	$t_{cp}' = \frac{1}{4P(1+d)(a-d)^2} \left\{ \sqrt{(1+a)(1+d)} [(2-a+d)(1-d^2)+2ad(1+d)+2d(1+a)] - 2[(1-a^2)(1-d^2)+2d(1+a)^2] \right\}$
III _a	$t_{cp}'^* = \frac{1}{4P^*(a-d)[\lambda_f(1+a)-(1+d)]} \left\{ \frac{[2(1+d)]}{\pi\lambda_f} [2\lambda_f(1+ad)-(1+d^2)] \cos^{-1} \left[\frac{(2+a+d)-2\lambda_f(1+a)}{a-d} \right] \pm \right.$ $\frac{2(1+d^2)}{\pi} \sqrt{(1+a)[\lambda_f(2+a+d)-\lambda_f^2(1+a)-(1+d)]} -$ $\left. \frac{\lambda_f}{\pi} \sqrt{\frac{1+a}{1+d}} [(2-a+d)(1-d^2)+2ad(1+d)+2d(1+a)] \cos^{-1} \left[\frac{2(1+d)-\lambda_f(2+a+d)}{\lambda_f(a-d)} \right] \right\}$
III _b	$r_{cp}^* = \frac{1}{6P^*(1-\lambda_f)[\lambda_f(1+a)-(1+d)]} \left\{ \frac{3(a-d)[2a\lambda_f-(a+d)]}{\pi} \cos^{-1} \left[\frac{(2+a+d)-2\lambda_f(1+a)}{a-d} \right] \pm \right.$ $\left. \frac{2[2\lambda_f(1-2a)-(2-3a-d)]}{\pi} \sqrt{(1+a)[\lambda_f(2+a+d)-\lambda_f^2(1+a)-(1+d)]} \right\}$

TABLE II.—COMPONENT PARTS OF EQUATIONS USED IN CALCULATING CHARACTERISTICS OF DEFLECTED CONTROLS HAVING TAPERED PLAN FORMS—Concluded

(c) Geometric Parameters $S_f = \frac{\beta b r^2 (a-d)(1+\lambda_f)}{2(1-\lambda_f)}$ and $2M_a = \frac{\beta^2 b r^2 (a-d)^2 (1-\lambda_f^2)}{3(1-\lambda_f)^3 \sqrt{1+\beta^2 a^2}}$

Region (fig. 1)	S_L/S_f	$S_L \bar{y}/b_p S_f$	$S_L \bar{x}/2M_a$
I	$\frac{2(a-d)}{(1-\lambda_f^2)(1-d^2)}$	$\frac{S_L}{S_f} \frac{2(a-d)(t_{cp}'-d)}{3(1-\lambda_f)(1+d^2)}$	$\frac{S_L}{S_f} \frac{1-\lambda_f^2}{1-\lambda_f^3} \left[\frac{(1+ad)-(a-d)t_{cp}'}{1+d^2} \right]$
I _a	$\frac{\lambda_f(1-d)-(1-a)}{(1-\lambda_f^2)(1-d)}$	$\frac{S_L}{S_f} \frac{2(a-d)(t_{cp}'-d)}{3(1-\lambda_f)(1+d^2)}$	$\frac{S_L}{S_f} \frac{1-\lambda_f^2}{1-\lambda_f^3} \left[\frac{(1+ad)-(a-d)t_{cp}'}{1+d^2} \right]$
I _b	$\frac{\lambda_f(1-d)-(1-a)}{(1+\lambda_f)(a-d)}$	$\frac{2}{3} \frac{S_L}{S_f}$	$\frac{S_L}{S_f} \frac{1-\lambda_f^2}{1-\lambda_f^3} \left[\frac{(1-\lambda_f)(r_{cp}-a)}{a-d} \right]$
I _c	-----	-----	$\frac{(a-d)[(1+ad)-(a-d)t_{cp}']}{(1-\lambda_f^2)(1-d)(1+d^2)}$
II	$\frac{2\lambda_f^2(a-d)}{(1-\lambda_f^2)(1-d^2)}$	$\frac{S_L}{S_f} \left[1 - \frac{2\lambda_f(a-d)(t_{cp}'+d)}{3(1-\lambda_f)(1+d^2)} \right]$	$\frac{S_L}{S_f} \lambda_f \frac{1-\lambda_f^2}{1-\lambda_f^3} \left[\frac{(1+ad)+(a-d)t_{cp}'}{1+d^2} \right]$
II _a	-----	-----	$\frac{\lambda_f^2[\lambda_f(1+a)-(1+d)][(1+ad)+(a-d)t_{cp}']}{(1-\lambda_f^2)(1+d)(1+d^2)}$
II _b	-----	-----	$\frac{(1-\lambda_f)^2[\lambda_f(1+a)-(1+d)](r_{cp}+a)}{(1-\lambda_f^2)(a-d)^2}$
II _c	-----	-----	$\frac{\lambda_f^2(a-d)[(1+ad)+(a-d)t_{cp}']}{(1-\lambda_f^2)(1+d)(1+d^2)}$
III	$\frac{\lambda_f^2(a-d)}{(1-\lambda_f^2)(1+d)}$	$\frac{S_L}{S_f} \left[1 - \frac{2\lambda_f(a-d)(t_{cp}'+d)}{3(1-\lambda_f)(1+d^2)} \right]$	$\frac{\lambda_f^2(a-d)[(1+ad)+(a-d)t_{cp}']}{(1-\lambda_f^2)(1+d)(1+d^2)}$
III _a	-----	-----	$\frac{\lambda_f^2[\lambda_f(1+a)-(1+d)][(1+ad)+(a-d)t_{cp}']}{(1-\lambda_f^2)(1+d)(1+d^2)}$
III _b	-----	-----	$\frac{(1-\lambda_f)^2[\lambda_f(1+a)-(1+d)](r_{cp}+a)}{(1-\lambda_f^2)(a-d)^2}$
Two-dimensional	$\frac{\left(\frac{1-a}{1-d}\right) - \lambda_f^2 \left(\frac{1+a}{1+d}\right)}{(1-\lambda_f^2)}$	$\left\{ \frac{(1+a)(1-\lambda_f) - \frac{(a-d)^3}{(1-\lambda_f)^2(1-d)^2}}{\frac{[(1+d) - \lambda_f(1+a)]^3}{(1-\lambda_f)^2(1+d)^2}} \right\} \frac{1}{3(a-d)(1+\lambda_f)}$	$\frac{\left(\frac{1-a}{1-d}\right)^2 - \lambda_f^2 \left(\frac{1+a}{1+d}\right)^2}{2(1-\lambda_f^2)}$

TABLE III.—COMPONENT PARTS OF EQUATIONS USED IN CALCULATING CHARACTERISTICS OF DEFLECTED CONTROLS HAVING UNTAPERED PLAN FORMS

(a) Average Pressure Ratio and Center-Of-Pressure Ray Location

[Values of P and t_{cp} (or r_{cp}) for regions II, II_a, II_b, and II_c are obtained by substituting $-a$ for a in equations for regions I, I_a, I_b, and I_c, respectively]

Region (fig. 1)	Average pressure ratio	Center-of-pressure ray location (t_{cp} or r_{cp})
I	$P = \frac{1}{2}$	$t_{cp} = \frac{8a + (1+a^2)}{4(1-a^2)}$
I _a	$P^* = \frac{1}{(1+a)[1-(1-a)A_f'] \left\{ \frac{\sqrt{(1-a^2)[1+2aA_f'-(1-a^2)A_f'^2]} - \frac{(1-a^2)A_f'-a}{\pi} \cos^{-1} [(1-a^2)A_f'-a] \right\}}$	$t_{cp}^* = \frac{1}{4P^*(1+a)(1-a^2)[1-(1-a)A_f']} \left\{ \frac{(7a^2-2a^4+1)-2A_f'(1-a^2)^2[2a+(1+a^2)A_f']}{\pi} \cos^{-1} [(1-a^2)A_f'-a] + \frac{[a(7-a^2)+(1-a^4)A_f']\sqrt{(1-a^2)[1+2aA_f'-(1-a^2)A_f'^2]}}{\pi} \right\}$
I _b	$P^* = \frac{1}{1-(1-a)A_f'} \left\{ \frac{1}{\pi} \cos^{-1} [(1-a^2)A_f'-a] + \frac{A_f'\sqrt{1-a^2}}{\pi} \log_e \left \frac{1+aA_f'-\sqrt{1+2aA_f'-(1-a^2)A_f'^2}}{A_f'} \right \right\}$	$r_{cp}^* = \frac{1}{2P^*[1-(1-a)A_f']} \left\{ \frac{1+2aA_f'}{\pi A_f'} \cos^{-1} [(1-a^2)A_f'-a] - \frac{\sqrt{(1-a^2)[1+2aA_f'-(1-a^2)A_f'^2]}}{\pi} + \frac{aA_f'\sqrt{1-a^2}}{\pi} \log_e \left \frac{1+aA_f'-\sqrt{1+2aA_f'-(1-a^2)A_f'^2}}{A_f'} \right \right\}$
I _c	$P = \frac{1}{(1+a)} \left(1 + \frac{a}{\pi} \cos^{-1} a - \frac{\sqrt{1-a^2}}{\pi} \right)$	$t_{cp} = \frac{1}{4P(1+a)(1-a^2)} \left[8a + (1+a^2) + \frac{7a^2-2a^4+1}{\pi} \cos^{-1} a - \frac{a(7-a^2)\sqrt{1-a^2}}{\pi} \right]$
III	$P = \frac{1}{2}$	$t_{cp} = \frac{5-8a-3a^2}{8(1+a)}$
III _a	$P^* = \frac{1}{2[1-(1+a)A_f']} \left\{ \frac{2\sqrt{A_f'(1+a)[1-A_f'(1+a)]}}{\pi} - \frac{2A_f'(1+a)-1}{\pi} \cos^{-1} [2A_f'(1+a)-1] \right\}$	$t_{cp}^* = \frac{1}{16P^*(1+a)[1-(1+a)A_f']} \left\{ \frac{3-8a-5a^2-8A_f'(1+a)^2[A_f'(1+a^2)-2a]}{\pi} \cos^{-1} [2A_f'(1+a)-1] - 2 \left[\frac{5a^2+8a-3-2A_f'(1+a)(1+a^2)}{\pi} \right] \sqrt{A_f'(1+a)[1-A_f'(1+a)]} \right\}$
III _b	$P^* = \frac{1}{1-(1+a)A_f'} \left\{ \frac{1}{\pi} \cos^{-1} [2A_f'(1+a)-1] - \frac{2\sqrt{A_f'(1+a)[1-(1+a)A_f']}}{\pi} \right\}$	$r_{cp}^* = \frac{1}{6P^*A_f'[1-(1+a)A_f']} \left\{ \frac{3(1-2aA_f')}{\pi} \cos^{-1} [2A_f'(1+a)-1] - \frac{2[1+2A_f'(1-2a)]}{\pi} \sqrt{A_f'(1+a)[1-(1+a)A_f']} \right\}$

(b) Geometric Parameters $S_f = \frac{\beta b_f^2}{A_f'}$ and $2M_a = \frac{\beta^2 b_f^3}{A_f'^2 \sqrt{1+\beta^2 a^2}}$

Region (fig. 1)	S_L/S_f	$S_L \bar{y}/b_f S_f$	$S_L \bar{x}/2M_a$
I	$\frac{1}{A_f'(1-a^2)}$	$\frac{S_L}{S_f} \frac{(1+4a)}{6A_f'(1-a^2)}$	$\frac{2}{3} \frac{S_L}{S_f}$
I _a	$\frac{1-A_f'(1-a)}{2A_f'(1-a)}$	$\frac{S_L}{S_f} \frac{2(t_{cp}-a)}{3A_f'(1+a^2)}$	$\frac{2}{3} \frac{S_L}{S_f}$
I _b	$\frac{1-A_f'(1-a)}{2}$	$\frac{2}{3} \frac{S_L}{S_f}$	$\frac{2}{3} \frac{S_L}{S_f} A_f'(r_{cp}-a)$
I _c	-----	-----	$\frac{1}{3A_f'(1-a)}$
II	$\frac{1}{A_f'(1-a^2)}$	$\frac{S_L}{S_f} \left[1 - \frac{(1-4a)}{6A_f'(1-a^2)} \right]$	$\frac{2}{3} \frac{S_L}{S_f}$
II _a	-----	-----	$\frac{1-A_f'(1+a)}{3A_f'(1+a)}$
II _b	-----	-----	$\frac{A_f'(r_{cp}+a)[1-A_f'(1+a)]}{3}$
II _c	-----	-----	$\frac{1}{3A_f'(1+a)}$
III	$\frac{1}{2A_f'(1+a)}$	$\frac{S_L}{S_f} \left[1 - \frac{5}{12A_f'(1+a)} \right]$	$\frac{2}{3} \frac{S_L}{S_f}$
III _a	-----	-----	$\frac{1-A_f'(1+a)}{3A_f'(1+a)}$
III _b	-----	-----	$\frac{A_f'(r_{cp}+a)[1-A_f'(1+a)]}{3}$
Two-dimensional	$\frac{A_f'(1-a^2)-1}{A_f'(1-a^2)}$	$\frac{3A_f'(1-a)(1-a^2)[A_f'(1+a)-1]-4a}{6A_f'^2(1-a^2)^2}$	$\frac{3A_f'(1-a^2)-4}{6A_f'(1-a^2)}$

TABLE IV.—EXAMPLE OF NUMERICAL INTEGRATION OF PRESSURES ALONG AN INCLINED SECTION INTERSECTING WING-ROOT MACH CONE

$1-p'$							
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
n	$1-(1)$	$\frac{1-p'}{K_1 \times (1)}$	$\left[\frac{(2)}{(3)}\right]^2$	$\frac{1-p'}{\theta^2 \times (4)}$	$\frac{K_2}{(5)}$	$(6)-1$	$\frac{\cos^{-1}(7)}{\pi}$
0.1	0.9	0.98	0.8434	0.7892	1.9007	0.9007	0.1431
.2	.8	.96	.6944	.8264	1.8151	.8151	.1967
.3	.7	.94	.5545	.8614	1.7414	.7414	.2342
.4	.6	.92	.4253	.8937	1.6784	.6784	.2627
.5	.5	.90	.3086	.9229	1.6253	.6253	.2850
.6	.4	.88	.2066	.9484	1.5816	.5816	.3024
.7	.3	.86	.1217	.9696	1.5470	.5470	.3158
.8	.2	.84	.0567	.9858	1.5216	.5216	.3253
.9	.1	.82	.0149	.9963	1.5056	.5056	.3313
1.0	0	.80	0	1.0000	1.5000	.5000	.3333

$$\theta = 0.50$$

$$K_1 = \frac{\tan \eta}{\beta} = 0.20$$

$$\theta^2 = 0.25$$

$$K_2 = 2(1-\theta^2) = 1.50$$

MULTIPLIERS																	
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)						
n	1-p''	t _{ep}															
		p*															
		(13)	(14)	(15)	(16)	(17)	(18)										
		Incr. Σ (11) $\times$ (12)	(12)-(13) $K_1 \times$ (13)	(14)	(15)	(16)	(17)										
		$\Sigma(1-p'') \times$ (1) to (10)															
		1.0	0.9	0.8	0.7	0.6	0.5	0.4	0.3	0.2	n=0.1						
		0.1	0.1431	0.130786	0.037500	-0.004167	0.004167	0	0	0	0.009797	0.000626	0.009171	0.000672	0.9482	10.0000	0.0980
		.2	.1967	-.045340	.079167	.054167	-.020833	0	0	0	.017154	.026951	.023732	.026307	.9021	5.0000	.1348
		.3	.2342	0	-.020833	.054167	.079167	0	0	0	.021650	.048601	.039940	.046869	.8522	3.3333	.1620
		.4	.2627	0	.004167	-.004167	.037500	.037500	0	0	.024891	.073492	.056091	.070012	.8012	2.5000	.1837
		.5	.2850	0	0	.079167	.054167	-.020833	0	0	.027430	.106922	.023764	.071158	.7493	2.0000	.2018
.6	.3024	0	0	0	.004167	-.004167	.037500	.037500	.029407	.130329	.045951	.084378	.6965	1.6667	.2171		
.7	.3158	0	0	0	0	0	0	0	.030940	.161269	.066075	.095194	.6430	1.4286	.2305		
.8	.3253	0	0	0	0	0	.079167	.054167	.032082	.193351	.090148	.103203	.5887	1.2500	.2417		
.9	.3313	0	0	0	0	0	-.020833	.054167	.079167	.032861	.226212	.118092	.108120	.202594	.5337	.2513	
1.0	.3333	0	0	0	0	0	.004167	-.004167	.037500	.033266	.259478	.149689	.109789	.229540	.4783	.2595	

MOMENT											
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
0.1	0.1431	0.006667	0	0	0	0	0	0	0	0.000626	0
.2	.1967	-.001667	.000833	0	0	0	0	0	0	.002593	0
.3	.2342	0	.023750	.016250	0	0	0	0	0	.005442	0
.4	.2627	0	-.008333	.021667	0	0	0	0	0	.003740	0
.5	.2850	0	0	.002083	.018750	.018750	.002083	0	0	.012303	0
.6	.3024	0	0	0	.047500	.032500	-.012500	0	0	.016187	0
.7	.3158	0	0	0	0	.037917	.055417	0	0	.021024	0
.8	.3253	0	0	0	0	-.003333	.030000	.033333	-.006667	.024073	0
.9	.3313	0	0	0	0	0	0	.060000	.060000	.027944	0
1.0	.3333	0	0	0	0	0	0	-.003333	.041667	.031597	0

TABLE V.—EXAMPLE OF NUMERICAL INTEGRATION OF PRESSURES ALONG A STREAMWISE SECTION INTERSECTING WING-ROOT MACH CONE—Concluded

(1)	(2)	(3)	(4)	(5)	(6)
r'	$[1+(1)]^2$	$\frac{\theta^2 + K_1 \times (2)}{K_1 \times (2)}$	$(2) - \theta^2$	$\frac{(3)}{(4)}$	$\frac{\cos^{-1}(5)}{\pi}$
2.0	9.00	4.750	8.75	0.5429	0.3173
3.0	16.00	8.250	15.75	.5238	.3245
4.0	25.00	12.750	24.75	.5152	.3277
5.0	36.00	18.250	35.75	.5105	.3295
6.0	49.00	24.750	48.75	.5077	.3305
7.0	64.00	32.250	63.75	.5059	.3312
8.0	81.00	40.750	80.75	.5046	.3316
9.0	100.00	50.250	99.75	.5038	.3319
10.0	121.00	60.750	120.75	.5031	.3322

$$\theta = 0.50$$

$$\theta^2 = 0.25$$

$$K_1 = (1 - 2\theta^2) = 0.50$$

AREA	r'	$1 - P'$	MULTIPLIERS										t_{cp}					P''	
			(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)		(16)
			$r' = 2.0$	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	$\Sigma(1 - P') \times$ (1) to (9)	Incr. Σ (10) _{area}	Incr. Σ (10) _{mom.}	(11) + (12)	$\frac{(11)}{(13)}$	$\frac{1}{r'}$		$(11) \times (15)$
1.0	0.2952	0.375000	-0.041667	0.041667	0	0	0	0	0	0	0.23825	0.23825	0.13403			0.4770	0.50000	0.2731	
2.0	.3173	.791667	.541667	-.208333	0	0	0	0	0	0	.30795	.54620	.59893	1.14513	0.3822	.33333	.2893	.2985	
3.0	.3245	-.208333	.541667	.791667	0	0	0	0	0	0	.32169	.80789	1.40307	2.27096	.3183	.25000	.2985	.3045	
4.0	.3277	.041667	-.375000	.791667	-.041667	.041667	0	0	0	0	.32598	1.19387	2.54540	3.73927	.2745	.20000	.3088	.3119	
5.0	.3295	0	0	.791667	.541667	-.208333	0	0	0	0	.32869	1.52256	4.02470	5.54726	.2408	.16667	.3144	.3163	
6.0	.3305	0	0	0	-.208333	.541667	.791667	0	0	0	.33005	1.85291	5.84000	7.69261	.2146	.12500	.3163	.3179	
7.0	.3312	0	0	0	.041667	-.375000	.791667	.375000	-.041667	.041667	.33085	2.18346	7.90066	10.17412	.1936	.11111	.3179	.3190	
8.0	.3316	0	0	0	0	0	0	.791667	.541667	-.208333	.33141	2.51487	10.47630	12.90117	.1763	.10000	.3190	.3200	
9.0	.3319	0	0	0	0	0	0	-.208333	.541667	.791667	.33175	2.84662	13.29623	16.14285	.1619	.10000	.3200	.3210	
10.0	.3322	0	0	0	0	0	0	.041667	-.041667	.375000	.33205	3.17867	16.45070	19.62937	.1519	.10000	.3210	.3220	

MOMENT																	
1.0	.2052	0.375000	-0.041667	0.041667	0	0	0	0	0	0	0.13403						
2.0	.3173	1.583333	1.083333	-.416667	0	0	0	0	0	0	.46490						
3.0	.3245	-.625000	1.625000	2.375000	0	0	0	0	0	0	.80414						
4.0	.3277	.166667	-.166667	1.500000	1.500000	-.166667	.166667	0	0	0	1.14233						
5.0	.3295	0	0	0	3.958333	2.708333	-1.041667	0	0	0	1.47930						
6.0	.3305	0	0	0	-1.250000	3.250000	4.750000	0	0	0	1.81530						
7.0	.3312	0	0	0	.291667	-.291667	2.625000	2.625000	-.291667	.291667	2.15065						
8.0	.3316	0	0	0	0	0	0	6.333333	4.333333	-1.666667	2.48564						
9.0	.3319	0	0	0	0	0	0	-1.875000	4.875000	7.125000	2.81993						
10.0	.3322	0	0	0	0	0	0	.416667	-.416667	3.750000	3.15447						

MACH CONF

(1)	(2)	(3)	(4)	(5)	(6)	(7)
n	$K_2 \times (1)$	$K_4 - (2)$	$K_5 \times (1)$	$K_1 - (4)$	$\frac{(3)}{(5)}$	$\frac{\cos^{-1}(6)}{\pi}$
0.1	0.27	1.23	0.03	1.47	0.8367	0.1844
.2	.54	.96	.06	1.44	.6867	.2677
.3	.81	.69	.09	1.41	.4884	.3372
.4	1.08	.42	.12	1.38	.3044	.4016
.5	1.35	.15	.15	1.35	.1111	.4646
.6	1.62	-.12	.18	1.32	-.0909	.5290
.7	1.89	-.39	.21	1.29	-.3023	.5978
.8	2.16	-.66	.24	1.26	-.5238	.6755
.9	2.43	-.93	.27	1.23	-.7561	.7729
1.0	2.70	-1.20	.30	1.20	-1.0000	1.0000

$$\begin{aligned} \varphi &= 0.50 \\ K_1 &= \frac{\tan \eta}{\beta} = 0.20 \\ K_2 &= (2 + \varphi + K_1) = 2.70 \\ K_3 &= (\varphi - K_1) = 0.30 \\ K_4 &= (1 + \varphi) = 1.50 \end{aligned}$$

AREA		MULTIPLIERS										t_{cp}	P^*								
		(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)			(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)
n	$1-P'$	$n=0.1$	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0	$\Sigma(1-P') \times (1) \text{ to } (10)$	$\text{Incr. } \Sigma (11)_{area}$	$\text{Incr. } \Sigma (11)_{mom.}$	$(12)-(13)$	$(12)+(13)$	$\frac{(14)}{(15)}$	$\frac{1}{n}$	$\frac{(12) \times (17)}{(17)}$		
0.1	0.1844	0.113301	0.037500	-0.004167	0.004167	0	0	0	0	0	0	0.012065	0.012065	0.000766	0.011209	0.012218	0.9248	10.0000	0.1207		
0.2	0.2667	-0.032975	0.079167	0.054167	0	0	0	0	0	0	0	0.022757	0.034822	0.004243	0.030579	0.035671	0.8573	5.0000	0.1741		
0.3	0.3372	0	-0.020833	0.054167	0.079167	0	0	0	0	0	0	0.030324	0.065146	0.011891	0.053255	0.067524	0.7887	3.3333	0.2172		
0.4	0.4016	0	0.004167	-0.004167	0.037500	0.041667	-0.008333	0	0	0	0	0.036946	0.102092	0.024885	0.077207	0.107069	0.7211	2.5000	0.2552		
0.5	0.4646	0	0	0	0	0.066667	0.066667	0	0	0	0	0.045209	0.145391	0.044426	0.100905	0.154276	0.6544	2.0000	0.2908		
0.6	0.5290	0	0	0	0	-0.008333	0.041667	0.037500	-0.004167	0.004167	0	0.049669	0.195090	0.071793	0.125267	0.200419	0.5886	1.6667	0.3251		
0.7	0.5978	0	0	0	0	0	0	0.079167	0.054167	-0.020833	0	0.056311	0.251371	0.108464	0.142907	0.273004	0.5233	1.4286	0.3591		
0.8	0.6755	0	0	0	0	0	0	-0.020833	0.054167	0.079167	-0.038494	0.065546	0.314917	0.156182	0.158735	0.346153	0.4586	1.2500	0.3936		
0.9	0.7729	0	0	0	0	0	0	0.004167	-0.004167	0.037500	0.121105	0.072212	0.387129	0.217654	0.109475	0.430660	0.3935	1.1111	0.4301		
1.0	1.0000	0	0	0	0	0	0	0	0	0	0.017389	0.084988	0.472117	0.208483	0.173634	0.531814	0.3265	1.0000	0.4721		

[illegible]

TABLE VII.—EXAMPLE OF NUMERICAL INTEGRATION OF PRESSURES ALONG A STREAMWISE SECTION INTERSECTING WING-TIP MAC CONE

(1)	(2)	(3)	(4)	(5)
r'	$K_1 - (1)$	$K_1 + (1)$	$\frac{(2)}{(3)}$	$\frac{\cos^{-1}(4)}{\pi}$
0.1	1.40	1.60	0.8750	0.1609
.2	1.30	1.70	.7647	.2229
.3	1.20	1.80	.6667	.2877
.4	1.10	1.90	.5789	.3035
.5	1.00	2.00	.5000	.3333
.6	.90	2.10	.4286	.3590
.7	.80	2.20	.3636	.3815
.8	.70	2.30	.3043	.4016
.9	.60	2.40	.2500	.4106
1.0	.50	2.50	.2000	.4359

$$\mu = \underline{0.50}$$

$$K_1 = (1 + g) = 1.50$$

		MULTIPLIERS															t, p	P^*	
	r'	$1-P'$	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)
																	</		

[illegible]

TABLE VII.—EXAMPLE OF NUMERICAL INTEGRATION OF PRESSURES ALONG A STREAMWISE SECTION INTERSECTING WING-TIP MACH CONE—Concluded

(1)	(2)	(3)	(4)	(5)
r'	$K_1 - (1)$	$K_1 + (1)$	$\frac{(2)}{(3)}$	$\frac{\cos^{-1}(4)}{\pi}$
2.0	-0.50	3.50	-0.1429	0.5456
3.0	-1.50	4.50	-0.3333	.6082
4.0	-2.50	5.50	-0.4545	.6502
5.0	-3.50	6.50	-0.5385	.6810
6.0	-4.50	7.50	-0.6000	.7048
7.0	-5.50	8.50	-0.6471	.7240
8.0	-6.50	9.50	-0.6842	.7398
9.0	-7.50	10.50	-0.7143	.7533
10.0	-8.50	11.50	-0.7391	.7647

$$g = 0.50$$

$$K_1 = (1 + g) = 1.50$$

		MULTIPLIERS										t_{cp}				P^*	
	r'	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
	$1 - P'$		3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0	$\Sigma(1 - P') \times$ (1) to (9)	Incr. Σ (10) _{area}	Incr. Σ (10) _{mem.}	(11)+(12)	$\frac{(11)}{(13)}$	$\frac{1}{P'}$	(11)×(15)
		$r' = 2.0$															
1.0	0.4359	0.375000	-0.041667	0.041667	0	0	0	0	0	0	0.49578	0.31010	0.18241				
2.0	.5456	.791667	.541667	-.208333	0	0	0	0	0	0	0.49578	.80588	.93798	1.74386	0.4621	0.50000	0.4029
3.0	.6082	-.208333	.541667	.791667	0	0	0	0	0	0	.57072	1.38560	2.39084	3.77644	.3669	.33333	.4619
4.0	.6502	.041667	-.041667	.375000	.375000	-.041667	.041667	0	0	0	.62981	2.01541	4.60144	6.61685	.3046	.25000	.5039
5.0	.6810	0	0	0	.791667	.541667	-.208333	0	0	0	.66628	2.68169	7.60253	10.28422	.2008	.20000	.5363
6.0	.7048	0	0	0	-.208333	.541667	.791667	0	0	0	.69338	3.37507	11.41797	14.79304	.2282	.16667	.5625
7.0	.7240	0	0	0	0	-.041667	.375000	.375000	-.041667	.041667	.71468	4.08975	16.06526	20.15501	.2029	.14286	.5843
8.0	.7398	0	0	0	0	0	0	.791667	.541667	-.208333	.73210	4.82185	21.55735	26.37920	.1828	.12500	.6027
9.0	.7533	0	0	0	0	0	0	-.208333	.541667	.791667	.74673	5.56858	27.90570	33.47428	.1664	.11111	.6187
10.0	.7647	0	0	0	0	0	0	.041667	-.041667	.375000	.75917	6.32775	35.11875	41.44650	.1527	.10000	.6328

MOMENT																	
1.0	0.4359	0.375000	-0.041667	0.041667	0	0	0	0	0	0	0.18241						
2.0	.5456	1.583333	1.083333	-.416667	0	0	0	0	0	0	.75557						
3.0	.6082	-.625000	1.625000	2.375000	0	0	0	0	0	0	1.45286						
4.0	.6502	.166667	-.166667	1.500000	1.500000	.166667	0	0	0	0	2.21060						
5.0	.6810	0	0	0	3.958333	-1.041667	0	0	0	0	3.00109						
6.0	.7048	0	0	0	-1.250000	4.750000	0	0	0	0	3.81544						
7.0	.7240	0	0	0	.291667	2.625000	2.625000	-.291667	.291667	.291667	4.64729						
8.0	.7398	0	0	0	0	0	0	4.333333	-1.666667	-1.666667	5.49209						
9.0	.7533	0	0	0	0	0	0	6.333333	4.333333	4.333333	6.34835						
10.0	.7647	0	0	0	0	0	0	-1.875000	4.875000	7.125000	7.21305						

TABLE VIII.—CONTROL-SURFACE CHARACTERISTICS NOT INCLUDED IN FIGURES

(a) Tapered Controls

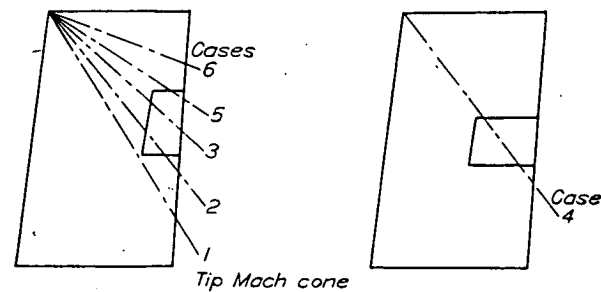
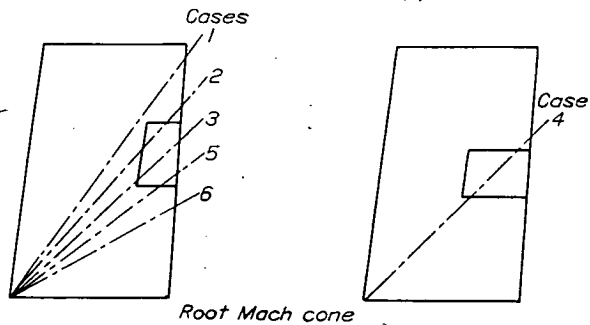
a	d	λ_f	l/λ_f	Wing-tip controls			Inboard controls	a	d	λ_f	l/λ_f	Wing-tip controls			Inboard controls
				$\beta C_{L_4}'$	$\beta C_{L_4}'$	$\beta C_{m_4}'$	$\beta C_{L_4}'$					$\beta C_{L_4}'$	$\beta C_{L_4}'$	$\beta C_{m_4}'$	$\beta C_{L_4}'$
-0.95	-0.80	0.95	0.95	-0.2678	-----	-----	-----	0.20	-0.80	0	0	-0.0474	0.1164	-0.1875	-0.0474
	-0.90	0.95	0.95	-0.2778	-----	-----	-----		-0.80	.20	0	-0.0680	.1139	-0.1865	-0.0661
	-0.80	0.95	0.95	-0.2162	-----	-----	-----		-0.80	.40	0	-0.1162	.1053	-0.1794	-0.1102
	-0.85	0.95	0.95	-0.1335	-----	-----	-----		-0.80	.60	0	-0.2056	.0846	-0.1555	-0.2106
	-0.95	0.95	0.95	-----	0.2236	-0.2751	-----		-0.95	0	0	-0.7606	.2236	-1.3644	-0.7606
	-0.95	0.95	0.95	-----	0.2232	-0.2751	-----		-0.95	.20	0	-0.9921	.2184	-1.3578	-0.9917
-0.80	-0.80	0.95	0.95	-0.1076	-----	-----	-----	.40	-0.80	.40	0	-1.4204	.1997	-1.3085	-1.4551
	-0.85	0.95	0.95	-0.2325	-----	-----	-----		-0.80	0	0	-----	.1164	-0.2133	-----
	-0.95	0.95	0.95	-0.2184	-----	-----	-----		-0.80	.20	0	-----	.1128	-0.2118	-----
	-0.95	0.95	0.95	-0.2184	-----	-----	-----		-0.80	.40	0	-----	.1369	-0.2004	-0.1422
	-0.95	0.95	0.95	-0.2184	-----	-----	-----		-0.95	0	0	-----	.9058	-0.5823	-0.9058
	-0.95	0.95	0.95	-0.2184	-----	-----	-----		-0.95	.20	0	-----	-1.1787	-0.5715	-1.1793
-0.60	-0.80	0.95	0.95	-0.3349	-----	-----	-----	.60	-0.95	.40	0	-----	-1.6605	-0.4907	-1.7248
	-0.85	0.95	0.95	-0.1796	-----	-----	-----		-0.60	0	0	-----	-----	-0.0927	-----
	-0.95	0.95	0.95	-0.2420	-----	-----	-----		-0.60	.20	0	-----	-----	-0.0917	-----
	-0.95	0.95	0.95	-0.2420	-----	-----	-----		-0.60	.40	0	-----	-----	-0.0845	-----
	-0.95	0.95	0.95	-0.2420	-----	-----	-----		-0.80	0	0	-----	-0.0819	-0.2392	-0.0819
	-0.95	0.95	0.95	-0.2420	-----	-----	-----		-0.80	.20	0	-----	-0.1148	-0.2366	-0.1106
-0.40	-0.80	0.95	0.95	-0.3773	-----	-----	-----	.80	-0.80	.40	0	-----	-0.1820	-0.0927	-0.1742
	-0.85	0.95	0.95	-0.6670	-----	-----	-----		-0.95	0	0	-----	-1.0510	-0.2236	-1.0510
	-0.95	0.95	0.95	-0.6670	-----	-----	-----		-0.95	.20	0	-----	-1.3642	-0.2123	-1.3669
	-0.95	0.95	0.95	-0.6670	-----	-----	-----		-0.95	.40	0	-----	-1.8747	-0.1737	-1.9945
	-0.95	0.95	0.95	-0.6670	-----	-----	-----		.75	.95	0	-----	-----	-----	.1405
	-0.95	0.95	0.95	-0.6670	-----	-----	-----		.60	.95	0	-----	-----	-----	.2229
-0.20	-0.80	0.95	0.95	-1.4710	-----	-----	-----	.95	-0.60	0	0	-----	-----	-0.1009	-----
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		-0.60	.20	0	-----	-----	-0.0990	-----
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		-0.60	.40	0	-----	-----	-0.0361	-----
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		-0.80	0	0	-----	-0.0991	-0.2650	-----
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		-0.80	.20	0	-----	-0.1397	-0.2602	-----
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		-0.80	.40	0	-----	-0.2121	-0.2252	-----
0	-0.80	0.95	0.95	-1.4710	-----	-----	-----	.95	-0.95	0	0	-----	-1.1963	-0.2180	-----
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		-0.95	.20	0	-----	-1.5463	-0.1922	-----
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		-0.95	.40	0	-----	-2.0117	-0.1762	-----
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		.90	.95	0	-----	-----	-----	.4493
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		.80	.95	0	-----	-----	-----	.4359
	-0.80	0.95	0.95	-1.4710	-----	-----	-----		-0.60	0	0	-----	-----	-0.1070	-0.0132

(b) Untapered Controls

a	A_f'	Wing-tip controls			Inboard controls	a	A_f'	Wing-tip controls			Inboard controls
		$\beta C_{L_4}'$	$\beta C_{L_4}'$	$\beta C_{m_4}'$	$\beta C_{L_4}'$			$\beta C_{L_4}'$	$\beta C_{L_4}'$	$\beta C_{m_4}'$	$\beta C_{L_4}'$
-0.95	0.8	-1.1421	0.1878	-0.0879	-1.2498	0	6.0	0.0320	0.0669	-0.0330	0.0349
	2.0	-0.4242	.2093	-0.1022	-0.4328		8.0	.0327	.0676	-0.0335	.0349
	4.0	-0.1619	.2164	-0.1070	-0.1605		10.0	.0331	.0681	-0.0337	.0349
	6.0	-0.0720	.2188	-0.1086	-0.0698		.8	.0125	.0446	-0.0182	.0449
	8.0	-0.0265	.2200	-0.1094	-0.0244		2.0	.0272	.0601	-0.0282	.0303
	10.0	-0.0009	.2207	-0.1099	-0.0029		4.0	.0317	.0657	-0.0319	.0375
-0.80	.8	-0.1061	.0962	-0.0447	-0.1034	.20	6.0	.0330	.0675	-0.0332	.0369
	2.0	-0.0117	.1083	-0.0528	-0.0665		8.0	.0337	.0685	-0.0338	.0366
	4.0	-0.0225	.1123	-0.0555	-0.0259		10.0	.0341	.0690	-0.0341	.0364
	6.0	-0.0342	.1137	-0.0564	-0.0366		.8	.0128	.0415	-0.0158	.0608
	8.0	-0.0402	.1143	-0.0568	-0.0420		2.0	.0289	.0603	-0.0275	.0472
	10.0	-0.0437	.1147	-0.0571	-0.0452		4.0	.0341	.0682	-0.0328	.0426
-0.60	.8	-0.0190	.0702	-0.0323	-0.0075	.40	6.0	.0356	.0709	-0.0346	.0411
	2.0	-0.0173	.0804	-0.0391	-0.0232		8.0	.0362	.0722	-0.0354	.0404
	4.0	-0.0302	.0839	-0.0414	-0.0334		10.0	.0366	.0730	-0.0360	.0399
	6.0	-0.0346	.0850	-0.0421	-0.0368		.8	.0120	.0379	-0.0131	.0948
	8.0	-0.0369	.0856	-0.0425	-0.0385		2.0	.0300	.0603	-0.0257	.0641
	10.0	-0.0382	.0859	-0.0427	-0.0395		4.0	.0384	.0736	-0.0345	.0539
-0.40	.8	-0.0005	.0592	-0.0268	-0.0154	.60	6.0	.0406	.0782	-0.0376	.0505
	2.0	-0.0226	.0694	-0.0336	-0.0290		8.0	.0415	.0804	-0.0391	.0487
	4.0	-0.0302	.0728	-0.0358	-0.0336		10.0	.0420	.0818	-0.0400	.0477
	6.0	-0.0328	.0739	-0.0366	-0.0351		.8	.0087	.0321	-0.0087	.2198
	8.0	-0.0341	.0745	-0.0370	-0.0358		2.0	.0296	.0589	-0.0222	.1228
	10.0	-0.0349	.0748	-0.0372	-0.0363		4.0	.0441	.0804	-0.0343	.0905
-0.20	.8	-0.0078	.0527	-0.0233	-0.0263	.80	6.0	.0508	.0921	-0.0420	.0797
	2.0	-0.0245	.0638	-0.0307	-0.0319		8.0	.0535	.0982	-0.0461	.0743
	4.0	-0.0301	.0675	-0.0332	-0.0338		10.0	.0549	.1018	-0.0485	.0711
	6.0	-0.0319	.0688	-0.0340	-0.0344		.8	-0.0048	.0149	-0.0032	1.4733
	8.0	-0.0328	.0694	-0.0344	-0.0347		2.0	.0242	.0526	-0.0157	.6564
	10.0	-0.0334	.0698	-0.0346	-0.0349		4.0	.0440	.0809	-0.0282	.3841
0	.8	-0.0110	.0482	-0.0205	-0.0349	.95	6.0	.0564	.0998	-0.0368	.2933
	2.0	-0.0258	.0611	-0.0291	-0.0349		8.0	.0657	.1145	-0.0438	.2479
	4.0	-0.0305	.0655	-0.0320	-0.0349		10.0	.0732	.1269	-0.0501	.2207

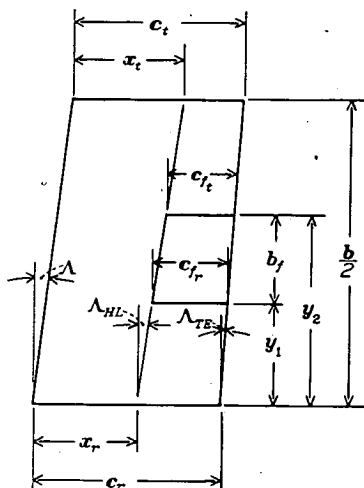
TABLE IX.—COMPUTING FORM FOR C_{h_a}

(a) Dimensions and Preliminary Computing Form



$$M = 1.80$$

$$\beta = \sqrt{M^2 - 1} = 1.4967$$

Root case 3Tip case 3

$$g = \frac{\tan \Lambda}{\beta} = 0.5995$$

$$a = \frac{\tan \Lambda_{HL}}{\beta} = 0.3990$$

$$d = \frac{\tan \Lambda_{TE}}{\beta} = 0.3489$$

$$K_1 = \beta y_2 = 7.8577$$

$$K_2 = \beta y_1 = 2.9934$$

$$K_3 = \beta \left(\frac{b}{2} - y_1 \right) = 5.9868$$

$$K_4 = \beta \left(\frac{b}{2} - y_2 \right) = 1.1225$$

$$K_5 = \beta y_2^2 = 41.2528$$

$$K_6 = \frac{1}{\beta(1-d)} = 1.0262$$

$$K_7 = \frac{1}{\beta(1-a)} = 1.1117$$

$$K_8 = \beta y_1^2 = 5.9868$$

$$K_9 = \beta \left(\frac{b}{2} - y_1 \right)^2 = 23.9472$$

$$K_{10} = \frac{1}{\beta(1+d)} = 0.4953$$

$$K_{11} = \frac{1}{\beta(1+a)} = 0.4776$$

$$K_{12} = \beta \left(\frac{b}{2} - y_2 \right)^2 = 0.8419$$

Tapered flap:

$$2M_a 3\sqrt{1+\beta^2 a^2} = b_f \frac{c_{f_r}^3 - c_{f_t}^3}{c_{f_r} - c_{f_t}} = 5.2159$$

Untapered flap:

$$2M_a 3\sqrt{1+\beta^2 a^2} = 3b_f c_{f_r}^2 = \underline{\hspace{2cm}}$$

TABLE IX.—COMPUTING FORM FOR C_{h_a} —Concluded(b) Form for summing $(PS_L \bar{x})^*$ of triangular segments of conical-flow region

	Figures for determining columns (2) and (3)	Region	Column (6)=0 for cases in table IX(a)	(1) Enter curve at following value of η or r'	(2) P^*	(3) t_{ep}	(4) $3\bar{x}\sqrt{1+\beta^2 a^2}$	(5) $2S_L$	(6) $(2)\times(4)\times(5)$
Root Mach cone	Figure 8	1	3, 5, 6	$\frac{c_r}{K_1} - (1-d) = \underline{\hspace{1cm}}$	<u> </u>	<u> </u>	$\frac{2K_1(1-at_{ep})}{t_{ep}} - 3x_r = \underline{\hspace{1cm}}$	$(1)\times K_1 = \underline{\hspace{1cm}}$	0
		2	2, 3, 4, 5, 6	$\frac{x_r}{K_1} - (1-a) = \underline{\hspace{1cm}}$	<u> </u>	<u> </u>	$3x_r - \frac{2K_1(1-at_{ep})}{t_{ep}} = \underline{\hspace{1cm}}$	$(1)\times K_1 = \underline{\hspace{1cm}}$	0
	Figure 7 $\frac{\tan \eta}{\beta} = d$	3	6	$1 - \frac{K_2}{c_r} (1-d) = \underline{0.6102}$	<u>0.258</u>	<u>0.736</u>	$\frac{2c_r(1-at_{ep})}{(1-d)t_{ep}} - 3x_r = \underline{-2.4965}$	$(1)\times c_r^2 K_2 = \underline{15.6547}$	-10.0832
		4	3, 5, 6	$1 - \frac{K_1}{c_r} (1-d) = \underline{\hspace{1cm}}$	<u> </u>	<u> </u>	$3x_r - \frac{2c_r(1-at_{ep})}{(1-d)t_{ep}} = \underline{\hspace{1cm}}$	$(1)\times c_r^2 K_2 = \underline{\hspace{1cm}}$	0
	Figure 7 $\frac{\tan \eta}{\beta} = a$	5	4, 5, 6	$1 - \frac{K_2}{x_r} (1-a) = \underline{0.5502}$	<u>.245</u>	<u> </u>	$x_r = \underline{4.0000}$	$(1)\times x_r^2 K_2 = \underline{9.7865}$	9.5908
		6	2, 3, 4, 5, 6	$1 - \frac{K_1}{x_r} (1-a) = \underline{\hspace{1cm}}$	<u> </u>	<u> </u>	$-x_r = \underline{\hspace{1cm}}$	$(1)\times x_r^2 K_2 = \underline{\hspace{1cm}}$	0
	Figure 8	7	6	$\frac{c_r}{K_2} - (1-d) = \underline{1.0192}$	<u>.300</u>	<u>.645</u>	$3x_r - \frac{2K_2(1-at_{ep})}{t_{ep}} = \underline{5.1074}$	$(1)\times K_1 = \underline{6.1017}$	9.3941
		8	4, 5, 6	$\frac{x_r}{K_2} - (1-a) = \underline{0.7353}$	<u>.277</u>	<u>.711</u>	$\frac{2K_2(1-at_{ep})}{t_{ep}} - 3x_r = \underline{-5.9688}$	$(1)\times K_2 = \underline{4.4021}$	-7.2778
Tip Mach cone	Figure 10	1	3, 5, 6	$\frac{c_t}{K_3} - (1+d) = \underline{\hspace{1cm}}$	<u> </u>	<u> </u>	$\frac{2K_3(1+at_{ep})}{t_{ep}} - 3x_t = \underline{\hspace{1cm}}$	$(1)\times K_3 = \underline{\hspace{1cm}}$	0
		2	2, 3, 4, 5, 6	$\frac{x_t}{K_3} - (1+a) = \underline{\hspace{1cm}}$	<u> </u>	<u> </u>	$3x_t - \frac{2K_3(1+at_{ep})}{t_{ep}} = \underline{\hspace{1cm}}$	$(1)\times K_3 = \underline{\hspace{1cm}}$	0
	Figure 9 $\frac{\tan \eta}{\beta} = d$	3	6	$1 - \frac{K_4(1+d)}{c_t} = \underline{0.4494}$	<u>.280</u>	<u>.662</u>	$\frac{2c_t(1+at_{ep})}{(1+d)t_{ep}} - 3x_t = \underline{-0.9520}$	$(1)\times c_t^2 K_4 = \underline{1.6833}$	- .4487
		4	3, 5, 6	$1 - \frac{K_3(1+d)}{c_t} = \underline{\hspace{1cm}}$	<u> </u>	<u> </u>	$3x_t - \frac{2c_t(1+at_{ep})}{(1+d)t_{ep}} = \underline{\hspace{1cm}}$	$(1)\times c_t^2 K_4 = \underline{\hspace{1cm}}$	0
	Figure 9 $\frac{\tan \eta}{\beta} = a$	5	4, 5, 6	$1 - \frac{K_4(1+a)}{x_t} = \underline{0.2862}$	<u>.224</u>	<u> </u>	$x_t = \underline{2.2000}$	$(1)\times x_t^2 K_4 = \underline{0.6616}$	.3260
		6	2, 3, 4, 5, 6	$1 - \frac{K_3(1+a)}{x_t} = \underline{\hspace{1cm}}$	<u> </u>	<u> </u>	$-x_t = \underline{\hspace{1cm}}$	$(1)\times x_t^2 K_4 = \underline{\hspace{1cm}}$	0
	Figure 10	7	6	$\frac{c_t}{K_4} - (1+d) = \underline{1.1010}$	<u>.314</u>	<u>.609</u>	$3x_t - \frac{2K_4(1+at_{ep})}{t_{ep}} = \underline{2.0179}$	$(1)\times K_3 = \underline{0.9185}$	.5820
		8	4, 5, 6	$\frac{x_t}{K_4} - (1+a) = \underline{0.5609}$	<u>.236</u>	<u>.751</u>	$\frac{2K_4(1+at_{ep})}{t_{ep}} - 3x_t = \underline{-2.7150}$	$(1)\times K_4 = \underline{0.4722}$	- .3025
$C_{h_a} = \frac{-2}{57.3\beta\sqrt{1-\rho^2}} \left(1 - \frac{\sum (6)}{2M_\infty^3\sqrt{1+\beta^2 a^2}} \right) = \underline{-0.0194}$									$\sum (6) = 1.7358$

REPORT 1042

SOME EFFECTS OF NONLINEAR VARIATION IN THE DIRECTIONAL-STABILITY AND DAMPING-IN-YAWING DERIVATIVES ON THE LATERAL STABILITY OF AN AIRPLANE¹

By LEONARD STERNFIELD

SUMMARY

A theoretical investigation has been made to determine the effect of nonlinear stability derivatives on the lateral stability of an airplane. Motions were calculated on the assumption that the directional-stability and the damping-in-yawing derivatives are functions of the angle of sideslip. The application of the Laplace transform to the calculation of an airplane motion when certain types of nonlinear derivatives are present is described in detail. The types of nonlinearities assumed correspond to the condition in which the values of the directional-stability and damping-in-yawing derivatives are zero for small angles of sideslip.

The results of the investigation indicated that under certain conditions the nonlinear stability derivatives assumed in the analysis caused a motion which had different rates of damping for the large and small amplitudes of motion, with very little damping at the small amplitudes. In general, the period of the resultant oscillation increased with time.

INTRODUCTION

Recent flight tests of several airplanes designed for high-speed high-altitude flight have indicated neutrally damped lateral oscillations of small amplitude generally referred to as snaking. Upon examination of the flight records, the decrement of the oscillatory motion is found in some cases to be different for the large and small amplitudes of motion with a neutrally stable oscillation occurring at the small amplitudes. One of the explanations offered for the cause of this type of motion is that some of the stability derivatives are nonlinear; that is, the derivatives have different values for the large and small amplitudes of motion. The nonlinearity could be caused by boundary-layer effects or flow separation due to poor fairing at the junction of the tail surfaces.

The present report represents a preliminary investigation of the effect of the presence of two nonlinear stability derivatives, the directional-stability derivative $C_{n\beta}$ and the damping-in-yawing derivative C_{nr} , on the motion of an airplane. These derivatives were selected for the analysis since the damping of the oscillation is a function of C_{nr} and

since C_{nr} depends upon the $C_{n\beta}$ contributed by the tail. The derivatives $C_{n\beta}$ and C_{nr} were both assumed to be functions of the sideslip angle β . Calculations were made of the airplane motion due to a disturbance in sideslip for three different types of nonlinearities.

SYMBOLS AND COEFFICIENTS

ϕ	angle of roll, radians
ψ	angle of yaw, radians
β	angle of sideslip, radians except where noted in figures (v/V)
r	yawing angular velocity, radians per second ($d\psi/dt$)
p	rolling angular velocity, radians per second ($d\phi/dt$)
v	sideslip velocity along the Y-axis, feet per second
V	airspeed, feet per second
ρ	mass density of air, slugs per cubic foot
q	dynamic pressure, pounds per square foot ($\frac{1}{2} \rho V^2$)
b	wing span, feet
S	wing area, square feet
W	weight of airplane, pounds
m	mass of airplane, slugs (W/g)
g	acceleration due to gravity, feet per second per second
μ_b	relative density factor ($m/\rho S b$)
η	inclination of principal longitudinal axis of airplane with respect to flight path, positive when principal axis is above flight path at the nose, degrees
γ	angle of flight path to horizontal axis, positive in climb, degrees
k_{X_0}	radius of gyration in roll about principal longitudinal axis, feet
k_{Z_0}	radius of gyration in yaw about principal vertical axis, feet
K_{X_0}	nondimensional radius of gyration in roll about principal longitudinal axis (k_{X_0}/b)

¹ Supersedes NACA TN 2233, "Some Effects of Nonlinear Variation in the Directional-Stability and Damping-in-Yawing Derivatives on the Lateral Stability of an Airplane" by Leonard Sternfield, 1950.

K_{z_0}	nondimensional radius of gyration in yaw about principal vertical axis (k_{z_0}/b)
K_x	nondimensional radius of gyration in roll about longitudinal stability axis ($\sqrt{K_{x_0}^2 \cos^2 \eta + K_{z_0}^2 \sin^2 \eta}$)
K_z	nondimensional radius of gyration in yaw about vertical stability axis ($\sqrt{K_{z_0}^2 \cos^2 \eta + K_{x_0}^2 \sin^2 \eta}$)
K_{xz}	nondimensional product-of-inertia parameter ($(K_{z_0}^2 - K_{x_0}^2) \sin \eta \cos \eta$)
C_L	trim lift coefficient ($\frac{W \cos \gamma}{qS}$)
C_l	rolling-moment coefficient ($\frac{\text{Rolling moment}}{qSb}$)
C_n	yawing-moment coefficient ($\frac{\text{Yawing moment}}{qSb}$)
C_Y	lateral-force coefficient ($\frac{\text{Lateral force}}{qS}$)
$C_{l_\beta} = \frac{\partial C_l}{\partial \beta}$	
$C_{n_\beta} = \frac{\partial C_n}{\partial \beta}$	
$C_{Y_\beta} = \frac{\partial C_Y}{\partial \beta}$	
$C_{n_r} = \frac{\partial C_n}{\partial \left(\frac{rb}{2V}\right)}$	
$C_{n_p} = \frac{\partial C_n}{\partial \left(\frac{pb}{2V}\right)}$	
$C_{l_p} = \frac{\partial C_l}{\partial \left(\frac{pb}{2V}\right)}$	
$C_{Y_p} = \frac{\partial C_Y}{\partial \left(\frac{pb}{2V}\right)}$	
$C_{Y_r} = \frac{\partial C_Y}{\partial \left(\frac{rb}{2V}\right)}$	
$C_{l_r} = \frac{\partial C_l}{\partial \left(\frac{rb}{2V}\right)}$	
C_{n_c}	yawing-moment constant
t	time, seconds
s_b	nondimensional time parameter based on span (Vt/b)
D_b	differential operator ($\frac{d}{ds_b}$)
σ	operator in Laplace transformation
$T_{1/2}$	time for amplitude of oscillation to damp to one-half its original value, seconds

The subscript 0 is used to indicate initial conditions and a bar is used to denote variables in the operational equations.

ANALYSIS

NONLINEAR STABILITY DERIVATIVES

The assumptions made with regard to the nonlinearity of the stability derivatives C_{n_β} and C_{n_r} are shown in figure 1.

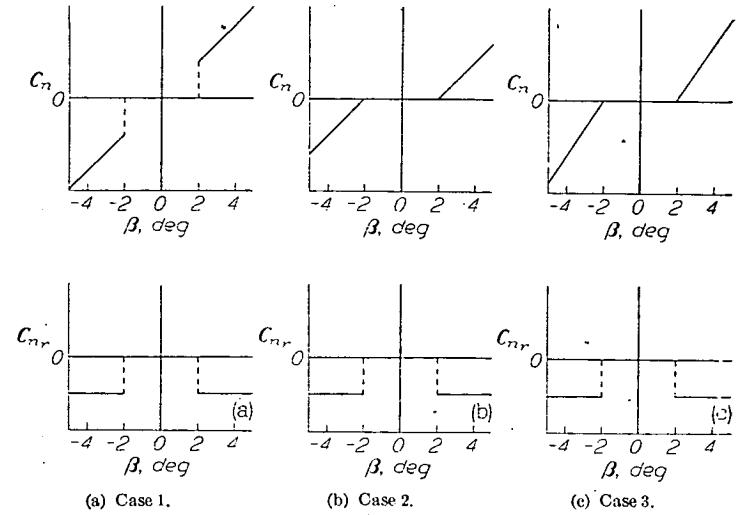


FIGURE 1.—Three types of nonlinear stability derivatives assumed in the analysis.

For all three cases presented in the figure, C_n is equal to zero for $-2^\circ < \beta < 2^\circ$, a region which is subsequently referred to as a dead spot. Thus when the airplane is within the dead spot, the value of the directional stability derivative C_{n_β} is zero. Since the damping-in-yawing derivative C_{n_r} is a direct function of C_{n_β} contributed by the tail, C_{n_r} was also assumed to be zero for values of $-2^\circ < \beta < 2^\circ$. In the region outside of the dead spot, each one of the cases represents a different type of variation of C_n with β in order to simulate the effect of several possible flow conditions on the side force acting on the vertical surface. For cases 1 and 2, $C_{n_\beta} = 0.28$ and for case 3, $C_{n_\beta} = 0.41$. The corresponding value of C_{n_r} for all three cases is -0.39 . It should be noted in figure 1 that for cases 2 and 3, $C_n = 0$ at β of 2° and -2° , whereas for case 1, C_n has a finite value at β of 2° and -2° .

METHOD OF CALCULATING MOTION

Since the nonlinearities shown in figure 1 can be treated as linear derivatives of different values within and outside of the dead spot, the airplane motion is calculated on the basis of classical linear theory. The equations of motion and the general method of calculating the motion of an airplane are given in references 1 and 2. The methods of references 1 and 2 are based on the Laplace transformation which inherently takes into account the initial conditions of the problem. Because the Laplace transformation considers the initial displacements and initial velocities of the problem, this method is directly applicable to the calculation of the motion of an airplane which has nonlinear derivatives similar to the derivatives presented in figure 1.

The nondimensional linearized lateral equations of motion, referred to the stability axes, are for rolling, for yawing, and for sideslipping, respectively:

$$\left. \begin{aligned} 2\mu_b(K_x^2 D_b^2 \phi + K_{xz} D_b^2 \psi) &= C_{l_\beta} \beta + \frac{1}{2} C_{l_p} D_b \phi + \frac{1}{2} C_{l_r} D_b \psi \\ 2\mu_b(K_z^2 D_b^2 \psi + K_{xz} D_b^2 \phi) &= C_{n_\beta} \beta + \frac{1}{2} C_{n_p} D_b \phi + \frac{1}{2} C_{n_r} D_b \psi + C_{n_c} \\ 2\mu_b(D_b \beta + D_b \psi) &= C_{Y_\beta} \beta + \frac{1}{2} C_{Y_p} D_b \phi + C_L \phi + \frac{1}{2} C_{Y_r} D_b \psi + (C_L \tan \gamma) \psi \end{aligned} \right\} \quad (1)$$

The Laplace transformation of equations (1), with the use of the symbol σ for the operator, is

$$\left. \begin{aligned} \left(2\mu_b K_X^2 \sigma^2 - \frac{1}{2} C_{l_p} \sigma\right) \bar{\phi} + \left(2\mu_b K_{XZ} \sigma^2 - \frac{1}{2} C_{l_r} \sigma\right) \bar{\psi} - C_{l_\beta} \bar{\beta} &= 2\mu_b K_X^2 [\sigma \phi_0 + (D_b \phi)_0] - \frac{1}{2} C_{l_p} \phi_0 + 2\mu_b K_{XZ} [\sigma \psi_0 + (D_b \psi)_0] - \frac{1}{2} C_{l_r} \psi_0 \\ \left(2\mu_b K_{XZ} \sigma^2 - \frac{1}{2} C_{n_p} \sigma\right) \bar{\phi} + \left(2\mu_b K_Z^2 \sigma^2 - \frac{1}{2} C_{n_r} \sigma\right) \bar{\psi} - C_{n_\beta} \bar{\beta} &= 2\mu_b K_{XZ} [\sigma \phi_0 + (D_b \phi)_0] - \frac{1}{2} C_{n_p} \phi_0 + 2\mu_b K_Z^2 [\sigma \psi_0 + (D_b \psi)_0] - \frac{1}{2} C_{n_r} \psi_0 + \frac{C_{n_c}}{\sigma} \\ \left(-\frac{1}{2} C_{Y_p} \sigma - C_L\right) \bar{\phi} + \left(2\mu_b \sigma - \frac{1}{2} C_{Y_r} \sigma - C_L \tan \gamma\right) \bar{\psi} + (2\mu_b \sigma - C_{Y_\beta}) \bar{\beta} &= -\frac{1}{2} C_{Y_p} \phi_0 + \left(2\mu_b - \frac{1}{2} C_{Y_r}\right) \psi_0 + 2\mu_b \beta_0 \end{aligned} \right\} \quad (2)$$

Equations (2) represent three simultaneous algebraic equations which can be solved for $\bar{\beta}$, $\bar{\phi}$, $\bar{\psi}$, and their derivatives by the method of determinants. For example,

$$\bar{\beta} = \frac{\bar{\Delta}_1}{\bar{\Delta}} = \frac{f(\sigma)}{F(\sigma)} \quad (3)$$

where $\bar{\Delta}$ is the characteristic lateral-stability equation

$$(A\sigma^4 + B\sigma^3 + C\sigma^2 + D\sigma + E)\sigma$$

and

$$\bar{\Delta}_1 = A_1\sigma^4 + B_1\sigma^3 + C_1\sigma^2 + D_1\sigma + E_1$$

The expressions for A , B , C , D , and E , in terms of the mass and aerodynamic parameters of the airplane, are given on pages 27 and 28 of reference 1. The coefficients of the $\bar{\Delta}_1$ equation are

$$A_1 = \beta_0 [8\mu_b^3 (K_X^2 K_Z^2 - K_{XZ}^2)]$$

$$B_1 = \phi_0 [4\mu_b^2 C_L (K_X^2 K_Z^2 - K_{XZ}^2)] + (D_b \phi)_0 [2\mu_b^2 C_{Y_p} (K_X^2 K_Z^2 - K_{XZ}^2)] + \psi_0 [4\mu_b^2 C_L \tan \gamma (K_X^2 K_Z^2 - K_{XZ}^2)] + (D_b \psi)_0 [2\mu_b^2 (C_{Y_r} - 4\mu_b) (K_X^2 K_Z^2 - K_{XZ}^2)] + \beta_0 [2\mu_b^2 (K_{XZ} C_{l_r} - K_X^2 C_{n_r}) + 2\mu_b^2 (K_{XZ} C_{n_p} - K_Z^2 C_{l_p})]$$

$$C_1 = \phi_0 [\mu_b C_L (K_{XZ} C_{l_r} - K_X^2 C_{n_r}) + \mu_b C_L (K_{XZ} C_{n_p} - K_Z^2 C_{l_p})] + (D_b \phi)_0 \left[\frac{1}{2} \mu_b K_X^2 (C_{n_p} C_{Y_r} - C_{n_r} C_{Y_p}) + 2\mu_b^2 K_X^2 (2K_Z^2 C_L - C_{n_p}) + 2\mu_b^2 K_{XZ} (C_{l_p} - 2K_{XZ} C_L) + \frac{1}{2} \mu_b K_{XZ} (C_{l_r} C_{Y_p} - C_{l_p} C_{Y_r}) \right] + \psi_0 [\mu_b C_L \tan \gamma (K_{XZ} C_{l_r} - C_{n_r} K_X^2) + \mu_b C_L \tan \gamma (K_{XZ} C_{n_p} - K_Z^2 C_{l_p})] + (D_b \psi)_0 \left[\frac{1}{2} \mu_b K_{XZ} (C_{n_p} C_{Y_r} - C_{n_r} C_{Y_p}) + \frac{1}{2} \mu_b K_Z^2 (C_{l_r} C_{Y_p} - C_{l_p} C_{Y_r}) + 4\mu_b^2 C_L \tan \gamma (K_X^2 K_Z^2 - K_{XZ}^2) + 2\mu_b^2 (K_Z^2 C_{l_p} - K_{XZ} C_{n_p}) \right] + \beta_0 \left[\frac{1}{2} \mu_b (C_{l_p} C_{n_r} - C_{l_r} C_{n_p}) \right] + C_{n_c} [-\mu_b K_{XZ} C_{Y_p} + \mu_b K_X^2 C_{Y_r} - 4\mu_b^2 K_X^2]$$

$$D_1 = \phi_0 \left[\frac{1}{4} C_L (C_{l_p} C_{n_r} - C_{l_r} C_{n_p}) \right] + (D_b \phi)_0 [\mu_b C_L \tan \gamma (K_X^2 C_{n_p} - K_{XZ} C_{l_p}) + \mu_b C_L (K_{XZ} C_{l_r} - K_X^2 C_{n_r})] + \psi_0 \left[\frac{1}{4} C_L \tan \gamma (C_{l_p} C_{n_r} - C_{l_r} C_{n_p}) \right] + (D_b \psi)_0 [\mu_b C_L (K_Z^2 C_{l_r} - K_{XZ} C_{n_r}) + \mu_b C_L \tan \gamma (K_{XZ} C_{n_p} - K_Z^2 C_{l_p})] + C_{n_c} \left[\frac{1}{4} C_{Y_p} C_{l_r} - 2\mu_b K_{XZ} C_L + 2\mu_b K_X^2 C_L \tan \gamma + \mu_b C_{l_p} - \frac{1}{4} C_{l_p} C_{Y_r} \right]$$

$$E_1 = C_{n_c} \left(\frac{1}{2} C_L C_{l_r} - \frac{1}{2} C_{l_p} C_L \tan \gamma \right)$$

The solution of equation (3), which will result in a time history of β as a function of s_b , is obtained from the Heaviside expansion theorem (reference 3):

$$\beta = \sum_{n=1}^m \frac{f(\lambda_n)}{F'(\lambda_n)} e^{\lambda_n s_b} \quad (4)$$

where λ_n are the roots of $F(\sigma)$ set equal to zero. Similar solutions are derived for ϕ , ψ , $D_b \phi$, $D_b \psi$, and $D_b \beta$. The time scale is readily converted from s_b units to t units by the equation $t = \frac{b}{V} s_b$.

The values of the stability derivatives and mass characteristics used in the calculations are presented in table I. The table is divided into two columns which differ only in the values of $C_{n\beta}$ and C_{n_r} of the airplane for the cases where the

TABLE I

STABILITY DERIVATIVES AND MASS CHARACTERISTICS
OF THE AIRPLANE CONSIDERED IN THE ANALYSIS

Derivative or characteristic	Outside of dead spot	Within dead spot
W/S , lb/ft ²	80	80
μ_s	101.1	101.1
ρ , slugs/ft ³	0.00089	0.00089
V , ft/sec.....	753	753
C_L	0.318	0.318
b , ft.....	27.7	27.7
γ , deg.....	0	0
K_Z^2	0.0573	0.0573
K_X^2	0.0069	0.0069
C_{l_p} , per radian.....	-0.462	-0.462
C_{n_p} , per radian.....	-0.0155	-0.0155
C_{l_r} , per radian.....	-0.126	-0.126
C_{r_p} , per radian.....	0	0
C_{r_r} , per radian.....	0	0
η , deg.....	-2.0, 0, 2.0	-2.0, 0, 2.0
C_{n_r} , per radian.....	-0.392	0
C_{n_β} (cases 1 and 2), per radian.....	0.28	0
C_{n_β} (case 3), per radian.....	0.41	0

airplane is either outside of or within the dead spot. From the analytical solution of the motion, based on the mass and aerodynamic characteristics of the first column of table I and an initial condition of $\beta=5^\circ$, the time history of β was computed for several values of s_b until the value of s_b for which $\beta=2^\circ$ was reached. For values of s_b greater than the s_b which results in $\beta=2^\circ$, this analytical solution is incorrect since the airplane has now entered into the dead spot and the values of C_{n_β} and C_{n_r} are zero. Thus, a new solution must be calculated with the use of the values given in the second column of table I with new initial conditions. The new initial conditions are determined by substituting the value of s_b at which $\beta=2^\circ$ in the original analytical solutions of ϕ , ψ , $D_b\phi$, $D_b\psi$, and $D_b\beta$. Once these initial conditions are known, another set of analytical solutions are computed for β , ϕ , ψ , and their derivatives from equations (3) and (4). This procedure is followed every time β crosses through 2° or -2° . The final resultant motion in sideslip is the sum of all the analytical solutions in β , each one of which is correct only for a particular interval of time.

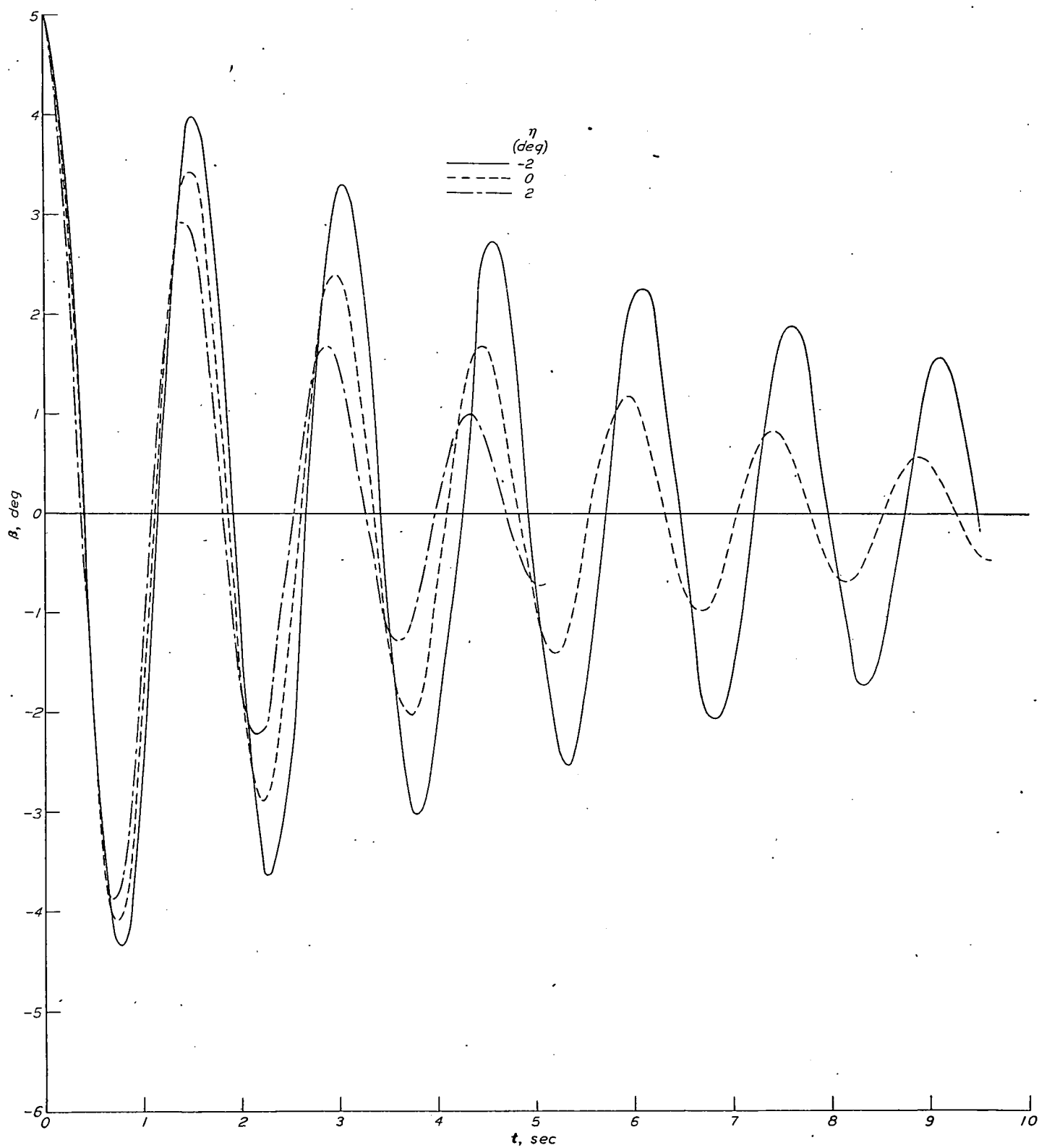
The constant C_{n_c} is introduced into the yawing-moment equation of equations (1), since the value of the yawing-moment coefficient due to sideslip is $C_{n_\beta}\beta + C_{n_c}$ for the condition of the airplane having the dead spot in cases 2 and 3 of figure 1. The values of C_{n_c} are $|0.00977|$ and $|0.0143|$ for cases 2 and 3, respectively. The sign of C_{n_c} is opposite to that of β . For case 1 of figure 1, $C_{n_c}=0$.

It is apparent that the procedure employed is a time-consuming process and subject to the possibility of many computational errors due to the magnitude of the computations. The final solution can be obtained, however, in a relatively short time through the use of automatic digital computing machines.

RESULTS AND DISCUSSION

The effect of the nonlinear stability derivatives on the lateral motion was investigated for the airplane described by the mass and aerodynamic characteristics given in table I, with three different values for the damping of the lateral oscillation as calculated on the basis of derivatives constant with amplitude. Since the damping was varied arbitrarily by assuming different values for the angle of inclination of the principal longitudinal axis of the airplane to the flight path η , three values of η , -2° , 0° , and 2° , were selected which correspond to a damping of the lateral oscillation, expressed in terms of $T_{1/2}$, of 5.6, 3.0, and 1.8, respectively. The motion of the airplane in sideslip, due to an initial disturbance in sideslip of 5° , for the three values of η is shown in figure 2. Since these motions are calculated on the assumption of derivatives constant with amplitude, the amplitudes of the motion decrease exponentially with time and will eventually reduce to zero. As can be noted in the first column of table I, the C_{n_β} for cases 1 and 2 is 0.28; whereas the C_{n_β} for case 3 is 0.41. The motions presented in figure 2 are for $C_{n_\beta}=0.28$; however, the motions for $C_{n_\beta}=0.41$ would exhibit oscillations of approximately the same damping and a slightly smaller period.

The motions of the airplane in sideslip, showing the effect of the nonlinearities illustrated in figures 1(a), 1(b), and 1(c), are presented in figures 3 to 5, respectively. In all cases, an initial disturbance in sideslip of 5° was assumed. The pronounced effect of the nonlinearities on the lateral motion is noted by a comparison of figure 2 and either one of figures 3, 4, or 5. In all three figures (figs. 3 to 5) the motion for $\eta=2^\circ$, the most stable case, approaches a constant value. The analytical solution of the motion for the case of $\eta=2^\circ$ in figure 3 indicates that, within the dead spot, the airplane will oscillate at a period of 6.56 seconds and $T_{1/2}=3.38$ seconds and will eventually approach the value of $\beta=-0.0092^\circ$. Similar motions would be obtained for the case of $\eta=2^\circ$ in figures 4 and 5. As η is decreased, the damping of the oscillatory motion depends upon the nonlinearity assumed and the values of η . In figure 3, the motion for $\eta=0^\circ$ damps at a slow rate at the large amplitudes until the oscillation reaches an amplitude of approximately 2.4° where the damping of the oscillation is zero. The period of the oscillation increases from 1.5 to 1.85 seconds. For the case of $\eta=-2^\circ$, a very lightly damped oscillation is apparent within the first few seconds and the airplane may be considered to be neutrally stable at an amplitude of $\mp 4.5^\circ$. In figures 4 and 5 the motion for $\eta=0^\circ$ clearly indicates that the damping is decreasing as the amplitude decreases and the period of the oscillation increases; for $\eta=-2^\circ$, the oscillatory motion is slightly unstable. A neutrally stable oscillation would be expected to occur in figures 4 and 5 for the combinations of a value of η between 0° and -2° and the dead spot assumed in the calculations or for $\eta=-2^\circ$ and a smaller dead spot.

FIGURE 2.—Calculated motion of an airplane due to an initial disturbance in sideslip for several values of η .

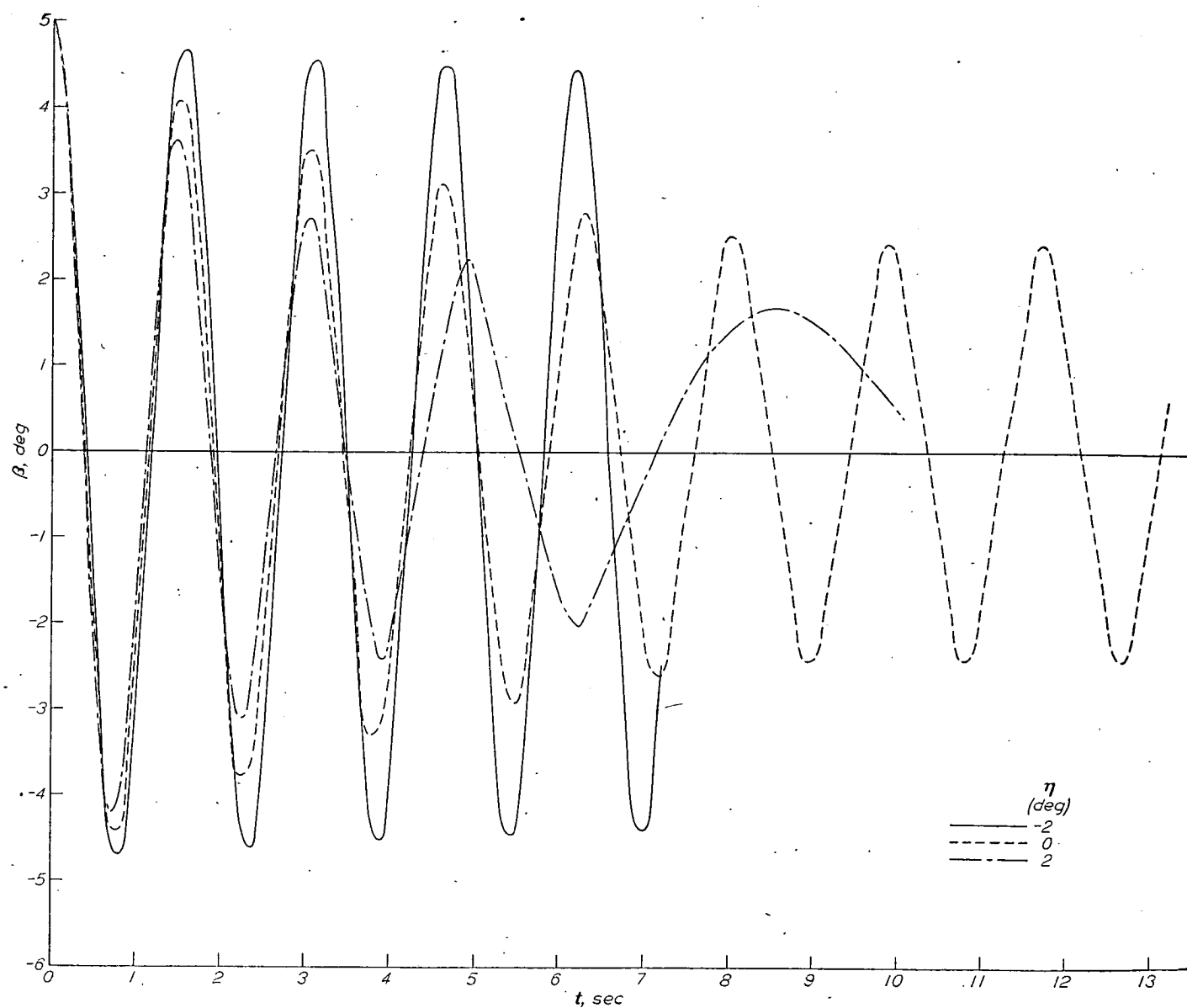


FIGURE 3.—The effect of the nonlinear derivatives described in figure 1(a) on the motion of an airplane.

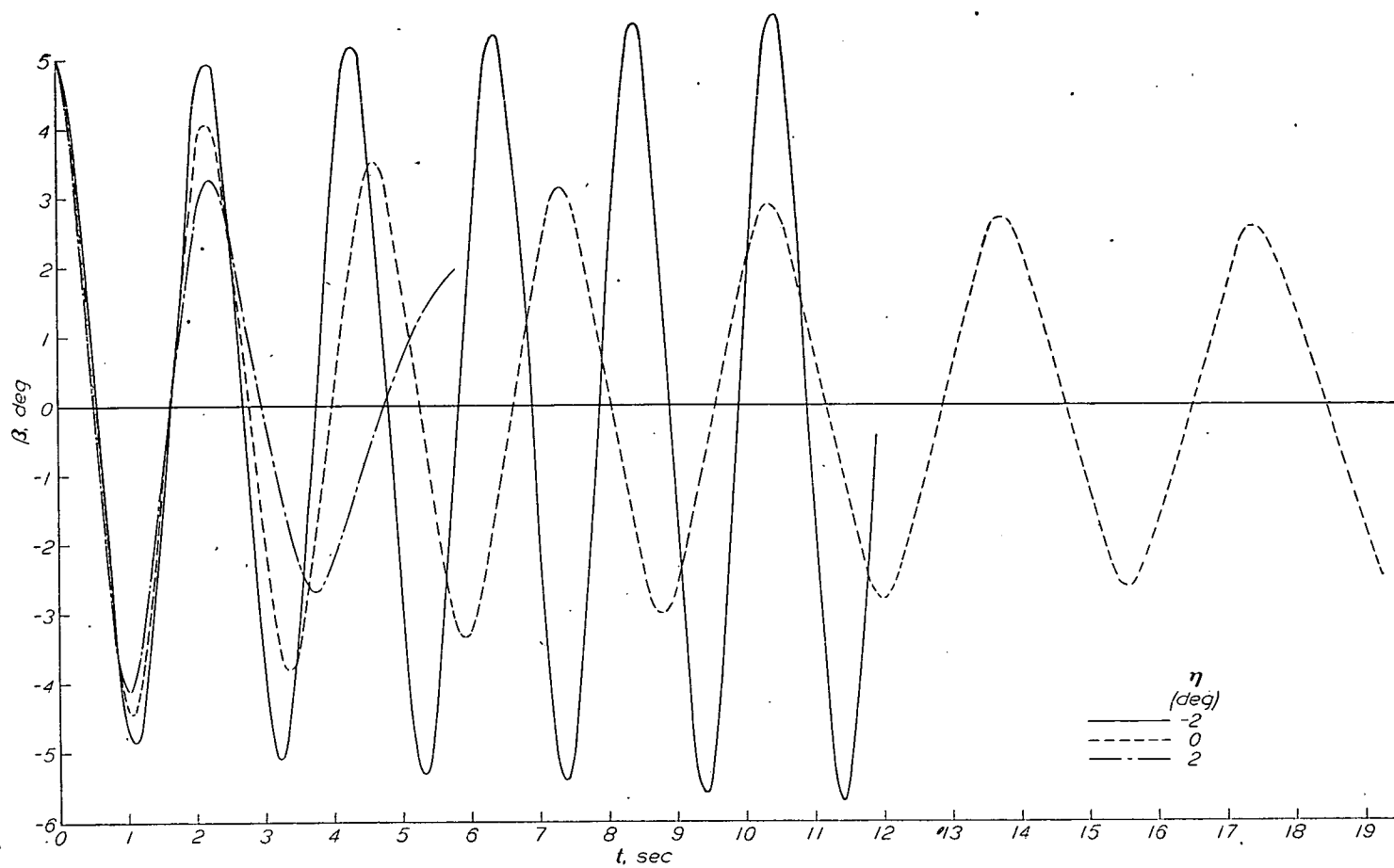


FIGURE 4.—The effect of the nonlinear derivatives described in figure 1(b) on the motion of an airplane.

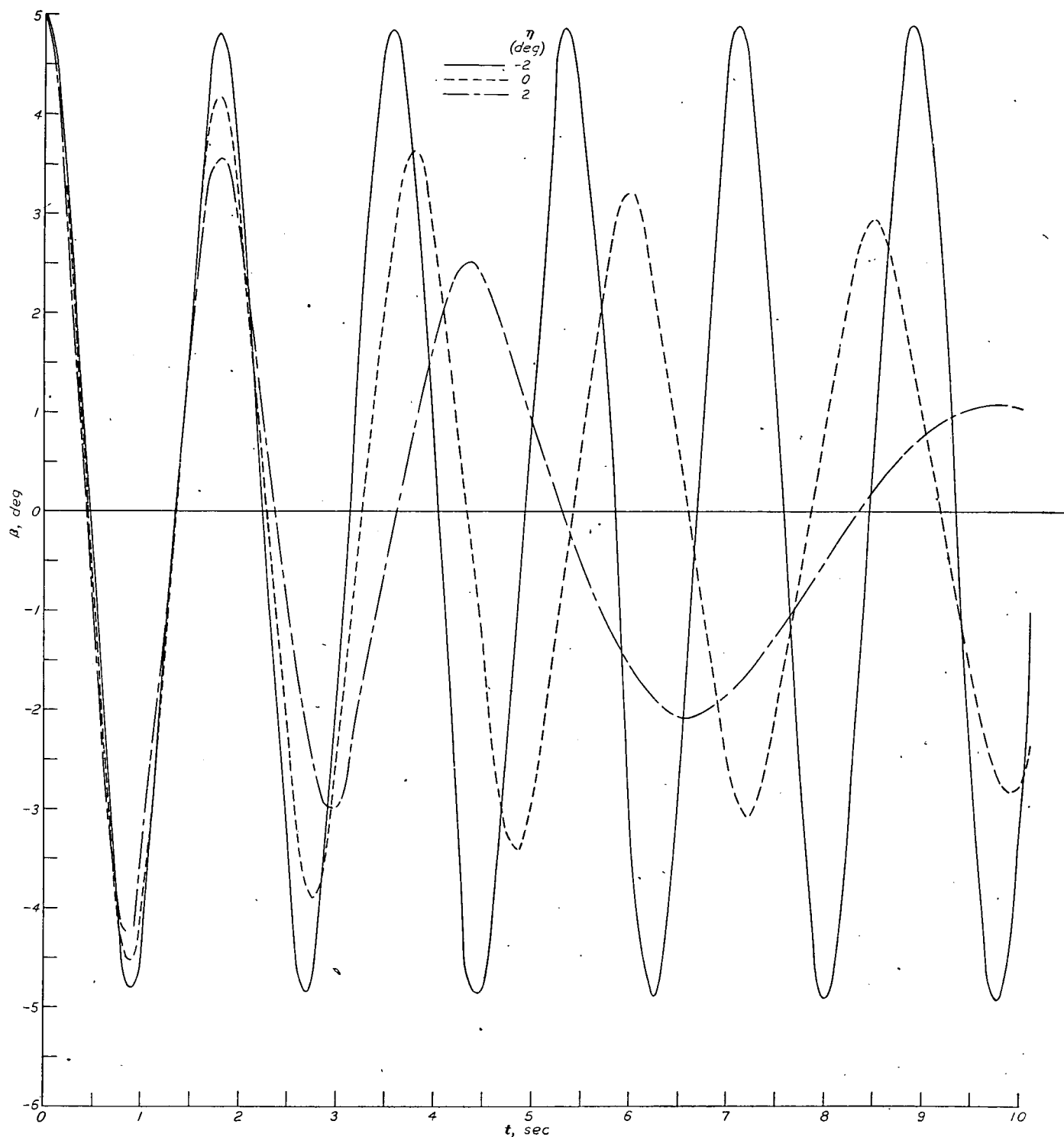


FIGURE 5.—The effect of the nonlinear derivatives described in figure 1(c) on the motion of an airplane.

In general, the results indicate that the damping of the lateral oscillation calculated with the use of derivatives constant with amplitude is a determining factor in the type of motion obtained where nonlinear derivatives are present. As the inherent damping of the lateral oscillation decreases, a smaller dead spot will result in a neutrally stable oscillation. Obviously, if the inherent damping is zero, a neutrally stable oscillation already exists with zero dead spot.

Some additional calculations were made for the case where the airplane is disturbed within the dead spot. The motions for an initial condition of $\beta = 1^\circ$ were computed for $\eta = -2^\circ$ and 0° with the assumption of the nonlinearity described in figure 1 (b). The results are presented in figure 6. It should be noted that the only difference between figures 4 and 6 is the initial condition assumed in the calculations. In figure 6, the motion for $\eta = -2^\circ$ is unstable and gradually approaches

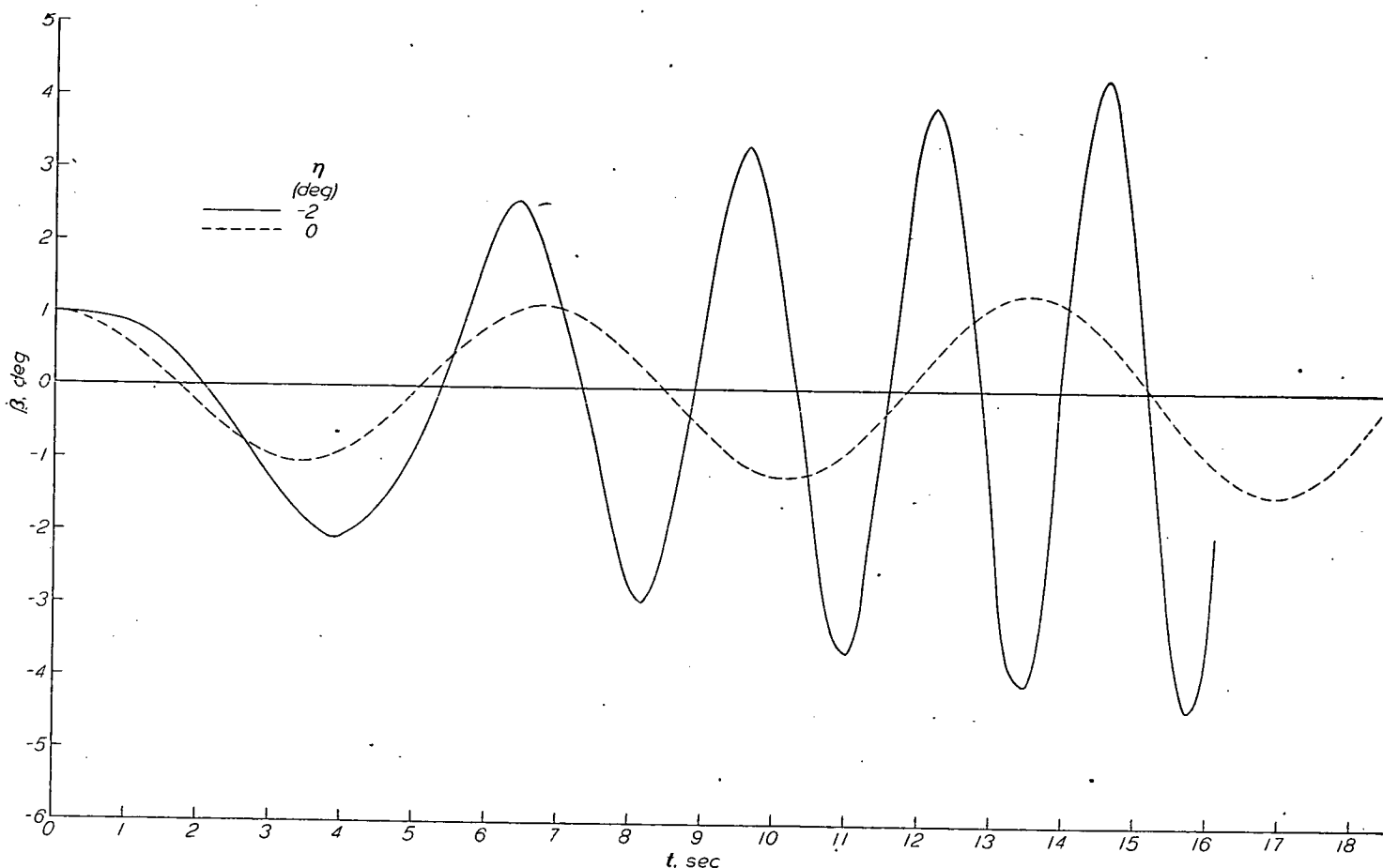


FIGURE 6.—The effect of the nonlinear derivatives described in figure 1(b) on the motion of an airplane. Initial disturbance in sideslip of 1° .

the amplitude and period of the motion for the case of $\eta = -2^\circ$ in figure 4. The motion for $\eta = 0^\circ$ in figure 6 is slightly unstable and will probably increase until its amplitude and period are in close agreement with the motion for the case of $\eta = 0^\circ$ in figure 4. Calculations have indicated that the oscillatory motion of the airplane within the dead spot will double amplitude about every 4 seconds for $\eta = -2^\circ$ and about every 30 seconds for $\eta = 0^\circ$. If the motion is unstable within the dead spot, either the airplane motion will be neutrally stable with an amplitude equal to or greater than the amplitude of the dead spot or the motion will be unstable.

The loss in damping and the increase in period which appeared in some of the lateral oscillations in figures 3 to 5 can be attributed to the type of nonlinearity assumed. From classical dynamic stability theory, it is well known that the damping of the oscillation is a function of C_{nr} , and the period of the oscillation is a function of $C_{n\beta}$. If the airplane is considered as a mass-spring dashpot system, $C_{n\beta}$ is the equivalent spring constant of the system and C_{nr} corresponds to the damping constant contributed by the dashpot. Thus as $C_{n\beta}$ is reduced the period increases, and as C_{nr} is reduced the damping decreases.

213637—53—65

CONCLUDING REMARKS

The results of the investigation made to determine the effect of nonlinearities assumed in the analysis on the lateral stability indicate that under certain conditions a motion is obtained which has different rates of damping for the large and small amplitudes of motion, with very little damping at the small amplitudes. In general, the period of the resultant oscillation increases with time.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., September 19, 1950.

REFERENCES

1. Mokrzycki, G. A.: Application of the Laplace Transformation to the Solution of the Lateral and Longitudinal Stability Equations. NACA TN 2002, 1950.
2. Murray, Harry E., and Grant, Frederick C.: Method of Calculating the Lateral Motions of Aircraft Based on the Laplace Transform. NACA TN 2129, 1950.
3. Churchill, Ruel V.: Modern Operational Mathematics in Engineering. McGraw-Hill Book Co., Inc., 1944.

REPORT 1043

A NUMERICAL METHOD FOR THE STRESS ANALYSIS OF STIFFENED-SHELL STRUCTURES UNDER NONUNIFORM TEMPERATURE DISTRIBUTIONS¹

By RICHARD R. HELDENFELS

SUMMARY

A numerical method is presented for the stress analysis of stiffened-shell structures of arbitrary cross section under nonuniform temperature distributions. The method is based on a previously published procedure that is extended to include temperature effects and multicell construction. The application of the method to practical problems is discussed, and an illustrative analysis is presented of a two-cell box beam under the combined action of vertical loads and a nonuniform temperature distribution.

INTRODUCTION

The effects of nonuniform temperature distributions, such as those produced by aerodynamic heating, are becoming of greater concern in the design of modern high-speed aircraft. The structural effects of temperature changes and the results of some analyses of a simplified structure under nonuniform distributions of temperature have been discussed in reference 1. The analytical methods considered in reference 1 were found, however, to yield inaccurate values for the secondary stresses in complicated structures, and in such cases some type of numerical approach is desirable. Numerical methods, however, usually require extensive and tedious calculations and they should be used only when satisfactory results cannot be obtained from a simplified analysis.

Several numerical methods of stress analysis have been presented in the literature, but none contains provisions for temperature changes. In the present report, one such method, the numerical procedure of reference 2, has been extended to include the effects of a nonuniform distribution of temperature. In addition, the equations developed permit the analysis of a stiffened-shell structure of arbitrary cross section with any number of internal cells. The application of the method is discussed and illustrated by analysis of a two-cell box beam under the combined action of vertical loads and a nonuniform temperature distribution.

DESCRIPTION OF THE NUMERICAL METHOD

BASIC THEORY

The structure analyzed is an idealized representation of a multicell stiffened-shell structure (see fig. 1) and has the

following characteristics:

(1) The basic unit is a rectangular panel bounded on two parallel sides by extensionally flexible stringers and on the other two sides by rigid bulkheads.

(2) The panels consist of sheet material and are assumed to carry shear stress only. The shear stress is constant within a given panel.

(3) The stringers run parallel to the direction of the primary stresses and carry axial load only.

(4) The bulkheads lie perpendicular to the stringers and are rigid in their own plane but offer no resistance to warping out of their plane.

(5) The structure is loaded only at the bulkheads.

(6) Material properties, cross-sectional dimensions, and temperature distribution do not vary along the length of a given bay.

With these assumptions about the basic elements of the structure, any type of stiffened shell can be analyzed, provided taper is excluded. The state of stress in such a structure can then be defined by suitable stress-strain relations and two types of displacements:

(1) Stringer displacements, which are displacements, at the end of a bay, of each flexible stringer in a direction parallel to the stringer

(2) Bay displacements, which are translations and rotations of the plane of each cross section defined by the rigid bulkheads

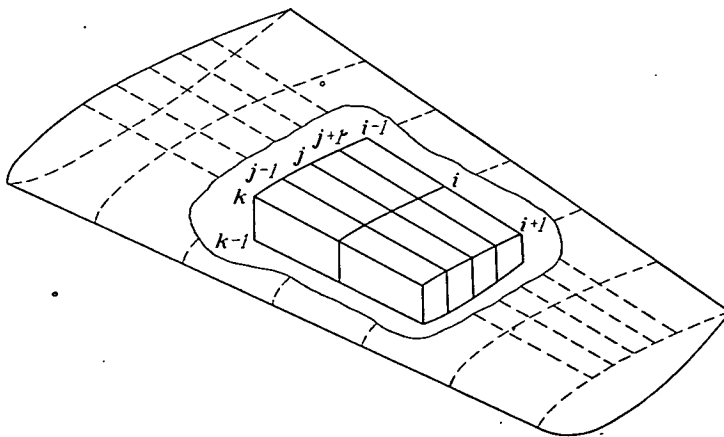


FIGURE 1.—Typical multicell, stiffened-shell wing structure.

¹ Supersedes NACA TN 2241, "A Numerical Method for the Stress Analysis of Stiffened-Shell Structures under Nonuniform Temperature Distributions" by Richard R. Heldenfels, 1950.

Once the stress-strain relations are established for the components of the idealized structure, equations of equilibrium can be used to obtain relationships between the displacements. The equilibrium equation for the forces on an individual stringer yields an expression for the stringer displacement of any panel point in terms of the surrounding stringer displacements and the displacements of the two adjacent bays. From this general expression, equations equal in number to the unknown stringer displacements are obtained. The additional equations required for the determination of the bay displacements are obtained from the equations of equilibrium of the shear forces on the cross sections. These equations then completely define the displacements of the structure. In most cases the number of equations is so large that a direct solution would be impractical and it has been found expedient to solve them by the recommended iteration procedure described in the next section. The required equations are derived in detail in the appendix.

SOLUTION OF EQUATIONS BY ITERATION

Matrix iteration often provides the easiest and quickest solution to the equations, and the procedure recommended is as follows:

The equations to be solved can be written in matrix notation as

$$[B]\{d\} = \{c\} \quad (1)$$

For purposes of iteration, these equations are rearranged to give

$$\{d\} = [C]\{d\} + \{c\} \quad (2)$$

where

$$[C] = [U] - [B]$$

$[B]$ square matrix of coefficients of general equations with diagonal terms reduced to unity

$[U]$ unit matrix

$\{d\}$ column matrix of stringer and bay displacements

$\{c\}$ column matrix of constant terms in general equation; these terms arise from applied load and thermal expansion

Initial approximate values of stringer and bay displacements $\{d_0\}$ are then selected. These values may be determined in any convenient manner; however, subsequent operations can be simplified, as explained in the appendix, if these values are chosen to correspond to elementary theory. Next, the initial displacement values are substituted into the righthand side of equation (2) to obtain a second approximation $\{d_1\}$ to the displacements

$$\{d_1\} = [C]\{d_0\} + \{c\} \quad (3)$$

and the differences between the second and initial approximate displacement values are computed from the equation

$$\{\Delta d_1\} = \{d_1\} - \{d_0\} \quad (4)$$

The iteration process is then begun by using these displacement differences. The n th difference is defined as

$$\{\Delta d_n\} = \{d_n\} - \{d_0\} \quad (5)$$

and it can be easily verified that the use of these differences leads to the following matrix equation:

$$\{\Delta d_n\} = [C]\{\Delta d_{n-1}\} + \{\Delta d_1\} \quad (6)$$

The iteration process consists of a series of solutions of equation (6), each successive solution yielding a better approximation to the displacement differences than the previous one. The process is continued until successive solutions of equation (6) yield the same result, that is, until

$$\{\Delta d_n\} = \{\Delta d_{n-1}\} \quad (7)$$

The final displacements are then determined from the final differences by using equation (5) and the initial values.

When equation (6) is being iterated, improved values should be used as soon as they are obtained; that is, each individual difference Δd_n should be substituted into the $\{\Delta d_{n-1}\}$ matrix immediately after calculation rather than at the end of the cycle. In this manner, each new value determined receives the benefit of all previous work and convergence is speeded.

The iteration of differences reduces the work required to obtain a solution because smaller numbers are involved. However, it is essential that no errors be made in the determination of the first differences $\{\Delta d_1\}$ since a single significant error will render the whole solution useless.

CONVERGENCE OF THE ITERATION PROCESS

In order to obtain more rapid convergence of the iteration process, bay displacements and loads are referred to the principal shear axes of each bay. The use of these axes greatly simplifies the equations for bay displacements by making each bay displacement independent of all other bay displacements and thus a function of the stringer displacements only. In addition, a special correction cycle is periodically introduced to bring the stringer forces on each cross section into equilibrium with the applied loads. Mathematically, the correction cycle is a special cycle that uses a certain combination of the basic equations. Its success in the particular case of the numerical method of stress analysis is a result of its physical significance, and in that respect it is similar to Southwell's "group relaxations" (reference 3).

The optimum frequency of application of the correction cycle depends largely on the characteristics of each individual problem and must be determined on a basis of experience with the method. If this frequency cannot be determined from previous experience, it can be approximated satisfactorily by one that permits the disturbances to spread their significant effect over the structure between correction cycles.

The application of the correction cycle begins at a station where the displacements are known and then proceeds outboard. The corrections required to bring the first bay into equilibrium are determined, and the stringer displacements at its outboard end are changed accordingly before the corrections required by the second bay are calculated.

EFFECT OF INTRODUCING NONUNIFORM TEMPERATURE DISTRIBUTIONS

The preceding method is applicable to any type of stress problem. Nonuniform temperature distributions do not affect the general procedure but merely change the details. These effects are of two types: A change in the effective structure due to changes in elastic properties of the material with temperature and thermal stresses resulting from restrained thermal expansion. The changes in elastic properties are easily handled if the moduli are treated as variables during the derivation of the equations. Their effect is analogous to that of variations in stringer area and panel thickness. The presence of thermal expansion requires modification of the stress-strain relationships for the stringers but does not affect those for the panels. The equations for stringer displacements contain thermal-expansion terms that are analogous to the applied-load terms. Bay-displacement equations are unaffected by thermal expansion, but thermal-expansion terms appear in the equations used for the correction cycle. If a difference solution is iterated, the elementary solution should include the distributions of thermal strain associated with the primary thermal stresses, which may be obtained from the equations derived in the appendix.

DISCUSSION OF THE NUMERICAL METHOD

The application of a method, such as that just described, always poses a number of questions; for example, what are some of the limitations of the method, would it be advantageous to use some other method of analysis, and how should the structure be idealized? Some of the factors requiring consideration, other than those mentioned in the previous section, are therefore now discussed.

VALIDITY OF BASIC ASSUMPTIONS

The assumptions upon which the method is based are commonly accepted in the analysis of stiffened shells. Comparison of theoretical and experimental results has established the fact that these assumptions will yield good results in most cases. Two important assumptions—that the bulkheads are rigid in their own plane and that the shear stress is constant in a given panel—may, however, introduce significant errors into the analysis in some cases. These assumptions are therefore examined in detail.

The assumption of rigid bulkheads is satisfactory as long as the primary stresses run perpendicular to the bulkheads, but, as demonstrated in reference 1, this assumption may not be good when dealing with problems involving thermal

stress. Large temperature gradients along the length of the structure or across the depth of a bulkhead distort the real bulkhead and make the assumption of rigidity inapplicable. In many cases, however, these effects are small and the assumption yields satisfactory results.

The numerical method could be extended to include the effects of bulkhead flexibility. Such an analysis, however, is very cumbersome and tedious and, if the equations are solved by iteration, the process is often very slowly convergent. Therefore, these extensions are not considered herein.

The assumption of constant shear stress in a given panel simplifies the development of the equations, and it yields satisfactory results if the bulkheads are reasonably close together. Cases arise, however, in which the assumption will lead to unreasonable results because the assumed constant shear stress is a poor approximation to a shear stress which should be changing rapidly in the spanwise direction. This situation is usually accompanied by slow convergence of the iteration process. This difficulty, however, can be minimized by reducing the bulkhead spacing of the idealized structure since it occurs only when the total shear stiffness of the panels joined to a stringer exceeds the extensional stiffness of that stringer.

IDEALIZATION OF AN ACTUAL STRUCTURE

The idealization process described in reference 2 is straightforward. However, it provides an opportunity for the stress analyst to exercise his engineering judgment and thus to simplify the analysis. By restricting the analysis to only a part of the structure or by using a comparatively simple idealized structure, the time required for the analysis can be substantially reduced. Such simplifications, however, can reduce the value of the results, and a compromise between speed and exactness is required.

The number and location of the idealized stringers completely define the stress-distribution shapes obtainable from the analysis. (For example, in an idealized shell of n stringers, there are n possible types of independent normal-stress distributions, three of which can be determined from elementary theory, the remaining $n-3$ being statically indeterminate.) Stringer location is thus an important part of the idealization process and in conventional problems the locations should be selected after consideration of the characteristics of the actual structure, the nature of the expected results, and the time available for the analysis. When non-uniform temperature distributions are involved, the shape of the temperature distribution should also be considered because the thermal-stress and temperature distributions will have similar shapes and the analysis will yield good results only if the idealized structure permits a stress distribution of that shape.

The bulkhead spacing usually is the same in the idealized and actual structures, but the idealized spacing should never be so large that trouble is caused by the assumption of

constant shear stress. A proper bulkhead spacing is one for which the extensional stiffness of each stringer element is greater than the sum of the shear stiffnesses of the adjacent panels so that no negative terms appear on the right-hand side of the equations for the stringer displacements.

CALCULATING PROCEDURES

All the calculations required by the numerical procedure (determining the coefficients of the equations, solving by iteration, and computing the stresses) are routine and involve only simple arithmetic. The calculations can be easily arranged in tabular form so that the bulk of the work can be done by modern automatic computing machinery or by a computer who does not need to have a knowledge of the structural theories involved.

In any problem that involves extensive numerical work, errors are very apt to occur. One of the advantages of the numerical method described herein is that a number of checking procedures can be devised to check the various steps in the calculations. No attempt is made to describe the many possible checks; a few, however, have been indicated in the illustrative example.

Solution of the equations by simple iteration also possesses another advantage with regard to errors. Values obtained from successive cycles of iteration show trends that can be observed by an experienced computer, and errors can be detected by their effects on these trends. Errors that do appear during the iteration process eventually work themselves out but may adversely affect the rate of convergence.

APPLICATION OF THE NUMERICAL METHOD

DESCRIPTION OF THE PROBLEM

The application of the numerical method is illustrated by an analysis of the idealized two-cell box beam shown in figure 2. The cross section is symmetrical about the horizontal center line and the beam is untapered; however, the stringer areas and sheet thicknesses vary from bay to bay.

The box beam is loaded by four concentrated vertical loads applied at the bulkhead stations along the inner web; in addition, it is subjected to the arbitrarily selected distribution

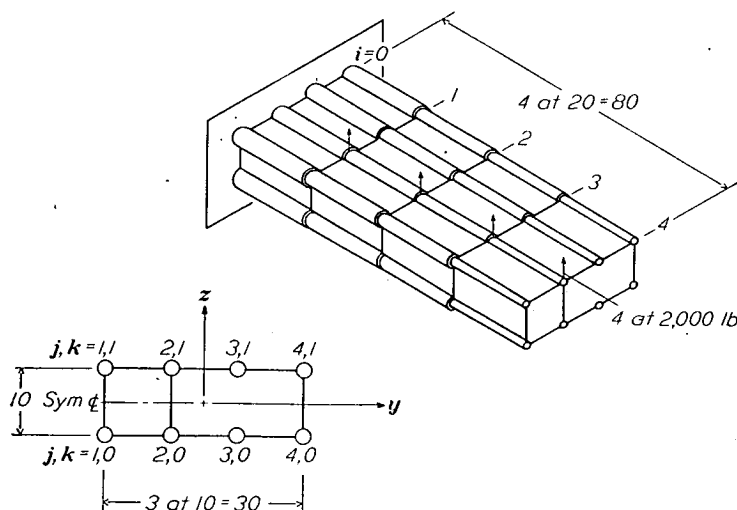


FIGURE 2.—Idealized structure used in illustrative example. (Stringer areas and skin thicknesses are listed in tables I and II.)

of temperature increase shown in figure 3. The temperature is highest at the tip and along the front web and decreases in both spanwise and chordwise directions, but it is constant across the depth of the beam. The beam is assumed to be constructed of 75S-T6 aluminum alloy which has the variation of elastic properties with temperature increase shown in figure 4. These data are the same as those used in reference 1.

It is assumed that no thermal stresses were present at 60° F. Since the method of analysis involves the assumption that no changes in temperature distribution occur over each element, the temperature used in the calculations was the temperature at the center of the element concerned.

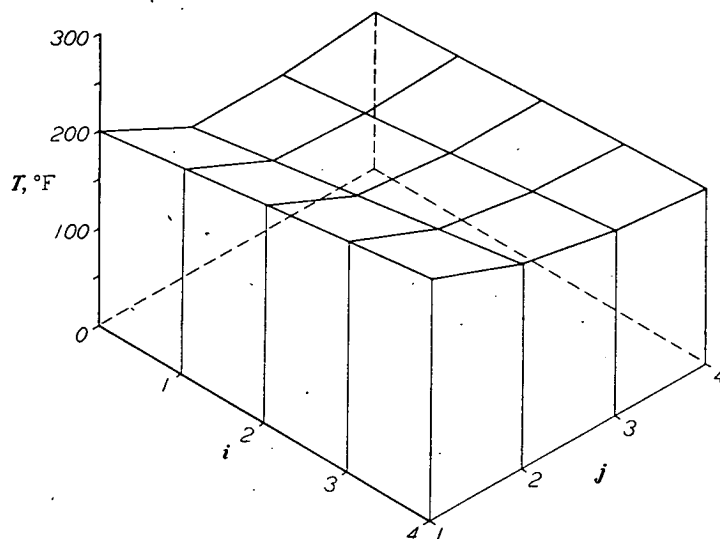


FIGURE 3.—Distribution of temperature increase.

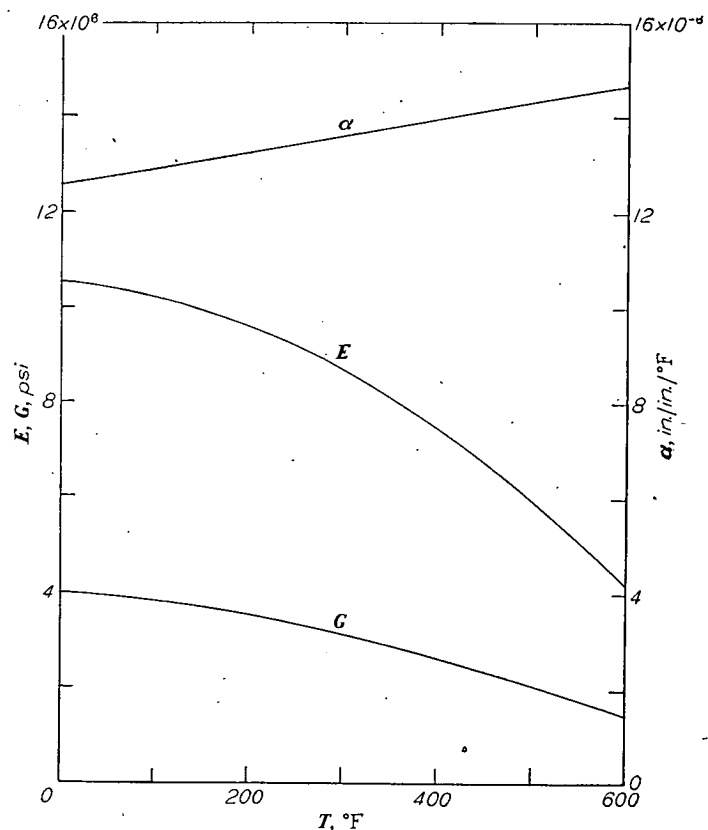


FIGURE 4.—Variation of elastic properties of 75S-T6 aluminum alloy with temperature increase (reference 1).

DETAILS OF THE ANALYSIS

Since the structure and the temperature distribution are symmetrical about the horizontal center line, the analysis can be restricted to one cover. Two solutions are required however, to determine the total stress since the thermal-stress system is symmetrical about the horizontal center line, but the load-stress system is antisymmetrical. In this way, the analysis requires the solution of two sets of equations (one of 20, the other of 24) which can be solved more easily than the set of 44 needed for a single analysis of the complete box.

The computations required are given in tabular form with most tables containing two parts—one related to the load stresses and the other related to the thermal stresses. The final solution is obtained by the superposition of these two solutions. The rectangular cross section and its symmetry permit several simplifications of the general equations. In each case the equations used are listed. The notation is described in the appendix. Methods used to check the calculations are also given in the tables. The checking methods used were determined from mathematical relationships existing between the coefficients of the equations and from equilibrium of forces.

Tables I and II present the physical characteristics and stiffness parameters of the individual stringers and panels. Table III gives the location of the principal shear axes of each bay and the coefficients of the bay-displacement equations. The location of the principal inertia axes of each bay, the coefficients used in the correction cycle, and the initial stringer displacements are given in table IV. The coefficients of the stringer-displacement equations are tabulated in table V. Table VI contains the $[C]$ matrices used for the iteration. The rows and columns have been interchanged in order that the matrix multiplications required will consist of the cumulative multiplication of the adjacent numbers in two columns. Table VII is a similar arrangement of data required for the correction cycle and also lists each correction determined. The displacements obtained from each cycle of iteration are given in table VIII and the correction cycles are indicated. Table IX contains the calculation of each type of stress and the superposition required to obtain the total stresses.

The numerical calculations in this example were done by a computer who had previous experience with the method. The following times were required:

Setting up the equations (table I to table VII).....	3 days
Solving the equations (table VIII).....	4 days
Computing stresses (table IX).....	1 day

In this example, the displacements were computed to six decimal places (five or six significant figures) in order that the stress would be accurate to 1 psi and thus would provide accurate equilibrium checks. Most practical problems will not require such numerical accuracy and a smaller number of decimal places should be used in order to speed the solution. It is estimated that the time required to solve this example could have been reduced by one-half if the number of decimal places had been reduced from six to four. This reduction would have given stresses accurate to 100 psi or about 1 percent of the maximum stress.

RESULTS OF THE CALCULATION

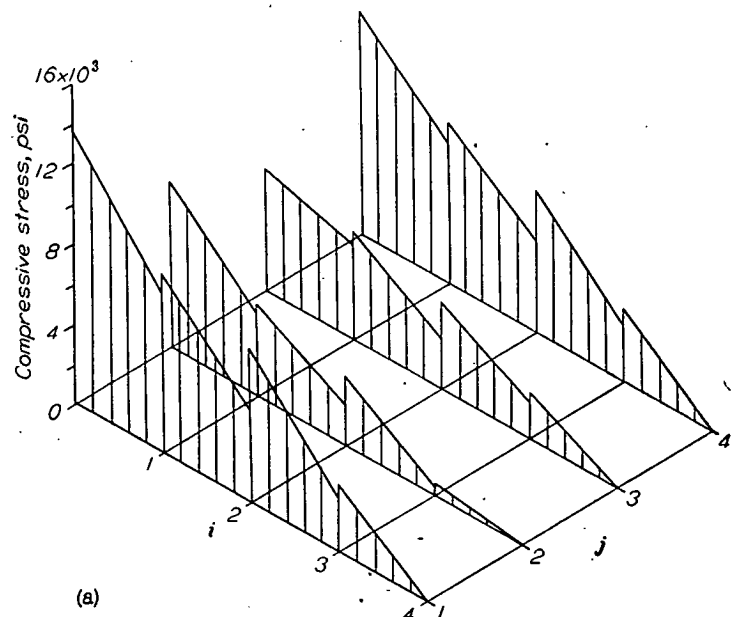
The results of the calculation are shown graphically in figure 5 by spanwise and chordwise plots of the stringer stresses in the top and bottom covers and a spanwise plot of shear stresses in the webs. The spanwise plots have a jagged appearance because stringer areas and sheet thicknesses are assumed constant in each bay with an abrupt change at the bulkheads. The dashed lines in the plots of stringer stresses are the values obtained from an elementary analysis.

CONCLUDING REMARKS

A numerical method for the stress analysis of stiffened-shell structures under nonuniform temperature distributions has been presented. The method is not applicable to the solution of all structural problems involving temperature effects because it requires extensive and tedious calculations and because the basic assumptions of bulkheads rigid in their own plane and constant shear stress in a given panel occasionally lead to unsatisfactory results. It is, however, a powerful tool for the solution of many structural problems because:

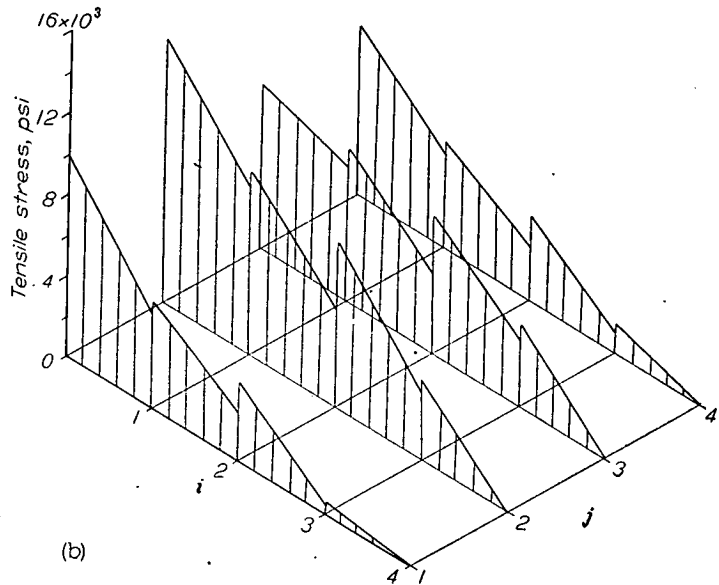
- (1) It is a means for accurately determining all types of secondary stresses in complicated structures that cannot be satisfactorily analyzed by simplified methods.
- (2) It is sufficiently flexible to cope with a wide variety of structural problems involving nonuniform temperature distributions.
- (3) It involves only simple arithmetic that can be handled by automatic computing machinery.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., September 12, 1950.

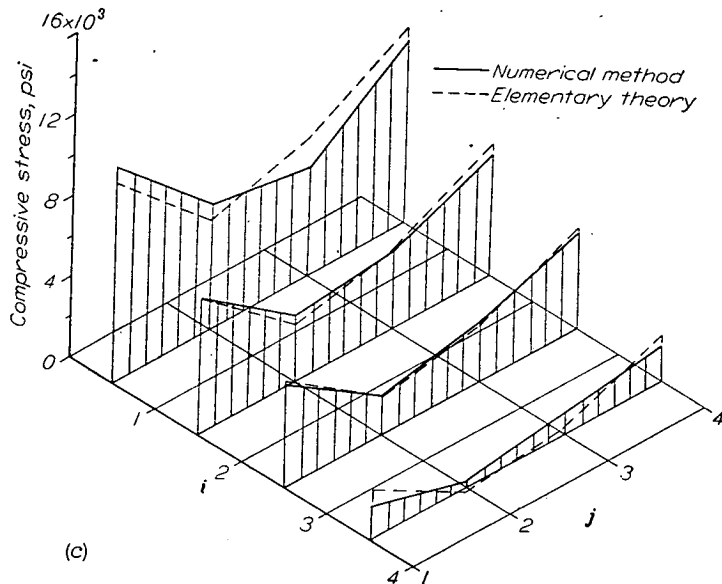


(a) Spanwise distribution of upper-surface stringer stress.

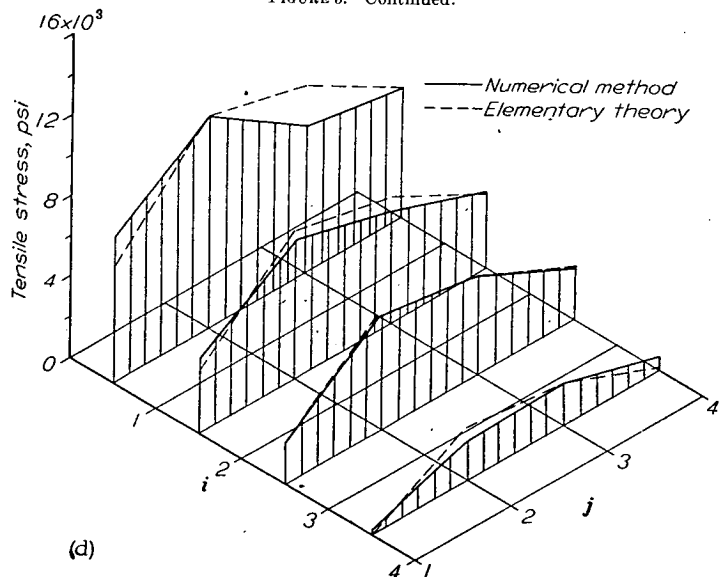
FIGURE 5.—Calculated stress distribution in the idealized structure.



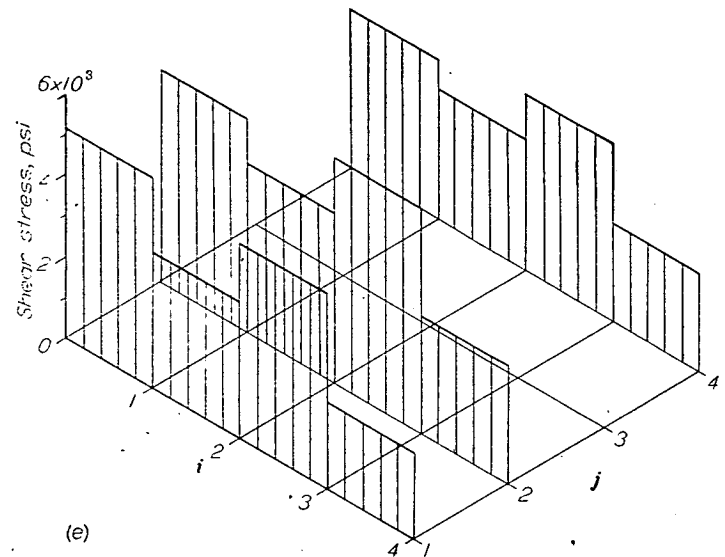
(b) Spanwise distribution of lower-surface stringer stress.
FIGURE 5.—Continued.



(c) Chordwise distribution of upper-surface normal stress.
FIGURE 5.—Continued.



(d) Chordwise distribution of lower-surface normal stress.
FIGURE 5.—Continued.



(e) Spanwise distribution of web shear stress.
FIGURE 5.—Concluded.

APPENDIX

DERIVATION OF GENERAL EQUATIONS

The general equations required for the numerical analysis of a stiffened shell of arbitrary cross section with any number of internal cells and under a nonuniform temperature distribution are developed. The basic assumptions and a general description of the method have been given previously and are not repeated.

SYMBOLS

A	cross-sectional area of stringer, square inches
b	width of panel on k grid line, inches
E	modulus of elasticity, psi
F	applied force, pounds
G	modulus of rigidity, psi
h	width of panel on j grid line, inches
I	moment of inertia, inches ⁴
J	shear stiffness parameters
l, m	coordinates of a special set of axes
L	length of bay, inches
M	applied moment, inch-pounds
P	axial load in stringer, positive for tensile load, pounds
Q	area moment, inches ³
r	normal distance to panel on k grid line, positive in positive z -direction, inches
T	temperature increment, measured from temperature of zero thermal stress which is 60° F in the example presented, degrees Fahrenheit
t	panel thickness, inches
u, v, w	displacements in x -, y -, and z -directions, respectively, inches
x, y, z	rectangular coordinate axes
α	coefficient of thermal expansion, inches per inch per degree Fahrenheit
β	angular rotation used in correction cycle, radians
γ	shear strain, radians
δ, Δ	increment

ϵ	normal strain, inches per inch
θ	angular rotation about x -axis, radians
λ	rotation of special set of axes, degrees or radians
ρ	normal distance to panel on j grid line, positive in positive y -direction, inches
σ	normal stress, positive for tensile stress, psi
τ	shear stress, positive in direction of associated coordinate axis when tensile stress on cross section is in positive x -direction, psi
ϕ	angle between normal line r and z -axis, degrees or radians
ψ	angle between normal line ρ and y -axis, degrees or radians

Subscripts:

i, j, k	grid system
x, y, z	coordinate axes
v, w, θ	bay displacements
0	initial value
n	cycle of iteration

A prime refers to the principal shear axes and two primes refer to the principal inertia axes. A bar over a symbol indicates an average value at the center of a bay.

NOTATION

The notation employed is illustrated in figure 6. The system adopted for designating parts of the structure is as follows:

Bulkheads divide the length of the structure into a number of bays. The subscript i is used to designate a given bulkhead or the bay between the $i-1$ and i th bulkheads.

The stringers and panels in a given cross section form the basis of a grid work which can be used to designate these elements. These grid lines are not necessarily straight, parallel, or perpendicular but follow the panels. Those grid lines that are approximately parallel to the z -axis are designated by the subscript j ; those approximately parallel to the y -axis are designated by the subscript k .

With this system, points and stringers can be uniquely located as follows:

The point on the i th bulkhead at the intersection of the

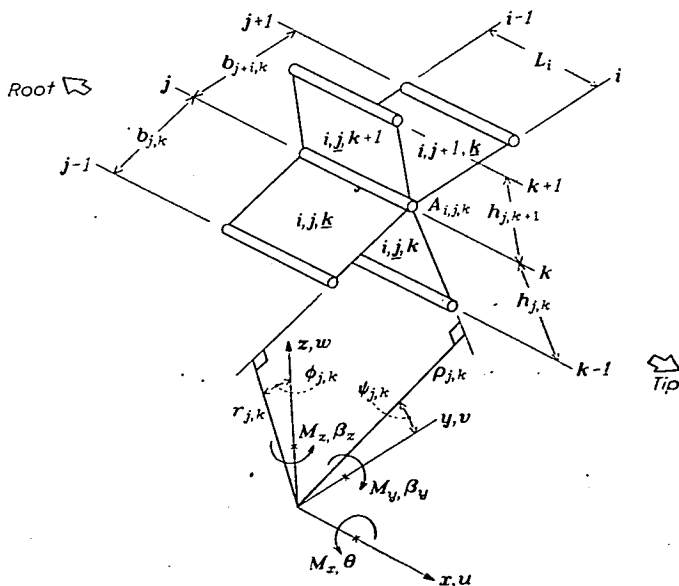


FIGURE 6.—Notation and coordinate system.

j th and k th grid lines is designated by the subscripts i, j, k .

The stringer in the i th bay at the intersection of the j th and k th grid lines is designated by the subscripts i, j, k .

In order to locate a panel, the grid line on which it lies must be known. This notation consists of underlining the appropriate subscript; for example:

The panel in the i th bay on the j th grid line and between the $k-1$ and k th grid lines is designated by the subscripts $i, \underline{j}, k$.

The panel in the i th bay on the k th grid line and between the $j-1$ and j th grid lines is designated by the subscripts $i, j, \underline{k}$.

The grid lines and bulkheads are numbered such that the numbers increase in the directions of the positive coordinate axes.

STRESS-STRAIN RELATIONSHIPS

The shear strain in a given panel is constant and is defined by the following relationships which depend upon the location of the panel:

$$\begin{aligned}\gamma_{i, \underline{j}, k} &= \left(\frac{\tau}{G} \right)_{i, \underline{j}, k} \\ &= \frac{1}{2b_{j,k}} (u_{i,j,k} + u_{i-1,j,k} - u_{i,j-1,k} - u_{i-1,j-1,k}) + \\ &\quad \frac{\Delta v_i}{L_i} \cos \phi_{j,k} + \frac{\Delta w_i}{L_i} \sin \phi_{j,k} - \frac{\Delta \theta_i}{L_i} r_{j,k}\end{aligned}\quad (A1a)$$

$$\begin{aligned}\gamma_{i, j, \underline{k}} &= \left(\frac{\tau}{G} \right)_{i, j, \underline{k}} \\ &= \frac{1}{2h_{j,k}} (u_{i,j,k} + u_{i-1,j,k} - u_{i,j,k-1} - u_{i-1,j,k-1}) - \\ &\quad \frac{\Delta v_i}{L_i} \sin \psi_{j,k} + \frac{\Delta w_i}{L_i} \cos \psi_{j,k} + \frac{\Delta \theta_i}{L_i} \rho_{j,k}\end{aligned}\quad (A1b)$$

When the shear strains are being computed, the normal distances r and ρ must be given their proper signs.

The constant shear stress produces a linearly varying strain in the stringer and its average value at the center of the bay is

$$\bar{\epsilon}_{i, j, k} = \frac{u_{i,j,k} - u_{i-1,j,k}}{L_i} = \left(\frac{\bar{P}}{AE} + \alpha T \right)_{i, j, k} \quad (A2)$$

Note that the thermal expansion is included in the relationship between stringer stress and strain.

EQUILIBRIUM OF INDIVIDUAL STRINGERS

If a half-bay length of stringer on each side of point (i, j, k) is isolated, the force system of figure 7 is obtained, and the following equilibrium equation can be written:

$$\begin{aligned}-\left(\frac{\tau t L}{2} \right)_{i, j, \underline{k}} + \left(\frac{\tau t L}{2} \right)_{i, j+1, \underline{k}} - \left(\frac{\tau t L}{2} \right)_{i, \underline{j}, k} + \left(\frac{\tau t L}{2} \right)_{i, \underline{j}, k+1} - \\ \left(\frac{\tau t L}{2} \right)_{i+1, j, \underline{k}} + \left(\frac{\tau t L}{2} \right)_{i+1, j+1, \underline{k}} - \left(\frac{\tau t L}{2} \right)_{i+1, \underline{j}, k} + \left(\frac{\tau t L}{2} \right)_{i+1, \underline{j}, k+1} + \\ \bar{P}_{i+1, j, k} - \bar{P}_{i, j, k} + (F_z)_{i, j, k} = 0\end{aligned}\quad (A3)$$

Substituting equations (A1) and (A2) into equation (A3) yields the following equation for the stringer displacement of point (i, j, k) in terms of the displacements of the adjacent bays and stringers:

$$\begin{aligned}
 u_{i,j,k} = & \frac{1}{\sum S_{i,j,k}} \left\{ u_{i-1,j-1,k} \left(\frac{GtL}{4b} \right)_{i,j,k} + u_{i-1,j,k-1} \left(\frac{GtL}{4h} \right)_{i,j,k} + \right. \\
 & u_{i-1,j,k} \left[\left(\frac{AE}{L} \right)_{i,j,k} - \left(\frac{GtL}{4b} \right)_{i,j,k} - \left(\frac{GtL}{4b} \right)_{i,j+1,k} - \left(\frac{GtL}{4h} \right)_{i,j,k} - \left(\frac{GtL}{4h} \right)_{i,j,k+1} \right] + u_{i-1,j,k+1} \left(\frac{GtL}{4h} \right)_{i,j,k+1} + \\
 & u_{i-1,j+1,k} \left(\frac{GtL}{4b} \right)_{i,j+1,k} + u_{i,j-1,k} \left[\left(\frac{GtL}{4b} \right)_{i,j,k} + \left(\frac{GtL}{4b} \right)_{i+1,j,k} \right] + u_{i,j,k-1} \left[\left(\frac{GtL}{4h} \right)_{i,j,k} + \left(\frac{GtL}{4h} \right)_{i+1,j,k} \right] + \\
 & u_{i,j,k+1} \left[\left(\frac{GtL}{4h} \right)_{i,j,k+1} + \left(\frac{GtL}{4h} \right)_{i+1,j,k+1} \right] + u_{i,j+1,k} \left[\left(\frac{GtL}{4b} \right)_{i,j+1,k} + \left(\frac{GtL}{4b} \right)_{i+1,j+1,k} \right] + u_{i+1,j-1,k} \left(\frac{GtL}{4b} \right)_{i+1,j,k} + \\
 & u_{i+1,j,k-1} \left(\frac{GtL}{4h} \right)_{i+1,j,k} + u_{i+1,j,k} \left[\left(\frac{AE}{L} \right)_{i+1,j,k} - \left(\frac{GtL}{4b} \right)_{i+1,j,k} - \left(\frac{GtL}{4b} \right)_{i+1,j+1,k} - \left(\frac{GtL}{4h} \right)_{i+1,j,k} - \left(\frac{GtL}{4h} \right)_{i+1,j,k+1} \right] + \\
 & u_{i+1,j,k+1} \left(\frac{GtL}{4h} \right)_{i+1,j,k+1} + u_{i+1,j+1,k} \left(\frac{GtL}{4b} \right)_{i+1,j+1,k} + \\
 & \Delta v_i \left[\left(\frac{Gt \cos \phi}{2} \right)_{i,j+1,k} - \left(\frac{Gt \cos \phi}{2} \right)_{i,j,k} - \left(\frac{Gt \sin \psi}{2} \right)_{i,j,k+1} + \left(\frac{Gt \sin \psi}{2} \right)_{i,j,k} \right] + \\
 & \Delta v_{i+1} \left[\left(\frac{Gt \cos \phi}{2} \right)_{i+1,j+1,k} - \left(\frac{Gt \cos \phi}{2} \right)_{i+1,j,k} - \left(\frac{Gt \sin \psi}{2} \right)_{i+1,j,k+1} + \left(\frac{Gt \sin \psi}{2} \right)_{i+1,j,k} \right] + \\
 & \Delta w_i \left[\left(\frac{Gt \sin \phi}{2} \right)_{i,j+1,k} - \left(\frac{Gt \sin \phi}{2} \right)_{i,j,k} + \left(\frac{Gt \cos \psi}{2} \right)_{i,j,k+1} - \left(\frac{Gt \cos \psi}{2} \right)_{i,j,k} \right] + \\
 & \Delta w_{i+1} \left[\left(\frac{Gt \sin \phi}{2} \right)_{i+1,j+1,k} - \left(\frac{Gt \sin \phi}{2} \right)_{i+1,j,k} + \left(\frac{Gt \cos \psi}{2} \right)_{i+1,j,k+1} - \left(\frac{Gt \cos \psi}{2} \right)_{i+1,j,k} \right] - \\
 & \Delta \theta_i \left[\left(\frac{Gtr}{2} \right)_{i,j+1,k} - \left(\frac{Gtr}{2} \right)_{i,j,k} - \left(\frac{Gt\rho}{2} \right)_{i,j,k+1} + \left(\frac{Gt\rho}{2} \right)_{i,j,k} \right] - \\
 & \Delta \theta_{i+1} \left[\left(\frac{Gtr}{2} \right)_{i+1,j+1,k} - \left(\frac{Gtr}{2} \right)_{i+1,j,k} - \left(\frac{Gt\rho}{2} \right)_{i+1,j,k+1} + \left(\frac{Gt\rho}{2} \right)_{i+1,j,k} \right] + (AE\alpha T)_{i,j,k} - (AE\alpha T)_{i+1,j,k} + (F_z)_{i,j,k} \} \quad (A4)
 \end{aligned}$$

where

$$\begin{aligned}
 \sum S_{i,j,k} = & \left(\frac{AE}{L} \right)_{i,j,k} + \left(\frac{GtL}{4b} \right)_{i,j,k} + \left(\frac{GtL}{4b} \right)_{i,j+1,k} + \left(\frac{GtL}{4h} \right)_{i,j,k} + \left(\frac{GtL}{4h} \right)_{i,j,k+1} + \\
 & \left(\frac{AE}{L} \right)_{i+1,j,k} + \left(\frac{GtL}{4b} \right)_{i+1,j,k} + \left(\frac{GtL}{4b} \right)_{i+1,j+1,k} + \left(\frac{GtL}{4h} \right)_{i+1,j,k} + \left(\frac{GtL}{4h} \right)_{i+1,j,k+1}
 \end{aligned}$$

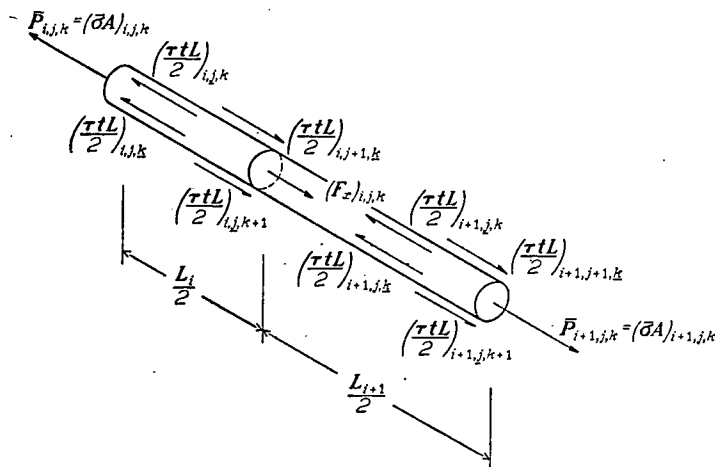


FIGURE 7.—Forces on stringer.

Equation (A4) involves no assumptions regarding equality of structural dimensions, temperatures, or elastic properties about point (i, j, k) . If any element is missing, the associated stiffness goes to zero and the general equation is still applicable. Since AE and Gt always appear as products, the variation of elastic properties with temperature is equivalent to changes in the stringer areas and sheet thicknesses of the effective structure. Furthermore, the thermal-expansion terms appear in the same manner as axial loads applied to the stringers. Thus, if desired, the effects of a nonuniform temperature distribution can be determined by applying a set of equivalent loads to a new effective structure.

BAY SHEAR AND TORQUE EQUILIBRIUM

The equations for the bay displacements (v, w, θ) can be obtained from equilibrium of the shear forces on the bay

cross section

$$(F_y)_i - \sum_j \sum_k [(\tau t b \cos \phi)_{i,j,k} - (\tau t h \sin \psi)_{i,j,k}] = 0 \quad (A5a)$$

$$(F_z)_i - \sum_j \sum_k [(\tau t b \sin \phi)_{i,j,k} + (\tau t h \cos \psi)_{i,j,k}] = 0 \quad (A5b)$$

$$(M_x)_i + \sum_j \sum_k [(\tau t b r)_{i,j,k} - (\tau t h \rho)_{i,j,k}] = 0 \quad (A5c)$$

Substitution of equations (A1) for the shear stresses in equations (A5) results in

$$(J_{vv})_i \Delta v_i + (J_{vw})_i \Delta w_i - (J_{\theta v})_i \Delta \theta_i = (F_y)_i + \sum_j \sum_k (u_{i,j,k} + u_{i-1,j,k}) \left[\left(\frac{Gt \cos \phi}{2} \right)_{i,j+1,k} - \left(\frac{Gt \cos \phi}{2} \right)_{i,j,k} - \left(\frac{Gt \sin \psi}{2} \right)_{i,j,k+1} + \left(\frac{Gt \sin \psi}{2} \right)_{i,j,k} \right] \quad (A6a)$$

$$(J_{vw})_i \Delta v_i + (J_{ww})_i \Delta w_i - (J_{\theta w})_i \Delta \theta_i = (F_z)_i + \sum_j \sum_k (u_{i,j,k} + u_{i-1,j,k}) \left[\left(\frac{Gt \sin \phi}{2} \right)_{i,j+1,k} - \left(\frac{Gt \sin \phi}{2} \right)_{i,j,k} + \left(\frac{Gt \cos \psi}{2} \right)_{i,j,k+1} - \left(\frac{Gt \cos \psi}{2} \right)_{i,j,k} \right] \quad (A6b)$$

$$-(J_{\theta v})_i \Delta v_i - (J_{\theta w})_i \Delta w_i + (J_{\theta \theta})_i \Delta \theta_i = (M_x)_i - \sum_j \sum_k (u_{i,j,k} + u_{i-1,j,k}) \left[\left(\frac{Gtr}{2} \right)_{i,j+1,k} - \left(\frac{Gtr}{2} \right)_{i,j,k} - \left(\frac{Gt\rho}{2} \right)_{i,j,k+1} + \left(\frac{Gt\rho}{2} \right)_{i,j,k} \right] \quad (A6c)$$

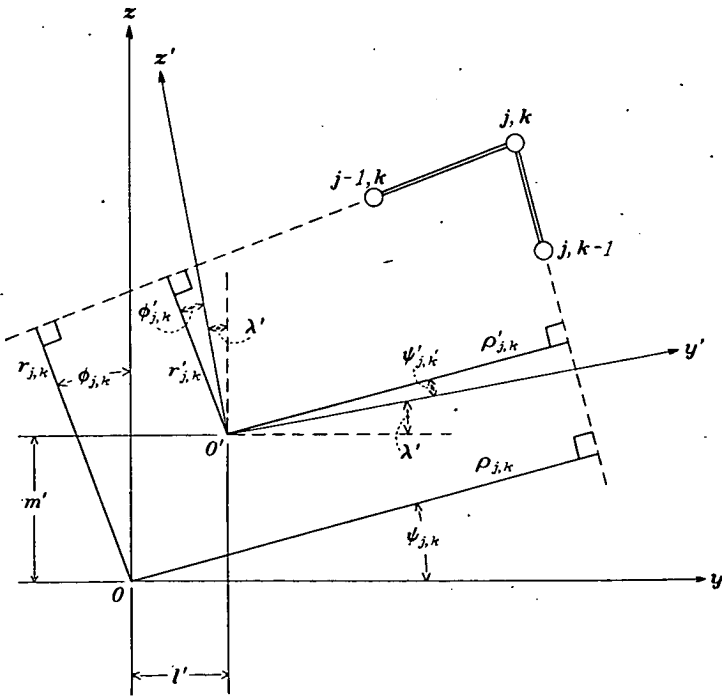


FIGURE 8.—Notation used to locate principal shear axes.

where

$$(J_{vv})_i = \sum_j \sum_k \left[\left(\frac{Gtb \cos^2 \phi}{L} \right)_{i,j,k} + \left(\frac{Gth \sin^2 \psi}{L} \right)_{i,j,k} \right]$$

$$(J_{vw})_i = \sum_j \sum_k \left[\left(\frac{Gtb \sin \phi \cos \phi}{L} \right)_{i,j,k} - \left(\frac{Gth \sin \psi \cos \psi}{L} \right)_{i,j,k} \right]$$

$$(J_{ww})_i = \sum_j \sum_k \left[\left(\frac{Gtb \sin^2 \phi}{L} \right)_{i,j,k} + \left(\frac{Gth \cos^2 \psi}{L} \right)_{i,j,k} \right]$$

$$(J_{\theta v})_i = \sum_j \sum_k \left[\left(\frac{Gtbr \cos \phi}{L} \right)_{i,j,k} + \left(\frac{Gth \rho \sin \psi}{L} \right)_{i,j,k} \right]$$

$$(J_{\theta w})_i = \sum_j \sum_k \left[\left(\frac{Gtbr \sin \phi}{L} \right)_{i,j,k} - \left(\frac{Gth \rho \cos \psi}{L} \right)_{i,j,k} \right]$$

$$(J_{\theta \theta})_i = \sum_j \sum_k \left[\left(\frac{Gtbr^2}{L} \right)_{i,j,k} + \left(\frac{Gth \rho^2}{L} \right)_{i,j,k} \right]$$

Equations (A6) can be simplified by eliminating the coupling terms if the axes used in the computations are the principal shear axes of the cross section. These axes are defined such that

$$J_{vw}' = J_{\theta v}' = J_{\theta w}' = 0 \quad (A7)$$

The relationship between the location and orientation of points and panels in two systems of coordinates, arbitrary axes (x, y, z) and the principal shear axes (x', y', z') , is shown in figure 8 and given by the following equations:

$$y' = (z - m') \sin \lambda' + (y - l') \cos \lambda' \quad (A8a)$$

$$z' = (z - m') \cos \lambda' - (y - l') \sin \lambda' \quad (A8b)$$

$$\phi' = \phi - \lambda' \quad (A9a)$$

$$\psi' = \psi - \lambda' \quad (A9b)$$

$$r' = r + l' \sin \phi - m' \cos \phi \quad (A10a)$$

$$\rho' = \rho - l' \cos \psi - m' \sin \psi \quad (A10b)$$

Then the location of the principal shear axes is

$$\tan 2\lambda' = \frac{2J_{vw}}{J_{vv} - J_{ww}} \quad (A11a)$$

$$l' = -\frac{J_{vv}J_{\theta w} - J_{vw}J_{\theta v}}{J_{vv}J_{ww} - J_{vw}^2} \quad (A11b)$$

$$m' = \frac{J_{ww}J_{\theta v} - J_{vw}J_{\theta w}}{J_{vv}J_{ww} - J_{vw}^2} \quad (A11c)$$

and, with respect to these axes,

$$J_{vv}' = J_{vv} \cos^2 \lambda' + J_{ww} \sin^2 \lambda' + 2J_{vw} \sin \lambda' \cos \lambda' \quad (A12a)$$

$$J_{ww}' = J_{vv} \sin^2 \lambda' + J_{ww} \cos^2 \lambda' - 2J_{vw} \sin \lambda' \cos \lambda' \quad (A12b)$$

$$J_{\theta \theta}' = J_{\theta \theta} + l' J_{\theta w} - m' J_{\theta v} \quad (A12c)$$

$$F_y' = F_z \sin \lambda' + F_y \cos \lambda' \quad (A13a)$$

$$F_z' = F_z \cos \lambda' - F_y \sin \lambda' \quad (\text{A13b})$$

$$M_z' = M_z + m' F_y - l' F_z \quad (\text{A13c})$$

When referred to the principal shear axes, the equations for the bay displacements become

$$\Delta v_i' = \left(\frac{1}{J_{vv'}} \right)_i \left\{ (F_y')_i + \sum_j \sum_k (u_{i,j,k} + u_{i-1,j,k}) \left[\left(\frac{Gt \cos \phi'}{2} \right)_{i,j+1,k} - \left(\frac{Gt \cos \phi'}{2} \right)_{i,j,k} - \left(\frac{Gt \sin \psi'}{2} \right)_{i,j,k+1} + \left(\frac{Gt \sin \psi'}{2} \right)_{i,j,k} \right] \right\} \quad (\text{A14a})$$

$$\Delta w_i' = \left(\frac{1}{J_{ww'}} \right)_i \left\{ (F_z')_i + \sum_j \sum_k (u_{i,j,k} + u_{i-1,j,k}) \left[\left(\frac{Gt \sin \phi'}{2} \right)_{i,j+1,k} - \left(\frac{Gt \sin \phi'}{2} \right)_{i,j,k} + \left(\frac{Gt \cos \psi'}{2} \right)_{i,j,k+1} - \left(\frac{Gt \cos \psi'}{2} \right)_{i,j,k} \right] \right\} \quad (\text{A14b})$$

$$\Delta \theta_i' = \left(\frac{1}{J_{\theta\theta'}} \right)_i \left\{ (M_z')_i - \sum_j \sum_k (u_{i,j,k} + u_{i-1,j,k}) \left[\left(\frac{Gtr'}{2} \right)_{i,j+1,k} - \left(\frac{Gtr'}{2} \right)_{i,j,k} - \left(\frac{Gtp'}{2} \right)_{i,j,k+1} + \left(\frac{Gtp'}{2} \right)_{i,j,k} \right] \right\} \quad (\text{A14c})$$

BAY THRUST AND MOMENT EQUILIBRIUM

The equations obtained from equations (A4) and (A14) are sufficient in themselves to define completely the displacements of the structure. However, if the equations are solved by iteration, it is helpful to employ a periodic correction cycle based on the gross equilibrium of axial loads in the cross section

$$(F_z)_i - \sum_j \sum_k (\bar{P})_{i,j,k} = 0 \quad (\text{A15a})$$

$$(\bar{M}_y)_i - \sum_j \sum_k (\bar{P}z)_{i,j,k} = 0 \quad (\text{A15b})$$

$$(\bar{M}_z)_i + \sum_j \sum_k (\bar{P}y)_{i,j,k} = 0 \quad (\text{A15c})$$

It can be shown that equations (A15) are satisfied by the solution of equations (A4) and (A14); however, they are not likely to be satisfied by the displacement values obtained from any given cycle of iteration. In reference 2 it was demonstrated that convergence of the iterative process can be speeded if the displacement values are periodically corrected so that the stringer displacements satisfy equations (A15).

The corrections applied to the stringer displacements are a planar distribution over the cross section and are determined as follows:

$$(u_{i,j,k})_{n+1} = (u_{i,j,k})_n + \Delta u_{i,j,k} \quad (\text{A16})$$

where

$$\Delta u_{i,j,k} = \delta u_i + \beta_{z,i} y_{j,k} + \beta_{y,i} z_{j,k}$$

Substituting equation (A16) into (A15) yields

$$(AE)_i \delta u_i + (EQ_z)_i \beta_{z,i} + (EQ_y)_i \beta_{y,i} = (LF_z)_i + L_i \sum_j \sum_k \left[(AE\alpha Tz)_{i,j,k} - (u_{i,j,k} - u_{i-1,j,k}) \left(\frac{AE}{L} \right)_{i,j,k} \right] \quad (\text{A17a})$$

$$(EQ_y)_i \delta u_i + (EI_{yz})_i \beta_{z,i} + (EI_{yy})_i \beta_{y,i} = (L\bar{M}_y)_i + L_i \sum_j \sum_k \left[(AE\alpha Ty)_{i,j,k} - (u_{i,j,k} - u_{i-1,j,k}) \left(\frac{AEy}{L} \right)_{i,j,k} \right] \quad (\text{A17b})$$

$$(EQ_z)_i \delta u_i + (EI_{zz})_i \beta_{z,i} + (EI_{yz})_i \beta_{y,i} = -(L\bar{M}_z)_i + L_i \sum_j \sum_k \left[(AE\alpha Tz)_{i,j,k} - (u_{i,j,k} - u_{i-1,j,k}) \left(\frac{AEz}{L} \right)_{i,j,k} \right] \quad (\text{A17c})$$

where

$$(AE)_i = \sum_j \sum_k (AE)_{i,j,k} \quad (EI_{yy})_i = \sum_j \sum_k (AEz^2)_{i,j,k}$$

$$(EQ_z)_i = \sum_j \sum_k (AEy)_{i,j,k} \quad (EI_{zz})_i = \sum_j \sum_k (AEy^2)_{i,j,k}$$

$$(EQ_y)_i = \sum_j \sum_k (AEz)_{i,j,k} \quad (EI_{yz})_i = \sum_j \sum_k (AEyz)_{i,j,k}$$

These equations can be simplified by elimination of the coupling terms if the computations are referred to the equivalent principal inertia axes of the cross section. These axes are referred to as equivalent because the variation of modulus of elasticity over the cross section is taken into consideration. These axes (x'' , y'' , z'') are defined such that

$$EQ_z'' = EQ_y'' = EI_{yz}'' = 0 \quad (\text{A18})$$

and then the following relationships are applicable:

$$y'' = (z - m'') \sin \lambda'' + (y - l'') \cos \lambda'' \quad (\text{A19a})$$

$$z'' = (z - m'') \cos \lambda'' - (y - l'') \sin \lambda'' \quad (\text{A19b})$$

$$\tan 2\lambda'' = \frac{2(EI_{yz} - AE l'' m'')}{(EI_{yy} - AE m''^2) - (EI_{zz} - AE l''^2)} \quad (\text{A20a})$$

$$l'' = \frac{EQ_z}{AE} \quad (\text{A20b})$$

$$m'' = \frac{EQ_y}{AE} \quad (\text{A20c})$$

$$EI_{yy}'' = (EI_{yy} - AE m''^2) \cos^2 \lambda'' + (EI_{zz} - AE l''^2) \sin^2 \lambda'' - 2(EI_{yz} - AE l'' m'') \sin \lambda'' \cos \lambda'' \quad (\text{A21a})$$

$$EI_{zz}'' = (EI_{yy} - AE m''^2) \sin^2 \lambda'' + (EI_{zz} - AE l''^2) \cos^2 \lambda'' + 2(EI_{yz} - AE l'' m'') \sin \lambda'' \cos \lambda'' \quad (\text{A21b})$$

$$F_x'' = F_x \quad (\text{A22a})$$

$$\bar{M}_y'' = \bar{M}_z \sin \lambda'' + \bar{M}_y \cos \lambda'' \quad (\text{A22b})$$

$$\bar{M}_z'' = \bar{M}_z \cos \lambda'' - \bar{M}_y \sin \lambda'' \quad (\text{A22c})$$

A further simplification of the correction cycle can be made by eliminating the load and temperature terms on the right-hand side of equations (A17). This elimination can be accomplished by iterating the difference between the exact solution and one which satisfies statics (equations (A15)) but not necessarily continuity. The iteration of differences has an additional advantage in that smaller numbers, and consequently less work, are required to obtain a solution.

An examination of equations (A17) indicates that they can be satisfied by a planar distribution of strain corresponding

to the elementary analysis of reference 1. Then the initial values of stringer displacements u can be defined as follows:

$$u_{i,j,k} = u_{i-1,j,k} + (\epsilon_0 L)_{i,j,k} \quad (A23)$$

where

$$(\epsilon_0 L)_{i,j,k} = (\delta u_i'')_0 + (\beta_{z,i}'')_0 y_{j,k}'' + (\beta_{y,i}'')_0 z_{j,k}''$$

and, with respect to the equivalent principal inertia axes,

$$(\delta u_i'')_0 = \left(\frac{L}{AE} \right)_i \left[(F_z'')_i + \sum_j \sum_k (AE \alpha T)_{i,j,k} \right] \quad (A24a)$$

$$(\beta_{z,i}'')_0 = \left(\frac{L}{EI_{zz}''} \right)_i \left[-(\bar{M}_z'')_i + \sum_j \sum_k (AE \alpha T y'')_{i,j,k} \right] \quad (A24b)$$

$$(\beta_{y,i}'')_0 = \left(\frac{L}{EI_{yy}''} \right)_i \left[(\bar{M}_y'')_i + \sum_j \sum_k (AE \alpha T z'')_{i,j,k} \right] \quad (A24c)$$

The corresponding values of the bay displacements are obtained from equations (A14) and (A23).

Then the correction-cycle equations applicable to the iterated differences are as follows:

$$\delta u_i'' = - \left(\frac{1}{AE} \right)_i \sum_j \sum_k (\Delta u_{i,j,k} - \Delta u_{i-1,j,k}) (AE)_{i,j,k} \quad (A25a)$$

$$\beta_{z,i}'' = - \left(\frac{1}{EI_{zz}''} \right)_i \sum_j \sum_k (\Delta u_{i,j,k} - \Delta u_{i-1,j,k}) (AE y'')_{i,j,k} \quad (A25b)$$

$$\beta_{y,i}'' = - \left(\frac{1}{EI_{yy}''} \right)_i \sum_j \sum_k (\Delta u_{i,j,k} - \Delta u_{i-1,j,k}) (AE z'')_{i,j,k} \quad (A25c)$$

Equations (A25) provide corrections to the stringer displacements u only. These corrections remove any unbalanced moment or thrust on the cross section but add unbalanced shear forces which are removed by correcting the

bay displacements (v, w, θ) . The corrected bay displacements are obtained from the corrected stringer displacements by application of equation (A14). These two operations constitute the complete correction cycle that brings the stringer loads into equilibrium with the external loads without changing the shear stress in any panel.

REFERENCES

1. Heldenfels, Richard R.: The Effect of Nonuniform Temperature Distributions on the Stresses and Distortions of Stiffened-Shell Structures. NACA TN 2240, 1950.
2. Duberg, John E.: A Numerical Procedure for the Stress Analysis of Stiffened Shells. Jour. Aero. Sci., vol. 16, no. 8, Aug. 1949, pp. 451-462.
3. Southwell, R. V.: Relaxation Methods in Engineering Science. A Treatise on Approximate Computations. The Clarendon Press (Oxford), 1940.

TABLE I.—STRINGER PROPERTIES

①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩
i	L	j	T	E	αT	A	AE	$\frac{AE}{L}$	$AE \alpha T$
				*	**		⑥×⑦	$\frac{⑧}{⑨}$	⑥×⑩
1	20	1	206	9.55640×10 ⁸	2.7285×10 ⁻⁶	1.11	10.60760×10 ⁸	530,380	23,943
	20	2	158	9.89078	2.066,0	1.15	11.37440	568,720	23,500
	20	3	155	9.90937	2,025.2	.74	7.33293	366,647	14,851
	20	4	162	9.86558	2,120.6	1.01	9.96424	498,212	21,130
2	20	1	219	9.45386	2,910.7	0.98	9.26478	463,239	26,967
	20	2	173	9.79376	2,271.3	1.05	10.23345	514,173	23,357
	20	3	165	9.84635	2,161.6	.66	6.49859	324,930	14,047
	20	4	167	9.83339	2,189.0	.89	8.75172	437,586	19,158
3	20	1	231	9.35468	3,080.0	0.59	5.51925	275,963	16,999
	20	2	187	9.69703	2,464.3	.62	6.01219	300,610	14,816
	20	3	175	9.78031	2,298.8	.38	3.71652	185,826	8,544
	20	4	172	9.80044	2,257.6	.54	5.29224	264,612	11,948
4	20	1	244	9.24233	3,264.4	0.46	4.25147	212,574	13,878
	20	2	202	9.53691	2,672.7	.52	4.98520	249,260	13,324
	20	3	185	9.71125	2,436.7	.30	2.91338	145,669	7,099
	20	4	177	9.76674	2,326.3	.42	4.10203	205,102	9,543

* $E = (10.5 - 0.00147 T - 0.0000151 T^2) \times 10^8$

** $\alpha T = (12.52 T + 0.00352 T^2) \times 10^{-6}$

TABLE II.—PANEL PROPERTIES

Cover panels $i,j,1$										Webs: panels $i,j,1$									
①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯	⑰	⑱	⑲	⑳
i	L	j	T	G	t	b	$\frac{Gt}{2}$	$\frac{Gtb}{L}$	$\frac{GLL}{4b}$	$\frac{2}{tL}$	T	G	t	h	$\frac{Gt}{2}$	$\frac{Gth}{L}$	$\frac{GLL}{4h}$	$\frac{2}{tL}$	
				*			$\frac{⑧ \times ⑩}{2}$	$\frac{2⑧ \times ⑩}{⑨}$	$\frac{⑨ \times ⑩}{2⑨}$	$\frac{2}{⑩ \times ⑩}$		*			$\frac{⑮ \times ⑰}{2}$	$\frac{2⑮ \times ⑰}{⑱}$	$\frac{⑱ \times ⑰}{2⑱}$	$\frac{2}{⑱ \times ⑱}$	
1	20	1	206	3.57892×10 ⁸	0.040	10	71,578	71,578	71,578	2.5	206	3.49967×10 ⁸	0.064	10	111,989	111,989	111,989	1.56250	
	20	2	158	3.65560	.040	10	73,112	73,112	73,112	2.5	158	3.65265	.051	10	93,143	93,143	93,143	1.96078	
	20	3	155	3.64969	.040	10	72,994	72,994	72,994	2.5		3.64075	.051	10	92,839	92,839	92,839	1.96078	
	20	4	162		.040	10				2.5	162			10					
2	20	1	219	3.53336	0.040	10	70,667	70,667	70,667	2.5	219	3.45443	0.064	10	110,542	110,542	110,542	1.56250	
	20	2	173	3.61955	.040	10	72,391	72,391	72,391	2.5	173	3.60722	.051	10	91,984	91,984	91,984	1.96078	
	20	3	165	3.62869	.040	10	72,574	72,574	72,574	2.5	167	3.62565	.051	10	92,454	92,454	92,454	1.96078	
	20	4	166		.040	10				2.5				10					
3	20	1	231	3.48937	0.020	10	34,894	34,894	34,894	5.0	231	3.41122	0.032	10	54,580	54,580	54,580	3.12500	
	20	2	209	3.58211	.020	10	35,821	35,821	35,821	5.0	187	3.56287	.025	10	44,536	44,536	44,536	4.00000	
	20	3	181	3.60722	.020	10	36,072	36,072	36,072	5.0		3.61032	.025	10	45,129	45,129	45,129	4.00000	
	20	4	173		.020	10				5.0	172			10					
4	20	1	244	3.44018	0.020	10	34,402	34,402	34,402	5.0	244	3.36287	0.032	10	53,806	53,806	53,806	3.12500	
	20	2	223	3.53999	.020	10	35,400	35,400	35,400	5.0	202	3.51326	.025	10	43,916	43,916	43,916	4.00000	
	20	3	194	3.58211	.020	10	35,821	35,821	35,821	5.0		3.59474	.025	10	44,934	44,934	44,934	4.00000	
	20	4	181		.020	10				5.0	177			10					

* $G = (4.0 - 0.00144 T - 0.000048 T^2) \times 10^8$

TABLE III.—PRINCIPAL SHEAR AXES AND BAY DISPLACEMENT EQUATIONS

Load problem*														Temperature problem**									
i	j	$J_{uv} = J_{vu}'$	v	$\frac{z}{2}, \frac{z'}{2}, \frac{z''}{2}$	J_{uv}	v'	y'	J_{uv}	$F_u = F_u'$	M_u	M_u'	$\left(\frac{Gt}{2}\right)^{i+1}_i - \left(\frac{Gt}{2}\right)^i_{i-1}$	$\left(\frac{Gt^2}{2}\right)^{i+1}_i - \left(\frac{Gt^2}{2}\right)^i_{i-1}$	$\left(\frac{Gt^3}{2}\right)^{i+1}_i - \left(\frac{Gt^3}{2}\right)^i_{i-1}$	$\Delta w'$ [C]	$\Delta w''$ [c]	$\Delta \theta'$ [C]	$\Delta \theta''$ [c]	$J_{uv} = J_{vu}'$	$\Delta \theta'$ [C]	$\Delta \theta''$ [c]	Temperature problem**	
1	1	$\sum_{j=1}^4 \textcircled{5}$			$\sum_{j=1}^4 \textcircled{25} + \sum_{j=1}^4 \textcircled{25} \times \textcircled{25}$	$-\frac{\textcircled{2}}{\textcircled{5}}$	$\textcircled{25} - \textcircled{25}$					$\textcircled{13} - \textcircled{13}$	$5\textcircled{3}$	$5\textcircled{3}$	$\frac{2\textcircled{26}}{\textcircled{5}}$	$\frac{\textcircled{26}}{\textcircled{5}}$	$\frac{\textcircled{1}}{\textcircled{5}}$	$\frac{\textcircled{1}}{\textcircled{5}}$	$\frac{4}{2} \sum_{j=1}^4 \textcircled{10}$				
2	2	$\sum_{j=1}^4 \textcircled{5}$			$\sum_{j=1}^4 \textcircled{5} + \sum_{j=1}^4 \textcircled{5} \times \textcircled{25}$	$-\frac{\textcircled{2}}{\textcircled{5}}$	$\textcircled{25} - \textcircled{25}$					$\textcircled{13} - \textcircled{13}$	$5\textcircled{3}$	$5\textcircled{3}$	$\frac{2\textcircled{26}}{\textcircled{5}}$	$\frac{\textcircled{26}}{\textcircled{5}}$	$\frac{\textcircled{1}}{\textcircled{5}}$	$\frac{\textcircled{1}}{\textcircled{5}}$	$\frac{4}{2} \sum_{j=1}^4 \textcircled{10}$				
3	3	$\sum_{j=1}^4 \textcircled{5}$			$\sum_{j=1}^4 \textcircled{5} + \sum_{j=1}^4 \textcircled{5} \times \textcircled{25}$	$-\frac{\textcircled{2}}{\textcircled{5}}$	$\textcircled{25} - \textcircled{25}$					$\textcircled{13} - \textcircled{13}$	$5\textcircled{3}$	$5\textcircled{3}$	$\frac{2\textcircled{26}}{\textcircled{5}}$	$\frac{\textcircled{26}}{\textcircled{5}}$	$\frac{\textcircled{1}}{\textcircled{5}}$	$\frac{\textcircled{1}}{\textcircled{5}}$	$\frac{4}{2} \sum_{j=1}^4 \textcircled{10}$				
4	4	$\sum_{j=1}^4 \textcircled{5}$			$\sum_{j=1}^4 \textcircled{5} + \sum_{j=1}^4 \textcircled{5} \times \textcircled{25}$	$-\frac{\textcircled{2}}{\textcircled{5}}$	$\textcircled{25} - \textcircled{25}$					$\textcircled{13} - \textcircled{13}$	$5\textcircled{3}$	$5\textcircled{3}$	$\frac{2\textcircled{26}}{\textcircled{5}}$	$\frac{\textcircled{26}}{\textcircled{5}}$	$\frac{\textcircled{1}}{\textcircled{5}}$	$\frac{\textcircled{1}}{\textcircled{5}}$	$\frac{4}{2} \sum_{j=1}^4 \textcircled{10}$				

*For load problem:

$$J_{uv} = \sum_{j=1}^4 \left(\frac{Gt^j}{L} \right)_{i,j}$$

$$J_{uv} = - \sum_{j=1}^4 \left(\frac{Gt^j}{L} \right)_{i,j}$$

$$J_{uv} = \sum_{j=1}^4 \left[2 \left(\frac{Gt^j}{L} \right)_{i,j} + \left(\frac{Gt^j}{L} \right)_{i,j} \right]$$

**For temperature problem.

$$J_{uv} = 2 \sum_{j=1}^4 \left(\frac{Gt^j}{L} \right)_{i,j}$$

Checks:

$$J_{uv} = - \sum_{j=1}^4 \sum_{i=1}^4 \sum_{k=1}^4 \sum_{l=1}^4 \Delta w' = 0$$

$$J_{uv} = J_{uv} + J_{vu}$$

$$\sum_{j=1}^4 \sum_{i=1}^4 \sum_{k=1}^4 \sum_{l=1}^4 \Delta w' = 0$$

$$\sum_{j=1}^4 \sum_{i=1}^4 \sum_{k=1}^4 \sum_{l=1}^4 \Delta \theta' = 0$$

TABLE IV.—PRINCIPAL INERTIA AXES, CORRECTION CYCLE, AND INITIAL STRINGER DISPLACEMENTS

Temperature problem**									
Load problem*									
i	j	AEz''	$(EI_{yy})_i$	$\beta_{y,i}$	$\bar{M}_y = \bar{M}_y''$	$(u_{i,i})_0$	$(\beta_{y,i})_0$	$(u_{i,i})_0$	$(\beta_{y,i})_0$
		$5 \textcircled{0}$	$50 \sum_{j=1}^4 \textcircled{0}$	$2 \textcircled{0}$ $5 \textcircled{0}$	$2 \textcircled{0}$ $5 \textcircled{0}$	$5 \textcircled{0}$ $5 \textcircled{0}$	$5 \textcircled{0}$ $5 \textcircled{0}$	$5 \textcircled{0}$ $5 \textcircled{0}$	$5 \textcircled{0}$ $5 \textcircled{0}$
1	1	$\begin{Bmatrix} 153.03800 \times 10^6 \\ 256.87200 \\ 336.66405 \\ 449.82120 \end{Bmatrix}$	$\begin{Bmatrix} 1,963.7770 \times 10^6 \\ 1,738.9270 \\ 1,027.0105 \\ 812.6040 \end{Bmatrix}$	$\begin{Bmatrix} -0.054016 \\ -0.037421 \\ -0.037341 \\ -0.060740 \end{Bmatrix}$	$\begin{Bmatrix} -320,000 \\ -180,000 \\ -80,000 \\ -20,000 \end{Bmatrix}$	$\begin{Bmatrix} -0.016285 \\ -0.016285 \\ -0.016285 \\ -0.016285 \end{Bmatrix}$	$\begin{Bmatrix} -0.003250026 \\ -0.002069052 \\ -0.001557920 \\ -0.000492245 \end{Bmatrix}$	$\begin{Bmatrix} -0.016285 \\ -0.016285 \\ -0.016285 \\ -0.016285 \end{Bmatrix}$	$\begin{Bmatrix} -0.003250026 \\ -0.002069052 \\ -0.001557920 \\ -0.000492245 \end{Bmatrix}$
2	2	$\begin{Bmatrix} 146.32390 \\ 251.41725 \\ 332.49295 \\ 443.75390 \end{Bmatrix}$	$\begin{Bmatrix} 1,738.9270 \\ 1,027.0105 \\ 812.6040 \\ 604.120 \end{Bmatrix}$	$\begin{Bmatrix} -0.03248 \\ -0.039103 \\ -0.037350 \\ -0.060299 \end{Bmatrix}$	$\begin{Bmatrix} -180,000 \\ -80,000 \\ -20,000 \\ -20,000 \end{Bmatrix}$	$\begin{Bmatrix} -0.026640 \\ -0.026640 \\ -0.026640 \\ -0.026640 \end{Bmatrix}$	$\begin{Bmatrix} -0.002069052 \\ -0.001557920 \\ -0.000492245 \\ -0.000492245 \end{Bmatrix}$	$\begin{Bmatrix} -0.026640 \\ -0.026640 \\ -0.026640 \\ -0.026640 \end{Bmatrix}$	$\begin{Bmatrix} -0.002069052 \\ -0.001557920 \\ -0.000492245 \\ -0.000492245 \end{Bmatrix}$
3	3	$\begin{Bmatrix} 127.59030 \\ 230.06095 \\ 318.68260 \\ 426.46120 \end{Bmatrix}$	$\begin{Bmatrix} 1,027.0105 \\ 812.6040 \\ 604.120 \\ 426.46120 \end{Bmatrix}$	$\begin{Bmatrix} -0.03741 \\ -0.035411 \\ -0.036188 \\ -0.051605 \end{Bmatrix}$	$\begin{Bmatrix} -80,000 \\ -20,000 \\ -20,000 \\ -20,000 \end{Bmatrix}$	$\begin{Bmatrix} -0.034430 \\ -0.034430 \\ -0.034430 \\ -0.034430 \end{Bmatrix}$	$\begin{Bmatrix} -0.001557920 \\ -0.000492245 \\ -0.000492245 \\ -0.000492245 \end{Bmatrix}$	$\begin{Bmatrix} -0.034430 \\ -0.034430 \\ -0.034430 \\ -0.034430 \end{Bmatrix}$	$\begin{Bmatrix} -0.001557920 \\ -0.000492245 \\ -0.000492245 \\ -0.000492245 \end{Bmatrix}$
4	4	$\begin{Bmatrix} 121.25735 \\ 224.92900 \\ 314.56680 \\ 420.51015 \end{Bmatrix}$	$\begin{Bmatrix} 812.6040 \\ 604.120 \\ 426.46120 \\ 314.56680 \end{Bmatrix}$	$\begin{Bmatrix} -0.052320 \\ -0.061348 \\ -0.038522 \\ -0.050480 \end{Bmatrix}$	$\begin{Bmatrix} -20,000 \\ -20,000 \\ -20,000 \\ -20,000 \end{Bmatrix}$	$\begin{Bmatrix} -0.038591 \\ -0.038591 \\ -0.038591 \\ -0.038591 \end{Bmatrix}$	$\begin{Bmatrix} -0.000492245 \\ -0.000492245 \\ -0.000492245 \\ -0.000492245 \end{Bmatrix}$	$\begin{Bmatrix} -0.038591 \\ -0.038591 \\ -0.038591 \\ -0.038591 \end{Bmatrix}$	$\begin{Bmatrix} -0.000492245 \\ -0.000492245 \\ -0.000492245 \\ -0.000492245 \end{Bmatrix}$

*For load problem:

$$\beta_{y,i}'' = -\left(\frac{2}{EI_{yy}}\right) \sum_{j=1}^4 (u_{i,j,1} - u_{i-1,j,1}) (AEz'')_{i,j,1}$$

**For temperature problem:

$$\delta u_i'' = -\left(\frac{2}{AE}\right) \sum_{j=1}^4 (u_{i,j,1} - u_{i-1,j,1}) (AE)_{i,j,1}$$

$$\beta_{y,i}'' = -\left(\frac{2}{EI_{yy}}\right) \sum_{j=1}^4 (u_{i,j,1} - u_{i-1,j,1}) (AEz'')_{i,j,1}$$

Checks:

$$(\beta_{y,i}')_0 = \left(\frac{L}{EI_{yy}}\right) \sum_{j=1}^4 (\bar{M}_y'')_i$$

$$(\delta u_i')_0 = 2 \left(\frac{L}{AE}\right) \sum_{j=1}^4 (AEz'')_{i,j,1}$$

$$(\beta_{y,i}')_0 = 2 \left(\frac{L}{EI_{yy}}\right) \sum_{j=1}^4 (AEz'')_{i,j,1}$$

$$(\beta_{y,i}')_0 = 2 \left(\frac{L}{EI_{yy}}\right) \sum_{j=1}^4 (AEz'')_{i,j,1}$$

TABLE V.—STRINGER-DISPLACEMENT EQUATIONS

(a) Load problem *

i	j	$\sum S$	(24)	(25)	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)	(38)	(39)	(40)	
			$u_{i-1,j-1}$	$u_{i-1,j}$	$u_{i-1,j+1}$	$\frac{2u_{i-1,j} + 2u_{i-1,j+1}}{4}$	$u_{i,j-1}$	$u_{i,j}$	$u_{i,j+1}$	$\frac{2u_{i,j} + 2u_{i,j+1}}{4}$	$u_{i+1,j-1}$	$u_{i+1,j}$	$u_{i+1,j+1}$	$\frac{2u_{i+1,j} + 2u_{i+1,j+1}}{4}$	$\Delta u_{i,j}'$	$\Delta u_{i+1,j}'$	$\Delta \theta_{i,j}'$	$\Delta \theta_{i+1,j}'$	$\left(\frac{GL}{h}\right)_{i,j-1} \frac{1}{\sum S}$	$\left(\frac{GL}{h}\right)_{i+1,j} \frac{1}{\sum S}$
1	1	1,580,925 1,740,895 982,648 1,451,952	0 0.041116 0.074403 0.050273	0.145535 0.135504 0.224435 0.164978	0.045276 0.041997 0.074282 0	0.089976 0.083579 0.148138 0	0 0.081708 0.148072 0.100257	0.108473 0.107500 0.183143 0.124043	0.044700 0.041583 0.073553 0	0.029697 0.027521 0.049570 0	0.044700 0.041583 0.073553 0	0.029697 0.027521 0.049570 0	-0.070838 -0.053503 -0.000600 -0.869324	-0.059002 -0.052837 -0.000600 -0.863065	-0.657179 -0.127008 -0.000600 -0.869324	0.659002 0.128284 0.000600 -0.863065	0.283350 0.214012 0 0.25704	0.270880 0.211348 0 0.254702		
	2	1,175,007 1,301,596 727,614 1,086,010	0 0.054203 0.099491 0.066826	0.145946 0.143783 0.247336 0.165840	0.050142 0.055617 0.099742 0	0.089839 0.083138 0.149318 0	0 0.081101 0.148722 0.100041	0.112252 0.108103 0.156584 0.127330	0.029697 0.027521 0.049570 0	0.044700 0.041583 0.073553 0	0.029697 0.027521 0.049570 0	0.044700 0.041583 0.073553 0	-0.094078 -0.070670 -0.001258 -1.153883	-0.046451 -0.034216 -0.001258 -0.415555	0.877243 0.171540 0.001258 -1.153883	0.430917 0.081071 -0.001258 -0.562239	0.377310 0.136865 0 0.340528	0.185804 0.136865 0 0.166220		
	3	774,605 867,291 474,609 721,733	0 0.040233 0.075475 0.049980	0.170292 0.162371 0.240057 0.191597	0.045047 0.041302 0.076004 0	0.089460 0.082119 0.151478 0	0 0.079899 0.150052 0.099612	0.091092 0.105646 0.156862 0.110031	0.044412 0.040817 0.075475 0	0.044412 0.040817 0.075475 0	0.044412 0.040817 0.075475 0	0.044412 0.040817 0.075475 0	-0.070462 -0.059463 -0.002644 -0.846015	-0.059463 -0.050636 -0.002644 -0.846015	0.653662 0.121662 -0.002644 -0.846015	0.648157 0.122248 -0.004435 -0.839626	0.281847 0.209403 0 0.250115	0.277850 0.202543 0 0.240034		
	4	354,588 408,894 216,890 330,791	0 0.064548 0.163216 0.108289	0.198902 0.225184 0.343252 0.240070	0.097020 0.087001 0.165157 0	0.097020 0.087001 0.165158 0	0 0.084548 0.163216 0.108289	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	-0.151742 -0.205772 -0.007705 -1.831930	0 0 0 0	1.415914 0.205772 -0.007705 -1.831930	0 0 0 0	0.600969 0.431719 0 0.543352	0 0 0 0	

$$\begin{aligned}
 & \frac{1}{\sum S} \left\{ u_{i-1,j-1} \left[\left(\frac{GL}{4b} \right)_{i,j-1} + \left(\frac{GL}{4b} \right)_{i,j} \right] + u_{i,j-1} \left[\left(\frac{GL}{4b} \right)_{i,j-1} - \left(\frac{GL}{4b} \right)_{i,j} \right] + \right. \\
 & u_{i-1,j} \left[\left(\frac{AE}{L} \right)_{i,j-1} - \left(\frac{GL}{4b} \right)_{i,j-1} - 2 \left(\frac{GL}{4b} \right)_{i,j} \right] + u_{i,j} \left[\left(\frac{AE}{L} \right)_{i,j} - \left(\frac{GL}{4b} \right)_{i,j} - 2 \left(\frac{GL}{4b} \right)_{i,j+1} \right] - \Delta w_i' \left(\frac{GL}{2} \right)_{i,j-1} + \\
 & \left. u_{i-1,j+1} \left(\frac{GL}{4b} \right)_{i,j+1} + u_{i,j+1} \left[\left(\frac{GL}{4b} \right)_{i,j+1} + \left(\frac{GL}{4b} \right)_{i,j+1} \right] + u_{i+1,j-1} \left[\left(\frac{GL}{2} \right)_{i+1,j-1} - \left(\frac{GL}{2} \right)_{i+1,j} \right] \right\} \\
 & \frac{1}{\sum S} \left\{ \left(\frac{AE}{L} \right)_{i,j-1} + \left(\frac{AE}{L} \right)_{i,j} + \left(\frac{GL}{4b} \right)_{i,j-1} + \left(\frac{GL}{4b} \right)_{i,j} + \left(\frac{GL}{4b} \right)_{i,j+1} + 2 \left(\frac{GL}{4b} \right)_{i,j+1} \right\}
 \end{aligned}$$

Check: (1)+(2)+(3)+(4)+(5)+(6)+(7)+(8)+(9)+(10)+(11)+(12)+(13)+(14)+(15)+(16)+(17)+(18)+(19)+(20)=1

TABLE V.—STRINGER-DISPLACEMENT EQUATIONS—Concluded

(b) Temperature problem *

(90)	(91)	(92)	(93)	(94)	(95)	(96)	(97)	(98)	(99)	(100)	(101)	(102)	(103)
i	j	$\sum S$	$u_{i-1,j-1}$	$u_{i-1,j}$	$u_{i-1,j+1}$	$u_{i,j-1}$	$u_{i,j+1}$	$u_{i+1,j-1}$	$u_{i+1,j}$	$u_{i+1,j+1}$	$\Delta v'_i$	$\Delta v'_{i+1}$	$\{c\}$
		..	$\frac{(20)_{i,j}}{(92)}$	$\frac{(9)_{i,j} - (20)_{i,j} - (20)_{i,j+1}}{(92)}$	$\frac{(20)_{i,j+1}}{(92)}$	$\frac{(20)_{i,j} + (20)_{i+1,j}}{(92)}$	$\frac{(20)_{i,j+1} + (20)_{i+1,j+1}}{(92)}$	$\frac{(20)_{i+1,j}}{(92)}$	$\frac{(9)_{i+1,j} - (20)_{i+1,j} - (20)_{i+1,j+1}}{(92)}$	$\frac{(20)_{i+1,j+1}}{(92)}$	$\frac{(42)_{i,j}}{(92)}$	$\frac{(42)_{i+1,j}}{(92)}$	$\frac{(10)_{i,j} - (10)_{i+1,j}}{(92)}$
1	1	1, 135, 864	0	0.403923	0.063016	0	0.125231	0	0.345615	0.062214	0.063016	0.062214	0.001740
	2	1, 370, 641	.052222	.309366	.053341	.103780	.106157	.051558	.270760	.052815	.001119	.001258	.000104
	3	982, 648	.074403	.224435	.074283	.148072	.148138	.073669	.183143	.073856	-.000120	.000186	.000818
	4	1, 081, 366	.067502	.393223	0	.134615	0	.067113	.337547	0	-.067502	-.067113	.001824
2	1	844, 763	0	0.464713	0.083653	0	0.124959	0	0.285369	0.041306	0.083653	0.041306	0.011800
	2	1, 028, 556	.068705	.360812	.070381	.102630	.105208	.033925	.223512	.034826	.001676	.000898	.008304
	3	727, 614	.099191	.247336	.099742	.148722	.149318	.049231	.156584	.049576	.000252	.000345	.007563
	4	510, 844	.089504	.450163	0	.133991	0	.044487	.281854	0	-.089504	-.044487	.008892
3	1	557, 833	0	0.432153	0.062553	0	0.124224	0	0.319400	0.061671	0.062553	0.061671	0.005595
	2	690, 387	.050543	.332994	.051885	.100373	.103161	.049830	.259938	.051276	.001338	.001446	.002161
	3	474, 609	.075475	.240057	.076004	.150062	.151478	.074588	.156862	.075475	.000529	.000887	.003045
	4	541, 607	.066602	.421966	0	.132740	0	.066138	.312553	0	-.066602	-.066138	.004440
4	1	246, 976	0	0.721414	0.139293	0	0.139293	0	0	0	0.139293	0	0.056192
	2	319, 062	.107822	.562455	.110950	.107822	.110950	0	0	0	.003128	0	.041760
	3	216, 890	.163217	.343252	.165158	.163216	.165157	0	0	0	.001941	0	.032731
	4	240, 923	.148682	.702635	0	.148682	0	0	0	0	-.148682	0	.039610

$$*u_{i,j,1} = \frac{1}{\sum S} \left\{ u_{i-1,j-1,1} \left(\frac{GtL}{4b} \right)_{i,j,1} + u_{i,j-1,1} \left[\left(\frac{GtL}{4b} \right)_{i,j,1} + \left(\frac{GtL}{4b} \right)_{i+1,j,1} \right] + u_{i+1,j-1,1} \left(\frac{GtL}{4b} \right)_{i+1,j,1} + u_{i-1,j,1} \left[\left(\frac{AE}{L} \right)_{i,j,1} - \left(\frac{GtL}{4b} \right)_{i,j,1} - \left(\frac{GtL}{4b} \right)_{i,j+1,1} \right] + \Delta v'_i \left[\left(\frac{Gt}{2} \right)_{i,j+1,1} - \left(\frac{Gt}{2} \right)_{i,j,1} \right] + \right. \\ \left. u_{i+1,j,1} \left[\left(\frac{AE}{L} \right)_{i+1,j,1} - \left(\frac{GtL}{4b} \right)_{i+1,j,1} - \left(\frac{GtL}{4b} \right)_{i+1,j+1,1} \right] + u_{i-1,j+1,1} \left(\frac{GtL}{4b} \right)_{i-1,j+1,1} + u_{i,j+1,1} \left[\left(\frac{GtL}{4b} \right)_{i,j+1,1} + \left(\frac{GtL}{4b} \right)_{i+1,j+1,1} \right] + \Delta v'_{i+1} \left[\left(\frac{Gt}{2} \right)_{i+1,j+1,1} - \left(\frac{Gt}{2} \right)_{i+1,j,1} \right] + \right. \\ \left. u_{i+1,j+1,1} \left(\frac{GtL}{4b} \right)_{i+1,j+1,1} + (AE\alpha T)_{i,j,1} - (AE\alpha T)_{i+1,j,1} \right\}$$

$$**\sum S = \left(\frac{AE}{L} \right)_{i,j,1} + \left(\frac{AE}{L} \right)_{i+1,j,1} + \left(\frac{GtL}{4b} \right)_{i,j,1} + \left(\frac{GtL}{4b} \right)_{i,j+1,1} + \left(\frac{GtL}{4b} \right)_{i+1,j,1} + \left(\frac{GtL}{4b} \right)_{i+1,j+1,1}$$

Check: (93) + (94) + (95) + (96) + (97) + (98) + (99) + (100) = 1

TABLE VI.—MATRIX OF COEFFICIENTS

(a) Load problem

	u_{111}	u_{121}	u_{131}	u_{141}	u_{211}	u_{221}	u_{231}	u_{241}	u_{311}	u_{321}	u_{331}	u_{341}
u_{111}	0	0.081708	0	0	0.145946	0.054293	0	0	0	0	0	0
u_{121}	.089976	0	.148072	0	.060142	.143783	.099491	0	0	0	0	0
u_{131}	0	.083579	0	.100257	0	.055617	.247336	.066826	0	0	0	0
u_{141}	0	0	.148138	0	0	0	.099742	.165840	0	0	0	0
u_{211}	0.108473	0.040592	0	0	0	0.081101	0	0	0.170292	0.040233	0	0
u_{221}	.044700	.107500	.073669	0	.089839	0	.148722	0	.045047	.162371	.075475	0
u_{231}	0	.041583	.183143	.049984	0	.083138	0	.100041	0	.041302	.240057	.049980
u_{241}	0	0	.073856	.124043	0	0	.149318	0	0	0	.076004	.191597
u_{311}	0	0	0	0	0.112262	0.026809	0	0	0	0.079899	0	0
u_{321}	0	0	0	0	.029697	.108193	.049231	0	.089460	0	.150062	0
u_{331}	0	0	0	0	0	.027521	.156584	.033215	0	.082119	0	.099612
u_{341}	0	0	0	0	0	0	.049576	.127330	0	0	.151478	0
u_{411}	0	0	0	0	0	0	0	0	0.091092	0.039666	0	0
u_{421}	0	0	0	0	0	0	0	0	.044412	.105646	.074588	0
u_{431}	0	0	0	0	0	0	0	0	0	.040817	.156862	.049632
u_{441}	0	0	0	0	0	0	0	0	0	0	.075475	.110031
$\Delta w_1'$	-.070838	-.053503	0	-.063941	0	0	0	0	0	0	0	0
$\Delta w_2'$	-.069922	-.052837	0	-.063676	-.094078	-.070670	0	-.085132	0	0	0	0
$\Delta w_3'$	0	0	0	0	-.046451	-.034216	0	-.041555	-.070462	-.051351	0	-.062529
$\Delta w_4'$	0	0	0	0	0	0	0	0	-.069463	-.050636	0	-.062258
$\Delta \theta_1'$	0.657179	0.127908	0.000600	-.0.869324	0	0	0	0	0	0	0	0
$\Delta \theta_2'$	.652002	.128254	-.000931	-.863065	.877243	.171540	-.001258	-1.153883	0	0	0	0
$\Delta \theta_3'$	0	0	0	0	.430917	.081071	-.001725	-.562239	.653662	.121669	-.002644	-.846015
$\Delta \theta_4'$	0	0	0	0	0	0	0	0	.648157	.122248	-.004435	-.839626
c	0	0	0	0	0	0	0	0	0	0	0	0

	u_{411}	u_{421}	u_{431}	u_{441}	$\Delta w_1'$	$\Delta w_2'$	$\Delta w_3'$	$\Delta w_4'$	$\Delta \theta_1'$	$\Delta \theta_2'$	$\Delta \theta_3'$	$\Delta \theta_4'$
u_{411}	0	0	0	0	-.0.751678	-.0.749488	0	0	0.036202	0.036204	0	0
u_{421}	0	0	0	0	-.625182	-.623662	0	0	.007760	.007842	0	0
u_{431}	0	0	0	0	0	0	0	0	.000020	-.000032	0	0
u_{441}	0	0	0	0	-.623142	-.626850	0	0	-.043982	-.044014	0	0
u_{211}	0	0	0	0	0	-.0.749488	-.0.756768	0	0	0.036204	0.036210	0
u_{221}	0	0	0	0	0	-.623662	-.617504	0	0	.007842	.007546	0
u_{231}	0	0	0	0	0	0	0	0	0	-.000032	-.000090	0
u_{241}	0	0	0	0	0	-.626850	-.625728	0	0	-.044014	-.043666	0
u_{311}	0.198992	0.084548	0	0	0	0	-.0.756768	-.0.754346	0	0	0.036210	0.036220
u_{321}	.097020	.225184	.163216	0	0	0	-.617504	-.615690	0	0	.007546	.007648
u_{331}	0	.087001	.343262	.108289	0	0	0	0	0	0	-.000090	-.000142
u_{341}	0	0	.165157	.240070	0	0	-.625728	-.629962	0	0	-.043666	-.043716
u_{411}	0	0.084548	0	0	0	0	0	-.0.754346	0	0	0	0.036220
u_{421}	.097020	0	.163216	0	0	0	0	-.615690	0	0	0	.007648
u_{431}	0	.087001	0	.108289	0	0	0	0	0	0	0	-.000142
u_{441}	0	0	.165158	0	0	0	0	-.629962	0	0	0	-.043716
$\Delta w_1'$	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta w_2'$	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta w_3'$	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta w_4'$	-.151742	-.107930	0	-.135838	0	0	0	0	0	0	0	0
$\Delta \theta_1'$	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta \theta_2'$	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta \theta_3'$	0	0	0	0	0	0	0	0	0	0	0	0
$\Delta \theta_4'$	1.415914	.260572	-.009705	-.1.831930	0	0	0	0	0	0	0	0
c	0	0	0	0	0.026848	0.020340	0.027731	0.014020	-.0.000345	-.0.000266	-.0.000354	-.0.000182

TABLE VI.—MATRIX OF COEFFICIENTS—Concluded

(b) Temperature problem

	u_{111}	u_{121}	u_{131}	u_{141}	u_{211}	u_{221}	u_{231}	u_{241}	u_{311}	u_{321}
u_{111}	0	0.103780	0	0	0.464713	0.068705	0	0	0	0
u_{121}	.125231	0	.148072	0	.083653	.360812	.099491	0	0	0
u_{131}	0	.106157	0	.134615	0	.070381	.247336	.089504	0	0
u_{141}	0	0	.148138	0	0	0	.099742	.450163	0	0
u_{211}	0.345615	0.051558	0	0	0	0.102630	0	0	0.432153	0.050543
u_{221}	.062214	.270760	.073669	0	.124959	0	.148722	0	.062553	.332994
u_{231}	0	.052815	.183143	.067113	0	.105208	0	.133991	0	.051885
u_{241}	0	0	.073856	.337547	0	0	.149318	0	0	0
u_{311}	0	0	0	0	0.285369	0.033925	0	0	0	0.100373
u_{321}	0	0	0	0	.041306	.223512	.049231	0	.124224	0
u_{331}	0	0	0	0	0	.156584	.044487	0	0	.103161
u_{341}	0	0	0	0	0	.034826	.049576	.281854	0	0
u_{411}	0	0	0	0	0	0	0	0	0.319400	0.049830
u_{421}	0	0	0	0	0	0	0	0	.061671	.259938
u_{431}	0	0	0	0	0	0	0	0	0	.051276
u_{441}	0	0	0	0	0	0	0	0	0	0
$\Delta p_1'$	0.063016	0.001119	-0.000120	-0.067502	0	0	0	0	0	0
$\Delta p_2'$	.062214	.001235	.000156	-.067113	.083653	.001676	.000252	-.089504	0	0
$\Delta p_3'$	0	0	0	0	.041306	.000898	.000345	-.044487	.062553	.001338
$\Delta p_4'$	0	0	0	0	0	0	0	0	.061671	.001446
c	0.001740	0.000104	0.000818	0.001824	0.011800	0.008304	0.007563	0.008892	0.005595	0.002161

	u_{331}	u_{341}	u_{411}	u_{421}	u_{431}	u_{441}	$\Delta p_1'$	$\Delta p_2'$	$\Delta p_3'$	$\Delta p_4'$
u_{111}	0	0	0	0	0	0	0.328816	0.327720	0	0
u_{121}	0	0	0	0	0	0	.007047	.007995	0	0
u_{131}	0	0	0	0	0	0	-.000542	.000849	0	0
u_{141}	0	0	0	0	0	0	-.335321	-.336564	0	0
u_{211}	0	0	0	0	0	0	0	0.327720	0.326763	0
u_{221}	.075475	0	0	0	0	0	0	.007995	.008681	0
u_{231}	.240057	.066602	0	0	0	0	0	.000849	.002350	0
u_{241}	.076004	.421966	0	0	0	0	0	-.336564	-.337794	0
u_{311}	0	0	0.721414	0.107822	0	0	0	0	0.326763	0.325706
u_{321}	.150062	0	.139293	.562455	.163217	0	0	0	.008681	.009449
u_{331}	0	.132740	0	.110950	.343252	.148682	0	0	.002350	.003986
u_{341}	.151478	0	0	0	.165158	.702635	0	0	-.337794	-.339140
u_{411}	0	0	0	0.107822	0	0	0	0	0	0.325706
u_{421}	.074588	0	.139293	.562455	.163216	0	0	0	0	.009449
u_{431}	.156862	.066138	0	.110950	0	.148682	0	0	0	.003986
u_{441}	.075475	.312553	0	0	.165157	0	0	0	0	-.339140
$\Delta p_1'$	0	0	0	0	0	0	0	0	0	0
$\Delta p_2'$	0	0	0	0	0	0	0	0	0	0
$\Delta p_3'$	.000529	-.066602	0	0	0	0	0	0	0	0
$\Delta p_4'$	.000887	-.066138	.139293	.003128	.001941	-.148682	0	0	0	0
c	0.003045	0.004440	0.056192	0.041760	0.032731	0.039610	0	0	0	0

TABLE VII.—CORRECTION CYCLE

(a) Load problem

	$\beta_{y,1''}$	$\beta_{y,2''}$	$\beta_{y,3''}$	$\beta_{y,4''}$
u_{111}	-0.054016	0.053248	-----	-----
u_{121}	-.057921	.059103	-----	-----
u_{131}	-.037341	.037350	-----	-----
u_{141}	-.050740	.050299	-----	-----
u_{211}	-----	-0.053248	0.053741	-----
u_{221}	-----	-.059103	.055541	-----
u_{231}	-----	-.037350	.036188	-----
u_{241}	-----	-.050299	.051505	-----
u_{311}	-----	-----	-0.053741	0.052320
u_{321}	-----	-----	-.055541	.061348
u_{331}	-----	-----	-.036188	.035852
u_{341}	-----	-----	-.051505	.050482
u_{411}	-----	-----	-----	-0.052320
u_{421}	-----	-----	-----	-.061348
u_{431}	-----	-----	-----	-.035852
u_{441}	-----	-----	-----	-.050480
$z_{1,1''}$	5	5	5	5
5th cycle	-35.25	-62.43	-85.11	-97.77
10th cycle	-.7006	-1.198	1.799	-2.156
14th cycle	.09770	.1949	.1669	.2673

(b) Temperature problem

	$\delta u_1''$	$\beta_{x,1''}$	$\delta u_2''$	$\beta_{x,2''}$	$\delta u_3''$	$\beta_{x,3''}$	$\delta u_4''$	$\beta_{x,4''}$
u_{111}	-0.270057	0.029722	0.266240	-0.029618	-----	-----	-----	-----
u_{121}	-.289578	.009505	.295514	-.009780	-----	-----	-----	-----
u_{131}	-.186687	-.008325	.186749	.008413	-----	-----	-----	-----
u_{141}	-.253677	-.030952	.251497	.030984	-----	-----	-----	-----
u_{211}	-----	-----	-0.266240	0.029618	0.268705	-0.029564	-----	-----
u_{221}	-----	-----	-.295514	.009780	.292703	-.009645	-----	-----
u_{231}	-----	-----	-.186749	-.008413	.180939	.007983	-----	-----
u_{241}	-----	-----	-.251497	-.030984	.257653	.031226	-----	-----
u_{311}	-----	-----	-----	-----	-0.268705	0.029564	0.261595	-0.029254
u_{321}	-----	-----	-----	-----	-.292703	.009645	.306742	-.010188
u_{331}	-----	-----	-----	-----	-.180939	-.007983	.179262	.008139
u_{341}	-----	-----	-----	-----	-.257653	-.031226	.252400	.031303
u_{411}	-----	-----	-----	-----	-----	-----	-0.261595	0.029254
u_{421}	-----	-----	-----	-----	-----	-----	-.306742	.010188
u_{431}	-----	-----	-----	-----	-----	-----	-.179262	-.008139
u_{441}	-----	-----	-----	-----	-----	-----	-.252400	-.031303
$v_{1,1''}$	-----	-14.239858	-----	-14.235020	-----	-14.275390	-----	-14.224672
$v_{1,2''}$	-----	-4.239858	-----	-4.235020	-----	-4.275390	-----	-4.224672
$v_{1,3''}$	-----	5.760142	-----	5.764980	-----	5.724610	-----	5.775328
$v_{1,4''}$	-----	15.760142	-----	15.764980	-----	15.724610	-----	15.775328
5th cycle	-33	1.951	-64	3.713	-94	5.286	-98	5.604
9th cycle	0	-.0199	0	.05887	-2	.07768	-2	.06513

TABLE VIII.—SUCCESSIVE VALUES OF DISPLACEMENT

(a) Load problem

	Initial values	1st cycle	Difference	2d cycle	3d cycle	4th cycle	5th cycle (correction)	6th cycle	7th cycle	8th cycle	9th cycle	10th cycle (correction)	11th cycle	12th cycle	13th cycle	14th cycle (correction)	Total value	Check cycle
u_{11}	-0.016295	-0.017584	-1289 $\times 10^{-6}$	-1471	-1845	-2034	-2210	-2283	-2318	-2334	-2341	-2345	-2345	-2350	-2351	-2351	-0.013046	-0.018645
u_{12}	-0.016295	-0.016616	-173	-183	-309	-375	-531	-570	-571	-583	-585	-589	-590	-590	-590	-590	-0.016885	-0.016885
u_{13}	-0.016295	-0.016336	2659	3092	3407	3510	3534	3588	3623	3641	3647	3643	3647	3649	3649	3649	-0.012845	-0.012845
u_{14}	-0.016295	-0.016304	-9	419	621	705	739	788	818	831	837	833	837	839	840	840	-0.015654	-0.015654
u_{21}	-0.026640	-0.027979	-1339	-1719	-2280	-2543	-2855	-2960	-3012	-3036	-3048	-3054	-3040	-3092	-3064	-3063	-0.029703	-0.029703
u_{22}	-0.026640	-0.028113	-173	-19	-183	-237	-369	-490	-612	-618	-621	-627	-629	-629	-630	-630	-0.027269	-0.027269
u_{23}	-0.026640	-0.024009	2631	3750	4132	4312	4000	4094	4135	4154	4163	4157	4161	4163	4165	4165	-0.022475	-0.022475
u_{24}	-0.026640	-0.026748	-108	620	907	1051	739	831	875	895	904	898	904	905	907	908	-0.025732	-0.025732
u_{31}	-0.034430	-0.035007	-1177	-1574	-2101	-2345	-2771	-2890	-2945	-2971	-2983	-2992	-2999	-3002	-3003	-3002	-0.037432	-0.037432
u_{32}	-0.034430	-0.034637	-36	-36	-25	-197	-323	-460	-654	-670	-674	-683	-685	-685	-685	-685	-0.035115	-0.035115
u_{33}	-0.034430	-0.032134	2296	3459	3995	4160	3734	3837	3884	3905	3914	3905	3910	3912	3913	3914	-0.030516	-0.030516
u_{34}	-0.034430	-0.034428	2	733	1144	1324	898	990	1051	1074	1083	1074	1080	1082	1084	1085	-0.033344	-0.033344
u_{41}	-0.038801	-0.037801	-910	-1232	-1718	-1940	-2429	-2508	-2560	-2584	-2594	-2605	-2611	-2613	-2615	-2614	-0.039505	-0.039505
u_{42}	-0.038801	-0.036944	-53	149	67	39	145	508	519	525	529	540	541	541	542	541	-0.037432	-0.037432
u_{43}	-0.038801	-0.035236	1655	2955	3302	3475	2886	3094	3133	3149	3156	3145	3150	3151	3152	3153	-0.033738	-0.033738
u_{44}	-0.038801	-0.036036	-45	832	1194	1353	864	977	1021	1041	1049	1038	1042	1044	1046	1047	-0.035844	-0.035844
$\Delta u_1'$	0.059438	0.059438	0	950	1193	1382	1676	1705	1719	1726	1737	1736	1737	1737	1737	1737	0.061174	0.061174
$\Delta u_2'$	0.059438	0.059438	0	1866	2441	2960	3700	3769	3802	3817	3845	3845	3847	3849	3849	3849	0.110053	0.110053
$\Delta u_3'$	0.059438	0.059438	0	1079	1460	1821	3069	4053	4091	4109	4155	4150	4155	4157	4157	4157	0.154025	0.154025
$\Delta u_4'$	0.059438	0.059438	0	1061	1382	1676	3473	3540	3576	3584	3650	3645	3650	3651	3651	3651	0.160310	0.160310
$\Delta u_1''$	-0.000345	-0.000345	0	-73	-96	-107	-107	-113	-116	-117	-117	-117	-118	-118	-118	-118	-0.000463	-0.000463
$\Delta u_2''$	-0.000345	-0.000345	0	-163	-221	-248	-248	-262	-268	-271	-271	-273	-273	-274	-274	-274	-0.000540	-0.000540
$\Delta u_3''$	-0.000345	-0.000345	0	-179	-250	-285	-285	-302	-310	-314	-314	-316	-317	-317	-317	-317	-0.000671	-0.000671
$\Delta u_4''$	-0.000345	-0.000345	0	-170	-241	-275	-275	-292	-300	-303	-303	-305	-306	-307	-307	-307	-0.000490	-0.000490

(b) Temperature problem

	Initial values	1st cycle	Difference	2d cycle	3d cycle	4th cycle	5th cycle (correction)	6th cycle	7th cycle	8th cycle	9th cycle (correction)	Total value	Check cycle
u_{11}	0.050256	0.050710	454 $\times 10^{-6}$	680	734	803	742	756	757	756	756	0.051011	0.051011
u_{12}	0.046582	0.046079	-503	-723	-810	-856	-897	-917	-917	-917	-917	0.045665	0.045665
u_{13}	0.042907	0.042735	-172	-345	-384	-412	-434	-435	-435	-435	-435	0.042472	0.042472
u_{14}	0.039232	0.039548	317	492	522	556	554	562	562	562	562	0.039794	0.039794
u_{21}	0.104643	0.105679	1036	1286	1544	1594	1477	1487	1487	1487	1486	0.106129	0.106129
u_{22}	0.096480	0.095376	-1110	-1386	-1594	-1664	-1744	-1757	-1757	-1759	-1759	0.094727	0.094727
u_{23}	0.088330	0.087856	-474	-723	-840	-852	-895	-895	-895	-895	-895	0.087435	0.087435
u_{24}	0.080174	0.080925	751	943	1089	1120	1115	1118	1118	1118	1119	0.081293	0.081293
u_{31}	0.162990	0.163540	550	1654	1746	1751	1582	1589	1589	1589	1586	0.164577	0.164577
u_{32}	0.149633	0.149052	-581	-1441	-1682	-1751	-1808	-1872	-1872	-1872	-1874	0.147759	0.147759
u_{33}	0.136286	0.136048	-238	-831	-885	-869	-963	-962	-961	-961	-963	0.135323	0.135323
u_{34}	0.122934	0.123322	388	1008	1197	1204	1193	1195	1195	1196	1195	0.124120	0.124120
u_{41}	0.225490	0.227888	2389	2986	3054	3030	2852	2856	2857	2857	2854	0.228353	0.228353
u_{42}	0.20129	0.203755	2374	2916	3058	3131	3253	3252	3252	3252	3254	0.202875	0.202875
u_{43}	0.186768	0.185593	-1175	-1701	-1704	-1724	-1790	-1789	-1789	-1789	-1791	0.184977	0.184977
u_{44}	0.167402	0.169155	1753	2148	2160	2154	2144	2146	2147	2147	2146	0.169548	0.169548
$\Delta u_1'$	0.003675	0.003675	0	54	61	61	52	54	54	54	54	0.003725	0.003725
$\Delta u_2'$	0.011832	0.011832	0	144	184	184	143	147	147	147	147	0.011977	0.011977
$\Delta u_3'$	0.021508	0.021508	0	214	270	270	188	188	188	188	188	0.021695	0.021695
$\Delta u_4'$	0.032718	0.032718	0	330	370	370	253	255	255	255	255	0.032972	0.032972

TABLE IX.—STRESSES

Load problem*				Temperature problem**				Total			
(104)	(105)	(106)	(107)	(108)	(109)	(110)	(111)	(112)	(113)	(114)	(115)
i, j	$u_{i,j,1}$	$\Delta w_i'$	$\Delta w_i'$	$\bar{P}_{i,j,1}$	$\bar{\sigma}_{i,j,1}$	$\left(\frac{rL}{2}\right)_{i,j,1}$	$\left(\frac{rL}{2}\right)_{i,j,1}$	$\tau_{i,j,1}$	$\tau_{i,j,1}$	$\tau_{i,j,1}$	$\tau_{i,j,1}$
1	1 2 3 4	-0.018645 -0.018855 -0.028455 -0.015654	-0.061174 -0.000463	-8.989 -9.603 -4.710 -7.799	-8.909 -8.350 -6.355 -7.722	3.321 2.659 2.019	3.321 2.659 2.019	-1.888 2.470 -1.304	-1.701 2.148 -1.201	-1.16 -39 77	-1.16 -39 77
2	1 2 3 4	-0.029703 -0.027260 -0.022475 -0.025732	-0.110056 -0.000540	-5.122 -5.339 -3.129 -4.410	-5.227 -5.085 -4.741 -4.955	2.224 2.126 1.650	2.224 2.126 1.650	-1.434 1.870 1.563 -999	-1.463 1.781 1.853 -1.122	-337 108 229	-337 108 229
3	1 2 3 4	-0.037433 -0.035115 -0.036515 -0.033344	-0.154025 -0.000671	-2.133 -2.359 -1.404 -2.014	-3.615 -3.805 -3.934 -3.730	1.535 1.377 1.088	1.535 1.377 1.088	-870 1.126 1.355 -612	-1.475 1.816 1.934 -1.133	-228 70 157	-228 70 157
4	1 2 3 4	-0.039506 -0.037432 -0.037338 -0.035843	-0.160310 -0.000490	-441 -578 -469 -513	-959 -1.112 -1.563 -1.221	676 723 601	676 723 601	-321 414 134 -227	-698 796 447 -540	-321 -93 227	-321 -93 227

$$* \left(\frac{rL}{2} \right)_{i,j,k} = \left(\frac{GL}{4b} \right)_{i,j,k} (u_{i,j,k} + u_{i-1,j,k} - u_{i,j-1,k} - u_{i-1,j-1,k}) - \left(\frac{GL}{2} \right)_{i,j,k} \Delta \theta_i'$$

$$\left(\frac{rL}{2} \right)_{i,j,k} = 2 \left(\frac{GL}{4b} \right)_{i,j,k} (u_{i,j,k} + u_{i-1,j,k}) + \left(\frac{GL}{2} \right)_{i,j,k} \Delta w_i' + \left(\frac{GL}{2} \right)_{i,j,k} \Delta \theta_i'$$

$$\bar{P}_{i,j,k} = \left(\frac{AE}{L} \right)_{i,j,k} (u_{i,j,k} - u_{i-1,j,k})$$

Checks:

$$(109)_i - (109)_{i+1,j} = (111)_i + 1 - (111)_{i+1,j} + (111)_{i,j} - (113)_{i+1,j} - (113)_i$$

$$10 \sum_{j=1}^4 (109)_i - (109)_{i+1,j} = 4 \sum_{j=1}^4 (113)_i - (113)_i$$

$$10 \sum_{j=1}^4 (111)_i - \sum_{j=1}^4 (113)_i = -40$$

Checks:

$$(117)_i - (117)_{i+1,j} = (119)_i + 1 - (119)_{i+1,j} + (119)_{i,j} - (119)_{i+1,j}$$

$$\sum_{j=1}^4 (119)_i - \sum_{j=1}^4 (119)_{i+1,j} = 0$$

$$\sum_{j=1}^4 (119)_i - \sum_{j=1}^4 (119)_{i+1,j} = 0$$

$$\sum_{j=1}^4 (119)_i - \sum_{j=1}^4 (119)_{i+1,j} = 0$$

REPORT 1044

THE METHOD OF CHARACTERISTICS FOR THE DETERMINATION OF SUPERSONIC FLOW OVER BODIES OF REVOLUTION AT SMALL ANGLES OF ATTACK ¹

By ANTONIO FERRI

SUMMARY

The method of characteristics has been applied for the determination of the supersonic-flow properties around bodies of revolution at a small angle of attack. The system developed considers the effect of the variation of entropy due to the curved shock and determines a flow that exactly satisfies the boundary conditions in the limits of the simplifications assumed. Two practical methods for numerical calculations are given.

INTRODUCTION

For the determination of aerodynamic properties of bodies of revolution at supersonic speeds, two methods have been used: a method that uses the small-disturbances theory and a method that uses the characteristics theory. Both methods are successful in the determination of the flow properties for bodies at zero angle of attack, but the precision of the small-disturbances theory decreases when a body of revolution at an angle of attack is considered.

For bodies of revolution having supersonic flow everywhere, the theory of characteristics can also be used at an angle of attack.

The method of characteristics for the determination of the flow field around bodies of revolution at an angle of attack was first used by Ferrari (reference 1) in 1936. Ferrari considers the flow as potential flow and develops a method for the analysis of the flow field around a body that in the approximation of potential flow appears to be general and can be applied to bodies of any shape and with any angle of attack. In the determination of the flow properties along the first characteristic surface from which the analysis starts, however, Ferrari analyzes the flow around a cone of revolution, and in this part of the analysis only small values of angle of attack are considered.

Sauer in 1942 (reference 2) considers the same problems and shows that, for small values of angle of attack, the analysis of the flow field around a body of revolution can be made by applying the characteristics method only in one meridian plane; and, therefore, Sauer uses characteristic lines in place of the characteristic surfaces considered by Ferrari. Sauer, in the development of his system, is interested essentially in the analysis of the flow around circular cones; and when the method is applied to bodies of revolution of shapes different from cones, the boundary conditions are

no longer satisfied. The flow obtained from the solution used, also at small angles of attack, wets a body that is not a body of revolution. The body can be obtained from the body of revolution considered initially by curving its axis of symmetry. Sauer also assumes that the flow is potential flow. With this assumption, the flow must be considered as potential flow for the case of the body at zero angle of attack also; therefore, all the effects of entropy gradients are neglected.

The flow field around circular cones at small angles of attack has been analyzed in a more exact form by Stone and Ferri. (See references 3 to 5.) In his analysis, Stone considers the flow as rotational flow and, therefore, takes into account the effect of entropy gradients on the velocity distribution. This effect exists only when the cone has an angle of attack and, at low Mach numbers, is small but of the same order as the effect of other parameters that are considered in the analysis. Ferri considers correctly the entropy distribution at the surface of the cone and found the existence of a vortical layer at the surface of the cone across which the pressure does not change although density and velocity change.

Here, the method of characteristics is extended to the analysis of the flow field around a body of revolution at small angles of attack for the case of rotational flow. The effect of entropy gradients about bodies of revolution even at small angles of attack can be important because the entropy gradients that exist in the stream for small angles of attack are due to the variation of curvature of the shock existing at zero angle of attack also, together with the fact that the shock surface does not have axial symmetry with respect to the direction of the undisturbed velocity.

The method presented permits the determination of a flow that in the assumption of small angles of attack exactly satisfies the boundary conditions and, therefore, wets the body of revolution considered. This method is given in a form that permits its application to practical problems and requires either numerical or numerical and graphical calculations of the same type as the calculations used for the analysis of the flow around bodies at zero angle of attack. The method can be applied to cases in which the entropy variations can be neglected or are zero. In these cases the terms that contain the entropy variations become zero.

¹ Supersedes NACA TN 1809, "The Method of Characteristics for the Determination of Supersonic Flow over Bodies of Revolution at Small Angles of Attack" by Antonio Ferri, 1949.

SYMBOLS

x, y, θ	cylindrical coordinates (fig. 1)
r, ψ, θ	polar coordinates (fig. 8)
V	local velocity (function of x, y, θ)
u, v, w	velocity components in cylindrical coordinates (u along x -axis, v along y -axis, and w normal to meridian plane)
v_r, v_n, w	velocity components in polar coordinates (v_r along r , v_n normal to r in meridian plane, and w normal to meridian plane)
V_i	limiting velocity corresponding to adiabatic expansion to zero pressure
p	pressure
ρ	density
γ	ratio of specific heats
a	speed of sound ($a^2 = \gamma \frac{p}{\rho}$)
α	angle of attack of body
β	Mach angle ($\sin \beta = \frac{a}{V}$)
ϕ	angle between velocity V and x -axis
η	angle between the axis of the cone tangent to the shock and the axis of the body
σ	angle at the apex of the cone tangent to the shock
λ_a, λ_b	tangents to the characteristic surfaces in the meridian plane $\theta = \text{Constant}$
v_N	velocity component normal to the shock surface
v_T	velocity component along the generatrix of the cone tangent to the shock
w	velocity component tangent to the cross section of the cone tangent to the shock
Ω	angle between the tangent to the shock and the axis of the body
ΔS	entropy variation for unit mass
n	normal to the streamline in the plane $\theta = \text{Constant}$
N	normal to the surface of the shock
H, L, K, Z	coefficients defined by equations (24)
A	coefficient defined by equation (41)
Q	coefficient defined by equation (45)
D_1, D_2	coefficients defined by equations (55)
A_1, A_2, P_1, P_2, T	coefficients defined by equations (60)
R	radius of the hodograph diagram
Subscripts:	
0	free-stream flow quantities
1	flow quantities for the condition of zero angle of attack
2	flow quantities related to the effect of angle of attack as defined in equations (5) and (11)

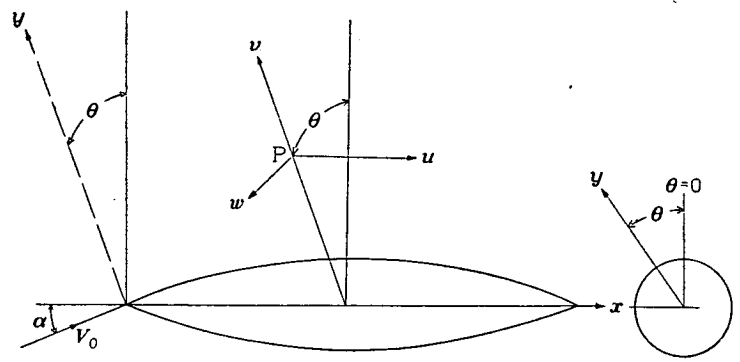


FIGURE 1.—Cylindrical coordinates and corresponding velocity components.

The prime (') represents quantities in front of the shock and the double prime (') represents quantities behind the shock.

EQUATION OF MOTION FOR FLOW AROUND A BODY OF REVOLUTION AT A SMALL ANGLE OF ATTACK

Consider a cylindrical coordinate system in which the x -axis is coincident with the axis of the body of revolution, the y -axis is normal to the x -axis in any meridian plane, and the position of every meridian plane is defined by the angle θ measured with respect to the meridian plane that contains the direction of the undisturbed velocity (fig. 1).

Euler's equations of motion for steady flow in cylindrical coordinates are

$$-\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v + \frac{\partial u}{\partial \theta} w \quad (1a)$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial y} = \frac{\partial v}{\partial x} u + \frac{\partial v}{\partial y} v + \frac{\partial v}{\partial \theta} w - \frac{w^2}{y} \quad (1b)$$

$$-\frac{1}{\rho} \frac{\partial p}{\partial \theta} = \frac{\partial w}{\partial x} u + \frac{\partial w}{\partial y} v + \frac{\partial w}{\partial \theta} w + \frac{vw}{y} \quad (1c)$$

The continuity equation in cylindrical coordinates can be expressed in the form

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v y)}{y \partial y} + \frac{\partial(w \rho)}{y \partial \theta} = 0 \quad (2)$$

whereas the law of conservation of energy can be written in the form

$$\frac{\gamma}{\gamma-1} \left(\frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial x} \right) = - \left(u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial x} + w \frac{\partial w}{\partial x} \right) \quad (3a)$$

$$\frac{\gamma}{\gamma-1} \left(\frac{1}{\rho} \frac{\partial p}{\partial y} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial y} \right) = - \left(u \frac{\partial u}{\partial y} + v \frac{\partial v}{\partial y} + w \frac{\partial w}{\partial y} \right) \quad (3b)$$

$$\frac{\gamma}{\gamma-1} \left(\frac{1}{\rho} \frac{\partial p}{\partial \theta} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial \theta} \right) = - \left(u \frac{\partial u}{\partial \theta} + v \frac{\partial v}{\partial \theta} + w \frac{\partial w}{\partial \theta} \right) \quad (3c)$$

If the density is eliminated from equation (2) by means of equations (1) and (3) and the quantity a is introduced as defined by

$$a^2 = \gamma \frac{p}{\rho}$$

the following equation can be obtained:

$$\begin{aligned} & \frac{\partial u}{\partial x} \left(1 - \frac{u^2}{a^2}\right) + \frac{\partial v}{\partial y} \left(1 - \frac{v^2}{a^2}\right) + \frac{\partial w}{\partial \theta} \left(1 - \frac{w^2}{a^2}\right) + \frac{v}{y} - \\ & \frac{vu}{a^2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) - \frac{uw}{a^2} \left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial \theta}\right) - \frac{vw}{a^2} \left(\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial y}\right) = 0 \end{aligned} \quad (4)$$

In this analysis only small angles of attack will be considered, and, therefore, only the first-order effect of the angle of attack will be determined; whereas, the quantities of the same or higher order than the square of the angle of attack will be neglected. In this approximation the velocity components of the flow around the body can be expressed in the following form (references 1 to 4):

$$u = u_1 + \alpha u_2 \cos \theta \quad (5a)$$

$$v = v_1 + \alpha v_2 \cos \theta \quad (5b)$$

$$w = \alpha w_2 \sin \theta \quad (5c)$$

where u , v , and w are functions of the three coordinates x , y , and θ ; whereas u_1 , v_1 , u_2 , v_2 , and w_2 are functions only of the coordinates x and y of any meridian plane. The quantity α is the angle of attack of the body, the quantities with subscript 1 are the quantities existing at the position (x, y) for the body considered at zero angle of attack at the same Mach number, and the quantities with subscript 2 are functions that take into account the effect of angle of attack.

It will be shown in the following considerations that the form assumed in equations (5) for the velocity components permits the boundary conditions to be satisfied in the simplifications assumed. For small angles of attack equation (4) becomes

$$\frac{\partial u}{\partial x} \left(1 - \frac{u^2}{a^2}\right) + \frac{\partial v}{\partial y} \left(1 - \frac{v^2}{a^2}\right) + \frac{v}{y} - \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \frac{uv}{a^2} + \frac{\partial w}{\partial \theta} = 0 \quad (6)$$

Equation (6) is similar in form to the corresponding equation for the case of the body at zero angle of attack and differs only in the term $\frac{\partial w}{y \partial \theta}$. In order to analyze the differences between this expression and the expression for the axial symmetrical case and in order to obtain another relation that defines the quantity w , the relation between rotation of the flow and entropy gradient will now be introduced.

Between rotation of flow and entropy the following relation exists:

$$\text{curl } \mathbf{V} \times \mathbf{V} = \frac{a^2}{\gamma R} \text{grad } S \quad (7)$$

or for small angles of attack

$$\frac{\partial S}{\partial x} \frac{a^2}{\gamma R} = -v \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (8a)$$

$$\frac{\partial S}{\partial y} \frac{a^2}{\gamma R} = u \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \quad (8b)$$

$$\frac{\partial S}{y \partial \theta} \frac{a^2}{\gamma R} = v \left(\frac{\partial w}{\partial y} - \frac{\partial v}{y \partial \theta}\right) + \frac{wv}{y} - u \left(\frac{\partial u}{y \partial \theta} - \frac{\partial w}{\partial x}\right) \quad (8c)$$

If n is the normal, in the meridian plane $\theta = \text{Constant}$, to the local tangent to the streamline, then

$$\begin{aligned} \frac{\partial S}{\partial n} &= -\frac{\partial S}{\partial x} \frac{v}{V} + \frac{\partial S}{\partial y} \frac{u}{V} \\ &= \frac{\gamma R}{a^2} V \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \end{aligned} \quad (9)$$

whereas from equation (8c), when equations (5) are used, it follows that

$$\frac{1}{y} \frac{a^2}{\gamma R} \frac{\partial S}{\partial \theta} = \left[v \left(\frac{\partial w_2}{\partial y} + \frac{v_2}{y}\right) + \frac{vw_2}{y} + u \left(\frac{u_2}{y} + \frac{\partial w_2}{\partial x}\right) \right] \alpha \sin \theta \quad (10)$$

Therefore, the entropy can be expressed in the form

$$\Delta S = \Delta S_1 + \alpha \Delta S_2 \cos \theta \quad (11)$$

where ΔS_1 and ΔS_2 are functions only of x and y . Equation (11) is valid outside of the vortical layer existing at the surface of the body (see reference 5); whereas the entropy at the surface of the body is constant and equal to the value existing at the conical tip. Because no pressure gradient exists across the vortical layer, the pressure will be determined outside of the layer and equation (11) will be used in all the flow field. The presence of the layer must be considered for the velocity and density distribution at the surface of the body. At the surface of the body the entropy is constant and equal to $\Delta S_1 - \alpha \Delta S_2$. (The meridian plane $\theta = 0$ is defined as in figure 1.) From equation (11) it results

$$v \frac{\partial w_2}{\partial y} + u \frac{\partial w_2}{\partial x} = -\frac{vw_2 + vv_2 + uu_2}{y} - \frac{a^2}{y\gamma R} \Delta S_2 \quad (12)$$

Equation (6) can be written in the following form:

$$\frac{\partial u}{\partial x} \left(1 - \frac{u^2}{a^2}\right) + \frac{\partial v}{\partial y} \left(1 - \frac{v^2}{a^2}\right) + \frac{v}{y} - \frac{2uv}{a^2} \frac{\partial v}{\partial x} + \frac{uv}{V\gamma R} \frac{\partial S}{\partial n} + \frac{\partial w}{y \partial \theta} = 0 \quad (13)$$

Equation (13) together with equation (12) defines the law of motion of the flow around the body at small angles of attack. These equations will be used as a basis for the calculation of the flow field by the method of characteristics to be treated in a subsequent section.

CONDITIONS AT THE SHOCK FRONT

Equations (5) and (11) represent a stream that wets a body of revolution at a small angle of attack. In order to satisfy the boundary conditions at the surface of the body, the functions u_2 , v_2 , and w_2 must be properly selected. Equations (5) and (11) must, however, satisfy the boundary conditions at the shock surface also in order to be a solution of the problem. It is necessary, therefore, to show that a shock surface can exist across which the undisturbed stream inclined at α with respect to the axis of the body is transformed into a flow represented by equations (5) and (11).

In order to show that the shock boundary conditions can be satisfied, the following procedure will be employed. A shock surface distorted in a manner to be described is assumed. Then, the free-stream velocity ahead of the shock will be resolved into three components: v_N normal to the shock in the plane $\theta = \text{Constant}$, v_T tangent to the shock in the plane $\theta = \text{Constant}$, and w perpendicular to the plane $\theta = \text{Constant}$. Similarly, the flow behind the shock will be resolved into three components. In addition, each component of the flow behind the shock will be divided into two terms: one term for zero angle of attack and one term for the difference due to the angle of attack (for example, $u = u_1 + \alpha u_2 \cos \theta$). Then, the conditions of equilibrium at the shock will be imposed, and it will be shown that the terms u_2 , v_2 , and w_2 at the shock are independent of θ when the angle of attack is small as initially assumed; hence, the distorted shock is consistent with the field of flow behind it. Such a shock surface can be obtained by deforming the shock surface produced by the body when the angle of attack α is zero in the following way (fig. 2):

When $\alpha = 0$, the shock surface is a surface of revolution in axis with the body; therefore, if OP, OP' is the curve intersection of the shock with the meridian plane $\theta = 0$, then for $\alpha = 0$ the tangent AQ at any point Q of the curve OP is the generatrix of a circular cone having the vertex at a point A of the axis and tangent along the circle QQ' to the shock surface. The shock surface, therefore, can be considered as a surface envelope of circular cones having the axis coincident with the axis of the body but having variable

cone angle and variable position of the apex A along the axis AB of the body. For the case of $\alpha \neq 0$ the shock surface is not a surface of revolution but can still be considered, for small angles of attack, as the envelope of the same circular cones considered for the case $\alpha = 0$. These cones have the same apexes and the same cone angles as the cases for $\alpha = 0$ but do not have the axis of symmetry AB coincident with the axis of the body AB although they are rotated in the plane $\theta = 0$ with respect to the body axis. The angle η , through which each axis of the cones must rotate in the plane $\theta = 0$ with respect to the axis of the body, is not constant but varies for each cone considered. For example, the cone AQQ' tangent to the shock surface for $\alpha = 0$, when $\alpha \neq 0$, must be rotated by an angle η to the position AQ_1Q_1' ; the axis AB_1 remains in the plane $\theta = 0$.

The shock surface so generated is consistent with the flow represented by equations (5) and (11), and this can be shown in the following way:

Consider a point P of the shock produced by the body at an angle of attack, and consider the cone tangent to the shock at the point P (fig. 3). Call σ the semiangle of the cone with respect to its axis of symmetry. The axis of this cone is inclined at an angle η with respect to the axis of the body and lies in the plane $\theta = 0$.

The uniform velocity V_0 ahead of the shock is decomposed in the three components: v_N' in the direction PB normal to the shock, v_T' in the direction AP along the generatrix of the cone, and w' in the direction normal to the plane APB . These components are, at small angles of attack,

$$v_N' = V_0 \sin \sigma - V_0(\alpha - \eta) \cos \sigma \cos \theta \quad (14a)$$

$$v_T' = V_0 \cos \sigma + V_0(\alpha - \eta) \sin \sigma \cos \theta \quad (14b)$$

$$w' = -V_0(\alpha - \eta) \sin \theta \quad (14c)$$

Strictly, in equations (14) ψ must be written in place of θ ; but, for small angles of attack in equations (14), the difference between ψ and θ can be neglected. Indeed,

$$\psi = \delta + \theta'$$

where δ is of the order of α , and θ' differs from θ by a quantity of the order of α .

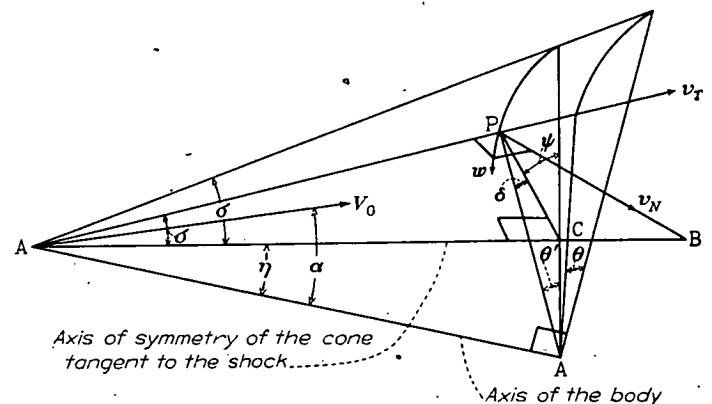


FIGURE 3.—The velocity components in front of and behind the shock.

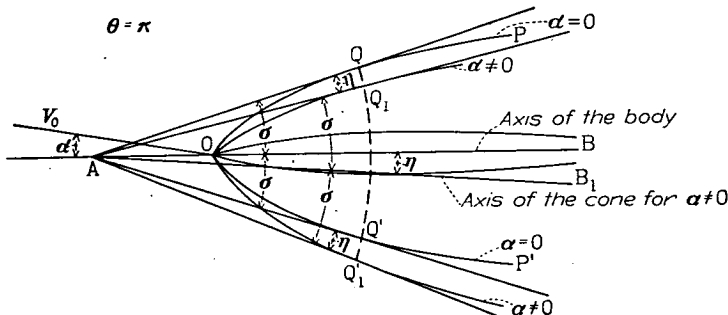


FIGURE 2.—The shock surface at $\alpha = 0$ and $\alpha \neq 0$.

The velocity components behind the shock are (fig. 3)

$$v_N'' = u \sin \sigma - v \cos \sigma + \eta \cos \theta (v \sin \sigma + u \cos \sigma) \quad (15a)$$

$$v_T'' = u \cos \sigma + v \sin \sigma + \eta \cos \theta (v \cos \sigma - u \sin \sigma) \quad (15b)$$

$$w'' = w + \eta (u - v \cot \sigma) \sin \theta \quad (15c)$$

where u , v , and w are the velocity components behind the shock in cylindrical coordinates in axis with the body at the point P considered.

The velocity components u , v , and w at the point P can be expressed in the form given by equations (5), in which the components u_1 and v_1 are the quantities obtained at the point P for the condition of $\alpha=0$ and are direct functions only of x and y . The point P, however, is a point of the shock, and its coordinates x and y change when the coordinate θ changes; therefore, the velocity components u_1 and v_1 at P also change with θ_1 . In order to separate the part of the components u , v , and w dependent on θ from the part independent of θ , the velocity components u_1 and v_1 at P will now be expressed as a function of the flow properties at a point P_1 near P, having a constant value of x and y for every value of θ .

Now, it has been assumed that the angle of the cone σ tangent to the shock at the point P is equal to the angle of the cone tangent to the shock for the condition of zero angle of attack at the point P_1 (fig. 4). The point P_1 is obtained on the shock by rotating the cone APQ tangent to the shock for the condition $\alpha=0$ through an angle η around the axis AN normal to the plane $\theta=0$ at the apex A of the cone. Because for the condition of zero angle of attack the velocity components u_1 and v_1 are independent of the coordinate θ , the velocity components u_1 and v_1 at P (x_P, y_P) (fig. 4) are equal to the velocity components at $P_2(x_P, y_P)$ in the plane AP_1C . Therefore, if ΔN is the distance P_2P_1 ,

$$u_{1P} = u_{1P_1} + \left(\frac{\partial u_1}{\partial N} \right)_{P_1} \Delta N \quad (16a)$$

$$v_{1P} = v_{1P_1} + \left(\frac{\partial v_1}{\partial N} \right)_{P_1} \Delta N \quad (16b)$$

where (fig. 4)

$$\Delta N = \frac{x_{P_1} \eta}{\cos \sigma} \cos \theta \quad (17)$$

Substituting equations (5), (16), and (17) in equations (15) results in

$$v_N'' = (u_1 \sin \sigma - v_1 \cos \sigma)_{P_1} + \alpha \cos \theta (u_2 \sin \sigma - v_2 \cos \sigma)_{P_1} + \eta \cos \theta (u_1 \cos \sigma + v_1 \sin \sigma)_{P_1} + \frac{x_{P_1} \eta}{\cos \sigma} \cos \theta \left(\frac{\partial u_1}{\partial N} \sin \sigma - \frac{\partial v_1}{\partial N} \cos \sigma \right)_{P_1} \quad (18a)$$

$$v_T'' = (u_1 \cos \sigma + v_1 \sin \sigma)_{P_1} + \alpha \cos \theta (u_2 \cos \sigma + v_2 \sin \sigma)_{P_1} - \eta \cos \theta (u_1 \sin \sigma - v_1 \cos \sigma)_{P_1} + \frac{x_{P_1} \eta}{\cos \sigma} \cos \theta \left(\frac{\partial u_1}{\partial N} \cos \sigma + \frac{\partial v_1}{\partial N} \sin \sigma \right)_{P_1} \quad (18b)$$

$$w'' = \alpha w_{2P_1} \sin \theta + \eta (u_1 - v_1 \cot \sigma)_{P_1} \sin \theta \quad (18c)$$

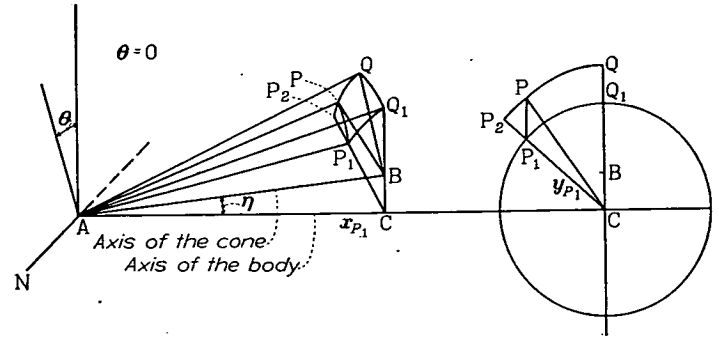


FIGURE 4.—The velocity components behind the shock for $\alpha \neq 0$ as a function of the velocity components behind the shock for $\alpha=0$.

For the condition of zero angle of attack at P_1

$$v_{N_1}'' = u_1 \sin \sigma - v_1 \cos \sigma \quad (19a)$$

$$v_{T_1}'' = u_1 \cos \sigma + v_1 \sin \sigma \quad (19b)$$

and for the condition of the equilibrium at the shock at zero angle of attack

$$V_0 \cos \sigma = v_{T_1}'' \quad (20a)$$

$$v_{N_1}'' V_0 \sin \sigma = \frac{\gamma-1}{\gamma+1} (V_0^2 - v_{T_1}''^2) \quad (20b)$$

At the point P for the case of a small angle of attack,

$$(v_N' v_N'')_P = \frac{\gamma-1}{\gamma+1} (V_0^2 - v_T''^2)_P \quad (21a)$$

$$v_{TP}'' = v_{TP}' \quad (21b)$$

$$w'' = w' \quad (21c)$$

If equations (19) are used, equations (18a) and (18b) can be written in the form

$$(v_N'')_P = (v_{N_1}'')_{P_1} + \alpha \cos \theta (v_{N_2}'')_{P_1} + \eta \cos \theta (v_{T_1}'')_{P_1} +$$

$$\frac{x_{P_1} \eta}{\cos \sigma} \cos \theta \left(\frac{\partial v_{N_1}''}{\partial N} \right)_{P_1}$$

$$(v_T'')_P = (v_{T_1}'')_{P_1} + \alpha \cos \theta (v_{T_2}'')_{P_1} - \eta \cos \theta (v_{N_1}'')_{P_1} +$$

$$\frac{x_{P_1} \eta}{\cos \sigma} \cos \theta \left(\frac{\partial v_{T_1}''}{\partial N} \right)_{P_1}$$

Therefore, from equations (14), (20), and (21),

$$-(\alpha - \eta) V_0 \cos \sigma (v_{N_1}'')_{P_1} +$$

$$V_0 \sin \sigma \left(\alpha v_{N_2}'' + \eta v_{T_1}'' + \frac{x_{P_1} \eta}{\sin \sigma} \frac{\partial v_{N_1}''}{\partial N} \right)_{P_1} = -\frac{2(\gamma-1)}{\gamma+1} (\alpha - \eta) V_0^2 \cos \sigma \sin \sigma \quad (22a)$$

$$(\alpha - \eta) V_0 \sin \sigma = (v_{T_2}'')_{P_1} \alpha - \eta (v_{N_1}'')_{P_1} +$$

$$\frac{x_{P_1} \eta}{\sin \sigma} \left(\frac{\partial v_{T_1}''}{\partial N} \right)_{P_1} \quad (22b)$$

or

$$\left(\frac{v_{T_2}''}{V_0}\right)_{P_1} = \left(1 - \frac{\eta}{\alpha}\right) \sin \sigma + \frac{\eta}{\alpha} \left(\frac{v_{N_1}''}{V_0}\right)_{P_1} - \frac{\eta}{\alpha} \frac{x_{P_1}}{\sin \sigma} \frac{1}{V_0} \left(\frac{\partial v_{T_1}''}{\partial N}\right)_{P_1} \quad (23a)$$

$$\left(\frac{v_{N_2}''}{V_0}\right)_{P_1} = \frac{1}{\tan \sigma} \left[-2 \left(\frac{\gamma-1}{\gamma+1}\right) \left(1 - \frac{\eta}{\alpha}\right) \sin \sigma + \left(1 - \frac{\eta}{\alpha}\right) \left(\frac{v_{N_1}''}{V_0}\right)_{P_1} - \frac{\eta}{\alpha} \sin \sigma - \frac{\eta}{\alpha} \frac{x_{P_1}}{\cos \sigma} \frac{1}{V_0} \left(\frac{\partial v_{N_1}''}{\partial N}\right)_{P_1} \right] \quad (23b)$$

$$\left(\frac{w_2}{V_0}\right)_{P_1} = -\left(1 - \frac{\eta}{\alpha}\right) + \left(\frac{-u_1 + v_1 \cot \sigma}{V_0}\right)_{P_1} \frac{\eta}{\alpha} \quad (23c)$$

In equations (23) the coordinate θ does not appear; therefore, for the shock considered the functions u_2 , v_2 , and w_2 are independent of θ , and equations (5) represent a flow condition in agreement with the conditions at the shock.

The ratio η/α which appears in equations (23) is independent of α ; therefore, for a given point P_1 , η/α remains constant in all the range of angle of attack in which the simplifications assumed are valid. (Indeed, u_2 , v_2 , and w_2 are also independent of the angle of attack (equations (5)). The values of u_2 , v_2 , w_2 , and η/α must therefore be determined only for one value of the angle of attack.

METHOD OF CHARACTERISTICS FOR FLOW AROUND A BODY OF REVOLUTION AT A SMALL ANGLE OF ATTACK

In this section the method of characteristics is applied to equation (13) to establish equations which will permit the flow field behind the shock to be calculated by a point-by-point process. If the flow is anywhere supersonic, equations (12) and (13) permit the determination of the flow around a body of revolution at a small angle of attack by using the method of characteristics. Equation (13) can be written in the following form:

$$H \frac{\partial u}{\partial x} + L \frac{\partial v}{\partial y} + 2K \frac{\partial w}{\partial x} + Z = 0$$

where

$$\left. \begin{aligned} H &= 1 - \frac{u^2}{a^2} \\ L &= 1 - \frac{v^2}{a^2} \\ K &= -\frac{uv}{a^2} \\ Z &= \frac{uv}{V} \frac{1}{\gamma R} \frac{\partial S}{\partial n} + \frac{\partial w}{y \partial \theta} + \frac{v}{y} \end{aligned} \right\} \quad (24)$$

If ϕ is the angle between the velocity V and the x -axis and β is the Mach angle,

$$\tan \phi = \frac{\sqrt{v^2 + w^2}}{u}$$

$$\sin \beta = \frac{a}{V}$$

or for small angles of attack

$$\tan \phi = \frac{v}{u}$$

and

$$\sin \beta = \frac{a}{\sqrt{v^2 + u^2}}$$

The tangent to the line intersection of a characteristic surface with the meridian plane $\theta = \text{Constant}$ is

$$\left. \begin{aligned} \lambda_a &= \tan(\phi + \beta) = \frac{K}{H} - \frac{1}{H} \sqrt{K^2 - HL} \\ \lambda_b &= \tan(\phi - \beta) = \frac{K}{H} + \frac{1}{H} \sqrt{K^2 - HL} \end{aligned} \right\} \quad (25)$$

where λ_a is the tangent to a line corresponding to the characteristic surface of the first family and λ_b is the tangent to a line corresponding to the characteristic surface of the second family. The terms λ_a and λ_b are solutions of the equations (reference 6)

$$H\lambda^2 - 2K\lambda + L = 0 \quad (26)$$

Because u , v , V , and a can be considered to be given by an equation of the type of equations (5), ϕ and β can also be written in the form

$$\phi = \phi_1 + \alpha \phi_2 \cos \theta$$

$$\beta = \beta_1 + \alpha \beta_2 \cos \theta$$

The characteristic surfaces are not, therefore, surfaces of revolution but can be obtained, as was true for the case of the shock, as an envelope of circular cones with their apexes at the axis of the body and their axis of symmetry in the plane $\theta = 0$ and inclined with the axis of the body.

The determination of the u and v components of the velocity in any point of the flow can be obtained from equation (13) by performing a transformation in order to obtain a law of variation along the characteristic lines (reference 6). Indeed, for every point of any meridian plane (for example, of the meridian plane $\theta = 0$, or $\theta = \pi$) two characteristic lines can be obtained as the intersection of two characteristic surfaces with the meridian plane. Along these lines the variation of the u and v velocity components is determined by the equations of characteristics that can be derived from equation (13). Assume that at two points P_1 and P_2 (fig. 5) of the meridian plane $\theta = \text{Constant}$ (for example, $\theta = 0$, or $\theta = \pi$) the velocity components are known. From equations (25) the tangents to the characteristic surfaces in this meridian plane can be drawn and the velocity components u and v at the point P_3 , intersection of the two tangents, can be obtained in the first approximation.

The equations of characteristics can be obtained by analyzing equation (13) along the characteristic lines given by equation (25) in the following way: If du and dv are the variations along the characteristic lines,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + (\lambda_a, \lambda_b) \frac{\partial u}{\partial y}$$

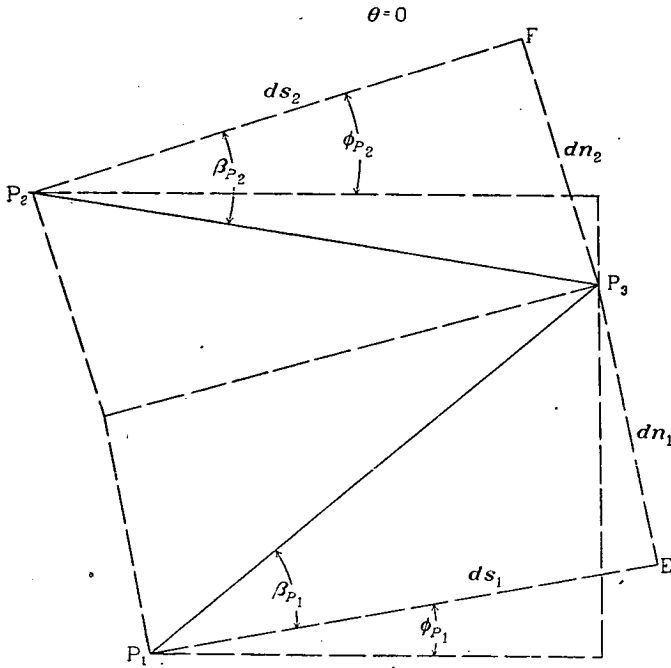


FIGURE 5.—The analysis of the flow with the characteristics system.

or (see equation (9))

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + (\lambda_a, \lambda_b) \frac{\partial v}{\partial x} - (\lambda_a, \lambda_b) \frac{a^2}{\gamma R V} \frac{\partial S}{\partial n}$$

$$\frac{dv}{dx} = \frac{\partial v}{\partial x} + (\lambda_a, \lambda_b) \frac{\partial v}{\partial y} \quad (27a)$$

then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \lambda \frac{dv}{dx} - \lambda^2 \frac{\partial v}{\partial y} - \lambda \frac{a^2}{\gamma R V} \frac{\partial S}{\partial n} \quad (27b)$$

If equations (27a) and (27b) are substituted in equation (13) and equation (26) is used, along the characteristic line of the first family defined by

$$\frac{dy}{dx} = \lambda_a = \tan(\beta + \phi) \quad (28a)$$

there results

$$\frac{du}{dx} + \lambda_b \frac{dv}{dx} - \frac{a^2}{\gamma R V} \frac{dS}{dn} \left(\frac{K}{H} - \lambda_a \right) + \left(\frac{v}{y} + \frac{\partial w}{y \partial \theta} \right) \frac{1}{H} = 0 \quad (28b)$$

and along the characteristic line of the second family defined by

$$\frac{dy}{dx} = \lambda_b = \tan(\phi - \beta) \quad (28c)$$

there results

$$\frac{du}{dx} + \lambda_a \frac{dv}{dx} - \frac{a^2}{\gamma R V} \frac{dS}{dn} \left(\frac{K}{H} - \lambda_b \right) + \left(\frac{v}{y} + \frac{\partial w}{y \partial \theta} \right) \frac{1}{H} = 0 \quad (28d)$$

Equations (28b) and (28d) contain the term $\frac{\partial w}{y \partial \theta}$, but at small angles of attack

$$\frac{\partial w}{y \partial \theta} = \frac{w_2 \alpha}{y} \cos \theta = \frac{w}{y} \cot \theta$$

and, therefore, $\frac{\partial w}{y \partial \theta}$ is known at the points P_1 and P_2 . The value of the entropy is also known at the points P_1 and P_2 and, therefore, the value of $\frac{dS}{dn}$ can be determined (reference 6)

$$\frac{dS}{dn} = \frac{\Delta S_{P_2} - \Delta S_{P_1}}{(x_{P_3} - x_{P_1}) \left[\frac{\sin \beta}{\cos(\beta + \phi)} \right]_{P_1} + (x_{P_3} - x_{P_2}) \left[\frac{\sin \beta}{\cos(\phi - \beta)} \right]_{P_2}} \quad (29)$$

From equations (28) and (29) the values of u and v can be determined in the first approximation for the point P_3 . In order to determine the value of w at P_3 , the following Procedure can be used:

If s is the projection of the streamline in the meridian plane considered (fig. 5) and P_3 is a point near P_1 and P_2 ,

$$\frac{dw_2}{ds} = \frac{\partial w_2}{\partial y} \sin \phi + \frac{\partial w_2}{\partial x} \cos \phi$$

$$= \frac{\partial w_2}{\partial y} \frac{v}{V} + \frac{\partial w_2}{\partial x} \frac{u}{V}$$

or from equation (12)

$$\frac{\partial w_2}{\partial s} = - \frac{v w_2 + v v_2 + u u_2}{y V} - \frac{\sin^2 \beta V \Delta S_2}{\gamma R y} \quad (30)$$

Now (fig. 5)

$$w_{2E} = w_{2P_1} + \left(\frac{dw_2}{ds} \right)_{P_1} (x_{P_3} - x_{P_1}) \left[\frac{\cos \beta}{\cos(\beta + \phi)} \right]_{P_1} \quad (31a)$$

$$w_{2F} = w_{2P_2} + \left(\frac{dw_2}{ds} \right)_{P_2} (x_{P_3} - x_{P_2}) \left[\frac{\cos \beta}{\cos(\phi - \beta)} \right]_{P_2} \quad (31b)$$

and

$$\left(\frac{dw_2}{dn} \right)_E = \frac{w_{2F} - w_{2E}}{(x_{P_3} - x_{P_1}) \left[\frac{\sin \beta}{\cos(\phi + \beta)} \right]_{P_1} + (x_{P_3} - x_{P_2}) \left[\frac{\sin \beta}{\cos(\phi - \beta)} \right]_{P_2}} \quad (32)$$

Therefore,

$$w_{2P_3} = w_{2E} + \left(\frac{\partial w_2}{\partial n} \right)_E (x_{P_3} - x_{P_1}) \left[\frac{\sin \beta}{\cos(\phi + \beta)} \right]_{P_1} \quad (33)$$

The values of u , v , w , ΔS , u_1 , v_1 , and ΔS_1 are known at the points P_1 and P_2 ; therefore, the values of u_2 , v_2 , w_2 , and ΔS_2 at the same points can be calculated from equations (5) and (11). (The values of u_1 , v_1 , and ΔS_1 at those points are known from the determination of the flow for $\alpha=0$.) Therefore, from equations (30) to (33) the value of w_2 at P_3 can be determined.

After the velocity components u , v , and w at P_3 have been determined in the first approximation, a second approximation can be determined by assuming the average values between the corresponding values at the points P_2 and P_3 or P_1 and P_3 for all the coefficients. After the velocity components at a point P_3 have been obtained, the velocity components at any other point having the same x and y coordinates as

P_3 but a different coordinate θ can be calculated from equations (5).

For practical calculations, equations (28) can be transformed in the following form:

$$\lambda_a = \tan(\beta + \phi) \quad (34a)$$

$$\frac{dV}{V} - d\phi \tan \beta - \left[\left(\frac{\sin \phi}{y} + \frac{\partial w}{yV \partial \theta} \right) \tan \beta - \frac{\sin^2 \beta}{\gamma R} \frac{dS}{dn} \right] \frac{\sin \beta}{\cos(\phi + \beta)} dx = 0 \quad (34b)$$

$$\lambda_b = \tan(\phi - \beta) \quad (34c)$$

$$\frac{dV}{V} + d\phi \tan \beta - \left[\left(\frac{\sin \phi}{y} + \frac{\partial w}{yV \partial \theta} \right) \tan \beta + \frac{\sin^2 \beta}{\gamma R} \frac{dS}{dn} \right] \frac{\sin \beta}{\cos(\phi - \beta)} dx = 0 \quad (34d)$$

$$\frac{1}{V} \frac{dw_2}{ds} = -\frac{1}{y} \frac{V_2}{V} - \frac{w_2}{Vy} \sin \phi - \frac{\sin^2 \beta}{\gamma R} \frac{\Delta S_2}{y} \quad (34e)$$

and

$$\phi = \phi_1 + \alpha \phi_2 \cos \theta \quad (35)$$

where

$$\phi_2 = \frac{v_2}{V_1} \cos \phi_1 - \frac{u_2}{V_1} \sin \phi_1 \quad (36)$$

and

$$V = V_1 + \alpha V_2 \cos \theta \quad (37)$$

where

$$V_2 = u_2 \cos \phi_1 + v_2 \sin \phi_1 \quad (38)$$

At the surface of the body outside of the vortical layer the calculations are similar to the case of zero angle of attack because the entropy at the surface of the body is known in every meridian plane and the value of θ is given. Equation (34e) gives the variation of w_2 along the body outside of the layer; therefore, the value of w_2 can be obtained directly from another point on the body in the same meridian plane.

At the surface of the shock the system of calculations is similar to the system for zero angle of attack. In figure 6 the point P_3 is at the intersection of the tangents to the first characteristic surface at P_1 and to the shock at P_2 in the meridian plane $\theta = \text{Constant}$. The equations of the shock and equation (34b) must be verified at P_3 , which is assumed as a point of the shock in the first approximation.

In the plane $\theta = 0$, w is zero and the values of V , ΔS , and ϕ behind the shock are functions only of the value of Ω ; and for any value of Ω , the values of V , ΔS , and ϕ can be obtained from the equations of the shock

$$\frac{\cos(\Omega - \alpha)}{\cos(\Omega - \phi)} = \frac{V}{V_0} = \frac{V/V_1}{V_0/V_1} \quad (39a)$$

$$\frac{1}{\tan(\phi - \alpha)} = \left[\frac{\gamma + 1}{2} \frac{M_0^2}{M_0^2 \sin^2(\Omega - \alpha) - 1} - 1 \right] \tan(\Omega - \alpha) \quad (39b)$$

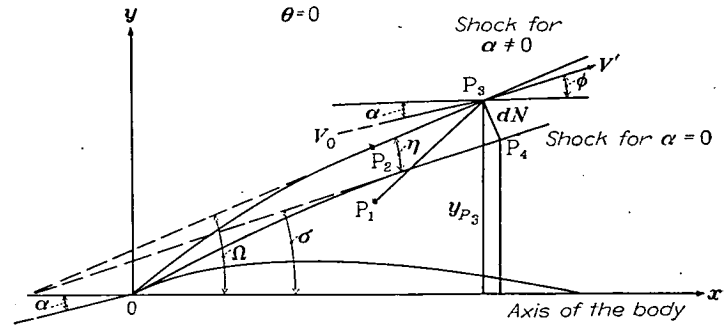


FIGURE 6.—Determination of a point of the shock.

$$\Delta S = \frac{R}{\gamma - 1} \log_e \gamma \left(\frac{2}{\gamma + 1} \right)^{\gamma + 1} \left[M_0^2 \sin^2(\Omega - \alpha) - \frac{\gamma - 1}{2\gamma} \right] \left[\frac{1}{M_0^2 \sin^2(\Omega - \alpha)} + \frac{\gamma - 1}{2} \right]^\gamma \quad (39c)$$

and

$$\frac{V_1^2}{V_0^2} = 1 + \frac{2}{\gamma - 1} \frac{1}{M_0^2} \quad (39d)$$

If the plane $\theta = \pi$ is considered, the sign of α in equations (39) must be reversed.

From equations (39) the values of V and ΔS can be determined as a function of ϕ ; then $\frac{dV}{d\phi}$ and $\frac{d\Delta S}{d\phi}$ as a function of ϕ can be evaluated. Now, if ϕ_{P_1} is the direction of the velocity at P_1 , the velocity at P_3 will have the direction

$$\phi_{P_3} = \phi_{P_1} + \Delta \phi$$

Therefore, the velocity at P_3 must correspond to a deviation across the shock of $\phi_{P_3} - \alpha$ and can be expressed as

$$V_{P_3} = V_{\phi_{P_1}} + \left(\frac{dV}{d\phi} \right)_{\phi_{P_1}} \Delta \phi$$

where $V_{\phi_{P_1}}$ is the velocity behind the shock corresponding to the direction ϕ_{P_1} . In a similar way,

$$\Delta S_{P_3} = \Delta S_{\phi_{P_1}} + \left(\frac{d\Delta S}{d\phi} \right)_{\phi_{P_1}} \Delta \phi$$

Therefore, equation (34b) at the point P_1 becomes

$$\begin{aligned} \frac{V_{\phi_{P_1}}}{V_{P_1}} - 1 + \frac{1}{V_{P_1}} \left(\frac{\partial V}{\partial \phi} \right)_{\phi_{P_1}} \Delta \phi - \tan \beta_{P_1} \Delta \phi - \\ \left[\left(\frac{\sin \phi}{y} + \frac{w_2 \alpha \cos \theta}{yV} \right) \frac{\sin \beta \tan \beta}{\cos(\beta + \phi)} \right]_{P_1} dx + \\ \left[\Delta S_{\phi_{P_1}} - \Delta S_{P_1} + \left(\frac{d\Delta S}{d\phi} \right)_{\phi_{P_1}} \Delta \phi \right] \frac{\sin^2 \beta_{P_1}}{\gamma R} = 0 \end{aligned} \quad (40)$$

In equation (40), $\Delta \phi$ is the only unknown and, therefore, can be determined. From the value of ϕ the value of Ω_{P_3}

and the value of V_{P_3} can be determined; a second approximation for the position of P_3 and its value of the velocity can be calculated if the corresponding average values between P_3 and P_1 are assumed for ϕ , β , and all the coefficients of equation (40):

The value of w at P can be obtained from equation (23c) in which η is given by figure 6 as

$$\eta = \frac{dN}{y_{P_3}} \sin \Omega$$

where $\eta = \Omega - \sigma$ for $\theta = 0$ and $\eta = \sigma - \Omega$ for $\theta = \pi$. The value of σ corresponding to the point P_4 on the shock for $\alpha = 0$ is given by the relation

$$\frac{y_{P_4}}{\sin \sigma} = \frac{y_{P_3}}{\sin \Omega}$$

and $y_{P_4} = f(\sigma_{P_4})$ is the curve that represents the shock for $\alpha = 0$

PRACTICAL APPLICATION OF THE CHARACTERISTIC SYSTEM

GRAPHICAL NUMERICAL METHOD

The analytical part of the characteristic system used for determining the flow field about a body of revolution at an angle of attack is similar to the system used for a body of revolution at zero angle of attack (reference 6), but the practical numerical application is slightly more involved. In equation (34e) the values of V_2 and ΔS_2 must be known in order to determine the value of w_2 and must be determined from equations (37), (36), and (11), where the values of V_1 and ΔS_1 are considered known in the entire flow field and given by the determination from the case of zero angle of attack. In the practical case, however, the values of V_1 , ϕ_1 , and ΔS_1 have been obtained with the characteristic system only in a finite number of points at the intersections of the characteristic net, and the characteristic net for the case of zero angle of attack is different from the net used for the case of a body with a small angle of attack. In order, therefore, to obtain the values of V_1 and ΔS_1 at the intersections of the characteristic lines for the case with a given angle of attack, a lengthy interpolation of the values V_1 and ΔS_1 would be necessary if the two characteristic nets for zero angle of attack and for a given angle of attack were constructed independently.

In order to reduce the numerical work to a minimum, the two following methods can be used, the first of which is practical when a graphical numerical calculation is performed, whereas the second can be more convenient when automatic computing machines are used.

In both cases the calculations start with the determination of the flow at an angle of attack around a cone when the body considered is a pointed-nose body of revolution or with the determination of the shock at the lip of the body if the body is an open-nose body of revolution. (The tangent to the

shock at the lip can be determined with the two-dimensional theory.) The flow around a cone at an angle of attack has been determined and tabulated in reference 3; whereas the flow for zero angle of attack has been tabulated in reference 7. A different method for determining the flow around a circular cone at an angle of attack is given in the appendix. It can be assumed, therefore, that the flow along the first characteristic line of the first family at the end of the conical region in the plane $\theta = \text{Constant}$ (for example, $\theta = \pi$) is known (fig. 7).

For the practical numerical calculations a value of the angle of attack must be selected. In order to obtain higher precision, it is convenient to select a relatively high value of the angle of attack because in this way the differences between V and V_1 and ΔS and ΔS_1 are large and, therefore, can be determined with sufficient precision.

Usually, when the determination of the flow field for the case of zero angle of attack is made with a graphical numerical process, in order to avoid numerical errors of computations, the value of the intensity and direction of the velocity are plotted as a function of the position along the characteristic lines for both families of characteristic lines. The velocity distribution and the entropy-variation distribution along the characteristic lines and along the surface of the body for the case of zero angle of attack can therefore be considered known. If the distribution is not given, the values of V_1 and ΔS_1 must be determined as a function of x along each characteristic line of a given family (for example, of the second family) along the body.

Then the construction of the characteristic net for the selected angle of attack must start by drawing the first characteristic line $P_0P_2P_8$ over the design of the characteristic net for zero angle of attack (fig. 7).

From equations (34c), (34d), and (34e) the flow at P_1 can be determined. From P_1 and P_2 the point P_3 can be obtained in the first approximation as the intersection of the tangents at P_2 and P_1 to the characteristic lines. By using equations (34b), (34d), and (29), V , ϕ , and ΔS can be obtained in P_3

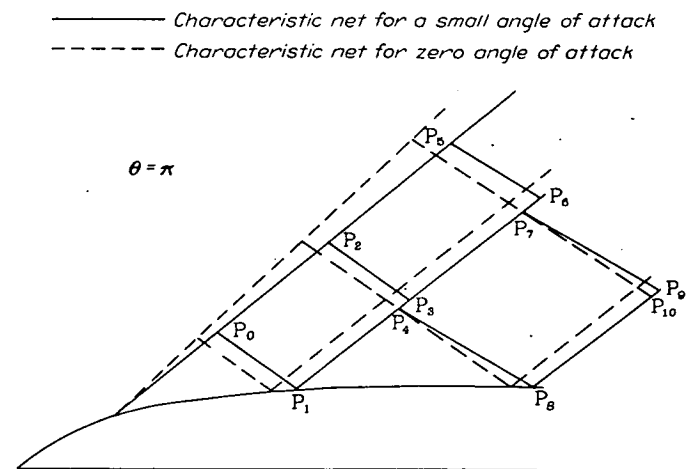


FIGURE 7.—Scheme of the characteristic net.

as for the case of zero angle of attack (reference 6). From the variations $\frac{dV}{dx}$, $\frac{d\phi}{dx}$, and $\frac{d\Delta S}{dx}$ along the line P_1P_3 the values of V , ϕ , and ΔS at the point P_4 can be obtained, where P_4 is obtained from the intersection of the characteristic line P_1P_3 with a characteristic line of the second family in the net for zero angle of attack. At the point P_4 , V_1 , ΔS_1 , and ϕ_1 are known and, therefore, ϕ_2 , V_2 , ΔS_2 , and w_2 can be obtained. From the values obtained from the first approximation a second approximation can be obtained. From P_4 and P_5 the point P_6 can be determined in a similar way, and the flow at P_7 can be calculated. By proceeding in a similar way, all the flow field can be analyzed.

NUMERICAL METHOD

The equation of motion (13) can be transformed by means of equations (5) and (11) in a system of equations that permits a numerical determination of the quantities V_2 , ϕ_2 , and ΔS_2 . This system is numerically more involved; however, the characteristic net determined for zero angle of attack is used. For a small angle of attack,

$$\frac{1}{a^2} = \frac{1}{a_1^2} \left[1 + \frac{\gamma-1}{a_1^2} (u_1 u_2 + v_1 v_2) \alpha \cos \theta \right]$$

where

$$\frac{\gamma-1}{a_1^2} (u_1 u_2 + v_1 v_2) = \frac{\gamma-1}{a_1^2} V_1 V_2 = \frac{\gamma-1}{\sin^2 \beta} \frac{V_2}{V_1} \equiv A \quad (41)$$

Therefore,

$$\frac{1}{a^2} = \frac{1}{a_1^2} (1 + A \alpha \cos \theta)$$

Substituting equations (5) in equations (6) results in the following expression if higher-order terms are neglected:

$$\begin{aligned} & \frac{\partial u_1}{\partial x} \left(1 - \frac{u_1^2}{a_1^2} \right) + \frac{\partial v_1}{\partial y} \left(1 - \frac{v_1^2}{a_1^2} \right) - \frac{u_1 v_1}{a_1^2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) + \frac{v_1}{y} \\ &= -\alpha \cos \theta \left[\frac{v_2}{y} + \frac{w_2}{y} - \frac{\partial u_1}{\partial x} \left(\frac{2u_2 u_1 + u_1^2 A}{a_1^2} \right) + \right. \\ & \quad \frac{\partial u_2}{\partial x} \left(1 - \frac{u_1^2}{a_1^2} \right) - \frac{\partial v_1}{\partial y} \left(\frac{2v_2 v_1 + v_1^2 A}{a_1^2} \right) + \frac{\partial v_2}{\partial y} \left(1 - \frac{v_1^2}{a_1^2} \right) - \\ & \quad \left. \frac{u_1 v_1 A + u_1 v_2 + v_1 u_2}{a_1^2} \left(\frac{\partial u_1}{\partial y} + \frac{\partial v_1}{\partial x} \right) - \frac{u_1 v_1}{a_1^2} \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right) \right] \quad (42) \end{aligned}$$

Because the left-hand side of equation (42) must be zero for the conditions at zero angle of attack,

$$\frac{\partial u_2}{\partial x} \left(1 - \frac{u_1^2}{a_1^2} \right) + \frac{\partial v_2}{\partial y} \left(1 - \frac{v_1^2}{a_1^2} \right) + \frac{v_2}{y} + \frac{w_2}{y} - \frac{u_1 v_1}{a_1^2} \left(\frac{\partial u_2}{\partial y} + \frac{\partial v_2}{\partial x} \right) + Q = 0 \quad (43)$$

where

$$Q = -\frac{1}{2} \left(\frac{\partial u_1^2}{\partial x} + \frac{\partial v_1^2}{\partial x} \right) \frac{2u_2 + u_1 A}{a_1^2} - \frac{1}{2} \left(\frac{\partial v_1^2}{\partial y} + \frac{\partial u_1^2}{\partial y} \right) \frac{2v_2 + v_1 A}{a_1^2} - \left(\frac{\partial v_1}{\partial x} - \frac{\partial u_1}{\partial y} \right) \frac{u_1 v_2 - u_2 v_1}{a_1^2} \quad (44)$$

or, from equations (9) and (36),

$$Q = -V_1 \frac{\partial V_1}{\partial x} \frac{2u_2 + u_1 A}{a_1^2} - V_1 \frac{\partial V_1}{\partial y} \frac{2v_2 + v_1 A}{a_1^2} - \frac{\phi_2 V_1}{\gamma R} \frac{\partial S_1}{\partial n_1} \quad (45)$$

The value of all the coefficients at the points P_1 and P_2 in equation (43) can be considered known because $\frac{\partial V_1}{\partial x}$ and $\frac{\partial V_1}{\partial y}$ can be considered known from the calculations for the case of zero angle of attack. Therefore, equation (43) can be considered an equation in which the characteristic lines are equal to the characteristic lines for zero angle of attack because the coefficients of the partial derivatives $\frac{\partial u_2}{\partial x}$, $\frac{\partial u_2}{\partial y}$, $\frac{\partial v_2}{\partial x}$, and $\frac{\partial v_2}{\partial y}$ are the same in both cases. Thus,

$$\left(\frac{dy}{dx} \right) = \lambda_{1a} = \tan (\phi_1 + \beta_1) \quad (46a)$$

$$\left(\frac{dy}{dx} \right) = \lambda_{1b} = \tan (\phi_1 - \beta_1) \quad (46b)$$

Equation (43) can be transformed by introducing the entropy gradient $\frac{\partial S_2}{\partial n}$, and the equation of motion along each characteristic line can be obtained. From equations (9) and (11) by means of equations (5) and (37), the following relation can be obtained:

$$\begin{aligned} \frac{\partial S}{\partial n} &= \frac{\partial S_1}{\partial n} + \frac{\partial S_2}{\partial n} \alpha \cos \theta \\ &= -\left(\frac{\partial S_1}{\partial x} + \frac{\partial S_2}{\partial x} \alpha \cos \theta \right) \frac{v_1 + v_2 \alpha \cos \theta}{V_1 + V_2 \alpha \cos \theta} + \\ & \quad \left(\frac{\partial S_1}{\partial y} + \frac{\partial S_2}{\partial y} \alpha \cos \theta \right) \frac{u_1 + u_2 \alpha \cos \theta}{V_1 + V_2 \alpha \cos \theta} \end{aligned}$$

or, for small angles of attack,

$$\begin{aligned} \frac{\partial S}{\partial n} &= \frac{\partial S_1}{\partial n_1} + \frac{\partial S_2}{\partial n_1} \alpha \cos \theta + \left[-\frac{\partial S_1}{\partial x} (v_2 V_1 - v_1 V_2) + \right. \\ & \quad \left. \frac{\partial S_1}{\partial y} (u_2 V_1 - u_1 V_2) \right] \frac{\alpha \cos \theta}{V_1^2} \quad (47) \end{aligned}$$

From equations (36) and (38)

$$u_2 = V_2 \cos \phi_1 - \phi_2 V_1 \sin \phi_1 \quad (48a)$$

$$v_2 = V_2 \sin \phi_1 + \phi_2 V_1 \cos \phi_1 \quad (48b)$$

and

$$v_1 = V_1 \sin \phi_1$$

$$u_1 = V_1 \cos \phi_1$$

Therefore,

$$\begin{aligned} & -\frac{\partial S_1}{\partial x} (v_2 V_1 - v_1 V_2) + \frac{\partial S_1}{\partial y} (u_2 V_1 - u_1 V_2) \\ & = -\phi_2 V_1^2 \left(\frac{\partial S_1}{\partial x} \frac{u_1}{V_1} + \frac{\partial S_1}{\partial y} \frac{v_1}{V_1} \right) \end{aligned}$$

Because the term in parentheses on the right-hand side of this equation represents the variation of entropy along the streamline, which is zero, equation (47) becomes

$$\frac{\partial S}{\partial n} = \frac{\partial S_1}{\partial n_1} + \frac{\partial S_2}{\partial n_1} \alpha \cos \theta \quad (49)$$

Then, from equations (9), (41), and (49)

$$\frac{\partial v_2}{\partial x} - \frac{\partial u_2}{\partial y} = \frac{a_1^2}{\gamma R V_1} \frac{\partial S_2}{\partial n_1} - \left(\frac{V_2}{V_1} + A \right) \frac{a_1^2}{\gamma R V_1} \frac{\partial S_1}{\partial n_1} \quad (50)$$

The equations of motion along the characteristic lines defined by equations (46) can be obtained by means of transformations similar to those of equations (27) and are

$$du_2 + \lambda_{1b} dv_2 + D_1 dx = 0 \quad (51)$$

$$\lambda_{1a} = \frac{dy}{dx} = \tan(\phi_1 + \beta_1) \quad (52)$$

$$du_2 + \lambda_{1a} dv_2 + D_2 dx = 0 \quad (53)$$

$$\lambda_{1b} = \frac{dy}{dx} = \tan(\phi_1 - \beta_1) \quad (54)$$

where

$$\begin{aligned} D_1 = & \left(\frac{w_2}{y} + \frac{v_2}{y} + Q \right) \frac{a_1^2}{a_1^2 - u_1^2} + \\ & \left(\frac{u_1 v_1}{a_1^2 - u_1^2} + \lambda_{1a} \right) \frac{V_1}{\gamma R} \sin^2 \beta_1 \left[\frac{\partial S_2}{\partial n_1} - \frac{V_2}{V_1} \left(1 + \frac{\gamma - 1}{\sin^2 \beta_1} \right) \frac{\partial S_1}{\partial n_1} \right] \end{aligned} \quad (55a)$$

$$\begin{aligned} D_2 = & \left(\frac{w_2}{y} + \frac{v_2}{y} + Q \right) \frac{a_1^2}{a_1^2 - u_1^2} + \\ & \left(\frac{u_1 v_1}{a_1^2 - u_1^2} + \lambda_{1b} \right) \frac{V_1}{\gamma R} \sin^2 \beta_1 \left[\frac{\partial S_2}{\partial n_1} - \frac{V_2}{V_1} \left(1 + \frac{\gamma - 1}{\sin^2 \beta_1} \right) \frac{\partial S_1}{\partial n_1} \right] \end{aligned} \quad (55b)$$

In equations (51) and (53) the coefficients D_1 and D_2 contain the derivatives $\frac{\partial V_1}{\partial x}$ and $\frac{\partial V_1}{\partial y}$ that must be obtained from the analysis of the case with zero angle of attack. Now, for every point P the variation of V_1 along the characteristic line of the first family for the case of zero angle of attack is

$$\left(\frac{dV_1}{dx} \right)_{\lambda_{1a}} = \frac{\partial V_1}{\partial x} + \lambda_{1a} \frac{\partial V_1}{\partial y} \quad (56a)$$

whereas along the characteristic line of the second family

$$\left(\frac{dV_1}{dx} \right)_{\lambda_{1b}} = \frac{\partial V_1}{\partial x} + \lambda_{1b} \frac{\partial V_1}{\partial y} \quad (56b)$$

At every point P given by the intersection of two characteristic lines λ_{1a} and λ_{1b} in the characteristic net, the values $\left(\frac{dV_1}{dx} \right)_{\lambda_{1a}}$ and $\left(\frac{dV_1}{dx} \right)_{\lambda_{1b}}$ are known, having been obtained from the evaluation of the following equations (reference 5):

$$\frac{dy}{dx} = \lambda_{1a} = \tan(\phi_1 + \beta_1) \quad (57a)$$

$$\begin{aligned} & \frac{1}{V_1} \left(\frac{dV_1}{dx} \right)_{\lambda_{1a}} - \tan \beta_1 \left(\frac{d\phi_1}{dx} \right) - \frac{\sin \phi_1 \sin \beta_1 \tan \beta_1}{\cos(\phi_1 + \beta_1)} \frac{1}{y} + \\ & \frac{dS_1}{dn_1} \frac{1}{\gamma R} \frac{\sin^3 \beta_1}{\cos(\phi_1 + \beta_1)} = 0 \end{aligned} \quad (57b)$$

$$\frac{dy}{dx} = \lambda_{1b} = \tan(\phi_1 - \beta_1) \quad (57c)$$

$$\begin{aligned} & \frac{1}{V_1} \left(\frac{dV_1}{dx} \right)_{\lambda_{1b}} + \tan \beta_1 \left(\frac{d\phi_1}{dx} \right) - \frac{1}{y} \frac{\sin \phi_1 \sin \beta_1 \tan \beta_1}{\cos(\phi_1 - \beta_1)} - \\ & \frac{dS_1}{dn_1} \frac{1}{\gamma R} \frac{\sin^3 \beta_1}{\cos(\phi_1 - \beta_1)} = 0 \end{aligned} \quad (57d)$$

Therefore, the values

$$\frac{\partial V_1}{\partial x} = \frac{\lambda_{1b}}{\lambda_{1b} - \lambda_{1a}} \left(\frac{dV_1}{dx} \right)_{\lambda_{1a}} - \frac{\lambda_{1a}}{\lambda_{1b} - \lambda_{1a}} \left(\frac{dV_1}{dx} \right)_{\lambda_{1b}} \quad (58a)$$

and

$$\frac{\partial V_1}{\partial y} = \frac{1}{\lambda_{1b} - \lambda_{1a}} \left[\left(\frac{dV_1}{dx} \right)_{\lambda_{1b}} - \left(\frac{dV_1}{dx} \right)_{\lambda_{1a}} \right] \quad (58b)$$

can be calculated directly for every point of intersection of the characteristic line (equations (57a) and (57c)).

After substituting the expressions of equations (48) and (58) in equations (51) and (53) after some simplifications and trigonometric transformations, the following equations can be obtained:

$$\lambda_{1a} = \frac{dy}{dx} = \tan(\phi_1 + \beta_1) \quad (59a)$$

$$\frac{dV_2}{V_1} - \tan \beta_1 d\phi_2 + \left[\frac{\sin^3 \beta_1}{\cos(\phi_1 + \beta_1)} \frac{1}{\gamma R} \frac{\partial S_2}{\partial n_1} + \phi_2 A_1 + \frac{V_2}{V_1} P_1 - \frac{w_2}{V_1} \frac{1}{y} \frac{\tan \beta_1 \sin \beta_1}{\cos(\phi_1 + \beta_1)} \right] dx = 0 \quad (59b)$$

$$\lambda_{1b} = \frac{dy}{dx} = \tan(\phi_1 - \beta_1) \quad (59c)$$

$$\frac{dV_2}{V_1} + \tan \beta_1 d\phi_2 - \left[\frac{\sin^3 \beta_1}{\cos(\phi_1 - \beta_1)} \frac{1}{\gamma R} \frac{\partial S_2}{\partial n_1} + \phi_2 A_2 - \frac{V_2}{V_1} P_2 + \frac{w_2}{V_1} \frac{1}{y} \frac{\tan \beta_1 \sin \beta_1}{\cos(\phi_1 - \beta_1)} \right] dx = 0 \quad (59d)$$

where

$$A_1 = \frac{1}{\cos \beta_1 \cos(\phi_1 + \beta_1)} \left[\frac{\sin \beta_1}{y} \sin(\phi_1 - \beta_1) + \frac{\sin^4 \beta_1}{\gamma R} \frac{\partial S_1}{\partial n_1} - \frac{1}{V_1} \left(\frac{dV_1}{dx} \right)_{\lambda_{1b}} \frac{\cos(\phi_1 - \beta_1)}{\sin \beta_1} \right] \quad (60a)$$

$$A_2 = \frac{1}{\cos \beta_1 \cos(\phi_1 - \beta_1)} \left[\frac{\sin \beta_1}{y} \sin(\phi_1 + \beta_1) - \frac{\sin^4 \beta_1}{\gamma R} \frac{\partial S_1}{\partial n_1} - \frac{1}{V_1} \left(\frac{dV_1}{dx} \right)_{\lambda_{1a}} \frac{\cos(\phi_1 + \beta_1)}{\sin \beta_1} \right] \quad (60b)$$

$$P_1 = \frac{1}{V_1} \left(\frac{dV_1}{dx} \right)_{\lambda_{1a}} \left(\frac{T}{\cos^2 \beta_1} - 1 \right) - T \left[\frac{2 \sin^3 \beta_1}{\gamma R \cos(\phi_1 + \beta_1)} \frac{\partial S_1}{\partial n_1} - \frac{1}{V_1} \left(\frac{dV_1}{dx} \right)_{\lambda_{1b}} \frac{\cos(\phi_1 - \beta_1)}{\cos^2 \beta_1 \cos(\phi_1 + \beta_1)} \right] \quad (60c)$$

$$P_2 = \frac{1}{V_1} \left(\frac{dV_1}{dx} \right)_{\lambda_{1b}} \left(\frac{T}{\cos^2 \beta_1} - 1 \right) + T \left[\frac{2 \sin^3 \beta_1}{\gamma R \cos(\phi_1 - \beta_1)} \frac{\partial S_1}{\partial n_1} + \frac{1}{V_1} \left(\frac{dV_1}{dx} \right)_{\lambda_{1a}} \frac{\cos(\phi_1 + \beta_1)}{\cos^2 \beta_1 \cos(\phi_1 - \beta_1)} \right] \quad (60d)$$

$$T = 1 + \frac{\gamma - 1}{2 \sin^2 \beta_1} \quad (60e)$$

The coefficients A_1 and A_2 and the value of $\frac{dS_1}{dn_1}$ have been determined for the flow at zero angle of attack and

$$\begin{aligned} \frac{dS_2}{dn_1} &= \frac{S_{2P_2} - S_{2P_1}}{(x_{P_3} - x_{P_2}) \left[\frac{\sin \beta_1}{\cos(\phi_1 - \beta_1)} \right]_{P_2} + (x_{P_3} - x_{P_1}) \left[\frac{\sin \beta_1}{\cos(\phi_1 + \beta_1)} \right]_{P_1}} \\ &= \frac{dS_1}{dn_1} \frac{S_{2P_2} - S_{2P_1}}{S_{1P_2} - S_{1P_1}} \end{aligned} \quad (61)$$

The practical use of equations (59) is identical to the use of the corresponding equations (57) for the case of zero angle of attack. (See reference 6.)

CONCLUDING REMARKS

The method of characteristics has been applied to bodies of revolution at a small angle of attack. Only the first-order effects of the angle of attack have been considered. The system developed takes into account the effects of the entropy variations on the flow phenomena and determines a flow that exactly satisfies the boundary conditions within the limits of the simplifications assumed.

The application of the method to practical problems has been discussed and two systems are given. The first method is numerical and analytical and requires less numerical computation but requires the construction of another characteristic net; whereas the second method is only numerical and uses the characteristic net and some of the numerical computations made for the calculations for zero angle of attack.

LANGLEY AERONAUTICAL LABORATORY,

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,

LANGLEY FIELD, VA., November 22, 1948.

APPENDIX

DETERMINATION OF FLOW PROPERTIES AROUND A CIRCULAR CONE AT A SMALL ANGLE OF ATTACK

Assume a polar coordinate system r, ψ, θ . Call v_r the velocity in radial direction, v_n the velocity in normal direction to r in the meridian plane $\theta = \text{Constant}$, and w the component normal to the meridian plane (fig. 8); that is,

$$v_r = \frac{dr}{dt}$$

$$v_n = \frac{r d\psi}{dt}$$

$$w = \frac{r d\theta \sin \psi}{dt}$$

If the phenomenon is conical,

$$\frac{\partial v_r}{\partial r} = 0$$

$$\frac{\partial v_n}{\partial r} = 0$$

$$\frac{\partial w}{\partial r} = 0$$

and

$$\frac{\partial p}{\partial r} = 0$$

$$\frac{\partial \rho}{\partial r} = 0$$

Therefore, Euler's equations are

$$\frac{v_n}{r} \frac{\partial v_r}{\partial \psi} + \frac{w}{r \sin \psi} \frac{\partial v_r}{\partial \theta} - \frac{v_n^2 + w^2}{r} = 0 \quad (62a)$$

$$\frac{v_n}{r} \frac{\partial v_n}{\partial \psi} + \frac{w}{r \sin \psi} \frac{\partial v_n}{\partial \theta} + \frac{1}{\rho r} \frac{\partial p}{\partial \psi} + \frac{v_r v_n - w^2 \cot \psi}{r} = 0 \quad (62b)$$

$$\frac{v_n}{r} \frac{\partial w}{\partial \psi} + \frac{w}{r \sin \psi} \frac{\partial w}{\partial \theta} + \frac{1}{r \rho \sin \psi} \frac{\partial p}{\partial \theta} + \frac{v_r w + v_n w \cot \psi}{r} = 0 \quad (62c)$$

The continuity equation is

$$2\rho v_r \sin \psi + v_n \sin \psi \frac{\partial \rho}{\partial \psi} + \rho \sin \psi \frac{\partial v_n}{\partial \psi} + v_n \rho \cos \psi + w \frac{\partial \rho}{\partial \theta} + \frac{\partial w}{\partial \theta} = 0 \quad (63)$$

and the energy equation is

$$\frac{\gamma}{\gamma-1} \left(\frac{1}{\rho} \frac{\partial p}{\partial \theta} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial \theta} \right) = - \left(v_r \frac{\partial v_r}{\partial \theta} + v_n \frac{\partial v_n}{\partial \theta} + w \frac{\partial w}{\partial \theta} \right) \quad (64a)$$

$$\frac{\gamma}{\gamma-1} \left(\frac{1}{\rho} \frac{\partial p}{\partial \psi} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial \psi} \right) = - \left(v_r \frac{\partial v_r}{\partial \psi} + v_n \frac{\partial v_n}{\partial \psi} + w \frac{\partial w}{\partial \psi} \right) \quad (64b)$$

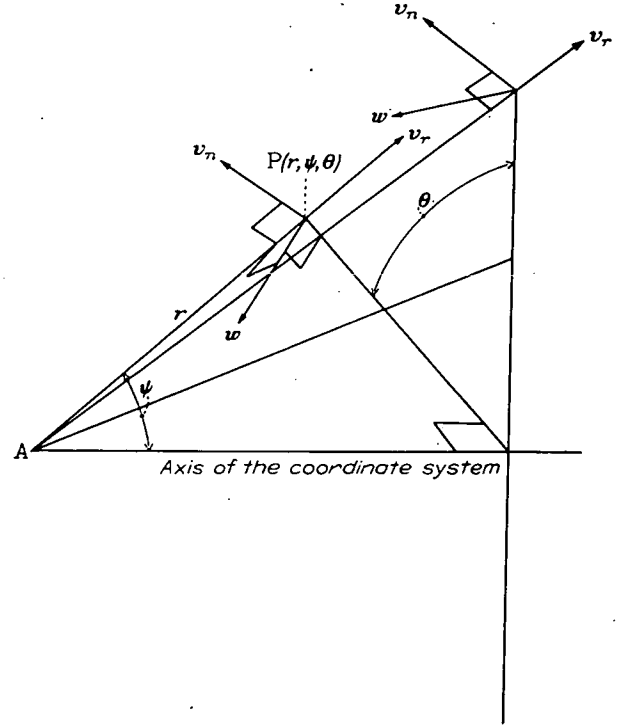


FIGURE 8.—Polar coordinates and velocity components.

Combining equations (62) to (64) results in

$$\begin{aligned} v_r \left(2 - \frac{v_n^2 + w^2}{a^2} \right) + v_n \cot \psi + \frac{\partial v_n}{\partial \psi} \left(1 - \frac{v_n^2}{a^2} \right) + \\ \frac{\partial w}{\sin \psi \partial \theta} \left(1 - \frac{w^2}{a^2} \right) - \frac{2wv_n}{a^2} \frac{\partial v_n}{\sin \psi \partial \theta} + \\ \frac{wv_n}{a^2} \left(\frac{\partial v_n}{\sin \psi \partial \theta} - \frac{\partial w}{\partial \psi} \right) = 0 \end{aligned} \quad (65)$$

For small angles of attack the velocity components can be expressed in the form (references 1 and 3)

$$\left. \begin{aligned} v_r &= v_{r_1} + \alpha v_{r_2} \cos \theta \\ v_n &= v_{n_1} + \alpha v_{n_2} \cos \theta \\ w &= \alpha w_2 \sin \theta \end{aligned} \right\} \quad (66)$$

when the second- and higher-order terms of the angle of attack have been neglected. Equation (65) at small angles of attack becomes

$$v_r \left(2 - \frac{v_n^2}{a^2} \right) + v_n \cot \psi + \frac{\partial v_n}{\partial \psi} \left(1 - \frac{v_n^2}{a^2} \right) + \frac{\partial w}{\sin \psi \partial \theta} = 0 \quad (67)$$

The shock is a circular conical shock having its axis inclined at an angle η with the axis of the cone. The quantities with subscript 1 are the quantities corresponding to the case of zero angle of attack. (Indeed, the cone is a particular body of revolution and, therefore, the considerations made for the case of bodies of revolution are still valid.)

From equations (62a), (62b), and (64b), there results at small angles of attack

$$\frac{1}{p} \frac{\partial p}{\partial \psi} - \frac{\gamma}{\rho} \frac{\partial \rho}{\partial \psi} = 0$$

In the meridian plane $\theta = \text{Constant}$, therefore, the transformation behind the shock is isentropic for small angles of attack. If $-\alpha \Delta S_2 \sin \theta$ is the variation of entropy in a direction normal to the meridian plane $\theta = \text{Constant}$ (reference 5),

$$\frac{\partial S}{\sin \psi \partial \theta} = -\alpha \Delta S_2 \sin \theta$$

where ΔS_2 is constant and independent of ψ outside of the vortical layer. From equations (62c) and (64a)

$$-\frac{a^2}{\gamma R} \Delta S_2 = v_n \frac{\partial w_2}{\partial \psi} \sin \psi + w_2 (v_r \sin \psi + v_n \cos \psi) + v_r v_{r_2} + v_n v_{n_2} \quad (68)$$

If the variation of entropy is small and the term ΔS_2 can be neglected, equation (68) becomes

$$w_2 \sin \psi = -v_{r_2} \quad (69)$$

Equation (67) can be written in the following form:

$$\left(v_r + \frac{\partial v_n}{\partial \psi} \right) \left(1 - \frac{v_n^2}{a^2} \right) = -v_r - v_n \cot \psi - \frac{\alpha w_2 \cos \theta}{\sin \psi} \quad (70)$$

By use of equation (67) and by considering the conditions for zero angle of attack, equation (70) becomes

$$\begin{aligned} & \left(v_{r_2} + \frac{\partial v_{n_2}}{\partial \psi} \right) \left(1 - \frac{v_{n_1}^2}{a_1^2} \right) \\ &= -v_{n_2} \left[\cot \psi + \left(\frac{2v_{n_1}}{a_1^2} + \frac{\gamma-1}{a_1^4} v_{n_1}^3 \right) \frac{v_{r_1} + v_{n_1} \cot \psi}{1 - \frac{v_{n_1}^2}{a_1^2}} \right] - \\ & \quad v_{r_2} \left(1 + \frac{\gamma-1}{a_1^4} \frac{v_{r_1} + v_{n_1} \cot \psi}{1 - \frac{v_{n_1}^2}{a_1^2}} v_{r_1} v_{n_1}^2 \right) - \frac{w_2}{\sin \psi} \quad (71) \end{aligned}$$

Equations (68), (70), and (71) permit the determination of the flow around the cone at an angle of attack by means of a step-by-step calculation when the calculation for $\alpha=0$ has been performed. Consider the hodograph plane uv , and consider the variation of velocity components v_r and v_n in a meridian plane $\theta = \text{Constant}$ (fig. 9). Assume that for a given value of ψ_a and θ_a the velocity components v_r , v_n , and

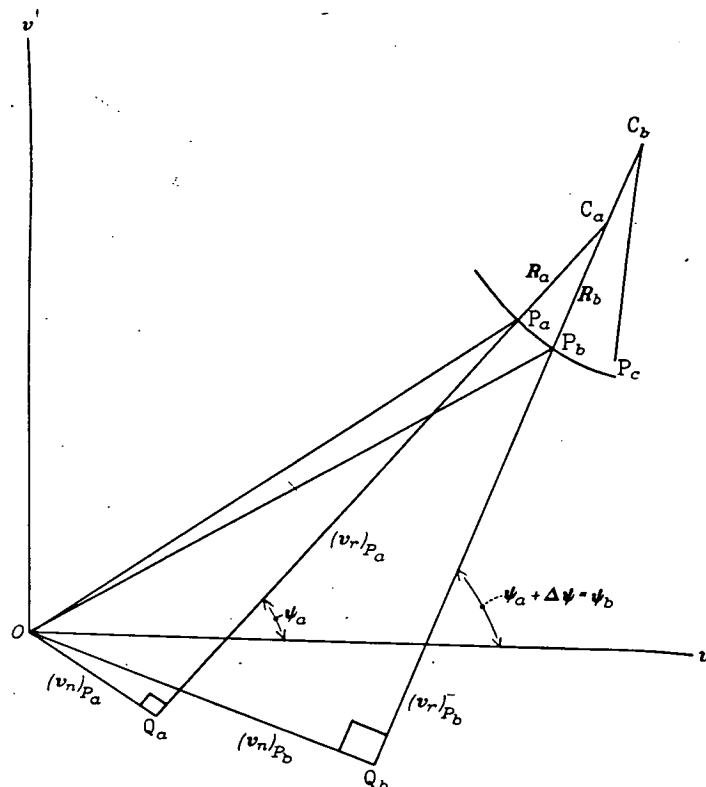


FIGURE 9.—The hodograph diagram.

w are known. Point P_a of the hodograph diagram represents the velocity vector OP_a corresponding to the velocity at every point of the space of coordinate ψ_a in the plane $\theta_a = \text{Constant}$; whereas OQ_a represents the values of $(v_n)_{P_a}$ and Q_aP_a represents the values of $(v_r)_{P_a}$. Now, the radius of curvature R_a of the hodograph diagram is along the line Q_aP_a and has a value given by (reference 1)

$$R_a = \left(v_r + \frac{\partial v_n}{\partial \psi} \right)_{P_a}$$

and, therefore, can be determined from equation (70). At P_0 the values of v_{r_1} and v_{n_1} are known from the calculation for $\alpha=0$; therefore, αv_{r_2} and αv_{n_2} can be determined from equations (66). Equation (71) can be used in place of equation (70) in the following way: The vectors OQ_a and Q_aP_a represent the values of v_{n_2} and v_{r_2} at P_a ; the vector OP_a in the hodograph diagram gives the values of V_2 ; therefore,

$$R_{2a} = \left(v_{r_2} + \frac{\partial v_{n_2}}{\partial \psi} \right)_{P_a}$$

can be obtained from equation (71).

Now at any point P_a the radius R_a , given from equation (70), or the radius R_{2a} , given from equation (71), is known; therefore, from the quantities at P_a the quantities at P_b of coordinate $\psi_b = \psi_a + \Delta \psi$ can be obtained by constructing a circle of center C_a (where $C_aP_a = R_a$ or R_{2a}) through the point

P_a until the point P_b along the line $C_a P_b$ which is a straight line from C_a , is inclined by $\psi_a + \Delta\psi$ with the u -axis. Therefore,

$$(v_n)_{\psi+\Delta\psi} = (v_n)_\psi \cos \Delta\eta + (R-v_r)_\psi \sin \Delta\eta$$

$$(v_r)_{\psi+\Delta\psi} = (v_n)_\psi \sin \Delta\eta - (R-v_r)_\psi \cos \Delta\eta + R_\psi$$

Inasmuch as the values of v_r and v_n at P_b have been obtained, the values of v_{r_2} and v_{n_2} can be determined by differences from the values for $\alpha=0$ with the use of equations (66). (If equation (71) is used, the values of v_{r_2} and v_{n_2} are obtained directly.) With equation (68) the value of $\frac{\partial w_2}{\partial \psi}$ can be calculated at P_a , and the value of w_2 at P_b can be obtained. Indeed, ΔS_2 is constant and has been determined from the conditions at the shock. In a similar way, all the hodograph diagram can be constructed. If necessary, for every point P_b a second approximation can be determined.

The calculation of all the flow field must start at the shock. For the calculations it is convenient to choose a coordinate system having the axis of the conical shock as the axis of polar coordinates. In this case, the velocity components v_r , v_n , and w behind the shock can still be expressed in the form of equations (66). Indeed, from equations (15) and (19),

$$\begin{aligned} v_{n_s} &= \left[v_{n_1} + \alpha \cos \theta \left(v_{n_2} + \frac{\eta}{\alpha} \frac{\partial v_{n_1}}{\partial \psi} \right) \right]_b \\ &= v_{n_{1_2}} + (\alpha - \eta) v_{n_{2_2}} \cos \theta \end{aligned} \quad (72a)$$

$$\begin{aligned} v_{r_s} &= \left[v_{r_1} + \alpha \cos \theta \left(v_{r_2} + \frac{\eta}{\alpha} v_{n_1} \right) \right]_b \\ &= v_{r_{1_2}} + (\alpha - \eta) v_{r_{2_2}} \cos \theta \end{aligned} \quad (72b)$$

$$\begin{aligned} w_s &= \alpha \sin \theta \left(w_2 - \frac{1}{\sin \psi} \frac{\eta}{\alpha} v_{n_1} \right)_b \\ &= (\alpha - \eta) w_{2_s} \sin \theta \end{aligned} \quad (72c)$$

where v_{n_s} , v_{r_s} , and w_s are the components referred to the axis of the conical shock, whereas the components v_{n_b} , v_{r_b} , and w_b are referred to the axis of the body. Indeed, η/α is constant.

The calculations start at the shock. After determining the flow field for zero angle of attack, the angle of the conical shock ψ_s is known and the velocity components v_{n_1} and v_{r_1} with respect to the axis of the shock for every value of ψ are also known. In order to determine the flow for the case of a small angle of attack, the direction of the undisturbed velocity must be rotated at a small angle $\alpha - \eta$ with respect to the axis of the shock (fig. 10). The value assumed for $\alpha - \eta$ fixes the value of α for which the calculations are performed. (This value of α is not yet known but is obtained as a result of the calculation.)

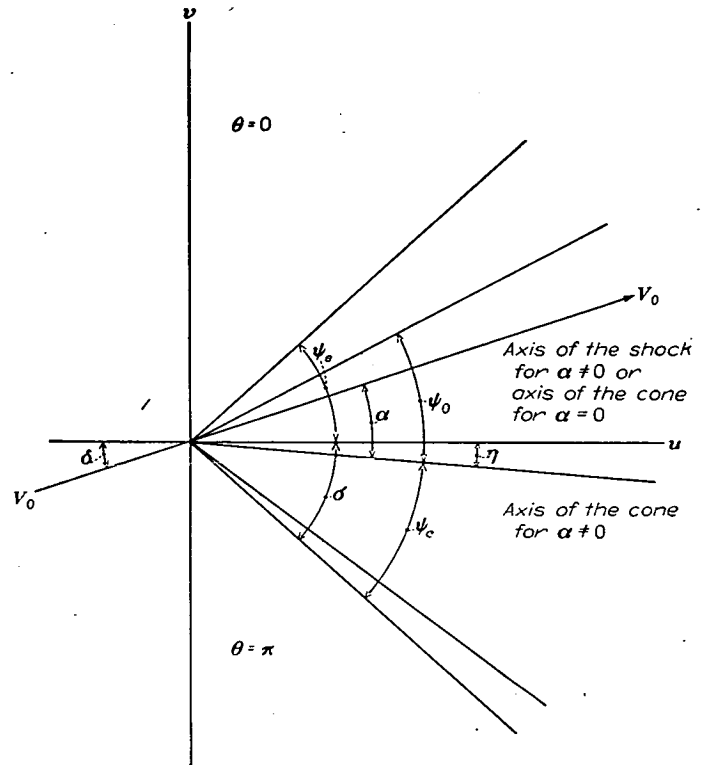


FIGURE 10.—Choice of the axis for the hodograph diagram.

For a unitary value of $\alpha - \eta$, the component v_{r_s} behind the shock can be determined from equation (14b) ($v_{r_s} = v_{r'}$) and the component w_s , from equation (14c) ($w' = w_s$); whereas v_{n_s} can be determined from equation (21a) ($v_{n_s} = -v_{N''}$) and $v_{N'}$ is given by equation (14a). The value of entropy $(\alpha - \eta)\Delta S_2 \cos \theta$ can also be determined from the equation of the shock, for example, from the difference between ΔS and ΔS_1 . When v_{r_s} , v_{n_s} , w_{2_s} , and ΔS_2 are known behind the shock, all the flow field can be obtained by means of equations (68) and (70) or (71). The hodograph diagram can be constructed, for example, in the plane $\theta = \pi$.

The axis u has been chosen in the direction of the undisturbed velocity for zero angle of attack that corresponds to the axis of the shock for $\alpha = 0$. For α the undisturbed velocity has been rotated at $\alpha - \eta$ with respect to the u -axis (fig. 10); therefore, the axis of the shock has not been changed. The velocity OP_a behind the shock of figure 9 must be decomposed into (1) a component $P_a Q_a$ inclined at ψ_s and corresponding to v_{r_s} if equation (70) is used or to $v_{r_{2_s}}$ if equation (71) is used and (2) a component $O_a Q_a$ corresponding to v_{n_s} or $v_{n_{2_s}}$.

In this way, the values of v_{r_1} and v_{n_1} that must be used in equations (68), (70), and (71) are the values obtained from the calculations for zero angle of attack at the same value of ψ (that is, $\psi = \psi_s$ for P_a). Because the calculations start at the shock, the construction of the hodograph diagram must be performed in the direction of decreasing values of ψ .

At the surface of the body for $\theta=\pi$, the component v_n must be zero; therefore, when the radius of the hodograph diagram passes at the origin of coordinates u and v , the corresponding value of ψ is equal to $\psi_0+\eta$ where ψ_0 is the angle of the cone (fig. 10). Because ψ_0 is known, the value of η and, therefore, of α can be determined.

The components v_r and v_n in the plane $\theta=0$ or $\theta=\pi$ do not change when, for the axis of reference, the axis of the body is assumed; but the corresponding value of ψ is increased at η (fig. 10). The value of w_2 changes; the value of w_2'' can be determined from the value of w_2 by means of equation (72c).

For practical calculations it is convenient to use nondimensional coefficients obtained by dividing all the velocity components by the limiting velocity V_i . The expression a/V_i can be obtained from equation (39d).

For small values of α , the values of v_{r_2} , v_{n_2} , w_2 , and η/α are independent of α and, therefore, the flow for every other value of α can be obtained from this determination. The calculations can be graphical or analytical.

REFERENCES

1. Ferrari, C.: Determination of the Pressure Exerted on Solid Bodies of Revolution with Pointed Noses Placed Obliquely in a Stream of Compressible Fluid at Supersonic Velocity. R.T.P. Translation No. 1105, British Ministry of Aircraft Production. (From Atti R. Accad. Sci. Torino, vol. 72, Nov.-Dec. 1936, pp. 140-163.)
2. Sauer, Robert: Supersonic Flow about Projectile Heads of Arbitrary Shape at Small Incidence. R.T.P. Translation No. 1573, British Ministry of Aircraft Production. (From Luftfahrtforschung, vol. 19, no. 4, May 1942, pp. 148-152.)
3. Staff of the Computing Section, Center of Analysis (Under Direction of Zdeněk Kopal): Tables of Supersonic Flow around Yawing Cones. Tech. Rep. No. 3, M.I.T., 1947.
4. Stone, A. H.: On Supersonic Flow past a Slightly Yawing Cone. Jour. Math. and Phys., vol. XXVII, no. 1, April 1948, pp. 67-81.
5. Ferri, Antonio: Supersonic Flow around Circular Cones at Angles of Attack. NACA Rep. 1045, 1951. (Supersedes NACA TN 2236.)
6. Ferri, Antonio: Application of the Method of Characteristics to Supersonic Rotational Flow. NACA Rep. 841, 1946. (Supersedes NACA TN 1135.)
7. Staff of the Computing Section, Center of Analysis (Under Direction of Zdeněk Kopal): Tables of Supersonic Flow around Cones. Tech. Rep. No. 1, M.I.T., 1947.

REPORT 1045

SUPERSONIC FLOW AROUND CIRCULAR CONES AT ANGLES OF ATTACK¹

By ANTONIO FERRI

SUMMARY

The flow around cones without axial symmetry and moving at supersonic velocity is analyzed. Singular points are shown to exist in the flow around the cone if no axial symmetry exists. The results of the analysis are applied to the determination of flow around circular cones at small angles of attack. The concept of a vortical layer around the cone at small angles of attack is introduced, and the correct values of the first-order terms of the velocity components are determined.

The method used is applied to cones at finite angles of attack, and it is shown that good agreement with experimental results can be obtained from the first-order theory if the complete equation for the pressure distribution is used.

INTRODUCTION

The flow around a cone having a circular cross section and moving at supersonic speed has been determined by means of the assumption of small disturbances or by means of more rigorous methods that consider the existence of the shocks. The latter methods can be applied for any Mach number larger than unity and have been developed by several authors, at first by assuming all the flow as potential flow (references 1 and 2) and later by also considering the variation of entropy due to the change in angle of attack (reference 3). By means of the development given in reference 3, values of flow properties around circular cones at an angle of attack have been tabulated in reference 4. The method has been extended in reference 5 to larger angles of attack.

In the method given in references 3, 4, and 5, the flow properties were considered continuous and were developed in Fourier series in terms of the angle of attack; however, the existence of a singular point at the surface of the cone was neglected. The derivatives of the flow properties were obtained by differentiating the Fourier series term by term, and the terms of the series that represent the derivatives were assumed to be of the same order as the corresponding terms of the integral quantities. For this reason, an erroneous distribution of the entropy at the surface of the cone was obtained.

In this report, the flow around the cone in the general case is discussed, the existence of singular points in the flow is proved, and a different procedure for determining the flow around cones at small, but finite, angles of attack is developed. This procedure shows the way in which the values tabulated in reference 4 can be used if a simple correction is introduced. The values obtained in this way are compared with experimental results at several values of angle of attack.

SYMBOLS

r, ψ, θ	polar coordinates (see fig. 1)
v_r	polar velocity component in radial direction (along r), referred to limiting velocity (see fig. 1)
v_n	polar velocity component normal to v_r in meridian plane $\theta = \text{Constant}$, referred to limiting velocity (fig. 1)
w	polar velocity component normal to meridian plane $\theta = \text{Constant}$, referred to limiting velocity (fig. 1)
t	time
p	pressure
ρ	density
S	entropy
γ	ratio of specific heats (c_p/c_v)
$R = c_p - c_v$	
c_p	specific heat at constant pressure
c_v	specific heat at constant volume
a	speed of sound
L	projection of streamline on sphere $r = \text{Constant}$ with center at center of the polar coordinate system
V	local velocity
V_1	undisturbed velocity, referred to limiting velocity
V_i	limiting velocity (velocity for expansion in the vacuum)
σ	semiapex angle of conical shock
δ	inclination of axis of conical shock with respect to free-stream direction
η	inclination of axis of conical shock with respect to axis of body
α	angle of attack
Subscripts:	
1	stream conditions
a	zero-order terms of Fourier series (part independent from angle of attack)
b	first-order terms of Fourier series (part proportional to angle of attack)
c	higher-order terms of Fourier series or quantities at surface of cone
e	quantities at external surface of vortical layer
s	quantities for polar coordinate system having axis coincident with axis of conical shock

¹Supersedes NACA TN 2236, "Supersonic Flow around Circular Cones at Angles of Attack" by Antonio Ferri, 1950.

A prime is used to designate the terms of zero order in the power series in $\Delta\theta$ for the quantities in the neighborhood of the meridian plane $\theta=\pi$; two primes are used to designate the factor of the term containing $\Delta\theta^2/2$ in the same power series.

THE FLOW FIELD FOR CONICAL FLOW WITHOUT AXIAL SYMMETRY

In order to analyze the flow field for conical flow without axial symmetry at supersonic speeds, assume a polar coordinate system (r, ψ, θ) . Call v_r the velocity component in the radial direction, v_n the velocity component in the direction normal to r in the meridian plane $\theta=\text{Constant}$, and w the component normal to the meridian plane (fig. 1); that is,

$$v_r = \frac{dr}{dt}$$

$$v_n = \frac{r d\psi}{dt}$$

$$w = \frac{r d\theta}{dt} \sin \psi$$

If the flow is conical,

$$\frac{\partial v_r}{\partial r} = 0 \quad \frac{\partial v_n}{\partial r} = 0 \quad \frac{\partial w}{\partial r} = 0$$

$$\frac{\partial p}{\partial r} = 0 \quad \frac{\partial \rho}{\partial r} = 0 \quad \frac{\partial S}{\partial r} = 0$$

For these conditions Euler's equations become

$$v_n \frac{\partial v_r}{\partial \psi} + \frac{w}{\sin \psi} \frac{\partial v_r}{\partial \theta} - v_n^2 - w^2 = 0 \quad (1a)$$

$$v_n \frac{\partial v_n}{\partial \psi} + \frac{w}{\sin \psi} \frac{\partial v_n}{\partial \theta} + \frac{1}{\rho} \frac{\partial p}{\partial \psi} + v_r v_n - w^2 \cot \psi = 0 \quad (1b)$$

$$v_n \frac{\partial w}{\partial \psi} + \frac{w}{\sin \psi} \frac{\partial w}{\partial \theta} + \frac{1}{\rho \sin \psi} \frac{\partial p}{\partial \theta} + v_r w + v_n w \cot \psi = 0 \quad (1c)$$

and the continuity equation becomes

$$2\rho v_r \sin \psi + v_n \sin \psi \frac{\partial \rho}{\partial \psi} + \rho \sin \psi \frac{\partial v_n}{\partial \psi} + v_n \rho \cos \psi + w \frac{\partial \rho}{\partial \theta} + \rho \frac{\partial w}{\partial \theta} = 0 \quad (2)$$

Because the energy in the flow is constant, the following relations must apply:

$$\frac{\gamma}{\gamma-1} \left(\frac{1}{\rho} \frac{\partial p}{\partial \theta} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial \theta} \right) = - \left(v_r \frac{\partial v_r}{\partial \theta} + v_n \frac{\partial v_n}{\partial \theta} + w \frac{\partial w}{\partial \theta} \right) \quad (3a)$$

$$\frac{\gamma}{\gamma-1} \left(\frac{1}{\rho} \frac{\partial p}{\partial \psi} - \frac{p}{\rho^2} \frac{\partial \rho}{\partial \psi} \right) = - \left(v_r \frac{\partial v_r}{\partial \psi} + v_n \frac{\partial v_n}{\partial \psi} + w \frac{\partial w}{\partial \psi} \right) \quad (3b)$$

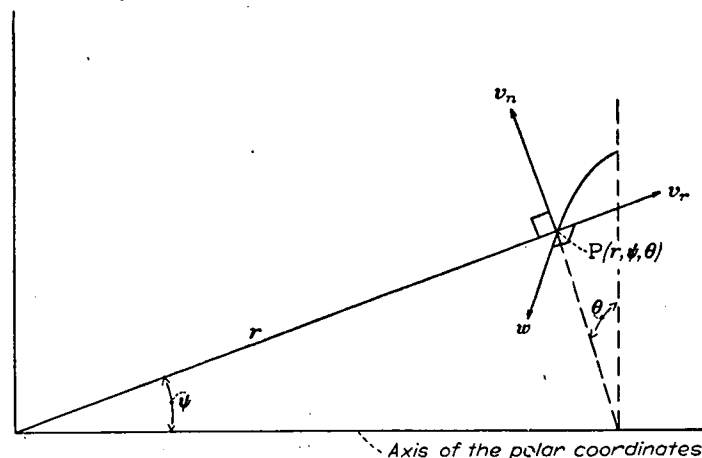


FIGURE 1.—The polar coordinate system.

Combining equations (1), (2), and (3) results in

$$v_r \left(2 - \frac{v_n^2 + w^2}{a^2} \right) + v_n \cot \psi + \frac{\partial v_n}{\partial \psi} \left(1 - \frac{v_n^2}{a^2} \right) + \frac{\partial w}{\sin \psi \partial \theta} \left(1 - \frac{w^2}{a^2} \right) - \frac{w v_n}{a^2} \left(\frac{\partial v_n}{\sin \psi \partial \theta} + \frac{\partial w}{\partial \psi} \right) = 0 \quad (4)$$

The entropy at any point of the flow can be expressed as

$$S = \frac{R}{\gamma-1} \log_e \frac{p}{p_1} \left(\frac{\rho_1}{\rho} \right)^\gamma + \text{Constant}$$

where p and ρ are the local quantities and p_1 and ρ_1 the stream quantities. Therefore,

$$\left. \begin{aligned} \frac{\gamma-1}{R} \frac{\partial S}{\partial \theta} &= \frac{1}{p} \frac{\partial p}{\partial \theta} - \gamma \frac{1}{\rho} \frac{\partial \rho}{\partial \theta} \\ \frac{\gamma-1}{R} \frac{\partial S}{\partial \psi} &= \frac{1}{p} \frac{\partial p}{\partial \psi} - \gamma \frac{1}{\rho} \frac{\partial \rho}{\partial \psi} \end{aligned} \right\} \quad (5)$$

Combining equations (5) with equations (1), (2), and (3) results in the following expressions:

$$\left. \begin{aligned} \frac{a^2}{\gamma R} \frac{\partial S}{\partial \theta} &= v_n \sin \psi \frac{\partial w}{\partial \psi} - v_r \frac{\partial v_r}{\partial \theta} - v_n \frac{\partial v_n}{\partial \theta} + v_r w \sin \psi + v_n w \cos \psi \\ \frac{a^2}{\gamma R} \frac{\partial S}{\partial \psi} &= -v_r \frac{\partial v_r}{\partial \psi} - w \frac{\partial w}{\partial \psi} + \frac{w}{\sin \psi} \frac{\partial v_n}{\partial \theta} + v_r v_n - w^2 \cot \psi \end{aligned} \right\} \quad (6)$$

Equations (6) combined with equation (1a) give

$$v_n \sin \psi \frac{\partial S}{\partial \psi} = -w \frac{\partial S}{\partial \theta} \quad (7)$$

Equation (7) is general for any conical flow and defines the lines of constant entropy, which correspond to the streamlines. In fact, if L is the streamline projection on the sphere given by $r=\text{Constant}$,

$$\frac{dS}{dL} = \frac{\partial S}{\partial \psi} \frac{d\psi}{dL} + \frac{\partial S}{\partial \theta} \frac{d\theta}{dL} = 0$$

and, from equation (7),

$$\left(\frac{d\theta}{d\psi}\right)_L = \frac{w}{v_n \sin \psi} \quad (8)$$

At the surface of any conical body the component of velocity in the direction normal to the surface is zero and the stream moves tangentially to the body; therefore, the entropy at the surface of the body must be constant or must change in a discontinuous way (in which case equations (7) and (8) are not valid).

THE PROPERTIES OF CONICAL FLOW WITHOUT AXIAL SYMMETRY

In order to analyze the properties of conical flow without axial symmetry, consider first a polar coordinate system having its axis coincident with the axis of a circular cone at a small angle of attack (fig. 2) and assume that the direction of the undisturbed velocity V_1 is in the plane $\theta=0$, $\theta=\pi$. In this case, the plane $\theta=0$, $\theta=\pi$ is a plane of symmetry of the flow and, in this plane,

$$w=0$$

$$v_n \neq 0$$

and

$$\frac{\partial v_r}{\partial \theta} = 0 \quad \frac{\partial v_n}{\partial \theta} = 0 \quad \frac{\partial S}{\partial \theta} = 0$$

Therefore, equation (7) shows that in the plane $\theta=0$, $\theta=\pi$ the entropy is constant. At the surface of the cone ($\psi=\psi_c$), the normal component v_n is zero and $w \neq 0$; therefore, equation (7) shows that the entropy remains constant also along the surface of the cone ($\psi=\psi_c$). Only at points A and B, (defined by $\theta=0$, $\theta=\pi$, and $\psi=\psi_c$) $v_n \rightarrow 0$ and $w \rightarrow 0$; therefore,

equation (7) is indeterminate. Because the body is at an angle of attack, the axis of conical shock does not remain coincident with the direction of the velocity V_1 , and the entropy in the plane $\theta=0$ must be different from the entropy in the plane $\theta=\pi$. The entropy at the cone surface therefore must be different from the entropy at the plane $\theta=0$, from the entropy at the plane $\theta=\pi$, or from both. In this case, where $w=0$ and $v_n=0$, a discontinuity of entropy must exist either at A or B or both points.

In order to find a relation between the value of the entropy at the surface of the cone and the values of the entropy in the meridian plane $\theta=0$, $\theta=\pi$, the following considerations can be used: In the meridian plane ($\theta=0$, $\theta=\pi$), w , $\frac{\partial S}{\partial \theta}$, and $\frac{\partial v_n}{\partial \theta}$ are zero because the plane is a plane of symmetry of the flow field and, from equation (7), $\frac{\partial S}{\partial \psi} = 0$. Therefore, in the plane of symmetry in the zone outside of the singular points A and B,

$$\frac{\partial}{\partial \psi} \left(\frac{\partial^2 S}{\partial \theta^2} \right) = - \frac{2}{v_n \sin \psi} \frac{\partial w}{\partial \theta} \frac{\partial^2 S}{\partial \theta^2} \quad (9)$$

For the case considered, the velocity component v_n is negative at the shock or at the Mach cone and remains negative throughout the field, until it becomes zero at the surface of the cone; therefore, the value of $\frac{\partial^2 S}{\partial \theta^2}$ tends to increase in absolute value as ψ decreases from the value corresponding to the value at the surface of the shock to the value at the surface of the cone when $\frac{\partial w}{\partial \theta}$ is negative and tends to decrease when $\frac{\partial w}{\partial \theta}$ is positive.

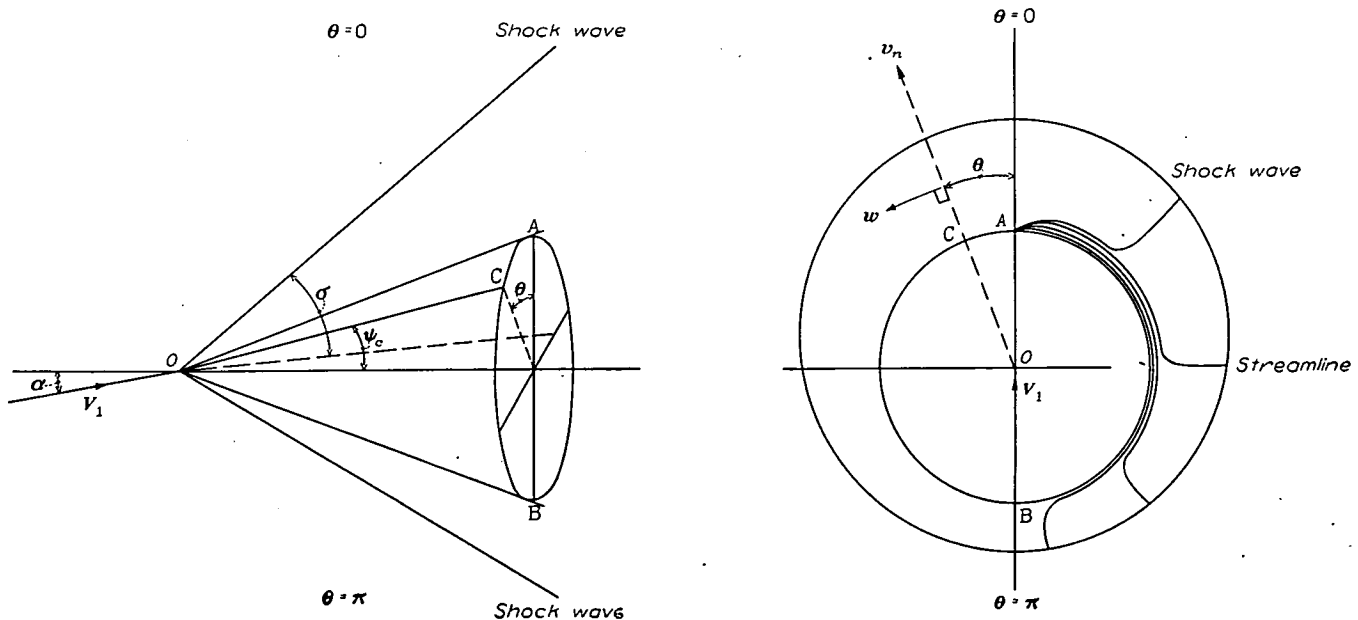


FIGURE 2.—The singular points at the surface of the cone.

Because the entropy remains constant along each streamline, the decrease of the absolute value of $\frac{\partial^2 S}{\partial \theta^2}$ as ψ decreases corresponds to a departure of projection of the streamlines on the sphere $r=\text{Constant}$ from the plane of symmetry; but, if $\frac{\partial}{\partial \psi} \left(\frac{\partial^2 S}{\partial \theta^2} \right)$ is of sign opposite to $\frac{\partial^2 S}{\partial \theta^2}$, the projection of the streamlines on the sphere $r=\text{Constant}$ tends to converge toward the plane of symmetry as ψ decreases from the value at the shock to the value at the surface of the cone.

Now, with the convention used in figure 2, the component w is negative throughout the field and is zero at $\theta=0$ and $\theta=\pi$. Therefore, $\frac{\partial w}{\partial \theta}$ is negative in the zone $\theta=0$ but is positive in the zone $\theta=\pi$, and the streamline projection tends to converge toward the zone of point A and diverge from the zone of point B. Because of the departure of the streamlines from the plane $\theta=\pi$, the entropy in zone B remains constant and, therefore, the entropy at the surface of the cone is equal to the entropy at the meridian plane $\theta=\pi$; at point A a discontinuity of entropy exists from the value corresponding to the plane $\theta=0$ to the larger value existing in the plane $\theta=\pi$. All the projections of the streamlines converge at point A where the entropy is not single-valued. Because v_n approaches zero near the cone, equation (8) shows that all the streamline projections tend to become parallel to the line $\psi=\text{Constant}$ in the zone near the cone and converge at A. The value of v_n in the meridian plane $\theta=0$ can change sign and can be positive in the neighborhood of the point A (case of large angles of attack). In this case the right-hand side of equation (9) changes sign and the singular point moves away from A in the meridian plane $\theta=0$ and occurs at the other point of the meridian plane where v_n is also zero. (At the shock or at the Mach wave v_n is negative; therefore, another singular point where $v_n=0$ must exist.)

It is interesting to observe that singular points must exist in any supersonic conical flow without axial symmetry. Considerations similar to those used for the case of circular cones at an angle of attack can be extended to other cases, and it can be shown that the streamlines that are tangent to the body start from points of the shock and meet the body at points where the component of velocity perpendicular to the radius and tangent to the body vanishes and has a positive derivative in the streamline direction (equiva-

lent to the condition of positive $\frac{\partial w}{\partial \theta}$). Convergency of streamlines occurs and, therefore, the points are singular at the points where this component vanishes and has a negative derivative in the direction tangent to the body, while the component normal to the body also vanishes and has a negative derivative in the direction normal to the body (equivalent to $v_n=0$ and $\frac{\partial v_n}{\partial \psi}$ negative).

For example, the conical body of figure 3 has two planes of symmetry, AA' and BB' when w and v_n are zero, but at the points BB', $\frac{\partial w}{\partial \theta}$ is positive, while at AA', it is negative.

Therefore, because $\frac{\partial v_n}{\partial \psi}$ is negative at AA' and BB', the points AA' are singular points and the entropy at the surface of the body is determined by the shock strength at the points CC'.

DETERMINATION OF THE FLOW AROUND CIRCULAR CONES AT SMALL ANGLES OF ATTACK

In order to determine the flow around circular cones at small angles of attack, consider a polar coordinate system, the axis of which is coincident with the body axis. At the surface of the body the velocity component v_n is zero and in the neighborhood of the body is very small; therefore, the terms v_n^2/a^2 can be neglected with respect to unity.

If the angle of attack is small, the component w is also small and the terms w^2/a^2 can also be neglected. On the basis of this approximation, in the neighborhood of the body equation (4) can be expressed as

$$2v_r + v_n \cot \psi + \frac{\partial v_n}{\partial \psi} + \frac{\partial w}{\sin \psi \partial \theta} = 0 \quad (10)$$

This equation permits a particular solution of this type chosen from physical considerations

$$\left. \begin{aligned} v_r &= v_{r_a} + \alpha v_{r_b} \cos \theta + \sum v_{r_c} \cos m\theta \\ v_n &= v_{n_a} + \alpha v_{n_b} \cos \theta + \sum v_{n_c} \cos m\theta \\ w &= \alpha w_b \sin \theta + \sum w_c \cos m\theta \end{aligned} \right\} \quad (11)$$

where $v_{r_{a,b,c}}$, $v_{n_{a,b,c}}$, and $w_{b,c}$ are functions only of ψ , are constant for constant values of ψ , and must be chosen in a form that satisfies the boundary conditions.

Consider now a conical shock having circular cross section and semiapex angle σ (fig. 4). Consider the cone inclined at an angle δ with respect to the undisturbed velocity with a polar coordinate system, the axis of which is coincident with the axis of the conical shock. If v_r , v_n , and w are the velocity components referred to the limiting velocity V_1 in the new coordinate system (r_s , ψ_s , θ_s), from the shock-wave relations the following equations result:

$$v_{r_s} = V_1 \cos \delta \cos \sigma + V_1 \sin \delta \sin \sigma \cos \theta_s$$

$$w_s = -V_1 \sin \delta \sin \theta_s$$

$$-v_{n_s} = \frac{\gamma-1}{\gamma+1} \frac{1-v_{r_s}^2-w_s^2}{V_1 \cos \delta \sin \sigma - V_1 \sin \delta \cos \sigma \cos \theta_s}$$

If δ is assumed to be small and terms of the order of δ^2 are neglected, these equations become

$$\left. \begin{aligned} v_{r_s} &= V_1 \cos \sigma + \delta V_1 \sin \sigma \cos \theta_s \\ w_s &= -V_1 \delta \sin \theta_s \\ v_{n_s} &= -\frac{\gamma-1}{\gamma+1} \left(\frac{1-V_1^2 \cos^2 \sigma}{V_1 \sin \sigma} \right) - \frac{\gamma-1}{\gamma+1} \delta \cos \sigma \left(-2V_1 + \frac{1-V_1^2 \cos^2 \sigma}{V_1 \sin^2 \sigma} \right) \cos \theta_s \end{aligned} \right\} \quad (12)$$

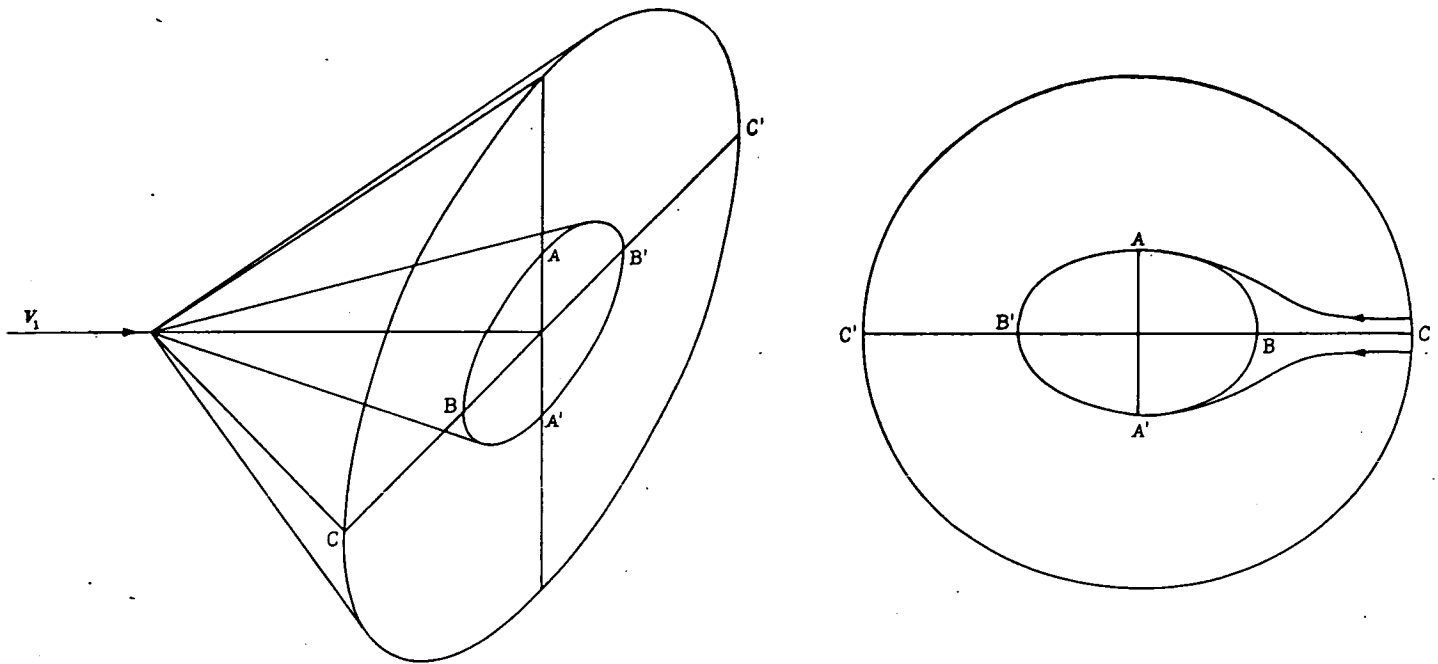


FIGURE 3.—The singular points at the surface of a cone in supersonic flow without axial symmetry.

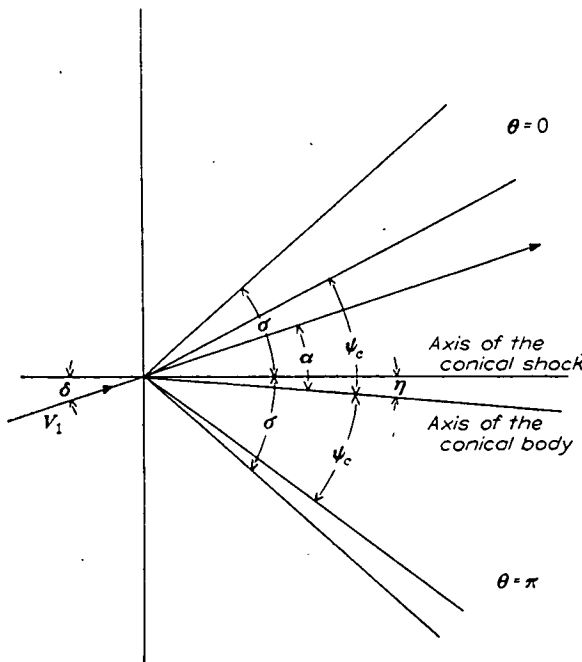


FIGURE 4.—The position of the conical shock with respect to the conical body.

The flow behind a circular conical shock inclined at a small angle δ with respect to the undisturbed stream can therefore be expressed in the form

$$\left. \begin{aligned} v_{rs} &= v_{ra_s} + \delta v_{rb_s} \cos \theta_s \\ v_{ns} &= v_{na_s} + \delta v_{nb_s} \cos \theta_s \\ w_s &= \delta w_{bs} \sin \theta_s \end{aligned} \right\} \quad (13)$$

where all the terms containing v_{rb} , v_{nb} , and w_b are small, that is, of the order of δ .

If the axis of the conical coordinates is rotated at an angle η of the same order as the angle δ and terms of the order of δ^2

are neglected, the velocity components v_r , v_n , and w referred to the new axis (fig. 4) become

$$\left. \begin{aligned} v_r &= v_{ra_s} + \left(\frac{\partial v_{ra_s}}{\partial \psi_s} \eta + \delta v_{rb_s} \right) \cos \theta \\ v_n &= v_{na_s} + \left(\frac{\partial v_{na_s}}{\partial \psi_s} \eta + \delta v_{nb_s} \right) \cos \theta \\ w &= \left(\delta w_{bs} - \frac{v_{na_s} \eta}{\sin \psi_s} \right) \sin \theta \end{aligned} \right\} \quad (14)$$

where θ , the coordinate referred to the new axis, is given by

$$\cos \theta = \cos \theta_s - \eta \cot \psi_s \sin^2 \theta_s$$

Equations (14) show that the flow behind a circular conical shock inclined at a small angle δ with respect to the direction of the undisturbed stream and at an angle η with respect to a conical body can also be expressed in the form given in equations (11) where, if the angles η and δ are small and the terms of the order of η^2 and δ^2 are neglected, only the terms having the subscripts a and b must be considered. Therefore, equations (11), when the terms with the subscript c are neglected, are valid for small angles of attack ($\alpha = \delta - \eta$), and a conical circular shock is consistent with the solution chosen for the flow around the body. At the surface of the cone the assumption that w^2 is small corresponds to the assumption that only the first-order terms of angle of attack are considered. The conical shock is inclined at an angle η with respect to the circular cone.

This analysis is similar to the analysis of references 2 and 3. No assumption, however, has been introduced for the entropy distribution; only the velocity components have been considered to be in the form of equations (11), and no limiting assumption has been introduced for the derivatives.

In reference 3, in addition to the equations

$$v_r = v_{r_a} + \alpha v_{r_b} \cos \theta \quad (15a)$$

$$v_n = v_{n_b} + \alpha v_{n_b} \cos \theta \quad (15b)$$

the expressions

$$w = \alpha w_b \sin \theta \quad (15c)$$

$$p = p_a + \alpha p_b \cos \theta \quad (16a)$$

$$\rho = \rho_a + \alpha \rho_b \cos \theta \quad (16b)$$

$$S = S_a + \alpha S_b \cos \theta \quad (16c)$$

have been used, and the derivatives of entropy, pressure, and density have been obtained from differentiation of equations (16). In this way a solution has been found which gives values of entropy that are variable along the cone surface and are constant in each meridian plane, while the entropy actually remains constant along the cone surface and changes in the meridian plane. An incorrect entropy distribution has therefore been obtained at the surface of the body.

In order to analyze in more detail the significance of equations (16) and their inconsistency with the approximation considered in references 2 and 3, consider equation (7). In the plane of symmetry $\theta=0$ or $\theta=\pi$, $\frac{\partial S}{\partial \psi}=0$; therefore, $S_a + \alpha S_b$ or $S_a - \alpha S_b$ of equation (16c) must remain constant.

Consider now the plane $\theta=\pi$ and express the entropy S in the form

$$S = S' - S'' \frac{\Delta \theta^2}{2} \quad (17)$$

which satisfies the condition of symmetry. Because of equation (7),

$$\frac{\partial S'}{\partial \psi} = 0 \quad (18)$$

and, from equations (16) and (17),

$$S' = S_a - \alpha S_b \quad (19)$$

and

$$\alpha S_b = -S'' \quad (20)$$

From equations (7) and (17),

$$v_n \sin \psi \left(\frac{\partial S'}{\partial \psi} - \frac{\partial S''}{\partial \psi} \frac{\Delta \theta^2}{2} \right) = w S'' \Delta \theta$$

However, $\frac{\partial S'}{\partial \psi} = 0$; therefore,

$$\frac{\Delta \theta}{2} \frac{\partial S''}{\partial \psi} = - \frac{w S''}{v_n \sin \psi}$$

or, since from equation (15c) $w = -\alpha w_b \Delta \theta$,

$$\frac{\partial S''}{\partial \psi} = - \frac{\alpha w_b S''}{v_n \sin \psi}$$

By use of equation (20), in the neighborhood of the plane $\theta=\pi$,

$$\alpha \frac{\partial S_b}{\partial \psi} = \frac{2 w_b S_b \alpha^2}{v_n \sin \psi} = - \frac{2 \alpha w_b S''}{v_n \sin \psi}$$

In reference 3, the term $\frac{w S''}{v_n \sin \psi}$ has been considered everywhere to be of the order of α^2 and has been neglected; hence, $\frac{\partial S_b}{\partial \psi} = 0$ and $\frac{\partial S_a}{\partial \psi} = 0$. However, neglecting this term is correct only when the ratio w_b/v_n is of an order different from $1/\alpha$ and, therefore, when $|v_n| \gg 0$. Near the surface of the cone, v_n approaches zero and, therefore, the term $\frac{\partial S''}{\partial \psi}$ becomes large and cannot be neglected. The extent of the field around the cone where the term $\frac{\partial S''}{\partial \psi}$ is of the order of α can be easily determined.

Consider a polar coordinate system having its axis coincident with the axis of the cone. At the surface of the body v_n is zero; therefore, in the neighborhood of the body the velocity component v_n can be expressed in the form

$$v_n = \left(\frac{\partial v_n}{\partial \psi} \right)_v \Delta \psi$$

or, by use of equation (10), in the form

$$v_n = - \left(2 v_r + \frac{\partial w}{\sin \psi \partial \theta} \right) (\Delta \psi)_v \quad (21)$$

Therefore, v_n is of the order of α when $(\Delta \psi)_v$ is of the order of α . In this conical layer of thickness $(\Delta \psi)_v$ of the order of α , $\frac{\partial S''}{\partial \psi}$ is also of the order of α or larger. In this layer, which can be called the vortical layer, the term $\frac{\partial S_a}{\partial \psi}$ also is of the order of α because from equations (18) and (19) $\frac{\partial S_a}{\partial \psi}$ can be shown to be of the same order as $\frac{\partial S''}{\partial \psi}$.

In order to investigate the effect of this vortical layer on the velocity and pressure distribution at the surface of the cone, consider equations (2), (4), (6), and (7). If the velocity components in the neighborhood of the meridian plane $\theta=\pi$ are expressed as

$$v_r = v_r' - \frac{\Delta \theta^2}{2} v_r'' \quad (22a)$$

$$v_n = v_n' - \frac{\Delta \theta^2}{2} v_n'' \quad (22b)$$

$$w = w'' \left(\Delta \theta - \frac{\Delta \theta^3}{3!} \right) \quad (22c)$$

the following expressions can be obtained:

$$\frac{\partial S''}{\partial \psi} = \frac{2w''S''}{v_n' \sin \psi} \quad (23)$$

$$\frac{\partial v_r''}{\partial \psi} = v_n'' - \frac{2w''}{v_n'} \left(\frac{v_r''}{\sin \psi} + w'' \right) \quad (24)$$

$$-\frac{\partial w''}{\partial \psi} = \frac{1}{v_n' \sin \psi} \left(v_r' v_r'' + v_n' v_n'' + w'' v_r' \sin \psi + v_n' w'' \cos \psi + \frac{S'' a'^2}{\gamma R} \right) \quad (25)$$

$$\frac{\partial v_n''}{\partial \psi} = \frac{1}{1 - \frac{v_n'^2}{a'^2}} \left[2B \left(\frac{v_n' v_n''}{a'^2} - A \frac{v_n'^2}{a'^2} \right) - v_n'' \cot \psi - v_r'' \left(2 - \frac{v_n'^2}{a'^2} \right) - \frac{w''}{\sin \psi} \left(\frac{2w''^2}{a'^2} - \frac{2v_n' v_n''}{a'^2} + \frac{2v_n' \sin \psi}{a'^2} \frac{\partial w''}{\partial \psi} \right) - \frac{2v_r' w''^2}{a'^2} \right] \quad (26)$$

where

$$A = \frac{-v_r' v_r'' - v_n' v_n'' + w''^2}{a'^2} \frac{\gamma - 1}{2} \quad (27)$$

$$B = -\frac{v_r' + v_n' \cot \psi + \frac{w''}{\sin \psi}}{1 - \frac{v_n'^2}{a'^2}} \quad (28)$$

and

$$\frac{\partial v_{r_a}}{\partial \psi} = v_{r_a} - \frac{a'^2}{v_{r_a}} \frac{1}{\gamma R} \frac{\partial S_a}{\partial \psi} \quad (29)$$

Therefore, all the derivatives of the velocity components of zero and first order are affected by the entropy variation in the meridian plane. The terms $\frac{\partial S_a}{\partial \psi}$ and $\frac{\partial S''}{\partial \psi}$ are of the order of α in a layer of thickness α near the surface of the cone. In this layer they change the value of $\frac{\partial p}{\partial \psi}$ of quantities of the order of α and the pressure at the surface of the cone of quantities of the order θ/α^2 . The only place where the terms $\frac{\partial S_a}{\partial \psi}$ and $\frac{\partial S''}{\partial \psi}$ are of zero order is near the surface of the cone in a layer of thickness α^2 , where v_n tends to the order θ/α^2 (equation (9)). Because the effect of this vortical layer existing at the surface of the cone on the pressure is of the order θ/α^2 , it can be considered in this approximation, which neglects terms of the order of α^2 or higher, that across this layer the pressure distribution remains constant, but an abrupt variation of entropy occurs; therefore, in this approximation the phenomenon can be represented as in references 3 and 4, where the entropy remains constant in every meridian plane until a vortical layer of infinitesimal thickness is reached at the surface of the cone across which a variation of entropy occurs from the value $S_a + \alpha S_b \cos \theta$ to the value $S_a - \alpha S_b$, that exists at the surface of the cone. Across the layer a variation of density and velocity components occurs and can be easily determined.

Let v_{r_e} , w_e , S_e , p_e , ρ_e , and a_e be the quantities at the external surface of the layer (these are the quantities tabulated in

reference 4) and v_{r_c} , w_c , S_c , p_c , and ρ_c be the quantities at the surface of the cone. Because it has been shown that

$$p_e = p_c \quad (30)$$

then

$$v_{r_e}^2 + w_e^2 - v_{r_c}^2 - w_c^2 = -\frac{2}{\gamma - 1} a_e^2 \frac{S_e - S_c}{c_p} \quad (31)$$

where $S_e - S_c$ is the entropy jump across the layer.

Now,

$$v_r = v_{r_a} + \alpha v_{r_b} \cos \theta$$

$$w = \alpha w_b \sin \theta$$

and in the plane $\theta = \pi$

$$v_{r_e} = v_{r_c}$$

$$w_e = w_c = 0$$

$$S_e = S_c$$

or

$$\left. \begin{aligned} (v_{r_a} - \alpha v_{r_b})_e &= (v_{r_a} - \alpha v_{r_b})_c = v_r' \\ (S_a - \alpha S_b)_e &= (S_a - \alpha S_b)_c = S' \end{aligned} \right\} \quad (32)$$

Therefore, if terms of the order of α^2 are neglected, equation (31) becomes

$$v_r'(v_{r_b})_e - v_r'(v_{r_b})_c = -\frac{1}{\gamma - 1} a^2 \frac{S_b}{c_p} \quad (33)$$

or

$$(v_{r_b})_c - (v_{r_b})_e = \frac{a^2}{(\gamma - 1)v_r'} \frac{S_b}{c_p} \quad (34)$$

However, from equation (1a), at the surface of the cone,

$$w = \frac{\partial v_r}{\sin \psi \partial \theta}$$

or

$$w_b = -\frac{v_{r_b}}{\sin \psi_c} \quad (35)$$

therefore, the values of v_{r_b} and w_b can be determined from

equation (34) and from the tables of reference 4 where

$$S_b = c_v \left(\frac{p_b}{p_a} - \gamma \frac{\rho_b}{\rho_a} \right) \quad (36)$$

and

$$\alpha^2 = \frac{\gamma-1}{2} (1 - v_r'^2) \quad (37)$$

The values v_{r_a} , v_{r_b} , and w_b having been determined, the values of v_r and w can be obtained at any part of the cone from equations (15).

THE NUMERICAL DETERMINATION OF THE FLOW FIELD AROUND CONES AT SMALL ANGLES OF ATTACK

The method presented permits the determination of the pressure distribution around the cones with the assumption that the terms of the order of α^2 can be neglected. The pressure at any point can be obtained from the equation

$$\frac{p}{p_1} = \left(\frac{1-V^2}{1-V_1^2} \right)^{\frac{\gamma}{\gamma-1}} e^{-\frac{1}{\gamma-1} \frac{\Delta S}{c_v}} \quad (38)$$

where V is the local velocity corresponding to the pressure p and ΔS is the increase of entropy with respect to the stream conditions where p_1 and V_1 exist. From equations (15), V^2 is found to be

$$V^2 = v_{r_a}^2 + \alpha^2 v_{r_b}^2 \cos^2 \theta + 2\alpha v_{r_a} v_{r_b} \cos \theta + \alpha^2 w_b^2 \sin^2 \theta \quad (39)$$

However, in equation (39) the terms of the order of α^2 having the form $2v_{r_a} v_{r_b} \cos m\theta$ have been neglected (see equations (11)). If all the terms of the order of α^2 are neglected, V^2 becomes

$$V^2 = v_{r_a}^2 + 2\alpha v_{r_a} v_{r_b} \cos \theta \quad (40)$$

Equations (39) and (40) are different in terms of the order of α^2 ; however, for finite angles of attack good agreement is obtained only if equation (39) in which some of the α^2 terms are retained is used in equation (38).

The reason for the better approximation given by equation (39) can be understood if the magnitude of the terms containing α^2 in the expression of V^2 is considered. Along the surface of the cone, αw_b is given by

$$\alpha w_b = -\frac{\alpha v_{r_b}}{\sin \psi_c} \quad (41)$$

For a finite value of α and a small value of ψ_c , $\alpha^2 w_b^2$ is of the same order as αv_{r_b} because $\sin^2 \psi_c$ is also small; therefore, $\alpha^2 w_b^2$ can have an effect on the velocity and pressure distribution of the same order as the term $\alpha v_{r_a} v_{r_b}$ which is the

only term retained in equation (40). Because the term $\alpha^2 w_b^2$ is correct and significant, it can still be retained also when other terms in α^2 are neglected and, therefore, equation (39) is the expression that must be used for finite angle of attack. (For example, for a 10° cone ($\psi_c = 10^\circ$) $\sin^2 \psi_c = \frac{1}{33}$ and $\frac{\alpha}{\sin^2 \psi_c} = 1$ for $\alpha = 1.75^\circ$, which can be considered a small angle.)

In reference 3, equation (16a) has been used in the derivation of the method; however, the use of this equation is not necessary, as is briefly shown in the following paragraph.

Consider a conical circular shock having the semiapex angle σ and inclined at an angle δ with respect to the undisturbed stream. In the neighborhood of the plane $\theta = \pi$ the velocity components can be expressed as in equations (22) and at the shock are given by (from equations (12))

$$v_r' = V_1 \cos \sigma - V_1 \delta \sin \sigma \quad (42)$$

$$v_r'' = -\delta V_1 \sin \sigma \quad (43)$$

$$w'' = V_1 \delta \quad (44)$$

$$v_n' = -\frac{\gamma-1}{\gamma+1} \frac{1-V_1^2 \cos^2 \sigma}{V_1 \sin \sigma} + \frac{\gamma-1}{\gamma+1} \delta \cos \sigma \left(-2V_1 + \frac{1-V_1^2 \cos^2 \sigma}{V_1 \sin^2 \sigma} \right) \quad (45)$$

$$v_n'' = \frac{\gamma-1}{\gamma+1} \delta \cos \sigma \left(-2V_1 + \frac{1-V_1^2 \cos^2 \sigma}{V_1 \sin^2 \sigma} \right) \quad (46)$$

and S' and S'' are given by

$$S' = c_v \log_e \left\{ 1 + \left(\frac{\gamma-1}{2} M_1^2 + 1 \right) [V_1^2 \sin^2 (\sigma + \delta) - v_n'^2] \right\} + (c_p - c_v) \log_e \frac{-v_n'}{V_1 \sin (\sigma + \delta)} \quad (47)$$

$$S'' = 2\delta \frac{V_1^2 \cos \sigma \sin \sigma - v_n' \frac{v_n''}{\delta}}{1 + \frac{\gamma-1}{2} M_1^2 + V_1^2 \sin^2 \sigma - v_n'^2} - (c_p - c_v) \delta \left(\frac{v_n''/\delta}{v_n'} + \frac{\cos \sigma}{\sin \sigma} \right) \quad (48)$$

All the velocity components are referred to the limiting velocity, and V_1 is the undisturbed velocity also referred to the limiting velocity.

In the meridian plane $\theta = \pi$ the entropy S' is constant;

therefore when $w=0$, from equation (1a),

$$\frac{\partial v_r'}{\partial \psi} = v_n' \quad (49)$$

If equation (4) is applied to the meridian plane $\theta=\pi$ and equation (21) is used, the radius of the hodograph diagram at any value of ψ smaller than σ is given by (reference 6)

$$(R)_\psi = - \left[\frac{v_r' + v_n' \cot \psi + \frac{w''}{\sin \psi}}{1 - \frac{v_n'^2}{a'^2}} \right] \quad (50)$$

where

$$\frac{2}{\gamma-1} a'^2 = 1 - v_r'^2 - v_n'^2 \quad (51)$$

Therefore, at the point $\psi-\Delta\psi$ in the meridian plane $\theta=\pi$,

$$(v_r')_{\psi-\Delta\psi} = -(v_n')_\psi \sin \Delta\psi + (v_r' - R)_\psi \cos \Delta\psi + (R)_\psi \quad (52)$$

and

$$(v_n')_{\psi-\Delta\psi} = (v_n')_\psi \cos \Delta\psi + (v_r' - R)_\psi \sin \Delta\psi \quad (53)$$

From equations (50) to (53), the velocity components v_r' and v_n' can be determined if the component w'' is known in the meridian plane $\theta=\pi$. Since $\frac{\partial w''}{\partial \psi}$ is given by equation (25), the value of w'' can be determined for any value of ψ . But v_{ra} and v_{na} are the quantities obtained for zero angle of attack at the same coordinate ψ of a coordinate system in axis with the conical shock, and

$$v_r'' = v_r' - v_{ra}$$

$$v_n'' = v_n' - v_{na}$$

therefore, the entire flow field can be determined until v_n'' becomes 0. The value of ψ for which $v_n''=0$ corresponds to $\psi_c + \alpha - \delta$ and, therefore, gives the value of α .

Considerations similar to those used for cones can be used for the characteristics method presented in reference 6. In this case the pressure at the surface of the body can be obtained from the complete equations, and the vortical layer must be considered in order to obtain the correct distribution of entropy. The application is the same because the entropy does not change along each streamline.

COMPARISON WITH EXPERIMENTAL RESULTS

In order to have an indication of the accuracy that can be expected from the first-order theory, theoretical results have been compared with some experimental results available. The theoretical results have been obtained by using the

values of reference 4 for the conditions outside the vortical layer, and the pressure at the surface of the cone has been determined in the following way:

From tables of reference 4 the value of δ/α has been determined (δ/ϵ of tables). The position ψ in the plane $\theta=\pi$ of the conical body in the shock coordinate system is

$$\psi = \psi_c + \alpha - \delta \quad (54)$$

The value of v_{ra} at $\psi_c + \alpha - \delta$ has been determined from the tables of reference 4 (v_r at ψ is given by $u_s - 2u_s \frac{(\alpha-\delta)^2}{2}$).

The value of v_r'' has been obtained from the tables in shock coordinates

$$[(v_{ra})_c]_\psi = -(x)_\psi - \left(\frac{\partial v_{ra}}{\partial \psi} \right)_\psi \frac{\delta}{\alpha} = - \left[x_{\psi_c} + (v_{na})_\psi \frac{\delta}{\alpha} \right] \quad (55)$$

Then $(v_{ra})_c$, $(v_{rb})_c$, and $(w_b)_c$ in the body coordinate system have been obtained by means of equations (32), (34), and (35).

From equation (36) S_b has been obtained (it has a negative value), and from tables of reference 7 S_a can be determined—for example, from the value of the angle of the shock obtained from reference 7—and

$$\Delta S = S_a - \alpha S_b \quad (56)$$

The pressure has been obtained from equation (38).

In figure 5 a comparison is presented for a cone of $\psi_c = 7.5^\circ$ at $M=1.6$ and four angles of attack. The experimental data are obtained from tests performed in the Langley 4- by 4-foot supersonic tunnel. For comparison the values given by reference 4 and by linear theory are also shown. In figure 6 a comparison is presented for a cone of $\psi_c = 10^\circ$ at $M=6.86$ and two angles of attack. The experimental data have been obtained from tests performed at the Langley 11-inch hypersonic tunnel. The agreement in both cases is good, even at angles of attack where it would be expected that higher-order terms would be important.

CONCLUDING REMARKS

The flow around cones without axial symmetry at supersonic velocity has been analyzed. Singular points which complicate the analysis of the flow field were shown to exist in the flow. The results of the analysis were applied to the determination of the flow around circular cones at an angle of attack. The concept of a vortical layer around the cone at small angles of attack has been introduced, and the correct values of the first-order terms of the velocity components were determined.

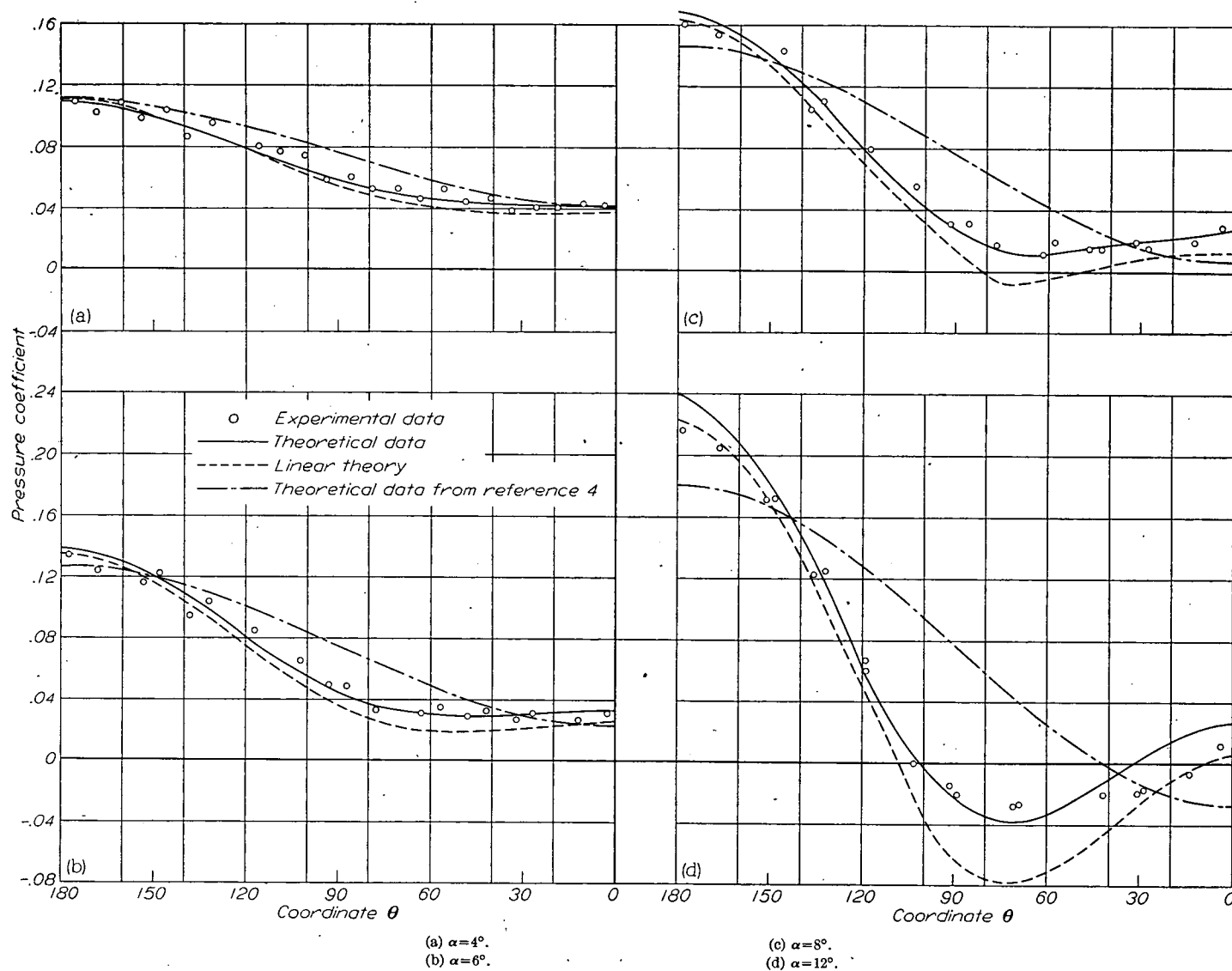


FIGURE 5.—Comparison of experimental pressure distribution over the surface of a cone with the theoretical pressure distribution. $\psi_s = 7.5^\circ$; $M = 1.6$. (Experimental data obtained from the Langley 4- by 4-foot supersonic tunnel.)

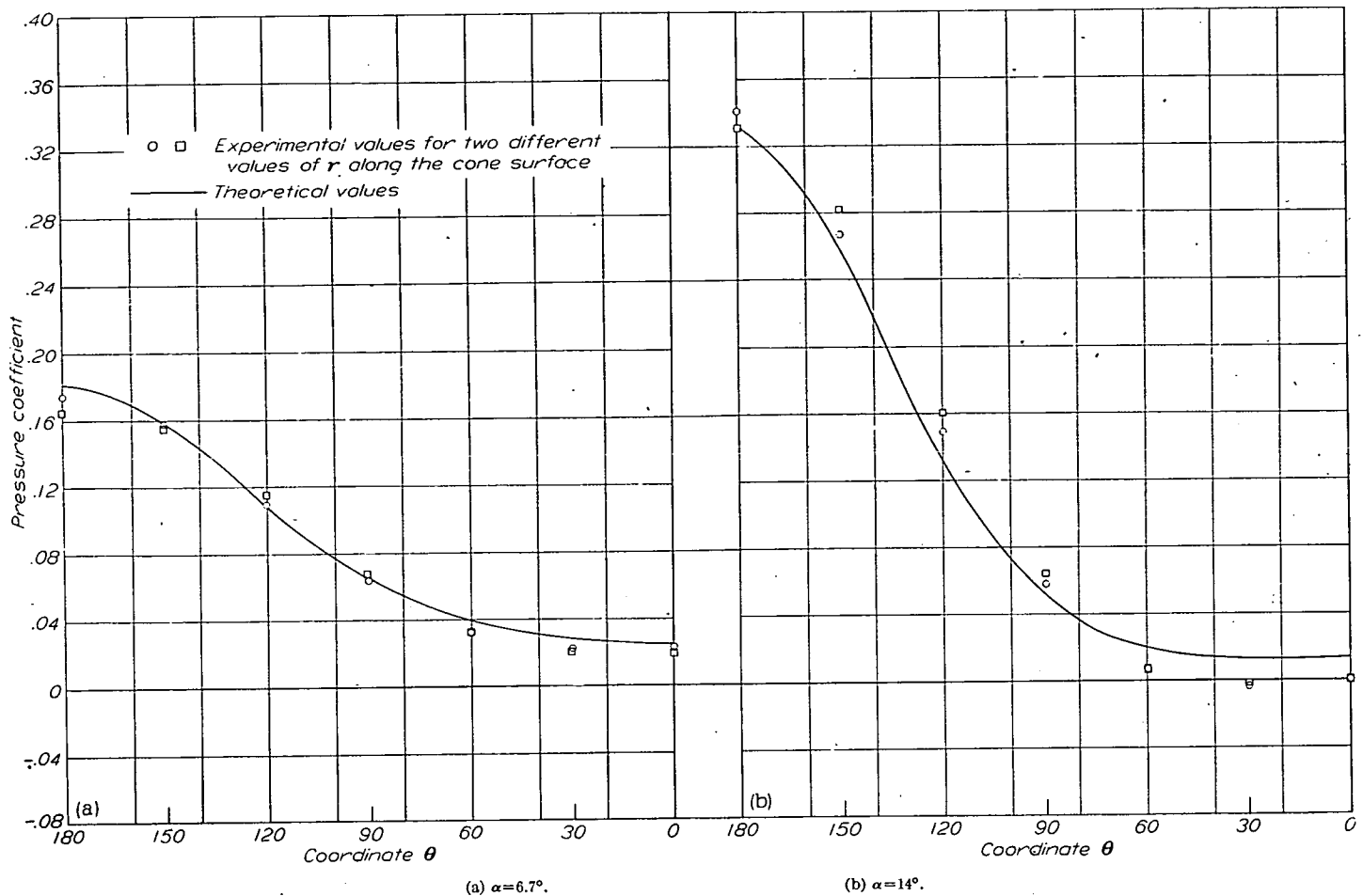


FIGURE 6.—Comparison of experimental pressure distribution over the surface of a cone with the theoretical pressure distribution. $\psi_c = 10^\circ$; $M = 6.86$. (Experimental values for two different values of r along the cone surface; data obtained from the Langley 11-inch hypersonic tunnel.)

The method determined was applied to cones at finite angle of attack, and it is shown that good agreement with experimental results can be obtained from the first-order theory if the complete equation for the pressure distribution is used. The analysis can be extended to the application of the characteristics method around bodies of revolution at small angles of attack.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., September 26, 1950.

REFERENCES

1. Ferrari, C.: Determination of the Pressure Exerted on Solid Bodies of Revolution with Pointed Noses Placed Obliquely in a Stream of Compressible Fluid at Supersonic Velocity. R.T.P. Translation No. 1105, British Ministry of Aircraft Production. (From Atti R. Accad. Sci. Torino, vol. 72, Nov.-Dec. 1936, pp. 140-163.)
2. Sauer, Robert: Supersonic Flow about Projectile Heads of Arbitrary Shape at Small Incidence. R.T.P. Translation No. 1573, British Ministry of Aircraft Production. (From Luftfahrtforschung, vol. 19, no. 4, May 1942, pp. 148-152.)
3. Stone, A. H.: On Supersonic Flow past a Slightly Yawing Cone. Jour. Math. and Phys., vol. XXVII, no. 1, April 1948, pp. 67-81.
4. Staff of the Computing Section, Center of Analysis (Under Direction of Zdeněk Kopal): Tables of Supersonic Flow around Yawing Cones. Tech. Rep. No. 3, M.I.T., 1947.
5. Staff of the Computing Section, Center of Analysis (Under Direction of Zdeněk Kopal): Tables of Supersonic Flow around Cones of Large Yaw. Tech. Rep. No. 5, M.I.T., 1949.
6. Ferri, Antonio: The Method of Characteristics for the Determination of Supersonic Flow over Bodies of Revolution at Small Angles of Attack. NACA Rep. 1044, 1951. (Supersedes NACA TN 1809.)
7. Staff of the Computing Section, Center of Analysis (Under Direction of Zdeněk Kopal): Tables of Supersonic Flow around Cones. Tech. Rep. No. 1, M.I.T., 1947.

REPORT 1046

A GENERAL INTEGRAL FORM OF THE BOUNDARY-LAYER EQUATION FOR INCOMPRESSIBLE FLOW WITH AN APPLICATION TO THE CALCULATION OF THE SEPARATION POINT OF TURBULENT BOUNDARY LAYERS¹

By NEAL TETERVIN and CHIA CHIAO LIN

SUMMARY

A general integral form of the boundary-layer equation is derived from the Prandtl partial-differential boundary-layer equation. The general integral equation, valid for either laminar or turbulent incompressible boundary-layer flow, contains the Von Kármán momentum equation, the kinetic-energy equation, and the Loitsianskii equation as special cases.

In an attempt to obtain a practical method for the calculation of the development of the turbulent boundary layer, use is made of the experimental finding that all the velocity profiles of the turbulent boundary layer form essentially a single-parameter family. The general equation is thereby changed to a simpler one from which an equation for the space rate of change of the shape parameter of the turbulent boundary layer can be obtained.

The resulting equation for the space rate of change of the velocity-profile parameter is restricted by the assumption that the velocity profiles of the turbulent boundary layer can be approximated by power profiles. Two of the resulting equations are used to calculate the distribution of the profile shape parameter over an airfoil for one experimentally determined pressure distribution. Although different assumptions were tried for the shearing stress across the boundary layer, the calculated distribution of the profile shape parameter did not agree exactly with the experimental distribution.

An examination is made of the effect of using the experimentally determined single-parameter family of velocity profiles instead of the power profiles on certain functions that occur in the equation for the space rate of change of the velocity-profile parameter. One calculation of the distribution of the profile shape parameter over an airfoil is also made for the experimentally determined pressure distribution by using the single-parameter family of velocity profiles found from experiment. A comparison of the results with those of a calculation made with the same assumptions except for the use of power profiles shows some difference near the separation point. It is believed, however, that the apparent lack of reliability of the specific equations used to make the calculations is caused mainly by the lack of precise knowledge concerning the surface shear and the distribution of the shearing stress across the turbulent boundary layer. The present analysis emphasizes the need for information concerning the shearing stresses in turbulent boundary layers.

INTRODUCTION

An outstanding problem in aerodynamic theory is to calculate whether the flow will separate from the surface of a specific body and, if so, where the separation will occur. The concept of the boundary layer and the equations that describe the flow in it, introduced by Prandtl (reference 1) and first worked out in some detail by Blasius (reference 2), reduce the problem to solving the Prandtl boundary-layer equation when the flow is laminar. Because of the mathematical difficulty of solving the equation, approximate methods were developed for the calculation of the properties of the laminar boundary layer (reference 3). In some of these methods, for example, the Pohlhausen method (reference 3) and the Wiegardt method (reference 4), a functional form is chosen for the velocity distribution through the boundary layer and is combined with either the Von Kármán momentum equation alone (reference 5) or with both the Von Kármán momentum equation and the kinetic-energy equation (reference 4). The result is the replacement of the Prandtl partial-differential equation by one ordinary differential equation in the Pohlhausen method and by two ordinary differential equations in the Wiegardt method. A solution of the ordinary differential equation or equations provides the boundary-layer velocity profiles along the body. These and other approximate methods that use only the Von Kármán momentum equation, or the momentum and kinetic-energy equations together, do not satisfy exactly the Prandtl boundary-layer equation.

Because the flow in the boundary layer is more often turbulent than laminar in cases encountered in engineering, the problem of calculating the separation point is of even more importance for turbulent than for laminar boundary layers. In spite of the importance of the problem, however, less progress has been made in the development of methods for the calculation of the behavior of turbulent boundary layers than for laminar boundary layers. The lack of progress stems from the absence of an explicit independent equation for the shearing stress that is accurate enough to lead to a description of the flow when used with the Prandtl equation.

The main attempts to obtain methods for the calculation of the behavior of the incompressible turbulent boundary

¹Supersedes NACA TN 2158, "A General Integral Form of the Boundary-Layer Equation for Incompressible Flow with an Application to the Calculation of the Separation Point of Turbulent Boundary Layers" by Neal Tetervin and Chia Chiao Lin, 1950.

layer in the presence of pressure gradients are those of references 6 to 12. The results of these attempts are unsatisfactory either because the assumptions upon which they rest are incorrect or because the equations used to make calculations were not derived from the boundary-layer equations.

The analysis of reference 6 is based on the assumption that the velocity profile is a single-valued function of the ratio of the pressure gradient to the skin friction, an assumption shown to be incorrect by later investigators (for example, see reference 12). In the analyses of references 7, 10, 11, and 12 the momentum equation is used, together with an auxiliary equation, to calculate the distribution of velocity profiles over a surface. In each of these four methods the auxiliary equation is not derived from the boundary-layer equations but is empirical.

In reference 8, the equation that gives the variation of the mixing length across a pipe (reference 3) was used to calculate the velocity profiles. The fact that the mixing-length distribution across the boundary layer is not the same as across pipes is shown in references 13 to 15.

Reference 9 does not provide a method for the calculation of the distribution of turbulent velocity profiles along a surface. It does, however, suggest that separation of the turbulent boundary layer always occurs when the numerical value of the nondimensional pressure gradient reaches an empirical constant.

The purpose of the present investigation is to begin with the boundary-layer equation for incompressible flow and to proceed as closely to a method for the calculation of the behavior of the turbulent boundary layer as the present knowledge of the turbulent boundary layer permits.

At first it might appear that the use of empirical auxiliary equations in methods for the calculation of the behavior of turbulent boundary layers can be avoided by developing a method similar to the Pohlhausen method which requires the solution of only the Von Kármán momentum equation. For turbulent flow, however, in contrast with laminar flow, the conditions on the behavior of the velocity profile at the surface that can be obtained from the boundary-layer equation seem to be of little or no use for the determination of the shape of the velocity profile across the boundary layer. This difference between laminar and turbulent flow makes inapplicable the Pohlhausen process in which a type of function is chosen to represent the velocity profiles, the function for the velocity profiles is combined with the Von Kármán momentum equation, and the resulting ordinary differential equation for the space rate of change of the profile shape parameter is solved.

An auxiliary equation for the calculation of the behavior of the turbulent boundary layer can, however, be obtained from the boundary-layer equation by making use of the

experimentally verifiable fact (references 7, 10, 11, 14, and 15) that all velocity profiles of the turbulent boundary layer form essentially a single-parameter family of curves. In the present analysis the Loitsianskii equation (reference 16) is generalized by multiplying the Prandtl boundary-layer equation not only by an arbitrary power of the velocity in the boundary layer but also by an arbitrary power of the distance from the surface. The resulting equation is then integrated across the boundary layer and provides a general integral form of the boundary-layer equation, valid for either laminar or turbulent flow. This general integral form of the boundary-layer equation reduces to the Loitsianskii equation when the distance from the surface is raised to the zeroth power, to the Von Kármán momentum equation when both the distance from the surface and the velocity are raised to the zeroth power, and to the kinetic-energy equation when the distance from the surface is raised to the zeroth power and the velocity is raised to the first power.

When use is made of the assumption of a single-parameter family of velocity profiles, the general integral form of the boundary-layer equation becomes a general equation for the rate of change along the surface of the velocity-profile shape parameter. This equation for the rate of change of the velocity-profile shape parameter is the desired auxiliary equation.

The assumption of the single-parameter family of velocity profiles changes the problem from one of finding a solution of a partial-differential equation, the Prandtl boundary-layer equation, to one of finding a solution of two simultaneous ordinary differential equations, the equation for the rate of change of the shape parameter and the Von Kármán momentum equation. The differential equation for the rate of change of the shape parameter, however, cannot result in a solution of the problem in the present analysis because a knowledge of the shearing stress is lacking. In the present analysis various assumptions are made for the distribution of shearing stress through the boundary layer, and the distribution of the shape parameter over the surface of an airfoil is then calculated. Because of the arbitrary assumptions for the shear distribution and the use of a flat-plate skin-friction formula, precise agreement between the calculated and experimentally obtained distributions of the shape parameter is not obtained.

The problem of finding the shearing stress in the turbulent boundary layer remains. It is believed, however, that, if suitable approximations are found for the shear and surface friction, the equations presented herein should enable the development of the turbulent boundary layer to be calculated with an accuracy sufficient for engineering purposes.

The present work was begun while Dr. Lin was temporarily at the Langley Laboratory and was continued by correspondence.

SYMBOLS

A	arbitrary positive integer in shear polynomial
a, b, c	coefficients in polynomial for $\int_0^1 \xi \frac{\partial g}{\partial \xi} d\xi$
B	exponent in expression for shear
c	reference chord
F_0, F_1	functional notation
$f = \frac{u}{U}$	
$g = \frac{\tau}{\tau_0}$	
g_1	derivative of shear polynomial for $\lambda=0$
g_2	coefficient of λ in expression for $\frac{\partial g}{\partial \xi}$
$\dot{H} = \frac{\delta^*}{\theta}$	
H_0	equilibrium value of H for $\omega=0$
$I = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1} (1-f^{m+1}) \left(\int_0^y \frac{\partial f}{\partial H} dy \right) dy$	
$I_1 = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y \frac{\partial(1-f)}{\partial x} dy \right] dy$	
$J = \frac{1}{\theta^{n+1}} \int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y (1-f) dy \right] dy$	
j, q, l	coefficients in polynomial for $\int_0^1 \xi^{\frac{H-1}{2}} \frac{\partial g}{\partial \xi} d\xi$
K	ratio of kinetic-energy thickness to momentum thickness $\left(\frac{1}{\theta} \int_0^\delta (1-f^2) f dy \right)$
$K' = \frac{dK}{dH}$	
k	function of H $\left(\frac{-(H-1)K}{K'} \right)$
$L = \frac{1}{\theta^{n+1}} \int_0^\delta f(1-f^{m-1}) y^n dy$	
$M = \frac{1}{\theta^{n+1}} \int_0^\delta y^n (1-f^{m+1}) dy$	
m	exponent of u in derivation of general equation
$N = \frac{1}{\theta^{n+1}} \int_0^\delta y^n (1-f^{m+1}) f dy$	
$N' = \frac{dN}{dH}$	
n	exponent of y in derivation of general equation
P	coefficient of ω in equation for $\theta \frac{dH}{dx}$
p	exponent in equation for power profiles $f = \xi^p$
p_1	static pressure
$Q = \frac{1}{\theta^n} \int_0^\delta y^{n-1} (1-f^{m+1}) dy$	

$R_\theta = \frac{\theta U \rho}{\mu}$	
r_0	radius of body of revolution
S	coefficient of ϕ in equation for $\theta \frac{dH}{dx}$
U	velocity parallel to surface and at outer edge of boundary layer
u	velocity parallel to surface and inside boundary layer, positive in direction of positive x
v	velocity perpendicular to surface and inside boundary layer, positive in direction of positive y
v_0	value of v at $y=0$
x	coordinate parallel to surface, positive in direction from leading to trailing edge
y	coordinate perpendicular to surface, positive outward from surface
δ	smallest value of y for which the difference $U-u$ is negligible
δ^*	displacement thickness $\left(\int_0^\delta (1-f) dy \right)$
$\xi = \frac{y}{\delta}$	
$\eta = \frac{y}{\theta}$	
θ	momentum thickness $\left(\int_0^\delta f(1-f) dy \right)$
$\lambda = \frac{\delta}{\tau_0} \frac{dp_1}{dx}$	
μ	viscosity
$\xi = -\frac{\omega}{\phi}$	
$\xi_0 = \frac{1}{H-1} \left(1 - \frac{2}{K} \int_0^1 f dg \right)$	
ρ	density
τ	shearing stress
τ_0	surface shearing stress
$\phi = \frac{\tau_0}{\rho U^2}$	
$\psi = \frac{v_0}{U}$	
$\omega = \frac{\theta}{U} \frac{dU}{dx}$	

ANALYSIS

DERIVATION OF GENERAL EQUATION

The general equation is derived for the body of revolution because the equation for two-dimensional flow can be obtained from this equation by letting the radius of a transverse section of the body of revolution become infinite.

The boundary-layer equation of motion for the body of revolution, also valid for two-dimensional flow, from reference 3 is

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{dp_1}{dx} + \frac{1}{\rho} \frac{\partial \tau}{\partial y} \quad (1)$$

After multiplying through by u^m , making use of the equation of continuity that is valid in the boundary layer of a body of revolution (reference 3)

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{r_0} \frac{dr_0}{dx} = 0$$

and noting that

$$\rho U \frac{dU}{dx} = -\frac{dp_1}{dx}$$

equation (1) becomes

$$\begin{aligned} \frac{u^{m+1}}{m+1} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{u}{r_0} \frac{dr_0}{dx} \right) + \frac{u}{m+1} \frac{\partial u^{m+1}}{\partial x} + \frac{v}{m+1} \frac{\partial u^{m+1}}{\partial y} \\ = u^m \left(U \frac{dU}{dx} + \frac{1}{\rho} \frac{\partial \tau}{\partial y} \right) \end{aligned} \quad (2)$$

After equation (2) is written in a form in which each term vanishes at the outer edge of the boundary layer, each term of the equation is multiplied by y^n and integrated from $y=0$ to $y=\delta$. The result (see appendix A for detailed development) is

$$\begin{aligned} (n+1)N \frac{d\theta}{dx} + \theta \left(\frac{dN}{dx} - nI_1 \right) + \frac{\theta}{U} \frac{dU}{dx} [N(m+2) - n(J-M) - \\ L(m+1)] + \frac{\theta}{r_0} \frac{dr_0}{dx} [N - n(J-M)] - nQ \frac{v_0}{U} \\ = -(m+1) \frac{\tau_0}{\rho U^2} \int_0^{\delta/\theta} f^m \eta^n \frac{\partial \tau}{\partial \eta} d\eta \end{aligned} \quad (3)$$

Equation (3) is the general integral form of the boundary-layer equation.

The Von Kármán momentum equation is obtained from equation (3) by letting $m=0$ and $n=0$, the equation for kinetic energy is obtained by letting $m=1$ and $n=0$, and the equation for moment of momentum is obtained by letting $m=0$ and $n=1$.

In the case $m=n=0$,

$$N\theta = \int_0^\delta (1-f)f dy = \theta$$

or

$$N=1$$

Also,

$$L\theta = \int_0^\delta (f-1) dy = -\delta^*$$

or

$$L = -\frac{\delta^*}{\theta} = -H$$

It can be easily verified that all the integrals, except Q , involved in equation (3) have finite integrands as n approaches 0. The limit nQ , however, approaches unity as

n approaches 0; thus

$$\begin{aligned} nQ\theta^n &= n \int_0^\delta y^{n-1} (1-f^{m+1}) dy \\ &= \left[y^n (1-f^{m+1}) \right]_0^\delta - \int_0^\delta y^n \frac{\partial}{\partial y} (1-f^{m+1}) dy \end{aligned}$$

The first term drops out if $\dot{n} \neq 0$. Then, by taking the limit $n \rightarrow 0$, the result is

$$\lim_{n \rightarrow 0} nQ\theta^n = 1$$

Hence, when $m=n=0$, equation (3) becomes

$$\frac{d\theta}{dx} + \frac{\theta}{U} \frac{dU}{dx} (H+2) + \frac{\theta}{r_0} \frac{dr_0}{dx} - \frac{v_0}{U} = \frac{\tau_0}{\rho U^2} \quad (4)$$

Equation (4) is the momentum equation for flow over a body of revolution with flow through the surface. For two-dimensional flow, equation (4) becomes

$$\frac{d\theta}{dx} + \frac{\theta}{U} \frac{dU}{dx} (H+2) - \frac{v_0}{U} = \frac{\tau_0}{\rho U^2} \quad (5)$$

when the value of $\frac{d\theta}{dx}$ from equation (4) is substituted into equation (3), the result is

$$\begin{aligned} \theta \left(\frac{dN}{dx} - nI_1 \right) + \frac{\theta}{U} \frac{dU}{dx} \{ m(N-L) - n[(J-M) + N(H+2)] - \\ L - NH \} - \frac{\theta}{r_0} \frac{dr_0}{dx} n[(J-M) + N] - \frac{v_0}{U} [nQ - N(n+1)] \\ = -\frac{\tau_0}{\rho U^2} \left[(m+1) \int_0^{\delta/\theta} f^m \eta^n \frac{\partial g}{\partial \eta} d\eta + (n+1)N \right] \end{aligned} \quad (6)$$

where

$$g = \frac{\tau}{\tau_0}$$

The assumptions contained in equation (6) are the usual boundary-layer assumptions. Equation (6) is valid for both laminar and turbulent flow.

FORM OF EQUATION (6) FOR SINGLE-PARAMETER FAMILY OF VELOCITY PROFILES

Equation (6) is now to be placed in a form valid when the velocity profiles form a single-parameter family of curves ($f=f(\eta, H)$). For this purpose the term I_1 of equation (6) is modified in the following manner:

By definition,

$$I_1 \dot{\theta}^{n+1} = \int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y \frac{\partial (1-f)}{\partial x} dy \right] dy$$

Because f depends only on η and H ,

$$\frac{\partial (1-f)}{\partial x} = \frac{\partial (1-f)}{\partial \eta} \frac{\partial \eta}{\partial x} + \frac{\partial (1-f)}{\partial H} \frac{dH}{dx}$$

From the definition of η

$$\frac{\partial \eta}{\partial x} = -\frac{\eta}{\theta} \frac{d\theta}{dx}$$

then

$$\int_0^y \frac{\partial(1-f)}{\partial x} dy = -\frac{1}{\theta} \frac{d\theta}{dx} \int_0^y \eta \frac{\partial(1-f)}{\partial \eta} dy + \frac{dH}{dx} \int_0^y \frac{\partial(1-f)}{\partial H} dy$$

or, after an integration by parts of the first term on the right-hand side, the result is

$$\int_0^y \frac{\partial(1-f)}{\partial x} dy = -\frac{1}{\theta} \frac{d\theta}{dx} \left[y(1-f) - \int_0^y (1-f) dy \right] - \frac{dH}{dx} \int_0^y \frac{\partial f}{\partial H} dy$$

The expression for $I_1 \theta^{n+1}$ then becomes

$$I_1 \theta^{n+1} = -\frac{1}{\theta} \frac{d\theta}{dx} \int_0^\delta y^n (1-f^{m+1}) dy + \frac{1}{\theta} \frac{d\theta}{dx} \int_0^\delta y^n f (1-f^{m+1}) dy + \frac{1}{\theta} \frac{d\theta}{dx} \int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y (1-f) dy \right] dy - \frac{dH}{dx} \int_0^\delta y^{n-1} (1-f^{m+1}) \left(\int_0^y \frac{\partial f}{\partial H} dy \right) dy$$

But

$$\int_0^\delta y^n (1-f^{m+1}) dy = M \theta^{n+1}$$

$$\int_0^\delta y^n (1-f^{m+1}) f dy = N \theta^{n+1}$$

and

$$\int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y (1-f) dy \right] dy = J \theta^{n+1}$$

Then with

$$\int_0^\delta y^{n-1} (1-f^{m+1}) \left(\int_0^y \frac{\partial f}{\partial H} dy \right) dy = I \theta^{n+1}$$

the expression for I_1 can be written as

$$I_1 = \frac{1}{\theta} \frac{d\theta}{dx} (-M + N + J) - I \frac{dH}{dx} \quad (7)$$

When the expression for I_1 from equation (7) is substituted into equation (6) and equation (4) for $\frac{d\theta}{dx}$ is used, the following equation is obtained:

$$\begin{aligned} \theta \frac{dH}{dx} \left(\frac{dN}{dH} + nI \right) &= \frac{\theta}{U} \frac{dU}{dx} [-n(H+1)(J-M) + L(m+1) + \\ &\quad N(H-m)] + \frac{\tau_0}{\rho U^2} \left[n(J-M) - N - \right. \\ &\quad \left. (m+1) \int_0^{\delta/\theta} f^m \eta^n \frac{\partial g}{\partial \eta} d\eta \right] + \\ &\quad \frac{v_0}{U} [n(J-M) - N + nQ] \end{aligned} \quad (8)$$

where $\frac{dN}{dx} = \frac{dN}{dH} \frac{dH}{dx}$ has been used. Equation (8) for $\theta \frac{dH}{dx}$ is applicable both to two-dimensional flow and to flow over a body of revolution.

In equation (8) all the integrals, except the one involving the shear ratio g , are functions of H , m , and n only. For

the present no restrictive assumptions regarding the shear are made. The form of the kinetic-energy equation for a single-parameter family of velocity profiles is obtained from equation (8) by placing $m=1$ and $n=0$ and dividing by $N'(H)=K'(H)$. Thus,

$$\theta \frac{dH}{dx} = \frac{K(H-1)}{K'} \frac{\theta}{U} \frac{dU}{dx} - \left(\frac{K+2}{K'} \int_0^{\delta/\theta} f \frac{\partial g}{\partial \eta} d\eta \right) \frac{\tau_0}{\rho U^2} - \left(\frac{K-1}{K'} \right) \frac{v_0}{U} \quad (9)$$

The symbol K represents the ratio of the kinetic-energy thickness $\int_0^\delta (1-f^2) f dy$ to the momentum thickness. Note that in the derivation of equation (9) from equation (8), the assumption of a single-parameter family of curves is not restricted to the case $\frac{v_0}{U}=0$.

RESTRICTION OF GENERAL EQUATION TO POWER PROFILES

The data in figure 1 show that the power profiles defined by $f=\zeta^p$ are a good approximation to the "standard" profiles derived by fairing experimental data (reference 10). Equation (8) can be further developed by using the assumption that $f=\zeta^p$. After some fairly lengthy calculations (see appendix B), equation (8) becomes

$$\begin{aligned} \theta \frac{dH}{dx} &= \frac{-4p(p+1)(2p+1)[p(m+2)+n+1]}{pm+n+1} \frac{\theta}{U} \frac{dU}{dx} + \\ &\quad \frac{2p[p(m+2)+n+1]}{pm+n} \left\{ (2p+1) + \right. \\ &\quad \left. [p(m+2)+n+1] \int_0^1 \zeta^{p(m+n)} \frac{\partial g}{\partial \zeta} d\zeta \right\} \frac{\tau_0}{\rho U^2} + \\ &\quad \frac{2p(p+1)[p(m+2)+n+1]}{p(m+1)+n} \frac{v_0}{U} \end{aligned} \quad (10)$$

As a first approximation the assumption has been made that $f=\zeta^p$ even when $\frac{v_0}{U} \neq 0$.

The occurrence of the arbitrary positive integers m and n in equations (8) and (10) requires an explanation. In order to determine why m and n appear, equation (8) is written in a different form. By making use of the definitions for N , I , J , M , L , and Q and integrating by parts where necessary in order to eliminate terms that contain η^{n-1} , the result is

$$\begin{aligned} \frac{dN}{dH} + nI &= -(m+1) \int_0^{\delta/\theta} \eta^n f^m \left(f \frac{\partial f}{\partial H} - \frac{\partial f}{\partial \eta} \int_0^\eta \frac{\partial f}{\partial H} d\eta \right) d\eta \\ &\quad - n(H+1)(J-M) + L(m+1) + N(H-m) \\ &= (m+1) \int_0^{\delta/\theta} \eta^n f^m \left[(H+1) \frac{\partial f}{\partial \eta} \int_0^\eta f d\eta + f^2 - 1 \right] d\eta \\ &\quad n(J-M) - N - (m+1) \int_0^{\delta/\theta} f^m \eta^n \frac{\partial g}{\partial \eta} d\eta \\ &= -(m+1) \int_0^{\delta/\theta} \eta^n f^m \left(\frac{\partial f}{\partial \eta} \int_0^\eta f d\eta + \frac{\partial g}{\partial \eta} \right) d\eta \end{aligned}$$

and

$$n(J-M)-N+nQ=(m+1)\int_0^{\delta/\theta}\eta^n f^m \frac{\partial f}{\partial \eta} \left(1 - \int_0^\eta f d\eta\right) d\eta$$

Equation (8) then becomes

$$\theta \frac{dH}{dx} = \frac{\int_0^{\delta/\theta} \eta^n f^m \left\{ \frac{\tau_0}{\rho U^2} \left(\frac{\partial f}{\partial \eta} \int_0^\eta f d\eta + \frac{\partial g}{\partial \eta} \right) + \frac{\theta}{U} \frac{dU}{dx} \left[1 - f^2 - (H+1) \frac{\partial f}{\partial \eta} \int_0^\eta f d\eta \right] + \frac{v_0}{U} \left[\frac{\partial f}{\partial \eta} \left(\int_0^\eta f d\eta - 1 \right) \right] \right\} d\eta}{\int_0^{\delta/\theta} \eta^n f^m \left(f \frac{\partial f}{\partial H} - \frac{\partial f}{\partial \eta} \int_0^\eta \frac{\partial f}{\partial H} d\eta \right) d\eta} \quad (11)$$

By using the assumption of a single-parameter family of curves directly in the partial-differential equation (1), the following ordinary differential equation is obtained:

$$\theta \frac{dH}{dx} = \frac{\frac{\tau_0}{\rho U^2} \left(\frac{\partial f}{\partial \eta} \int_0^\eta f d\eta + \frac{\partial g}{\partial \eta} \right) + \frac{\theta}{U} \frac{dU}{dx} \left[1 - f^2 - (H+1) \frac{\partial f}{\partial \eta} \int_0^\eta f d\eta \right] + \frac{v_0}{U} \left[\frac{\partial f}{\partial \eta} \left(\int_0^\eta f d\eta - 1 \right) \right]}{f \frac{\partial f}{\partial H} - \frac{\partial f}{\partial \eta} \int_0^\eta \frac{\partial f}{\partial H} d\eta} \quad (12)$$

The concept of a single-parameter family of velocity profiles is consistent with equation (1) and with particular functions for $\tau_0/\rho U^2$, g , and f when the right-hand side of equation (12) is independent of η . When the right-hand side of equation (12) is independent of η , the right-hand side of equation (11) is independent of m and n . Equations (11) and (12) are then identical.

To obtain an equation for $\theta \frac{dH}{dx}$ that does not contain either m or n or both, the functions $\tau_0/\rho U^2$, g , and f must therefore be such that the right-hand side of equation (12) is independent of η ; the solution of the equation for $\theta \frac{dH}{dx}$ then provides a solution of equation (1). Note that the problem is to find a solution not of equation (1) alone but of equation (1) and the independent relation for the shearing stress in turbulent boundary layers; this relation is at present unknown.

The nature of the approximation made in the present analysis, in order to obtain a specific equation for $\frac{dH}{dx}$, may be clarified by noting that a specific equation for $\frac{dH}{dx}$ is obtained from equation (8) by choosing the functions $\tau_0/\rho U^2$, g , and f and substituting an arbitrary positive integer for m and an arbitrary positive integer for n . The calculated distribution of H over a body for arbitrarily chosen functions for $\tau_0/\rho U^2$, g , and f is then consistent with the momentum equation and one of the integral equations for $\frac{dH}{dx}$. For example, if $m=1$ and $n=0$, both the momentum and the kinetic-energy equations are satisfied but no other ones. If $m=0$ and $n=1$, only the momentum and the moment of momentum equations are satisfied. In the present analysis only the momentum, the kinetic-energy, and the moment of momentum equations—equations which have familiar physical meaning—are used.

As noted previously, equation (11) is independent of m and n if the functions $\tau_0/\rho U^2$, g , and f are such that the right-hand side of equation (12) is independent of η . In this case a solution of equation (1) results and the functions $\tau_0/\rho U^2$, g , and f and the calculated distribution of H satisfy every particular equation obtainable from equation (11), (10), or (8) by assigning positive integers to m and n .

Note that m and n cannot both be made zero in equation (8) because $\frac{dN}{dH} + nI = 0$ for $m=n=0$. If m and n are both

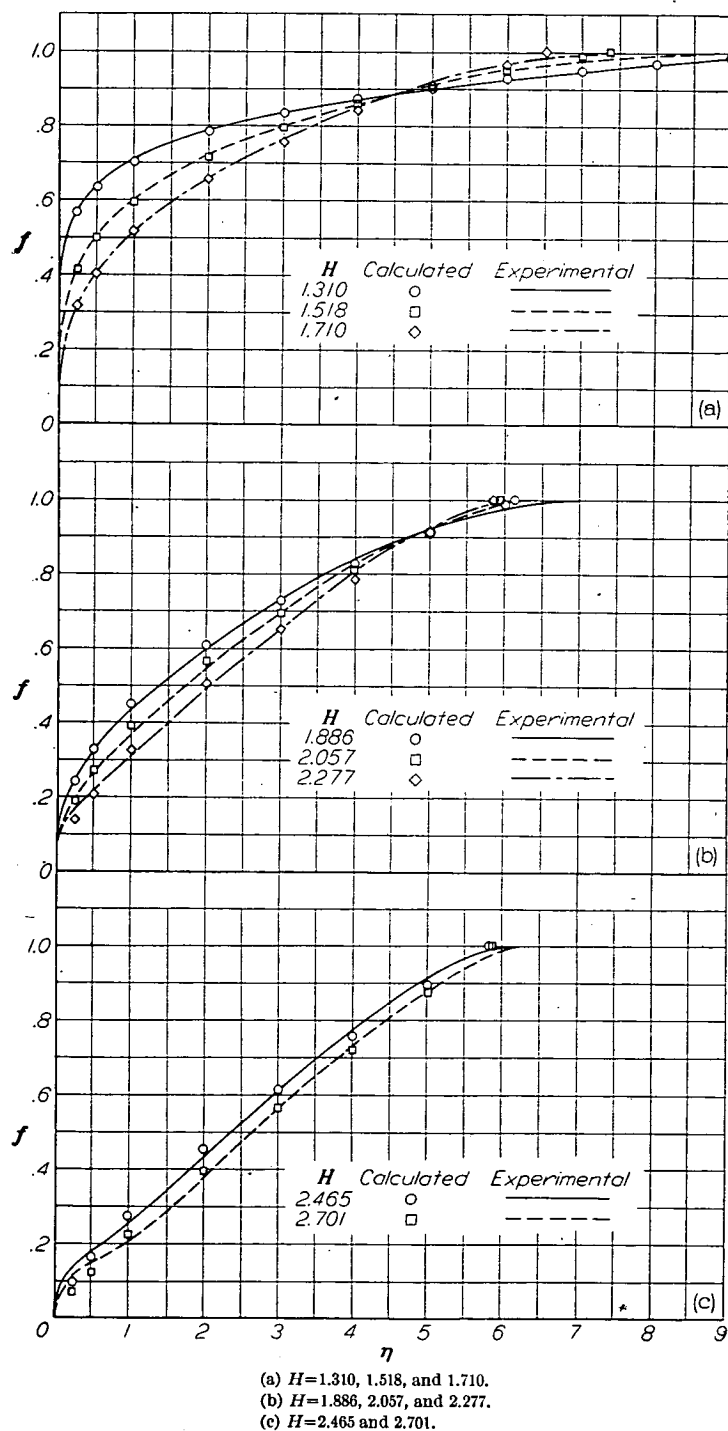


FIGURE 1.—Comparison of experimental and power velocity profiles.

zero, equation (8) becomes $0=0$. It is also noted that equations (8) and (10) are valid both for flow over a body of revolution and for two-dimensional flow.

For $m=1$ and $n=0$, equation (10) leads to the equation for kinetic energy

$$\theta \frac{dH}{dx} = -H(H-1)(3H-1) \frac{\theta}{U} \frac{dU}{dx} + (3H-1) \left(H + \frac{3H-1}{2} \int_0^1 \xi^p \frac{\partial g}{\partial \xi} d\xi \right) \frac{\tau_0}{\rho U^2} + \frac{(H+1)(3H-1)}{4} \frac{v_0}{U} \quad (13)$$

where the relation for power profiles $2p+1=H$ has been introduced. This form of the energy equation can also be obtained from equation (9) by noting that, from the definition of K and the equation for power profiles,

$$K = \frac{2(2p+1)}{3p+1} = \frac{4H}{3H-1}$$

A comparison of the values of K obtained from this formula and obtained from the standard profiles is given in figure 2.

The equation of moment of momentum for power profiles is obtained from equation (10) by letting $m=0$ and $n=1$; it is

$$\theta \frac{dH}{dx} = -\frac{H(H+1)(H^2-1)}{2} \frac{\theta}{U} \frac{dU}{dx} + (H^2-1) \left[H + (H+1) \int_0^1 \xi \frac{\partial g}{\partial \xi} d\xi \right] \frac{\tau_0}{\rho U^2} + (H^2-1) \frac{v_0}{U} \quad (14)$$

In this equation the term involving the shear distribution may be rewritten as follows:

$$\int_0^1 \xi \frac{\partial g}{\partial \xi} d\xi = - \int_0^1 g d\xi$$

It then involves the mean shear inside the boundary layer.

ATTEMPTS TO DERIVE A RELATION GOVERNING THE CHANGE OF THE FORM PARAMETER

In most of the recent analyses of the development of a turbulent boundary layer, an empirical relation governing the change of the form parameter H is usually introduced.

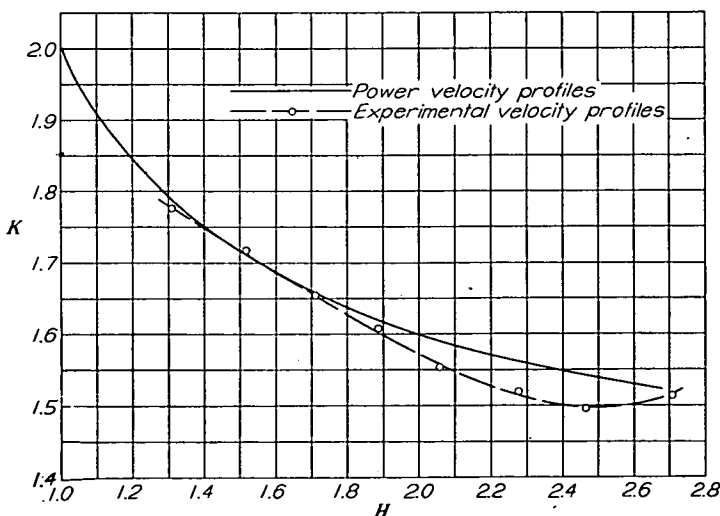


FIGURE 2.—Values of K for experimental and power velocity profiles.

It is clear that equation (10) automatically furnishes such relations if the shear distribution is known. In this section, three attempts are described to establish such a relation. These attempts are based on the following simple assumptions for the shear distribution:

(a) The shear distribution depends only on $\frac{\delta}{\tau_0} \frac{dp_1}{dx}$, which is equal to the Pohlhausen parameter multiplied by a factor (reference 3).

(b) The shear is constant across the boundary layer.

(c) The shear distribution depends only on the form parameter of the velocity distribution.

The first two assumptions are used either with the energy equation in the forms given by equations (9) and (13) or with equation (14) for the moment of momentum. The last assumption is used with equations (13) and (14) jointly.

(a) **Shear distribution depending only on the Pohlhausen parameter.**—The first assumption follows the original idea of the method of Von Kármán and Pohlhausen in using polynomial approximations together with the boundary conditions obtained by successive differentiation of the equations of motion (reference 3). Fediaevsky (reference 8) appears to have been the first to introduce it into the investigation of turbulent boundary layers. When the shear stress through the turbulent boundary layer is assumed to be a polynomial of fifth degree in $\xi = \frac{y}{\delta}$ satisfying the following boundary conditions:

at $y=0$

$$\tau = \tau_0 \quad \frac{\partial \tau}{\partial y} = \frac{dp_1}{dx} \quad \frac{\partial^2 \tau}{\partial y^2} = 0$$

at $y=\delta$

$$\tau = 0 \quad \frac{\partial \tau}{\partial y} = 0 \quad \frac{\partial^2 \tau}{\partial y^2} = 0$$

the following expression is obtained:

$$g = (1-\xi)^3 [1 + (3+\lambda)\xi + 3(2+\lambda)\xi^2] \quad (15)$$

The shear distribution g is a function of ξ and λ , where

$$\lambda = \frac{\delta}{\tau_0} \frac{dp_1}{dx} = -\frac{\theta}{U} \frac{dU}{dx} \frac{\delta}{\theta} \frac{\rho U^2}{\tau_0}$$

The particular boundary conditions at $y=0$ restrict this development to the case $\frac{v_0}{U} = 0$.

From the shear distribution (equation (15)) the calculation may be made of the coefficients P and S . The attempt to calculate P and S by using the standard profiles together with equations (9) and (15) was, however, unsuccessful for two reasons. First, the ratio δ/θ , which must be known, could not be accurately determined from the standard profiles. Second, for reasonable values of δ/θ , the calculated values of P were positive for values of H for which P should be negative.

The calculation of the part of P independent of the shear profile was then made both for the standard profiles and the power profiles by making use of the kinetic-energy equations (equations (9) and (13), respectively); the comparison is shown in figure 3. The closeness of the results suggests

that it is permissible to use power profiles as an approximation for calculating P and S . From equations (13) and (15),

$$P = -H(3H-1) \left[H-1 - \frac{96(3H-1)}{(H+5)(H+7)(H+9)} \right] \quad (16)$$

and

$$S = (3H-1) \left[H - \frac{240(3H-1)}{(H+5)(H+7)(H+9)} \right] \quad (17)$$

The functions P and S , given by equations (16) and (17), respectively, are shown in figure 4.

The fact that the equation

$$\theta \frac{dH}{dx} = P\omega + S\phi$$

where P and S are obtained from equations (16) and (17), respectively, does not predict the behavior of the turbulent boundary layer is shown as follows: Let $\omega=0$; then, for H greater than approximately 1.5, $\frac{dH}{dx}$ should be negative. Because S from equation (17) is positive for $H > 1.2$, it follows that $\frac{dH}{dx}$ is positive. This conclusion is incorrect;

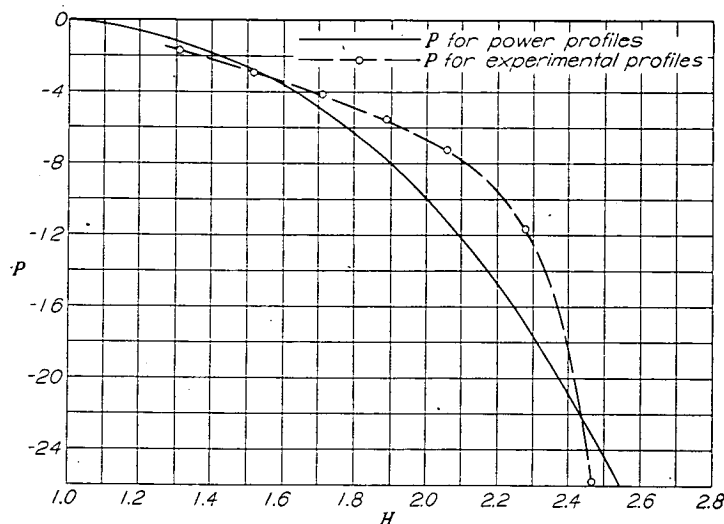


FIGURE 3.—Values of P from energy equation. For power profiles, $P = -H(H-1)(3H-1)$; for experimental profiles, $P = \frac{K(H-1)}{K'}$.

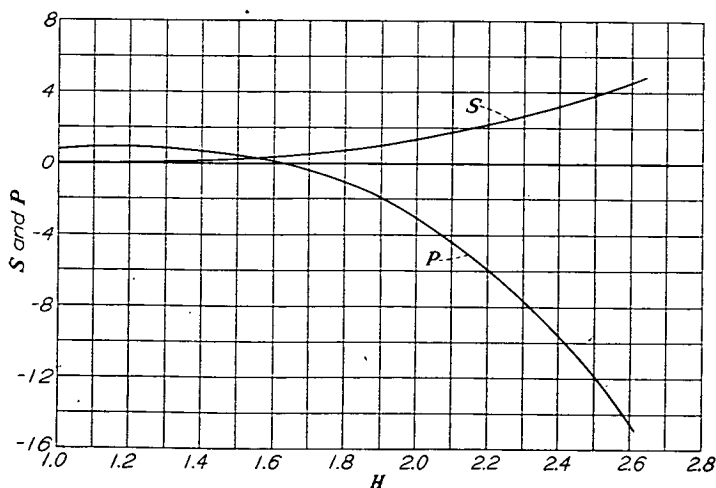


FIGURE 4.—Values of P from equation (16) and values of S from equation (17).

therefore, the function for S (equation (17)) is inconsistent with the known behavior of turbulent boundary layers.

To show that the function for P (equation (16)) is inconsistent with the known behavior of turbulent boundary layers, let $H \approx 1.4$. By making $\frac{dU}{dx}$ positive and large, $\frac{dH}{dx}$ becomes positive and large because P given by equation (16) is positive. For positive values of $\frac{dU}{dx}$, however, it is known that $\frac{dH}{dx}$ should be negative. The function for P (equation (16)) is therefore inconsistent with the known behavior of turbulent boundary layers.

In order to determine whether functions for P and S that do not result in obviously incorrect conclusions can be obtained by making the shear polynomial satisfy a greater number of boundary conditions at the outer edge of the boundary layer, the shear polynomial is generalized by writing

$$g = (1-\zeta)^A \left[1 + A\zeta + \frac{A(A+1)}{2} \zeta^2 \right] + \lambda \zeta (1-\zeta)^A (1+A\zeta) \quad (18)$$

The boundary conditions at the surface that are satisfied by equation (18) are

$$\begin{array}{lll} g=1 & \text{or} & \tau=\tau_0 \\ \frac{\partial g}{\partial \zeta}=\lambda & \text{or} & \frac{\partial \tau}{\partial y}=\frac{dp_1}{dx} \\ \frac{\partial^2 g}{\partial \zeta^2}=0 & \text{or} & \frac{\partial^2 \tau}{\partial y^2}=0 \end{array}$$

At $y=\delta$, the conditions that are satisfied are

$$\begin{array}{l} g=0 \\ \frac{\partial g}{\partial \zeta}=0 \\ \frac{\partial^2 g}{\partial \zeta^2}=0 \\ \dots \dots \\ \frac{\partial^{A-1} g}{\partial \zeta^{A-1}}=0 \end{array}$$

In order to evaluate the integral $\int_0^1 \zeta^p \frac{\partial g}{\partial \zeta} d\zeta$ in equation (13) the term $\frac{\partial g}{\partial \zeta}$ is written as

$$\frac{\partial g}{\partial \zeta} = g_1 + \lambda g_2$$

where

$$g_1 = (1-\zeta)^A [A + A(A+1)\zeta] - A(1-\zeta)^{A-1} \left[1 + A\zeta + \frac{A(A+1)}{2} \zeta^2 \right]$$

and, for $A \geq 1$,

$$g_2 = (1-\zeta)^A (1 + 2A\zeta) - \zeta A(1+A\zeta)(1-\zeta)^{A-1}$$

By using the expression for g_1 , the equation obtained for S is

$$S = 2(3p+1)^2 \left\{ \frac{2p+1}{3p+1} - \frac{A!p}{(A+p)(A+p-1)(A+p-2) \dots (p+1)} \left[\frac{1}{p} + \frac{A}{A+1+p} + \frac{A(A+1)(p+1)}{2(A+2+p)(A+1+p)} \right] \right\} \quad (19)$$

By using the expression for g_2 the equation for P is found to be

$$P = -2(2p+1)(3p+1)^2 \left\{ \frac{2p}{3p+1} - \frac{A!(p+1)}{(A+p)(A+p-1)(A+p-2) \dots (p+1)} \left[\frac{1}{A+1+p} + \frac{A(p+1)}{(A+2+p)(A+1+p)} \right] \right\} \quad (20)$$

To avoid positive values for S obtained from equation (19) for $H < 3$, it is found that A must be 1 in the expression for g_1 . It is also found that, to avoid positive values for P in equation (20) for $H > 0$, A must be ∞ in the expression for g_2 . The values for S and P then become

$$S = \frac{(3H-1)(H-1)(H-3)}{H+5} \quad (21)$$

$$P = -H(H-1)(3H-1) \quad (22)$$

The expression for P (equation (22)) is the same as the coefficient of $\frac{\theta}{U} \frac{dU}{dx}$ in equation (13); letting $A \rightarrow \infty$ makes the coefficient of λ in equation (18) become zero. The shear profile then contributes nothing to the coefficient of $\frac{\theta}{U} \frac{dU}{dx}$ in equation (13).

Equations (21) and (22) for S and P , respectively, were tested by making a computation of H and θ for the pressure distribution given in table I of reference 10. The computation began at $\frac{x}{c} = 0.075$ with the values given in table I of reference 10. The equations used are

$$\theta \frac{dH}{dx} = -H(H-1)(3H-1)\omega + \frac{(3H-1)(H-1)(H-3)}{H+5} \phi$$

and

$$\frac{d\theta}{dx} = -(H+2)\omega + \phi$$

The equation for $\frac{d\theta}{dx}$ is the Von Kármán momentum equation.

The equation for ϕ was obtained from reference 17 and is

$$\phi = \frac{0.006535}{R_\theta^{1/6}}$$

The calculated distribution of H along x was far from the experimental curve.

In an attempt to reduce the sensitivity of the equation for $\theta \frac{dH}{dx}$ to the shear distribution, the moment of momentum equation (equation (14)), in which the shear appears in the coefficient of ϕ only as a mean value, is used. When the generalized expression (equation (18)) is used for the shear distribution g , the result obtained is

$$\theta \frac{dH}{dx} = \left[-\frac{H(H-1)(H+1)^2}{2} + \frac{3H(H+1)^3}{(A+2)(A+3)} \right] \omega + (H^2-1) \left[H - \frac{3(H+1)}{A+3} \right] \phi \quad (23)$$

where $\psi=0$ as required by equation (18). To keep the coefficient of ω negative for all positive values of H , A must equal ∞ in the coefficient of ω . The shear distribution is then independent of the pressure gradient. To make the coefficient of ϕ negative for values of H near 3, A must have the smallest value that it can take; therefore, let $A=1$ in the coefficient of ϕ . Equation (23) then becomes

$$\theta \frac{dH}{dx} = -\frac{H(H^2-1)(H+1)}{2} \omega + \frac{(H^2-1)(H-3)}{4} \phi \quad (24)$$

A calculation for the example in table I of reference 10 with equation (24) resulted in a computed curve for H that was far from the experimental curve.

(b) **Assumption of constant shear across the boundary layer.**—All the computations of H have led to values of H much larger than the experimental values. Therefore, in order to reduce the calculated values of H it is necessary to increase S . In order to increase S , the assumption of constant shear across the boundary layer is made. For constant shear it can be shown that

$$\int_0^1 \xi \frac{\partial g}{\partial \xi} d\xi = -1$$

by letting $g = (1-\xi)^B$ and taking the limit of the integral as $B \rightarrow 0$. Equation (14), after the assumption of constant shear is introduced, becomes

$$\theta \frac{dH}{dx} = -\frac{H(H+1)(H^2-1)}{2} \omega - (H^2-1)\phi + (H^2-1)\psi$$

In order to make $\frac{dH}{dx} = 0$ at $H = 1.286$ for $\frac{dU}{dx} = 0$, the coefficient of ϕ was arbitrarily changed to $H^2 - 1.286^2$. The equation then becomes

$$\theta \frac{dH}{dx} = -\frac{H(H+1)(H^2-1)}{2} \omega - (H^2 - 1.653)\phi + (H^2-1)\psi \quad (25)$$

This equation was used for the computation of H with $\psi=0$, and the results for the example given in table I of reference 10 are shown in figure 5.

The assumption of constant shear across the boundary layer was also combined with the kinetic-energy equation. When the power profiles and the assumption of constant shear are used in equation (13), the kinetic-energy equation becomes

$$\theta \frac{dH}{dx} = -H(H-1)(3H-1)\omega - \frac{(H-1)(3H-1)}{2} \phi + \frac{(H+1)(3H-1)}{4} \psi$$

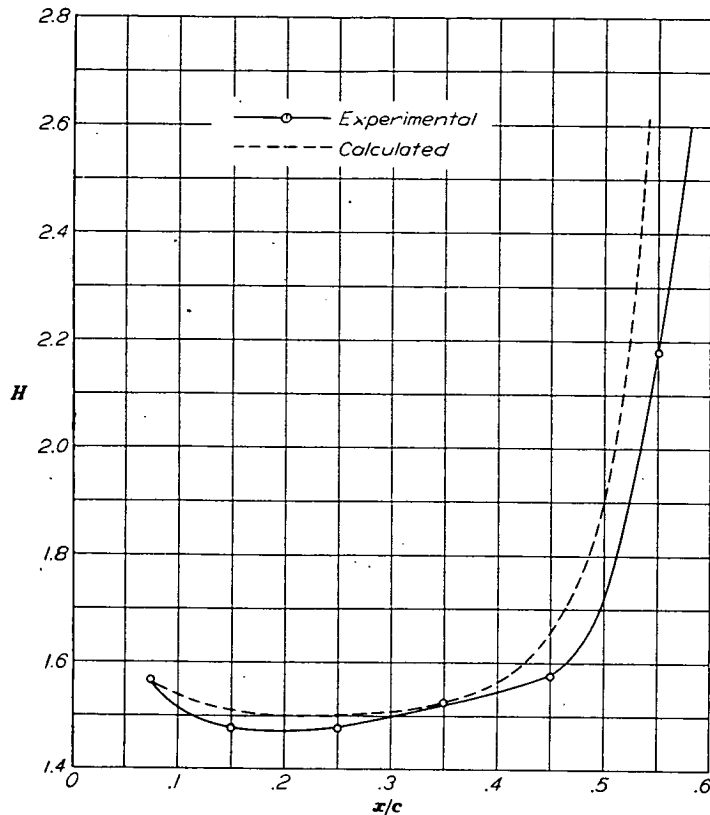


FIGURE 5.—Comparison of experimental values of H and values calculated by equation (25).
 $P = -\frac{H(H+1)(H^2-1)}{2}$; $S = H^2 - 1.653$; $\psi = 0$.

The function $-H(H-1)(3H-1)$ is shown in figure 3 and the function $-\frac{(H-1)(3H-1)}{2}$, in figure 6. When the standard profiles are substituted for the power profiles and the assumption of constant shear is made, the kinetic-energy equation (equation (9)) becomes

$$\theta \frac{dH}{dx} = \frac{K(H-1)}{K'} \omega - \frac{K-2}{K'} \phi - \frac{K-1}{K'} \psi$$

where the function $\frac{K(H-1)}{K'}$ is shown in figure 3 and the function $-\frac{K-2}{K'}$ is shown in figure 6. The results of these calculations of H (with $\psi = 0$) are shown in figure 7. In this case, the use of power profiles makes the result somewhat different from that obtained by using the standard profiles.

(c) **Determination of S by the simultaneous use of the energy and moment of momentum equations.**—It seems obvious that, if equations (13) and (14) were exact, the coefficients of ω , ϕ , and ψ in equation (13) would be equal to the coefficients of ω , ϕ , and ψ in equation (14). The ratio of the coefficients of ω is

$$\frac{-\frac{H(H+1)(H^2-1)}{2}}{-H(H-1)(3H-1)} = \frac{(H+1)^2}{2(3H-1)}$$

The curve of $\frac{(H+1)^2}{2(3H-1)}$ is given in figure 8 and is seen to be close to unity.

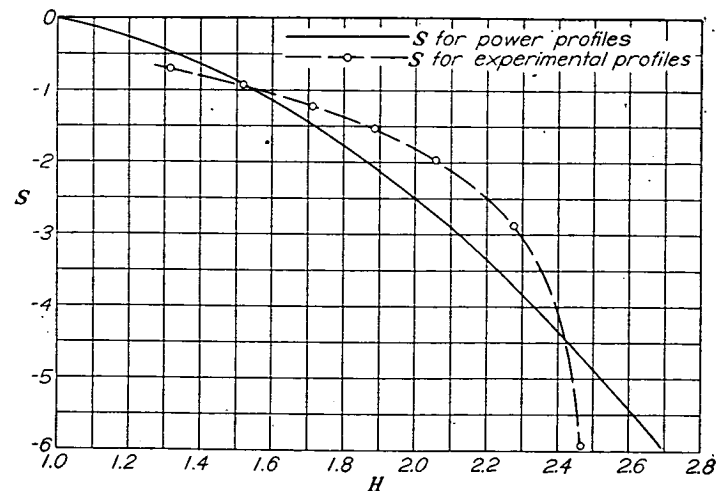


FIGURE 6.—Values of S from kinetic-energy equation with the assumption of constant shear.
 For power profiles, $S = -\frac{(H-1)(3H-1)}{2}$; for experimental profiles, $S = -\frac{K-2}{K'}$.

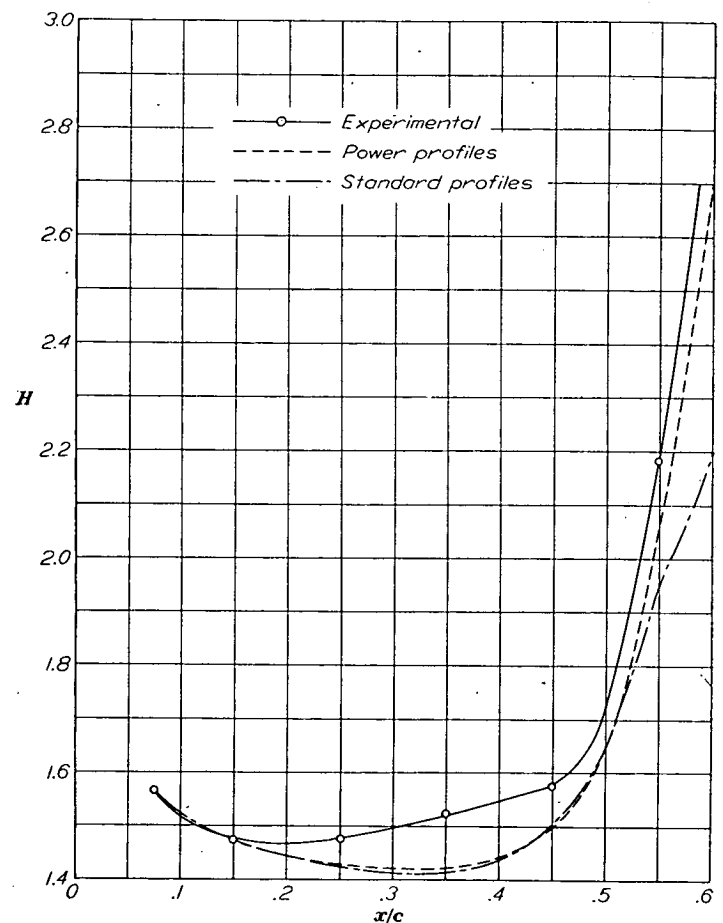


FIGURE 7.—Comparison of experimental values of H and values of H calculated by kinetic energy equation with constant shear across boundary layer.

The ratio of the coefficients of ψ is

$$\frac{4(H^2-1)}{(H+1)(3H-1)} = \frac{4(H-1)}{3H-1}$$

The curve of $\frac{4(H-1)}{3H-1}$ is also given in figure 8. The values are far from unity for small values of H but become equal to unity for $H=3$.

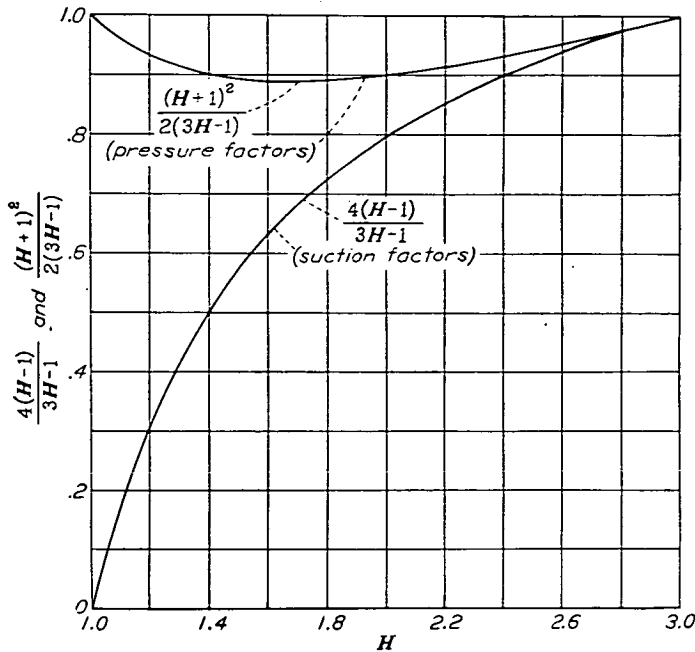


FIGURE 8.—Ratios of pressure and suction factors in energy and momentum equations.

Equating the coefficients of ϕ results in

$$\begin{aligned} (H^2-1) \left[H + (H+1) \int_0^1 \xi \frac{\partial g}{\partial \xi} d\xi \right] \\ = (3H-1) \left[H + \frac{3H-1}{2} \int_0^1 \xi^{\frac{H-1}{2}} \frac{\partial g}{\partial \xi} d\xi \right] \end{aligned}$$

or

$$\begin{aligned} (H^2-1)(H+1) \int_0^1 \xi \frac{\partial g}{\partial \xi} d\xi - \frac{(3H-1)^2}{2} \int_0^1 \xi^{\frac{H-1}{2}} \frac{\partial g}{\partial \xi} d\xi \\ = H^2(3-H) \end{aligned} \quad (26)$$

Now let

$$\int_0^1 \xi \frac{\partial g}{\partial \xi} d\xi = a + bH + cH^2$$

and

$$\int_0^1 \xi^{\frac{H-1}{2}} \frac{\partial g}{\partial \xi} d\xi = j + qH + lH^2$$

When the shear distribution is assumed to depend only on H , the integrals in equation (26) are functions of H alone. Because equation (26) is then an identity, the coefficients of the various powers of H can be equated to zero. The resulting equations are:

For H^0

$$-a - \frac{j}{2} = 0$$

for H^1

$$-a - b + 3j - \frac{q}{2} = 0$$

for H^2

$$a - b - c - \frac{9}{2}j + 3q - \frac{l}{2} = 3$$

for H^3

$$a + b - c - \frac{9}{2}q + 3l = -1$$

for H^4

$$b + c - \frac{9}{2}l = 0$$

for H^5

$$c = 0$$

The results obtained are:

$$a = \frac{28}{128}$$

$$b = -\frac{180}{128}$$

$$c = 0$$

$$j = -\frac{56}{128}$$

$$q = -\frac{32}{128}$$

$$l = -\frac{40}{128}$$

Therefore,

$$\int_0^1 \xi \frac{\partial g}{\partial \xi} d\xi = \frac{7-45H}{32}$$

and

$$\int_0^1 \xi^{\frac{H-1}{2}} \frac{\partial g}{\partial \xi} d\xi = -\frac{7+4H+5H^2}{16}$$

Equation (14), the momentum equation, becomes

$$\begin{aligned} \theta \frac{dH}{dx} = -\frac{H(H^2-1)(H+1)}{2} \omega - \\ \frac{(H-1)(3H-1)(7+22H+15H^2)}{32} \phi + (H^2-1)\psi \end{aligned} \quad (27)$$

and equation (13), the energy equation, becomes

$$\begin{aligned} \theta \frac{dH}{dx} = -H(H-1)(3H-1) \omega - \\ \frac{(H-1)(3H-1)(7+22H+15H^2)}{32} \phi + \frac{(H+1)(3H-1)}{4} \psi \end{aligned} \quad (28)$$

The variation of H with x for the initial values and the pressure distribution given in table I of reference 10 was computed by using a modified form of equation (28). In order that

$\frac{dH}{dx} = 0$ at values of H in agreement with experiment when $\omega = 0$, the coefficient of ϕ in equation (28) was replaced by

$$\frac{(H-H_0)(3H-H_0)(7+22H+15H^2)}{32}$$

where

$$H_0 = H_0(R_\theta)$$

The variation of H_0 with R_θ was calculated from the equation

$$\log_{10} H_0 = 0.5990 - 0.1980 \log_{10} R_\theta - 0.0189 (\log_{10} R_\theta)^2$$

which was derived to represent a faired curve through the

experimental data (see fig. 9); the data were obtained from reference 13 and from British results that are not generally available. The result of a computation of H for $\psi=0$ and with equation (28) modified as follows

$$\theta \frac{dH}{dx} = -H(H-1)(3H-1)\omega - \frac{(H-H_0)(3H-H_0)(7+22H+15H^2)}{32}\phi + \frac{(H+1)(3H-1)}{4}\psi \quad (29)$$

is given in figure 10.

Assumptions (b) and (c) lead to somewhat better results than assumption (a) although they are still not as satisfactory as those obtained from the purely empirical relations introduced in references 10 and 12. It is clear that this difference is caused partly by the inaccuracy of the simple assumptions about the shear distribution and can be improved by using better descriptions. However, in view of the limited present knowledge of the shear distribution, it does not seem worth while to make more complicated assumptions.

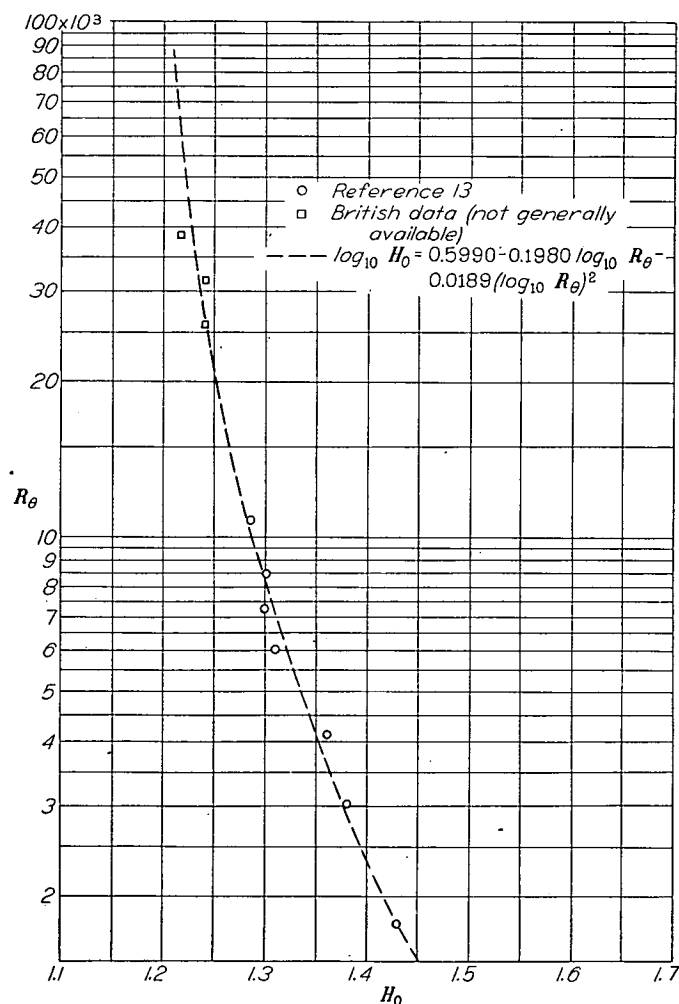


FIGURE 9.—Variation of H_0 with R_θ .

It may be noted that the final equations obtained for the change of the form parameter by the three assumptions are all of the form

$$\theta \frac{dH}{dx} = P(H)\omega + S(H)\phi$$

where $\omega = \frac{\theta}{U} \frac{dU}{dx}$ and $\phi = \frac{\tau_0}{\rho U^2}$. This form is used in reference 12, but a different form is used in reference 10.

INVESTIGATION OF ENERGY EQUATION

Since none of the three assumptions for the shear distribution results in a dependable equation for $\frac{dH}{dx}$, an investigation is made to determine whether a result common to the three assumptions—namely that the coefficient of $\frac{\tau_0}{\rho U^2}$ in the equations for $\frac{dH}{dx}$ is a function of H alone—is very far from true by using experimental data and the kinetic-energy equation without any assumption for the shear.

If no assumptions other than the boundary-layer assumptions are made and if in equation (6) $n=0$ and $m=1$, the result is

$$\theta \frac{dK}{dx} = -\phi \left(K + 2 \int_0^{\delta/\theta} f \frac{\partial g}{\partial \eta} d\eta \right) + \omega(H-1)K + \psi(1-K) \quad (30)$$

If the assumption of a single-parameter family of curves is

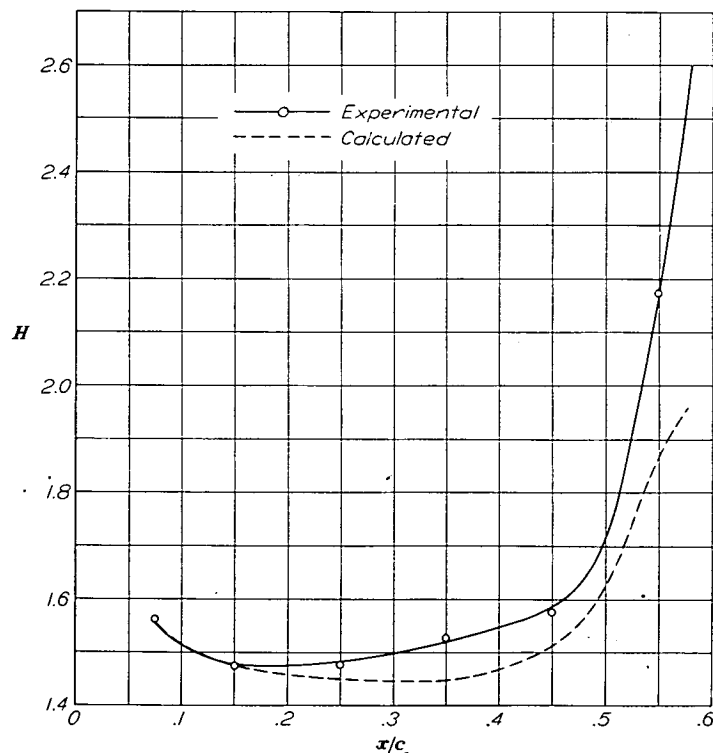


FIGURE 10.—Comparison of experimental values of H with values calculated by equation (29).

$$P = -H(H-1)(3H-1); S = -\frac{(H-H_0)(3H-H_0)(7+22H+15H^2)}{32}; \psi=0.$$

made ($f=f(\eta, H)$), then $K=K(H)$, and equation (30) becomes

$$\theta \frac{dH}{dx} = -\phi \frac{K+2 \int_0^{\delta/\theta} f \frac{\partial g}{\partial \eta} d\eta}{K'} + \omega \frac{H-1}{K'} K + \psi \frac{1-K}{K'}$$

or, for $\psi=0$,

$$\theta \frac{dH}{dx} = \phi k(H) (\xi + \xi_0) \quad (31)$$

where

$$k(H) = \frac{-(H-1)K}{K'}$$

$$\xi = -\frac{\omega}{\phi}$$

$$\xi_0 = \frac{1}{H-1} \left(1 - \frac{2}{K} \int_0^1 f dg \right)$$

If the assumption is made that $g=g(\eta, H)$, then $f=f(g, H)$ and $\int_0^1 f dg$ is a function of H only. Therefore, $\xi_0=\xi_0(H)$. Equation (31) then becomes

$$\theta \frac{dH}{dx} = \phi k(H) [\xi + \xi_0(H)] \quad (32)$$

In order to obtain an estimate of the quantity $1 - \frac{2}{K} \int_0^1 f dg$ under the assumption that $f=f(g, H)$, reference 12 is used. Equation (7) of reference 12 may be written as

$$\theta \frac{dH}{dx} = \phi k_1(H) [\xi - 2.065(H-1.4)] \quad (33)$$

where

$$k_1(H) = e^{5(H-1.4)}$$

Note that Garner's equation (equation (33)) has the form the kinetic-energy equation takes when the assumptions that $f=f(\eta, H)$ and that $g=g(\eta, H)$ are used in the kinetic-energy equation. The kinetic-energy equation (equation (31)) can also be placed in the form of equation (32) when the more general assumption that $g = \frac{\omega}{\phi} F_0(\eta, H) + F_1(\eta, H)$ is made for the shear distribution. For the purpose of obtaining an estimate of the value of $1 - \frac{2}{K} \int_0^1 f dg$, the quantity $\xi + \xi_0(H)$ in equation (32) is assumed to be identical with the quantity $\xi - 2.065(H-1.4)$ in equation (33). Then

$$\xi_0(H) = -2.065(H-1.4)$$

and for $H=1.5$, for example,

$$\xi_0(H) = -0.2065$$

therefore,

$$\frac{1}{0.5} \left(1 - \frac{2}{K} \int_0^1 f dg \right) = -0.2065$$

or

$$1 - \frac{2}{K} \int_0^1 f dg = -0.1032$$

Therefore $1 - \frac{2}{K} \int_0^1 f dg$ is the difference between two quantities, each of which is much larger than their difference. It follows that, in order to determine $\xi_0(H)$ for values of H not close to separation with any accuracy, f and g must be known with relatively good accuracy.

It may be noted that the moment of momentum equation is also sensitive to g . This sensitivity can be seen by writing the coefficient of ϕ in equation (14) as

$$(H^2-1) \left[(H+1) \left(1 - \int_0^1 g d\zeta \right) - 1 \right]$$

When it is noted that the integral $\int_0^1 g d\zeta$ is of the order of unity and that H lies between 1.2 and 2.6, the sensitivity of the coefficient of ϕ to g becomes clear.

In an attempt to determine whether ξ_0 is determined mainly by H , all the data that were used in reference 10 were used to compute ξ_0 by making use of equation (32) in the form

$$\xi_0(H) = \frac{\theta \frac{dH}{dx}}{\phi k(H)} - \xi$$

The surface-friction coefficient ϕ was calculated by the formula (from reference 17)

$$\phi = \frac{0.006535}{R_\theta^{1/6}}$$

and $k(H)$ was calculated by the expression obtained from the moment of momentum equation

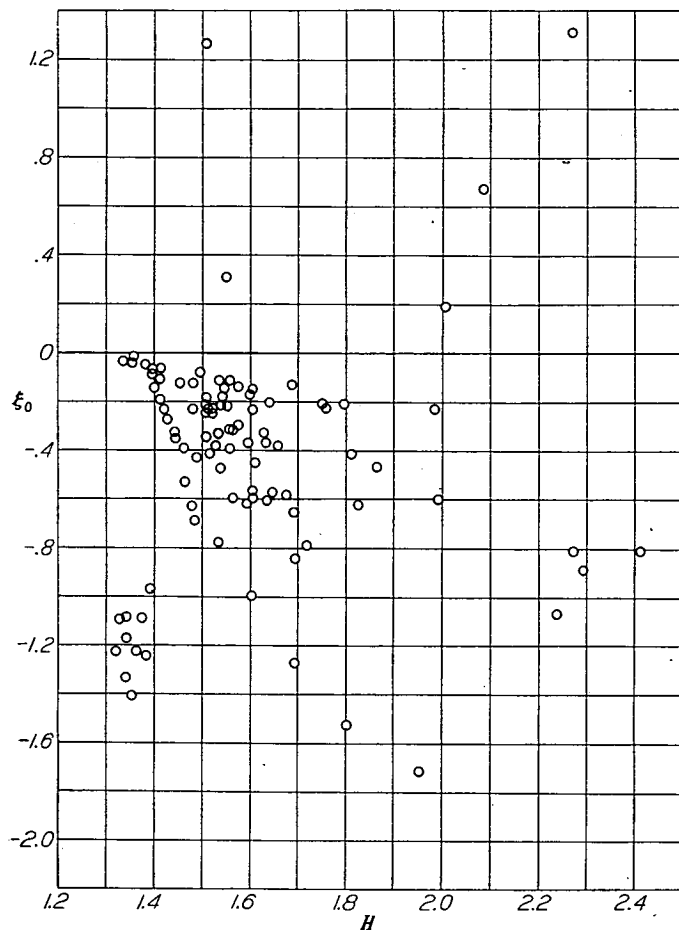
$$k(H) = \frac{H(H^2-1)(H+1)}{2}$$

The values of ξ_0 plotted against H are given in figure 11. The effort to determine whether ξ_0 is a function mainly of H is inconclusive. At least part of the scatter occurs because $\frac{dH}{dx}$ and $\frac{dU}{dx}$ were obtained from curves faired through experimental points. In addition, the calculation of ξ_0 requires

the subtraction of ξ from $\frac{\theta \frac{dH}{dx}}{\phi k(H)}$, an operation which further decreases the accuracy of the calculated values of ξ_0 .

DISCUSSION

Although equation (6) is valid whenever the boundary-layer assumptions are valid, the equations for $\frac{dH}{dx}$ that result after additional assumptions are made do not lead to

FIGURE 11.—Values of ξ_0 plotted against H .

good agreement with experiment. The first of the additional assumptions made is that all velocity profiles of the turbulent boundary layer belong to a single-parameter family of curves. The experimental data of references 7, 10, 11, 14, and 15 substantiate this assumption.

The second assumption is that the single-parameter family of curves can be approximated by power profiles. The data in figure 1, in which velocity profiles are compared, and also the data in figures 2 and 3, in which K and P are compared, show this assumption to be good, at least for $H < 1.8$.

From the data in figures 1 to 3, it is inferred that power profiles can be substituted for the standard velocity profiles without greatly affecting the calculated distribution of H against x for $H < 1.8$. To test this inference, the kinetic-energy equation was used with the assumption of constant shear across the boundary layer; the result is shown in figure 7. As expected from the data of figures 1 to 3, the effect of the substitution of power profiles for the standard profiles is noticeable only for $H > 1.8$. It thus appears that

the inaccuracy of the equations for $\frac{dH}{dx}$ that were tested is caused mainly by the surface-friction law that was used and by the assumed shear distributions rather than by the use of the power profiles.

The data of references 12 and 15 show skin frictions that increase strongly in the region upstream of the separation point before dropping to zero at the separation point. On the other hand, the skin-friction data presented in reference 14 indicate that the skin friction falls monotonically to zero as the separation point is reached. In the present analysis a skin-friction law obtained from experiments on flat plates is used. It is therefore probable that part of the inaccuracy in the equations used to calculate H is caused by the use of a relation for the skin friction that does not give correct values when there are pressure gradients along the surface.

The assumptions for the shear distribution that were made to obtain a specific equation for $\frac{dH}{dx}$ were

(a) The shear distribution depends only on the ratio of the pressure gradient to the skin friction $\frac{\delta}{\tau_0} \frac{dp_1}{dx}$ or $-\frac{\delta}{\theta} \frac{\omega}{\phi}$

(b) The shear is constant across the boundary layer

(c) The shear distribution depends only on the form parameter of the velocity distribution

Because none of these simple assumptions is derived from a knowledge of the details of the turbulent flow, it is not likely that any of them are valid. When it is recalled that the coefficient of ϕ in both the kinetic-energy and the moment of momentum equations is sensitive to the shear distribution, it is not surprising that a reliable equation for $\frac{dH}{dx}$ was not found.

In order to obtain a reliable equation for $\frac{dH}{dx}$ from equation

(8) it thus seems necessary to calculate the surface shear and the shear distribution across the boundary layer more accurately than in the present analysis. Efforts should therefore be made to understand the mechanics of turbulent shear flow sufficiently well to provide an independent relation for the shearing stress that will predict the behavior of turbulent boundary layers when used with the Prandtl boundary-layer equation (equation (1)).

CONCLUDING REMARKS

A general integral form of the boundary-layer equation is derived from the Prandtl partial-differential boundary-layer equation. The general integral equation, valid for either laminar or turbulent incompressible boundary-layer flow, contains the Von Kármán momentum equation, the kinetic-energy equation, and the Loitsianskii equation as special cases.

In an attempt to obtain a practical method for the calculation of the development of the turbulent boundary layer, use is made of the experimental finding that all the velocity profiles of the turbulent boundary layer form essentially a single-parameter family. The general equation is thereby changed to a simpler one from which an equation for the space rate of change of the shape parameter of the turbulent boundary layer can be obtained.

The resulting equation for the space rate of change of the velocity-profile parameter is restricted by the assumption that the velocity profiles of the turbulent boundary layer can be approximated by power profiles. Two of the resulting equations are used to calculate the distribution of the profile shape parameter over an airfoil for one experimentally determined pressure distribution. Although different assumptions were tried for the shearing stress across the boundary layer, the calculated distribution of the profile shape parameter did not agree exactly with the experimental distribution.

An examination is made of the effect of using the experimentally determined single-parameter family of velocity profiles instead of the power profiles on certain functions that occur in the equation for the space rate of change of the velocity-profile parameter. One calculation of the distribution of the profile shape parameter over an airfoil is also made

for the experimentally determined pressure distribution by using the single-parameter family of velocity profiles found from experiment. A comparison of the results with those of a calculation made with the same assumptions except for the use of power profiles shows some difference near the separation point. It is believed, however, that the apparent lack of reliability of the specific equations used to make the calculations is caused mainly by the lack of precise knowledge concerning the surface shear and the distribution of the shearing stress across the turbulent boundary layer. The present analysis emphasizes the need for information concerning the shearing stresses in turbulent boundary layers.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., May 22, 1950.

APPENDIX A

DETAILED DEVELOPMENT OF EQUATION (3)

Equation (2) can be written so that terms of the form $u^{m+1} - U^{m+1}$ appear explicitly; therefore, each term will vanish at the outer edge of the boundary layer. The resulting equation is

$$\begin{aligned} & \frac{1}{m+1} \left[\frac{\partial}{\partial x} (u^{m+1} - U^{m+1})u + \frac{\partial}{\partial y} (u^{m+1} - U^{m+1})v \right] + \\ & \frac{1}{m+1} \frac{\partial u U^{m+1}}{\partial x} + \frac{1}{m+1} \frac{\partial v U^{m+1}}{\partial y} + \\ & \frac{1}{m+1} \frac{(u^{m+1} - U^{m+1})u}{r_0} \frac{dr_0}{dx} + \frac{1}{m+1} \frac{u U^{m+1}}{r_0} \frac{dr_0}{dx} \\ & = u^m U \frac{dU}{dx} + \frac{u^m}{\rho} \frac{\partial \tau}{\partial y} \end{aligned} \quad (A1)$$

or, after simplification,

$$\begin{aligned} & -\frac{1}{m+1} \left[\frac{\partial}{\partial x} (U^{m+1} - u^{m+1})u + \frac{\partial}{\partial y} (U^{m+1} - u^{m+1})v \right] - \\ & \frac{1}{m+1} \frac{(U^{m+1} - u^{m+1})u}{r_0} \frac{dr_0}{dx} - (u^m U - u U^m) \frac{dU}{dx} = \frac{u^m}{\rho} \frac{\partial \tau}{\partial y} \end{aligned} \quad (A2)$$

Equation (A2) is now multiplied through by y^n and integrated with respect to y from $y=0$ to $y=\delta$. The resulting equation is

$$\begin{aligned} & -\frac{1}{m+1} \int_0^\delta y^n \frac{\partial}{\partial x} (U^{m+1} - u^{m+1})u \, dy - \\ & \frac{1}{m+1} \int_0^\delta y^n \frac{\partial}{\partial y} (U^{m+1} - u^{m+1})v \, dy - \\ & \frac{1}{m+1} \frac{1}{r_0} \frac{dr_0}{dx} \int_0^\delta y^n (U^{m+1} - u^{m+1})u \, dy - \\ & \frac{dU}{dx} \int_0^\delta y^n (u^m U - u U^m) \, dy = \frac{1}{\rho} \int_0^\delta u^m y^n \frac{\partial \tau}{\partial y} \, dy \end{aligned}$$

or, after simplification and substitution of the formula for the differentiation of a definite integral

$$\int_0^\delta y^n \frac{\partial}{\partial x} (U^{m+1} - u^{m+1})u \, dy = \frac{d}{dx} \int_0^\delta y^n (U^{m+1} - u^{m+1})u \, dy$$

the following equation results:

$$\begin{aligned} & -\frac{1}{m+1} \frac{d}{dx} U^{m+2} \int_0^\delta y^n \left[1 - \left(\frac{u}{U} \right)^{m+1} \right] \frac{u}{U} \, dy - \\ & \frac{1}{m+1} \int_0^\delta y^n \frac{\partial}{\partial y} (U^{m+1} - u^{m+1})v \, dy - \\ & \frac{1}{m+1} \frac{U^{m+2}}{r_0} \frac{dr_0}{dx} \int_0^\delta y^n \left[1 - \left(\frac{u}{U} \right)^{m+1} \right] \frac{u}{U} \, dy - \\ & U^{m+1} \frac{dU}{dx} \int_0^\delta \left[\left(\frac{u}{U} \right)^m - \frac{u}{U} \right] y^n \, dy = \frac{1}{\rho} \int_0^\delta u^m y^n \frac{\partial \tau}{\partial y} \, dy \end{aligned} \quad (A3)$$

By integration by parts,

$$\int_0^\delta y^n \frac{\partial}{\partial y} (U^{m+1} - u^{m+1})v \, dy = -n \int_0^\delta (U^{m+1} - u^{m+1})v y^{n-1} \, dy$$

and equation (A3) becomes

$$\begin{aligned} & -\frac{1}{m+1} \frac{d}{dx} U^{m+2} \int_0^\delta y^n \left[1 - \left(\frac{u}{U} \right)^{m+1} \right] \frac{u}{U} \, dy + \\ & \frac{n}{m+1} \int_0^\delta (U^{m+1} - u^{m+1})v y^{n-1} \, dy - \\ & \frac{1}{m+1} \frac{U^{m+2}}{r_0} \frac{dr_0}{dx} \int_0^\delta y^n \left[1 - \left(\frac{u}{U} \right)^{m+1} \right] \frac{u}{U} \, dy - \\ & U^{m+1} \frac{dU}{dx} \int_0^\delta \left[\left(\frac{u}{U} \right)^m - \frac{u}{U} \right] y^n \, dy = \frac{1}{\rho} \int_0^\delta u^m y^n \frac{\partial \tau}{\partial y} \, dy \end{aligned} \quad (A4)$$

The velocity v can be eliminated from the term

$$\int_0^\delta (U^{m+1} - u^{m+1}) v y^{n-1} dy$$

by the following development:

The velocity v may be written as

$$v = \int_0^y \frac{\partial v}{\partial y} dy + v_0$$

or, by use of the equation of continuity,

$$\begin{aligned} v &= - \int_0^y \frac{\partial u}{\partial x} dy - \frac{1}{r_0} \frac{dr_0}{dx} \int_0^y u dy + v_0 \\ &= \int_0^y \frac{\partial(U-u)}{\partial x} dy - y \frac{dU}{dx} + \frac{1}{r_0} \frac{dr_0}{dx} \int_0^y (U-u) dy - \frac{U}{r_0} \frac{dr_0}{dx} y + v_0 \\ &= \int_0^y \left[U \frac{\partial(1-\frac{u}{U})}{\partial x} + \frac{dU}{dx} \left(1-\frac{u}{U}\right) \right] dy - y \frac{dU}{dx} + \\ &\quad \frac{U}{r_0} \frac{dr_0}{dx} \int_0^y \left(1-\frac{u}{U}\right) dy - \frac{U}{r_0} \frac{dr_0}{dx} y + v_0 \\ v &= U \int_0^y \frac{\partial(1-\frac{u}{U})}{\partial x} dy + \frac{dU}{dx} \int_0^y \left(1-\frac{u}{U}\right) dy - y \left(\frac{dU}{dx} + \frac{U}{r_0} \frac{dr_0}{dx}\right) + \\ &\quad \frac{U}{r_0} \frac{dr_0}{dx} \int_0^y \left(1-\frac{u}{U}\right) dy + v_0 \end{aligned}$$

or, after terms are collected and f is substituted for u/U , the result is

$$v = U \int_0^y \frac{\partial(1-f)}{\partial x} dy + \left(\frac{dU}{dx} + \frac{U}{r_0} \frac{dr_0}{dx}\right) \left[\int_0^y (1-f) dy - y \right] + v_0$$

The term $\int_0^\delta (U^{m+1} - u^{m+1}) v y^{n-1} dy$ can now be written as

$$\begin{aligned} \int_0^\delta (U^{m+1} - u^{m+1}) v y^{n-1} dy &= \int_0^\delta (U^{m+1} - u^{m+1}) y^{n-1} \\ &\quad \left\{ U \int_0^y \frac{\partial(1-f)}{\partial x} dy + \left(\frac{dU}{dx} + \frac{U}{r_0} \frac{dr_0}{dx}\right) \left[\int_0^y (1-f) dy - y \right] + v_0 \right\} dy \\ \text{or} \\ \int_0^\delta (U^{m+1} - u^{m+1}) v y^{n-1} dy &= U^{m+2} \int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y \frac{\partial(1-f)}{\partial x} dy \right] dy + \left(\frac{dU}{dx} + \right. \\ &\quad \left. \frac{U}{r_0} \frac{dr_0}{dx} \right) U^{m+1} \int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y (1-f) dy - y \right] dy + \\ &\quad v_0 U^{m+1} \int_0^\delta y^{n-1} (1-f^{m+1}) dy \end{aligned}$$

Now let

$$\int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y \frac{\partial(1-f)}{\partial x} dy \right] dy = I_1 \theta^{n+1}$$

$$\int_0^\delta y^{n-1} (1-f^{m+1}) \left[\int_0^y (1-f) dy \right] dy = J \theta^{n+1}$$

$$\int_0^\delta y^n (1-f^{m+1}) dy = M \theta^{n+1}$$

and

$$\int_0^\delta y^{n-1} (1-f^{m+1}) dy = Q \theta^n$$

The term $\int_0^\delta (U^{m+1} - u^{m+1}) v y^{n-1} dy$ now becomes

$$\begin{aligned} \int_0^\delta (U^{m+1} - u^{m+1}) v y^{n-1} dy &= U^{m+2} I_1 \theta^{n+1} + \\ &\quad \left(\frac{dU}{dx} + \frac{U}{r_0} \frac{dr_0}{dx}\right) U^{m+1} (J - M) \theta^{n+1} + v_0 U^{m+1} Q \theta^n \end{aligned}$$

Now let

$$\int_0^\delta y^n (1-f^{m+1}) f dy = N \theta^{n+1}$$

and

$$\int_0^\delta (f - f^m) y^n dy = L \theta^{n+1}$$

Equation (A4) can then be written as

$$\begin{aligned} & - \frac{1}{m+1} \frac{d}{dx} (U^{m+2} N \theta^{n+1}) + \frac{n}{m+1} U^{m+2} I_1 \theta^{n+1} + \\ & \frac{n}{m+1} \left(\frac{dU}{dx} + \frac{U}{r_0} \frac{dr_0}{dx}\right) U^{m+1} (J - M) \theta^{n+1} + \frac{n}{m+1} v_0 U^{m+1} Q \theta^n - \\ & \frac{1}{m+1} \frac{U^{m+2}}{r_0} \frac{dr_0}{dx} N \theta^{n+1} + U^{m+1} \frac{dU}{dx} L \theta^{n+1} = \frac{U^m}{\rho} \int_0^\delta f^m y^n \frac{\partial \tau}{\partial y} dy \end{aligned} \quad (A5)$$

After $\frac{d}{dx} U^{m+2} N \theta^{n+1}$ is expanded and terms are collected, equation (A5) becomes

$$\begin{aligned} & (n+1) N \frac{d\theta}{dx} + \theta \left(\frac{dN}{dx} - n I_1\right) + \\ & \frac{\theta}{U} \frac{dU}{dx} [N(m+2) - n(J-M) - L(m+1)] + \\ & \frac{\theta}{r_0} \frac{dr_0}{dx} [N - n(J-M)] - n Q \frac{v_0}{U} = -(m+1) \frac{\tau_0}{\rho U^2} \int_0^{\delta/\theta} f^m \eta^n \frac{\partial \tau}{\partial \eta} d\eta \end{aligned}$$

where

$$\eta = \frac{y}{\theta}$$

APPENDIX B

SIMPLIFICATION OF TERMS IN EQUATION (8) FOR POWER PROFILES

CALCULATION OF $\frac{dN}{dH} + nI$

The definition of $N\theta^{n+1}$ is

$$N\theta^{n+1} = \int_0^\delta (1-f^{m+1}) f y^n dy$$

therefore,

$$N = \int_0^{\delta/\theta} (1-f^{m+1}) f \eta^n d\eta$$

and

$$\frac{dN}{dH} = \int_0^{\delta/\theta} \left[(1-f^{m+1}) \frac{\partial f}{\partial H} - f(m+1) f^m \frac{\partial f}{\partial H} \right] \eta^n d\eta$$

or

$$\frac{dN}{dH} = \int_0^{\delta/\theta} \frac{\partial f}{\partial H} [1 - (m+2)f^{m+1}] \eta^n d\eta$$

The definition of $I\theta^{n+1}$ is

$$I\theta^{n+1} = \int_0^\delta (1-f^{m+1}) \left(\int_0^y \frac{\partial f}{\partial H} dy \right) y^{n-1} dy$$

therefore,

$$I = \int_0^{\delta/\theta} (1-f^{m+1}) \left(\int_0^\eta \frac{\partial f}{\partial H} d\eta \right) \eta^{n-1} d\eta$$

Then

$$\begin{aligned} \frac{dN}{dH} + nI &= \int_0^{\delta/\theta} \frac{\partial f}{\partial H} [1 - (m+2)f^{m+1}] \eta^n d\eta + \\ &+ n \int_0^{\delta/\theta} (1-f^{m+1}) \left(\int_0^\eta \frac{\partial f}{\partial H} d\eta \right) \eta^{n-1} d\eta \end{aligned} \quad (B1)$$

By integration by parts,

$$\begin{aligned} &n \int_0^{\delta/\theta} (1-f^{m+1}) \left(\int_0^\eta \frac{\partial f}{\partial H} d\eta \right) \eta^{n-1} d\eta = \\ &- \int_0^{\delta/\theta} \left[(1-f^{m+1}) \frac{\partial f}{\partial H} - (m+1) f^m \frac{\partial f}{\partial \eta} \int_0^\eta \frac{\partial f}{\partial H} d\eta \right] \eta^n d\eta \end{aligned} \quad (B2)$$

When equation (B2) is substituted into equation (B1), the following equation is obtained:

$$\frac{dN}{dH} + nI = (m+1) \int_0^{\delta/\theta} \eta^n f^m \left(\frac{\partial f}{\partial \eta} \int_0^\eta \frac{\partial f}{\partial H} d\eta - f \frac{\partial f}{\partial H} \right) d\eta$$

Use is now made of the power-law assumption

$$f = \zeta^p = \eta^p \left(\frac{\theta}{\delta} \right)^p$$

Then

$$\frac{\partial f}{\partial H} = \frac{1-2p^2}{2(p+1)(2p+1)} \zeta^p + \frac{\zeta^p \log \zeta}{2}$$

and

$$\begin{aligned} &\frac{\partial f}{\partial \eta} \int_0^\eta \frac{\partial f}{\partial H} d\eta - f \frac{\partial f}{\partial H} \\ &= p \left\{ \frac{1-2p^2}{2(p+1)^2(2p+1)} \zeta^{2p} + \frac{1}{2} \left[\frac{\zeta^{2p} \log \zeta}{p+1} - \frac{\zeta^{2p}}{(p+1)^2} \right] \right\} - \\ &\quad \left[\frac{1-2p^2}{2(p+1)(2p+1)} \zeta^{2p} + \frac{\zeta^{2p} \log \zeta}{2} \right] \end{aligned}$$

After a lengthy manipulation, $\frac{dN}{dH} + nI$ is found to be

$$\frac{dN}{dH} + nI = -\frac{m+1}{2} \frac{(p+1)^n (2p+1)^n (pm+n)}{p^{n+1} [p(m+2)+n+1]^2}$$

where use has been made of the following equation:

$$\frac{\delta}{\theta} = \frac{(p+1)(2p+1)}{p}$$

CALCULATION OF N

The definition of $N\theta^{n+1}$ is

$$N\theta^{n+1} = \int_0^\delta (1-f^{m+1}) f y^n dy$$

When $f = \zeta^p$ is used, the equation for N is

$$N = \left(\frac{\delta}{\theta} \right)^{n+1} \frac{p(m+1)}{[p(m+2)+n+1](p+n+1)}$$

or

$$N = \frac{(m+1)(p+1)^{n+1}(2p+1)^{n+1}}{p^n [p(m+2)+n+1](p+n+1)}$$

CALCULATION OF J

From the definition of $J\theta^{n+1}$,

$$J\theta^{n+1} = \int_0^\delta (1-f^{m+1}) \left[\int_0^y (1-f) dy \right] y^{n-1} dy$$

When $f = \zeta^p$, the equation for J is

$$J = \left(\frac{\delta}{\theta} \right)^{n+1} \int_0^1 [1 - \zeta^{p(m+1)}] \left(\zeta - \frac{\zeta^{p+1}}{p+1} \right) \zeta^{n-1} d\zeta$$

or, after a lengthy manipulation,

$$\begin{aligned} J &= \frac{(p+1)^n (2p+1)^{n+1}}{p^{n-1}} (m+1) \\ &\quad \left\{ \frac{(p+1+n)[pm+n+2(p+1)] + p(m+1)+1+n}{(n+1)(p+1+n)[p(m+1)+n+1][p(m+1)+p+1+n]} \right\} \end{aligned}$$

CALCULATION OF M

From the definition of $M\theta^{n+1}$

$$M\theta^{n+1} = \int_0^{\delta} (1 - f^{m+1}) y^n dy$$

When $f = \zeta^p$, the equation for M is

$$M = (m+1) \frac{(p+1)^{n+1} (2p+1)^{n+1}}{p^n (n+1) [p(m+1) + n+1]}$$

CALCULATION OF L

From the definition of $L\theta^{n+1}$

$$L\theta^{n+1} = \int_0^{\delta} (1 - f^{m-1}) f y^n dy$$

When $f = \zeta^p$, the equation for L is

$$L = (m-1) \frac{(p+1)^{n+1} (2p+1)^{n+1}}{p^n (p+n+1) (pm+n+1)}$$

CALCULATION OF $\frac{N(H-m) - n(J-M)(H+1) + (m+1)L}{\frac{dN}{dH} + nI}$

From the expressions for J and M , the expression for $J-M$ is

$$J-M = \frac{(p+1)^n (2p+1)^{n+1}}{p^{n-1}} (m+1) \frac{(p+1+n) [pm+n+2(p+1)] + [p(m+1)+1+n]}{(n+1)(p+1+n) [p(m+1)+n+1] [p(m+1)+p+1+n]} - (m+1) \frac{(p+1)^{n+1} (2p+1)^{n+1}}{p^n (n+1) [p(m+1)+n+1]}$$

After a lengthy simplification, the result is

$$J-M = \frac{-(m+1)(p+1)^n (2p+1)^{n+1}}{p^n (p+1+n) [p(m+1)+p+1+n]}$$

or

$$-n(J-M)(H+1) = \frac{2n(m+1)(p+1)^{n+1} (2p+1)^{n+1}}{p^n (p+1+n) [p(m+1)+p+1+n]}$$

where $H=2p+1$ was used. The expression obtained for $N(H-m) + (m+1)L$ is

$$N(H-m) + (m+1)L = \frac{(m+1)(p+1)^{n+1} (2p+1)^{n+1} 2p(pm+m+n)}{p^n (p+n+1) [p(m+2)+n+1] (pm+n+1)}$$

and the expression obtained for

$$N(H-m) - n(J-M)(H+1) + (m+1)L$$

is

$$\begin{aligned} & N(H-m) - n(J-M)(H+1) + (m+1)L \\ &= \frac{2(m+1)(p+1)^{n+1} (2p+1)^{n+1} [n(pm+n+1) + p(pm+m+n)]}{p^n (p+1+n) [p(m+2)+n+1] (pm+n+1)} \end{aligned}$$

By substitution and simplification

$$\begin{aligned} & \frac{N(H-m) - n(J-M)(H+1) + (m+1)L}{\frac{dN}{dH} + nI} \\ &= \frac{-4p(p+1)(2p+1)[p(m+2)+n+1]}{pm+n+1} \end{aligned}$$

It can also be shown that

$$\begin{aligned} & \frac{-N + n(J-M) - (m+1) \int_0^{\delta/\theta} \eta^n f^m \frac{\partial g}{\partial \eta} d\eta}{\frac{dN}{dH} + nI} \\ &= \frac{2p}{pm+n} [p(m+2)+n+1] \left\{ 2p+1 + [p(m+2)+n+1] \int_0^1 \zeta^{p m+n} \frac{\partial g}{\partial \zeta} d\zeta \right\} \end{aligned}$$

EVALUATION OF $-N + n(J-M) + nQ$

From the results for N and $J-M$

$$-N + n(J-M) = \frac{-(m+1)(p+1)^n (2p+1)^{n+1}}{p^n [p(m+2)+n+1]}$$

For Q , the development is

$$Q\theta^n = \int_0^{\delta} (1 - f^{m+1}) y^{n-1} dy$$

and with $f = \zeta^p$, the following expression is obtained for $n \neq 0$:

$$Q = \frac{m+1}{n} \frac{(p+1)^n (2p+1)^n}{p^{n-1} [p(m+1)+n]}$$

Then, by substitution and simplification, for $n \neq 0$,

$$-N + n(J-M) + nQ = \frac{-(m+1)(p+1)^{n+1} (2p+1)^n (pm+n)}{p^n [p(m+2)+n+1] [p(m+1)+n]} \quad (B3)$$

If use is made of the previously derived result that $nQ=1$ for $n=0$, the following equation is obtained for $n=0$:

$$-N + n(J-M) + nQ = \frac{-m(p+1)}{p(m+2)+1} \quad (B4)$$

If n is placed equal to zero in equation (B3), equation (B4) results; therefore, equation (B3) is valid for $n=0$ as well as $n \neq 0$.

Then, for all values of n ,

$$\frac{-N + n(J-M) + nQ}{\frac{dN}{dH} + nI} = \frac{2(p+1)p[p(m+2)+n+1]}{p(m+1)+n}$$

REFERENCES

1. Prandtl, L.: Motion of Fluids with Very Little Viscosity. NACA TM 452, 1928.
2. Blasius, H.: The Boundary Layers in Fluids with Little Friction. NACA TM 1256, 1950.
3. Fluid Motion Panel of the Aeronautical Research Committee and Others: Modern Developments in Fluid Dynamics. Vol. I, S. Goldstein, ed., The Clarendon Press (Oxford), 1938.
4. Wieghardt, K.: On an Energy Equation for the Calculation of Laminar Boundary Layers. Repts. and Translations No. 89T, British M.A.P. Völkenrode, June 1, 1946. (Issued by Joint Intelligence Objectives Agency with File No. B.I.G.S.-65.) (Also available from CADO, Wright-Patterson Air Force Base, as ATI 33090.)
5. Von Kármán, Th.: On Laminar and Turbulent Friction. NACA TM 1092, 1946.
6. Buri, A.: A Method of Calculation for the Turbulent Boundary Layer with Accelerated and Retarded Basic Flow. R.T.P. Translation No. 2073, British Ministry of Aircraft Production. (From Thesis No. 652, Federal Technical College, Zurich, 1931.) (Also available from CADO, Wright-Patterson Air Force Base, as ATI 43493.)
7. Gruschwitz, E.: Die turbulente Reibungsschicht in ebener Strömung bei Druckabfall und Druckanstieg. Ing.-Archiv, Bd. II, Heft 3, Sept. 1931, pp. 321-346.
8. Fediaevsky, K.: Turbulent Boundary Layer of an Airfoil. NACA TM 822, 1937.
9. Kalikhman, L. E.: A New Method for Calculating the Turbulent Boundary Layer and Determining the Separation Point. Comptes Rendus Acad. Sci. USSR, vol. XXXVIII, no. 5-6, 1943, pp. 165-169.
10. Von Doenhoff, Albert E., and Tetervin, Neal: Determination of General Relations for the Behavior of Turbulent Boundary Layers. NACA Rep. 772, 1943. (Supersedes NACA ACR 3G13).
11. Kehl, A.: Investigations on Convergent and Divergent Turbulent Boundary Layers. R.T.P. Translation No. 2035, British Ministry of Aircraft Production. (From Ing.-Archiv, Bd. XIII, Heft 5, 1943, pp. 293-329.) (Also available from CADO, Wright-Patterson Air Force Base, as ATI 38429.)
12. Garner, H. C.: The Development of Turbulent Boundary Layers. R. & M. No. 2133, British A.R.C., 1944.
13. Schultz-Grunow, F.: New Frictional Resistance Law for Smooth Plates. NACA TM 986, 1941.
14. Dryden, Hugh L.: Some Recent Contributions to the Study of Transition and Turbulent Boundary Layers. NACA TN 1168, 1947.
15. Wieghardt, K., and Tillmann, W.: On the Turbulent Friction Layer for Rising Pressure. NACA TM 1314, 1951.
16. Loitsianskii, L. G.: Integral Methods in the Theory of the Boundary Layer. NACA TM 1070, 1944.
17. Falkner, V. M.: A New Law for Calculating Drag. The Resistance of a Smooth Flat Plate with Turbulent Boundary Layer. Aircraft Engineering, vol. XV, no. 169, March 1943, pp. 65-69.

REPORT 1047

THE STABILITY OF THE COMPRESSION COVER OF BOX BEAMS STIFFENED BY POSTS¹

By PAUL SEIDE and PAUL F. BARRETT

SUMMARY

An investigation is made of the buckling of the compression cover of post-stiffened box beams, subjected to end moments. Charts are presented for the determination of the minimum post axial stiffnesses and the corresponding compressive buckling loads required for the compression cover to buckle with nodes through the posts. Application of the charts to design and analysis and the limitations of their use are discussed.

If the flexural stiffness of the tension cover is equal to or less than that of the compression cover, buckle patterns with longitudinal nodes through the posts are not obtainable, no matter how stiff the posts are made or how closely they are spaced.

The weight of the required posts is found to be very low compared with the weight of the compression cover in an illustrative example, the ratio of the two being about 2½ percent.

INTRODUCTION

Post-type construction has recently gained some support in the aircraft industry. This construction has been incorporated in at least one operational aircraft and is being considered for others. The basic elements are relatively thick tension and compression surfaces connected in the interior by posts or rods. These posts serve to raise the buckling load of the surfaces by restricting the buckle patterns that may form. In reference 1 and in the present report, a compression surface supported by a rectangular array of rigid or effectively rigid posts is found to buckle as if continuously supported along either transverse or longitudinal lines through the posts.

If post-type construction is to reach an advanced stage of development, much research is needed. The large number of parameters involved, even in the simplest idealization of the problem, however, makes a complete investigation difficult. The scope of the present investigation has therefore been very much limited.

The problem as conceived in the present report is as follows: Two long, flat, rectangular plates of identical length and width, simply supported along their edges and of unequal thickness and material properties, are connected in the interior by identical axially elastic posts in rectangular array. The connection between the posts and plates is assumed to be one of simple support (of the ball-and-socket type, for instance) so that no resistance is offered to rotation of the plates. One plate is subjected to a longitudinal compressive load and the other, to an equal longitudinal tensile load. Both the longitudinal and transverse spacing

of the posts are uniform; the two spacings, however, may differ. (See fig. 1.) The minimum axial stiffness of the posts required for the compression surface to buckle with nodes through the posts and the corresponding buckling loads are to be determined.

The structure thus outlined is an idealization of a long post-stiffened box beam subjected to end bending moments. The numerical results of a theoretical analysis of this problem are presented herein in both tabular and graphical form (tables I to III and figs. 2 to 4). The analysis itself, presented in the appendix, is of a more general nature and may be used to investigate other problems in the design of plate structures stiffened by posts.

SYMBOLS

b	box-beam width
t	plate thickness
E	Young's modulus of elasticity of plate material
μ	Poisson's ratio of plate material
D	plate flexural stiffness per unit width $\left(\frac{Et^3}{12(1-\mu^2)}\right)$
D_T/D_C	ratio of flexural stiffnesses of tension and compression surfaces
R	plate flexural-stiffness ratio (D_T/D_C)
N_x	plate load per unit width, positive in compression surface for compression and in tension surface for tension
k	plate-load coefficient (buckling-load coefficient for compression plate) ($N_x b^2/\pi^2 D$)
N_{cr}	critical compressive load per unit width in compression surface for equal tensile load in tension surface
$N_{cr} b^2/\pi^2 D_C$	compressive-buckling-load parameter
M	number of longitudinal bays
N	number of transverse bays
L	longitudinal distance between posts
L/b	ratio of longitudinal distance between posts and box-beam width
β	aspect ratio of plate between adjacent lines of post (L/b)
F	axial stiffness of posts, force per unit deflection
$Fb^2/\pi^2 D_C$	post axial-stiffness parameter
S	post axial-stiffness parameter ($Fb^2/\pi^2 D_C$)
X, Y, Z	coordinate axes (see fig. 1)
x, y, z	distance along coordinate axes
w	plate deflection in Z -direction

¹ Supersedes NACA TN 2153, "The Stability of the Compression Cover of Box Beams Stiffened by Posts" by Paul Seide and Paul F. Barrett, 1950.

a_{mn}	general coefficient in Fourier series for w
i, j, m, n, r, s	integers
c	integer defining location of post in x -direction ($1 \leq c \leq (M-1)$)
d	integer defining location of post in y -direction ($1 \leq d \leq (N-1)$)
q	integer defining number of buckles in x -direction ($1 \leq q \leq (M-1)$)
p	integer defining number of buckles in y -direction ($1 \leq p \leq (N-1)$)

U	potential energy
V	strain energy
W	work done by external loads

Subscripts:

C	compression plate
T	tension plate
P	posts

The subscript cr when applied to L/b or β refers to the transition from buckling with longitudinal nodes through the posts to buckling with transverse nodes through the posts.

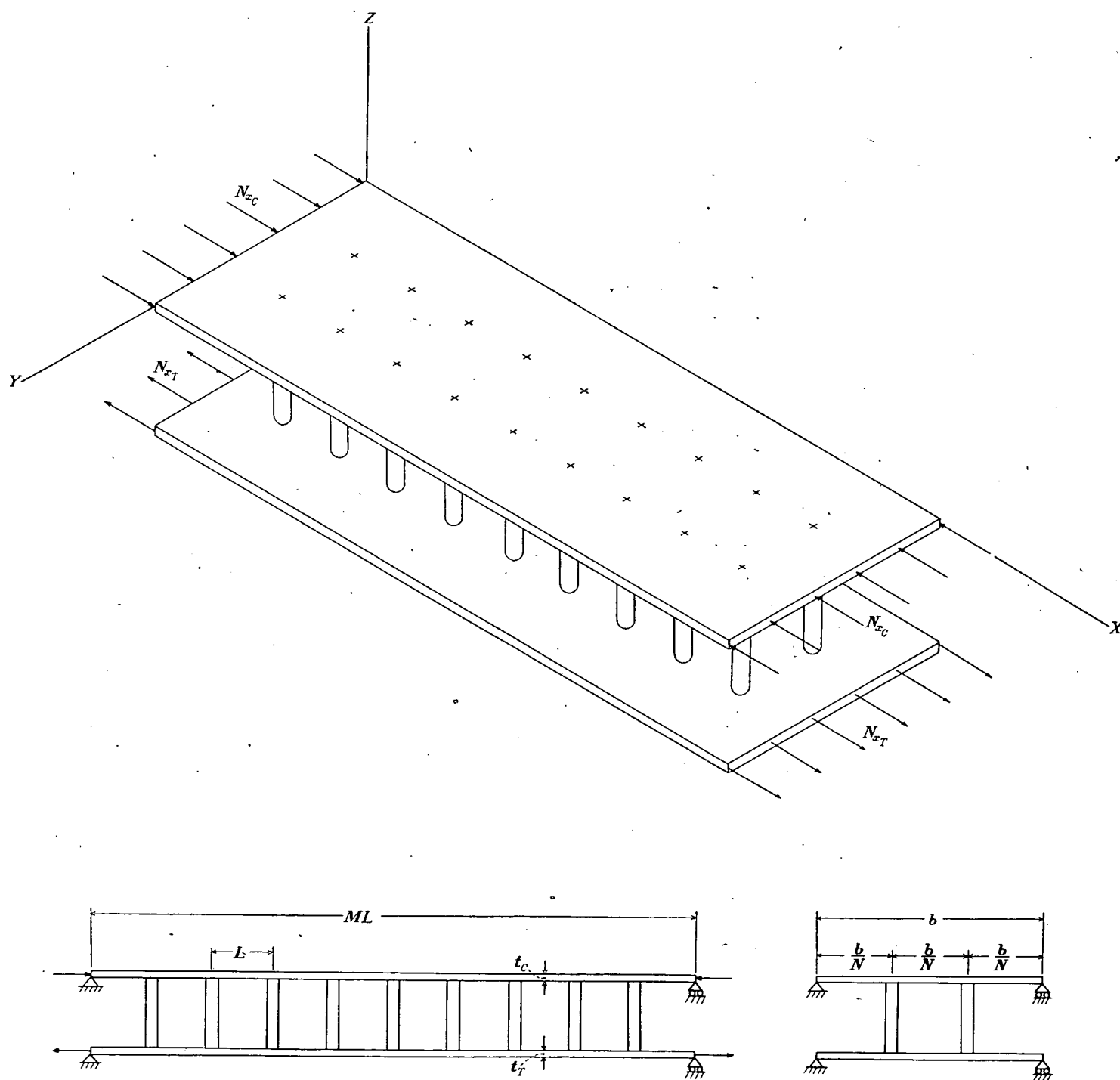
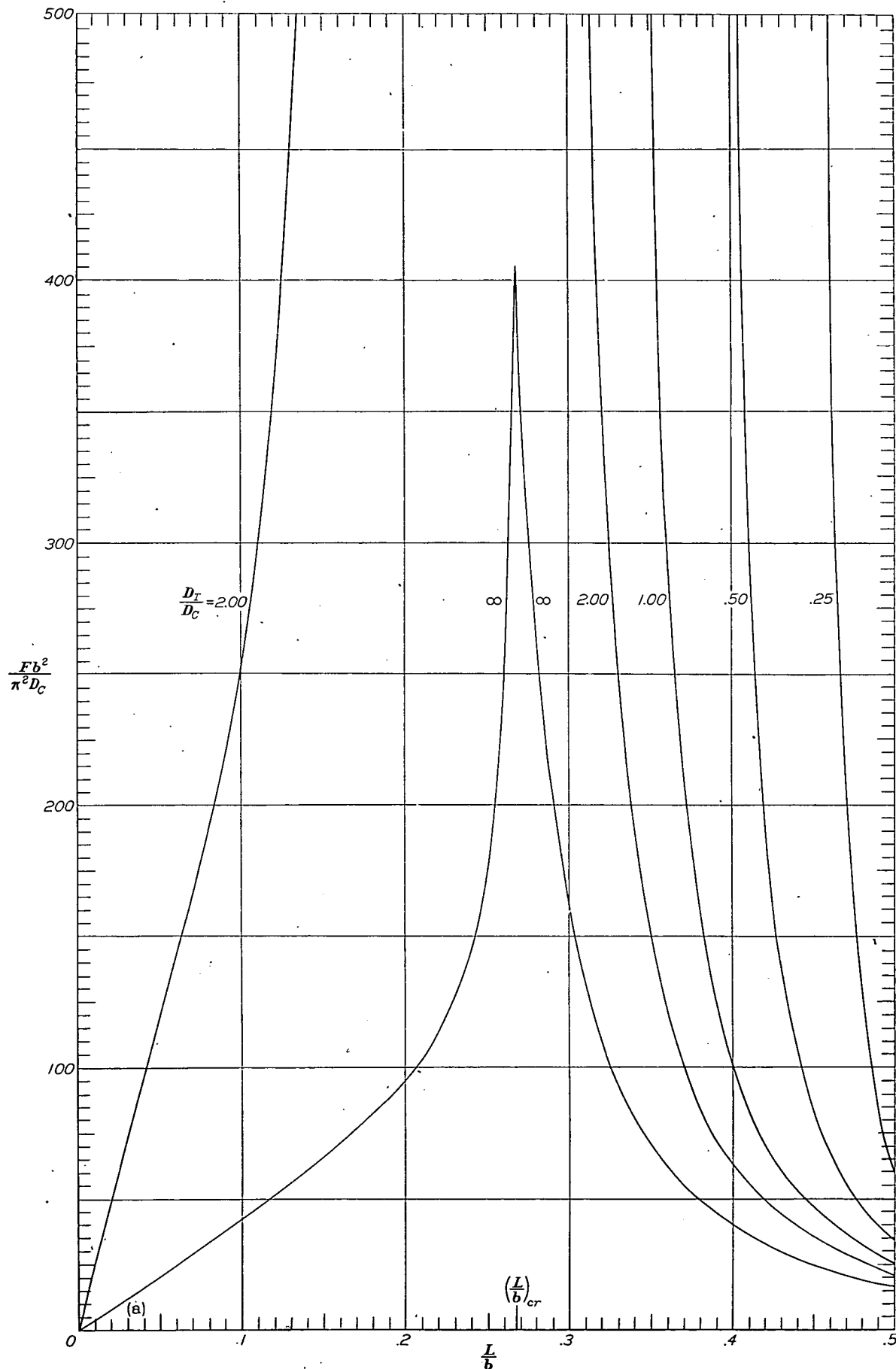
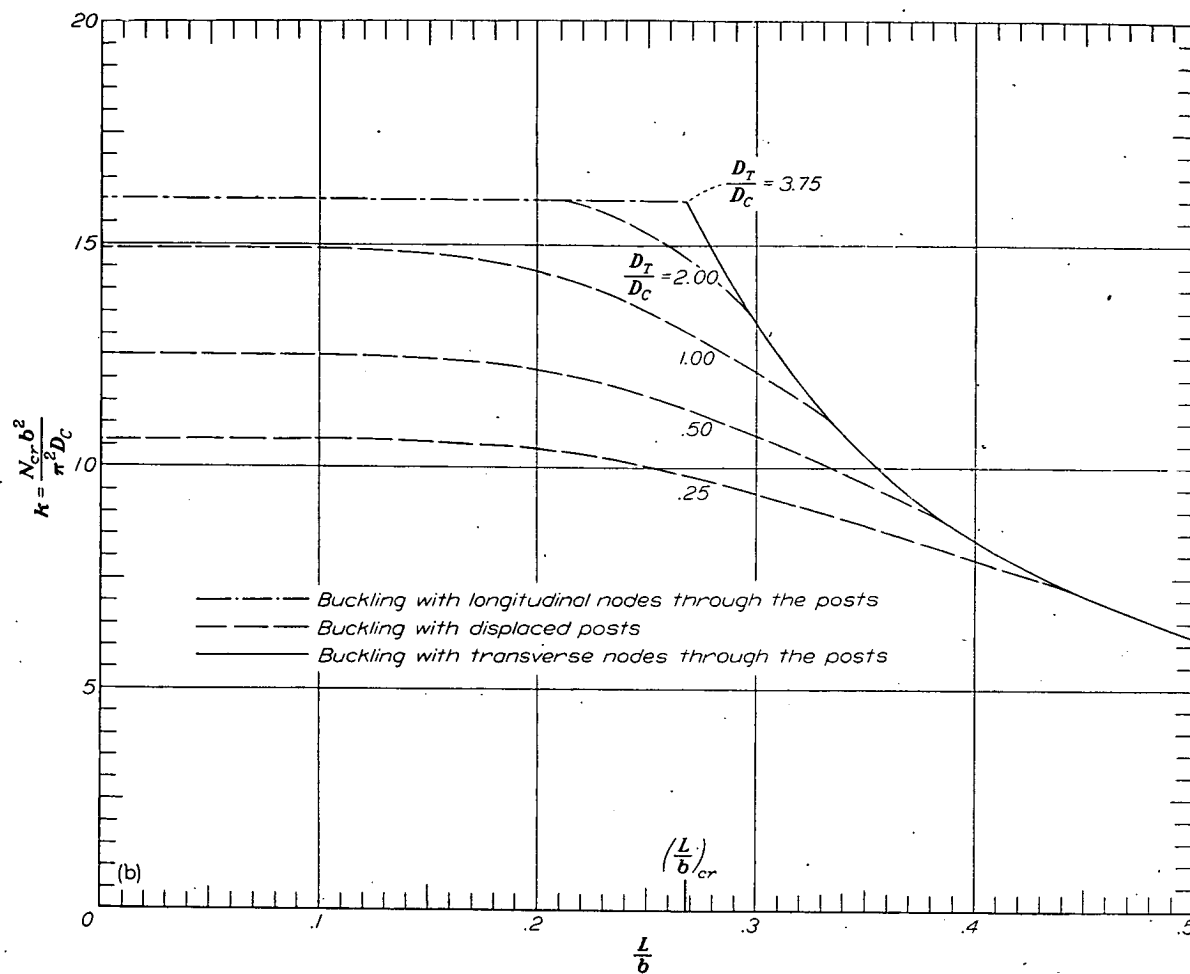


FIGURE 1.—Idealization of box beam stiffened by posts.



(a) Minimum post axial stiffness required or compression cover to buckle with nodes through the posts.

FIGURE 2.—One row of posts.



(b) Buckling loads for box beams with effectively rigid posts.

FIGURE 2.—Continued.

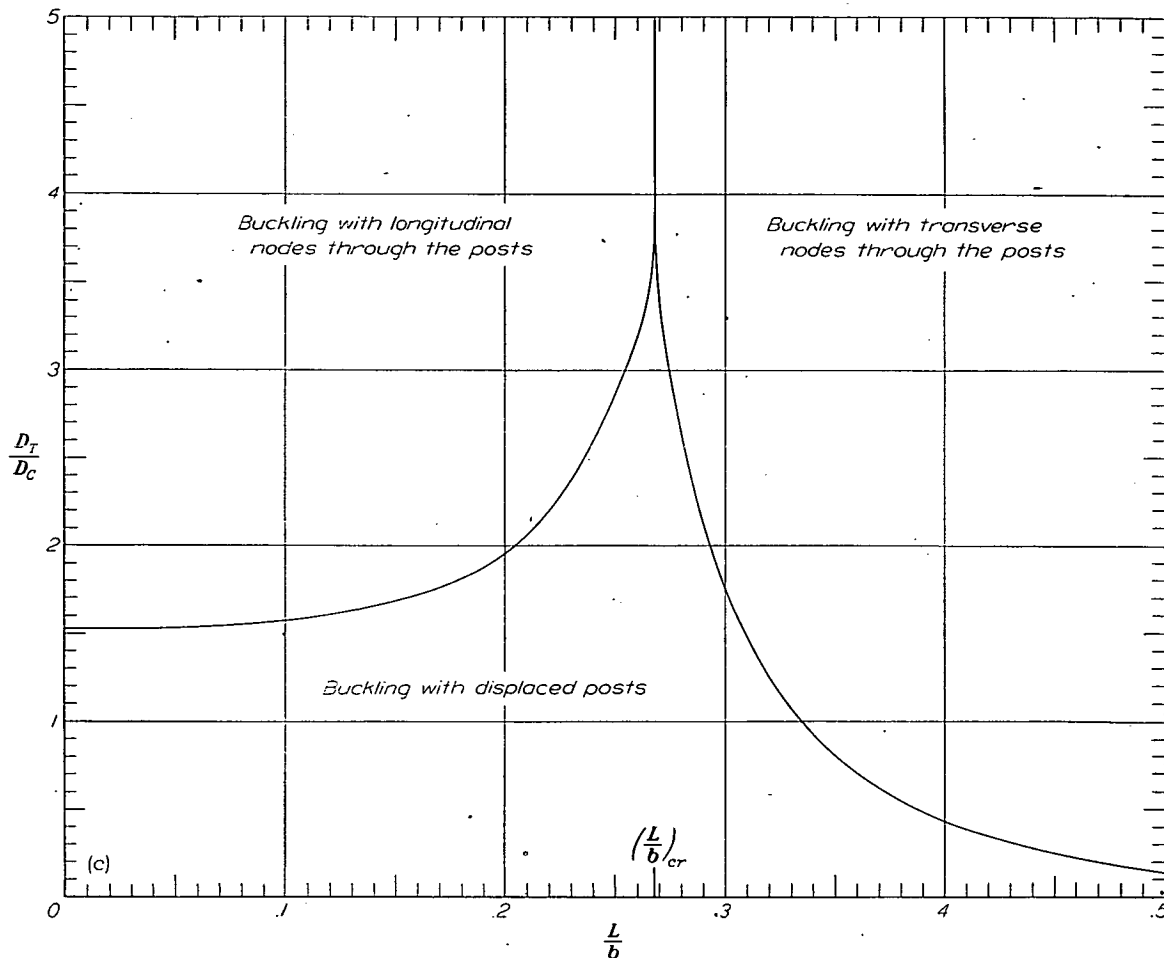
RESULTS AND DISCUSSION

The numerical results of the present investigation consist of values of the minimum post stiffnesses and the corresponding compressive buckling loads required for the compression surface of long post-stiffened box beams subjected to end bending moments to buckle with nodes through the posts. Because such buckle patterns do not transmit load to the posts, the posts are not compressed and are effectively rigid. Post axial stiffnesses greater than those presented herein will not serve to increase the buckling load of the structure. Combinations of the post axial-stiffness parameter and the compressive-buckling-load parameter, for different values of the ratio of the longitudinal distance between posts to the box-beam width and the ratio of the flexural stiffnesses of the tension and compression surfaces, are given in tables I to III and are presented graphically in figures 2 to 4 for one, two, and three rows of posts.

The most significant qualitative result to be read from the figures (especially figs. 2(c), 3(c), and 4(c)) is that buckling of the compression surface with longitudinal nodes through the posts is never obtainable for box beams in use in aircraft construction (for which the stiffness of the tension surface is at most equal to that of the compression surface),

even if the posts are infinitely stiff. Reference 1 shows that, if the tension surface and the posts are rigid, buckling of the compression surface occurs with longitudinal nodes through the posts when the ratio of the distance between the posts and the box-beam width is less than a certain value $\left(\frac{L}{b} < \left(\frac{L}{b}\right)_{cr}\right)$

and occurs with transverse nodes through the posts when this value is exceeded. These results are found to be valid when the flexural stiffness of the tension surface is as little as about 3.5 times that of the compression surface (the exact value of D_T/D_C depends on the number of rows of posts) and when the post axial stiffnesses are those given in figures 2 (a), 3 (a), and 4 (a). However, if the flexural-stiffness ratio is less than this value of about 3.5, neither longitudinal nor transverse nodes through the posts can be obtained for some values of L/b , even if the post axial stiffness is made infinite. The spread of these values of L/b is indicated by figures 2(c), 3(c), and 4(c); the buckling loads in this region when the posts are rigid are shown by the dashed-line curves of figures 2(b), 3(b), and 4(b). As the flexural-stiffness ratio decreases, the region in which the compression cover buckles with longitudinal nodes through the posts also decreases until, when the ratio is about 1.5 or less, the longitudinal-node buckle pattern is not obtainable.



(c) Buckling phenomena attainable for box beams with effectively rigid posts.

FIGURE 2.—Concluded.

LIMITATIONS ON USE OF CURVES IN ANALYSIS

The use of the curves of figures 2 to 4 for analysis is subject to many limitations. Inasmuch as the investigations of the present report are limited in scope, the buckling loads of plates having post stiffnesses less than the stiffness required for nodes to pass through the posts cannot be found in most cases. Certain conclusions, however, may be drawn from the present results to aid in the determination of whether a given load will be carried by a given plate-post system: namely,

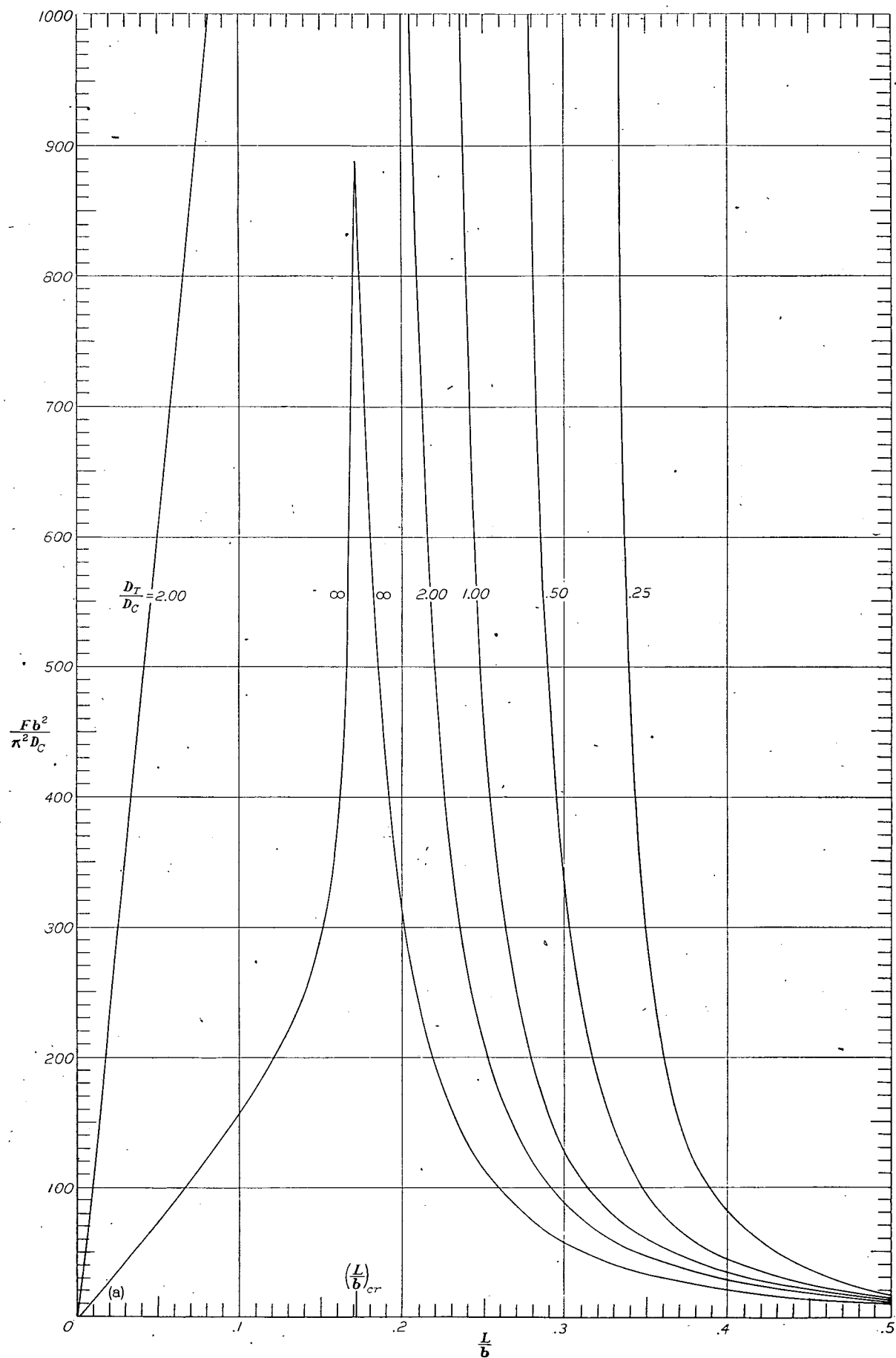
(1) If, for given values of the ratio of the flexural stiffnesses of the tension and compression surfaces and the ratio of longitudinal distance between posts and box-beam width, the required buckling-load coefficient lies above the appropriate curve of figures 2(b), 3(b), or 4(b), the load cannot be carried because these curves indicate the maximum buckling loads of long plate-post structures.

(2) If the buckling-load coefficient lies on or below the solid or dash-dotted part of the appropriate curve and the post axial-stiffness parameter is greater than or equal to the value required for nodes to pass through the posts, as given by figures 2(a), 3(a), or 4(a), the load can be carried.

(3) If the buckling-load coefficient lies on the appropriate solid or dash-dotted line curves, and the post axial-stiffness parameter is less than the required values, the load cannot be carried. Further investigation is required if the buckling-load coefficient lies below these curves.

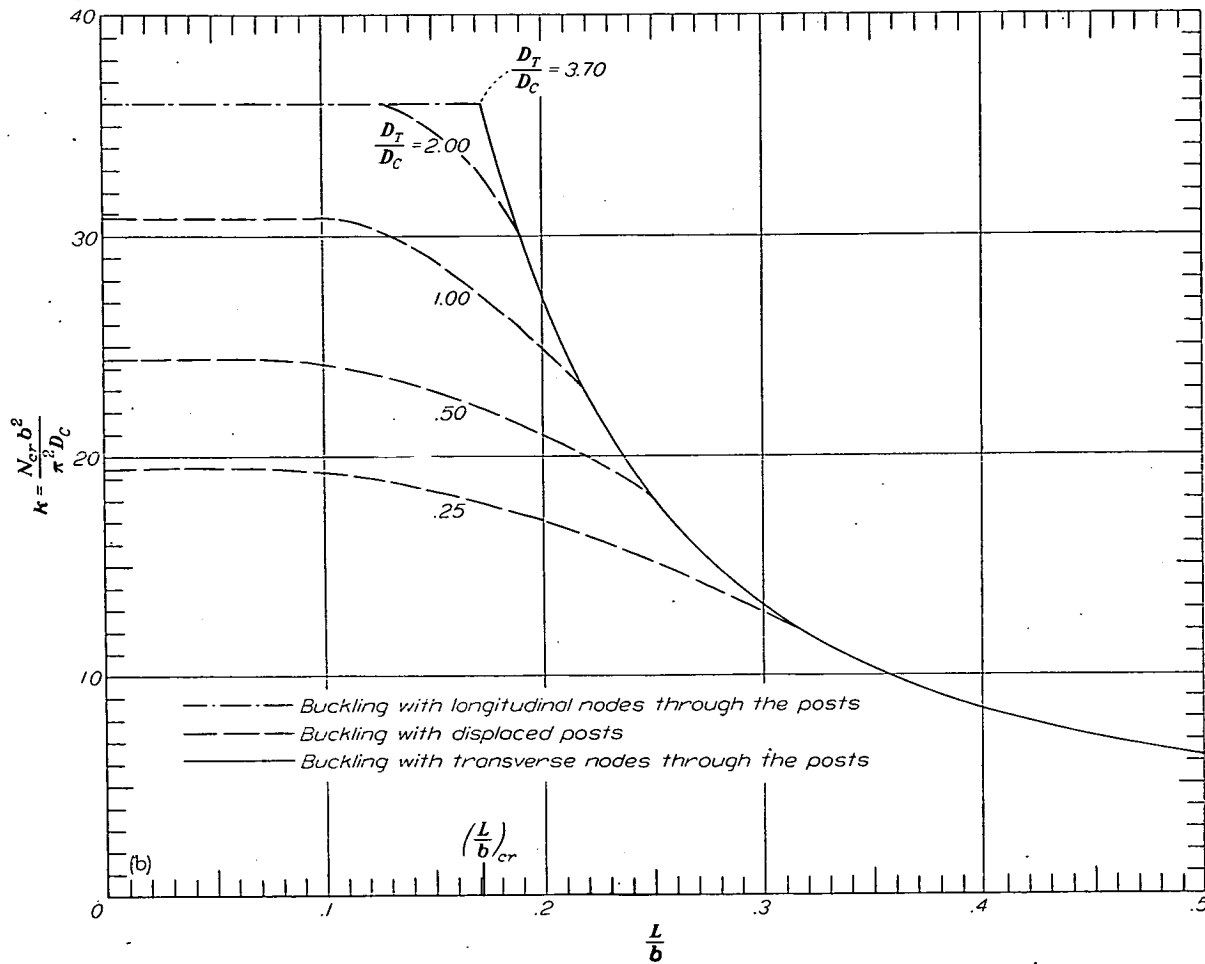
(4) If the buckling-load coefficient falls on the dashed part of the appropriate curve, the load can definitely not be carried because an infinite post stiffness would be needed. Further investigation is necessary if the buckling-load coefficient is below these curves.

Several simplifying assumptions that also may affect the applicability of the present results to the design and analysis of actual post-stiffened box beams have been made in the statement of the problem. The statement that the plates are flat and subjected to axial loading implicitly neglects the effects of bending of the plates and the resultant loads on the posts, due to the end moments. Possible buckling of the posts has not been considered but may be dealt with by making the moment of inertia of the post as large as possible. If hollow or built-up sections are used for this purpose, the thickness of the walls should be made large enough to prevent local buckling.



(a) Minimum post axial stiffness required for compression cover to buckle with nodes through the posts.

FIGURE 3.—Two rows of posts.



(b) Buckling loads for box beams with effectively rigid posts.

FIGURE 3.—Continued.

Because of the large number of computations involved, no investigations have been made to determine the adequacy of the present results in their application to plates of finite longitudinal length. It seems reasonable, however, in view of the results of references 2 to 4, to conclude that a finite plate having three posts or more in the longitudinal direction may be considered long, insofar as buckling with transverse nodes through the posts is concerned.

ILLUSTRATIVE DESIGN

As an illustration of the application of the various charts to a design problem, the posts are designed for a plate structure for which the following data are given:

Unit load, N_{cr} , pounds per inch.....	15,000
Compression-plate thickness, t_c , equals tension-plate thickness, t_T , inches.....	0.375
Plate width, b , inches.....	28
Vertical distance between plates, h , inches.....	5.5
Modulus of elasticity of plate material, E , psi.....	10.5×10^6
Poisson's ratio of plate material, μ	0.3

It is desired that the buckle pattern be such that nodal lines pass through the posts: This specification is equivalent to that of designing the ribs of skin-stringer construction so that buckling takes place between ribs.

The required compressive-buckling-load coefficient of the structure is given by $N_{cr}b^2/\pi^2D_c$ where

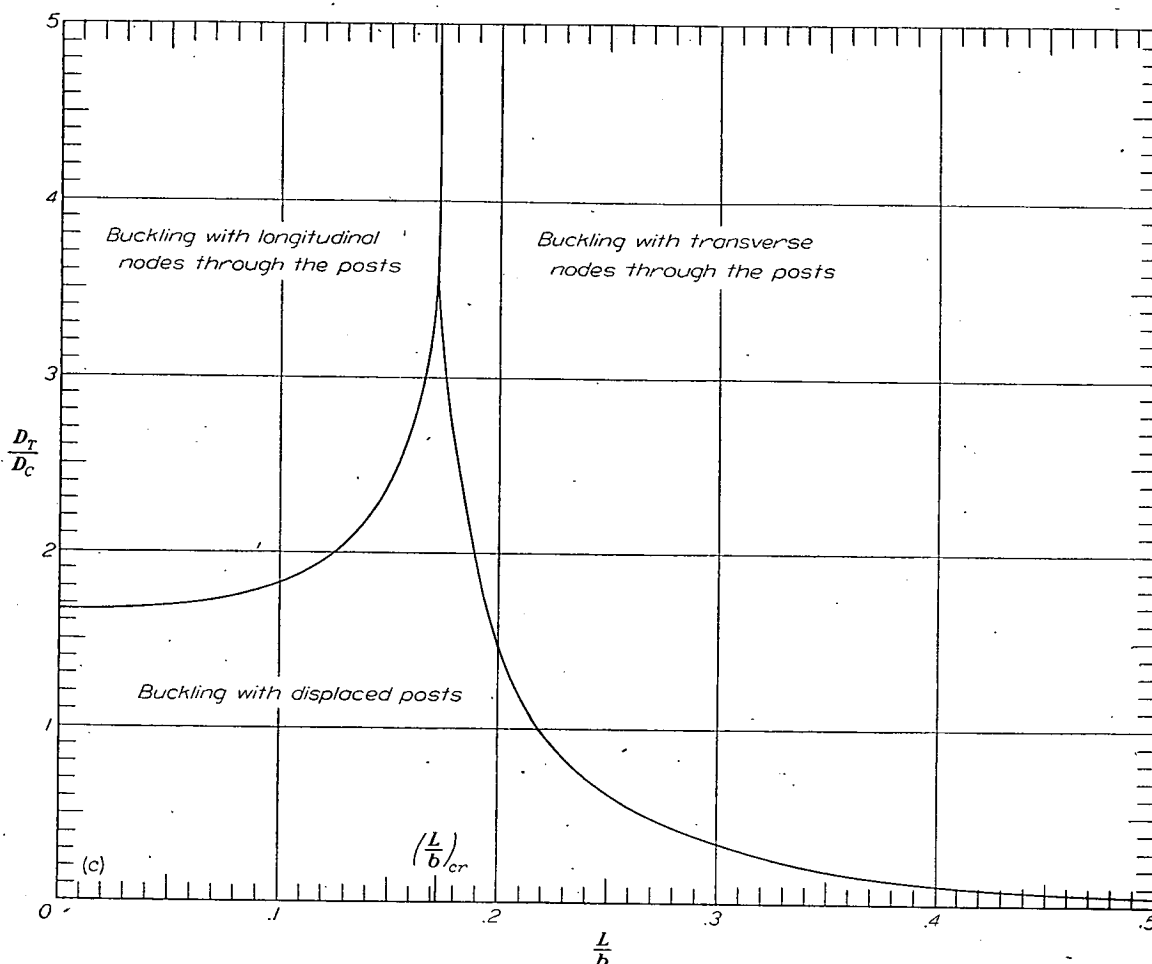
$$D_c = \frac{E_c t_c^3}{12(1-\mu_c^2)} \\ = \frac{10.5 \times 10^6 \times (0.375)^3}{12(1-0.3^2)} \\ = 50,700 \text{ pound-inches}$$

Then,

$$\frac{N_{cr}b^2}{\pi^2D_c} = \frac{15000 \times 28^2}{50700\pi^2} \\ = 23.5$$

Inasmuch as the thickness and material properties of the tension plate are identical with those of the compression plate, the values of D_T/D_c for the problem is unity.

The maximum buckling-load coefficient obtainable with any number of rows of posts is given by $\frac{N_{cr}b^2}{\pi^2D_c} = 4N^2$, the buckling-load coefficient of a long plate of width b/N . From figure 2(b) it can be seen that the required buckling-load coefficient may not be obtained with one row of posts. The coefficient could be obtained by using two rows of posts but the required buckle pattern of nodes passing through the



(c) Buckling phenomena attainable for box beams with effectively rigid posts.

FIGURE 3.—Concluded.

posts could not be obtained. At least three rows of posts must be used.

If three rows are used, the curves of figure 4(b) indicate that, for D_T/D_C equal to 1.00, a buckle pattern with transverse nodes through the posts is obtainable. With $N_{cr}b^2/\pi^2D_C$ equal to 23.5, the corresponding value of L/b is equal to 0.215. Hence, the maximum permissible longitudinal post spacing L is given by

$$\begin{aligned} L &= b \frac{L}{b} \\ &= 28 \times 0.215 \\ &= 6.0 \text{ inches} \end{aligned}$$

The minimum required post stiffness may now be obtained from figure 4 (a).

With L/b equal to 0.215 and D_T/D_C equal to 1.00, the corresponding value of Fb^2/π^2D_C is equal to 300. The post stiffness is then

$$\begin{aligned} F &= \frac{\pi^2 D_C}{b^2} \frac{Fb^2}{\pi^2 D_C} \\ &= \frac{50700\pi^2}{(28)^2} 300 \\ &= 191,000 \text{ pounds per inch} \end{aligned}$$

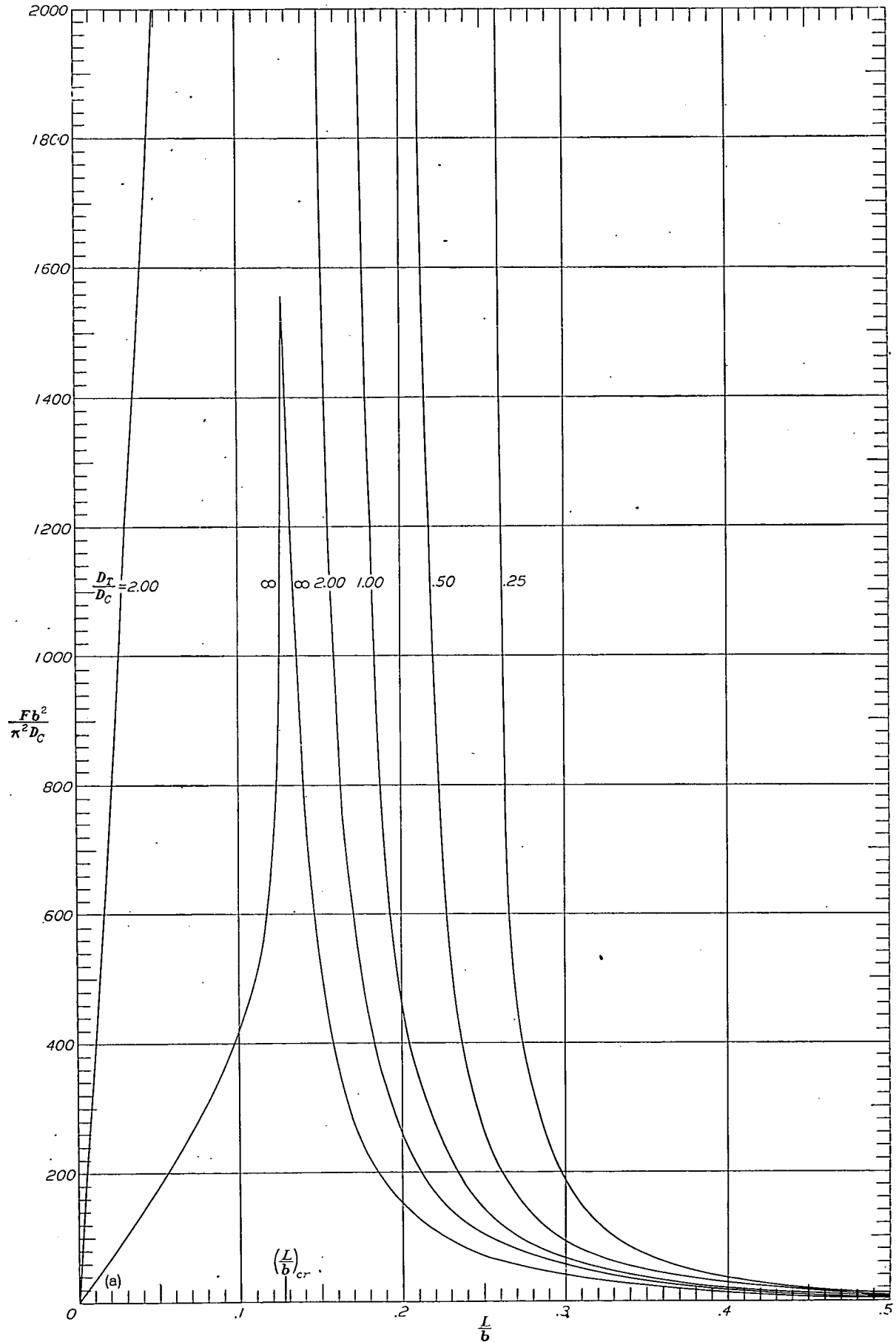
If the posts were of aluminum, the required area would be

$$\begin{aligned} A &= \frac{h}{E} F \\ &= \frac{5.5 \times 191000}{10.5 \times 10^6} \\ &= 0.100 \text{ square inch} \end{aligned}$$

The diameter of a circular rod would be

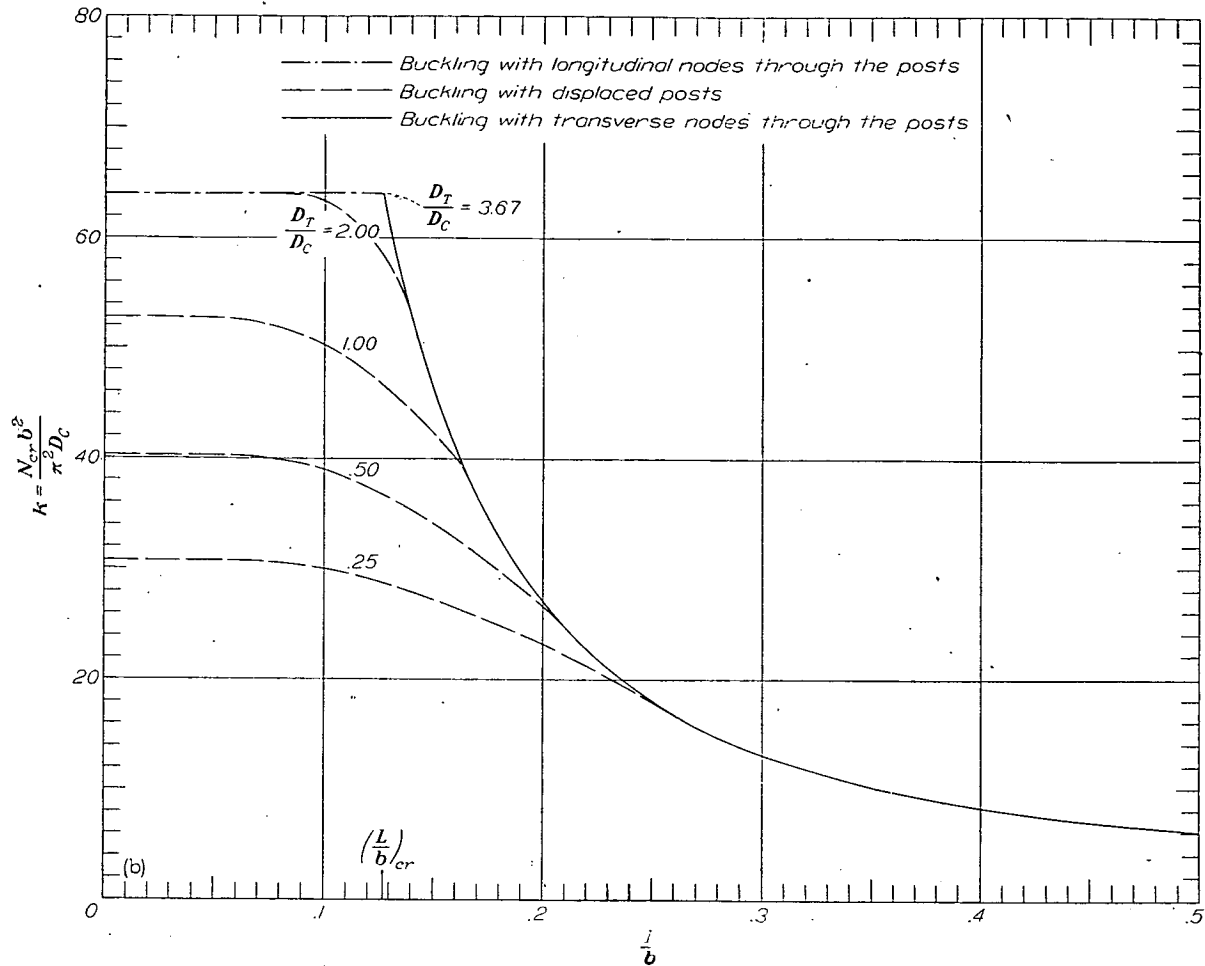
$$\begin{aligned} d &= \sqrt{\frac{4}{\pi} \times 0.100} \\ &= 0.36 \text{ inch} \end{aligned}$$

The ratio of the weights of posts and compression cover is 0.026. This ratio is appreciably lower than that of the weights of ribs and compression surface in conventional skin-stringer construction, given in reference 5 as about 0.5 for minimum weight design, and represents a great saving in stiffening material. The question of whether such low weight ratios are obtainable for actual box beams in which factors other than buckling of the compression cover must be considered in design cannot be completely answered until many designs utilizing post stiffening have been made.



(a) Minimum post axial stiffness required for compression cover to buckle with nodes through the posts.

FIGURE 4.—Three rows of posts.



(b) Buckling loads for box beams with effectively rigid posts.

FIGURE 4.—Continued.

CONCLUDING REMARKS

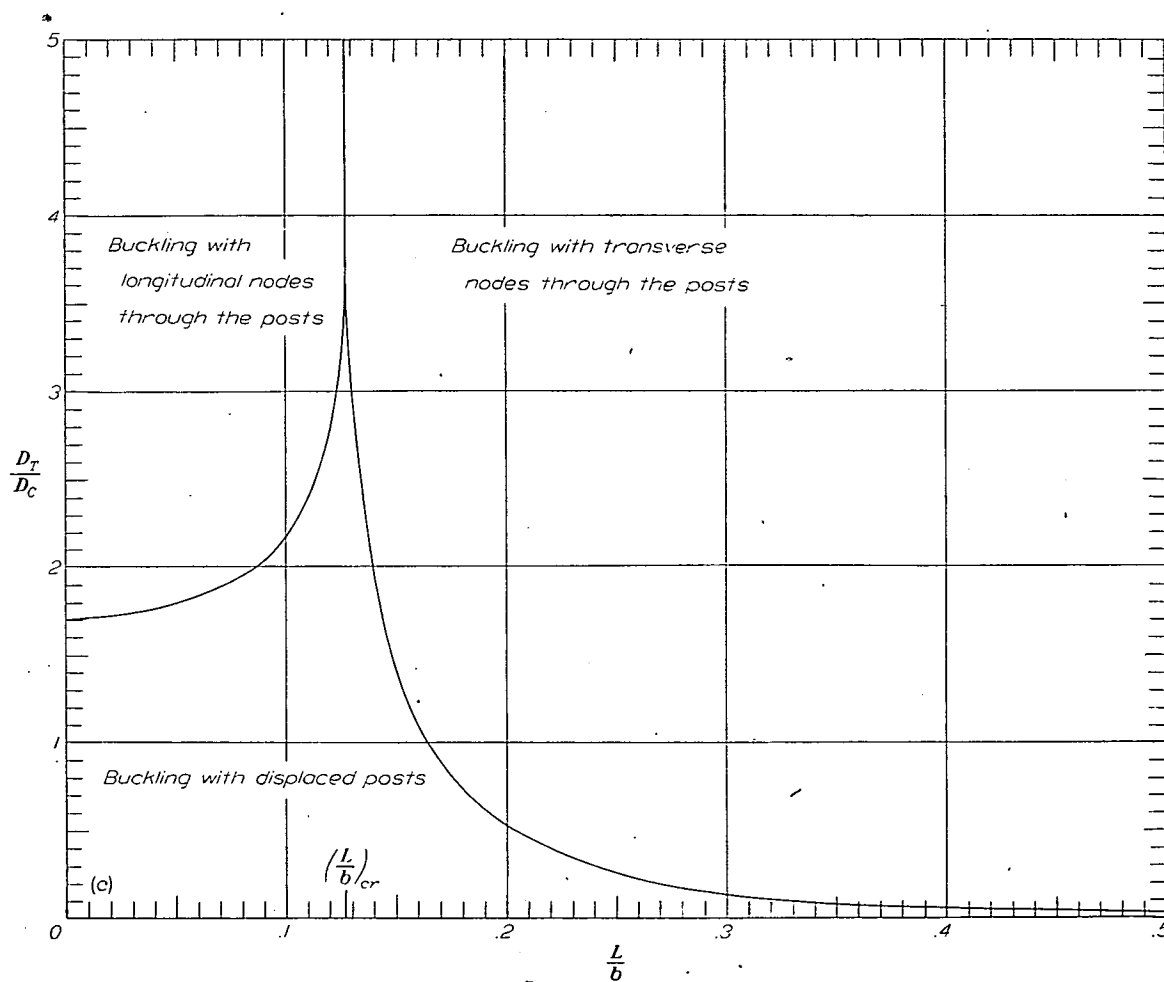
The minimum post axial stiffness and the corresponding buckling loads required for the compression cover of a box beam stiffened by posts and subjected to end moments to buckle with nodes through the posts have been determined. Where buckling with nodes through the posts is not obtainable, even with rigid posts, the buckling loads of box beams with rigid posts are given.

If the flexural stiffness of the tension cover is equal to or less than that of the compression cover, buckle patterns with longitudinal nodes are not obtainable, no matter how

stiff the posts are made or how closely they are spaced.

For the particular example worked in the present report the weight of the required posts is very low compared with the weight of the compression cover, the ratio of the two being about 2½ percent. A complete evaluation of post stiffening is not possible, however, until many more designs and tests have been made.

LANGLEY AERONAUTICAL LABORATORY,
 NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
 LANGLEY FIELD, VA., May 10, 1950.



(c) Buckling phenomena attainable for box beams with effectively rigid posts.

FIGURE 4.—Concluded.

APPENDIX

DETAILS OF ANALYSIS AND COMPUTATIONS

DERIVATION OF CRITERIA FOR STABILITY OF PLATE STRUCTURES STIFFENED BY POSTS

The following problem is more general than the one outlined in the introduction:

Two flat rectangular plates of equal finite length and width, simply supported along their edges and of unequal thickness and material properties, are connected in the interior by identical axially elastic rods in rectangular array. The connection between the posts and plates is one of simple support, of the ball-and-socket type, for instance, so that no resistance is offered to rotation of the plate. The plates are subjected to unequal axial loads. Both the longitudinal and transverse spacing of the rods are uniform; the two spacings, however, may differ. (See fig. 1.) The combination of loads that will cause the structure to buckle is to be found.

The Rayleigh-Ritz energy method used to analyze this problem consists of the following procedure. Fourier series to represent the deflected shape of the plates are first assumed. The energy of the various components of the structure and the work done by the external loads are then expressed in terms of the unknown Fourier coefficients, and the potential-energy expression of the structure is written. When the potential-energy expression is minimized with respect to each of the coefficients, a set of simultaneous equations is obtained which is manipulated in the manner of references 2 and 6 to yield finally the stability criteria for the structure.

The deflected shapes of the two buckled plates may be expressed generally by the double Fourier series

$$w_c = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn_c} \sin \frac{m\pi x}{ML} \sin \frac{n\pi y}{b} \quad (1)$$

$$w_T = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn_T} \sin \frac{m\pi x}{ML} \sin \frac{n\pi y}{b} \quad (2)$$

These expressions satisfy the condition of simple support on all four edges of each plate.

The energy stored in the buckled plates is then

$$\begin{aligned} V_C + V_T &= \frac{D_C}{2} \int_0^{ML} \int_0^b \left(\frac{\partial^2 w_C}{\partial x^2} + \frac{\partial^2 w_C}{\partial y^2} \right)^2 dx dy + \\ &\quad \frac{D_T}{2} \int_0^{ML} \int_0^b \left(\frac{\partial^2 w_T}{\partial x^2} + \frac{\partial^2 w_T}{\partial y^2} \right)^2 dx dy \\ &= \frac{\pi^4 b}{8(ML)^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[m^2 + \left(\frac{ML}{b} \right)^2 n^2 \right]^2 (D_C a_{mn_c}^2 + D_T a_{mn_T}^2) \end{aligned} \quad (3)$$

and that stored in the deformed posts is

$$\begin{aligned} V_P &= \sum_{c=1}^{M-1} \sum_{d=1}^{N-1} \frac{F}{2} (w_C - w_T)^2 \\ &= \frac{F}{2} \sum_{c=1}^{M-1} \sum_{d=1}^{N-1} \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn_c} - a_{mn_T}) \sin \frac{m\pi c}{M} \sin \frac{n\pi d}{N} \right]^2 \end{aligned} \quad (4)$$

The work done by the loads in moving with the ends of the plates is

$$\begin{aligned} W_C + W_T &= \frac{N_{x_c}}{2} \int_0^{ML} \int_0^b \left(\frac{\partial w_C}{\partial x} \right)^2 dx dy - \\ &\quad \frac{N_{x_T}}{2} \int_0^{ML} \int_0^b \left(\frac{\partial w_T}{\partial x} \right)^2 dx dy \\ &= \frac{\pi^2 b}{8ML} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 (N_{x_c} a_{mn_c}^2 - N_{x_T} a_{mn_T}^2) \end{aligned} \quad (5)$$

The potential energy is

$$\begin{aligned} U &= V_C + V_T + V_P - W_C - W_T \\ &= \frac{\pi^4 b}{8(ML)^3} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left[m^2 + \left(\frac{ML}{b} \right)^2 n^2 \right]^2 (D_C a_{mn_c}^2 + D_T a_{mn_T}^2) + \\ &\quad \frac{F}{2} \sum_{c=1}^{M-1} \sum_{d=1}^{N-1} \left[\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn_c} - a_{mn_T}) \sin \frac{m\pi c}{M} \sin \frac{n\pi d}{N} \right]^2 - \\ &\quad \frac{\pi^2 b}{8ML} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} m^2 (N_{x_c} a_{mn_c}^2 - N_{x_T} a_{mn_T}^2) \end{aligned} \quad (6)$$

The buckling load is determined by the condition that the potential energy be a minimum; that is,

$$\frac{\partial U}{\partial a_{ij_c}} = \frac{\partial U}{\partial a_{ij_T}} = 0 \quad \begin{aligned} (i=1, 2, \dots, \infty) \\ (j=1, 2, \dots, \infty) \end{aligned}$$

or

$$\begin{aligned} [(i^2 + j^2 M^2 \beta^2)^2 - k_c i^2 M^2 \beta^2] a_{ij_c} + \frac{4M^3 \beta^3 S}{\pi^2} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn_c} - \\ a_{mn_T}) \left(\sum_{c=1}^{M-1} \sin \frac{m\pi c}{M} \sin \frac{i\pi c}{M} \right) \left(\sum_{d=1}^{N-1} \sin \frac{n\pi d}{N} \sin \frac{j\pi d}{N} \right) = 0 \end{aligned} \quad (7)$$

and

$$\begin{aligned} [(i^2 + j^2 M^2 \beta^2)^2 + k_T i^2 M^2 \beta^2] a_{ij_T} - \frac{4M^3 \beta^3 S}{\pi^2 R} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn_c} - \\ a_{mn_T}) \left(\sum_{c=1}^{M-1} \sin \frac{m\pi c}{M} \sin \frac{i\pi c}{M} \right) \left(\sum_{d=1}^{N-1} \sin \frac{n\pi d}{N} \sin \frac{j\pi d}{N} \right) = 0 \end{aligned} \quad (8)$$

(i=1, 2, ... ∞)
(j=1, 2, ... ∞)

Equations (7) and (8) may be combined to yield

$$\begin{aligned} (a_{ij_c} - a_{ij_T}) + \frac{4M^3 \beta^3 S}{\pi^2} \left[\frac{1}{(i^2 + j^2 M^2 \beta^2)^2 - k_c i^2 M^2 \beta^2} + \right. \\ \left. \frac{1/R}{(i^2 + j^2 M^2 \beta^2)^2 + k_T i^2 M^2 \beta^2} \right] \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (a_{mn_c} - \\ a_{mn_T}) \left(\sum_{c=1}^{M-1} \sin \frac{m\pi c}{M} \sin \frac{i\pi c}{M} \right) \left(\sum_{d=1}^{N-1} \sin \frac{n\pi d}{N} \sin \frac{j\pi d}{N} \right) = 0 \end{aligned} \quad (9)$$

(i=1, 2, ... ∞)
(j=1, 2, ... ∞)

unless $a_{i,c} = a_{i,T}$, in which case

$$k_c = -k_T$$

$$= \left(\frac{i}{M\beta} + j^2 \frac{M\beta}{i} \right)^2 \quad \begin{matrix} (i=1, 2, \dots, \infty) \\ (j=1, 2, \dots, \infty) \end{matrix} \quad (10)$$

which indicates that both plates are subjected to their respective Euler buckling loads.

The remaining details of the analysis are similar to those of references 2 and 6 and are therefore omitted. In summary, the series

$$\left(\sum_{c=1}^{M-1} \sin \frac{m\pi c}{M} \sin \frac{i\pi c}{M} \right) \left(\sum_{d=1}^{N-1} \sin \frac{n\pi d}{N} \sin \frac{j\pi d}{N} \right)$$

may be evaluated by means of the table on page 8 of reference 2. Equations (9) are then seen to separate into independent sets, each of which may be manipulated to yield a stability criterion. The resulting criteria are

$$\frac{1}{S} - \frac{\beta^3 N}{\pi^2} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \frac{1}{k_c \beta^2 \left(2s + \frac{q}{M} \right)^2 - \left[\left(2s + \frac{q}{M} \right)^2 + (2rN + p)^2 \beta^2 \right]^2} +$$

$$\frac{\beta^3 N}{\pi^2 R} \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} \frac{1}{k_T \beta^2 \left(2s + \frac{q}{M} \right)^2 + \left[\left(2s + \frac{q}{M} \right)^2 + (2rN + p)^2 \beta^2 \right]^2} = 0 \quad (11)$$

$$\begin{matrix} (q=1, 2, \dots, M-1) \\ (p=1, 2, \dots, N-1) \end{matrix}$$

$$k_c = -k_T$$

$$= \left(\frac{s}{\beta} + r^2 \frac{M^2 \beta}{s} \right)^2 \quad \begin{matrix} (r=1, 3, 5, \dots, \infty) \\ (s=1, 3, 5, \dots, \infty) \end{matrix} \quad (12)$$

and

$$k_c = -k_T$$

$$= \left(\frac{s}{\beta} + r^2 \frac{M^2 \beta}{s} \right)^2 \quad \begin{matrix} (r=2, 4, 6, \dots, \infty) \\ (s=2, 4, 6, \dots, \infty) \end{matrix} \quad (13)$$

Equations (10) to (13) constitute the complete set of stability criteria for the problem.

COMPUTATION DETAILS

The computations may be divided into three main groups:

(a) The determination of the minimum post stiffness required for the compression cover plates to buckle with longitudinal or transverse nodes through the posts.

(b) In those regions where the required stiffness does not exist, the determination of the buckling load attainable with infinitely stiff posts.

(c) The determination of the boundaries of the various regions.

Equation (11), which was used to make the calculations, involves nine quantities, eight of which must be known in order that the ninth may be determined. These nine quantities are the post axial-stiffness parameter ($Fb^2/\pi^2 D_C$ or S), the ratio of the distance between posts and the box-beam width (L/b or β), the two axial-load parameters ($N_{x_c} b^2/\pi^2 D_C$ or k_c and $N_{x_T} b^2/\pi^2 D_T$ or k_T), the ratio of the flexural stiffnesses

of the tension and compression cover (D_T/D_C or R), the number of bays in the longitudinal and transverse directions (M and N), and the integers which indicate the number of buckles in the longitudinal and transverse directions (q and p). Values of these quantities are determined by the details of the problem to be solved. In all the computations for the present report, the box beam has been taken to be infinitely long.

Buckling with transverse nodes through the posts.—If the compression cover buckles with transverse nodes through the posts, the buckling load coefficient is

$$k_c = \left(\frac{1}{\beta} + \beta \right)^2$$

which is the compressive-buckling-load coefficient of a simply supported plate of length L and width b . The load coefficient in the tension cover was chosen for the present report as

$$k_T = \frac{k_c}{R}$$

in which case

$$N_{x_c} = N_{x_T} = N_{cr}$$

The minimum required post axial-stiffness parameter is completely determined by equation (11) for specified values of the cover flexural-stiffness ratio, the length-width ratio, and the number of rows of posts, if the number of buckles in the longitudinal and transverse directions are also known.

The ratio q/M is indicative of the number of buckles in a longitudinal bay of the buckled plate: It seemed reasonable, in view of the results of references 2 to 4, to conclude that q/M should be taken as unity, since it may be assumed that the transition from buckling with deformation of the posts to buckling with transverse nodes through the posts is continuous for structures with an infinite number of longitudinal bays and not abrupt as would be the case if the number of longitudinal bays were finite. One buckle occurs in the transverse direction and, therefore, p was taken equal to unity.

Buckling with longitudinal nodes through the posts.—When the ratio of the longitudinal spacing between posts to the box-beam width becomes small, the buckle pattern abruptly changes from one with transverse nodes through the posts to one with longitudinal nodes through the posts. The value of the width ratio at which this phenomenon occurs was found by equating the buckling loads for the two types of buckling because the two are equal at the point of transition. The compressive-buckling-load coefficient for a long simply supported plate of width b/N is given by

$$k_c = 4N^2$$

The critical length-width ratio was, therefore, found from the relationship

$$4N^2 = \left(\frac{1}{\beta} + \beta \right)^2$$

or

$$\beta_{cr} = N \left(1 - \sqrt{1 - \frac{1}{N^2}} \right)$$

The minimum post axial-stiffness parameter required for buckling to occur with longitudinal nodes through the posts can be found from equation (11) provided that the ratio q/M and the number of transverse buckles are known. The calculations of reference 7 for compressive buckling of plates with longitudinal stiffeners indicate that the change from buckling with deflection of the stiffeners to buckling with nodes at the stiffeners is abrupt and that only one transverse buckle should be assumed for the determination of minimum required flexural stiffness. It seemed logical that this condition should be assumed to hold also for plates stiffened by posts and, accordingly, p was taken equal to unity. The value of q/M was determined by graphically minimizing the post axial-stiffness parameter with respect to q/M ; that is, various values of q/M were chosen, the corresponding values of S were obtained from equation (11), and the correct value of S was determined by drawing a curve through the computed values and picking off the minimum.

Boundary between regions.—A third type of buckling is found to appear for relatively low values of the cover flexural-stiffness ratio. For instance, when R was equal to 2.00 the minimum post axial stiffness required for either longitudinal or transverse node buckle patterns was found to be unobtainable for a certain range of values of the length-width ratio. Further investigation showed that the post axial stiffness required for a longitudinal node buckle pattern became infinite at a certain value of the length-width ratio β less than β_{cr} and that the post axial stiffness required for a transverse node buckle pattern became infinite at some value of β greater than β_{cr} . Between these two values, only buckle patterns with deformation of the posts are obtainable. (If the posts are rigid, they displace axially but are not compressed.)

The maximum value of the cover flexural-stiffness ratio R at which this phenomenon is found to occur was determined by setting the post axial-stiffness parameter equal to infinity, the length-width ratio equal to β_{cr} , the load coefficients k_c and Rk_r equal to $\left(\frac{1}{\beta_{cr}} + \beta_{cr} \right)^2$, the ratio q/M and the number of transverse buckles equal to unity, and by solving equation (11) graphically. This procedure determines the value of R for which the spread of the range of β , in which

the third type of buckle pattern occurs, vanishes and thus determines the maximum value of R for which the phenomenon occurs.

The boundaries between the regions in which the various buckle patterns occur were found by placing the post axial stiffness equal to infinity and using a procedure similar to that of the previous sections to determine the length-width ratio β for various values of R .

When the cover flexural-stiffness ratio is approximately 1.5, the value of β marking the transition from buckling with longitudinal nodes through the posts and buckling with the displacement of the rigid posts is found to become equal to zero, and this fact thus indicates that the longitudinal node buckle pattern is unobtainable for values of R less than 1.5. Because the stability criterion becomes indeterminate when β is equal to zero, more accurate values of the cover flexural-stiffness ratio for which the phenomenon occurs were found by placing β equal to the low value of 0.01 and finding R graphically from equation (11).

Buckling loads attainable with rigid posts.—The buckling loads for various values of R and β in the region where the compression cover buckles with axial displacement of the rigid posts were found by graphically minimizing the loads determined from equation (11) with respect to q/M , with the post axial-stiffness parameter equal to infinity, and with one buckle in the transverse direction.

REFERENCES

1. Budiansky, Bernard: Compressive Buckling of Flat Rectangular Plates Supported by Rigid Posts. NACA RM L8I30b, 1948.
2. Budiansky, Bernard, Seide, Paul, and Weinberger, Robert A.: The Buckling of a Column on Equally Spaced Deflectional and Rotational Springs. NACA TN 1519, 1948.
3. Seide, Paul, and Eppler, John F.: The Buckling of Parallel Simply Supported Tension and Compression Members Connected by Elastic Deflectional Springs. NACA TN 1823, 1949.
4. Budiansky, Bernard, and Seide, Paul: Compressive Buckling of Simply Supported Plates with Transverse Stiffeners. NACA TN 1557, 1948.
5. Farrar, D. J.: The Design of Compression Structures for Minimum Weight. Jour. R.A.S., vol. 53, no. 467, Nov. 1949, pp. 1041-1052.
6. Cox, H. L., and Smith, H. E.: The Buckling of Grids of Stringers and Ribs. Proc. London Math. Soc., ser. 2, vol. 48, 1943.
7. Seide, Paul, and Stein, Manuel: Compressive Buckling of Simply Supported Plates with Longitudinal Stiffeners. NACA TN 1825, 1949.

TABLE I
DATA FOR ONE ROW OF POSTS

$R=\infty$			
β	S	k	Type of buckling (°)
0.75	2.4	4.340	A
.50	15.9	6.250	A
.45	24.4	7.141	A
.40	39.3	8.410	A
.37	54.9	9.441	A
.35	70.7	10.286	A
.33	94.6	11.292	A
.30	163.2	13.201	A
.28	266.1	14.834	A
.268	404.9	16.000	A or C
.225	117.7	16.000	C
.20	94.9	16.000	C
.15	65.0	16.000	C
.10	42.5	16.000	C
.05	20.9	16.000	C

$R=2$			
β	S	k	Type of buckling (°)
0.75	2.6	4.341	A
.50	20.6	6.250	A
.45	34.6	7.141	A
.40	63.4	8.410	A
.37	101.7	9.441	A
.35	151.2	10.286	A
.33	261.5	11.292	A
.31	659.3	12.502	A
.30	1715.4	13.201	A
.294	∞	13.6	A or B
.25	∞	13.3	B
.207	∞	16.000	B or C
.15	522.6	16.000	C
.10	253.7	16.000	C
.075	176.1	16.000	C
.05	118.9	16.000	C

$R=1$			
β	S	k	Type of buckling (°)
0.75	2.7	4.340	A
.50	24.4	6.250	A
.45	46.5	7.141	A
.40	102.8	8.410	A
.37	217.5	9.441	A
.35	544.4	10.286	A
.335	∞	11.0	A or B
.25	∞	11.6	B
.10	∞	14.9	B

$R=0.5$			
β	S	k	Type of buckling (°)
0.75	2.8	4.340	A
.54	22.2	5.721	A
.50	34.1	6.250	A
.45	84.3	7.141	A
.43	137.7	7.593	A
.41	337.3	8.117	A
.40	566.4	8.410	A
.388	∞	8.8	A or B
.25	∞	11.6	B
.10	∞	12.5	B
.05	∞	12.5	B

$R=0.25$			
β	S	k	Type of buckling (°)
0.75	3.0	4.340	A
.60	14.9	5.138	A
.54	32.4	5.721	A
.50	60.6	6.250	A
.48	134.4	6.571	A
.47	222.5	6.748	A
.46	434.1	6.937	A
.449	∞	7.2	A or B
.25	∞	10.0	B
.10	∞	10.6	B
.05	∞	10.6	B

- ° A Buckling with transverse nodes through the posts.
 B Buckling with deflection of the posts.
 C Buckling with longitudinal nodes through the posts.

TABLE II
DATA FOR TWO ROWS OF POSTS

$R=\infty$			
β	S	k	Type of buckling (°)
0.75	1.6	4.340	A
.50	9.8	6.250	A
.40	21.1	8.410	A
.35	32.3	10.286	A
.30	56.7	13.201	A
.27	84.9	15.970	A
.25	112.9	18.063	A
.22	193.8	22.710	A
.20	313.3	27.040	A
.18	598.6	32.897	A
.172	888.0	36.000	A or C
.15	292.3	36.000	C
.10	157.5	36.000	C
.05	75.2	36.000	C

$R=2$			
β	S	k	Type of buckling (°)
0.75	1.7	4.340	A
.50	11.9	6.250	A
.40	27.7	8.410	A
.35	46.2	10.286	A
.30	86.1	13.201	A
.27	143.4	15.970	A
.25	209.5	18.063	A
.22	490.8	22.710	A
.21	767.9	24.720	A
.20	1526.7	27.040	A
.189	∞	30.0	A or B
.15	∞	34.8	B
.126	∞	36.000	B or C
.10	1933.5	36.000	C
.075	918.7	36.000	C
.05	597.8	36.000	C
.025	282.7	36.000	C

$R=1$			
β	S	k	Type of buckling (°)
0.75	1.8	4.340	A
.50	13.4	6.250	A
.40	33.6	8.410	A
.35	59.6	10.286	A
.30	125.1	13.201	A
.27	248.6	15.970	A
.25	450.7	18.063	A
.235	1101.4	20.163	A
.219	∞	22.9	A or B
.15	∞	28.8	B
.10	∞	30.8	B
.05	∞	30.8	B

$R=0.5$			
β	S	k	Type of buckling (°)
0.75	1.9	4.340	A
.50	15.8	6.250	A
.40	44.0	8.410	A
.35	93.6	10.286	A
.30	330.1	13.201	A
.28	813.0	14.834	A
.27	7130.5	15.970	A
.268	∞	15.995	A or B
.20	∞	20.9	B
.15	∞	22.9	B
.10	∞	24.4	B
.05	∞	24.4	B

$R=0.25$			
β	S	k	Type of buckling (°)
0.75	1.9	4.340	A
.50	17.9	6.250	A
.45	35.0	7.141	A
.40	81.5	8.410	A
.37	148.8	9.441	A
.35	289.8	10.286	A
.33	1648.6	11.292	A
.325	∞	11.6	A or B
.20	∞	17.0	B
.15	∞	18.3	B
.10	∞	19.4	B
.05	∞	19.4	B

- ° A Buckling with transverse nodes through the posts.
 B Buckling with deflection of the posts.
 C Buckling with longitudinal nodes through the posts.

TABLE III
DATA FOR THREE ROWS OF POSTS

$R=\infty$			
β	S	k	Type of buckling (°)
0.75	1.2	4.340	A
.50	7.2	6.250	A
.34	26.1	10.766	A
.25	70.6	18.063	A
.20	153.1	27.040	A
.17	279.0	36.331	A
.15	527.5	46.467	A
.14	774.9	53.040	A
.127	1558.4	64.000	A or C
.10	420.4	64.000	C
.08	308.2	64.000	C
.04	144.1	64.000	C

$R=2$			
β	S	k	Type of buckling (°)
0.75	1.2	4.340	A
.50	8.6	6.250	A
.34	34.7	10.766	A
.25	104.1	18.063	A
.20	260.8	27.040	A
.17	595.2	36.331	A
.15	2295.5	46.467	A
.14	∞	53.0	A or B
.125	∞	59.1	B
.11	∞	62.3	B
.087	∞	64.000	B or C
.03	1151.2	64.000	C
.02	752.3	64.000	C
.01	357.3	64.000	C

$R=1$			
β	S	k	Type of buckling (°)
0.75	1.3	4.340	A
.50	9.6	6.250	A
.34	42.3	10.766	A
.29	76.5	13.975	A
.25	143.5	18.063	A
.20	456.2	27.040	A
.18	1253.7	32.897	A
.17	3473.9	36.331	A
.164	∞	39.2	A or B
.10	∞	50.3	B
.05	∞	52.8	B

$R=0.5$			
β	S	k	Type of buckling (°)
0.75	1.3	4.340	A
.50	10.9	6.250	A
.34	56.5	10.766	A
.32	69.2	11.868	A
.29	114.8	13.975	A
.27	167.7	15.970	A
.25	263.3	18.063	A
.23	534.8	20.957	A
.21	2097.5	24.720	A
.205	∞	25.8	A or B
.10	∞	38.9	B
.05	∞	40.3	B

$R=0.25$			
β	S	k	Type of buckling (°)
0.75	1.4	4.340	A
.50	12.7	6.250	A
.40	36.5	8.410	A
.34	87.5	10.766	A
.32	122.3	11.868	A
.29	247.4	13.975	A
.27	467.2	15.970	A
.26	1781.9	16.860	A
.254	∞	17.6	A or B
.175	∞	25.3	B
.10	∞	29.9	B
.05	∞	30.8	B

- A Buckling with transverse nodes through the posts.
 B Buckling with deflection of the posts.
 C Buckling with longitudinal nodes through the posts.

REPORT 1048

A STUDY OF EFFECTS OF VISCOSITY ON FLOW OVER SLENDER INCLINED BODIES OF REVOLUTION¹

By H. JULIAN ALLEN AND EDWARD W. PERKINS

SUMMARY

The observed flow field about slender inclined bodies of revolution is compared with the calculated characteristics based upon potential theory. The comparison is instructive in indicating the manner in which the effects of viscosity are manifest.

Based on this and other studies, a method is developed to allow for viscous effects on the force and moment characteristics of bodies. The calculated force and moment characteristics of two bodies of high fineness ratio are shown to be in good agreement, for most engineering purposes, with experiment.

INTRODUCTION

The problem of the longitudinal distribution of cross force on inclined bodies of revolution in inviscid, incompressible flow, which was primarily of interest to airship designers in the past, was treated simply and effectively by Max Munk (reference 1). Munk showed that the cross force per unit length on any body of revolution having high fineness ratio can be obtained by considering the flow in planes perpendicular to the axis of revolution to be approximately two-dimensional. By treating the problem in this manner, Munk showed that

$$f = q_0 \frac{dS}{dx} \sin 2\alpha \quad (1)$$

where

f cross force per unit length
 q_0 stream dynamic pressure
 dS/dx rate of change in body cross-sectional area with longitudinal distance along the body
 α angle of inclination

Tsien (reference 2) investigated the cross force on slender bodies of revolution at moderate supersonic speeds—a problem of more interest at the present to missile and supersonic aircraft designers—and showed that, to the order of the first power of the angle of inclination, the reduced Munk formula

$$f = 2q_0 \frac{dS}{dx} \alpha \quad (2)$$

was still applicable. This is not surprising when it is realized that the cross component of the flow field corresponds to a cross velocity

$$V_{v_0} = V_0 \sin \alpha$$

where V_0 is the stream velocity. Thus the cross component of velocity, and hence, the cross Mach number will, for small angles of inclination, have a small subsonic value so that the cross flow will be essentially incompressible in character.

Using equation (1) for the cross-force distribution, then, the total forces and moments experienced by a body in an inviscid fluid stream can be calculated. Comparison of the calculated and experimental characteristics of bodies has shown that the lift experienced exceeds the calculated lift in absolute value by an amount which is greater the greater the angle of attack; the center of pressure is farther aft than the calculations indicate, the discrepancy increasing with angle of attack; while the absolute magnitude of the moment about the center of volume is less than that calculated. It has long been known that these observed discrepancies are due primarily to the failure to consider the effects of viscosity in the flow.

Experience has demonstrated, notably in the development of airfoils, that the behavior of the boundary layer on a body is intimately associated with the nature of the pressure distribution that would exist on the body in inviscid flow. In particular, boundary-layer separation is associated with the gradient of pressure recovery on a body. The severity of the effect of such separation can be correlated, in part, with the magnitude of the total required pressure recovery indicated by inviscid theory. It is therefore to be expected that it will be of value to compare the actual pressure distribution on inclined bodies of revolution with that calculated on the assumption that the fluid is inviscid. For the purpose of this study, a simple method is developed for determining, for an inviscid fluid, the incremental pressure distribution resulting from inclined flow on a slender body of revolution.² The experimental incremental pressure distributions about an airship hull are compared with the corresponding distributions calculated by this method. The comparisons are instructive in indicating the manner in which the viscosity of the fluid influences the flow. In the light of this and other studies, a method for allowing for viscous effects on the force and moment characteristics of slender bodies is developed and the results compared with experiment.

² The problem of determining the pressure distribution on inclined bodies has been treated by other authors, but for several reasons these methods are not satisfactory for the present purposes. For example, Kaplan (reference 3) treated, in a thorough manner, the flow about slender inclined bodies, but the solution, which is expressed in Legendre polynomials, is unfortunately tedious to evaluate. On the other hand, Laitone (reference 4), by linearizing the equations of motion, obtained a solution for the pressure distribution on slender inclined bodies of revolution, but, as will be seen later, the solution is inadequate in the general case due to the linearization.

¹ Supersedes NACA TN 2044, "Pressure Distribution and Some Effects of Viscosity on Slender Inclined Bodies of Revolution" by H. Julian Allen, 1950.

SYMBOLS

A	reference area for body force and pitching-moment coefficient evaluation	S_b	body base area (at $x=L$)
A_p	plan-form area ($2 \int_0^L R dx$)	t	time
c_{d_c}	circular-cylinder section drag coefficient based on cylinder diameter	V_0	free-stream velocity
$c_{d_{\alpha=90^\circ}}$	local cross-flow drag coefficient at any x station based on body diameter	V_x	local axial velocity at body surface at any station x
C	constant of integration	V_{x_0}	axial component of the stream velocity ($V_0 \cos \alpha$)
$C_{d_{\alpha=90^\circ}}$	cross-flow drag coefficient ($\frac{\int_0^L 2c_{d_{\alpha=90^\circ}} R dx}{A_p}$)	V_{y_0}	cross-flow component of the stream velocity ($V_0 \sin \alpha$)
C_{D_F}	body foredrag coefficient ($\frac{\text{foredrag}}{q_0 A}$)	x	axial distance from bow of body to any body station
$C_{D_F(\alpha=0)}$	body foredrag coefficient at zero angle of inclination	x_m	axial distance from bow of body to pitching-moment center
ΔC_{D_F}	incremental foredrag coefficient due to inclination	$x_{\alpha=90^\circ}$	axial distance from bow of body to center of viscous cross force
C_L	body lift coefficient ($\frac{\text{lift}}{q_0 A}$)	X	reference length for moment coefficient evaluation
C_M	body pitching-moment coefficient about station x_m ($\frac{\text{pitching moment}}{q_0 A X}$)	y	ordinate in plane of inclination normal to axis of revolution
D'	mean body diameter ($\frac{A_p}{L}$)	z	ordinate normal to plane of inclination and to axis of revolution
f	local cross force (normal to body axis) at any station x on body	α	angle of body-axis inclination relative to free-stream-flow direction
L	body length	β	$\tan^{-1} \frac{dR}{dx}$
M_0	free-stream Mach number	ν	fluid kinematic viscosity
M_c	cross-flow Mach number ($M_0 \sin \alpha$)	θ	polar angle about axis of revolution measured from approach direction of the cross-flow velocity
p	local surface pressure	ρ	fluid mass density
p_0	free-stream static pressure	ϕ	velocity potential
$p_{\alpha=0}$	local surface pressure at zero angle of inclination		
P	local surface-pressure coefficient ($\frac{p-p_0}{q_0}$)		
$P_{\alpha=0}$	local surface-pressure coefficient at zero angle of inclination ($\frac{p_{\alpha=0}-p_0}{q_0}$)		
ΔP	incremental surface-pressure coefficient due to angle of inclination ($\frac{P-P_{\alpha=0}}{q_0}$)		
q_0	free-stream dynamic pressure		
Q	body volume		
r	polar radius about axis of revolution		
R	local body radius at any station x		
R_0	free-stream Reynolds number based on maximum body diameter		
R_c	cross-flow Reynolds number ($R_0 \sin \alpha$)		
R_c'	cross-flow Reynolds number based on diameter D'		
S	body cross-sectional area at station x		

PRESSURE DISTRIBUTION ON SLENDER INCLINED BODIES OF REVOLUTION

POTENTIAL FLOW THEORY

Consider the flow over the body of revolution shown in figure 1 which is inclined at an angle α to the stream of velocity V_0 . If the body is slender, the axial component velocity V_x at the body surface will not differ appreciably from the axial component V_{x_0} of the stream velocity. With this condition, it is clear that the cross flow may be treated

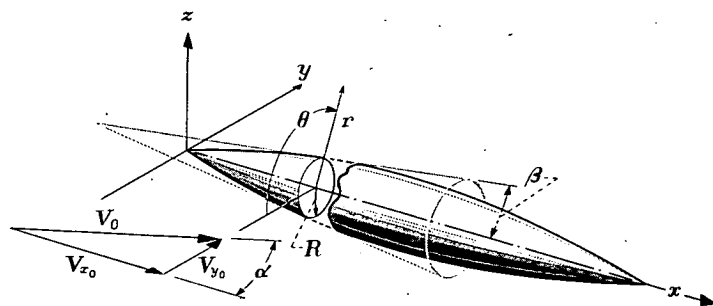


FIGURE 1.—Body of revolution in inclined flow field.

approximately by considering it to be two-dimensional in a plane which is parallel to the yz plane and is moving axially with the constant velocity V_{x_0} . In other words, the problem may be treated by determining the two-dimensional flow about a circular cylinder which is first growing (over the forebody) and then collapsing (over the afterbody) with time.

The velocity potential for the cross flow at any x station is given in polar coordinates as

$$\phi = -V_{x_0} \left(r + \frac{R^2}{r} \right) \cos \theta \quad (3)$$

which in this moving reference plane is a function of time.

Bernoulli's equation for an incompressible flow which changes with time is

$$\frac{p}{\rho} = -\frac{\partial \phi}{\partial t} - \frac{1}{2} \left[\left(\frac{\partial \phi}{\partial r} \right)^2 + \left(\frac{\partial \phi}{r \partial \theta} \right)^2 \right] + C \quad (4)$$

Now from equation (3)

$$\frac{\partial \phi}{\partial t} = -2 V_{x_0} \left(\frac{R}{r} \right) \frac{dR}{dt} \cos \theta \quad (5)$$

but

$$\frac{dR}{dt} = \frac{dx}{dt} \frac{dR}{dx} = V_{x_0} \tan \beta \quad (6)$$

so that equation (5) becomes

$$\frac{\partial \phi}{\partial t} = -2 V_{x_0} V_{x_0} \tan \beta \left(\frac{R}{r} \right) \cos \theta \quad (7)$$

Also, by differentiation of equation (3),

$$\left. \begin{aligned} \frac{\partial \phi}{\partial r} &= -V_{x_0} \cos \theta \left(1 - \frac{R^2}{r^2} \right) \\ \frac{\partial \phi}{r \partial \theta} &= V_{x_0} \sin \theta \left(1 + \frac{R^2}{r^2} \right) \end{aligned} \right\} \quad (8)$$

so that equation (4) for the pressure at any point in the flow field becomes

$$\begin{aligned} \frac{p}{\rho} &= 2 V_{x_0} V_{x_0} \tan \beta \left(\frac{R}{r} \right) \cos \theta - \frac{V_{x_0}^2}{2} \left\{ \cos^2 \theta \left[1 - \left(\frac{R}{r} \right)^2 \right]^2 + \right. \\ &\quad \left. \sin^2 \theta \left[1 + \left(\frac{R}{r} \right)^2 \right]^2 \right\} + C \end{aligned} \quad (9)$$

For

$$r \rightarrow \infty, p \rightarrow p_0$$

so

$$C = \frac{p_0}{\rho} + \frac{V_{x_0}^2}{2}$$

and hence equation (9) for the pressure at the surface of the body becomes for $r=R$

$$\frac{p-p_0}{\rho} = 2 V_{x_0} V_{x_0} \tan \beta \cos \theta + \frac{V_{x_0}^2}{2} (1 - 4 \sin^2 \theta) \quad (10)$$

and writing

$$V_{x_0} = V_0 \sin \alpha$$

$$V_{x_0} = V_0 \cos \alpha$$

the surface pressure in coefficient form becomes

$$P = \frac{p-p_0}{q_0} = 2 \tan \beta \cos \theta \sin 2\alpha + (1 - 4 \sin^2 \theta) \sin^2 \alpha \quad (11)$$

For bodies of moderate fineness ratio at zero angle of inclination, the surface pressure at any station, designated $p_{\alpha=0}$, will differ slightly from the static pressure p_0 but, if the fineness ratio is not too low, the pressure, $p_{\alpha=0}$, in any yz plane will be approximately constant for several body radii from the surface. Under the assumption that the pressure at the surface at zero inclination applies uniformly in the portion of the yz plane for which the major effects of the cross-flow distribution are felt, the change in pressure from p_0 to $p_{\alpha=0}$ will be additive to, but will not otherwise influence, the cross-flow pressure distribution. Hence for any station on a body of high fineness ratio for which at zero inclination the pressure is $p_{\alpha=0}$, the pressure coefficient distribution at this same station under inclined flow conditions will be; from equation (11),

$$P = P_{\alpha=0} + (2 \tan \beta \cos \theta) \sin 2\alpha + (1 - 4 \sin^2 \theta) \sin^2 \alpha \quad (12)$$

For very slender bodies at small angles of inclination

$$\tan \beta \cong \beta$$

$$\sin 2\alpha \cong 2\alpha$$

$$\sin^2 \alpha \cong \alpha^2$$

so that equation (12) becomes³

$$P = P_{\alpha=0} + (4 \cos \theta) \beta \alpha + (1 - 4 \sin^2 \theta) \alpha^2 \quad (13)$$

The cross force per unit length of the body is then found as

$$f = \int_0^{2\pi} p R \cos \theta d\theta = 2 q_0 \int_0^{\pi} P R \cos \theta d\theta + 2 \int_0^{\pi} p_0 R \cos \theta d\theta$$

³ Equation (13), for the case in which β is constant, reduces to that derived by Busemann (reference 5) for the flow over an inclined cone. Laitone's linearized solution (reference 4) for the pressure distribution over bodies at supersonic speeds agrees with equation (13) except that the α^2 term, of course, is absent. This linearized solution is inadequate in general since, for the cases of usual interest, the values of α are of the same order of magnitude as β , thus the α^2 term is as important as the α term.

and clearly

$$2 \int_0^\pi p_0 R \cos \theta d\theta = 0$$

Substituting $\dot{P}$ from equation (12) gives

$$f = 2Rq_0 P_{\alpha=0} \int_0^\pi \cos \theta d\theta + 4Rq_0 \tan \beta \sin 2\alpha \int_0^\pi \cos^2 \theta d\theta + 2Rq_0 \sin^2 \alpha \int_0^\pi (1 - 4 \sin^2 \theta) \cos \theta d\theta$$

The first and third integrals are zero, while the second integral yields

$$f = 2\pi Rq_0 \tan \beta \sin 2\alpha$$

and since

$$2\pi R \tan \beta = 2\pi R \frac{dR}{dx} = \frac{dS}{dx}$$

then

$$f = q_0 \frac{dS}{dx} \sin 2\alpha$$

which is equation (1) derived by Munk for the cross force on slender airship hulls and, in the form,

$$f = 2q_0 \frac{dS}{dx} \alpha$$

that derived by Tsien for the cross force, to the order of the first power of the angle of inclination, for slender bodies at moderate supersonic speeds. This development shows that these equations for the cross force are also correct to the second power of α for inviscid flow.

COMPARISONS WITH EXPERIMENT AND DISCUSSION OF THE EFFECTS OF VISCOSITY

In reference 6, a thorough investigation at low speeds was made of the pressure distribution over a hull model of the rigid airship "Akron." Incremental pressure distributions due to inclination calculated by equation (12) for four stations along the hull at three angles of attack are compared with the experimental values in figures 2(a) to 2(d). In each of the figures is shown a sketch of the airship which indicates the station at which the incremental pressure distributions apply. This comparison represents a severe test of the theoretical method of this report since the method was developed on the assumption that the fineness ratio of the body is very large, while for the case considered the fineness ratio is only 5.9.

At the more forward stations (figs. 2(a) and 2(b)), the agreement is seen to be essentially good⁴ but some discrep-

⁴ At stations extremely close to the bow the method must be inaccurate as evident from the work of Upson and Klikoff (reference 7).

ancy, particularly at values of θ near 180° , is evident which increases with increasing distance from the bow. Downstream of the maximum diameter section (figs. 2(c) and 2(d)) the discrepancy increases very rapidly.

The disagreement that exists at the afterbody stations results from effects of viscosity not considered in the theory, as will be seen from the following: R. T. Jones, in reference 8, showed that, for laminar flow on an infinitely long yawed cylinder of arbitrary cross section, the behavior of the component flow of a viscous fluid in planes normal to the cylinder axis was independent of the component flow parallel to the axis.⁵ For an inclined circular cylinder, then, viewed along the cylinder axis the viscous flow about the cylinder would appear identical to the flow about a circular cylinder section in a stream moving at the velocity $V_0 \sin \alpha$. Hence separation of the flow would occur in the yz plane as a result of the adverse pressure gradients that exist across the cylinder. Jones demonstrated that this behavior explained the cross forces on inclined right circular cylinders that were experimentally observed in reference 10. That such separation effects also occur on the inclined hull model of the "Akron" is also evident from the pressure distributions in figures 2(c) and 2(d).

While the treatment of reference 8 explains qualitatively the observed behavior of the flow field about the hull model considered, it cannot be used quantitatively for a low fineness ratio body such as the "Akron" for at least two reasons.

First, the influence of the term

$$2 \tan \beta \cos \theta \sin 2\alpha$$

of equation (12) is to distort the typical circular-cylinder pressure distribution, given by the term

$$(1 - 4 \sin^2 \theta) \sin^2 \alpha$$

so as to move the calculated position of minimum pressure away from the $\theta = 90^\circ$ point and to change the magnitude of the pressure to be recovered on the lee side of the body. Over the forward stations of the body, where $\tan \beta$ is positive, the position of minimum pressure lies between 90° and 180° and the theoretical pressure recovery is small and even zero at the most forward stations. For the rearward station where $\tan \beta$ is negative, the minimum pressure lies between 0° and 90° , and the theoretical pressure recovery is large and increases proceeding toward the stern. For the hull of the "Akron" model, the theoretical line of minimum pressure along the hull is shown in figure 3 for the angles of attack of 6° , 12° , and 18° .⁶ Since separation can only occur in an adverse

⁵ The recent work of A. P. Young and T. B. Booth (reference 9) indicated that this may be true for the turbulent flow case as well.

⁶ It is of interest to note in this figure that even for small angles of inclination the line of minimum pressure becomes oriented close to the direction of the axis of revolution, while at zero inclination it must, of course, be normal to this axis.

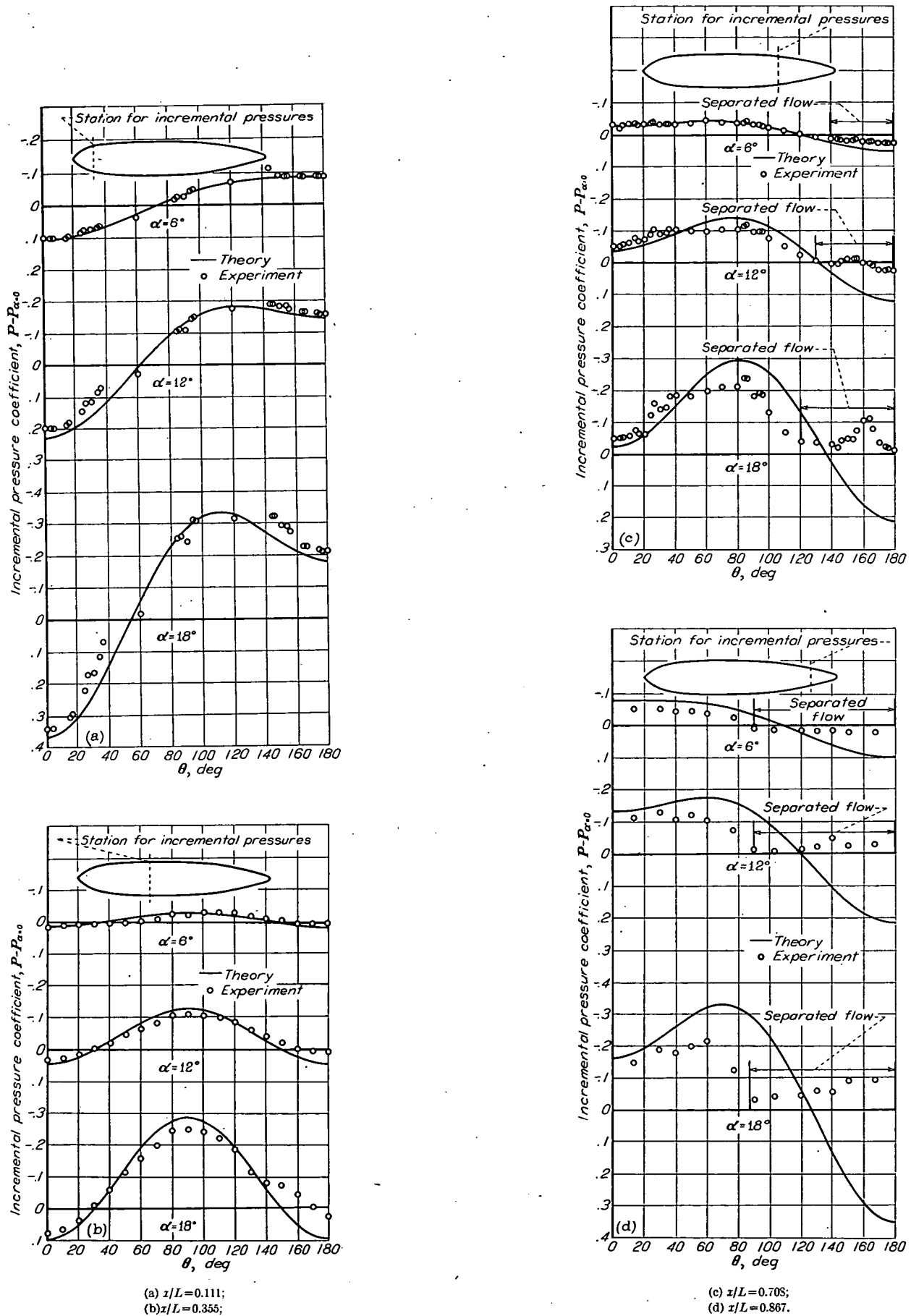


FIGURE 2—Calculated and experimental pressure distribution on a model hull of the U. S. S. Akron.

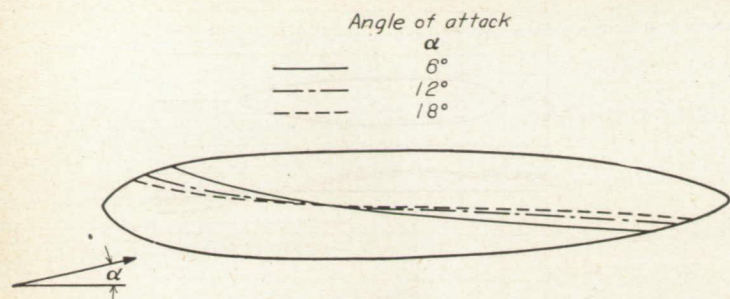


FIGURE 3.—Calculated lines of minimum pressures for a model hull of U. S. S. Akron at three angles of attack.

gradient, it is clear that the line of separation will roughly parallel the line of minimum pressures. Hence, the flow about forward stations will be, or will more nearly be, that calculated for a nonviscous fluid. Over the rearward stations the flow separation should tend to be even more pronounced than would occur on a right circular cylinder. That such is the case is shown by the flow studies on the ellipsoid of revolution of reference 11. In those studies, the flow on the model surface was investigated by lampblack and kerosene traces. The traces showed the line of separation followed the trend indicated above. From the foregoing, it is evident that the potential flow solution for the pressures on inclined bodies can only be expected to hold over the forebody, and that over the afterbody the pressure distribution, particularly on the lee side, will be importantly influenced by the fluid viscosity.

Second, it is evident that there exists a certain analogy between the cross flow at various stations along the body and the development with time of the flow about a cylinder starting from rest. This may be seen by considering the development of the cross flow with respect to a coordinate system that is in a plane perpendicular to the axis of the inclined body. Let the plane move downstream with a velocity V_0 and let the coordinate system move within the plane such that the axis of revolution of the body is always coincident with the x axis of the coordinate system. The cross velocity is then $V_0 \sin \alpha$. At any instant during the travel of the plane from the nose to the base of the body, the trace of the body in the plane will be a circle and the cross-flow pattern within the plane may be compared with the flow pattern about a circular cylinder. Neglecting, for the moment, the effect of the taper over the nose portions of the body, it might be anticipated that over successive downstream sections, the development of the cross flow with distance along the body as seen in this moving plane would appear similar to that which would be observed with the passage of time for a circular cylinder impulsively set in motion from rest with the velocity $V_0 \sin \alpha$. Thus the flow in the cross plane for the more forward sections should contain a pair of symmetrically disposed vortices on the lee side (cf. reference 12). These vortices should increase in strength as the plane moves rearward and eventually, if the body is long enough, should discharge to form a Kármán vortex street as viewed in the moving cross plane. Viewed in

this moving plane the vortices would appear to be shed and slip rearward in the wake, but viewed with respect to the stationary body the shed vortices would appear fixed. This process of the growth and eventual discharge of the lee-side vortices should occur over a shorter length of body the higher the angle of attack since the movement of the cylindrical trace in the cross plane at any given station is greater the greater the angle of attack. For a low fineness ratio body, however, the development of the lee-side vortices would be expected to have progressed no farther than the "symmetrical pair" case even at the highest angles of attack of interest. This is corroborated by the flow surveys of Harrington (reference 11).

For bodies of high fineness ratio, such as those used for supersonic missiles, it was clearly of interest to determine experimentally the nature of the anticipated growth and discharge of lee-side vortices. In the course of an investigation of a series of bodies with ogival noses and cylindrical afterbodies conducted in the Ames Laboratory 1- by 3-foot supersonic wind tunnels, it was determined that the growth and discharge of lee-side vortices did occur for such bodies at angle of attack as was evidenced in two ways. The schlieren picture for one of the bodies (fig. 4 (a)) showed a line on the lee side at the more forward stations which drifted away from the body surface and eventually branched into a series of lines trailing in the stream direction. The "line" at the more forward stations was indicated to be the cores of the symmetrical vortex pair, which in this side view would appear coincident. The branches were indicated to be the cores of the alternately shed vortices. In order to make the vortices visible in a more convincing manner, use was made of a technique which we have termed the "vapor screen" method. With this technique, the cross flow is made visible in the following manner (see fig. 5): A small amount of water vapor, which condenses in the wind-tunnel test section to produce a fine fog, is introduced into the tunnel air stream. A narrow plane of bright light, produced by a high-pressure mercury-vapor lamp, is made to shine through the glass window in a plane essentially perpendicular to the axis of the tunnel. In the absence of the model this plane appears as a uniformly lighted screen of fog particles. When the model is put in place at any arbitrary angle of attack, the result of any disturbances in the flow produced by the model which affects the amount of light scattered by the water particles in this lighted plane can be seen and photographed.

In figures 4 (b) and 4 (c) are shown photographs of the vapor screens corresponding to the indicated stations for the body of figure 4 (a). The photographs are three-quarter front views from a vantage point similar to that of the sketch of figure 5. In these photographs vortices made themselves evident as black dots on the vapor screens due to the absence of scattered light, which is believed to result from the action of the vortices in spinning the fine droplets of fog out of the fast turning vortex cores. Other details of the flow, particu-

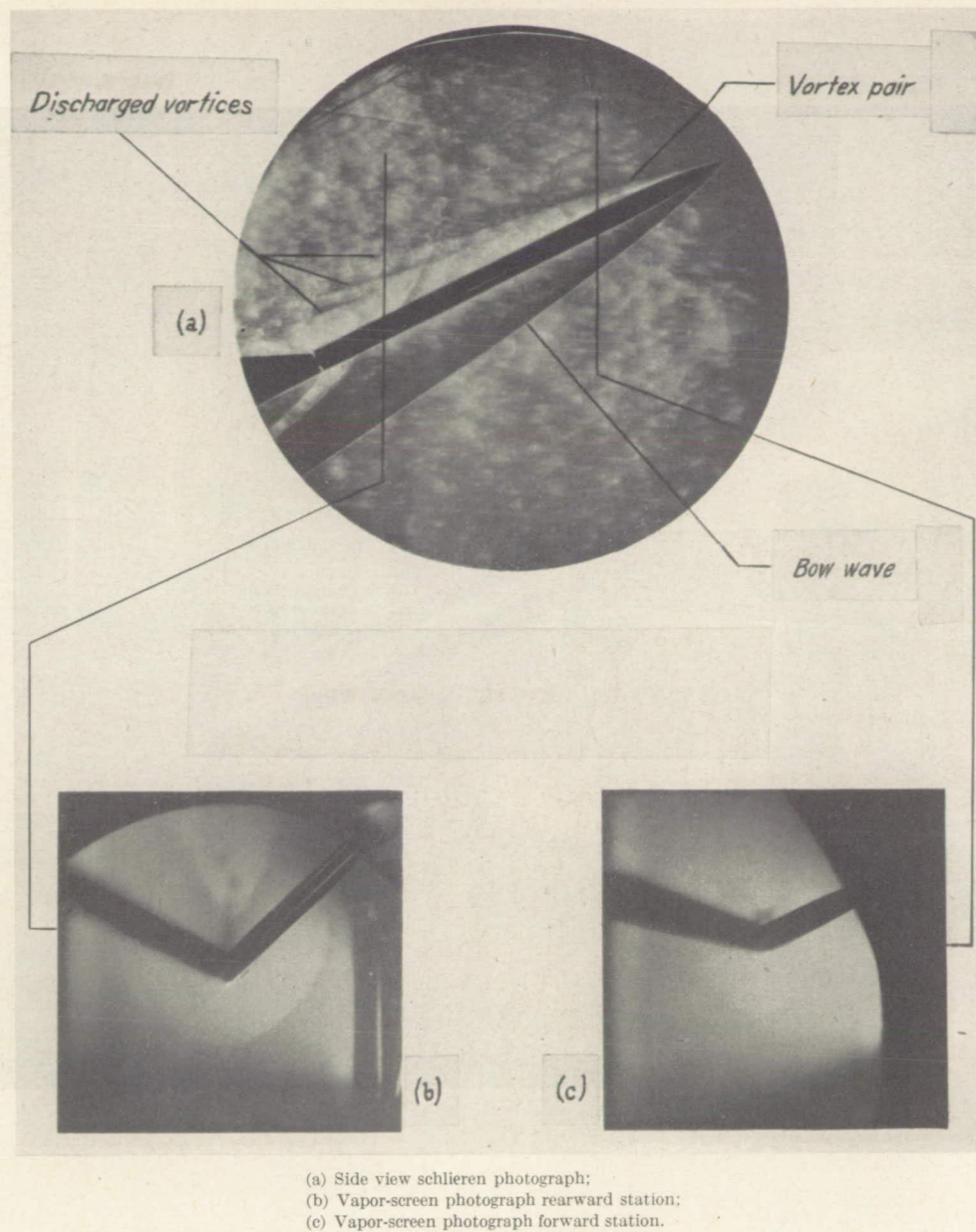


FIGURE 4.—Schlieren and vapor-screen photographs showing vortex configuration for an inclined body of revolution ($\alpha \approx 25^\circ$) at supersonic speed ($M \approx 2$).

larly shock waves, are evident as a change in light intensity. Figure 4 (c) shows the symmetrical vortex pair to exist as previously indicated at the more forward stations, while figure 4 (b) demonstrates that the vortices are shed at stations far removed from the bow. Other observations at different angles of attack demonstrated that the shedding of vortices began, as indicated previously, at the more forward stations the higher the angle of attack. It is of interest to point out that in these wind-tunnel tests the order of discharge of the vortices was aperiodically reversed. Thus, in any cross-flow plane, the discharged vortex closest to the body would at one instant be on one side of the body and at the next instant, or perhaps several seconds later, on the other. No regularity in this change in the distribution of the vortex street has been found.

METHOD FOR ESTIMATING FORCE AND MOMENT CHARACTERISTICS OF SLENDER INCLINED BODIES IN VISCOUS FLOW

For bodies of high fineness ratio at high angles of attack when the cross force is important, it is clear that the third term of equation (13) must predominate since β is small, so that the pressure distribution increment due to angle of attack will closely approximate the pressure distribution for a circular-cylinder section at a velocity equal to the cross component of velocity for the body. Moreover, except for the sections near the bow, development of the cross-flow boundary layer will have been sufficient to promote the flow that is characteristic of the steady-state flow for a circular-cylinder section at the Mach and Reynolds numbers corresponding to the cross velocity over the body.

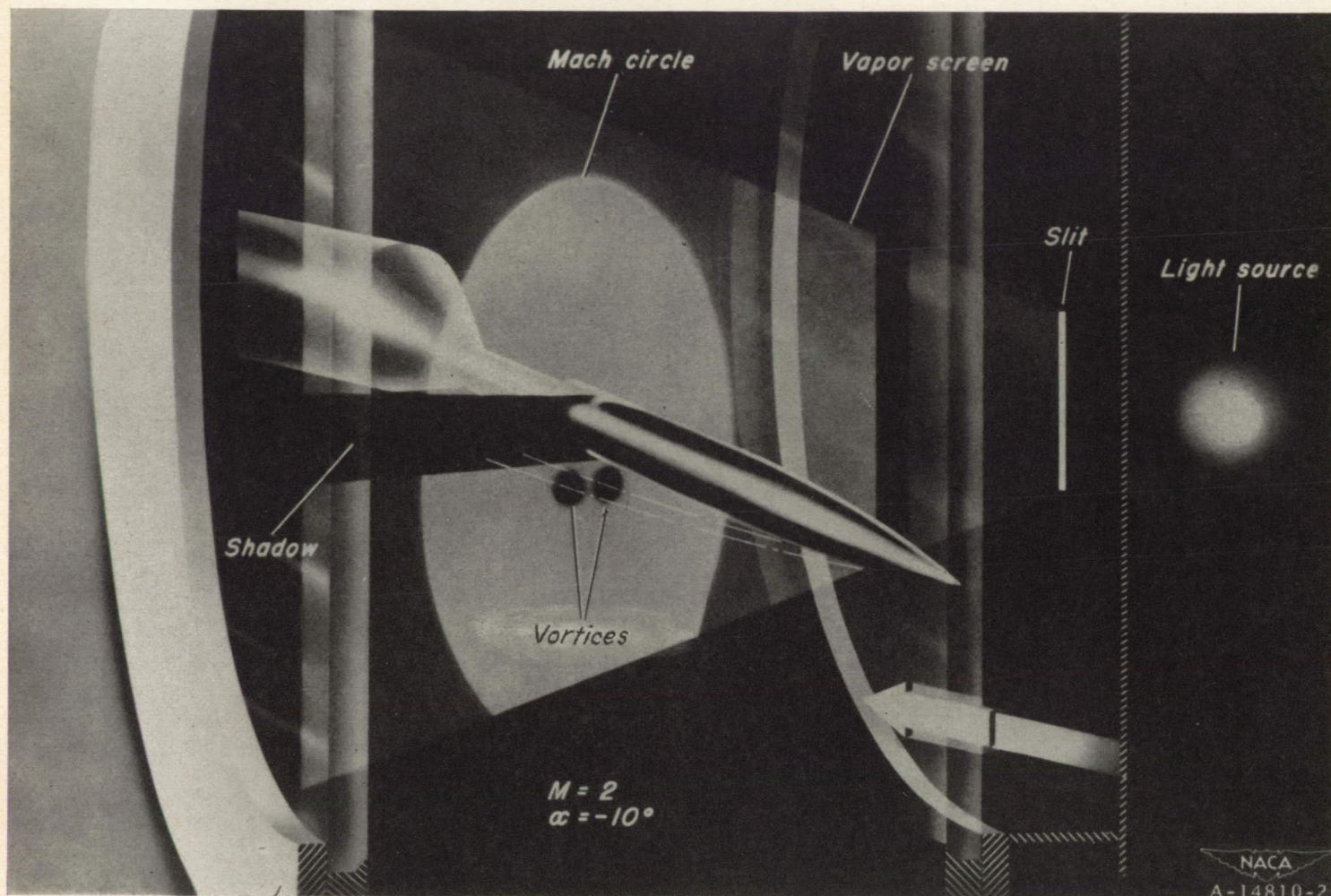


FIGURE 5.—Schematic diagram of vapor-screen apparatus showing vortices from a lifting body of revolution.

For the limiting case of a slender body, the appropriate value of the cross-wise drag coefficient is, of course, the value of the drag for an infinite cylinder. As will be shown later, experiments indicate that for actual bodies of finite length a somewhat smaller value should be used. Since the actual cross-wise drag coefficient of a body of finite length is always somewhat less than the coefficient for an infinite body, this reduced value suggests itself. Thus it might be expected that the viscous cross-force distribution on such a body could be calculated on the assumption that each circular element along the body experiences a cross force equal to the drag force the section would experience with the axis of revolution normal to a stream moving at the velocity $V_0 \sin \alpha$. This viscous contribution is given by

$$2Rc_{d\alpha=90^\circ} q_0 \sin^2 \alpha dx$$

where R is the body radius at x and $c_{d\alpha=90^\circ}$ is the local drag

coefficient at x for $\alpha=90^\circ$ corresponding to the Reynolds number

$$R_c = R_0 \sin \alpha$$

and the Mach number

$$M_c = M_0 \sin \alpha$$

As a first approximation to the total cross force we may add the potential cross force to the viscous contribution. The total cross force at x would then be ⁷

$$f = q_0 \frac{dS}{dx} \sin 2\alpha \cos \frac{\alpha}{2} + 2Rc_{d\alpha=90^\circ} q_0 \sin^2 \alpha$$

With this simple allowance for viscous effects the lift coefficient,⁸ the foredrag coefficient, and the pitching-moment

⁷ From the work of Ward (reference 13), it may be shown that the potential cross force is directed midway between the normal to the axis of revolution and the normal to the wind direction.

⁸ In the expression for C_L the contribution of the axial drag force $-C_{D_P}(\alpha=0) \cos^2 \alpha \sin \alpha$ is inconsequential small and has been ignored.

coefficient about an arbitrary moment center x_m from the nose are given by

$$\left. \begin{aligned} C_L &= \frac{S_b}{A} \sin 2\alpha \cos \frac{\alpha}{2} + C_{d_{\alpha=90^\circ}} \frac{A_p}{A} \sin^2 \alpha \cos \alpha \\ C_{D_F} &= C_{D_{F_{\alpha=0}}} \cos^3 \alpha + \frac{S_b}{A} \sin 2\alpha \sin \frac{\alpha}{2} + C_{d_{\alpha=90^\circ}} \frac{A_p}{A} \sin^3 \alpha \\ C_M &= \left[\frac{Q - S_b(L - x_m)}{AX} \right] \sin 2\alpha \cos \frac{\alpha}{2} + \\ &\quad C_{d_{\alpha=90^\circ}} \frac{A_p}{A} \left(\frac{x_m - x_{\alpha=90^\circ}}{X} \right) \sin^2 \alpha \end{aligned} \right\} \quad (14)$$

where

$$C_{d_{\alpha=90^\circ}} = \frac{\int_0^L c_{d_{\alpha=90^\circ}} 2R dx}{A_p}$$

and

$$x_{\alpha=90^\circ} = \frac{\int_0^L c_{d_{\alpha=90^\circ}} 2R x dx}{A_p C_{d_{\alpha=90^\circ}}}$$

where

- A_p plan-form area
- S_b base area
- Q body volume
- L body length
- A reference area for coefficient evaluation
- X reference length for pitching-moment-coefficient evaluation

Because of the approximate nature of the theory, it is not considered justified to retain the complex forms of these equations. Accordingly, it is assumed that, for the functions of the angle of attack, cosines may be replaced by unity and sines by the angles in radians to give⁹

$$\left. \begin{aligned} C_L &= 2 \left(\frac{S_b}{A} \right) \alpha + C_{d_{\alpha=90^\circ}} \left(\frac{A_p}{A} \right) \alpha^2 \\ \Delta C_{D_F} &= C_{D_F} - C_{D_{F_{\alpha=0}}} = \frac{S_b}{A} \alpha^2 + C_{d_{\alpha=90^\circ}} \left(\frac{A_p}{A} \right) \alpha^3 \\ C_M &= 2 \left[\frac{Q - S_b(L - x_m)}{AX} \right] \alpha + C_{d_{\alpha=90^\circ}} \left(\frac{A_p}{A} \right) \left(\frac{x_m - x_{\alpha=90^\circ}}{X} \right) \alpha^2 \end{aligned} \right\} \quad (15)$$

To assess the adequacy of these equations for predicting the force and moment characteristics of high fineness ratio bodies, two bodies of revolution were tested in the 1- by 3-foot supersonic wind tunnel at a Mach number of 1.98 from angles of attack of zero to more than 20° and in the 1- by 3½-foot high-speed wind tunnel from Mach numbers of 0.3 to 0.7 at 90° angle of attack to determine the cross-flow drag coefficient $C_{d_{\alpha=90^\circ}}$ and the center of application $x_{\alpha=90^\circ}$ of this force.¹⁰ The bodies investigated (see fig. 6)

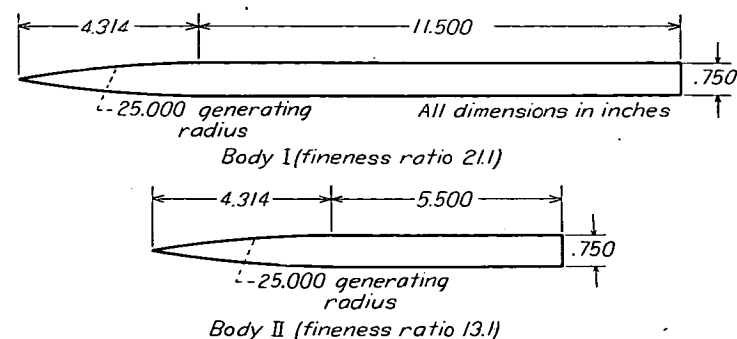


FIGURE 6.—Bodies of revolution.

each had a 33½-caliber ogival nose and constant diameter afterbody of such length as to make the fineness ratios 21.1 for body I and 13.1 for body II. Shown in figures 7 and 8 are the lift coefficient, foredrag-increment coefficient, pitching-moment coefficient about the bow, and center of pressure as a function of angle of attack for the two bodies as determined from the tests in the 1- by 3-foot wind tunnels. Also shown are the calculated characteristics (indicated by the solid-line curves) using the experimental values of $C_{d_{\alpha=90^\circ}}$ and $x_{\alpha=90^\circ}$ (see figs. 9 and 10) obtained from the 90° angle-of-attack tests in the 1- by 3½-foot wind tunnel as well as calculated characteristics obtained from potential theory (indicated by the dotted-line curves). The reference area A for coefficient evaluation for these data is the base area and the reference length X for moment-coefficient evaluation is the base diameter.

It is seen that for the higher fineness ratio body (body I), the calculated characteristics which include the allowance for viscous effects are in good agreement with experiment and that the potential theory is clearly inadequate at all but very small angles of attack. For the lower fineness ratio body (body II), the calculated allowances for viscous effects depart further from experiment than they do for body I, as would be expected, although again they are in much better agreement with experiment than are the calculated characteristics based only on potential theory.

While the comparisons made demonstrate that the indicated allowance for viscous effects is adequate for very high-fineness-ratio bodies, the calculated characteristics were, themselves, based on experimentally determined values of $C_{d_{\alpha=90^\circ}}$ and $x_{\alpha=90^\circ}$. For the method to be useful in design, of course, some means for calculating these parameters must exist. In many cases, there are available sufficient experimental drag data on cylinders to provide the required information. In the appendix one approximate method is given for determining the values of $C_{d_{\alpha=90^\circ}}$ and $x_{\alpha=90^\circ}$ for the bodies I and II previously considered.

The variations of the coefficients of lift, foredrag increment, and pitching moment and of the center-of-pressure position with angle of attack for the two bodies as estimated using the calculated cross-flow drag characteristics given in the appendix are shown in figures 7 and 8 (as the dashed-line curves). The estimated characteristics are seen to be in even better agreement with experiment than are the calculated variations using the experimental cross-flow drag characteristics.

⁹ In the expression for ΔC_{D_F} , the term $-C_{D_{F_{\alpha=0}}} \alpha^2$, which should properly appear on the right side of this equation, has been omitted since, for practical cases, its contribution is small.

¹⁰ Although the cross Reynolds numbers for the 1- by 3½-foot wind-tunnel test were almost twice that for the 1- by 3-foot wind-tunnel tests, the results are considered comparable since in the range of Reynolds numbers investigated the drag characteristics of circular cylinders are insensitive to change in Reynolds numbers.

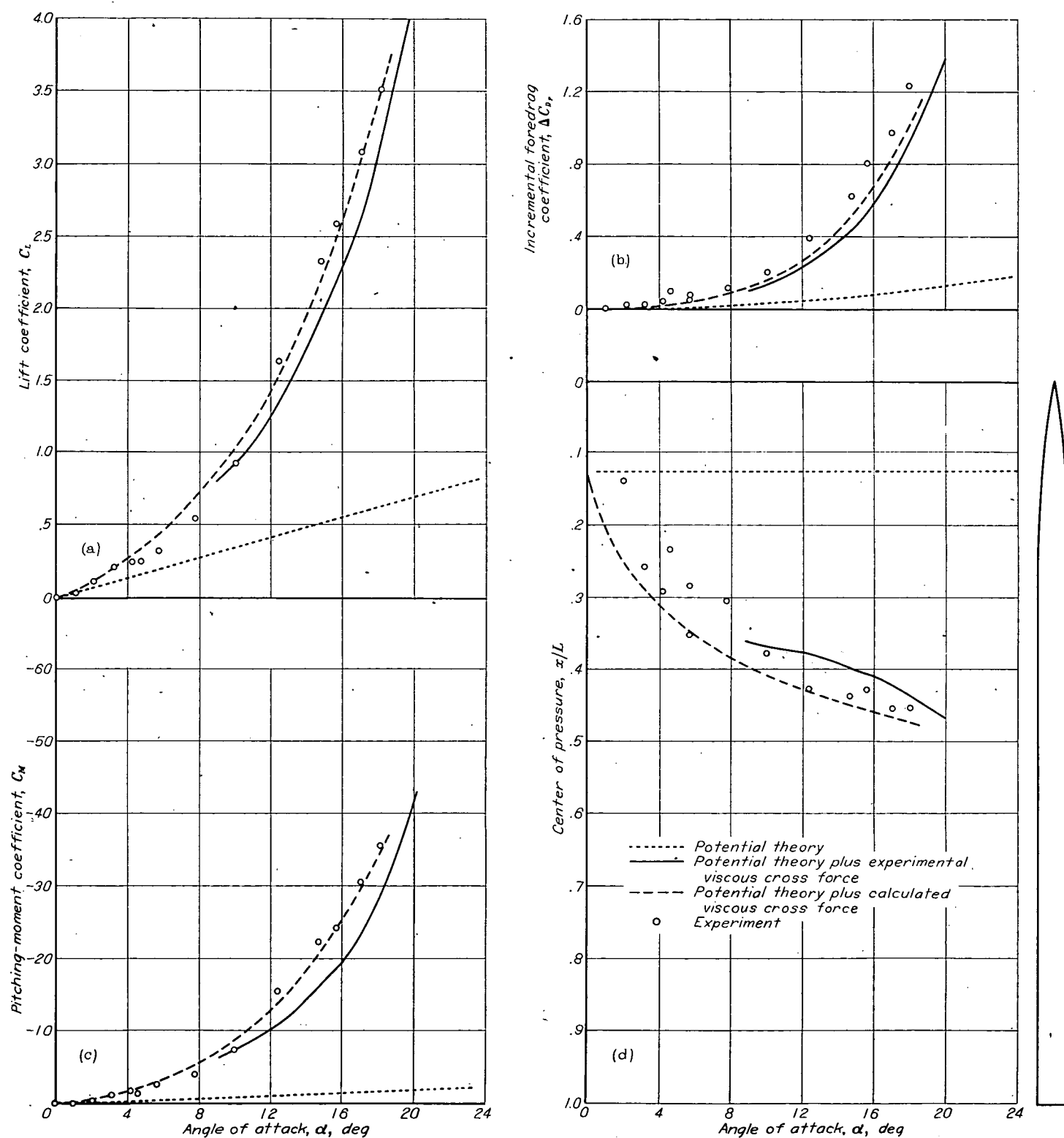


FIGURE 7.—Force, moment, and center-of-pressure characteristics for body I (fineness ratio 21.1).

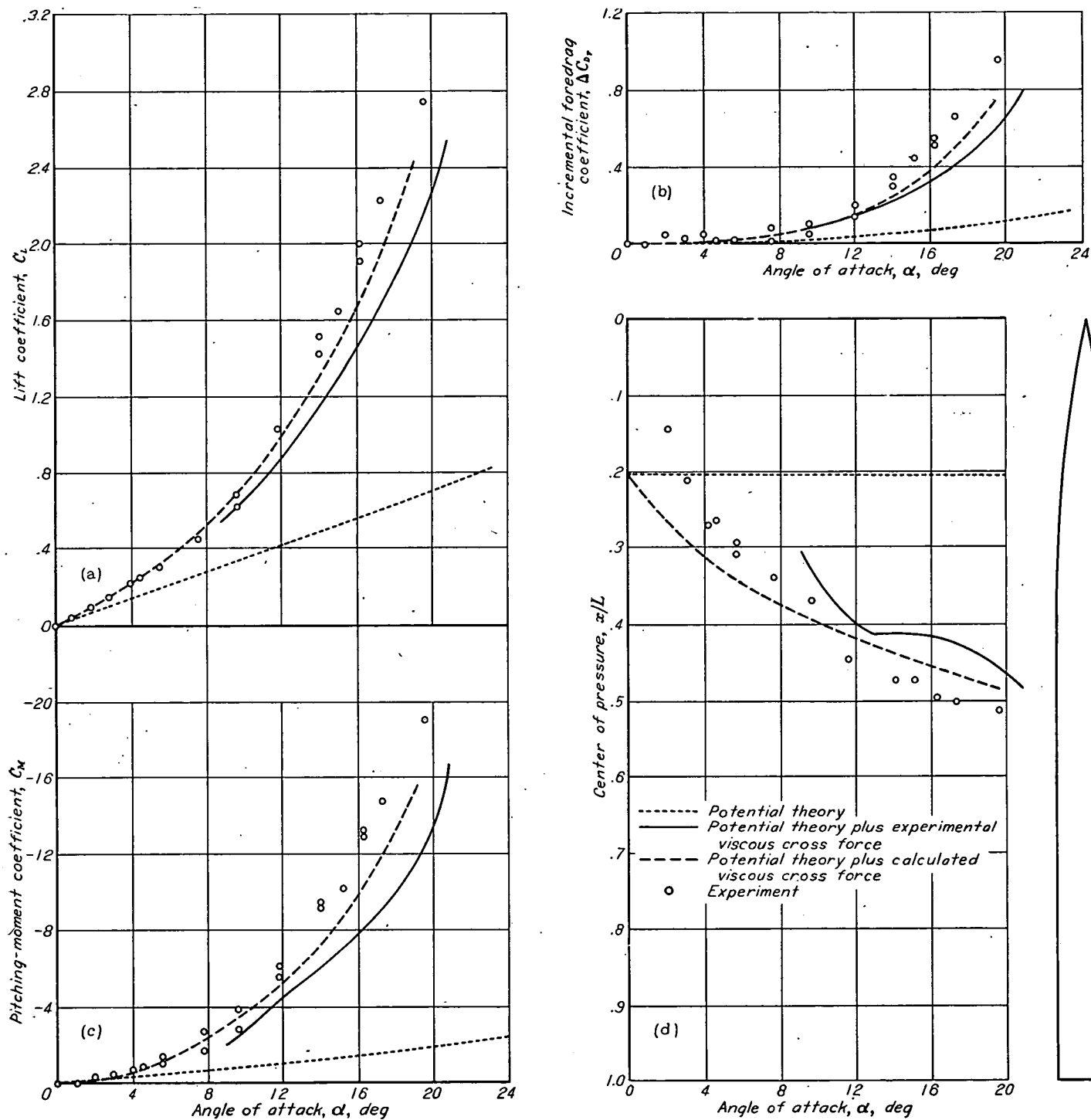


FIGURE 8.—Force, moment, and center-of-pressure characteristics for body II (fineness ratio 13.1).

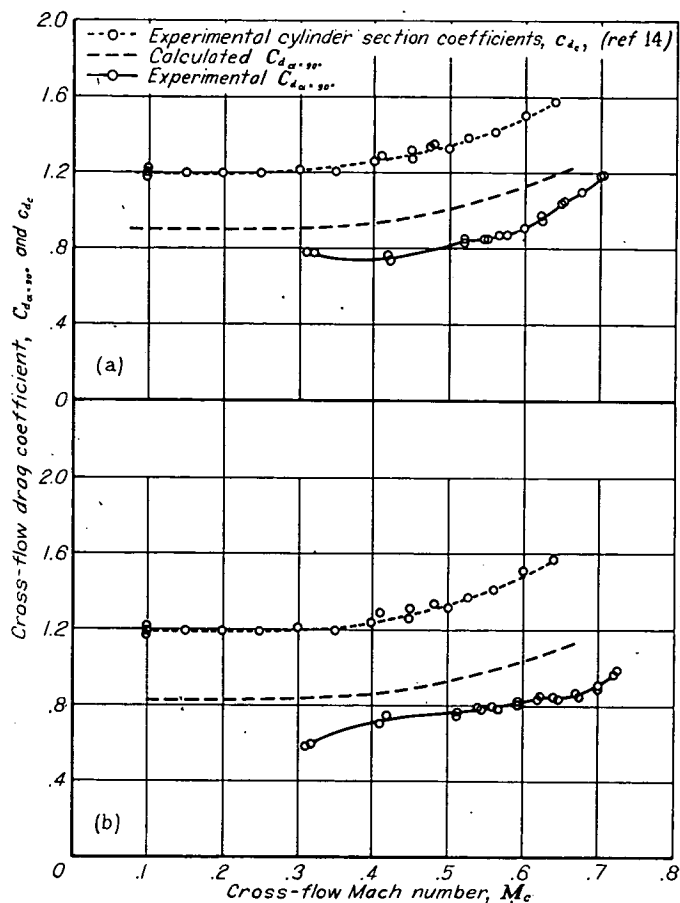


FIGURE 9.—Comparison of calculated cross-flow drag coefficients at 90° angle of attack with experimental values obtained from Ames 1-by $3\frac{1}{2}$ -foot wind tunnel for (a) body I and (b) body II.

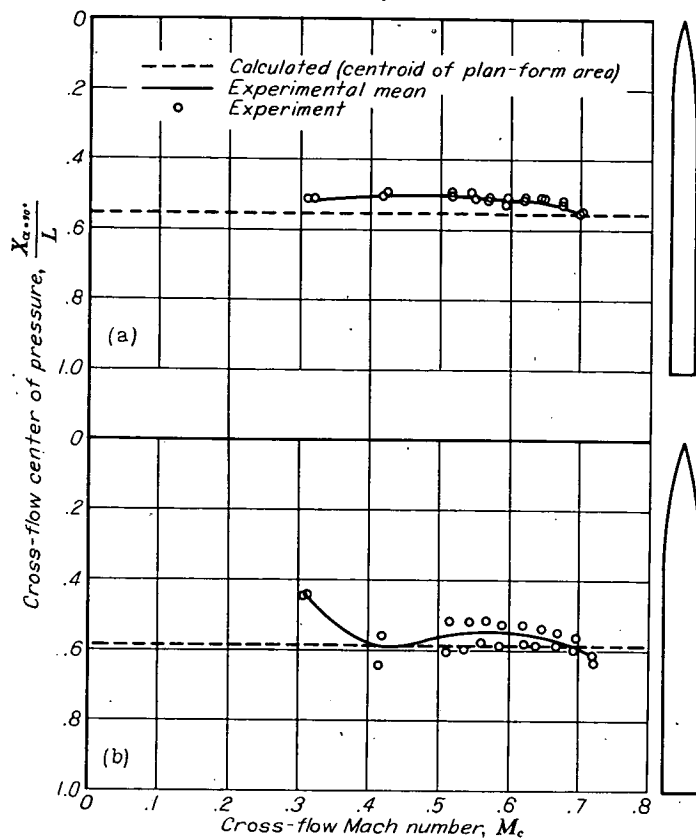


FIGURE 10.—Comparison of calculated centers of pressure at 90° angle of attack with experimental values obtained from Ames 1-by $3\frac{1}{2}$ -foot wind tunnel for (a) body I and (b) body II.

AMES AERONAUTICAL LABORATORY,
 NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
 MOFFETT FIELD, CALIF., Dec. 28, 1949.

APPENDIX

ESTIMATION OF CROSS-FLOW DRAG COEFFICIENT

A procedure which suggests itself for estimating the magnitude of the cross-flow drag coefficient $C_{d_{\alpha=90^\circ}}$ as a function of angle of attack for the two ogival-nosed bodies treated in the text is to consider them to have the same characteristics as circular cylinders of constant diameter.

$$D' = \frac{A_p}{L}$$

The cross-flow drag coefficient of this fictitious cylinder of finite length can then be approximated by first determining the drag coefficient, c_{d_c} , of a circular-cylinder section of diameter D' at the cross-flow Mach number

$$M_c = M_0 \sin \alpha$$

and cross-flow Reynolds number

$$Re_c' = \frac{V_0 D'}{\nu} \sin \alpha$$

and then correcting this drag coefficient for the effect of the finite fineness ratio, L/D' .

From references 14 and 15, it is found that for the values of D' corresponding to the two bodies considered the circular-cylinder section drag coefficients, c_{d_c} , are the same for both bodies and dependent only on the Mach number. The values at various cross Mach numbers are given in figure 9.

From reference 16, it is found that for a finite-length circular cylinder in the range of Reynolds numbers for which the cross-drag coefficient, as for the present cases, is 1.2 at low Mach numbers, the ratio of the drag of the circular cylinder of finite length to that for the circular cylinder of infinite length is 0.755 for body I and 0.692 for body II. Assuming that this ratio is independent of Mach number, the estimated values of $C_{d_{\alpha=90^\circ}}$ for the two bodies are as given in figure 9 wherein they may be compared with the experimental values obtained from the 1- by 3½-foot wind-tunnel tests.

The value of $x_{\alpha=90^\circ}$ is logically assumed to be the distance from the bow to the centroid of plan-form area. This

assumed position which is independent of Mach number is compared with the experimentally determined values from the 1- by 3½-foot wind-tunnel tests for the two bodies in figure 10.

REFERENCES

1. Munk, Max M.: The Aerodynamic Forces on Airship Hulls. NACA Rep. 184, 1924.
2. Tsien, Hsue-Shen: Supersonic Flow Over an Inclined Body of Revolution. *Jour. Aero. Sci.*, vol. 5, no. 12, Oct. 1938, pp. 480-483.
3. Kaplan, Carl: Potential Flow About Elongated Bodies of Revolution. NACA Rep. 516, 1935.
4. Laitone, E. V.: The Linearized Subsonic and Supersonic Flow About Inclined Slender Bodies of Revolution. *Jour. Aero. Sci.*, vol. 14, no. 11, Nov. 1947, pp. 631-642.
5. Busemann, Adolf: Infinitesimal Conical Supersonic Flow. NACA TM 1100, 1947.
6. Freeman, Hugh B.: Pressure-Distribution Measurements on the Hull and Fins of a 1/40-Scale Model of the U. S. Airship "Akron." NACA Rep. 443, 1932.
7. Upson, Ralph H., and Klikoff, W. A.: Application of Practical Hydrodynamics to Airship Design. NACA Rep. 405, 1931.
8. Jones, Robert T.: Effects of Sweepback on Boundary Layer and Separation. NACA Rep. 884, 1947. (Formerly NACA TN 1402)
9. Young, A. D., and Booth, T. B.: The Profile Drag of Yawed Wings of Infinite Span. The College of Aeronautics, Cranfield. Rep. 38, May, 1950.
10. Relf, E. F. and Powell, C. H.: Tests on Smooth and Stranded Wires Inclined to the Wind Direction, and a Comparison of Results on Stranded Wires in Air and Water. R. & M. No. 307, British A. R. C., 1917.
11. Harrington, R. P.: An Attack on the Origin of Lift of an Elongated Body. Daniel Guggenheim Airship Inst., Publication 2, 1935.
12. Schwabe, M.: Pressure Distribution in Nonuniform Two-Dimensional Flow. NACA TM 1039, 1943.
13. Ward, G. N.: Supersonic Flow Past Slender Pointed Bodies. *Quar. Jour. of Mech. and Applied Math.*, vol. 2, part I, Mar. 1949, pp. 75-97.
14. Lindsey, W. F.: Drag of Cylinders of Simple Shapes. NACA Rep. 619, 1938.
15. Stack, John: Compressibility Effects in Aeronautical Engineering. NACA ACR, Aug. 1941.
16. Goldstein, S.: Modern Developments in Fluid Dynamics. Oxford, The Clarendon Press, 1938, vol. II, pp. 419-421.

REPORT 1049

EXPERIMENTAL INVESTIGATION OF THE EFFECT OF VERTICAL-TAIL SIZE AND LENGTH AND OF FUSELAGE SHAPE AND LENGTH ON THE STATIC LATERAL STABILITY CHARACTERISTICS OF A MODEL WITH 45° SWEEPBACK WING AND TAIL SURFACES¹

By M. J. QUEIJO and WALTER D. WOLHART

SUMMARY

An investigation was made to determine the effects of vertical-tail size and length and of fuselage shape and length on the static lateral stability characteristics of a model with wing and vertical tails having the quarter-chord lines swept back 45°. The results indicate that the directional instability of the various isolated fuselages was about two-thirds as large as that predicted by classical theory. A reduction in area of vertical tails (geometric aspect ratio kept constant) attached to a given fuselage resulted in an increase in the effective aspect ratio of the vertical tail for the range of tail sizes considered. Simple analytical considerations indicate, however, that for tail sizes below the range investigated the opposite effect would be expected.

For the fuselage-tail combinations investigated, the tail effectiveness, usually decreased with increasing angle of attack, with the greatest rate of decrease occurring at angles of attack greater than about 16°.

The wing-fuselage interference for the midwing arrangements investigated was only slightly affected by the shape of the fuselage and tended to increase slightly the directional stability of the combination. The interference effects of the wing tended to decrease the vertical-tail effectiveness, particularly at high angles of attack. The large effects observed were attributed to a partially stalled condition of the wing.

INTRODUCTION

Recent advances in the understanding of the principles of high-speed flight have led to significant changes in the design of the principal components of airplanes. Two of the more important changes have been the incorporation of large amounts of sweep of the wing and tail surfaces and the elevation of the horizontal tail to a higher position. Much information is available on the influence of the wing, fuselage, and tail geometry on the static stability characteristics of the more conventional airplane designs (for example, references 1 and 2); however, little information is available on the influence of the various airplane components on the characteristics of airplanes having wings and tail surfaces with large amounts of sweep. In order to provide such

information, a series of investigations is being conducted in the Langley stability tunnel with a model having various interchangeable parts. The effects of changes in the size and location of the horizontal tail on the low-speed static lateral stability characteristics have been reported in reference 3. The effects on the static-lateral-stability derivatives of variations of vertical-tail size and length and of fuselage shape and length are presented herein. The data also have been used to determine interference effects between the wing and fuselage and the interference effects of the wing-fuselage combination on the vertical-tail effectiveness.

SYMBOLS AND COEFFICIENTS

The data presented herein are in the form of standard NACA coefficients of forces and moments which are referred to the stability axes, with the origin at the projection on the plane of symmetry of the quarter-chord point of the mean aerodynamic chord or at the midpoint of the fuselage. The positive directions of the forces, moments, and angular displacements are shown in figure 1. The coefficients and symbols are defined as follows:

- A aspect ratio (b^2/S)
 b span, measured perpendicular to fuselage center line, feet

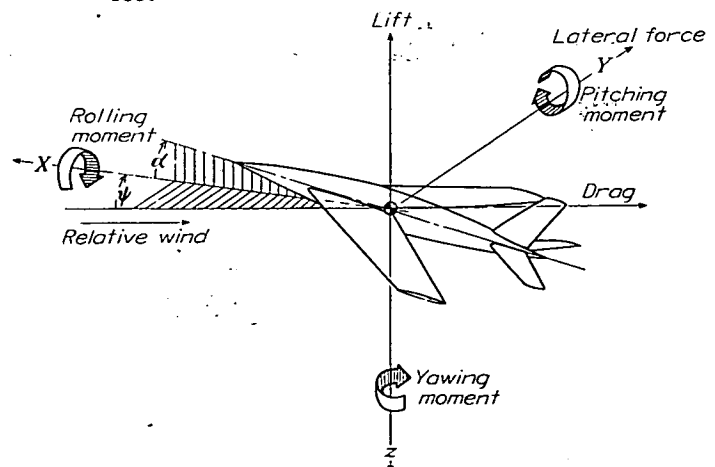


FIGURE 1.—System of axes used. Arrows indicate positive direction of angles, forces, and moments.

¹ Supersedes NACA TN 2163, "Experimental Investigation of the Effect of Vertical-Tail Size and Length and of Fuselage Shape and Length on the Static Lateral Stability Characteristics of a Model with 45° Sweptback Wing and Tail Surfaces" by M. J. Queijo and Walter D. Wolhart, 1950.

c	chord, measured parallel to plane of symmetry, feet
c_r	root chord, feet
c_t	tip chord, feet
$\bar{c}$	wing mean aerodynamic chord, feet $\left(\bar{c}_w = \frac{2}{S_w} \int_0^{b_w/2} c_w^2 dy\right)$
D_F	fuselage diameter at longitudinal station of aerodynamic center of vertical tail, feet
l	fuselage length, feet
l_v	tail length, distance from origin of axis $l/2$ to $\bar{c}/4$ of vertical tail, feet
q	dynamic pressure, pounds per square foot $\left(\frac{1}{2} \rho V^2\right)$
S	area, square feet
S_s	projected side area of fuselage, square feet
t	maximum thickness of fuselage, feet
V	velocity, feet per second
V_F	volume of fuselage, cubic feet
x	chordwise distance from leading edge of root chord to quarter-chord point of any chord, feet
$\bar{x}$	chordwise distance from leading edge of root chord to quarter-chord point of mean aerodynamic chord, feet $\left(\bar{x}_w = \frac{2}{S_w} \int_0^{b_w/2} c_w x_w dy\right)$
y	spanwise distance measured from the plane of symmetry, feet
$\bar{y}$	spanwise distance to quarter chord of mean aerodynamic chord, feet $\left(\bar{y}_w = \frac{2}{S_w} \int_0^{b_w/2} c_w y_w dy\right)$
z_v	perpendicular distance from fuselage center line to aerodynamic center of vertical tail, feet
α	angle of attack, degrees
λ	taper ratio $\left(\frac{c_t}{c_r}\right)$
η_Y, η_N	angle-of-attack correction factors to effectiveness of vertical tail in yaw
Λ	angle of sweepback of quarter-chord line, degrees
ρ	mass density, slugs per cubic foot
ψ	angle of yaw, degrees
C_L	lift coefficient $\left(\frac{\text{Lift}}{q S_w}\right)$
C_D	drag coefficient $\left(\frac{\text{Drag}}{q S_w}\right)$; $C_D = -C_X$ at $\psi = 0^\circ$
C_X	longitudinal-force coefficient $\left(\frac{\text{Longitudinal force}}{q S_w}\right)$
C_Y	lateral-force coefficient $\left(\frac{\text{Lateral force}}{q S_w}\right)$
C_m	pitching-moment coefficient $\left(\frac{\text{Pitching moment}}{q S_w \bar{c}_w}\right)$

C_n	yawing-moment coefficient $\left(\frac{\text{Yawing moment}}{q S_w b_w}\right)$
C_l	rolling-moment coefficient $\left(\frac{\text{Rolling moment}}{q S_w b_w}\right)$
$C_{Y_\psi} = \left(\frac{\partial C_Y}{\partial \psi}\right)_{\psi=0^\circ}$	
$C_{l_\psi} = \left(\frac{\partial C_l}{\partial \psi}\right)_{\psi=0^\circ}$	
$C_{n_\psi} = \left(\frac{\partial C_n}{\partial \psi}\right)_{\psi=0^\circ}$	
$(C_{L_\alpha})_v = \left(\frac{\partial (C_L)_v}{\partial \alpha}\right)_{\alpha=0^\circ}$	where $(C_L)_v$ is based on vertical-tail area
$\Delta_1 C_{Y_\psi}, \Delta_1 C_{n_\psi}, \Delta_1 C_{l_\psi}$	increments of coefficients caused by wing-fuselage interference; that is, $\Delta_1 C_{Y_\psi} = (C_{Y_\psi})_{W+F} - (C_{Y_\psi})_W - (C_{Y_\psi})_F$
$\Delta_2 C_{Y_\psi}, \Delta_2 C_{n_\psi}, \Delta_2 C_{l_\psi}$	increments of coefficients caused by wing-fuselage interference on vertical-tail effectiveness; that is, $\Delta_2 C_{Y_\psi} = [(C_{Y_\psi})_{W+F+v} - (C_{Y_\psi})_{W+F}] - [(C_{Y_\psi})_{F+v} - (C_{Y_\psi})_F]$

Subscripts and abbreviations:

W	wing
V	vertical tail; used with subscripts 1 to 5 to denote the various vertical tails (see fig. 2)
F	fuselage; used with subscripts 1 to 5 to denote the various fuselages (see fig. 3)
S	slat
e	effective
s	side area

APPARATUS AND TESTS

All parts of the models used in this investigation were constructed of mahogany. Sketches of the parts of the models are presented as figures 2, 3, and 4. The various vertical tails and fuselages will be referred to henceforth by the symbol and number assigned to them in figures 2 and 3. All vertical tails had 45° sweepback of the quarter-chord line, taper ratio of 0.6, and NACA 65A008 profiles (table I) in planes parallel to the fuselage center line. The ratios of tail area to wing area were chosen to cover a range representative of that used for current high-speed airplane configurations. The tails were mounted on the fuselages so that the tail length was always a constant percent of the fuselage length $\left(\frac{l_v}{l} = 0.42\right)$. The tail length was varied by changing the fuselage length. The three fuselages (fineness ratios of

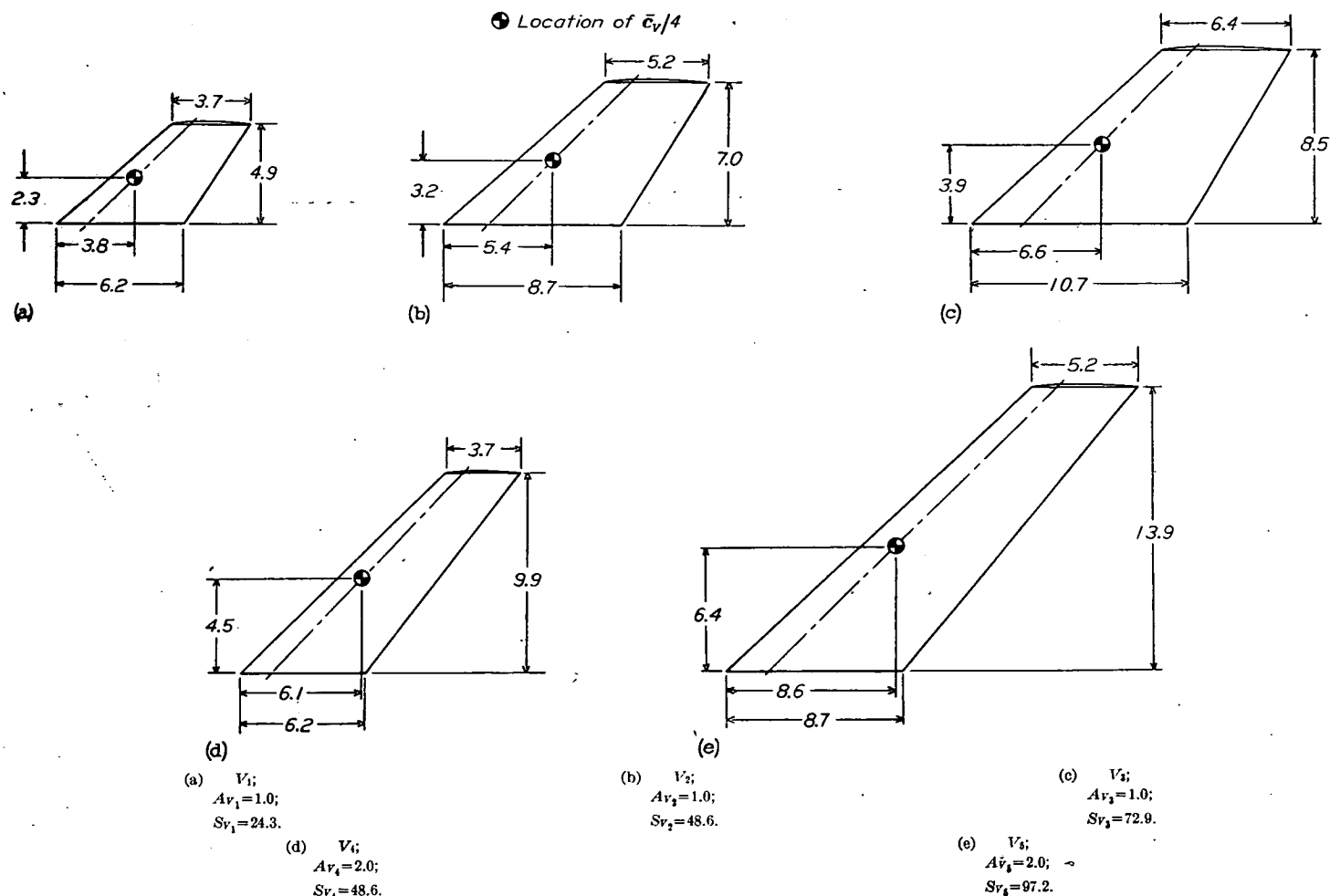


FIGURE 2.—Dimensions of vertical tails tested. $\lambda=0.6$; $\Lambda=45^\circ$; profile, NACA 65A008. All dimensions are in inches.

0, 6.67 and 10.0) of circular-arc profile used in the investigation are shown in figure 3. Two additional fuselages having the same fineness ratio as fuselage 2 (fineness ratio of 6.67) were used to determine the effects of fuselage nose and trailing-edge modifications. All fuselages had circular cross sections and all had the same maximum thickness. The coordinates of the fuselages are given in table II.

The wing had an aspect ratio of 4.0, taper ratio of 0.6, sweepback of 45° of the quarter-chord line, and NACA 65A008 profiles parallel to the plane of symmetry. The wing was mounted on the fuselage so that the quarter-chord point of the mean aerodynamic chord coincided with the fuselage mounting point (fig. 4). A summary of the geometric characteristics of the various model components is given in table III. A full-span slat, fitted to the wing for some tests with fuselage F_2 , had a chord which was 8 percent of the wing chord. (See fig. 4.) The slat was made by bending a strip of $\frac{1}{16}$ -inch-thick aluminum sheet to fit the

contour of the wing leading edge. Photographs of two of the model configurations are presented as figure 5.

Most of the tests of this investigation were conducted in the 6-foot-diameter rolling-flow test section of the Langley stability tunnel. Tests of configurations with fuselages F_4 and F_5 were conducted in the 6- by 6-foot curved-flow test section of the Langley stability tunnel. All tests were made at a dynamic pressure of 24.9 pounds per square foot, which corresponds to a Mach number of 0.13 and a Reynolds number of 0.71×10^6 based on the wing mean aerodynamic chord. The angle of attack of the model was varied from about -4° to approximately 32° for yaw angles of 0° and $\pm 5^\circ$.

CORRECTIONS

The angle of attack, longitudinal-force coefficient, and rolling-moment coefficient have been corrected for jet-boundary effects. No corrections have been applied for the effects of blocking, turbulence, or support-strut interference.

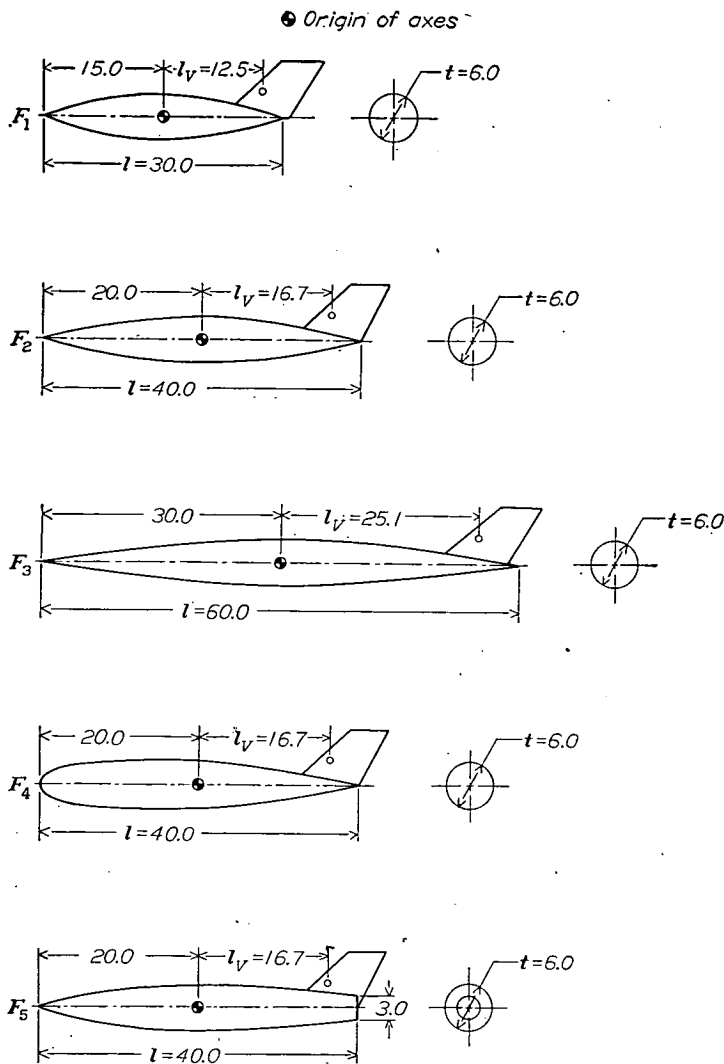


FIGURE 3.—Dimensions of fuselages; profile ordinates in table II. All dimensions are in inches.

At relatively large angles of attack (above about 20°) the vertical tail generally was in the wake of the support strut; hence, data dependent principally on the vertical-tail contribution probably are unreliable at angles of attack above about 20° . This unreliability is particularly true for data obtained with fuselage F_3 , and therefore these data are not presented.

METHODS OF ANALYSIS

The results of the present investigation are analyzed in terms of the individual contributions of the various parts and the more important interference effects. In accordance with conventional procedures (for example, see reference 2), the static-lateral-stability derivatives of a complete airplane

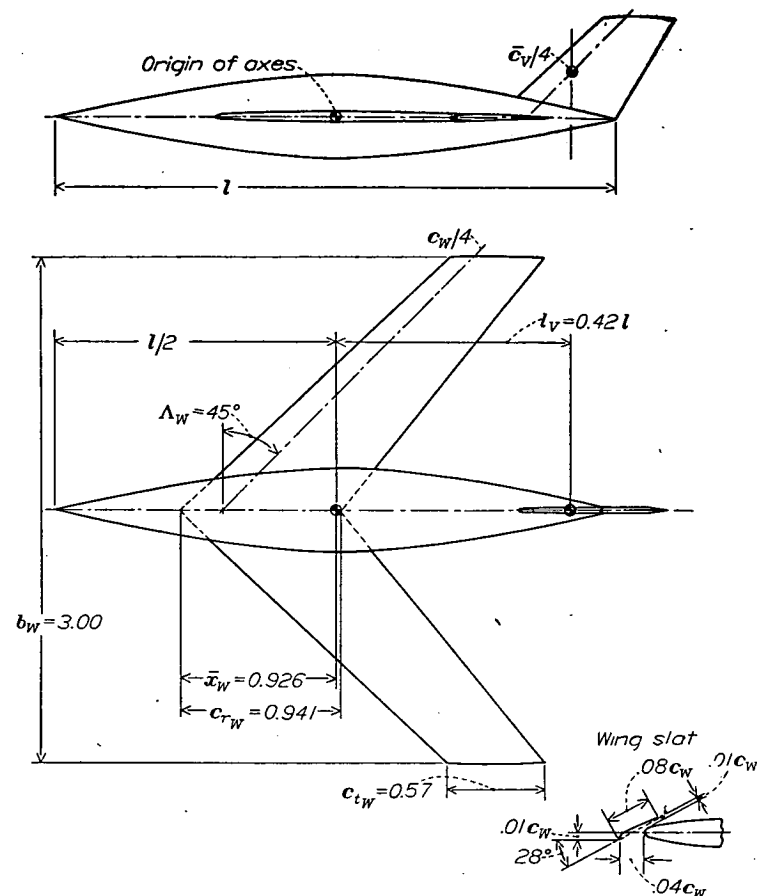


FIGURE 4.—Dimensions and location of wing and vertical tails. All dimensions are in feet.

can be expressed as

$$C_{Y\psi} = (C_{Y\psi})_F + (C_{Y\psi})_W + (C_{Y\psi})_V + \Delta_1 C_{Y\psi} + \Delta_2 C_{Y\psi} \quad (1)$$

$$C_{n\psi} = (C_{n\psi})_F + (C_{n\psi})_W + (C_{n\psi})_V + \Delta_1 C_{n\psi} + \Delta_2 C_{n\psi} \quad (2)$$

$$C_{l\psi} = (C_{l\psi})_F + (C_{l\psi})_W + (C_{l\psi})_V + \Delta_1 C_{l\psi} + \Delta_2 C_{l\psi} \quad (3)$$

The subscripts F and W refer to the derivatives of the isolated fuselage and of the isolated wing, respectively. In the general case, the subscript V refers to the contribution of the vertical tail when mounted on the fuselage and when in the presence of the horizontal tail. The present tests were made without a horizontal tail, since the effects of various horizontal-tail sizes and locations were investigated in reference 3. In the present report, therefore, the derivatives with the subscript V include both the effectiveness of the isolated vertical tail and the interference of the fuselage.

The vertical-tail contribution can be expressed analytically as follows:

$$(C_{Y\psi})_V = (C_{L\alpha})_V \frac{S_V}{S_W} \eta_Y \quad (4)$$

$$(C_{n\psi})_V = -(C_{L\alpha})_V \frac{l_V}{b_W} \frac{S_V}{S_W} \eta_N \quad (5)$$

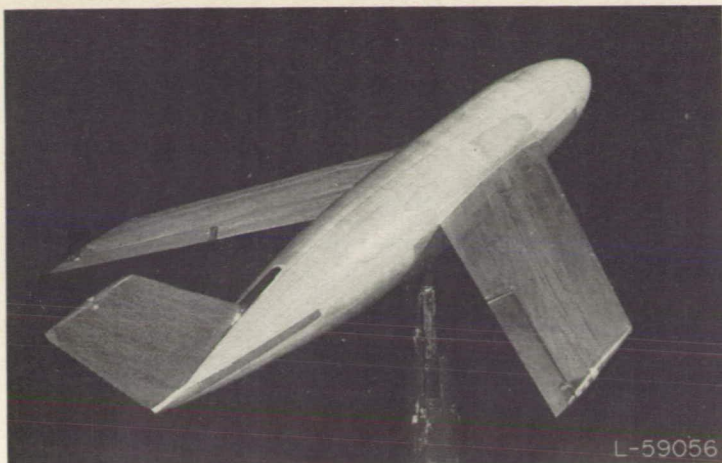
(a) Configuration $W+F_1+V_2$.

FIGURE 5.—View of model in the Langley stability tunnel.

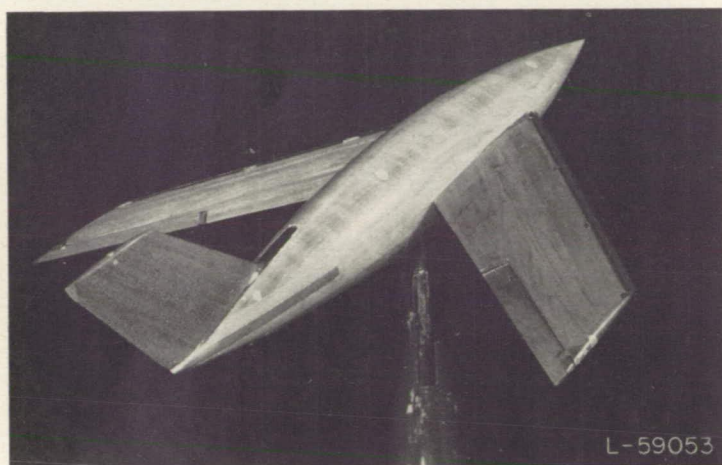
(b) Configuration $W_S+F_2+V_2$.

FIGURE 5.—Concluded.

$$(C_{l_\psi})_v = \left(\frac{z_v}{b_w} \cos \alpha - \frac{l_v}{b_w} \sin \alpha \right) \frac{S_v}{S_w} (C_{L_\alpha})_v \quad (6)$$

where $(C_{L_\alpha})_v$ is the effective vertical-tail lift-curve slope when the model is at zero angle of attack, and η_Y and η_N are correction factors which account for the variation in tail effectiveness with angle of attack. (A similar correction to C_{l_ψ} is neglected because it generally has been found to be very nearly 1.0.) Equations (4) to (6) are similar to equations given in reference 4, except that in the reference the factors η_Y and η_N are neglected. The results of the present tests are used for evaluating the factors η_Y and η_N and the effective aspect ratio A_{e_v} , corresponding to the vertical-tail lift-curve slope $(C_{L_\alpha})_v$.

Perhaps the most consistent approach to the problem of evaluating tail effectiveness would involve determination of A_{e_v} corresponding to $(C_{L_\alpha})_v$ as determined from equation (4). In order to make use of such values of A_{e_v} in the calculation of $(C_{n_\psi})_v$ and $(C_{l_\psi})_v$, effective, rather than geometric, values

of the tail length l_v and of the tail height z_v also would have to be known. From practical considerations, it has seemed most convenient to assume that the location of the vertical-tail center of pressure is given accurately by the geometric lengths l_v and z_v . Since the directional-stability parameter $(C_{n_\psi})_v$ is considered to be the most important of the three static-lateral-stability parameters, values of A_{e_v} , corresponding to $(C_{L_\alpha})_v$ as determined from equation (5), are obtained in the present analysis. The reliability of values of A_{e_v} so determined, when used to calculate $(C_{Y_\psi})_v$ and $(C_{l_\psi})_v$, is checked against the experimental results.

Since, at zero angle of attack, the factor η_N is 1.0, equation (5) can be rewritten as

$$(C_{L_\alpha})_v = -\frac{b_w S_w}{l_v S_v} (C_{n_\psi})_v$$

Values of A_{e_v} , corresponding to $(C_{L_\alpha})_v$, may be obtained from theory such as that of reference 5. A correction to A_{e_v} for the effect of the horizontal tail can be obtained from reference 3.

The increments prefixed by Δ_1 and Δ_2 express, respectively, the interference of the wing-fuselage combination and the interference of the wing-fuselage on the vertical-tail effectiveness; for example,

$$\Delta_1 C_{Y_\psi} = (C_{Y_\psi})_{W+F} - [(C_{Y_\psi})_W + (C_{Y_\psi})_F]$$

and

$$\Delta_2 C_{Y_\psi} = [(C_{Y_\psi})_{W+F+V} - (C_{Y_\psi})_{W+F}] - [(C_{Y_\psi})_{F+V} - (C_{Y_\psi})_F]$$

The interference increments usually are assumed to apply to airplanes having configurations which are somewhat similar to that of the model used in evaluating the increments. Of the various factors which affect the magnitudes of the interference increments, the height of the wing, relative to the center line of the fuselage, previously has been found to be one of the most important (reference 2). Since, for the present investigation, the wing was located on the center line of the fuselages, the results are considered applicable only to midwing or near-midwing arrangements.

RESULTS AND DISCUSSION

PRESENTATION OF RESULTS

The basic data obtained in this investigation are presented in figures 6 to 14. The longitudinal characteristics of the wing alone and of the wing with slat are given in figure 6. The static-lateral-stability parameters of the various configurations investigated are given in figures 7 to 14. A summary of the configurations investigated and of the figures that give data for these configurations is given in table IV. Most of the remaining figures (figs. 15 to 30) were made up from the data of figures 7 to 14 and present the data in a form more suitable for analysis.

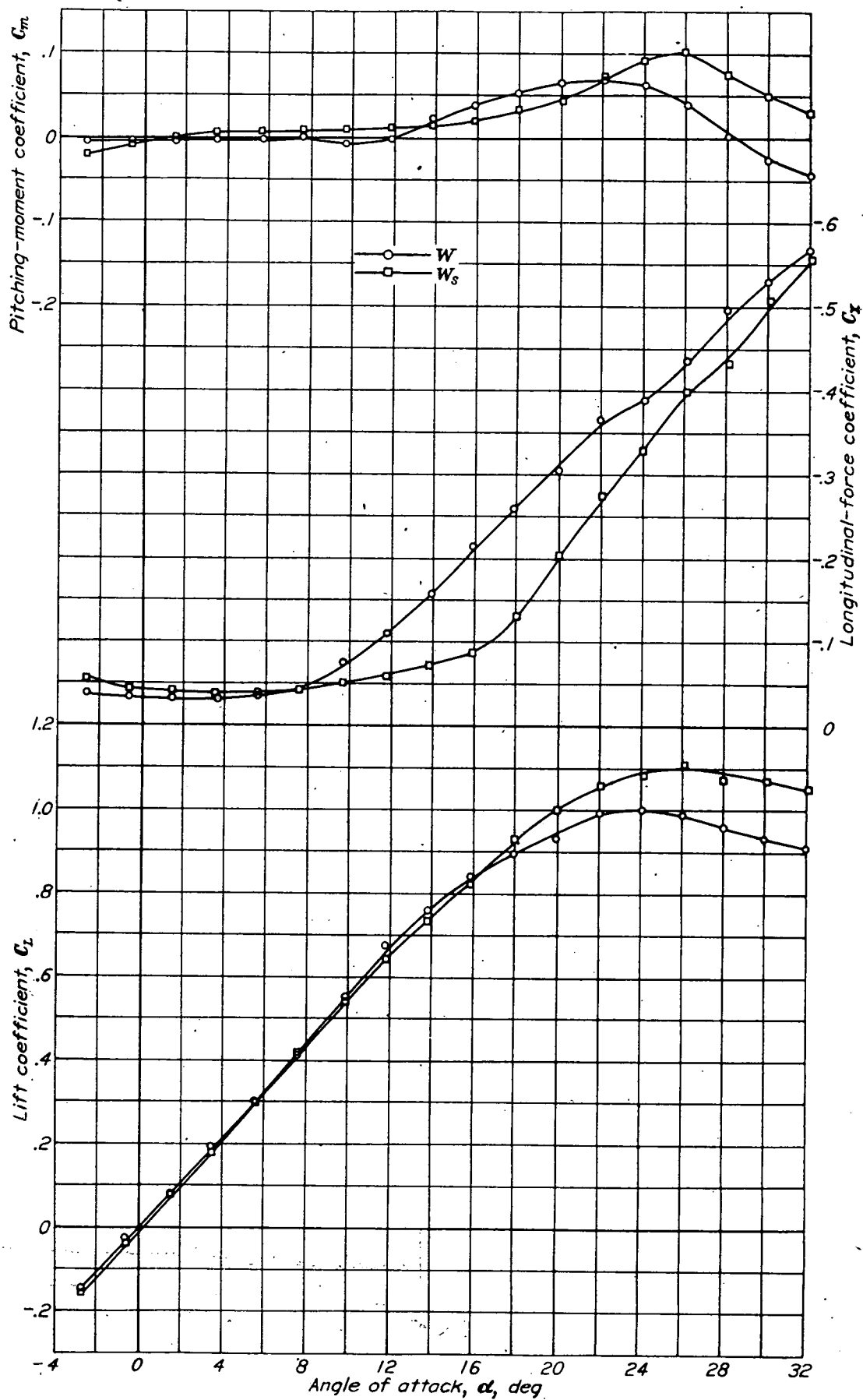


FIGURE 6.—Aerodynamic characteristics of the wing.

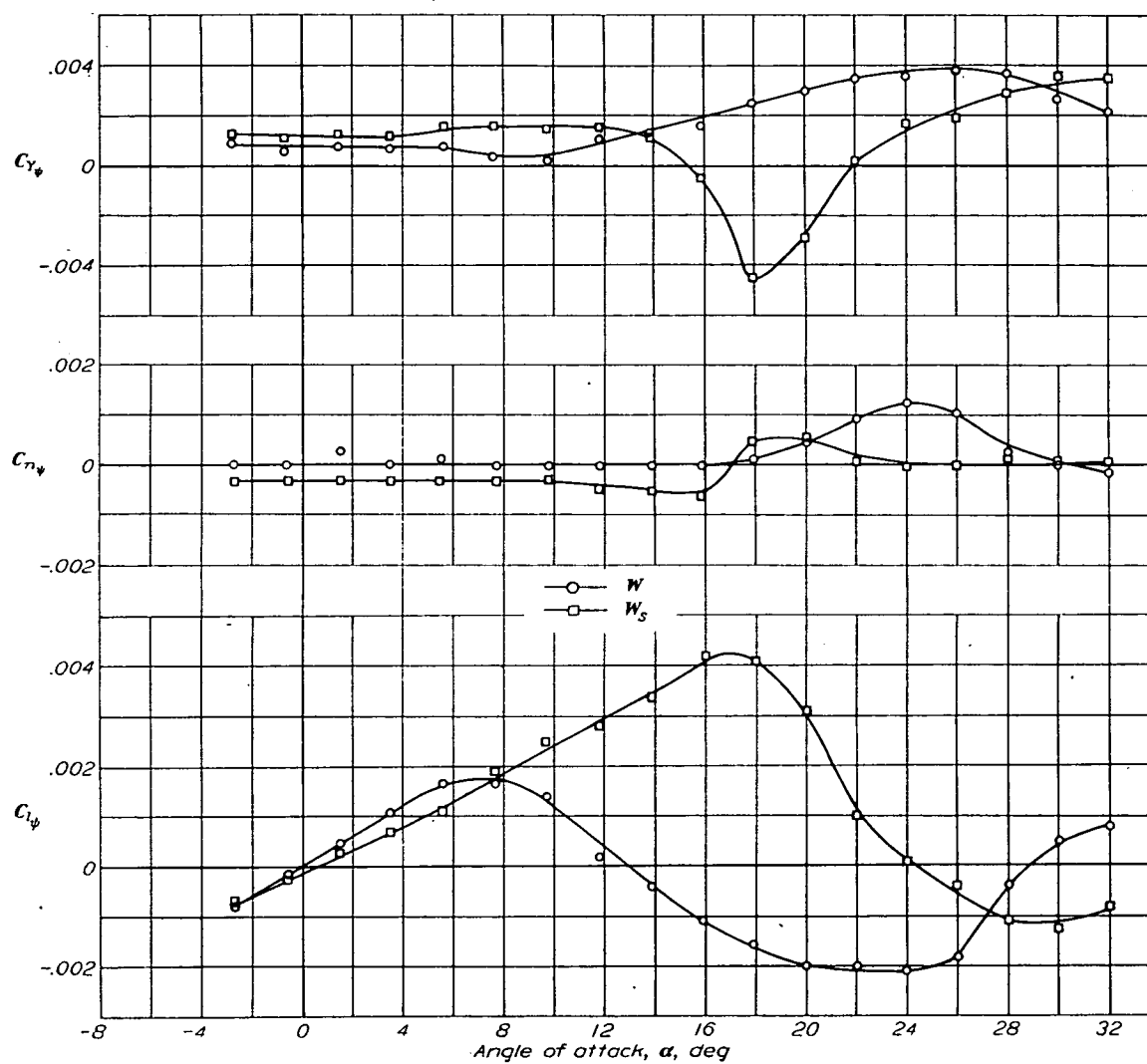
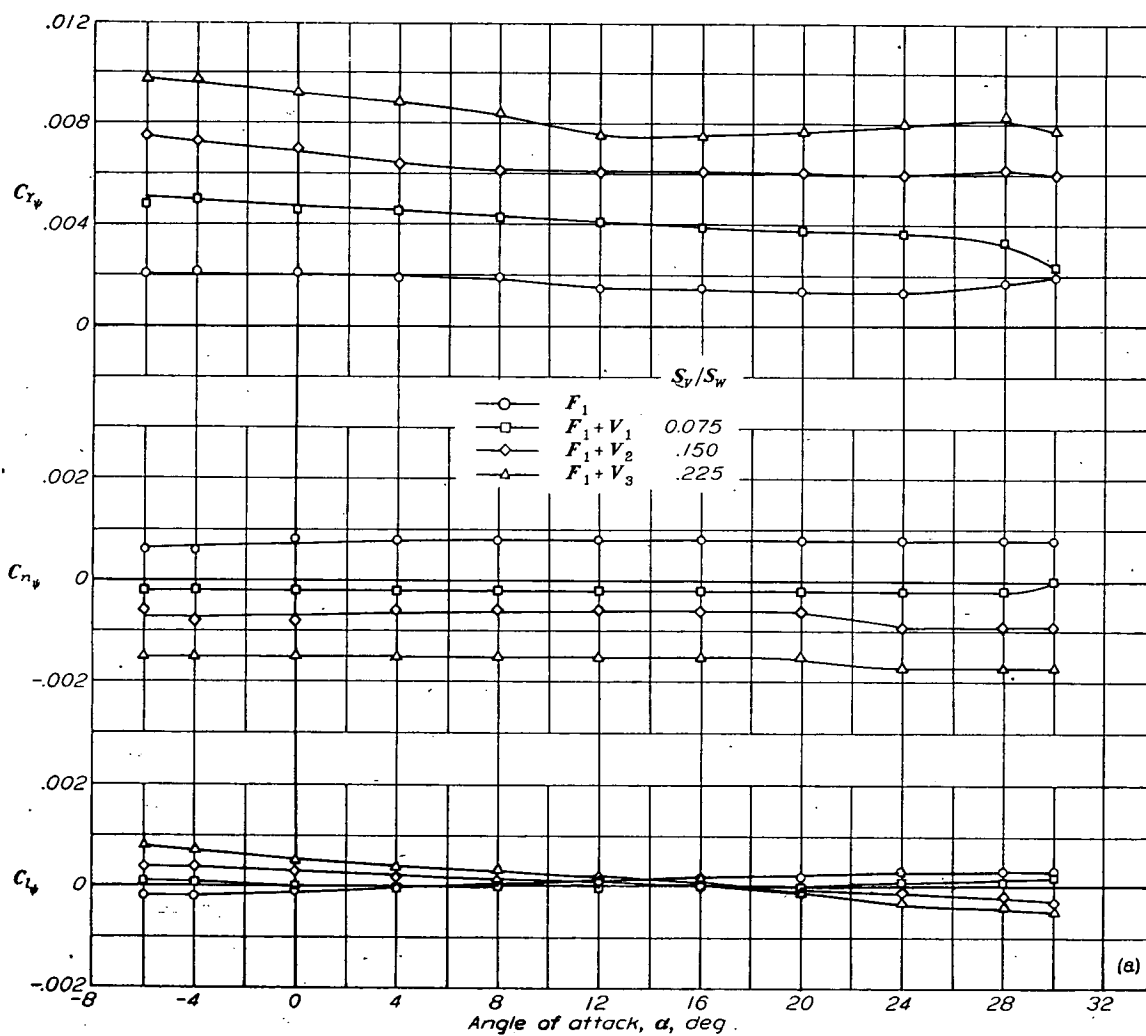
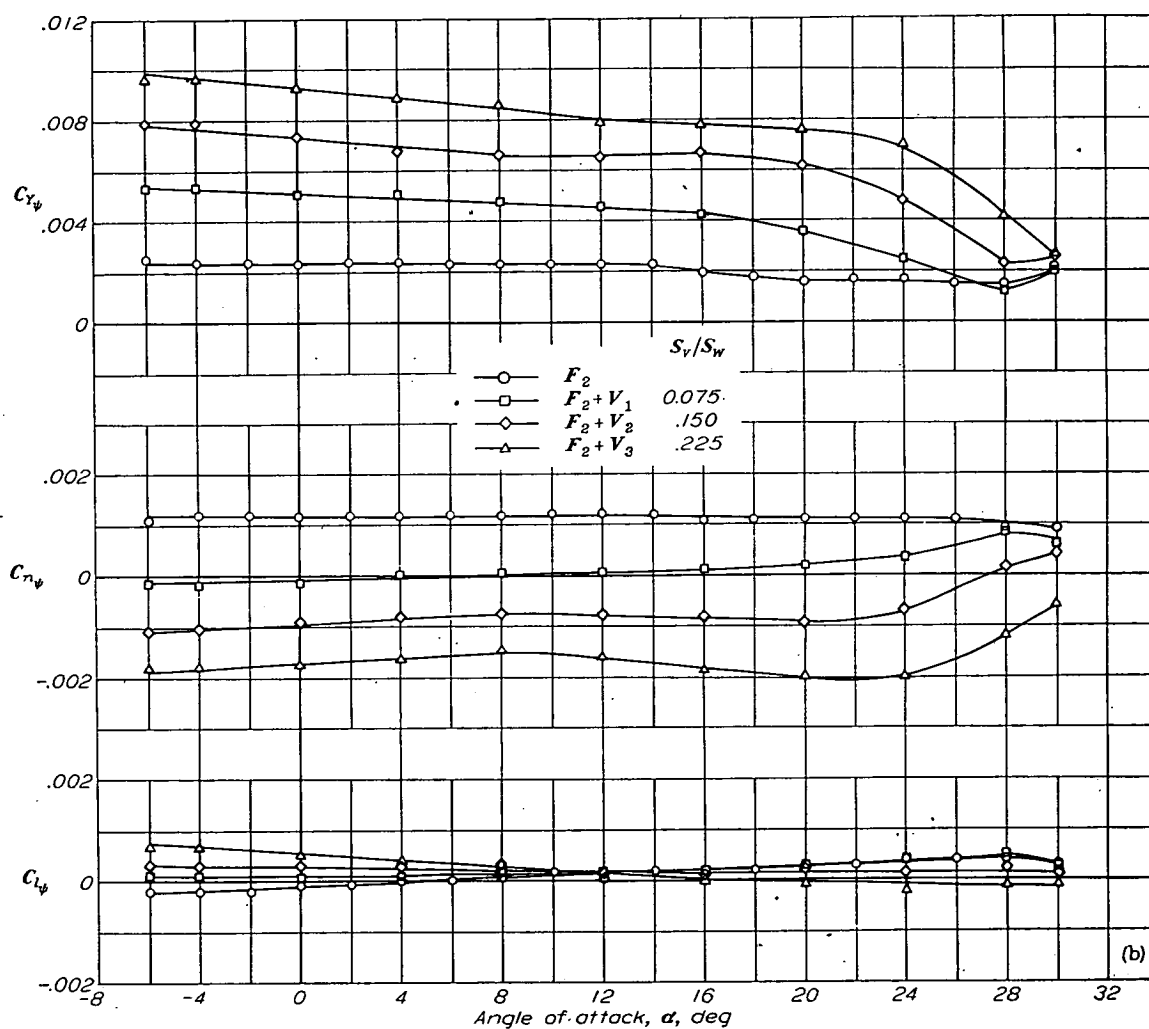


FIGURE 7.—Static lateral stability characteristics of the wing.



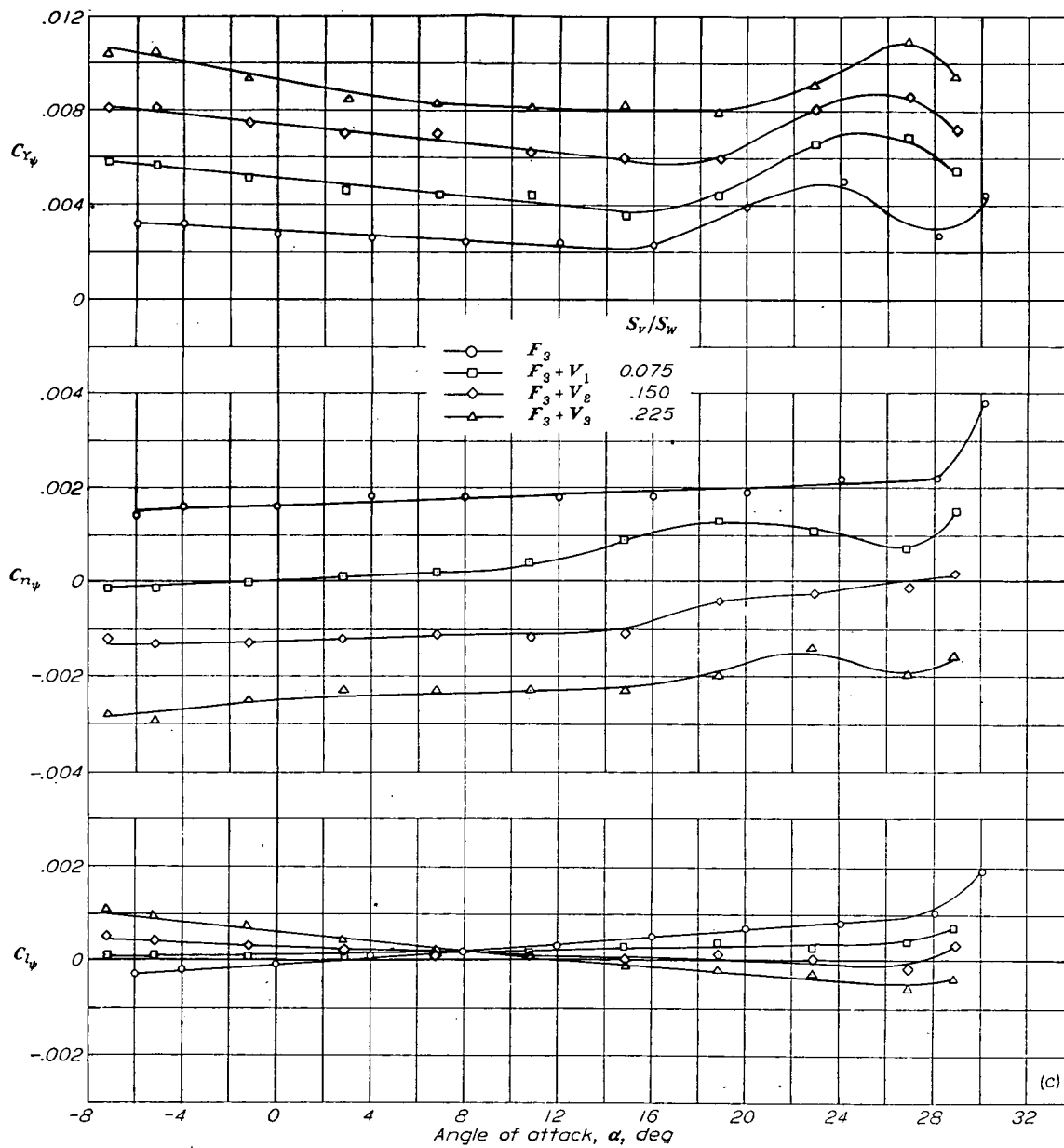
(a) Fuselage 1 (short).

FIGURE 8.—Effect of vertical tail on the static lateral stability characteristics. Wing off; $A_V=1.0$.

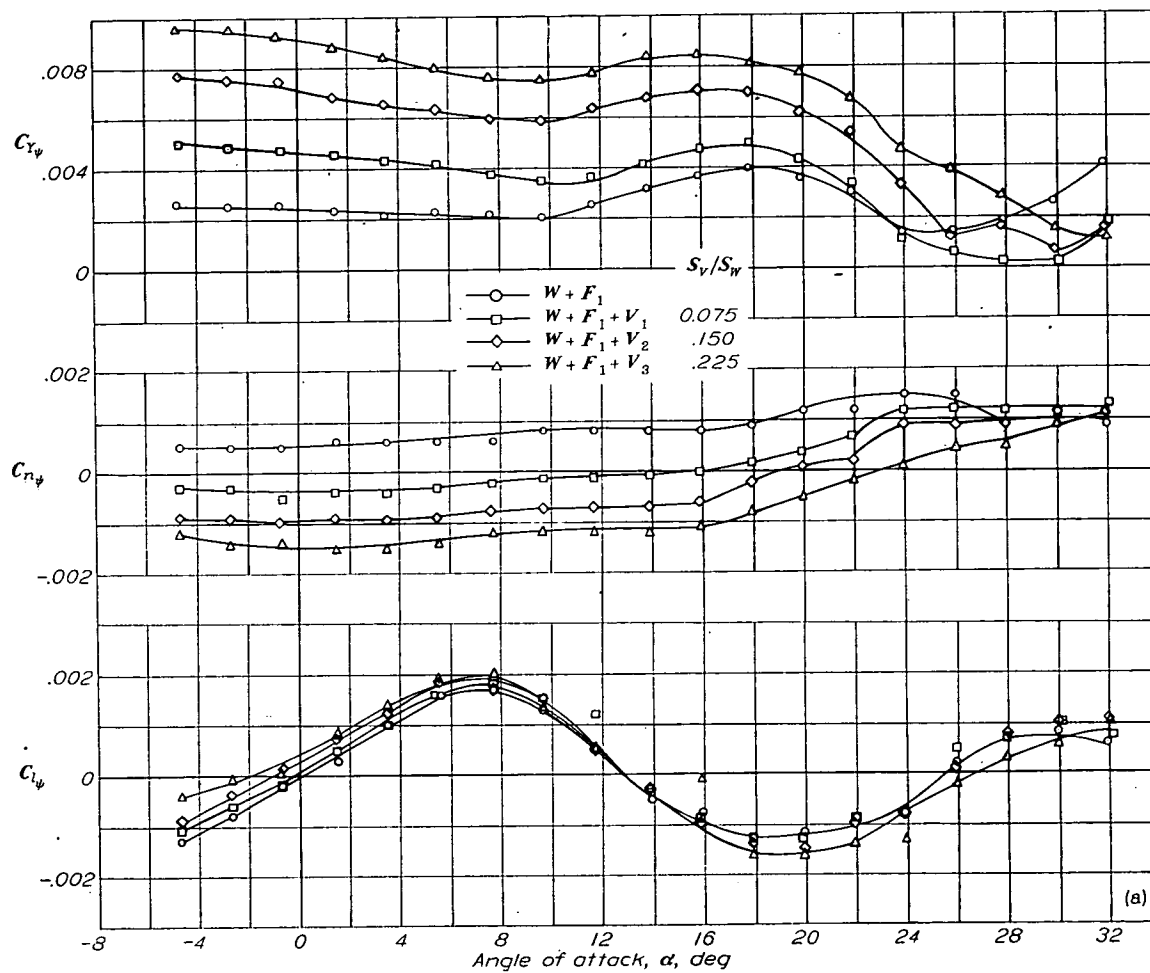


(b) Fuselage 2 (medium).

FIGURE 8.—Continued.

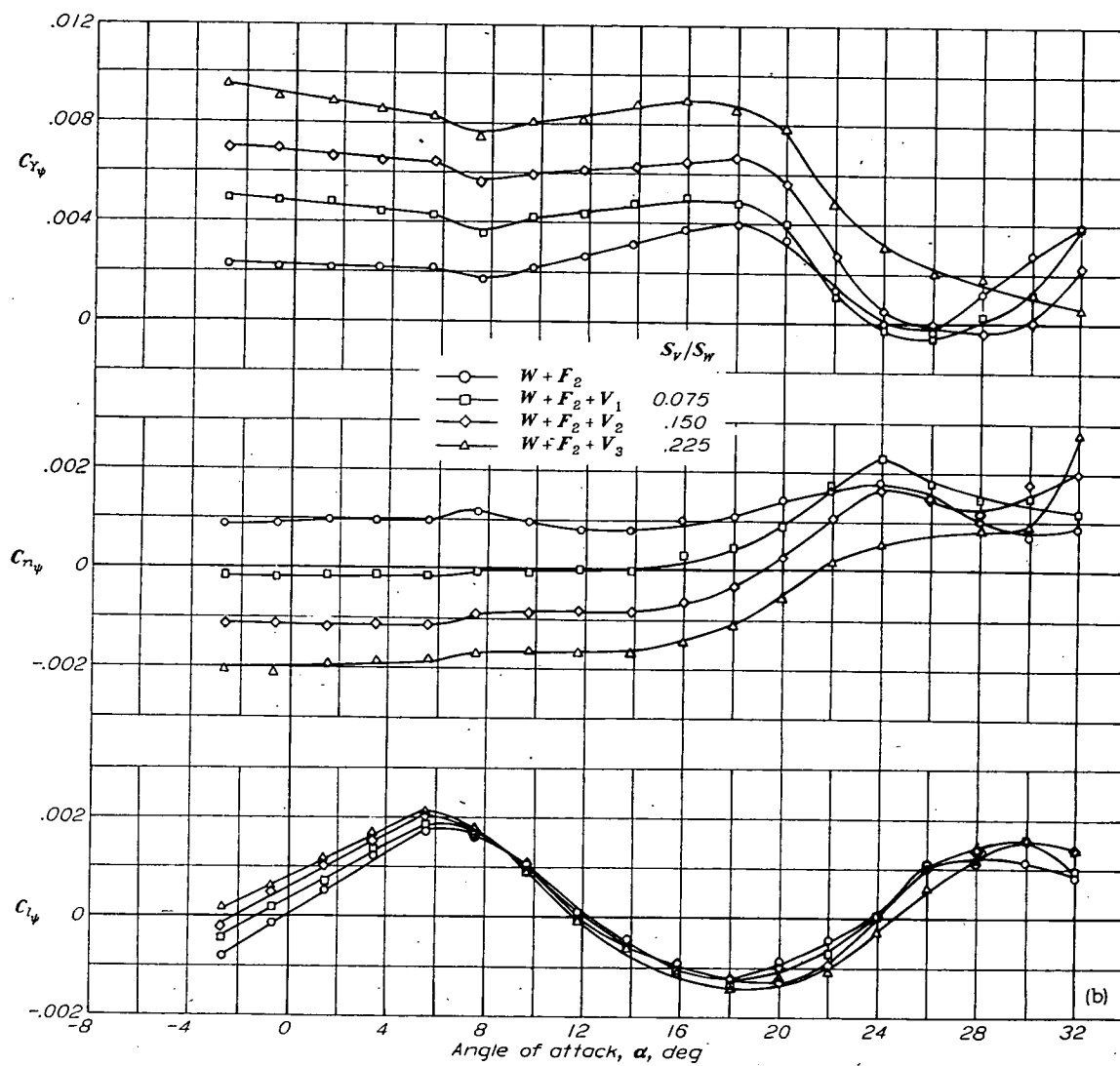


(c) Fuselage 3 (long).
FIGURE 8.—Concluded.

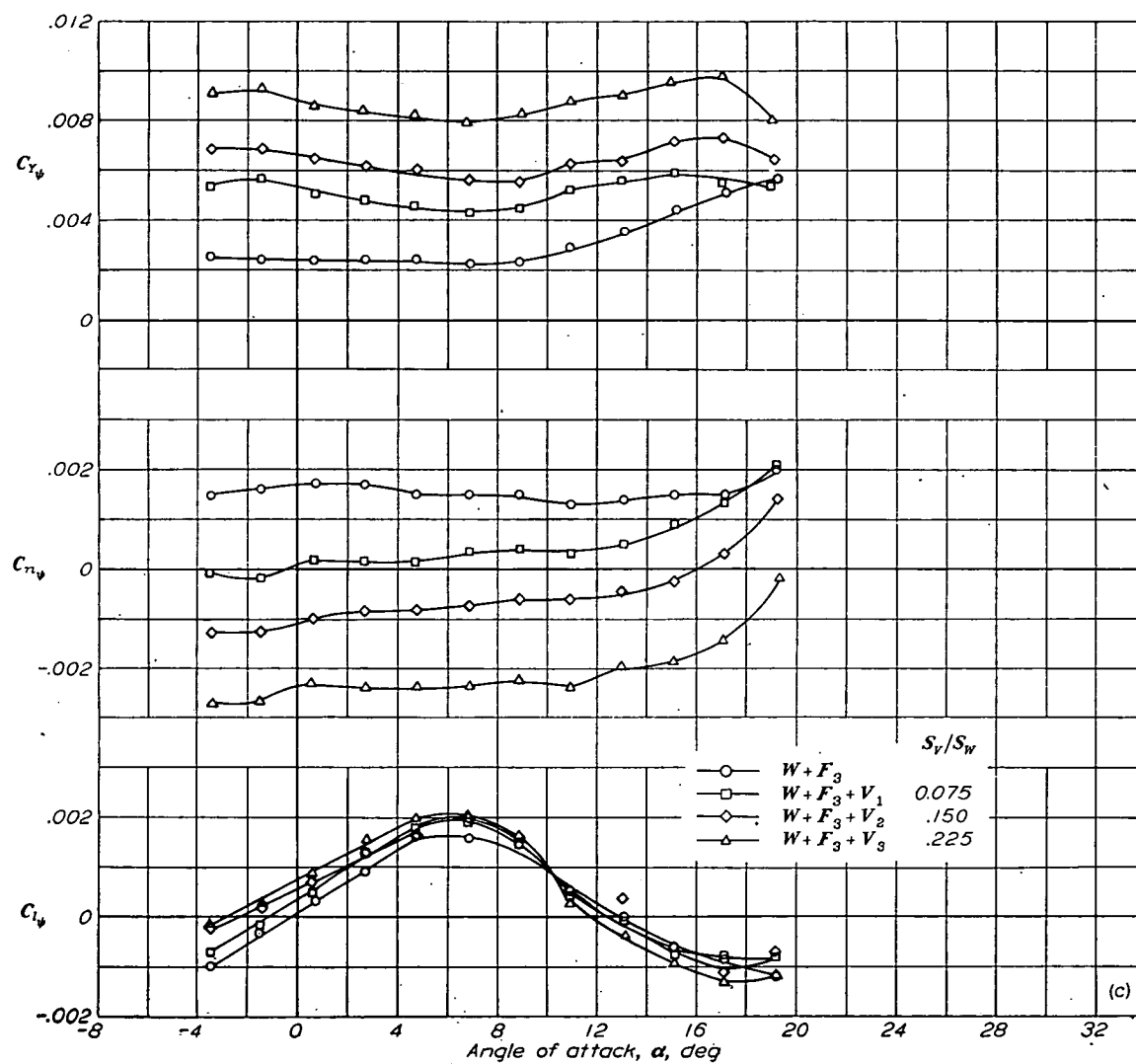


(a) Fuselage 1 (short).

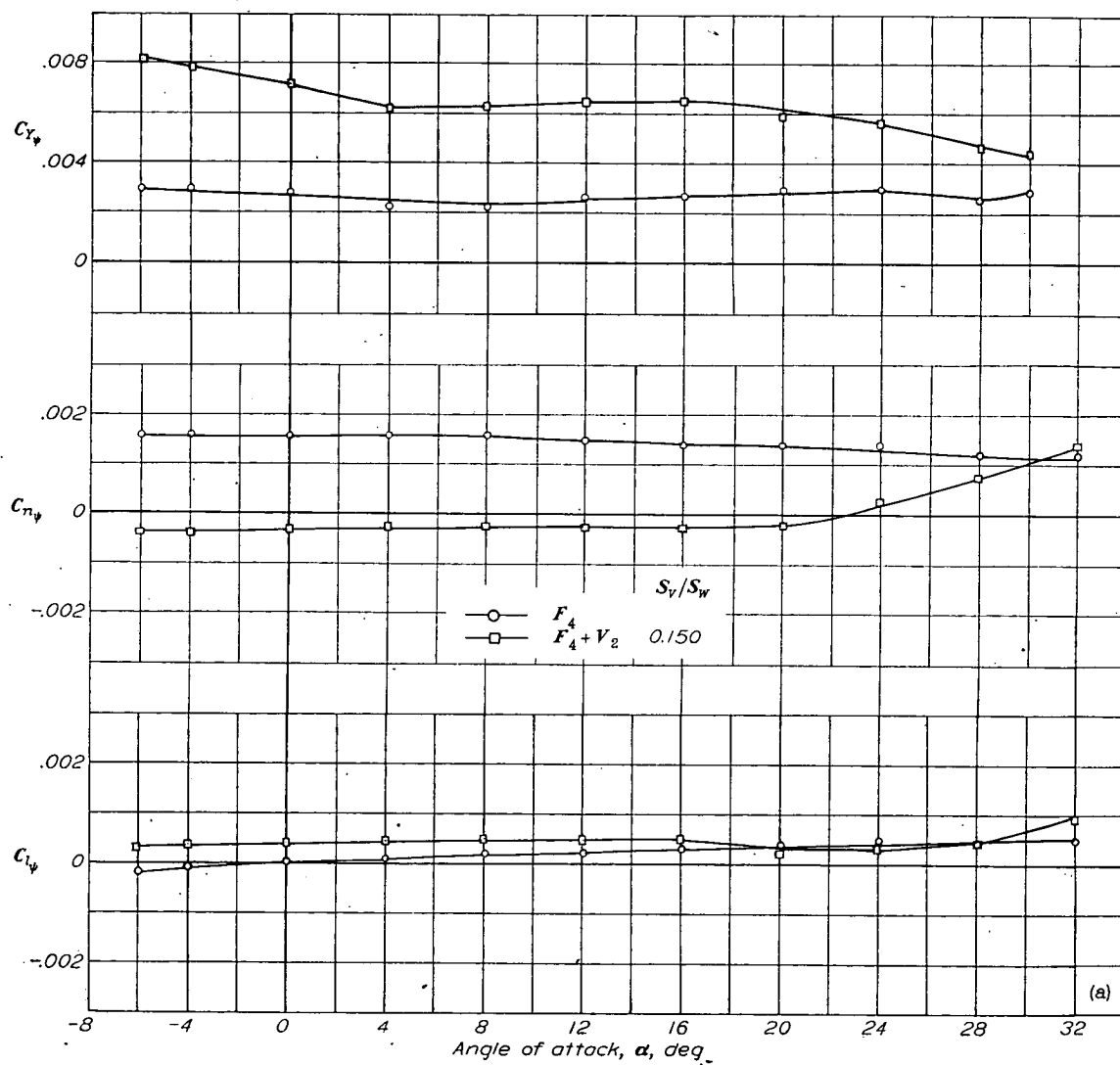
FIGURE 9.—Effect of vertical tail on the static lateral stability characteristics. Wing on; $A_v=1.0$.



(b) Fuselage 2 (medium).
FIGURE 9.—Continued.

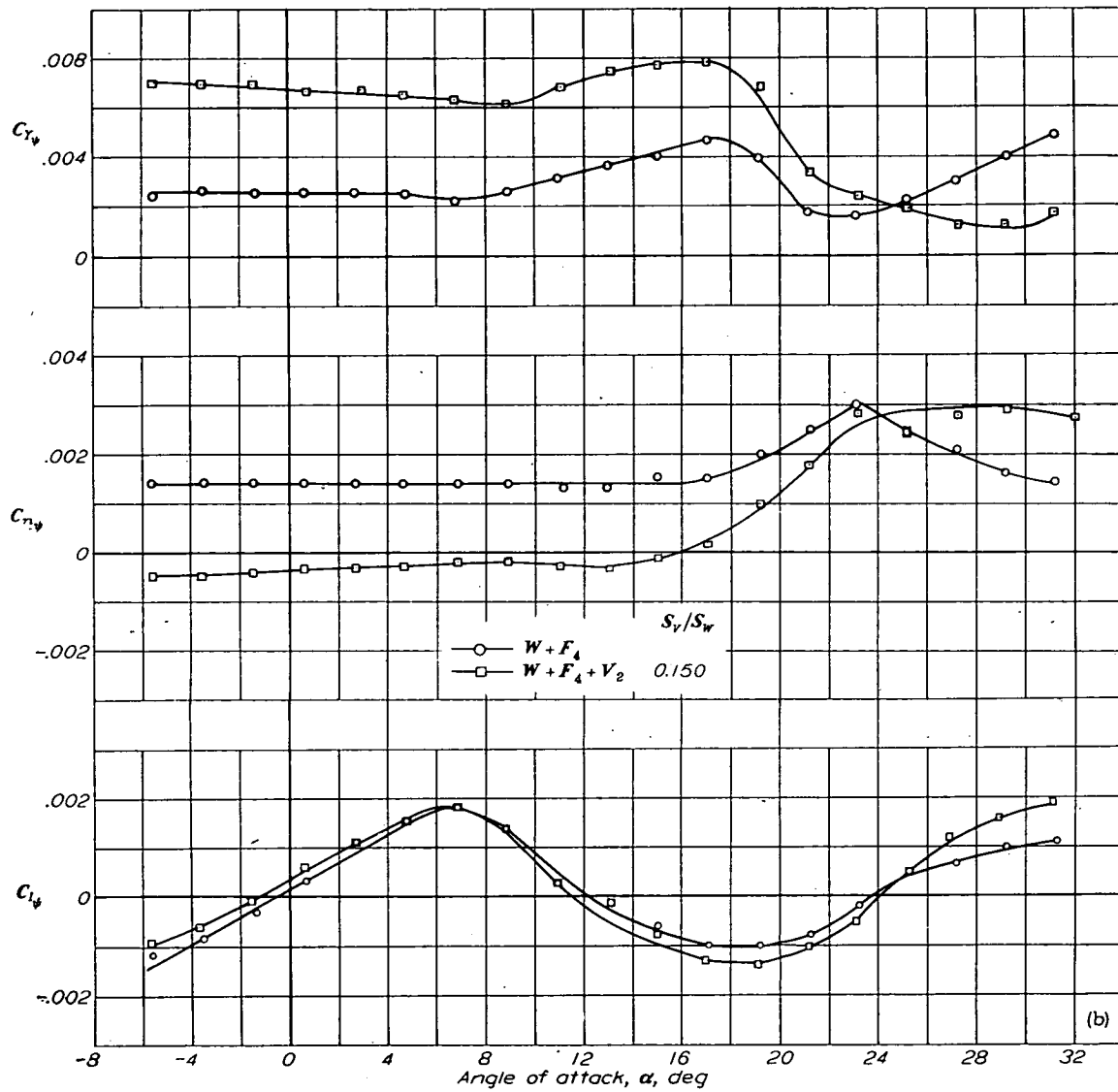


(c) Fuselage 3 (long).
FIGURE 9.—Concluded.



(a) Wing off.

FIGURE 10.—Static lateral stability characteristics obtained with the blunt-nose fuselage. $Ar=1.0$.



(b) Wing on.
FIGURE 10.—Concluded.

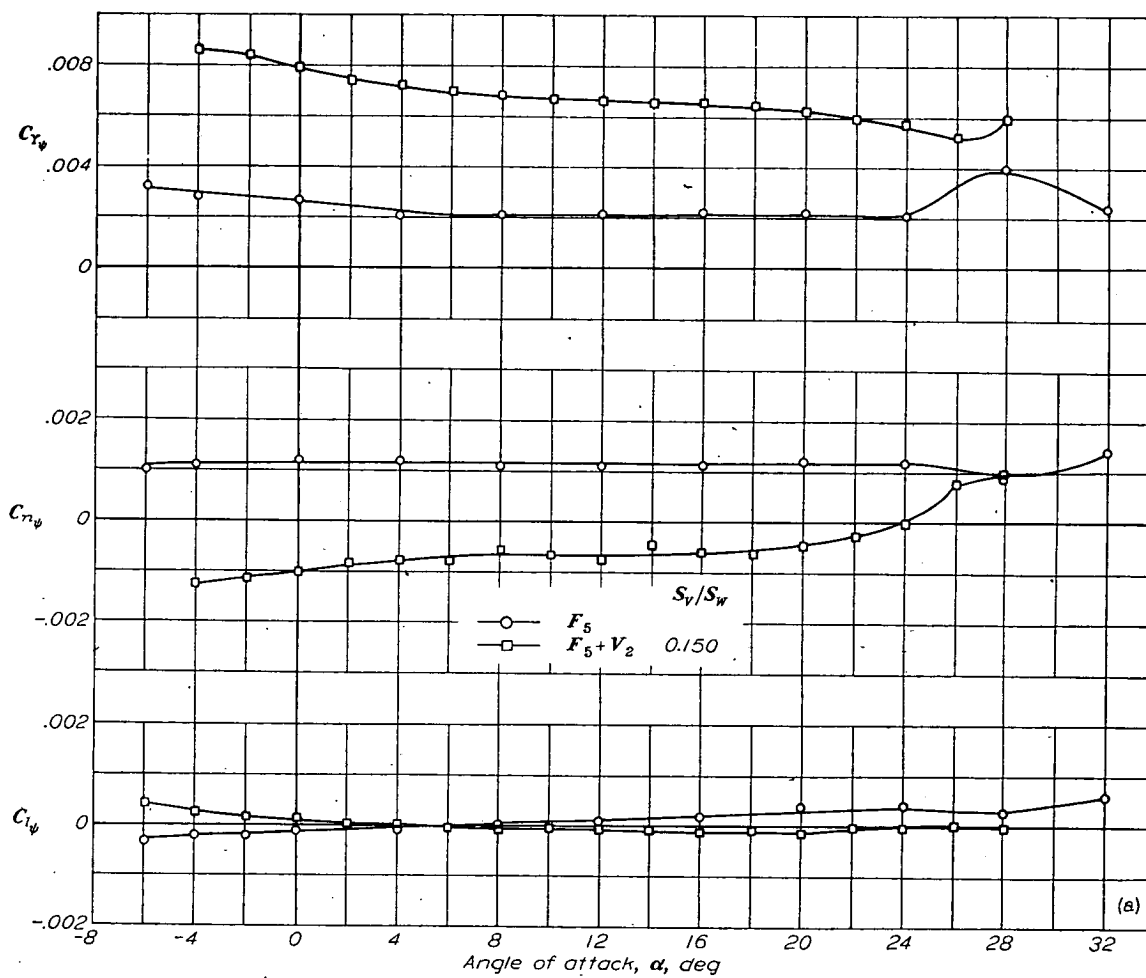
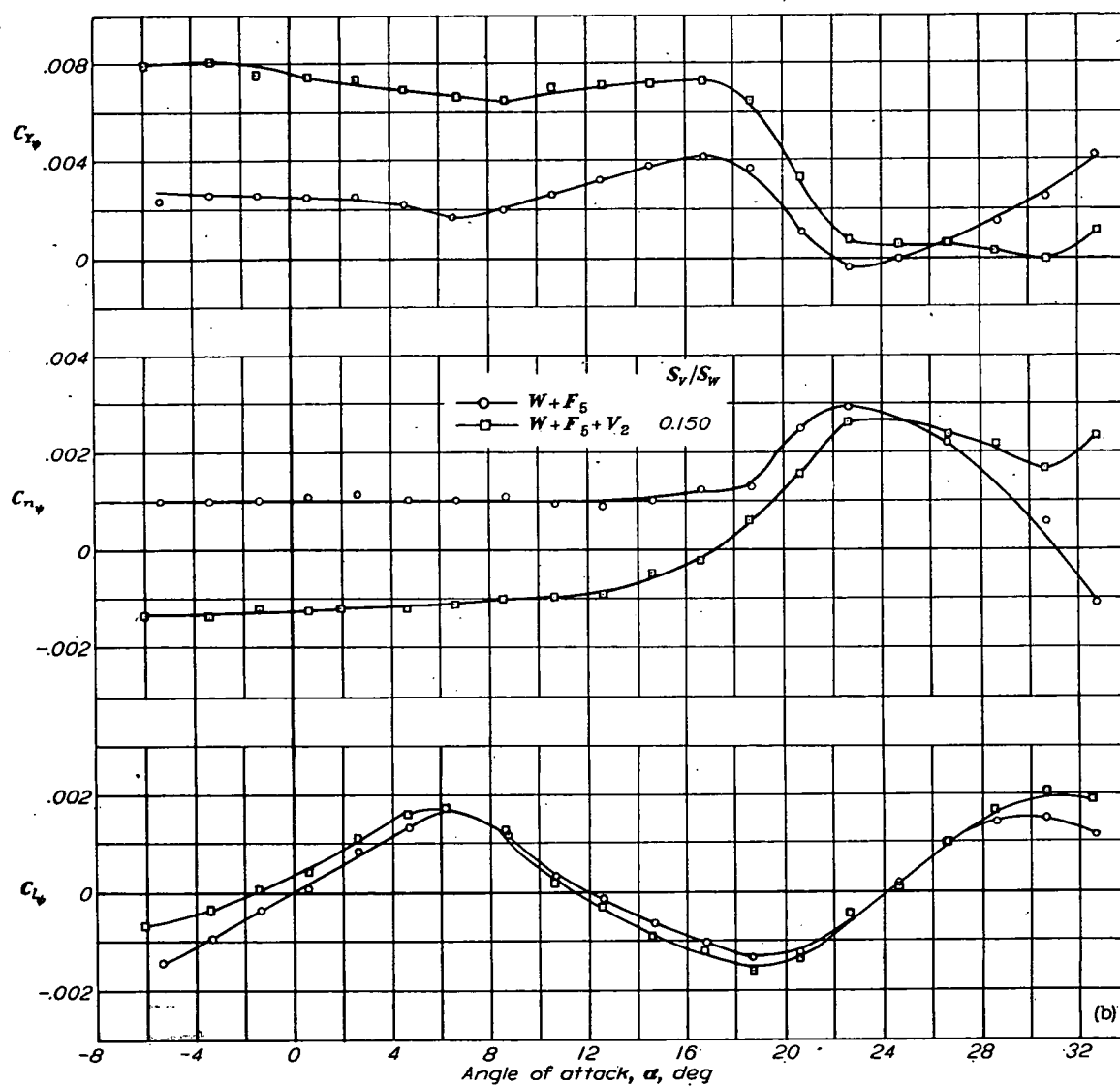


FIGURE 11.—Static lateral stability characteristics obtained with blunt-end fuselage. $Ar=1.0$.



(b) Wing on.

FIGURE 11.—Concluded.

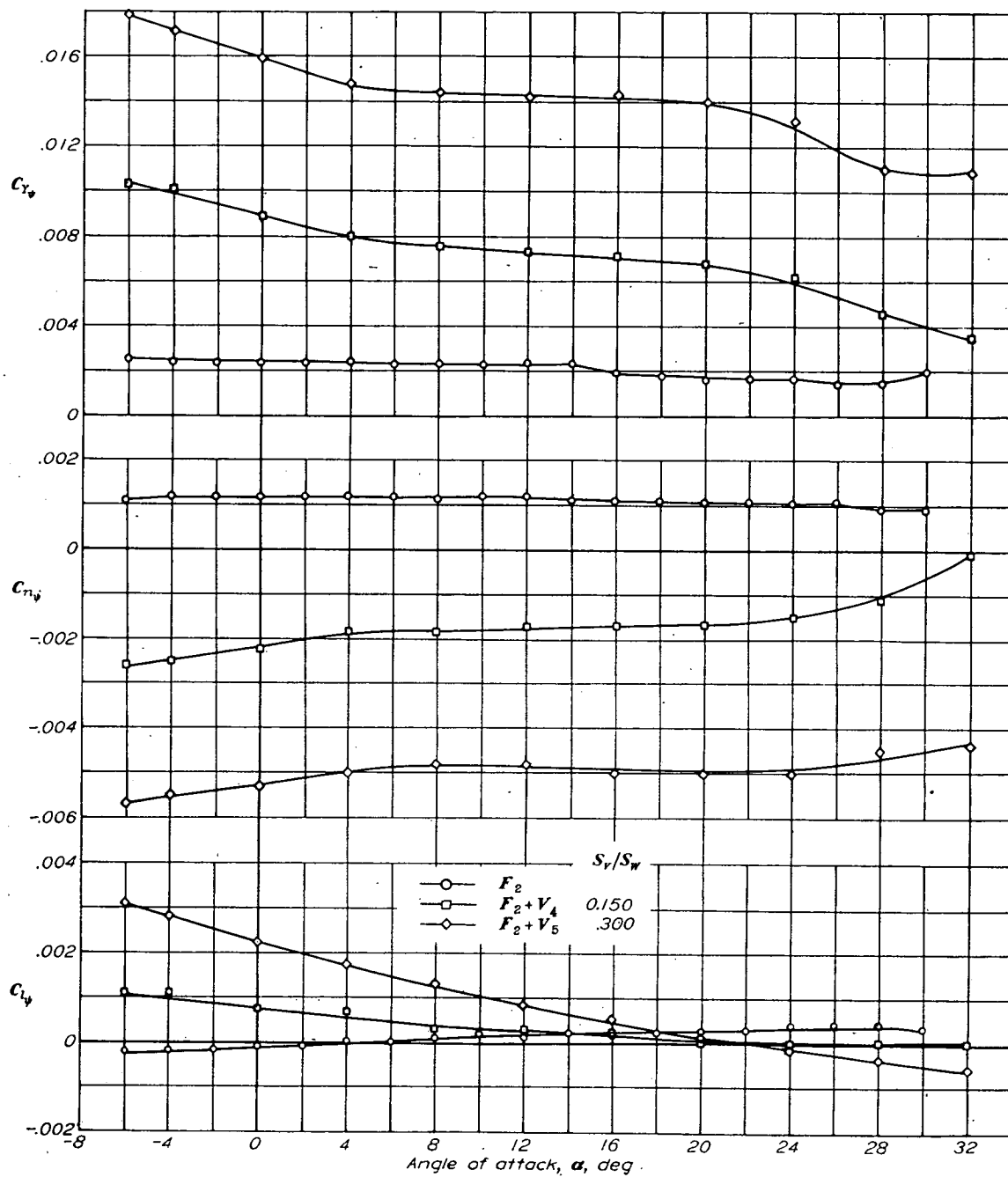


FIGURE 12.—Effect of vertical tail on the static lateral stability characteristics. Wing off; fuselage 2; $A_v=2.0$.

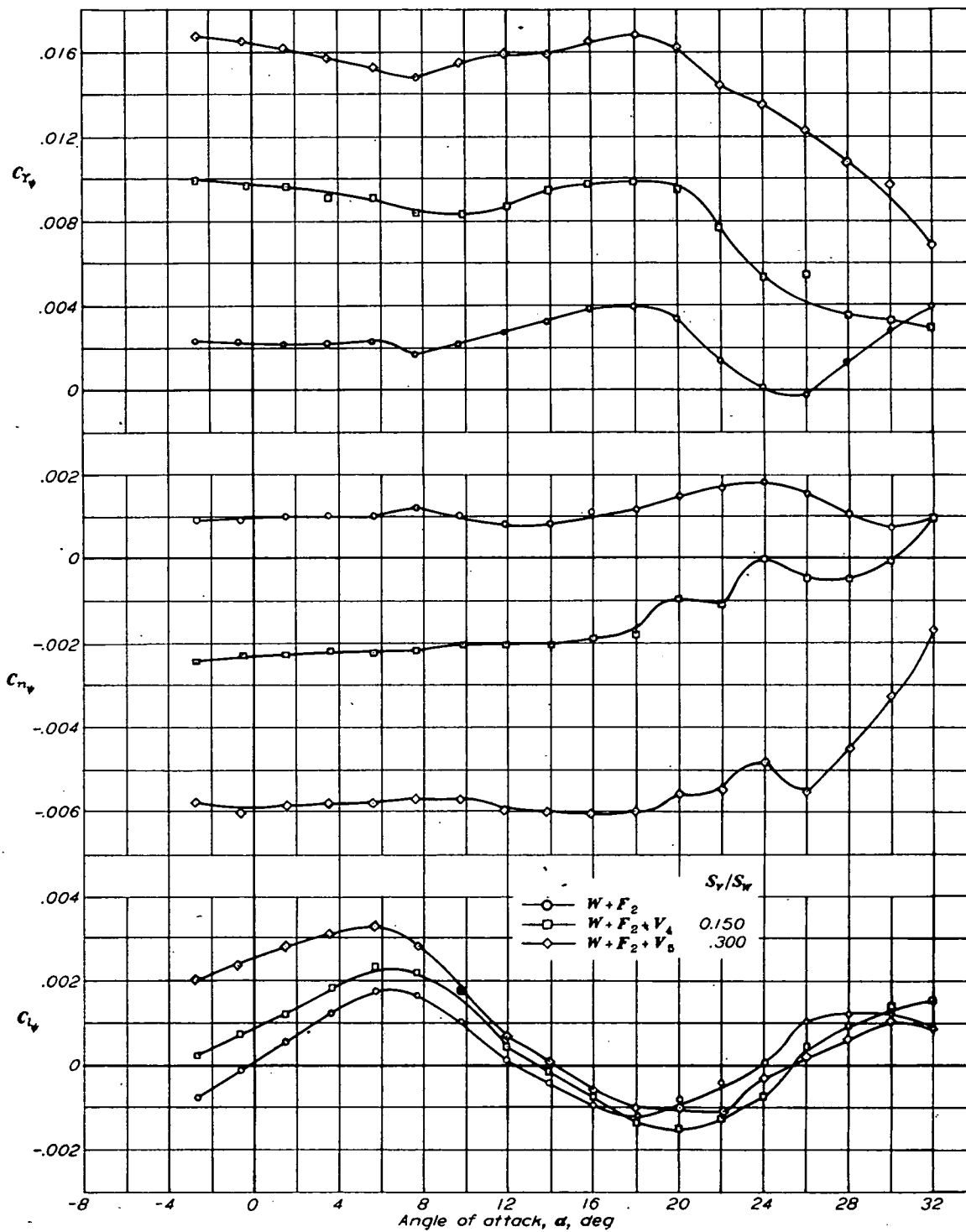
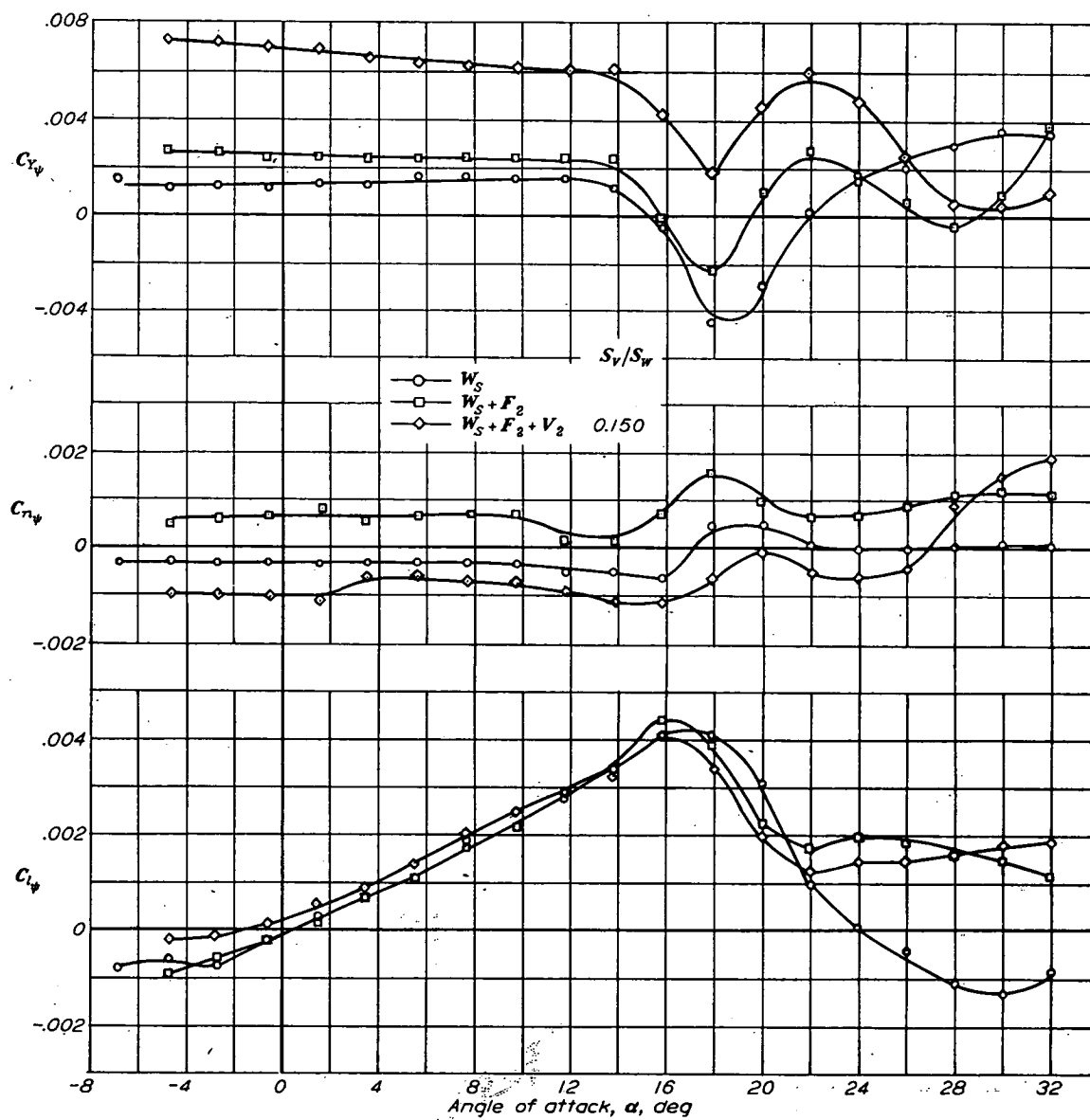


FIGURE 13.—Effect of vertical tail on the static lateral stability characteristics. Wing on; fuselage 2; $A_T=2.0$.

FIGURE 14.—Static lateral stability characteristics of configurations having wing leading-edge slat. $A_T=1.0$.

WING CHARACTERISTICS

The longitudinal aerodynamic characteristics of the wing alone (fig. 6) have been given in reference 3; hence, they are reviewed only briefly in this report. The plain wing stalled at about 24° angle of attack ($C_L=1.0$) and showed an aerodynamic-center position of $0.25\bar{c}_w$. The theory of reference 5 predicts an aerodynamic-center position of $0.26\bar{c}_w$. Addition of the $0.08c_w$ slat delayed the stall to about 26° angle of attack ($C_L=1.1$) but had no appreciable effect on the position of the aerodynamic center at low angles of attack. The slat caused an appreciable reduction in drag at angles of attack greater than about 8° .

Many of the aerodynamic parameters of a complete airplane are dependent to some extent on the character of the flow over the wing; hence, some consideration must be given to the angle-of-attack range over which flow does not separate from the wing. As pointed out in reference 6, an indication of the limit of this range can be obtained by locating the initial break in the plot of $C_D - \frac{C_L^2}{\pi A_w}$ against angle of attack.

A plot of this increment for the plain wing and for the wing with slat is given in figure 15. The figure shows breaks in the curves at about 7.7° and at about 16° for the wing alone and for the wing with slat, respectively. Corresponding breaks in the curves of the aerodynamic characteristics of combinations involving the wing and the wing with slat are to be expected at about these same angles of attack.

Investigations involving Reynolds number as a variable have shown that for smooth wings increases in Reynolds number tended to extend the angle-of-attack range before which initial breaks occurred in plots of aerodynamic parameters against angle of attack. For this reason, results obtained for configurations with slats might be expected to be somewhat similar to data for the plain wing at a higher Reynolds number than the test Reynolds number.

FUSELAGE CHARACTERISTICS

The important characteristics of the various fuselages are summarized in figure 16. In general, the parameters considered $(C_{Y_\psi})_F$ and $(C_{n_\psi})_F$ varied only slightly with angle of attack, and therefore the analysis has been limited to characteristics at $\alpha=0^\circ$.

In order that the results obtained may be applied conveniently to arbitrary airplane configurations, coefficients in terms of fuselage dimensions rather than wing dimensions are needed. This manner of expressing the coefficient is accomplished by plotting the quantities $(C_{Y_\psi})_F \frac{S_w}{S_s}$ and $(C_{n_\psi})_F \frac{S_w b_w}{V_F}$ against fuselage fineness ratio. The quantities plotted, therefore, are effectively a lateral-force coefficient based on fuselage side area S_s and a yawing-moment coefficient based on fuselage volume V_F .

Comparisons are made with the theory presented in reference 7. Although the theory, which is based on potential-flow considerations, predicts no side force, the experimental results show a positive side force which increases as the fineness ratio is decreased. The variations in fuselage shape considered, for a constant fineness ratio, have a negligible effect on the value of a lateral-force coefficient based on fuselage side area.

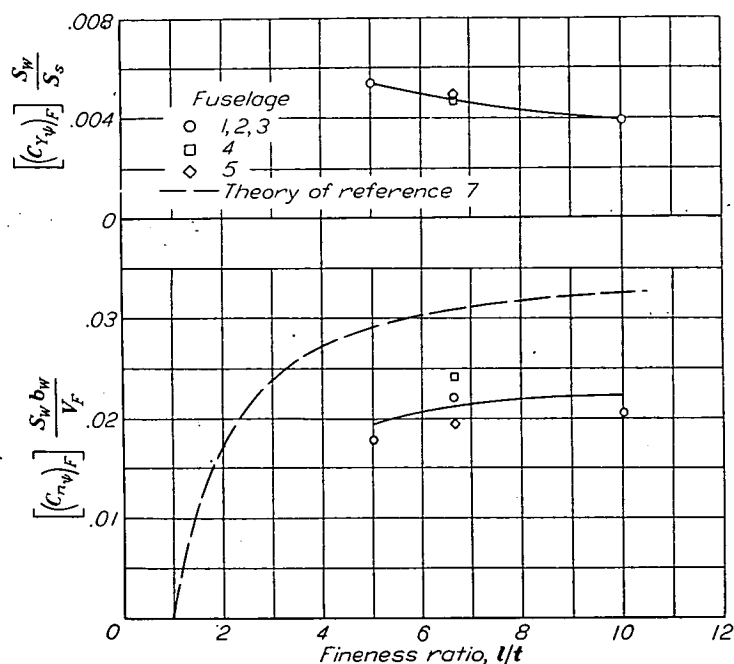


FIGURE 16.—Summary of fuselage contributions to C_{r_ψ} and C_{n_ψ} . $\alpha=0^\circ$.

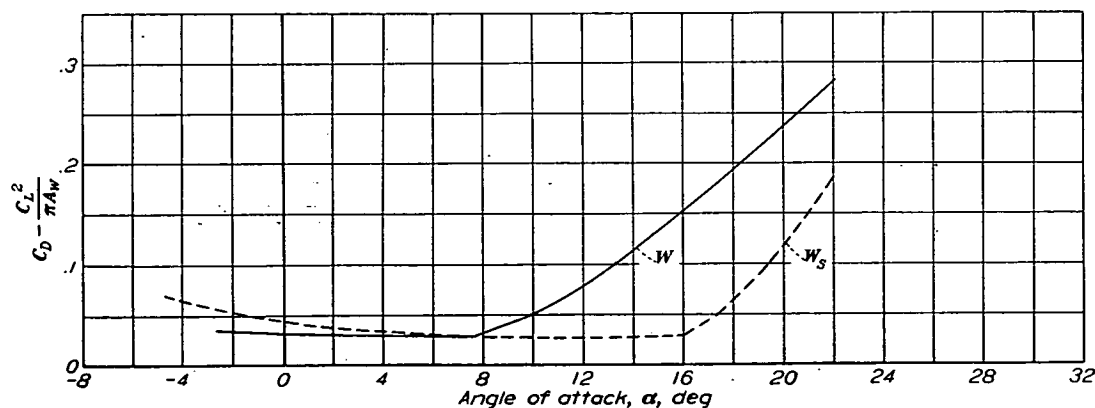


FIGURE 15.—Effect of wing leading-edge slat on the variation of $C_D - \frac{C_L^2}{\pi A_w}$ of the wing with angle of attack.

The experimental results obtained for the directional-stability parameter $(C_{N\psi})_F$ of the biconvex fuselages show about the same trend with variation in fineness ratio as that predicted by theory, although, quantitatively, the magnitude is only about two-thirds of that predicted by theory. For a constant fineness ratio, the variations in fuselage shape considered produced a rather large change in the magnitude of the directional-stability parameter based on fuselage volume. An increase in volume near the fuselage nose increased this parameter; whereas an increase in volume over the rear half of the fuselage decreased this parameter.

VERTICAL-TAIL EFFECTIVENESS

Effective aspect ratio.—As explained in the section entitled "Methods of Analysis," the effective aspect ratio of the vertical tail is obtained by calculating the tail lift-curve slope from experimental values of $(C_{N\psi})_V$ and then obtaining the corresponding aspect ratio from a theory of plain wings. The theory of reference 5 has been used herein, although it is realized that a swept vertical tail represents an unsymmetrical configuration to which the theory is not strictly applicable. The relationship, given by reference 5, between lift-curve slope and aspect ratio for wings having a sweep angle of 45° and a taper ratio of 0.6 is reproduced in figure 17. The results of the effective-aspect-ratio determinations are presented in figure 18 in the form of the ratio A_{eV}/A_V plotted against b_V/D_F for $\alpha=0^\circ$. The quantity b_V/D_F is the ratio of vertical-tail span to the fuselage diameter at the longitudinal location of the vertical-tail aerodynamic center and is regarded as a significant parameter for determining the influence of the fuselage on the vertical-tail effectiveness. An average curve is drawn through the data obtained with the tails of aspect ratio 1.0, and another curve, through the two points obtained with the tails of aspect ratio 2.0. The fairing of the average curve at low values of b_V/D_F has been

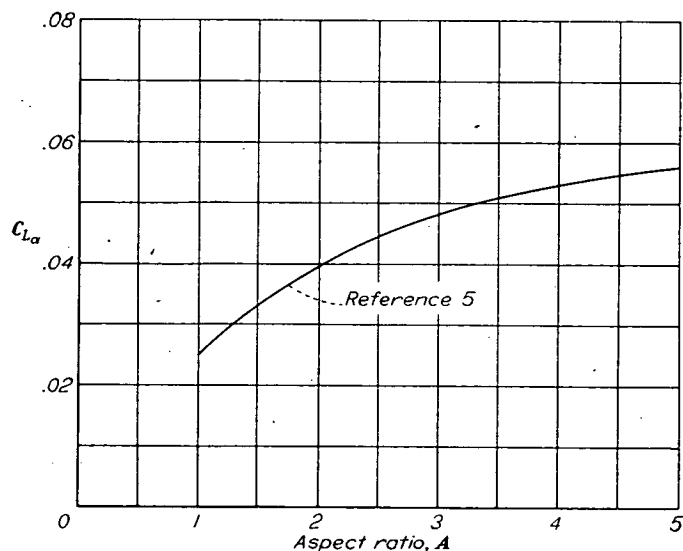


FIGURE 17.—Variation of lift-curve slope with aspect ratio for 45° sweptback wings with taper ratio 0.6.

guided by the shape of the calculated curve which represents reasonable maximum values of A_{eV}/A_V for given values of b_V/D_F . The calculated curve was determined by an equation derived on the assumption that the fuselage acts as an infinite end plate on the portion of the vertical tail protruding outside the fuselage. The equation of the curve is

$$\frac{A_{eV}}{A_V} = \frac{2\left(2\frac{b_V}{D_F} - 1\right)}{2\frac{b_V}{D_F} - \frac{1 - \lambda_V}{1 + \lambda_V}} \quad (7)$$

A reduction in area (geometric aspect ratio kept constant) of vertical tails attached to a given fuselage resulted in an increase in the effective aspect ratios of the vertical tails for the range of tail size investigated. The calculated curve indicates that for smaller tails the opposite would be true.

The experimental data show that the ratio A_{eV}/A_V approaches the value 1.0 as b_V/D_F becomes large. This variation is to be expected since an increase in b_V/D_F represents a decrease in the size of the end plate relative to the vertical tail. For very large values of b_V/D_F , the effective and geometric aspect ratios should be approximately equal. The values of A_{eV}/A_V given in figure 18 depend to some extent on the curve of $C_{L\alpha}$ against A from which the values of A_{eV} were obtained. The values of A_{eV} might have been slightly different had some variation of $C_{L\alpha}$ with A other than that of reference 5 been used. The data show some scatter at low values of b_V/D_F ; this scatter indicates that factors other than b_V/D_F enter into the determination of A_{eV}/A_V . The vertical-tail contribution to $C_{Y\psi}$ and $C_{N\psi}$ at $\alpha=0^\circ$ is shown in figure 19. Also shown in the figure are calculated curves of the parameters as determined by equations (4) and (5) and the use of average values of A_{eV} to determine $(C_{L\alpha})_V$. Ratios of A_{eV}/A_V of 1.25 and 1.45 were used for vertical tails having geometric aspect ratios of 1.0 and 2.0, respectively. The fact that reasonably good

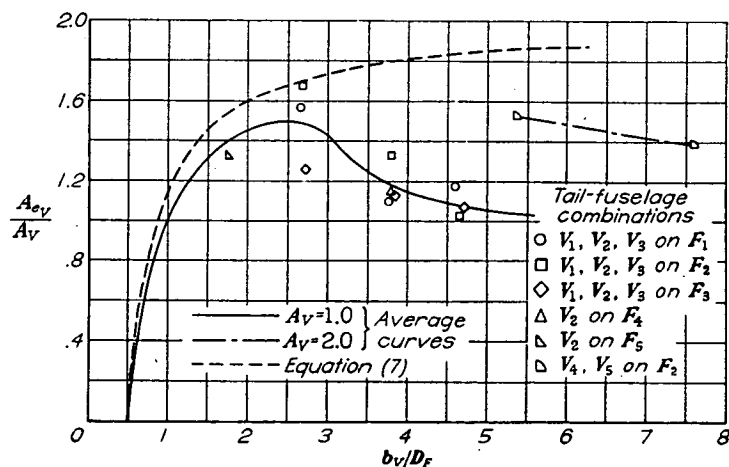


FIGURE 18.—Effective aspect ratio of vertical tails as influenced by the fuselage. $\alpha=0^\circ$.

agreement between the calculated curves and the experimental values of C_{n_ψ} was obtained is of only incidental interest, since the experimental results shown were originally used to determine appropriate values of the ratio A_{eV}/A_V . The scatter of the experimental points is indicative, however, of the accuracy that might be expected by use of average values of A_{eV}/A_V for arbitrary arrangements. The agreement between the calculated and experimental values of $(C_{Y_\psi})_V$ also is reasonably good. Therefore, the values of A_{eV}/A_V calculated from increments of $(C_{n_\psi})_V$ appear to be usable for predicting $(C_{Y_\psi})_V$ with reasonable accuracy at least for the arrangements investigated.

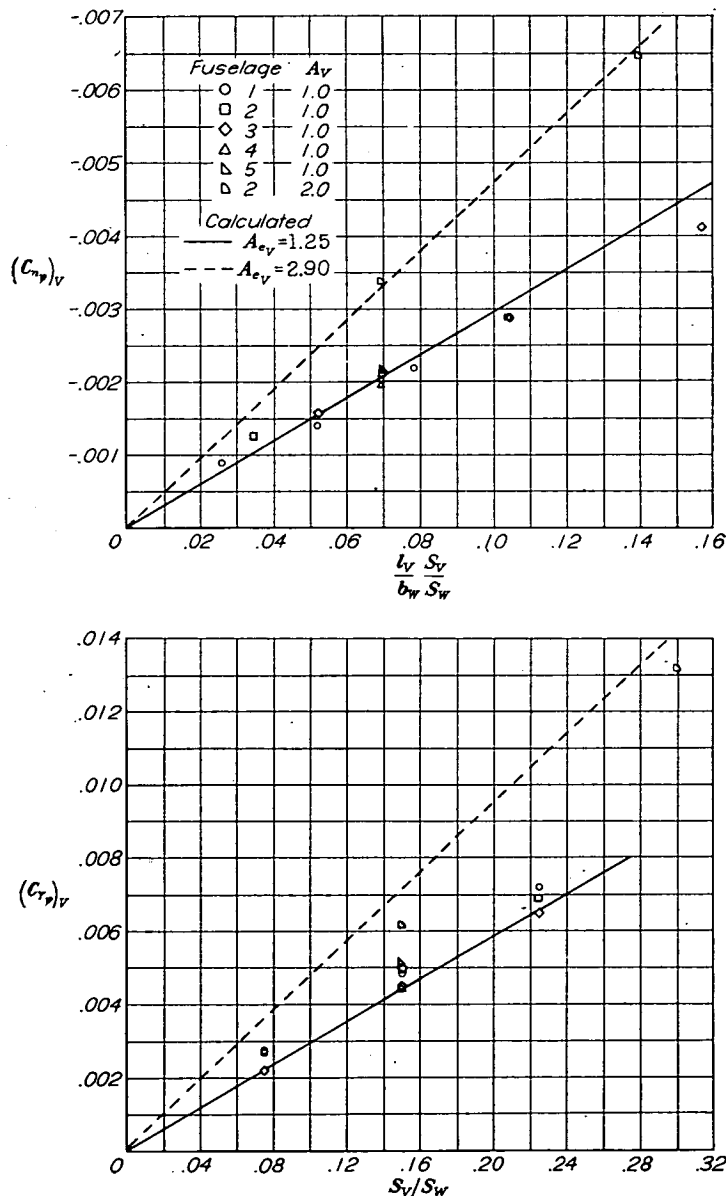


FIGURE 19.—Effect of tail area and tail length on vertical-tail contribution to C_{Y_ψ} and C_{n_ψ} , $\alpha = 0^\circ$.

The vertical-tail contribution to the derivative C_{l_ψ} can be separated into two parts as given by the two terms of the following equation:

$$(C_{l_\psi})_V = \frac{z_V}{b_W} \frac{S_V}{S_W} (C_{L_a})_V \cos \alpha - \frac{l_V}{b_W} \frac{S_V}{S_W} (C_{L_a})_V \sin \alpha$$

For small angles of attack the equation can be written as

$$(C_{L_a})_V = \frac{z_V}{b_W} \frac{S_V}{S_W} (C_{L_a})_V - \frac{l_V}{b_W} \frac{S_V}{S_W} (C_{L_a})_V \frac{\alpha}{57.3}$$

The first part of the equation is the increment of $(C_{l_\psi})_V$ at $\alpha = 0^\circ$, and the second part shows that the variation of $(C_{l_\psi})_V$ with α is given by

$$\frac{\partial (C_{l_\psi})_V}{\partial \alpha} = - \frac{l_V}{b_W} \frac{S_V}{S_W} \frac{(C_{L_a})_V}{57.3}$$

In analyzing the contribution of the vertical tail to C_{l_ψ} , consideration has been given to the increment of $(C_{l_\psi})_V$ at zero angle of attack and the rate of change of $(C_{l_\psi})_V$ with angle of attack. The experimental and calculated results for both of these effects are shown in figure 20 to be in fairly good agreement.

Angle-of-attack correction.—In the preceding section, the effective aspect ratio of the vertical tail mounted on the fuselage was determined at zero angle of attack. The effects of variations in angle of attack are now evaluated in terms of the correction factors to the vertical-tail contribution to C_{Y_ψ} and C_{n_ψ} , η_Y and η_N , respectively.

The variation of the factor η_Y with angle of attack is shown in figure 21 for three values of the ratio l_V/b_W . In each case an average curve is drawn through the data. The ratios l_V/b_W and S_V/S_W seem to cause no appreciable change in the variation of η_Y with α for values of α less than 6° .

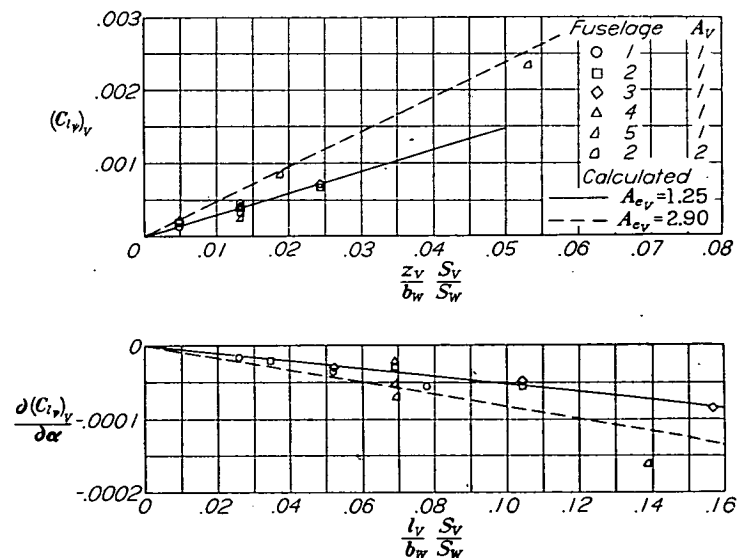


FIGURE 20.—Effect of tail area and length on the vertical-tail contribution to C_{l_ψ} , $\alpha = 0^\circ$.

At higher angles of attack, however, both l_v/b_w and S_v/S_w appear to affect the variation of η_Y with α , but not enough data were available to establish a definite relation between the various parameters. The effects of fuselage shape and vertical-tail aspect ratio on the variation of η_Y with α are shown in figure 22. Also given in the figure is the average curve from figure 21(b). It is seen that the curve fits the data reasonably well and that the variations in fuselage shape considered have very little effect on the variation of η_Y with α . Changes in vertical-tail aspect ratio appear to have some effect on the variation of η_Y with α ; nevertheless, the general trend shown by the average curve is still fairly accurate.

In general, it appears that the vertical-tail contribution to $C_{Y\psi}$ may be reduced as much as twenty percent as the angle of attack is increased from 0° to 15° and that this reduction usually increases rapidly at higher angles of attack.

The variation of the factor η_N with α is shown in figure 23 for several values of l_v/b_w and S_v/S_w . Average curves are drawn through each set of data. At low angles of attack the area ratio S_v/S_w appears to have a negligible effect on the variation of η_N with α ; however, it does have a large effect at angles of attack greater than about 8° and the effects increase with an increase of the l_v/b_w ratio. Fuselage shape and vertical-tail aspect ratio appear to have some effect on the variation of η_N with α (fig. 24), but the effects are not clearly defined by the data. In general, the average curve of figure 23 (b) fits the data of figure 24 reasonably well.

Except for the smallest vertical tail (V_1), the tail contributions to $C_{n\psi}$ tend to show a smaller decrease with angle of attack than had previously been noted for the tail contribution to $C_{Y\psi}$.

INTERFERENCE EFFECTS

Wing-fuselage interference.—The lateral-stability data of this investigation were used to determine wing-fuselage interference increments by the procedure explained under "Methods of Analysis." The increments are presented in figure 25 as functions of the angle of attack. Both $\Delta_1 C_{Y\psi}$ and $\Delta_1 C_{l\psi}$ show large variations with angle of attack and are of large magnitude at high angles of attack. The increment $\Delta_1 C_{n\psi}$ is rather small for all fuselage shapes investigated and tends to increase slightly the directional stability of the wing-fuselage combination over most of the angle-of-attack range.

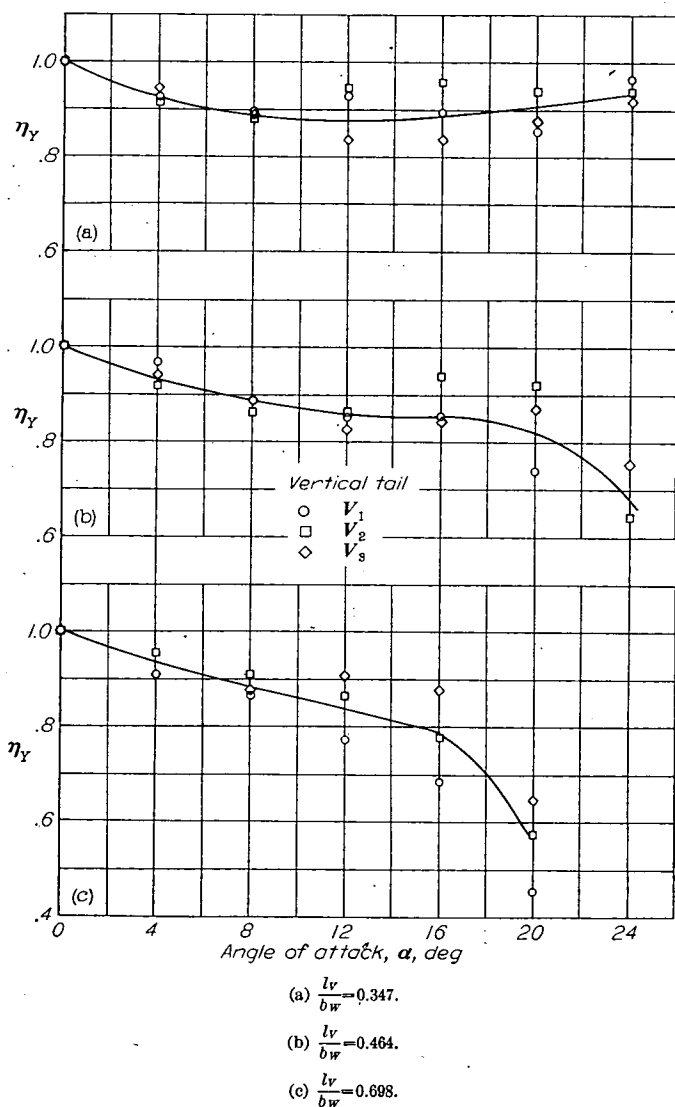


FIGURE 21.—Effect of tail area and length on the angle-of-attack correction to the vertical-tail contribution to $C_{Y\psi}$. Circular-arc fuselages.

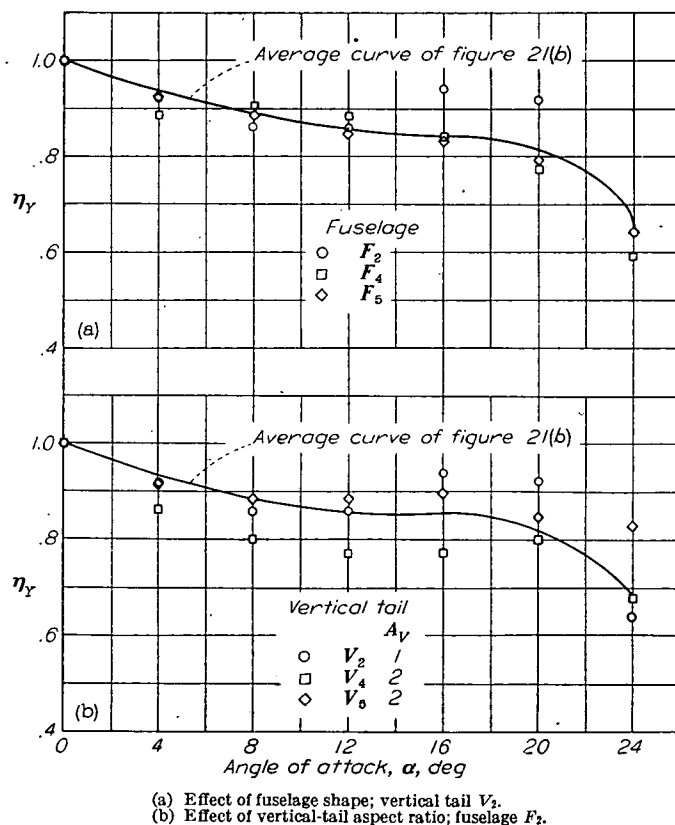


FIGURE 22.—Effect of fuselage shape and vertical-tail aspect ratio on the angle-of-attack correction to the vertical-tail contribution to $C_{Y\psi}$.

The average value of $\Delta_1 C_{n_\psi}$ is about -0.0002 up to 16° angle of attack.

Wing-fuselage interference on vertical-tail effectiveness.—Increments of $\Delta_2 C_{Y_\psi}$, $\Delta_2 C_{n_\psi}$, and $\Delta_2 C_{l_\psi}$ are shown in figures 26, 27, and 28, respectively, for various combinations of the circular-arc fuselages and the vertical tails of aspect ratio 1.0. The data are divided into groups of constant l_v/b_w ratio. An average curve was drawn through each set of data. In general, the data show little scatter about the faired curves. The addition of the wing almost invariably reduced the tail contribution to the directional stability for the arrangements investigated (fig. 27). The effect was negligible at very small angles of attack, but at 20° angle of attack a value of $\Delta_2 C_{n_\psi}$ of about 0.0020 was obtained with the largest fuselage (F_3 .) The large interference effects noted at high angles of attack probably result from the partially stalled condition of the wing at these attitudes. If stalling could be avoided, the

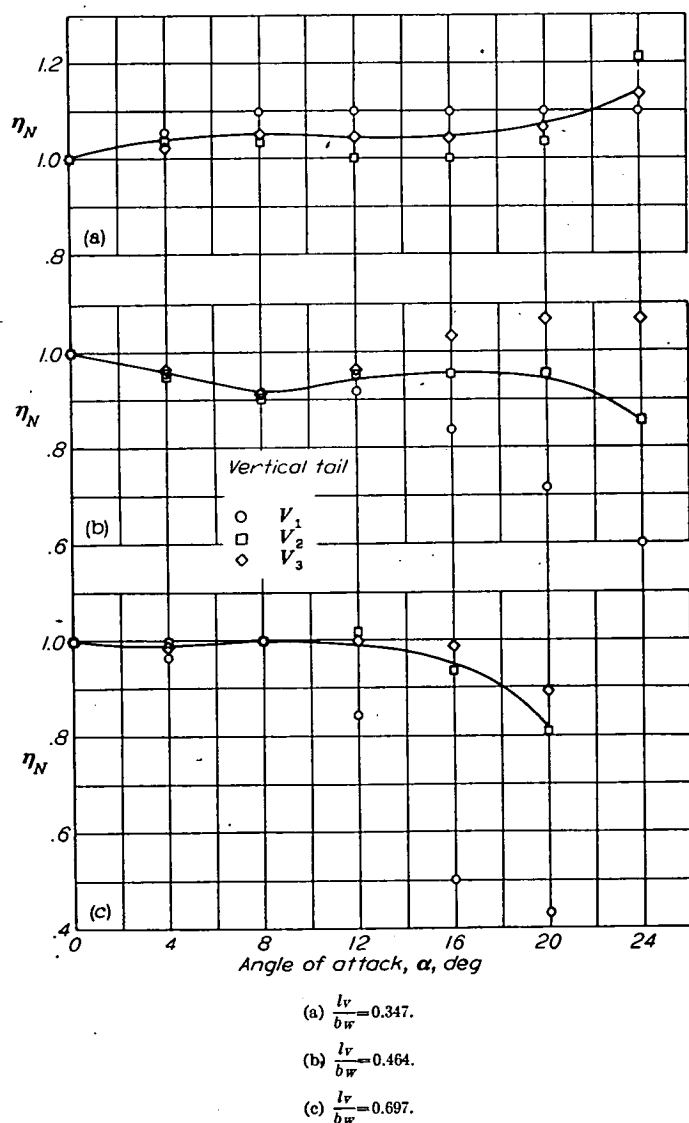


FIGURE 23.—Effect of tail area and length on the angle-of-attack correction to the vertical-tail contribution to C_{n_ψ} . Circular-arc fuselages.

interference effects undoubtedly would be considerably smaller.

The effects of fuselage shape on the increments of C_{Y_ψ} , C_{n_ψ} , and C_{l_ψ} caused by wing-fuselage interference on the vertical-tail effectiveness are indicated in figure 29. Also given in the figure are the average curves of the $\frac{l_v}{b_w} = 0.464$ data of figures 26 (b), 27 (b), and 28 (b). The figure indicates that variations in fuselage shapes considered have little effect on the interference increments and that the average curves fit the data quite well.

A comparison is given in figure 30 between the interference increments $\Delta_1 C_{n_\psi}$ and $\Delta_2 C_{n_\psi}$ for a model configuration with and without the wing slat. The model configuration was made up of the wing, fuselage F_2 , and vertical tail V_2 . The increment $\Delta_1 C_{n_\psi}$ for both configurations varied erratically with angle of attack and indicated no definite trends. The increment $\Delta_2 C_{n_\psi}$ for the model with the slat was larger (more positive) than for the wing without the slat up to about 20° , after which the opposite was true.

It should be pointed out again that the interference increments presented herein can be expected to apply fairly accurately only to midwing or near-midwing configurations since the height of the wing relative to the fuselage center line has been found to be an important factor in determining interference increments (reference 2).

CONCLUSIONS

The results of an investigation to determine the effects of vertical-tail size and length and of fuselage shape and length

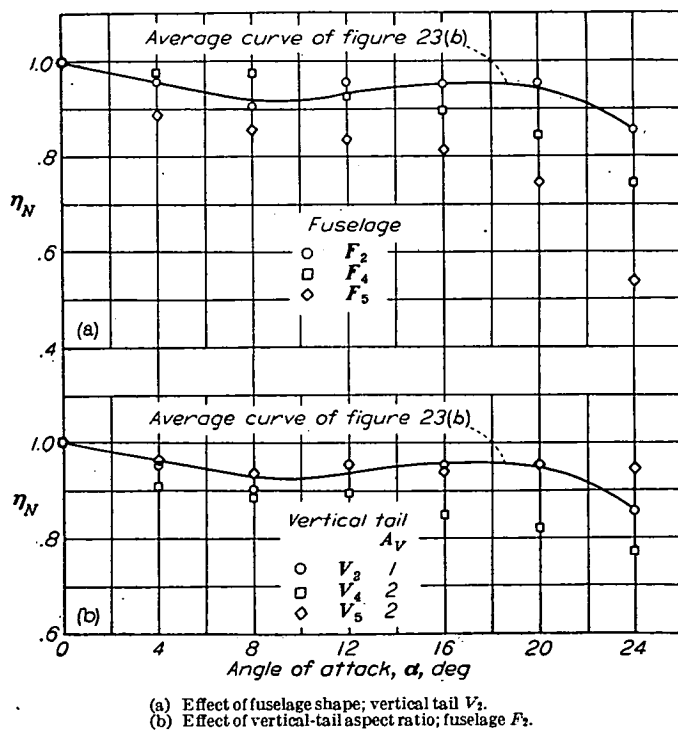


FIGURE 24.—Effect of fuselage shape and vertical-tail aspect ratio on the angle-of-attack correction to the vertical-tail contribution to C_{n_ψ} .

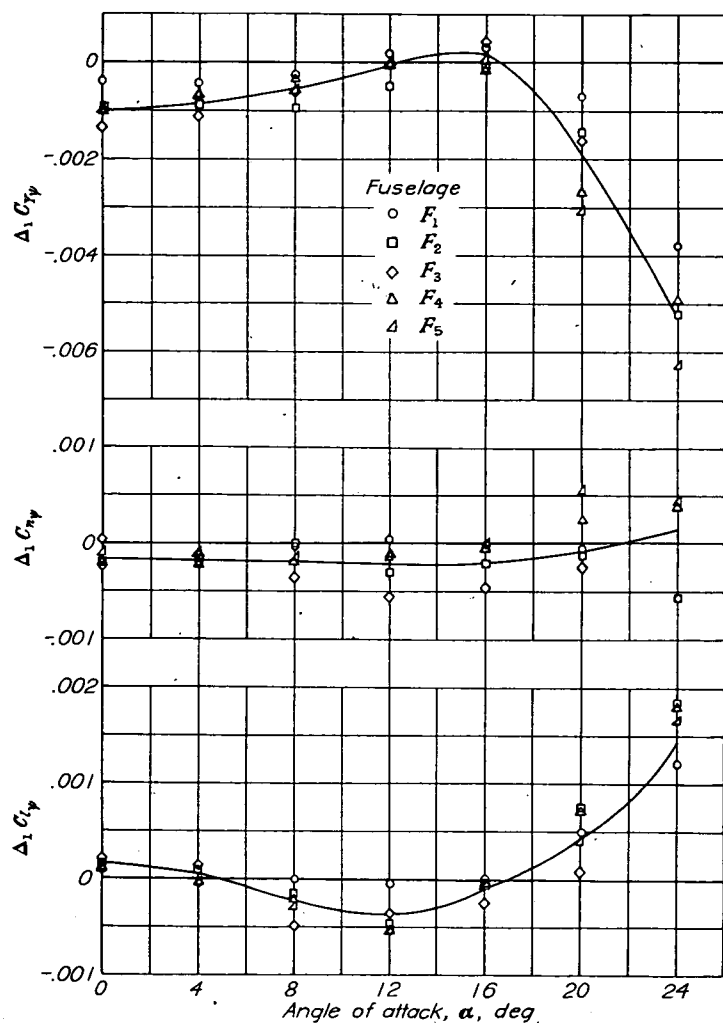


FIGURE 25.—Variation of increments of $C_{Y\psi}$, $C_{n\psi}$, and $C_{l\psi}$ caused by wing-fuselage interference with angle of attack.

on the lateral static stability characteristics of a model with a 45° sweptback wing indicate the following conclusions:

1. The directional instability of the various isolated fuselages was about two-thirds as large as that predicted by classical theory.

2. A reduction in area (geometric aspect ratio kept constant) of vertical tails attached to a given fuselage resulted in an increase in the effective aspect ratio of the vertical tails for the range of tail sizes considered. Simple analytical considerations indicate, however, that for tail sizes below the range investigated the opposite effect would be expected.

3. For the fuselage-tail combinations investigated, the tail effectiveness usually decreased with increasing angle of

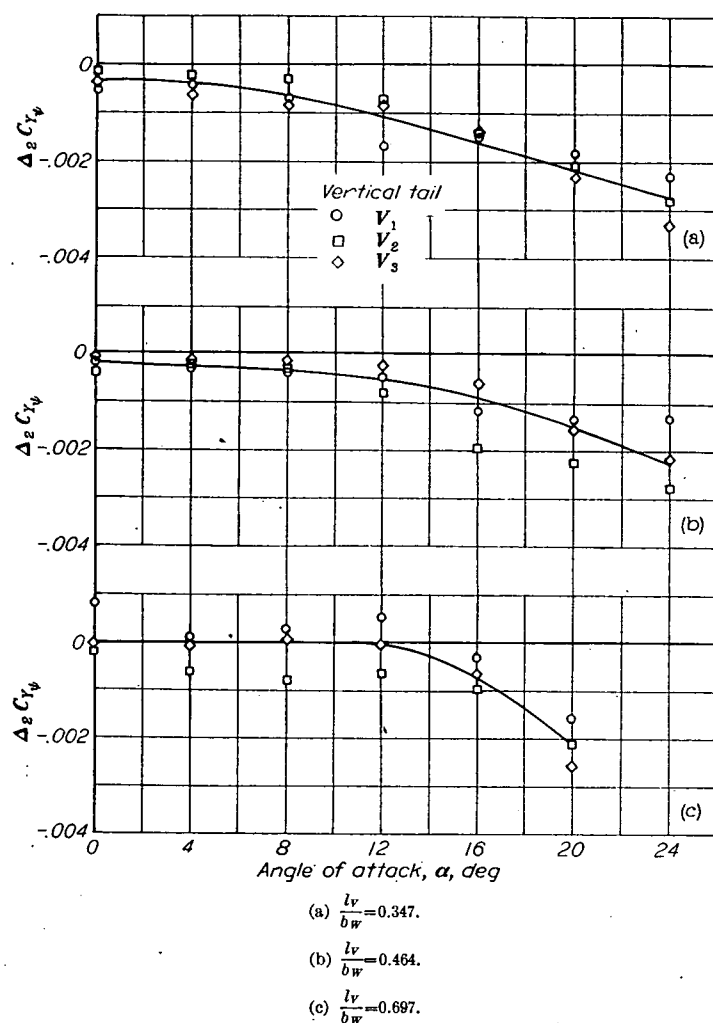


FIGURE 26.—Effect of the tail area and length on the increment of $C_{Y\psi}$ caused by wing-fuselage interference on vertical-tail effectiveness. Circular-arc fuselages.

attack, with the greatest rate of decrease occurring at angles of attack greater than about 16° .

4. The wing-fuselage interference for the midwing arrangements investigated was only slightly affected by the shape of the fuselage and the interference tended to increase slightly the directional stability of the combinations.

5. The interference effects of the wing tended to decrease the vertical-tail effectiveness, particularly at high angles of attack. The large effects observed were attributed to a partially stalled condition of the wing.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., June 5, 1950.

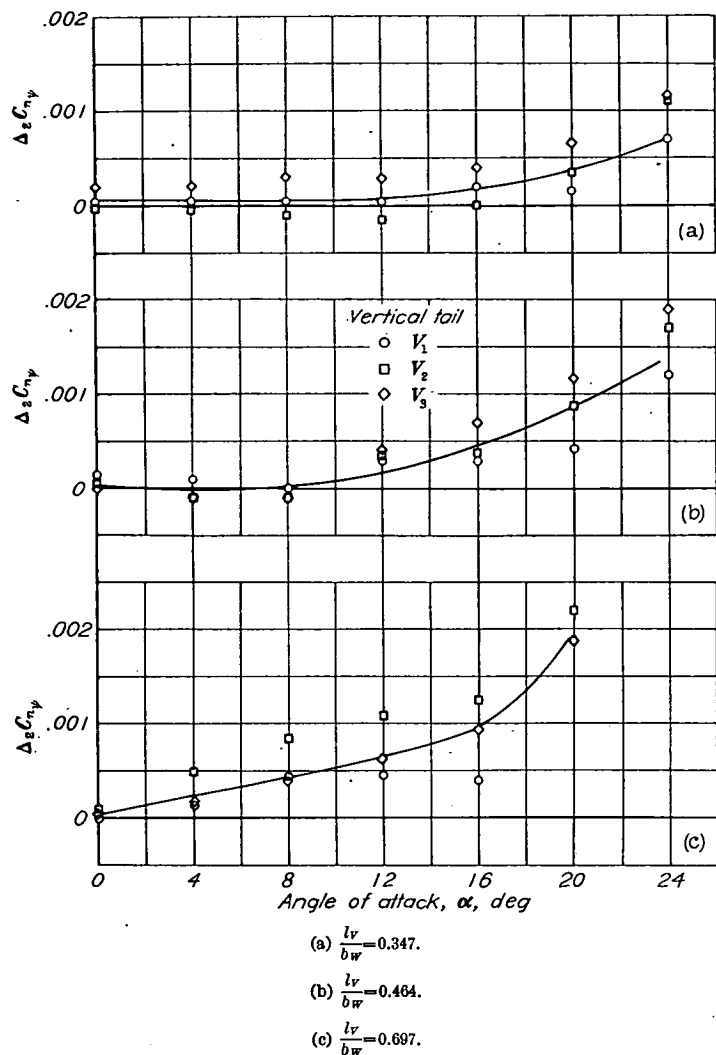


FIGURE 27.—Effect of tail area and length on the increment of $C_{n\psi}$ caused by the wing-fuselage interference on the vertical-tail effectiveness. Circular-arc fuselages.

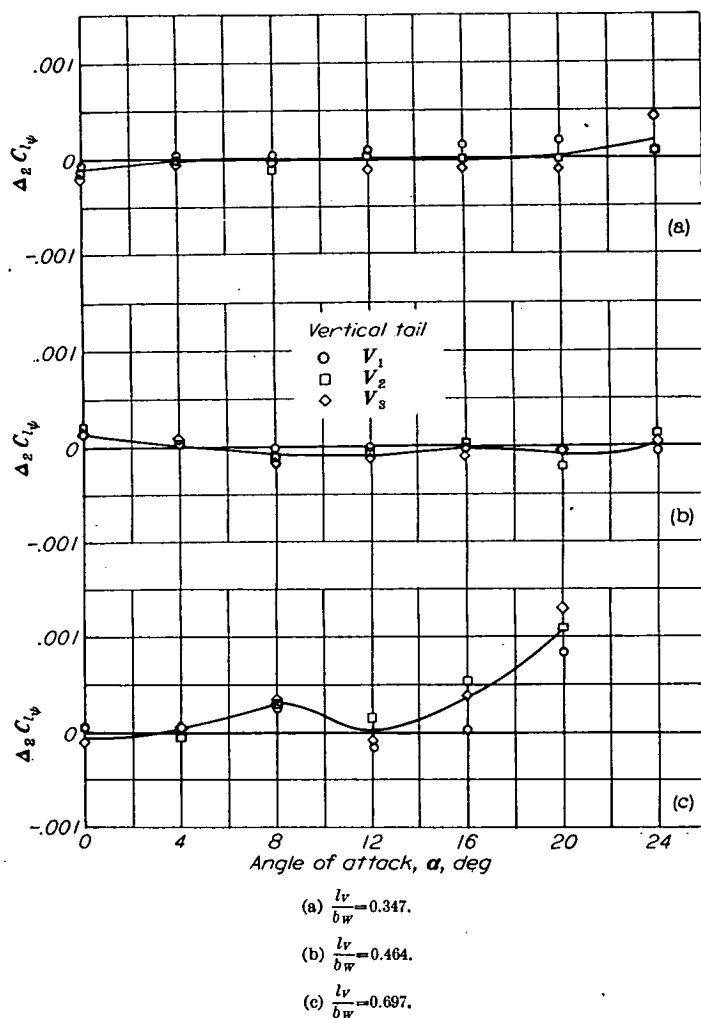


FIGURE 28.—Effect of tail area and length on the increment of $C_{l\psi}$ caused by the wing-fuselage interference on vertical-tail effectiveness. Circular-arc fuselages.

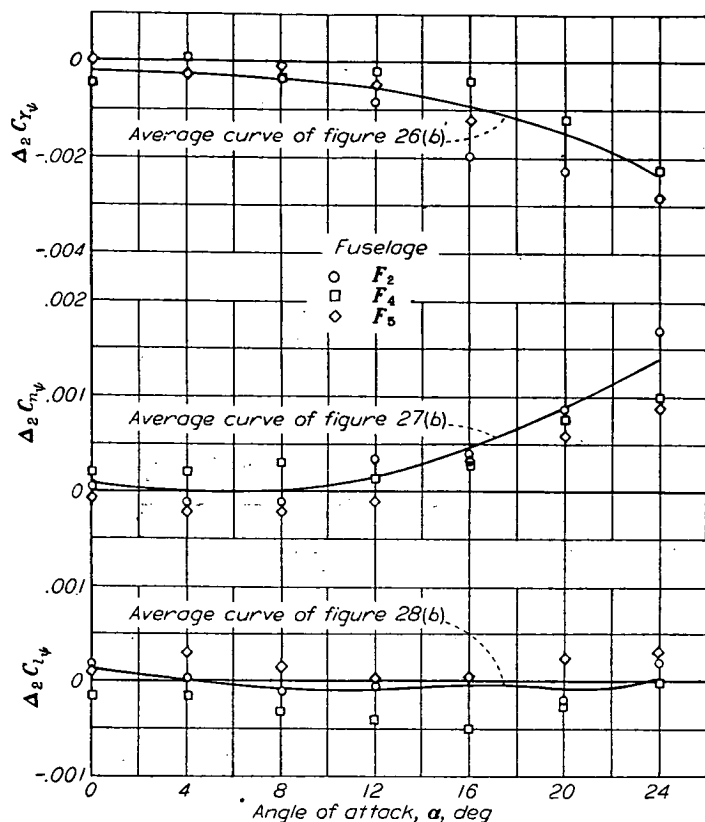


FIGURE 29.—Effect of fuselage shape on the increments of $C_{Y_{\psi}}$, $C_{n_{\psi}}$, and $C_{l_{\psi}}$ caused by the wing-fuselage interference on the vertical-tail effectiveness. Vertical tail V_2 .

REFERENCES

1. Jacobs, Eastman N., and Ward, Kenneth E.: Interference of Wing and Fuselage from Tests of 209 Combinations in the N.A.C.A. Variable-Density Tunnel. NACA Rep. 540, 1935.
2. House, Rufus O., and Wallace, Arthur R.: Wind-Tunnel Investigation of Effect of Interference on Lateral-Stability Characteristics of Four NACA 23012 Wings, an Elliptical and a Circular Fuselage, and Vertical Fins. NACA Rep. 705, 1941.
3. Brewer, Jack D., and Lichtenstein, Jacob H.: Effect of Horizontal

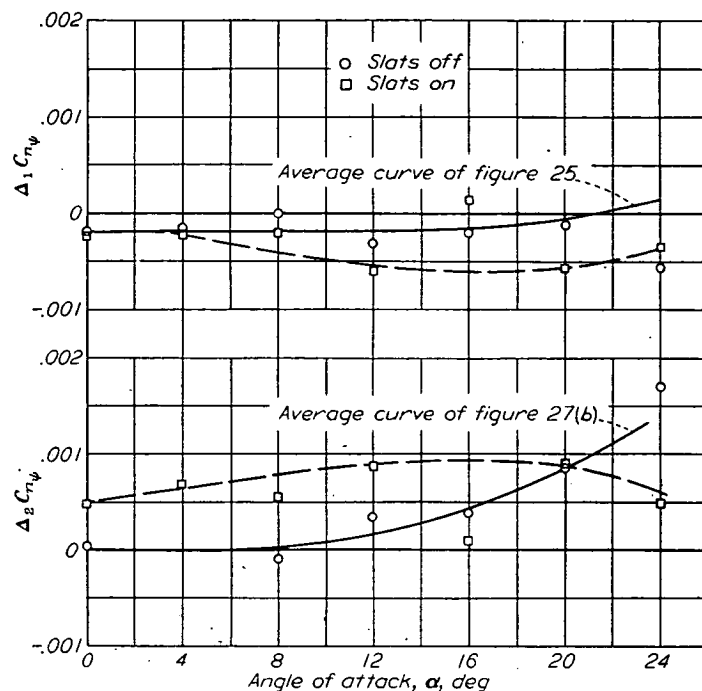


FIGURE 30.—Effect of leading-edge slat on interference increments. Fuselage F_2 ; vertical tail V_2 .

Tail on Low-Speed Static Lateral Stability Characteristics of a Model Having 45° Sweptback Wing and Tail Surfaces. NACA TN 2010, 1950.

4. Bamber, Millard J.: Effect of Some Present-Day Airplane Design Trends on Requirements for Lateral Stability. NACA TN 814, 1941.
5. DeYoung, John: Theoretical Additional Span Loading Characteristics of Wings with Arbitrary Sweep, Aspect Ratio, and Taper Ratio. NACA TN 1491, 1947.
6. Goodman, Alex, and Fisher, Lewis R.: Investigation at Low Speeds of the Effect of Aspect Ratio and Sweep on Rolling Stability Derivatives of Untapered Wings. NACA Rep. 968, 1950. (Supersedes NACA TN 1835.)
7. Munk, Max M.: The Aerodynamic Forces on Airship Hulls. NACA Rep. 184, 1924.

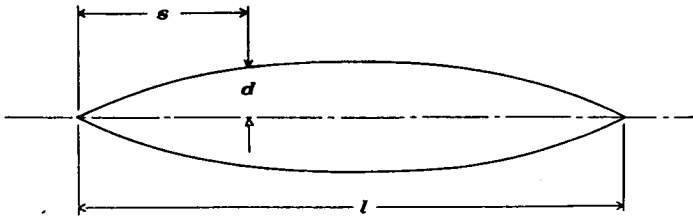
TABLE I.—COORDINATES FOR NACA 65A008 AIRFOIL

[Station and ordinates in percent airfoil chord]

Station	Ordinate
0	0
.50	.62
.75	.75
1.25	.95
2.50	1.30
5.0	1.75
7.5	2.12
10.0	2.43
15	2.93
20	3.30
25	3.59
30	3.79
35	3.93
40	4.00
45	3.99
50	3.90
55	3.71
60	3.46
65	3.14
70	2.76
75	2.35
80	1.90
85	1.43
90	.96
95	.49
100	.02

L. E. radius: 0.408

TABLE II.—FUSELAGE COORDINATES



s/l	d/l				
	Fuselage 1	Fuselage 2	Fuselage 3	Fuselage 4	Fuselage 5
0	0	0	0	0	0
.025	.010	.007	.005	.033	.007
.050	.020	.014	.010	.045	.014
.075	.029	.021	.014	.054	.021
.100	.037	.027	.018	.060	.027
.125	.045	.033	.022	.065	.033
.150	.052	.039	.026	.069	.039
.200	.065	.048	.032	.074	.048
.250	.076	.057	.038	.075	.057
.30	.085	.063	.042	.075	.063
.35	.091	.068	.046	.075	.068
.40	.096	.072	.048	.075	.072
.45	.099	.074	.049	.075	.074
.50	.100	.075	.050	.075	.075
.55	.099	.074	.049	.074	.075
.60	.096	.072	.048	.072	.073
.65	.091	.068	.046	.068	.072
.70	.085	.063	.042	.063	.069
.75	.076	.057	.038	.057	.066
.80	.065	.048	.032	.048	.062
.85	.052	.039	.026	.039	.057
.90	.037	.027	.018	.027	.051
.95	.020	.014	.010	.014	.045
1.00	0	0	0	0	.038

TABLE III.—PERTINENT GEOMETRIC CHARACTERISTICS OF MODEL

Wing:

Aspect ratio, A_W	4.0
Taper ratio, λ_W	0.6
Quarter-chord sweep angle, Λ_W , deg	45
Dihedral angle, deg	0
Twist, deg	0
NACA airfoil section	65A008
Area, S_W , sq ft	2.25
Span, b_W , ft	3.00
Mean aerodynamic chord, $\bar{c}_W$, ft	0.765

Fuselage:

	F_1	F_2	F_3	F_4	F_5
Length, ft	2.50	3.34	5.00	3.34	3.34
Fineness ratio	5.00	6.67	10.0	6.67	6.67
Volume, V_F , cu ft	0.267	0.350	0.526	0.448	0.385
Tail length, l_V , ft (all tails)	1.04	1.39	2.09	1.39	1.39
Tail-length ratio, l_V/b_W , (all tails)	0.347	0.464	0.697	0.464	0.464

Side area, S_s , sq ft

	0.833	1.11	1.67	1.30	1.25
--	-------	------	------	------	------

Vertical tail:

	V_1	V_2	V_3	V_4	V_5
Aspect ratio	1.0	1.0	1.0	2.0	2.0
Taper ratio	0.6	0.6	0.6	0.6	0.6

Quarter-chord sweep angle,

Λ_V , deg	45	45	45	45	45
-------------------	----	----	----	----	----

NACA airfoil section

	65A008	65A008	65A008	65A008	65A008
--	--------	--------	--------	--------	--------

Area, S_V , sq ft

	0.169	0.338	0.506	0.338	0.675
--	-------	-------	-------	-------	-------

Span, b_V , ft

	0.408	0.583	0.710	0.825	1.159
--	-------	-------	-------	-------	-------

Mean aerodynamic chord,

$\bar{c}_V$, ft	0.417	0.592	0.725	0.416	0.592
------------------	-------	-------	-------	-------	-------

Area ratio, S_V/S_W

	0.075	0.150	0.225	0.150	0.300
--	-------	-------	-------	-------	-------

TABLE IV.—CONFIGURATIONS INVESTIGATED

Wing off		Wing on	
Configura- tion (a)	Figure	Configura- tion (a)	Figure
-----	-----	W	6, 7
F_1 F_1+V_1 F_1+V_2 F_1+V_3	8(a)	$W+F_1$ $W+F_1+V_1$ $W+F_1+V_2$ $W+F_1+V_3$	9(a)
F_2 F_2+V_1 F_2+V_2 F_2+V_3	8(b), 12 8(b) 8(b) 8(b)	$W+F_2$ $W+F_2+V_1$ $W+F_2+V_2$ $W+F_2+V_3$	9(b), 13 9(b) 9(b) 9(b)
F_3 F_3+V_1 F_3+V_2 F_3+V_3	8(c)	$W+F_3$ $W+F_3+V_1$ $W+F_3+V_2$ $W+F_3+V_3$	9(c)
F_4 F_4+V_2	10(a)	$W+F_4$ $W+F_4+V_2$	10(b)
F_5 F_5+V_2	11(a)	$W+F_5$ $W+F_5+V_2$	11(b)
F_2+V_4 F_2+V_5	12	$W+F_2+V_4$ $W+F_2+V_5$	13
-----	-----	W_S	6, 7, 14
-----	-----	W_S+F_2	14
-----	-----	$W_S+F_2+V_2$	14

*Notation (for details, see table III and figs. 2 to 4):

W wing; with subscript S, wing with slat

F fuselage

V vertical tail

REPORT 1050

CONTENTS

	Page		Page
SUMMARY.....	1147	III—LIFT.....	
INTRODUCTION.....	1147	GENERAL PROCEDURE FOR CONICAL FLOWS.....	1172
I—METHOD OF THE SUPERPOSITION OF CONICAL FLOWS.....	1148	General Formula for the Lift Induced by a Single Tip Element.....	1172
SYSTEM OF NOTATION FOR CONICAL FLOWS.....	1148	General Formula for the Lift Induced by an Oblique Trailing-Edge Element.....	1173
BOUNDARY CONDITIONS FOR CANCELLATION OF LIFT.....	1149	WING WITH SUBSONIC LEADING EDGE.....	1173
CANCELLATION OF NONUNIFORM LIFT.....	1149	Uncorrected Lift.....	1173
II—LOADING ON WING WITH SUBSONIC LEADING EDGE.....	1150	Wing With Supersonic Trailing Edge (Tip Correction).....	1174
LOAD DISTRIBUTION OVER TRIANGULAR WING.....	1150	Wing With Subsonic Trailing Edge.....	1174
SWEEP-BACK WING WITH SUPERSONIC TRAILING EDGE (TIP CORRECTION).....	1150	Tip correction with subsonic trailing edge.....	1174
Elementary Solution for a Streamwise Tip.....	1150	Trailing-edge corrections.....	1175
Tip-Induced Correction to the Loading.....	1151	WING WITH INTERACTING LEADING AND TRAILING EDGES.....	1176
Value at the side edge.....	1151	Lift on Inboard Portion of Wing.....	1176
Drop in lift across tip Mach line.....	1151	Lift on Outer Portions of Wing.....	1177
SWEEP-BACK WING WITH SUBSONIC TRAILING EDGE.....	1153	Tip-Induced Correction to the Lift.....	1177
Primary Trailing-Edge Corrections.....	1153	APPLICATION OF LIFT FORMULAS.....	1179
Procedure for canceling lift in the wake region.....	1153	Cases Computed.....	1179
Symmetrical solution.....	1154	Summary of Computations.....	1180
Oblique solutions for the wake region.....	1154	Discussion of Results.....	1180
Correction of loading near the trailing edge.....	1155	IV—DRAG DUE TO LIFT.....	1180
Secondary Corrections.....	1155	V—SUMMARY OF FORMULAS.....	1181
Secondary corrections at the trailing edge.....	1155	APPENDIX A: SYMBOLS.....	1183
Secondary correction at the tip.....	1156	APPENDIX B: EVALUATION OF THE INTEGRAL IN EQUATION (26).....	1184
Numerical Example.....	1156	APPENDIX C: INTEGRATION FOR LOSS OF LIFT AT THE TIP OF WING WITH SUBSONIC LEADING EDGE.....	1185
SWEEP-BACK WINGS WITH INTERACTING TRAILING AND LEADING EDGES.....	1158	REFERENCES.....	1186
Leading-Edge Corrections.....	1158		
Elementary solution for the region ahead of the leading edge.....	1158		
Leading-edge correction to the loading.....	1159		
Further Corrections.....	1161		
Numerical Results (Without Tip Effect).....	1162		
Application of Two-Dimensional Formulas to Calculation of Load Distribution.....	1163		
Correlation of two-dimensional and swept-back-wing loadings.....	1163		
Numerical results.....	1165		
Discussion of the σ function.....	1167		
Calculation of tip effect.....	1169		
Numerical examples, tip effect.....	1171		

REPORT 1050

FORMULAS FOR THE SUPERSONIC LOADING, LIFT, AND DRAG OF FLAT SWEEPED-BACK WINGS WITH LEADING EDGES BEHIND THE MACH LINES

By DORIS COHEN

SUMMARY

The method of superposition of linearized conical flows has been applied to the calculation of the aerodynamic properties, in supersonic flight, of thin flat, swept-back wings at an angle of attack. The wings are assumed to have rectilinear plan forms, with tips parallel to the stream, and to taper in the conventional sense. The investigation covers the moderately supersonic speed range where the Mach lines from the leading-edge apex lie ahead of the wing. The trailing edge may lie ahead of or behind the Mach lines from its apex. The case in which the Mach cone from one tip intersects the other tip is not treated.

Formulas are obtained for the load distribution, the total lift, and the drag due to lift. For the cases in which the trailing edge is outside the Mach cone from its apex (supersonic trailing edge), the formulas are complete. For the wing with both leading and trailing edges behind their respective Mach lines, a degree of approximation is necessary. It has been found possible to give practical formulas which permit the total lift and drag to be calculated to within 2 or 3 percent of the accurate linearized-theory value. The local lift can be determined accurately over most of the wing; but the trailing-edge-tip region is treated only approximately.

Charts of some of the functions derived are included to facilitate computing, and several examples are worked out in outline.

INTRODUCTION

It is customary, in supersonic wing theory, to describe any straight segment of the boundary of a wing plan form as supersonic or subsonic accordingly as the segment lies outside or is contained within its foremost Mach cone; that is, as the component of the flight velocity normal to the edge is greater than or less than the speed of sound. These two circumstances result in fundamentally different types of flow over the surface. It is apparent that the real reference is not to a property of the wing plan form, but to a combination of plan-form geometry and the velocity of the wing relative to the speed of sound. Thus (see fig. 1) every swept-back wing, on entering the supersonic regime, has subsonic leading and, in most cases, subsonic trailing edges. At a higher Mach number, the same plan form may have subsonic leading edges and supersonic trailing edges. Finally, if the Mach number is increased sufficiently, both leading and trailing edges will become supersonic.

Interference effects also depend on the flight Mach number, since the extent of the various disturbance fields is determined by the angle between the Mach lines. Thus, no single concise formula or method of treatment has as yet been developed to predict, even approximately, the aerodynamic characteristics of an arbitrary wing plan form through the supersonic speed range.

The present report is concerned with the loading, lift, and drag, according to linearized theory, of thin, flat, swept-back wings with rectilinear boundaries and conventional taper. Various methods are available for the calculation of these properties when the leading edge is supersonic. Of these, the method of reference 1 is perhaps the most convenient. Formulas obtained by this method for the loading and lift-curve slope of wings with supersonic leading and trailing edges are presented in reference 2. In the following, therefore, the emphasis will be on the solution of the problems arising from the interaction of the flow fields in the presence of subsonic leading edges (figs. 1 (b), (c),

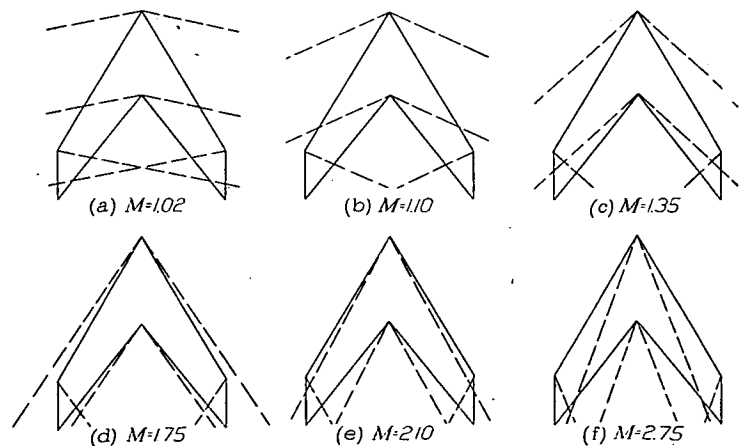


FIGURE 1.—A typical tapered swept-back wing at six supersonic Mach numbers, showing the Mach lines from the leading- and trailing-edge apices and from the tips.

and (d)). The case (fig. 1 (a)) in which the Mach number and aspect ratio are so low that interaction takes place between the tip flow fields will not be treated. An approximate solution to this problem may be found in reference 3.

When a wing with a subsonic leading edge is to be studied, considerable simplification of the problem may be achieved by making use of the solutions, available in reference 4

is the ratio of the slope of the ray t_a of that field to the slope of the Mach lines.

If the ratio of the slope of the leading edge to the slope of the Mach lines is

$$m = \beta \cot \Lambda \tag{3}$$

where Λ is the angle of sweepback, then at the leading edge $a=m$, and a ray from the leading-edge tip is designated by t_m . If s is the wing semispan, the leading-edge tip has the coordinates $\frac{\beta s}{m}$, s and any point x, y has the conical coordinate

$$t_m = \beta \frac{y-s}{x-\frac{\beta s}{m}} \tag{4}$$

in the field with apex at $\frac{\beta s}{m}$, s .

Other symbols referring to angular locations will be defined in the same way as needed. A summary of the symbols will be found in appendix A.

BOUNDARY CONDITIONS FOR CANCELLATION OF LIFT

The general problem of deriving the flow over a wing of finite dimensions from the known flow over an infinite wing is the problem of determining the induction effects due to the edges. These effects may be thought of as associated with the cancellation of the lifting pressure at the boundaries of the finite wing. In the linearized lifting-surface theory, they may be evaluated by the superposition of flow fields with negative lifting pressure over the portion of the infinite wing outside the boundaries of the finite plan form, provided the other boundary conditions are not disturbed. In the case of a flat wing at an angle of attack, the latter provision means that the canceling field must (1) induce no downwash within the boundaries of the finite wing and (2) introduce no new lifting pressure outside those boundaries.

In accordance with thin-airfoil theory, the boundary conditions will be satisfied in the horizontal plane rather than on the surface of the wing. Also, by thin-airfoil theory, the conditions on the lifting pressure are converted to conditions on the velocity field through the relation

$$\frac{\Delta p}{q} = 4 \left(\frac{u}{V} \right)_{z=+0} \tag{5}$$

In the simplest case, the lift to be canceled will be distributed uniformly over a semi-infinite region bounded by two straight lines. The boundary conditions of the problem may then be said to be conical with respect to the intersection of the two lines, which become "rays" of the canceling conical field. The boundary conditions on the canceling velocity field in this case may be summarized as follows:

- (1) The streamwise velocity u must approach values equal in magnitude and opposite in direction on the upper and lower surfaces of the horizontal plane.
- (2) In the horizontal plane, u must be constant over the infinite sector in which lift is to be canceled.
- (3) The vertical velocity w must be zero in the portion of the $z=0$ plane occupied by the projection of the finite wing.

(4) From equation (5), u must equal zero in the portion of the horizontal plane not covered by conditions (2) or (3).

(5) In supersonic flow there exists the additional condition that all the velocities must go to zero on the Mach cone from the apex of the field.

CANCELLATION OF NONUNIFORM LIFT

The foregoing are the general conditions for a uniformly loaded canceling flow field. Under the proper conditions, a nonuniform distribution of lift may be canceled by the superposition of a number of such fields. This procedure is best explained by a concrete example.

Consider the problem of a swept-back wing flying at a high Mach number such that, as in figure 1 (c), the Mach lines from the leading-edge apex intersect the tips of the wing. The method of deriving the swept-back wing from an infinite triangular wing in that case is indicated in figure 3. It may be noted at the start that, according to linear theory, the lift behind the supersonic trailing edge may be canceled in any way without affecting the velocities on the wing. Thus it remains only to consider the effect of canceling the lift outboard of the tips.

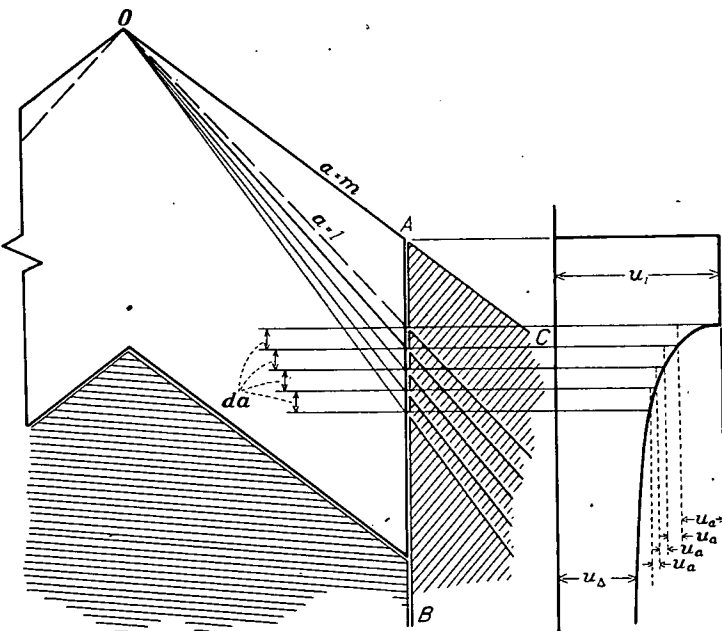


FIGURE 3.—Method of cancellation of lift beyond the tip when the leading-edge Mach line intersects the side edge of the wing.

An infinite triangular wing with supersonic leading edges has a load distribution which is constant over the portions of the wing between the leading edge and the Mach lines from the leading-edge apex (see fig. 4). This constant load may be canceled outboard of each of the tips of the swept-back wing by a single negatively loaded triangle of infinite extent, one side coinciding with the side edge of the wing and a second side coinciding with the extension of the leading edge. However, the area to be removed (region BAC, fig. 3) includes also a region over which the pressure varies, and is conical with respect to A, no one conical solution can satisfy the requirements of the problem. The problem is brought within the limitations of the conical solutions by

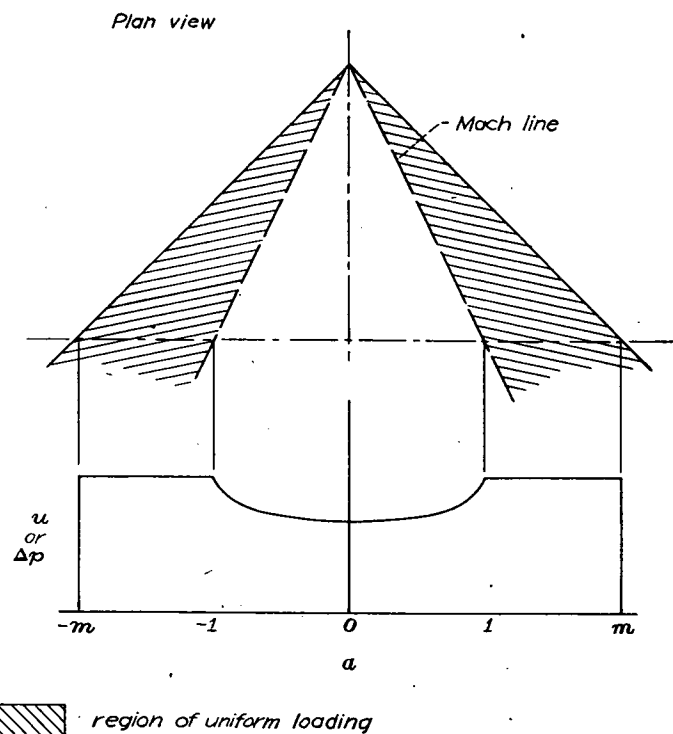


FIGURE 4.—Lift distribution on a triangular wing with supersonic leading edges.

considering the lift to be made up of an infinite number of constantly loaded, overlapping sectors of infinite extent. (See fig. 3.) These sectors are bounded on one side by the wing tip; the second side is the extension of a ray from apex 0 of the wing. Between the leading edge ($a=m$) and the leading-edge Mach line ($a=1$), no division of the field is necessary since the lift density is constant in that region.

If a sector with apex at A and angle $\tan^{-1} \frac{m}{\beta}$ is used to cancel this uniform lift, then the remaining superposed fields must be used where $a < 1$ (see fig. 4) to restore the difference between that lift and the loading on the triangular wing.

If u_1 is the streamwise component of the perturbation velocity corresponding to the constant loading ahead of the leading-edge Mach lines, and $u_\Delta(a)$ is the same velocity in the region between the Mach lines, then the magnitude of the u component of the velocity in the initial canceling sector will be $-u_1$, and on the remaining sectors (see fig. 3) minus the increment in $u_1 - u_\Delta$ corresponding to an increment in a , or $\frac{du_\Delta}{da} da$. (Note that this last quantity is positive, as required). To determine the total effect of canceling the loading outboard of the tip, the velocities induced by the latter infinitesimally loaded elements are integrated and added to the negative effect of the initial constant-load sector.

II—LOADING ON WING WITH SUBSONIC LEADING EDGE

LOAD DISTRIBUTION OVER TRIANGULAR WING

In the notation of this paper, the velocity distribution over a flat lifting triangle with leading edge behind the Mach lines may be written

$$u_\Delta = \frac{m u_0}{\sqrt{m^2 - a^2}} \quad (6)$$

where

$$u_0 = \frac{m V \alpha}{\beta E'(m)} \quad (7)$$

is the (constant) velocity along the center line $a=0$. In the expression for u_0 , $E'(m)$ is the complete elliptic integral of the second kind, of modulus $\sqrt{1-m^2}$. The load distribution is obtained from the velocity distribution by equation (5).

SWEPT-BACK WING WITH SUPERSONIC TRAILING EDGE (TIP CORRECTION)²

If the problem is now to find the loading on a swept-back wing with subsonic leading edges, but supersonic trailing edges, only the tip effects will modify the triangular-wing distribution. The calculation of the tip effect on a wing with subsonic leading edge ($m < 1$) is somewhat complicated by the fact that the pressure becomes infinite at the leading edge, but otherwise follows the procedure outlined in the preceding section.

It will first be necessary to present the expression for the previously described conical field with uniformly loaded sector to be used as the element in canceling the lift outboard of the tip.

ELEMENTARY SOLUTION FOR A STREAMWISE TIP

If s is the semispan of the wing, the apex of any element a (see the section on Notation) is at

$$x_a = \frac{\beta s}{a}, \quad y_a = s \quad (8)$$

and, from equation (2),

$$t_a = \beta \frac{y-s}{x - \frac{\beta s}{a}} \quad (9)$$

Then, if u_a is the constant perturbation-velocity component to be canceled over the region between the tip and the extension of the ray a , the previously listed boundary conditions for each of the required canceling fields may be written as follows (see fig. 5):

- (1) and (2) When $0 \leq t_a \leq a$, $u = \pm u_a$ (constant for the field)
- (3) When $t_a < 0$, $w = 0$
- (4) When $t_a > a$, $u = 0$
- (5) When $|t_a| \geq 1$, $u = v = w = 0$.

The solution of the supersonic flow equation satisfying the above boundary conditions has been derived in reference 4.³ In the xy plane, the streamwise component of the velocity is

$$u = \pm r.p. \frac{u_a}{\pi} \cos^{-1} \frac{a + t_a + 2at_a}{t_a - a} \quad (10)$$

The signs refer to the upper and lower surfaces, respectively.

In figure 5, the essential features of the solution are shown. At the top is a detail view of the wing side edge and shows the boundary conditions. In the center is a typical plot of the argument of the inverse cosine in equation (10), against t_a . Where this quantity is less than -1 (i. e., $0 \leq t_a \leq a$), the real part of the inverse cosine is π . Where the argument is greater than $+1$ ($t_a > a$ and $t_a < -1$), the

² Approximate formulas, valid when m is close to 1, have been presented for this case in reference 10.

³ The corresponding solutions for raked-in or raked-out tips may also be found in reference 4, or deduced from later sections in the present report.

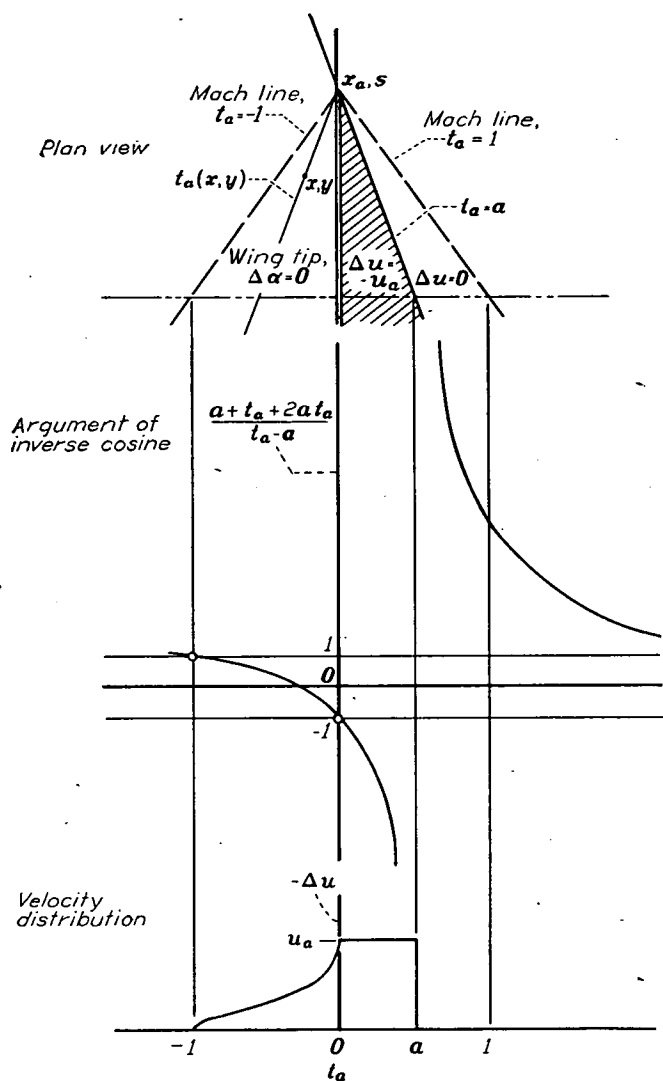


FIGURE 5.—Elementary solution for canceling lift at the tip.

real part of the inverse cosine is zero. On the wing ($-1 \leq t_a < 0$), the argument goes from $+1$ to -1 and the inverse cosine is real. Thus in canceling, or subtracting, the velocity u_a between $t_a=0$ and $t_a=a$, the increment in velocity

$$u(x, y, a) = -\frac{u_a}{\pi} \cos^{-1} \frac{a + t_a + 2at_a}{t_a - a} \quad (11)$$

is induced on the wing upper surface.

TIP-INDUCED CORRECTION TO THE LOADING

Following the procedure outlined in Part I, we proceed to determine the effect of canceling the lift outboard of the wing tip. Since the value of u_a for the initial canceling field $-u_a(m)$ and the value for the first incremental field $\frac{du_a}{da} da$ are both infinite when the leading edge is subsonic, it is first necessary to write the induced velocity at a point x, y as

$$(\Delta u)_{tp} = \lim_{a \rightarrow \infty} \left[\frac{-u_{\Delta}(a)}{\pi} \cos^{-1} \frac{a+t_a+2at_a}{t_a-a} + \frac{1}{\pi} \int_{a_0}^a \frac{du_{\Delta}}{da} \cos^{-1} \frac{a+t_a+2at_a}{t_a-a} da \right] \quad (12)$$

where the limit a_0 is the value of a corresponding to the rearmost sector including the point x, y in its Mach cone. The value of a_0 is found by setting t_a (equation (9)) equal to -1 . Thus, for the tip correction,

$$a_0 = \frac{\beta s}{x + \beta(y - s)} \quad (13)$$

This parameter will be additionally useful as the value of a at which the velocity correction given by equation (11) goes to zero and its derivative has a singularity.

Before performing the integration of equation (12), t_a must be replaced by its expression in terms of x , y , and a . Then integration of the second term by parts results in a term which, at the upper limit, exactly cancels the first term, and at the other limit is zero, leaving, after substitution for u_A ,

$$(\Delta u)_{\text{tp}} = \frac{-m(x + \beta y)u_0}{\pi \sqrt{s}} \int_{a_0}^m \frac{\sqrt{a_0(s-y)} da}{(ax - \beta y) \sqrt{(m^2 - a^2)(1+a)(a-a_0)}} \quad (14)$$

This integral is finite and can be evaluated in terms of elliptic integrals as follows:

$$(\Delta u)_{itp=u_0} \left[\sqrt{\frac{m\beta(s-y)}{2(x+\beta y)}} K_0 - \frac{mx}{\sqrt{m^2 x^2 - \beta^2 y^2}} \Lambda_0(k, \psi) \right] \quad (15)$$

where

$$\Lambda_0 = K_0 E(\psi, k') - (K_0 - E_0) F(\psi, k') \quad (16)$$

and K_0 and E_0 are $2/\pi$ times the complete elliptic integrals K and E of modulus

$$k = \sqrt{\frac{(m - a_0)(1 - m)}{2m(a_0 + 1)}}$$

In equation (16), $F(\psi, k')$ and $E(\psi, k')$ are the incomplete integrals with the complementary modulus $k' = \sqrt{1 - k^2}$ and argument

$$\psi = \sin^{-1} \sqrt{\frac{a_0(mx + \beta y)}{\beta s(a_0 + m)}}$$

The functions K_0 , E_0 and Λ_0 are tabulated in reference 11⁴ or may be computed from the tables of reference 12. A plot of Λ_0 is given in figure 6.

Value at the side edge.—At the tip, y is equal to s and the first term in equation (15) vanishes. In the second term, ψ becomes $\pi/2$ and $E(\psi, k')$ and $F(\psi, k')$ reduce to the complete integrals $E' = E(k')$ and $K' = K(k')$, respectively. Then, since, by Legendre's relation,

$$K'E - K'K + KE' = \pi/2$$

Λ_0 reduces to 1. The induced velocity correction is seen to be exactly equal to $-u_\Delta$, bringing the lift to zero at the wing tip.

Drop in lift across tip Mach line.—An interesting effect shows itself at the other limit of the tip region, that is, at the Mach line from the tip of the leading edge. Along this line only the influence of the leading-edge pressure is felt, so that

* The quantity K_0 is called F_0 in reference 11.

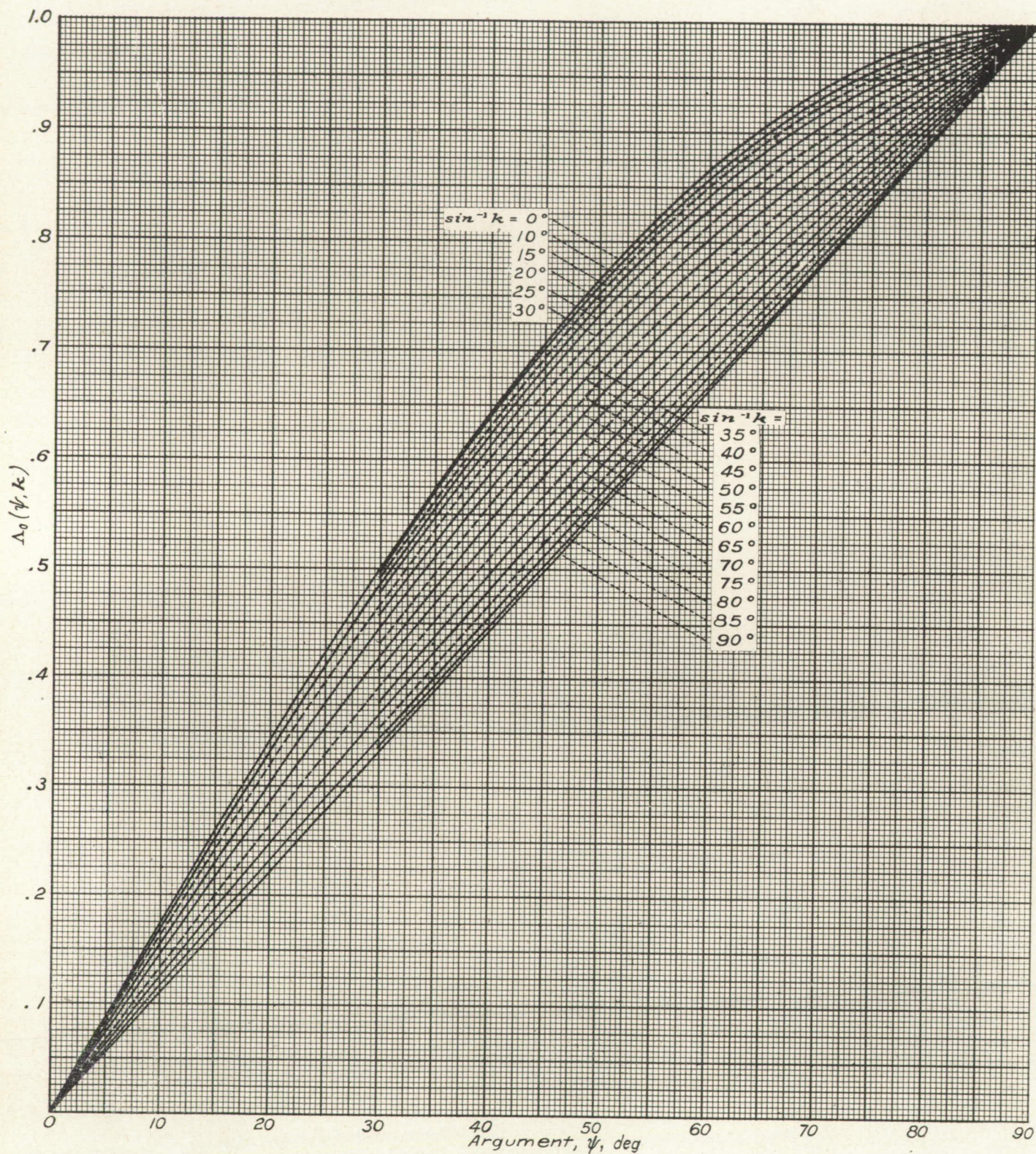


FIGURE 6.—The elliptic function $\Delta_0(\psi, k) = K_0 E(\psi, k) - (K_0 - E_0) F(\psi, k)$.

$\alpha_0 = m$. Then $k=0$, $k'=1$, $K=\pi/2$, $E(\psi, k')$ reduces to

$$\sin \psi = \sqrt{\frac{mx + \beta y}{2\beta s}}, \text{ and finally}$$

$$(\Delta u)_{tip} (t_m = -1) \equiv \Delta u^* = \frac{-u_0 \beta s}{\sqrt{2(x + \beta y)(mx - \beta s)}} \quad (17a)$$

or, since along the tip Mach line $\beta(s - y) = x - \frac{\beta s}{m}$,

$$\Delta u^* = \frac{-u_0 \sqrt{m\beta s}}{\sqrt{2(1+m)(mx - \beta s)}} \quad (17b)$$

This result indicates a finite drop in pressure across the Mach line from the tip, an effect which is associated with the cancellation of infinite pressure at the leading edge and consequently does not appear as long as the leading edge is ahead of the Mach lines. The ratio of the drop in lift across the tip Mach line to the uncorrected lift can be written

$$\frac{\Delta u^*}{u_a} = -\sqrt{\frac{(1+a)(m+a)}{2m(1+m)}} \quad (18)$$

This ratio is plotted against a/m in figure 7 and shows the percentage loss of lift at the tip to be very large. In fact, for any but the lowest-aspect-ratio wings, the lift remaining in that region is almost negligible. This effect, which should be of considerable practical interest, was first indicated in the results of reference 13 for the limiting case of $m=0$.

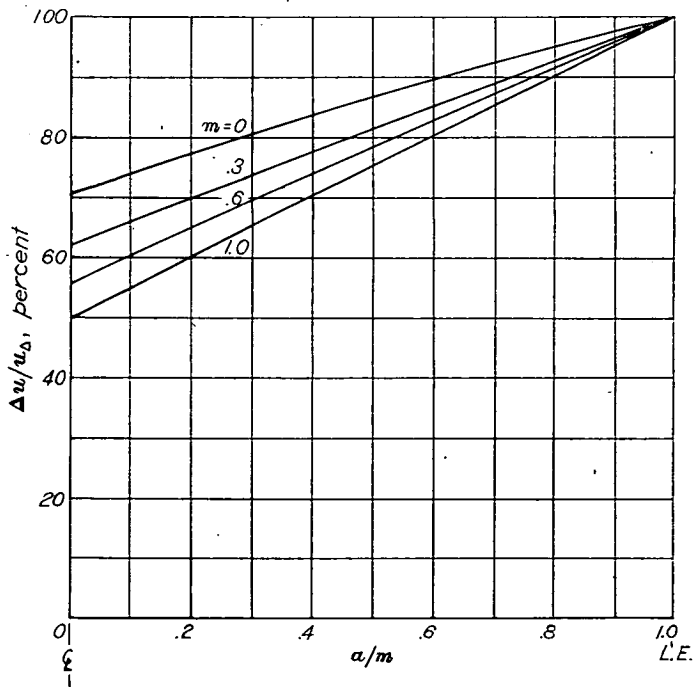


FIGURE 7.—Percent drop in lift across Mach line from tip.

SWEPT-BACK WING WITH SUBSONIC TRAILING EDGE

The tip-effect correction just derived applies equally to wings with supersonic or subsonic trailing edges. The effect of a subsonic trailing edge is calculated separately, and is primarily due to canceling the triangular-wing loading in the wake region. If, however, the triangular-wing loading

has been modified by the introduction of side edges, then this modification must also be taken into account when canceling the lift behind the trailing edge. In the conical-flow method, the various component flow fields must be canceled individually. The sections immediately following will discuss the cancellation of the triangular-wing loading; cancellation of the tip-induced components of velocity will be considered under the heading "Secondary Corrections."

PRIMARY TRAILING-EDGE CORRECTIONS

Procedure for canceling lift in the wake region.—The basic procedure is again to consider the load to be canceled to be built up by the superposition of uniformly loaded sectors, bounded on one side (see fig. 8) by the rays a , and on the other by the trailing edge of the wing. It is convenient at this point to introduce the parameter

$$m_t = \beta \times \cot (\text{angle of sweep of trailing edge})$$

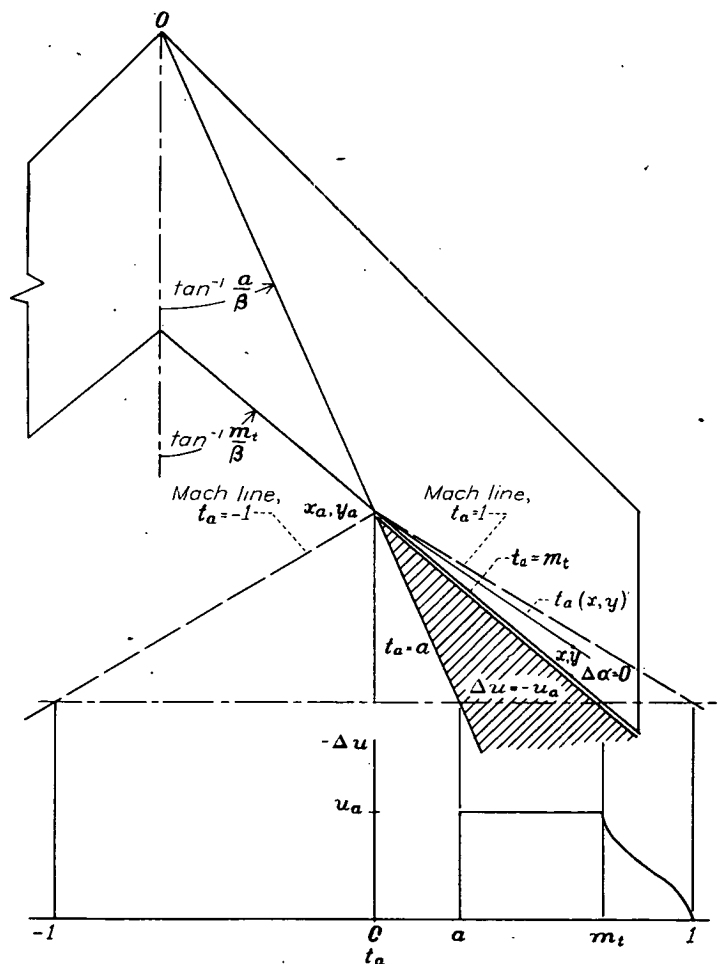


FIGURE 8.—Oblique constant-lift element (shaded) for cancellation of lift at subsonic trailing edge, and induced velocity distribution.

The boundary conditions to be satisfied by the u component of the elementary canceling velocity field are indicated for the right span in figure 8; each field must have constant velocity u_a when $a \leq t_a \leq m_t$ and zero streamwise velocity over the wake region, $-1 \leq t_a < a$. The concomitant vertical velocity must be zero on the wing surface. However, when a is small, the region $-1 \leq t_a < a$ will include a portion of the left-hand wing panel. Since in this region the u component of velocity has already been specified, the vertical velocity

will not, in general, be zero. Nor is it possible to modify the field to satisfy the boundary condition on the far wing, since the area involved is not conical with respect to the apex of the field.

The error involved in the foregoing procedure is minimized by the use of a symmetrical flow field to cancel the initial load u_0 at $a=0$, where a single conical field can be made to satisfy the boundary conditions exactly on both wing panels. This flow field (see fig. 9) would have its origin at the apex $c_0, 0$ of the trailing edge, and the constant-load region would extend over the entire wake region. Between the trailing edge of the wing and the Mach lines from $c_0, 0$ the induced downwash would be zero in the plane of the wing, while the pressure would vary as required to satisfy the fundamental flow equations.

In figure 9, a typical curve of u_Δ is shown, from which it can be seen that the load to be canceled is very nearly constant over a considerable fraction of the wake region. Cancellation of the velocity u_0 by the symmetrical field will consequently leave only a small variation in u to be canceled by the oblique fields described earlier in the section. The resulting violation of the flat-plate condition may be expected to be small,⁵ and will take place only over a small region near the tip of the trailing edge.

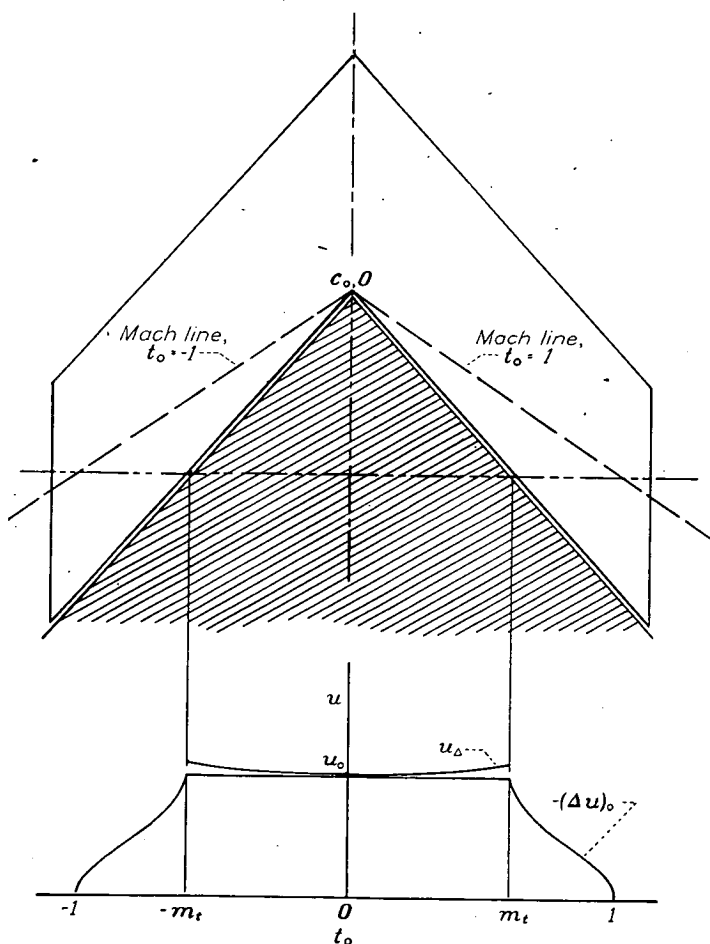


FIGURE 9.—Symmetrical field for cancellation of u_0 at subsonic trailing edge.

⁵ Calculations made to check this statement have shown the induced downwash angles to be less than 0.5 percent of the angle of attack, even in the most unfavorable circumstances.

Symmetrical solution.—For the symmetrical solution we define the conical variable

$$t_0 = \frac{\beta y}{x - c_0} \quad (19)$$

which is zero along the center line of the wing and equals $\pm m_t$ at either trailing edge. Then the boundary conditions to be satisfied in the xy plane may be summarized as follows:

$$\begin{aligned} -m_t \leq t_0 \leq +m_t, \quad u &= \pm u_0 \\ m_t < |t_0| \leq 1 \quad w &= 0 \end{aligned}$$

The required solution is given in reference 14. The u component in the xy plane is

$$r. p. \frac{\pm u_0}{K'(m_t)} F(\phi, \sqrt{1 - m_t^2})$$

where $K'(m_t)$ is the complete elliptic integral of the first kind of modulus $\sqrt{1 - m_t^2}$ and $F(\phi, \sqrt{1 - m_t^2})$ is the corresponding incomplete integral of argument

$$\phi = \sin^{-1} \sqrt{\frac{1 - t_0^2}{1 - m_t^2}}$$

The form of the induced velocity on the wing (see fig. 9) is very similar to the inverse cosine curves of the tip solutions.

On the wing, ϕ is real and the symbols $r. p.$ may be omitted. The velocity induced on the upper surface by cancellation of u_0 behind the trailing edge is therefore

$$(\Delta u)_0 = -\frac{u_0}{K'(m_t)} F(\phi, \sqrt{1 - m_t^2}) \quad (20)$$

Oblique solutions for the wake region.—The symbol t_a will be used as before to indicate a ray of the flow field with apex at x_a, y_a , the point of intersection of the ray a with the wing boundary—in this case the trailing edge. Along the trailing edge,

$$y_a = \frac{m_t}{\beta} (x_a - c_0)$$

Since $a = \beta (y_a/x_a)$, we may solve for x_a and y_a as functions of a and the constants m_t and c_0 :

$$x_a = \frac{m_t c_0}{m_t - a} \quad (21)$$

$$\beta y_a = \frac{m_t c_0 a}{m_t - a} \quad (22)$$

Then

$$t_a = \frac{\beta y (m_t - a) - m_t c_0 a}{x (m_t - a) - m_t c_0} \quad (23)$$

The boundary conditions to be satisfied by the elementary solution are (for $a > 0$)

$$a \leq t_a \leq m_t \quad u = \pm u_a$$

$$m_t < t_a \leq +1 \quad w = 0$$

$$-1 \leq t_a < a \quad u = 0$$

The solution satisfying these conditions may be obtained from the tip solutions by an oblique transformation. (See reference 4.) In the xy plane, the resulting expression for the u component is

$$r \cdot p \pm \frac{u_a}{\pi} \cos^{-1} \frac{(1-a)(t_a-m_t)-(m_t-a)(1-t_a)}{(1-m_t)(t_a-a)}$$

Then the velocity induced at any point x, y on the upper surface of the wing by the cancellation of the infinitesimal increment of perturbation velocity u_a over the sector bounded by the ray a and the trailing edge is

$$\frac{d\Delta u}{da} da = (\Delta u)_a = \frac{-u_a}{\pi} \cos^{-1} \frac{(1-a)(t_a-m_t)-(m_t-a)(1-t_a)}{(1-m_t)(t_a-a)} \quad (24)$$

Correction of loading near the trailing edge.—To determine the lift at any point x, y near the trailing edge of the wing, it is first necessary to determine the most rearward canceling sector a_0 that will influence that point. Setting t_a (equation (23)) equal to 1, we solve for

$$a_0 = m_t \frac{x - \beta y - c_0}{x - \beta y - m_t c_0} \quad (25)$$

Then the total correction to the triangular-wing velocity u_Δ obtained as a result of canceling that velocity behind the trailing edge is

$$(\Delta u)_{T.E.}(x, y) = (\Delta u)_0 + \int_0^{a_0} \frac{d\Delta u}{da} da \quad (26a)$$

The integral in the foregoing expression has been evaluated in terms of an incomplete elliptic integral of the third kind, which may be computed with the aid of the tables of references 11 and 15. Because it will be necessary to define several new functions it was thought better to present the results in an appendix (appendix B). For practical use, graphical or numerical integration may be preferred, in which case a convenient form is obtained by rewriting u_a as $(du_\Delta/da) da$, or du_Δ , in equation (24). Thus equation (26a) becomes

$$(\Delta u)_{T.E.}(x, y) = (\Delta u)_0 -$$

$$\frac{1}{\pi} \int_{u_0}^{u_\Delta(a_0)} \cos^{-1} \frac{(1-a)(t_a-m_t)-(m_t-a)(1-t_a)}{(1-m_t)(t_a-a)} du_\Delta \quad (26b)$$

where t_a and u_Δ must be evaluated for selected values of a between zero and a_0 . The integrand, of course, goes to zero at $u_\Delta(a_0)$. At points along the leading edge (in cases in which the leading edge extends into the zone of influence of the trailing edge), the integral takes on a somewhat simpler form, with the result that the entire trailing-edge correction at such points can be written

$$(\Delta u)_{T.E.}\left(x, \frac{mx}{\beta}\right) = -u_0 \left\{ \frac{F(\phi, \sqrt{1-m_t^2})}{K(\sqrt{1-m_t^2})} - \frac{1}{\pi} \cos^{-1} \frac{(t_0-m_t)-m_t(1-t_0)}{(1-m_t)t_0} + \frac{1}{\pi} \sqrt{\frac{2(t_0-m_t)}{(1-m_t)(1-m_t)t_0}} \left[F(\psi, k) - \frac{1-a_0}{m-a_0} E(\psi, k) \right] \right\} \quad (26c)$$

where the first term inside the braces is $\frac{(\Delta u)_0}{u_0}$ and, in the last term,

$$\psi = \sin^{-1} \sqrt{\frac{2a_0}{m+a_0}}$$

and

$$k = \sqrt{\frac{(1-m)(m+a_0)}{2m(1-a_0)}}$$

SECONDARY CORRECTIONS

The term "secondary corrections" is used here to designate the effect of canceling the lift introduced outside the boundaries of the wing in the process of canceling the original triangular-wing loading beyond the tips and behind the trailing edge. As previously mentioned, cancellation of lift at the tip introduces new (negative) components of lift to be canceled at the trailing edge. The original cancellation of lift behind the trailing edge, on the other hand, will introduce negative incremental pressures outboard of the tip and, under certain circumstances (see figs. 1 (a) and (b)), ahead of the leading edge. The distribution of lift to be canceled in each case is no longer part of a single conical field, but is composed of an infinite number of superposed conical fields originating at various points along the trailing edge or tip. In order to cancel these pressures accurately, it would be necessary to set up, for each of the original canceling elements, an infinity of positively loaded elements at the opposite boundary. Thus, each secondary correction would require a double integration for each point, and would obviously be quite tedious. The procedure is described in detail in references 7 and 8. The more recent work of Mirels (reference 5) offers an alternative method which, while no less tedious at the computational state, is somewhat easier to set up for computing. Nevertheless, the exact calculation of the secondary corrections, and of the succeeding corrections arising as the secondary corrections are in turn canceled at the opposite edges, appears feasible only with the aid of high-speed computing machinery.

These corrections may be thought of as a converging series, since in each case (except in the neighborhood of the leading edge) the induced effect is smaller than the canceled lift. Over most of the wing, the secondary correction is of the same order of magnitude as the tolerable error. Formulas for obtaining a major part of the secondary corrections can be given rather simply and should suffice to give results of practical accuracy in problems (fig. 1(c)) not involving leading-edge corrections. Problems of the type shown in figure 1(b) will be discussed in a later section.

Secondary corrections at the trailing edge.—The pressure differences induced by the tip are in the main due to cancellation of the infinite pressure at the leading edge. It should therefore be permissible, for the secondary corrections, to approximate the tip-correction field by a single conical field from the leading-edge tip. The lift associated with this field may then be canceled behind the wing (see fig. 10) by a single infinity of superposed fields, as was the original triangular-wing loading. If the values of $(\Delta u)_{tt}$ calculated for points x_b, y_b along the trailing edge are assumed to apply all along the corresponding rays $t_m(x_b, y_b)$ from the tip, then the

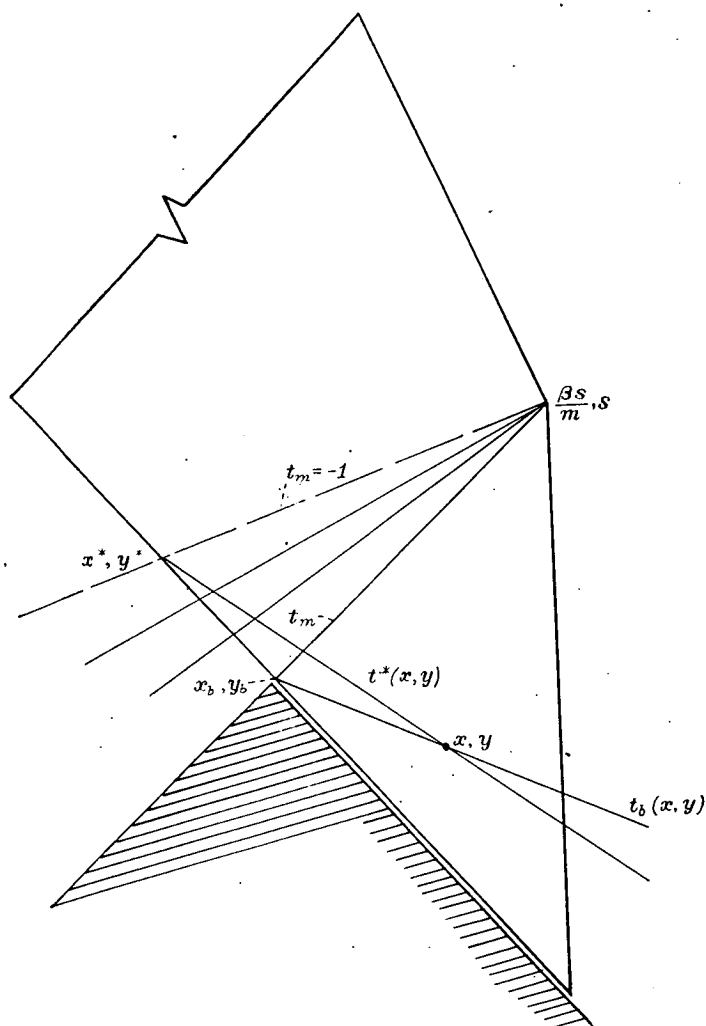


FIGURE 10.—Sketch for approximate cancellation of extraneous lift introduced behind the leading edge by the tip correction.

lifting pressure will be exactly canceled along the trailing edge and the remaining variation of pressure in the wake will have very little effect on the flow over the wing.

The cancellation fields are of the previously used oblique type, with a replaced by

$$t_m = \frac{\beta(y_b - s)}{x_b - \frac{\beta s}{m}} \quad (27)$$

Let the particular point at which the line $t_m = -1$ intersects the trailing edge be designated by x^*, y^* and other symbols referring to that point be similarly starred. Then the velocity induced at any point x, y on the wing by removal of $(\Delta u)_{tp}$ along the trailing edge will be (from equation (24))

$$\frac{-\Delta u^*}{\pi} \cos^{-1} \frac{2(t^* - m_i) - (m_i + 1)(1 - t^*)}{(1 - m_i)(t^* + 1)} - \frac{1}{\pi} \int_{-1}^{t_m(x_0, y_0)} \frac{d(\Delta u)_{tp}}{dt_m} \cos^{-1} \frac{(1 - t_m)(t_b - m_i) - (m_i - t_m)(1 - t_b)}{(1 - m_i)(t_b - t_m)} dt_m \quad (28)$$

where Δu^* is given by equation (17), t^* and t_b are calculated by

$$t^* = \frac{\beta(y - y^*)}{x - x^*} \quad (29)$$

and

$$t_b = \frac{\beta(y - y_b)}{x - x_b} \quad (30)$$

respectively, and x_0, y_0 is the point of intersection of the Mach forecone from x, y with the trailing edge.

The derivative $\frac{d}{dt_m} (\Delta u)_{tp}$ would have to be determined numerically or graphically from a plot of the calculated values of $(\Delta u)_{tp}$ against t_m . In order to avoid this procedure, it is preferable to rewrite expression (28) as

$$\frac{-\Delta u^*}{\pi} \cos^{-1} \frac{2(t^* - m_i) - (m_i + 1)(1 - t^*)}{(1 - m_i)(t^* + 1)} - \frac{1}{\pi} \int_{\Delta u^*}^{(\Delta u)_{tp}(x_0, y_0)} \cos^{-1} \frac{(1 - t_m)(t_b - m_i) - (m_i - t_m)(1 - t_b)}{(1 - m_i)(t_b - t_m)} d(\Delta u)_{tp} \quad (31)$$

and integrate by plotting the inverse cosine function against $(\Delta u)_{tp}$.

As long as the aspect ratio of the wing is greater than $1/\beta$ (a condition already imposed by the exclusion of the problem shown in fig. 1 (a)), Δu^* will be more than half $(\Delta u)_{tp}$ at any other point on the trailing edge. Since, moreover, the integral term in equation (31) has zero slope at the Mach cone ($t^* = 1$), while the first term starts with infinite slope, it is apparent that the secondary correction may be simplified still further by omitting the calculation of the integral. For points near the trailing edge, the loading can usually be faired to zero with sufficient accuracy.

Secondary correction at the tip.—A similar method of approximating the secondary correction at the tip cannot be formulated with equal confidence. Since, however, over most of the wing the symmetrical correction $(\Delta u)_o$ contributes the larger part of the total subsonic-trailing-edge effect, it will again be assumed that the entire effect constitutes a single conical field, with its apex at $c_0, 0$. The u velocity along each ray t_0 will have the value $(\Delta u)_{T.E.}(x_b, s)$ of the trailing-edge correction at the intersection x_b, s of the ray with the tip. The canceling fields will have the same form as those (equation (10)) used in deriving the primary tip correction, and the total approximate correction to the u velocity will be

$$-\frac{1}{\pi} \int_{t_0=1}^{(\Delta u)_{T.E.}(x_0, s)} \cos^{-1} \frac{t_0 + t_b + 2t_0 t_b}{t_b - t_0} d(\Delta u)_{T.E.} \quad (32)$$

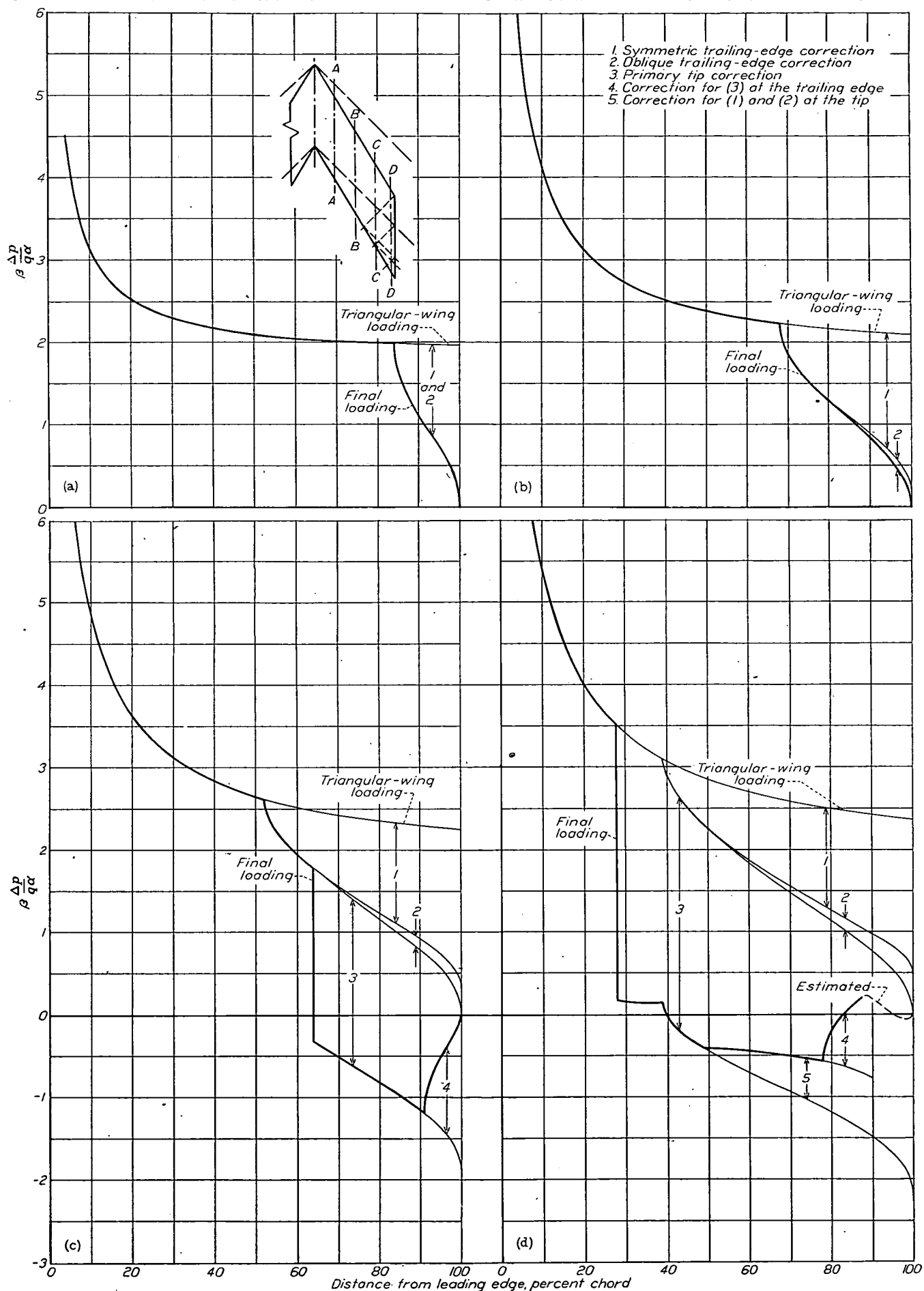
in which

$$t_b = \frac{\beta(y - s)}{x - x_b} \quad (33)$$

x_0 is the value of x_b which makes $t_b = -1$, and $(\Delta u)_{T.E.}$ is calculated for $x = x_b, y = s$ by equation (26b).

NUMERICAL EXAMPLE

Before proceeding to consider the problem of interaction between the leading and trailing edges, which introduces some radically different effects, the results so far obtained will be illustrated by a numerical example. The loading over an untapered wing, with $\beta \cot \Lambda = 0.6$ and reduced aspect ratio $\beta A = 1.92$, has been calculated at four spanwise stations: 25-, 50-, 75-, and 95-percent semispan. The wing plan form and section lift distributions are shown in figure 11.


 FIGURE 11.—Load distributions calculated for four streamwise sections of an untapered wing; $m=0.6$; $BA=1.92$.

The results of the calculations are presented in the form of values of $\beta(\Delta p/q\alpha)$ (equation (5)).

The various components of lift are presented separately as calculated. In figures 11 (a) and (b), the discontinuities in slope show the effect of the cancellation of the finite velocity u_0 at the trailing-edge apex. The integrated part of the trailing-edge correction (component 2) has zero slope at the Mach line. The two outboard sections (figs. 11 (c) and (d)) are intersected by the Mach cone from the tip, as indicated by the finite drop in the load curves. Cancellation of the finite tip effect at the trailing edge (component 4 in both figures) results in a sharp discontinuity in pressure gradient along the reflected Mach line, at 91-percent chord at $y/s=0.75$ and at 78-percent chord when $y/s=0.95$. The cancellation of the trailing-edge corrections at the tip, which affects only the last section shown, results in another break in the load curve at 49-percent chord. Further corrections enter at the rear of the section as a result of successive cancellations of the superposed pressures at the tip and trailing edge. Their effect has been only estimated.

SWEPT-BACK WINGS WITH INTERACTING TRAILING AND LEADING EDGES

When, as in figure 1 (b), the Mach cone from the trailing-edge apex includes a region ahead of the leading edge, the previously calculated trailing-edge corrections to u must be canceled in that region, since they represent a discontinuity in pressure which cannot be supported in the free stream. Thus there must be calculated a leading-edge correction, which is one of the previously defined secondary corrections. However, the location of the disturbed field ahead of the wing causes its influence on the wing to be so much more widespread than that of the other secondary corrections as to require more careful consideration. A new type of flow field is also required, as discussed in the following paragraphs.

LEADING-EDGE CORRECTIONS

Elementary solution for the region ahead of the leading edge.—In general, the elementary solution required for the cancellation of pressure in the plane of the wing ahead of the leading edge is one that:

1. Provides constant streamwise velocity over an infinite sector bounded on one side by the leading edge of the wing (extended) and on the other by an arbitrary ray extending outward into the stream from some point x_b, y_b on the leading edge. (See fig. 12.)
2. Induces no vertical velocity, or downwash, on the wing.
3. Induces no lift except on the wing and within the sector described in condition 1.

At first glance these conditions would appear to be satisfied by the oblique solutions used at the trailing edge, if properly oriented with respect to the wing, and the same form of solution might be expected to apply. In reference 4, however, it has been pointed out that the downwash connected with the latter solution remains constant over the wing only if the wing area does not include the line $y=\text{constant}$ extending downstream from the apex of the element. In the case of the leading-edge element this condition is violated (fig. 12) and an additional term is needed to bring the downwash to zero throughout the area of the wing affected by the element.

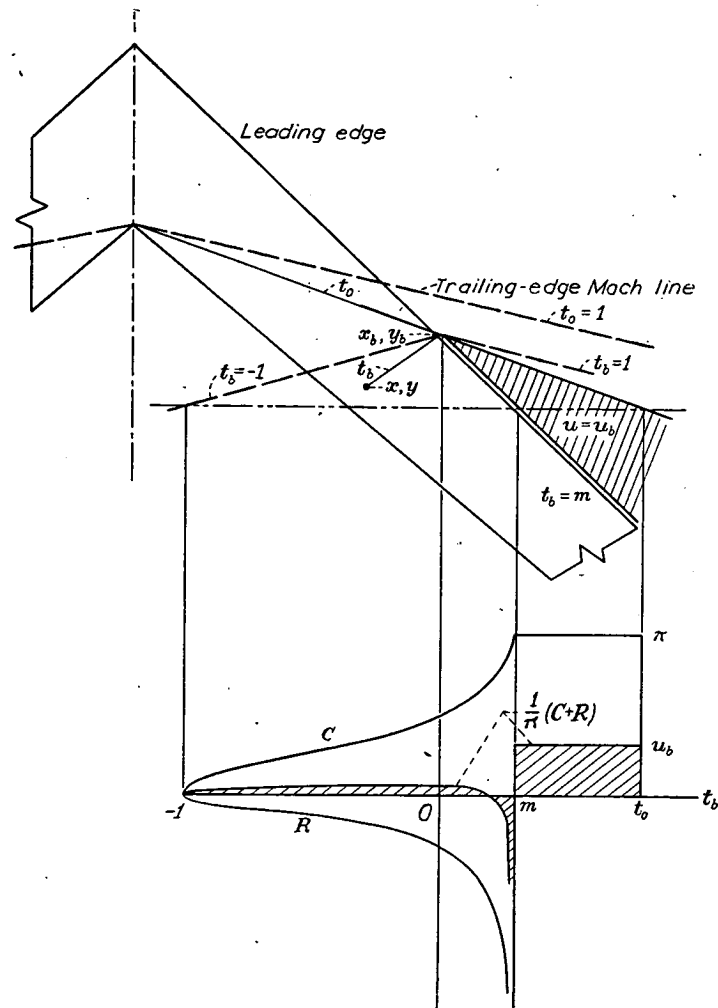


FIGURE 12—Leading-edge element and induced-velocity function.

The solution applicable to this case has been given in reference 4. The u component of the velocity in the plane of the wing is as follows:

$$u = r \cdot p \cdot \frac{u_b}{\pi} \left[\cos^{-1} \frac{(t_a - m)(1 + t_b) - (m - t_b)(1 + t_a)}{(1 + m)(t_b - t_a)} - \frac{2m}{(1 + m)t_a} \sqrt{(t_a - m)(1 + t_a)} \sqrt{\frac{1 + t_b}{m - t_b}} \right] \quad (34)$$

where u_b is the constant streamwise perturbation velocity over the element, and t_b refers as before to a ray from its apex. The ray bounding the element originates at a point on the trailing edge and has been designated, from equation (24), as t_a . When the correction is being made for the symmetrical trailing-edge element, t_a is replaced in equation (34) by t_0 .

For brevity, the two parts of the correction function will be referred to as

$$C(t_a) = r \cdot p \cdot \cos^{-1} \frac{(t_a - m)(1 + t_b) - (m - t_b)(1 + t_a)}{(1 + m)(t_b - t_a)} \quad (35)$$

and

$$R(t_a) = r \cdot p \cdot \frac{-2m}{(1 + m)t_a} \sqrt{(t_a - m)(1 + t_a)} \sqrt{\frac{1 + t_b}{m - t_b}} \quad (36)$$

The variation with t_b of these functions and the induced velocity (equation (34)) are illustrated in figure 12.

Leading-edge correction to the loading.—The single conical field of $(\Delta u)_0$ will be considered first. (See fig. 13.) The velocity field to be superposed ahead of the leading edge to cancel the velocity $(\Delta u)_0$ induced in the plane of the wing by the symmetrical solution (equation (20)) can be built up, as shown in figure 13, of overlapping constant-velocity sec-

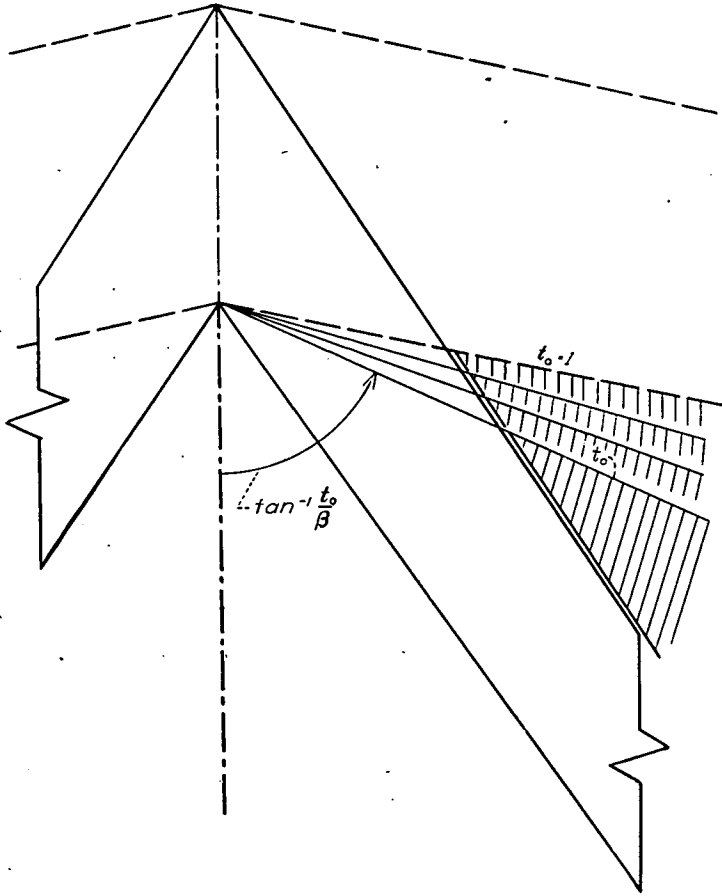


FIGURE 13—Cancellation of the pressure field introduced ahead of the leading edge in the course of canceling u_0 behind the trailing edge.

tors having one edge along the leading edge of the wing and one along the extended ray t_0 from the apex of the trailing edge. The magnitude of the constant velocity on each element is $\frac{d(\Delta u)_0}{dt_0} dt_0$ or, from equation (20),

$$\frac{u_0 dt_0}{K'(m_i) \sqrt{(1-t_0^2)(t_0^2-m_i^2)}} \quad (37)$$

Applying equation (34) to the cancellation of the symmetrical-correction velocities $(\Delta u)_0$ ahead of the wing results in the following induced velocity increment at any point (x, y) on the wing:

$$(\Delta_2 u)_0 = \frac{1}{\pi} \int_{\tau_0}^1 \frac{d(\Delta u)_0}{dt_0} [C(t_0) + R(t_0)] dt_0 \quad (38)$$

where τ_0 is that value of t_0 for which $t_0 = -1$, and designates the most rearward leading-edge element containing the point x, y within its Mach cone. In terms of x and y ,

$$\tau_0 = \frac{m(x+\beta y)}{(x+\beta y) - (1+m)c_0} \quad (39)$$

Integration of $\frac{d(\Delta u)_0}{dt_0} C(t_0) dt_0$ is not feasible by elementary means. For graphical integration it is advisable to rewrite $\frac{d(\Delta u)_0}{dt_0} dt_0$ as $d(\Delta u)_0$ to avoid the infinite value of the derivative at $t_0 = 1$.

The second term of the product in equation (38) can be integrated in closed form as follows:

$$\int_{\tau_0}^1 \frac{d(\Delta u)_0}{dt_0} R(t_0) dt_0 = \frac{-4m^{3/2}u_0 K(k)}{m_i(1+m)K'(m_i)} \sqrt{\frac{x+\beta y}{mx-\beta y}} Z(\psi, k) \quad (40)$$

where

$$k = \sqrt{\frac{2m_i(1-\tau_0)}{(1-m_i)(\tau_0+m_i)}}$$

and

$$Z(\psi, k) = E(\psi, k) - \frac{E(k)}{K(k)} F(\psi, k) \quad (41)$$

with

$$\psi = \sin^{-1} \sqrt{\frac{\tau_0+m_i}{2\tau_0}}$$

The function $Z(\psi, k)$ is tabulated in reference 16; a plot of $\frac{Z(\psi, k)}{k \sin \psi}$ against ψ is given in figure 14.

Similarly, for each oblique trailing-edge element a (see fig. 15), a canceling field can be built up ahead of the leading edge by the superposition of sectors bounded by the leading edge and by rays t_a from the apex x_a, y_a of the element a , and having a constant velocity of the magnitude

$$\frac{\partial(\Delta u)_a}{\partial t_a} dt_a = -\frac{1}{\pi} u_a \frac{\partial}{\partial t_a} \cos^{-1} \frac{(1-a)(t_a-m_i)-(m_i-a)(1-t_a)}{(1-m_i)(t_a-a)} dt_a \quad (42)$$

(from equation (24)). If the symbol $\Delta u_{L.E.}$ is used to designate the total leading-edge correction to the u component of velocity at any point, then the part due to canceling the field of a single oblique trailing-edge element a is

$$\frac{d\Delta u_{L.E.}}{da} da = \frac{1}{\pi} \int_{\tau_a}^1 \frac{\partial(\Delta u)_a}{\partial t_a} [C(t_a) + R(t_a)] dt_a \quad (43)$$

where

$$\tau_a = \frac{m(m_i-a)(x+\beta y) - m_i c_0(1+m)a}{(m_i-a)(x+\beta y) - m_i c_0(1+m)} \quad (44)$$

is the value of t_a for which $t_a = -1$ and the leading-edge correction function vanishes.

When the expression (equation (42)) for $\frac{\partial(\Delta u)_a}{\partial t_a}$ is substituted in equation (43), it is again impractical to attempt to write a closed expression for the integral of the first term $\frac{\partial(\Delta u)_a}{\partial t_a} C(t_a)$ of the product. The integral of the second term is

$$\begin{aligned} \int_{\tau_a}^1 \frac{\partial(\Delta u)_a}{\partial t_a} R(t_a) dt_a = & \\ & \frac{-4m}{\pi(1+m)} \frac{u_a}{a} \sqrt{\frac{(m_i-a)(1-m)(x+\beta y) - m_i c_0(1+m)(1-a)}{(1-m_i)(mx-\beta y)}} \times \\ & K(k_a) \left[\sqrt{1+a} \frac{Z(\psi_a, k_a)}{k_a \sin \psi_a} - \sqrt{1-a} \frac{Z(\psi_0, k_a)}{k_a \sin \psi_0} \right] \quad (45) \end{aligned}$$

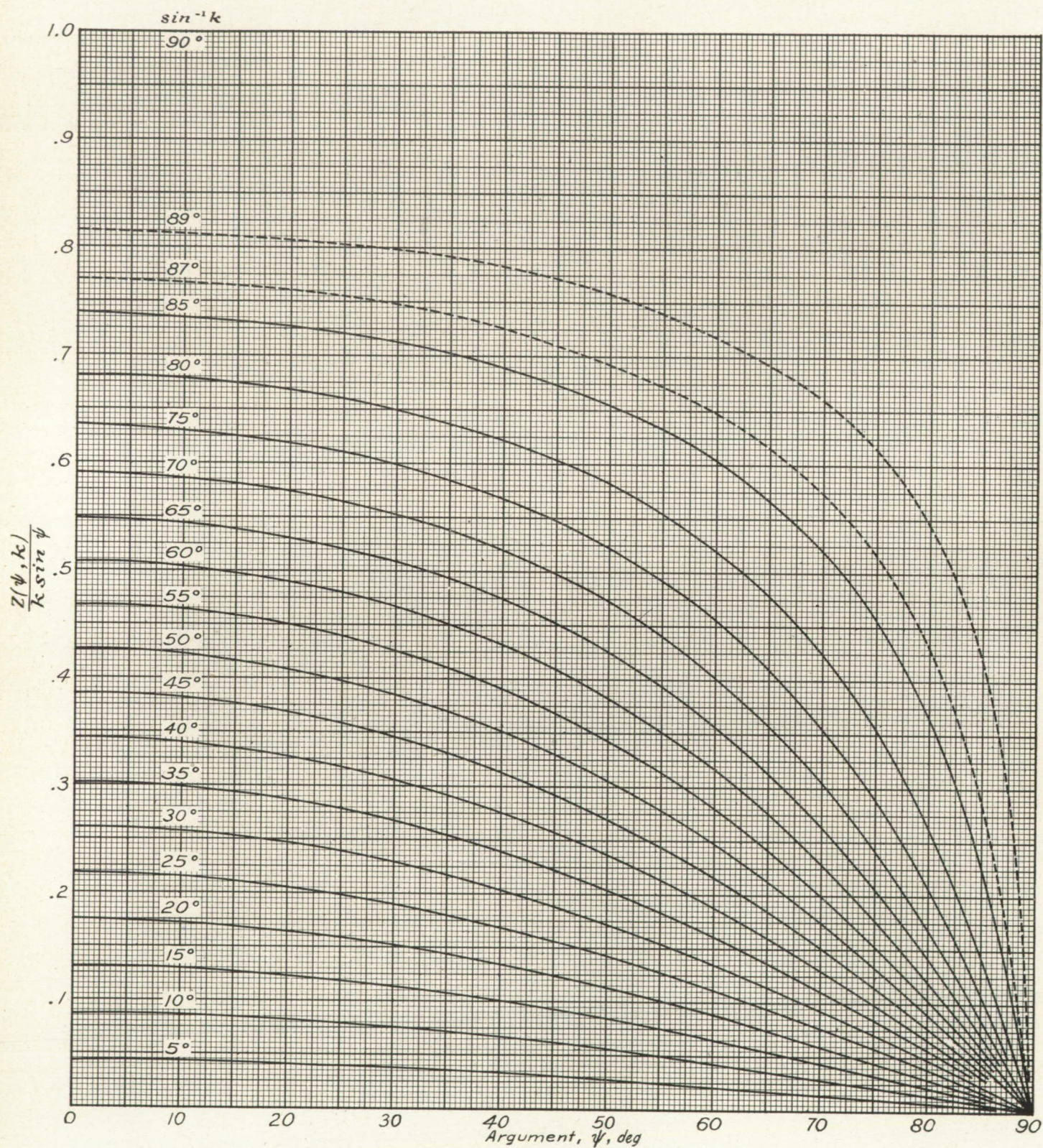


FIGURE 14.—The elliptic function $\frac{Z(\psi, k)}{k \sin \psi}$

where

$$k_a = \sqrt{\frac{(1+m_i)(1-\tau_a)}{(1-m_i)(1+\tau_a)}}$$

$$\psi_a = \sin^{-1} \sqrt{\frac{(m_i-a)(1+\tau_a)}{(\tau_a-a)(1+m_i)}}$$

$$\psi_0 = \sin^{-1} \sqrt{\frac{m_i(1+\tau_a)}{\tau_a(1+m_i)}}$$

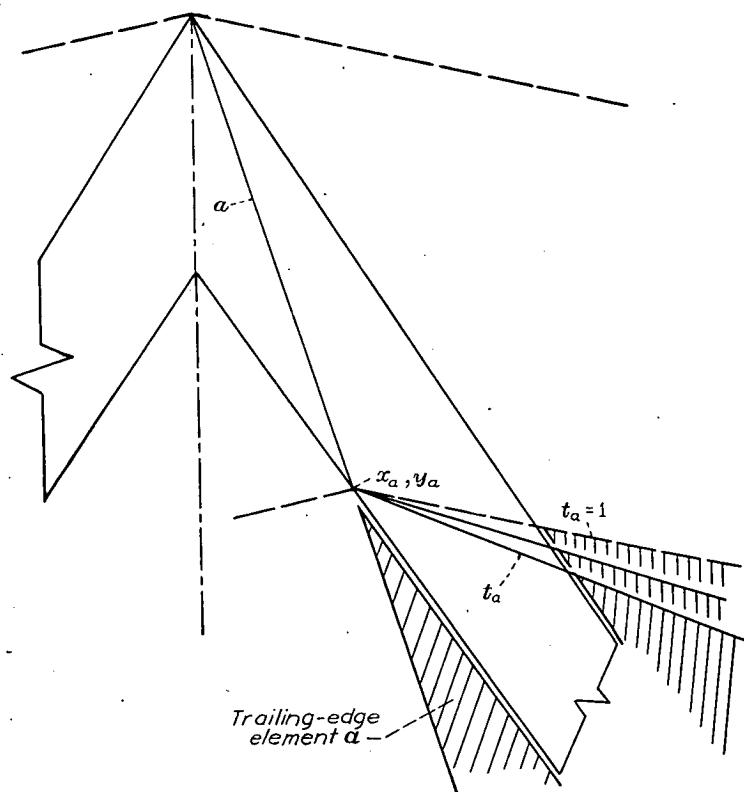


FIGURE 15.—Cancellation of the pressure field introduced ahead of the leading edge by a single oblique trailing-edge element.

Then the total leading-edge correction to the velocity u at any point x, y is

$$(\Delta u)_{L.E.} = (\Delta_2 u)_0 + \int_0^{a_0'} \frac{d(\Delta u)_{L.E.}}{da} da \quad (46)$$

where $(\Delta_2 u)_0$ is obtained from equations (38) and (40), $\frac{d(\Delta u)_{L.E.}}{da}$ from equations (43) and (45), and

$$a_0' = \frac{(1+m)c_0 - (1-m)(x+\beta y)}{(1+m)m_i c_0 - (1-m)(x+\beta y)} m_i \quad (47)$$

is the value of a at which $\tau_a(x, y, a)$ (equation (44)) is equal to 1.

The last term in equation (46) will seldom be found to contribute any significant amount to the loading, but will be needed in calculating the leading-edge thrust.

FURTHER CORRECTIONS

Omitting for the moment any specification of tip location, it is in any case necessary, as seen in figure 12, to consider the effect of a further cancellation necessitated by the excess

lift introduced behind the trailing edge by the leading-edge cancellation field. To compute this effect by the conical-flow method would be feasible only with the aid of high-speed computing machinery. The previously mentioned cancellation method of reference 5, being more direct, would be somewhat easier to use in this connection, but the calculations would still be very lengthy. It will be shown by numerical example that the effect of the first cancellation at the trailing edge of the leading-edge correction, which is initially quite small, may be estimated with adequate accuracy when the section loading is considered as a whole, provided the fraction of the chord affected is not too large.

If the product $\beta \cot \Lambda$ is low or the aspect ratio high, still further cancellations will be required (see fig. 16) at both leading and trailing edges. It is clear that calculation of the effect of these further cancellations by the conical-flow method is all but impossible. The doublet-distribution method of reference 5 does not appear to offer any considerable advantage in this application since, in canceling lift ahead of a subsonic leading edge, it is necessary to find not only the pressure distribution to be canceled, but the associated sidewash distribution as well.

It is apparent that an alternative method must be sought for describing the flow in the outboard regions of a high-aspect-ratio wing or a wing the sweep of which is large compared to the sweep of the Mach line. If the wing could be

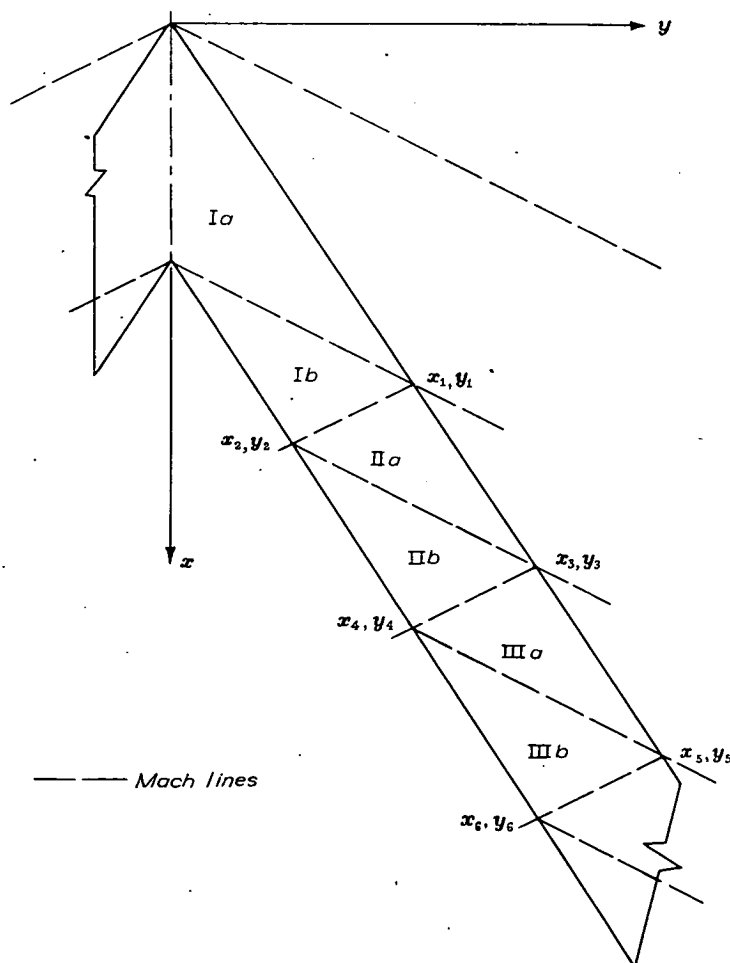


FIGURE 16.—Plan view of central portion of high-aspect-ratio wing, showing pattern of Mach lines arising at leading and trailing edges.

extended indefinitely, it is known that the flow must eventually approach the two-dimensional subsonic flow, in accordance with simple sweep theory. The question then arises, can the flow at a distance of the order of a semispan from the apex of the swept-back wing be related to the two-dimensional asymptotic flow? While the flow field appears to be too complex to obtain an answer to this question on analytical grounds, numerical values, presented in the following paragraph, suggest a practical approach.

NUMERICAL RESULTS (WITHOUT TIP EFFECT)

Load distributions have been calculated by the conical-flow method for three combinations of taper, sweep, and Mach number as follows:

	Untapered		Tapered
$m =$	0.2	0.4	0.4
$m_t =$	0.2	0.4	0.6

These values of m and m_t represent, by virtue of the Prandtl-Glauert transformation, a variety of sweep angles at Mach numbers between 1 and 2; as for example, 0.2 would be the value of m for a wing with 63° sweep of the leading edge at a Mach number of 1.075, or 75° sweep at a Mach number of

1.25. Similarly, $m=0.4$ would correspond to 45° of sweep at $M=1.08$, 60° at $M=1.22$, or 75° at $M=1.80$. The trailing-edge sweep angles at these latter Mach numbers, if $m_t=0.6$, are $34^\circ 13'$, 49° , and 68° , respectively.

Figure 17 presents the lift distributions at two stations of the tapered wing. Each component is plotted independently in order to show the magnitudes at the leading edge. Section A-A contains the intersection of the trailing-edge Mach line with the leading edge, so that the value of the leading-edge correction is zero at the leading edge of this section. At points farther back along the leading edge, as at $\beta y/c_0=0.8$, the correction is minus infinity. However, it is seen to increase to a small positive value within a fraction of the chord length at this station.

At both stations it is necessary to estimate the effect of cancellation of the leading-edge correction at the trailing edge to satisfy the Kutta condition. Cancellation would be carried out by means of oblique elements of the type used previously (equation (24)) in canceling lift at the trailing edge. The pressure to be canceled is initially (i. e., at x_2, y_2 (fig. 16)) zero. Then the lift induced on the wing by this cancellation may be presumed to have the same general shape as the oblique trailing-edge correction of figure

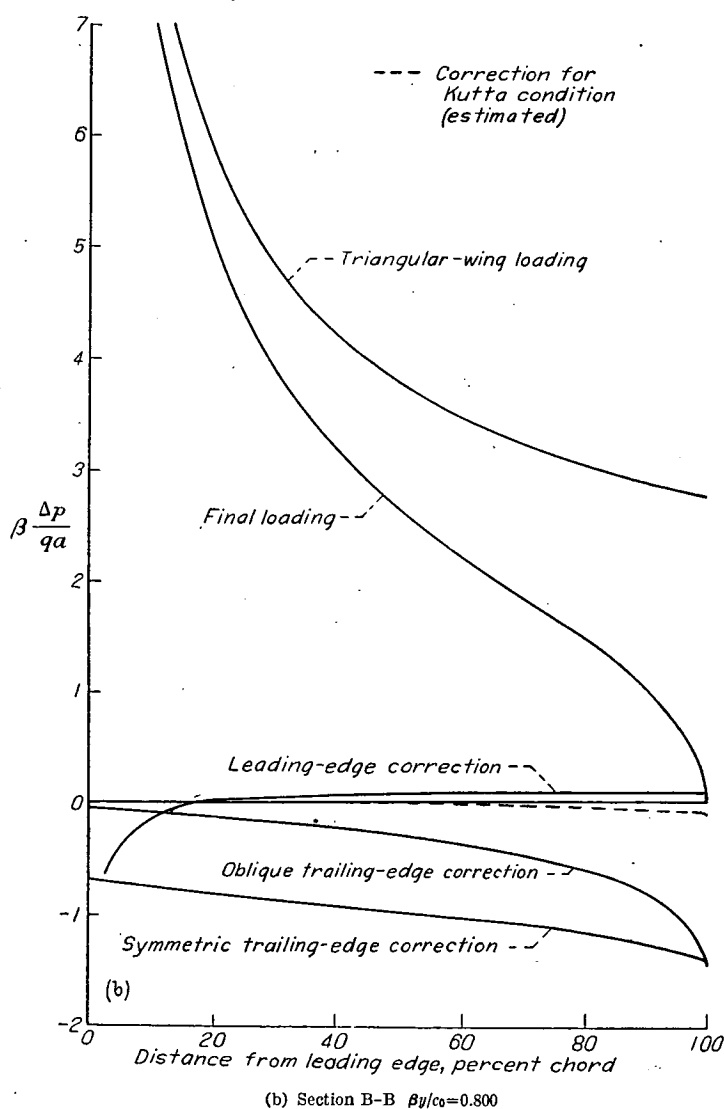
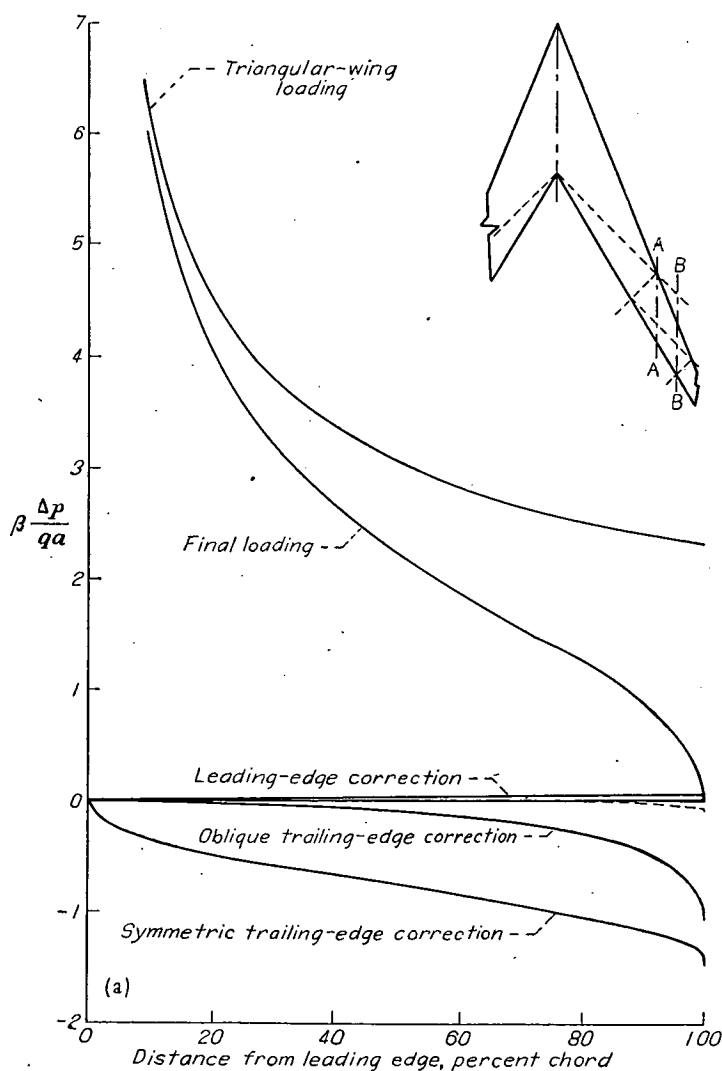
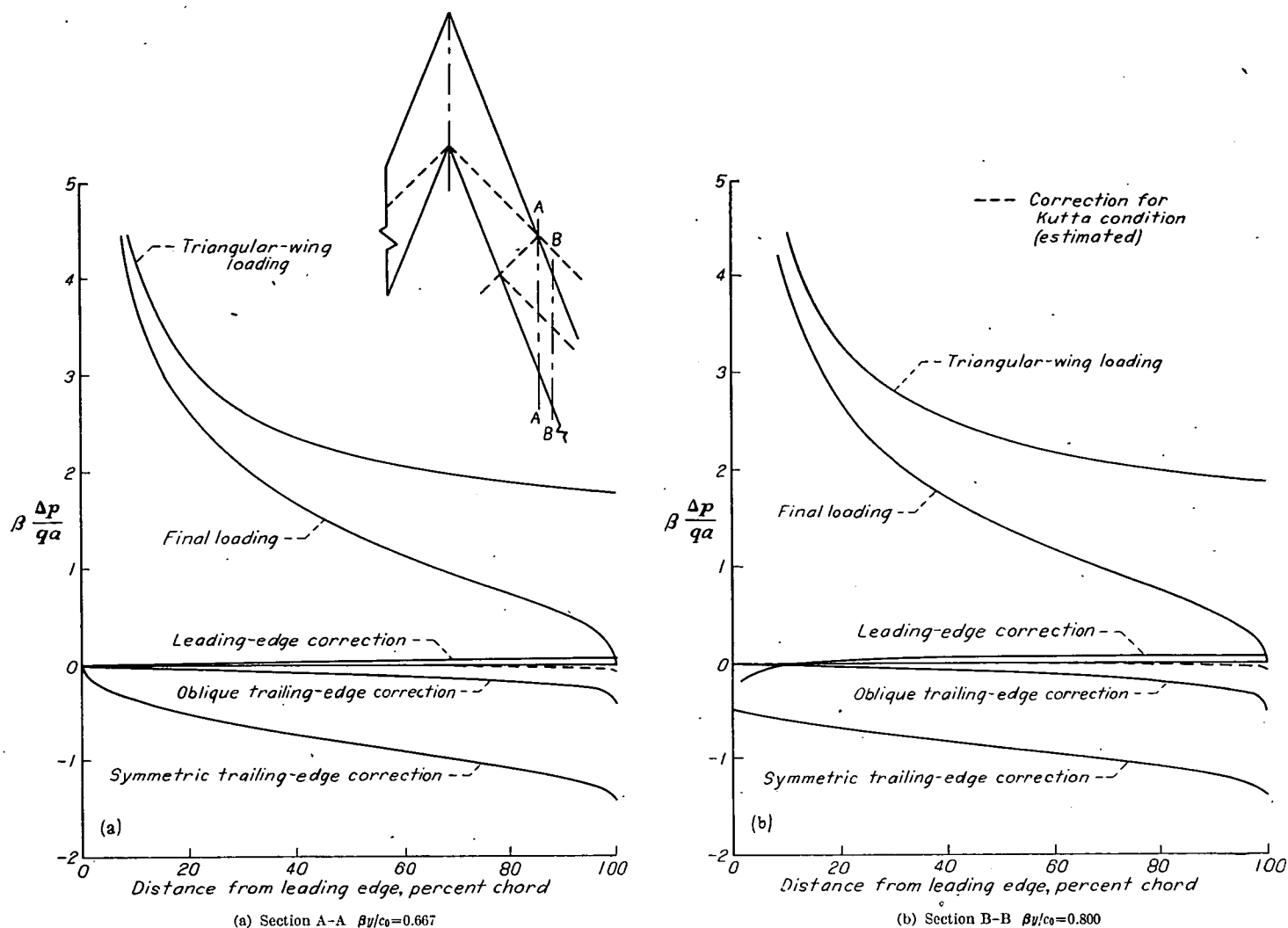


FIGURE 17.—Load distributions calculated by the conical-flows method for two streamwise sections of a tapered swept-back wing; $m=0.4$; $m_t=0.6$.


 FIGURE 18.—Load distributions calculated by the conical-flows method for two streamwise sections of an untapered wing; $m=0.4$.

11, falling along a modified inverse cosine curve from the value of the error at the trailing edge to zero, with zero slope, at the boundary of the region affected. With this boundary (the Mach line from the point x_2, y_2), it is possible to draw a satisfactory estimate (dotted curve) of the correction needed to bring the pressure once more to zero at the trailing edge.

The untapered wing with the same sweep ($m=0.4$) relative to the Mach lines is shown in figure 18, with the load distributions calculated at the same stations.

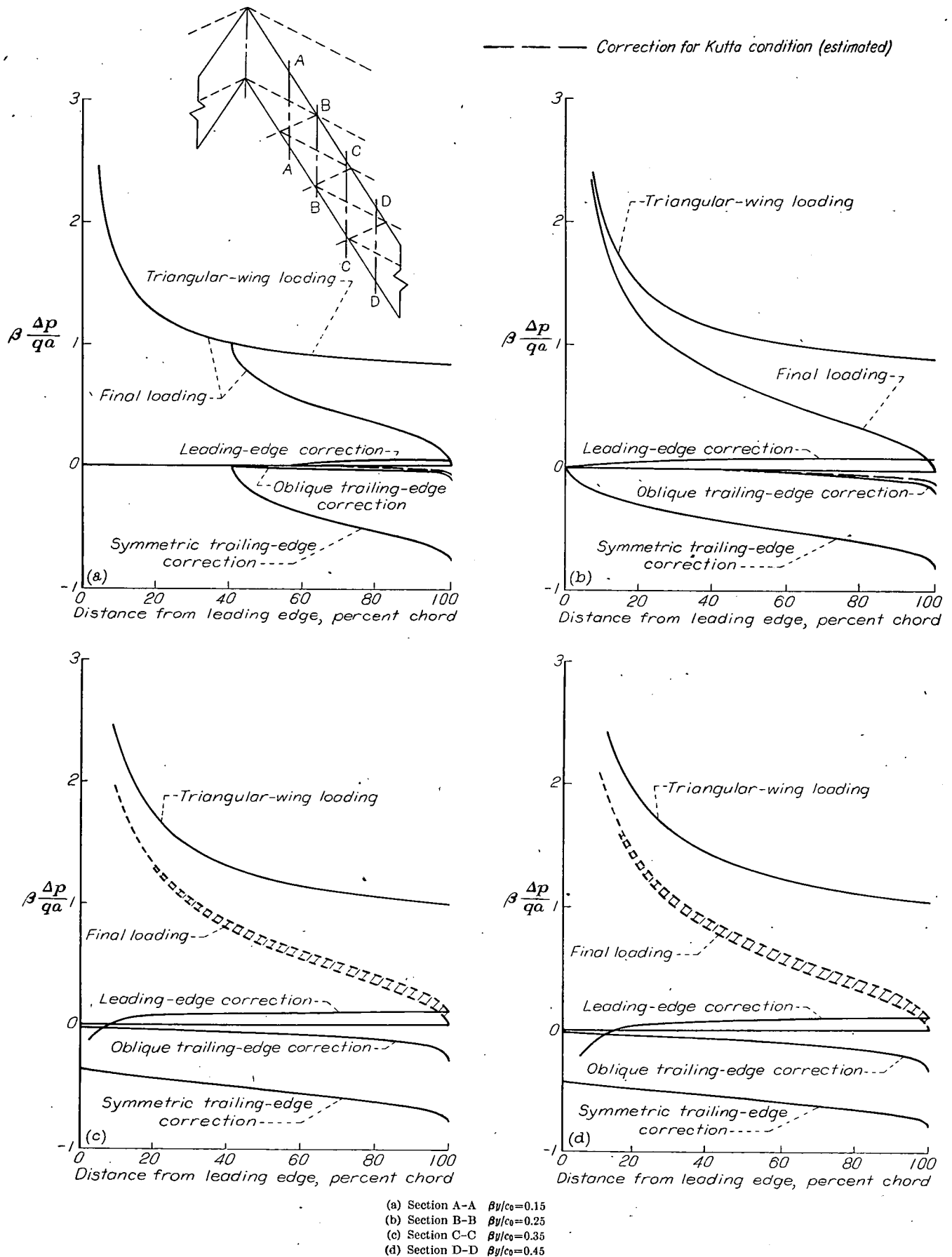
Four section lift distributions are presented (fig. 19) for $m=0.2$. At $\frac{\beta y}{c_0}=0.15$ only the rear 60 percent is influenced by the subsonic trailing edge. The reflection of this influence at the leading edge alters the pressure over the rear 40 percent of the section. At section B-B, the leading- and trailing-edge interaction affects the entire section. A further reflection of this effect at the trailing edge must be estimated.

At section C-C the influence of cancellation of the leading-edge correction at the trailing edge extends over the whole of the chord and any estimate of its magnitude would be necessarily arbitrary. Also, a second pair of reflections must be taken into account. The final pressure distribution has therefore been drawn as a band within which the true

curve may be shown to lie. Its height is the error introduced at the trailing edge by the first leading-edge correction, except very near the leading edge, where an infinite negative correction is known to be introduced by the second leading-edge correction. The calculations were also carried out for $\beta y/c_0=0.45$. The margin of uncertainty was found not to have increased by any appreciable amount. (See fig. 19 (d).)

APPLICATION OF TWO-DIMENSIONAL FORMULAS TO CALCULATION OF LOAD DISTRIBUTION

Correlation of two-dimensional and swept-back-wing loadings.—It is apparent from the calculated results that, whenever the plan form and the Mach number are such that the trailing-edge Mach line intersects the leading edge, the load distribution behind the Mach lines from the point of intersection resembles in shape the theoretical load distribution over an infinitely long flat plate in incompressible flow. However, as the results have been plotted, the quantitative agreement is not good, particularly in the case of the tapered wing. On the other hand, if the load distributions in cross sections normal to the stream are examined, a near proportionality of the curves is observed. In order to determine the factor of proportionality, it is only necessary to find the ratio of the strengths of the singularities at the

FIGURE 19.—Load distributions calculated by the conical-flows method for four streamwise sections of an untapered wing; $m=0.2$.

leading edge. Then an approximate expression for the loading on the outer portions of a high-aspect-ratio wing can be obtained by adjusting the two-dimensional loading by that factor.

Both the swept-back-wing and the subsonic two-dimensional loadings approach infinity as the reciprocal of the square root of the distance to the leading edge. In sections normal to the stream, the distance from any point x, y to the leading edge may be written $\frac{1}{\beta}(mx - \beta y)$. The value at the leading edge of the coefficient of $(mx - \beta y)^{-1/2}$ will be referred to as the strength of the leading-edge singularity.

The subsonic two-dimensional perturbation velocity has the form

$$u = B \sqrt{\frac{1-\eta}{\eta}} \quad (48)$$

where η is the distance to the leading edge, expressed as a fraction of the chord, and B is a constant. If the section of the swept-back wing is taken perpendicular to the stream (x constant), the chord length is

$$\frac{1}{\beta} [mx - m_i(x - c_0)] \quad (49)$$

and

$$\eta = \frac{mx - \beta y}{mx - m_i(x - c_0)} \quad (50)$$

Substitution for η in equation (48) gives

$$u = B \sqrt{\frac{\beta y - m_i(x - c_0)}{mx - \beta y}} \quad (51)$$

Then the strength of the leading-edge singularity in u is

$$B \sqrt{mx - m_i(x - c_0)} \quad (52)$$

The leading-edge singularity in the loading on the swept-back wing is initially (region I, fig. 16) that in the triangular-wing loading. Introduction of the leading-edge corrections to the load, in region II, reduces the strength of the singularity there through the terms $R(t_0)$ and $R(t_a)$. (The inverse-cosine function is always finite.) The coefficient of $(mx - \beta y)^{-1/2}$ in u_Δ is, from equation (6),

$$\frac{mxu_0}{\sqrt{mx + \beta y}}$$

reducing to

$$C_\Delta = u_0 \sqrt{\frac{mx}{2}} \quad (53)$$

at the leading edge.

From equations (40) and (45), decrements to this coefficient may be derived for the portion of the leading edge just behind the intersection x_1, y_1 with the trailing-edge Mach line, as follows:

$$(\Delta C)_0 = \frac{-4mu_0K(k)}{\pi m_i K'(m_i)} \sqrt{\frac{mx}{1+m}} Z(\psi, k) \quad (54)$$

and, for each value of a from 0 to that value a_0' which makes τ_a equal to one,

$$\frac{d\Delta C}{da} da = \frac{-4m}{\pi^2 \sqrt{1+m}} \frac{u_a}{a} \sqrt{\frac{(m_i - a)x - (1-a)m_i c_0}{1-m_i}} K(k_a) \times \left[\sqrt{1+a} \frac{Z(\psi_a, k_a)}{k_a \sin \psi_a} - \sqrt{1-a} \frac{Z(\psi_0, k_a)}{k_a \sin \psi_0} \right] \quad (55)$$

where τ_0 (equation (39)) and τ_a (equation (44)) reduce to

$$\tau_0 = \frac{mx}{x - c_0}$$

and

$$\tau_a = \frac{(m_i - a)mx - m_i c_0 a}{(m_i - a)x - m_i c_0}$$

and the arguments and moduli of the elliptic integrals follow as for equations (40) and (45).

The coefficient of $(mx - \beta y)^{-1/2}$ at the leading edge is, therefore, in region II, figure 16,

$$C_\Delta + (\Delta C)_0 + \int_0^{a_0'} \frac{d\Delta C}{da} da \quad (56)$$

with a_0' reducing to

$$a_0' \left(x, \frac{m}{\beta} x \right) = m_i \frac{c_0 - (1-m)x}{m_i c_0 - (1-m)x} \quad (57)$$

Equating the two coefficients, expressions (56) and (52), gives for any one section

$$B = \frac{1}{\sqrt{mx - m_i(x - c_0)}} \left[C_\Delta + (\Delta C)_0 + \int_0^{a_0'} \frac{d\Delta C}{da} da \right] \quad (58)$$

For convenience, a nondimensional coefficient

$$\sigma(x) = \frac{1}{V\alpha\sqrt{c_0}} \left[C_\Delta + (\Delta C)_0 + \int_0^{a_0'} \frac{d\Delta C}{da} da \right] \quad (59)$$

is defined, so that

$$B = V\alpha \frac{\sigma(x)\sqrt{c_0}}{\sqrt{mx - m_i(x - c_0)}}$$

By substituting for B in equation (51), the loading on the outer portions of a swept-back wing is obtained as

$$\frac{u}{V\alpha} = \sigma(x) \sqrt{\frac{c_0[\beta y - m_i(x - c_0)]}{[mx - m_i(x - c_0)](mx - \beta y)}} \quad (60)$$

Numerical results.—The closeness with which the foregoing procedure predicts the theoretical loading over swept-back wings is indicated by figures 20, 21, and 22, where the previously calculated load distributions are compared with those calculated by equation (60). Even in the case of the highly tapered wing, the agreement is seen to be good. At the most inboard section of the $m=0.2$ wing (fig. 19 (a)) there is, of course, no agreement over that portion, forward of the 60-percent-chord point, where the flow is essentially conical. At station B-B, however, the agreement is very good. At sections C-C and D-D, where the exact theoretical

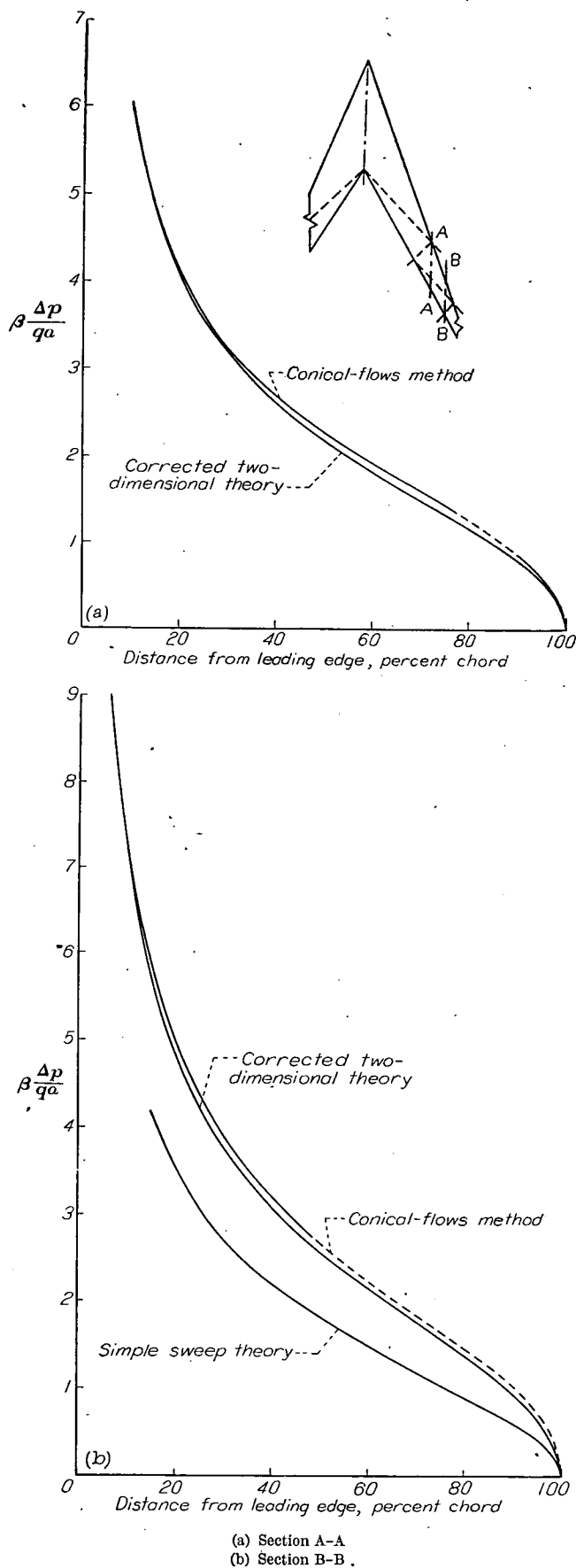


FIGURE 20.—Load distributions on the tapered wing as calculated by the conical-flows method, compared with the two-dimensional approximation.

loading had not been determined, the two-dimensional-type loading lies within the band prescribed by the conical-flow calculations. Since the discrepancy between the corrected two-dimensional loading and the exact theoretical distribution is already, at section B-B (fig. 22 (b)), less than the width of the bands in figures 22 (c) and (d) and must diminish to zero at infinity, it may be supposed that the corrected two-dimensional curve is at least as satisfactory an approximation to the correct curves at sections C-C and outboard as at section B-B. It is probably more satisfactory than can be obtained by a limited application of the conical-flow method.

The load distributions derived by simple sweep theory are included in the last part of each figure to show the magnitude of the plan-form effect and also, in the case of the untapered wings, the curves that the load distributions must

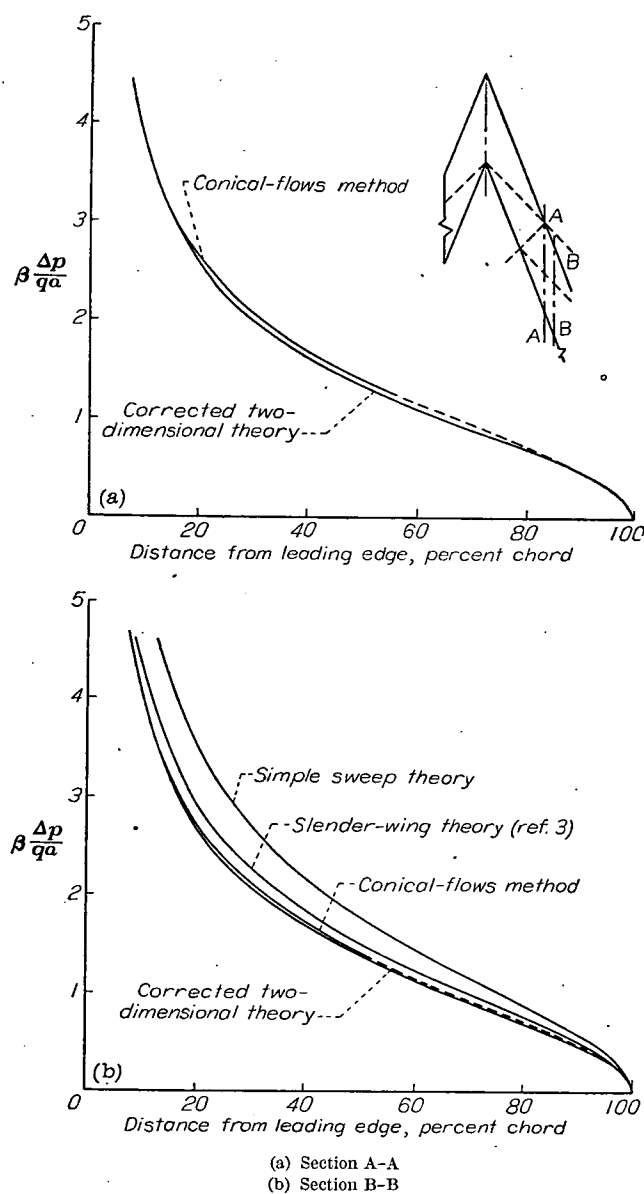
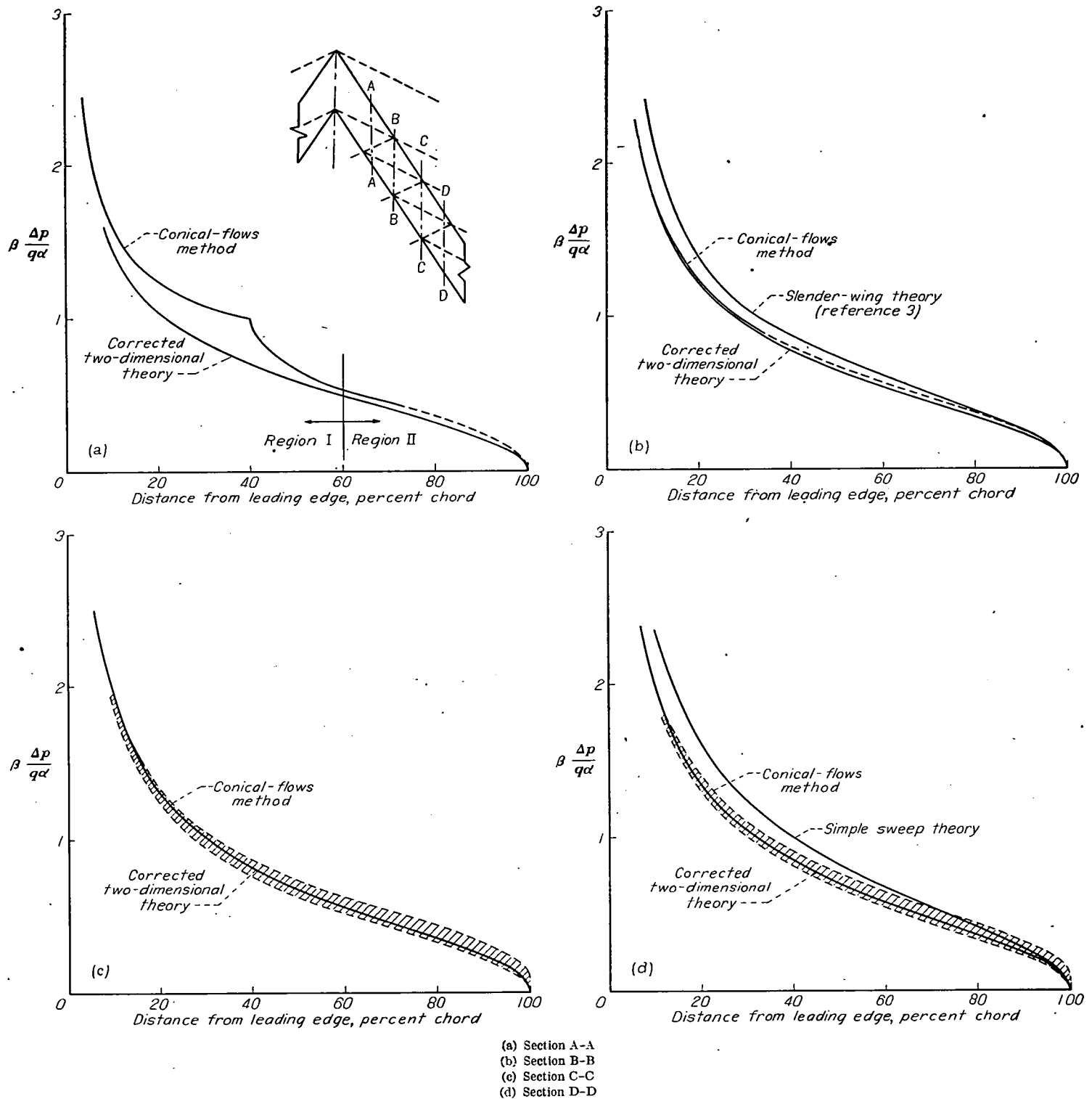


FIGURE 21.—Load distributions on the untapered wing, $m=0.4$, as calculated by the conical-flows method, compared with the two-dimensional approximation.


 FIGURE 22.—Load distributions on the untapered wing, $m=0.2$, as calculated by the conical-flows method, compared with the two-dimensional approximation.

approach as the distance from the plane of symmetry is increased. In figures 21 (b) and 22 (b), comparison is also made with results of the slender-wing theory of reference 3.

Discussion of the σ function.—In the calculation of the pressure coefficient at points toward the rear of most of the sections considered in figures 20, 21, and 22, it was necessary to find $\sigma(x)$ for values of x greater than x_3 (fig. 16). In deriving $\sigma(x)$, it was mentioned that expression (56) applied to region II. In region III, the strength of the leading-edge

singularity is affected by further modifications of the flow taking place in region IIb, so that additional terms in $\sigma(x)$ should be considered when x is greater than x_3 . Evaluation of these terms by presently known methods would require, as suggested earlier, the aid of high-speed computing machinery. However, the successive terms are all initially zero and enter with zero slope at x_3 , zero slope and curvature at x_5 , and so on, so that the three-term expression for σ given by equation (59) may be used with satisfactory accuracy

for some distance beyond the last value of x for which it is strictly valid. In practice, the third term in equation (58) may also be neglected for values of x only slightly greater than x_1 .

Charts have been prepared (fig. 23) giving $\left(\frac{\beta}{m} \sqrt{\frac{1-m}{m}}\right) \sigma$ as a function of $\frac{x-x_1}{c_0}$ for several values of the ratio m/m_t . This last parameter is the ratio of the tangents of the semi-apex angles of the leading and trailing edges and is constant

through the Mach number range for any one wing. The value of x_1 is readily determined:

$$x_1 = \frac{c_0}{1-m} \quad (61)$$

The curves were computed using equation (59) and are therefore exact only up to $x=x_3$ (shown by a vertical mark on each curve). Cross marks are drawn at the points $x=x_5$ to indicate a more practical limit to which use of the curves

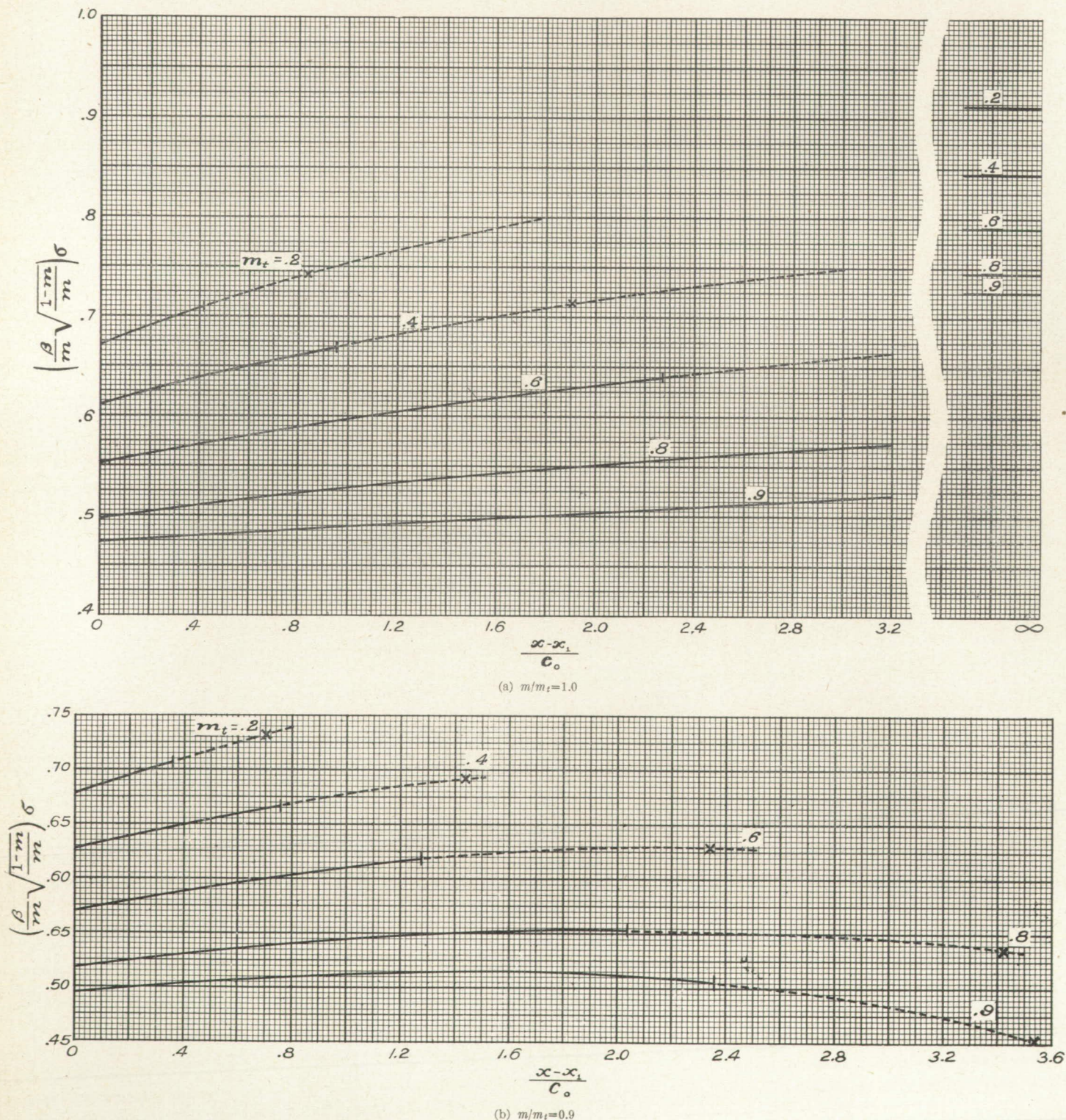


FIGURE 23.—Charts for determining σ , the strength of the leading-edge singularity.

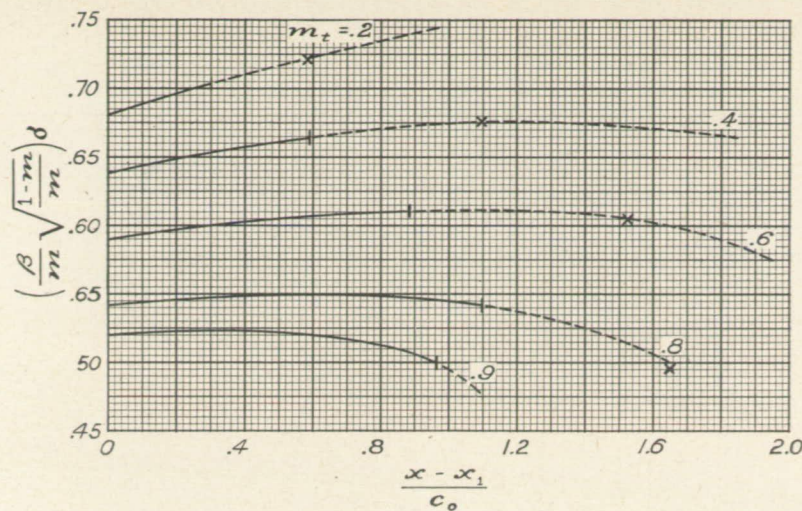
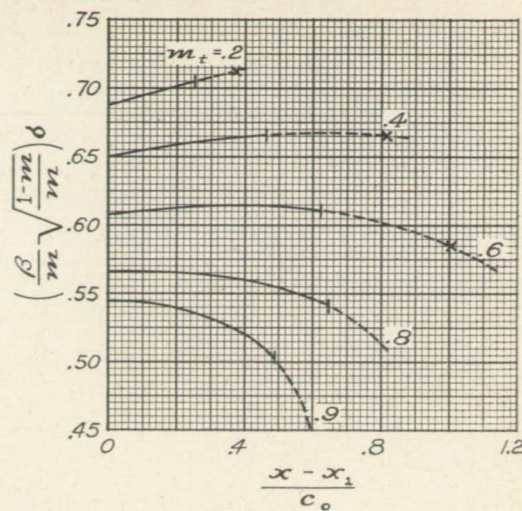
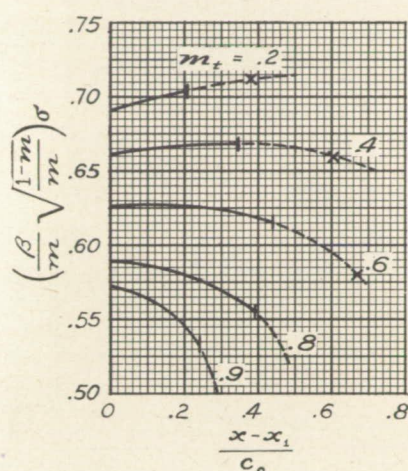
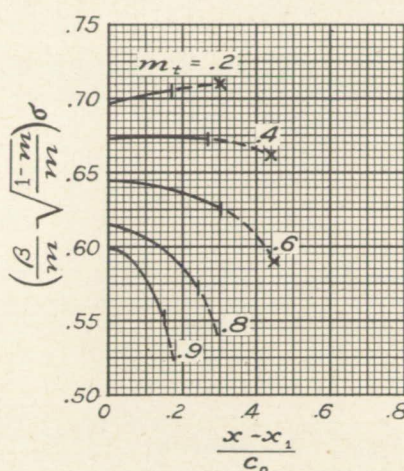
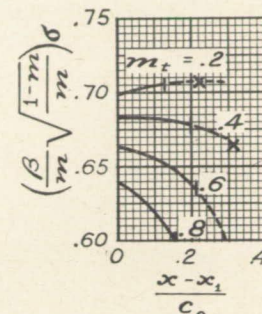

 (c) $m/m_t=0.8$

 (d) $m/m_t=0.7$

 (e) $m/m_t=0.6$

 (f) $m/m_t=0.5$

 (g) $m/m_t=0.4$

FIGURE 23—Continued.

may be extended. (These points are off the scale for $m_t=0.8$ and 0.9 in figure 23 (a).) When the wings are untapered ($m/m_t=1.0$), asymptotes

$$\frac{\beta \sigma(\infty)}{m} \sqrt{\frac{1-m}{m}} = \frac{1}{\sqrt{1+m}}$$

derived from simple sweep theory, may be drawn.

The curves, for the most part, are regular enough to permit interpolation within intervals of 0.2 in m_t . However, at $m_t=1.0$ the lines diminish to a point on the vertical axis; a curve for $m_t=0.9$ was therefore inserted in the charts for values of m/m_t equal to or greater than 0.5 . When m/m_t is less than 0.5 , $m=0.9$ represents, if the leading edge extends beyond x_1, y_1 , such extreme taper that the successive reflection of the Mach lines (at $x_3, x_5, \dots$) take place within a very small fraction of a chord length and no useful curve can be drawn. No curves are drawn for values of m_t smaller than 0.2 because of the tip-interference limitation mentioned in the introduction.

Calculation of tip effect.—The foregoing assumption of two-dimensional flow can be extended to give fairly simple

approximate formulas for the tip effect on a high-aspect-ratio wing. It is assumed that the velocity distribution to be canceled in the stream outboard of the tip is cylindrical; that is, is an extension of the velocity distribution calculated for the tip section along lines parallel to the leading edge. For this purpose the approximate load distribution given by equation (60) is used, still further simplified by assuming σ to remain constant at its value at the leading edge of the tip section. (Where the wing is tapering to a point and σ is changing very rapidly, the tip region is so small that the entire calculation of tip effects could probably be omitted.)

The assumption of constant σ results in a failure to cancel exactly the lift along the tip. The assumption of cylindrical flow, while reasonable for the untapered wing (compare fig 21 (a) with fig. 21 (b), for example) would appear to be too drastic for the tapered wing, where neither the chord nor the loading remains constant. However, as has been mentioned earlier, the major part of the tip effect results from the cancellation of the infinite pressure along the leading edge, and this part will be accurately calculated. The effect of the residual lift on the rearward portion of the tip section and in the stream should be small.

The distribution of perturbation velocity at the tip station $y=s$, with the simplification of constant σ , is, from equation (60), approximately

$$u(x_c, s) = \sigma_s V \alpha \sqrt{\frac{[\beta s - m_t(x_c - c_0)] c_0}{[m x_c - m_t(x_c - c_0)](m x_c - \beta s)}} \quad (62)$$

where x_c, s are the coordinates of a point on the tip and σ_s is the value of σ at the leading edge of the tip section.

This expression may be more conveniently written in terms of the parameter

$$\mu = 1 - \frac{m}{m_t} \quad (63)$$

and the variable

$$\xi_c = \frac{x_c - \frac{\beta s}{m}}{c_t} \quad (64)$$

which is the distance of x_c, s from the leading edge (see fig. 24) expressed as a fraction of the tip chord c_t . Since

$$c_t = \left(c_0 + \frac{\beta s}{m_t}\right) - \frac{\beta s}{m}$$

equation (62) may be written

$$u(\xi_c, s) = \frac{\sigma_s V \alpha}{\sqrt{m \lambda}} \sqrt{\frac{1 - \xi_c}{(1 - \mu \xi_c) \xi_c}} \quad (65)$$

where λ is the taper ratio c_t/c_0 .

If the velocity distribution u is assumed to be constant beyond $y=s$ along lines parallel to the leading edge, it may be canceled by the superposition of conical flow fields of which the constant-velocity regions have one edge along the tip and the other parallel to the leading edge, with apexes displaced along the tip by increments in ξ_c . The velocity induced at a point x, y by each such element would be (equation (11))

$$\frac{u_c}{\pi} \cos^{-1} \frac{m + t_c + 2mt_c}{t_c - m} \quad (66)$$

where

$$t_c = \frac{\beta(y - s)}{x - x_c} \quad (67)$$

and u_c is the velocity on each sector.

Following the procedure used in deriving equation (14), the corresponding equation may be written for the pressures induced by canceling the cylindrical flow

$$\left(\frac{\Delta u}{V \alpha}\right)_{tip} = \frac{-\sqrt{m(1+m)(x-x_0)}}{\pi V \alpha} \int_{\frac{\beta s}{m}}^{x_0} \frac{u(x_c, s) dx_c}{[(1+m)x - x_0 - m x_c] \sqrt{x_0 - x_c}} \quad (68)$$

where x_0, s is the intersection of the Mach forecone from x, y with the tip.

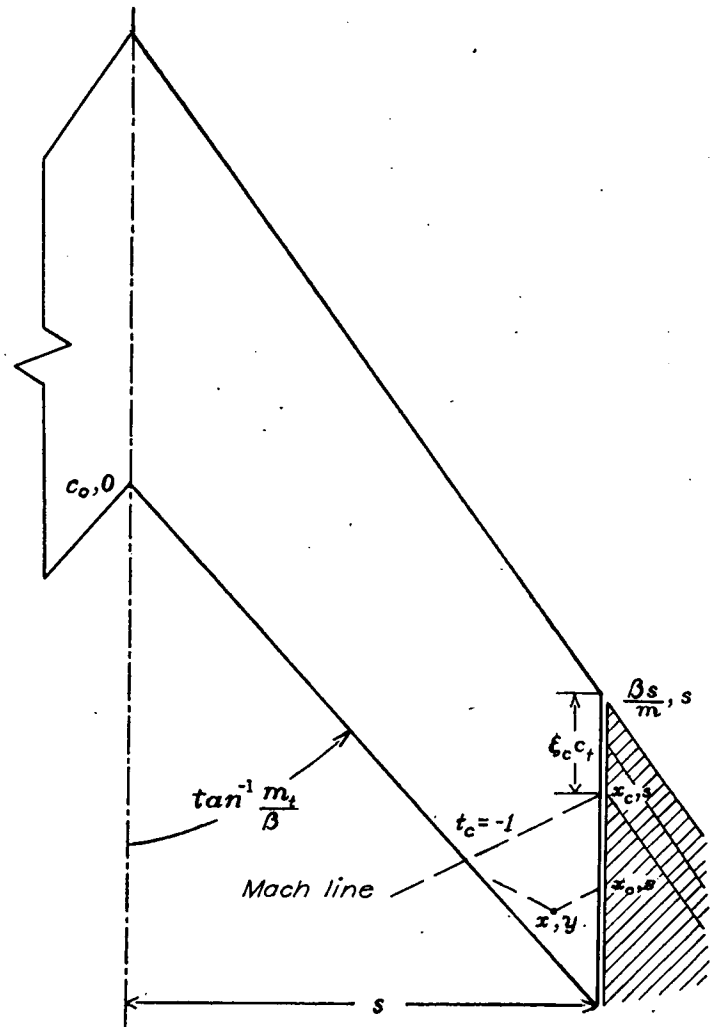


FIGURE 24.—Sketch for derivation of approximate tip correction to loading at x, y .

If the distances of x, y and x_0, s back of the leading edge, measured as fractions of the tip chord, are

$$\xi = \frac{1}{c_t} \left(x - \frac{\beta y}{m}\right) \quad (69)$$

and

$$\xi_0 = \frac{1}{c_t} \left(x_0 - \frac{\beta s}{m}\right) \quad (70)$$

it can be shown that

$$(1+m)(x-x_0) = m c_t (\xi - \xi_0) \quad (71)$$

from which equation (68) can be written (with the substitution for $u(x_c, s)$ from equation (65))

$$\left(\frac{\Delta u}{V \alpha}\right)_{tip} = -\frac{\sigma_s \sqrt{\xi - \xi_0}}{\pi \sqrt{m \lambda}} \int_0^{\xi_0} \frac{d\xi_c}{\xi - \xi_c} \sqrt{\frac{1 - \xi_c}{(\xi_0 - \xi_c)(1 - \mu \xi_c) \xi_c}} \quad (72)$$

In integrating equation (72), three cases must be distinguished: (1) $\xi < 1$ (always true for the untapered wing),

(2) $1 < \xi < \frac{1}{\mu}$ (when the point x, y lies more than a tip-chord length behind the leading edge), and (3) $\xi > \frac{1}{\mu}$

(a possible condition for some points near the trailing edge of a highly swept or tapered wing).

In the first case ($\xi < 1$)

$$\left(\frac{\Delta u}{V\alpha}\right)_{tip} = \frac{-\sigma_s}{\sqrt{m\lambda}} \left[\frac{m}{m_t(1-\mu\xi)} \sqrt{\frac{\xi-\xi_0}{1-\mu\xi_0}} K_0 + \sqrt{\frac{1-\xi}{\xi(1-\mu\xi)}} \Lambda_0(\psi_1, k) \right] \quad (73a)$$

where

$$k = \sqrt{\frac{m\xi_0}{m_t(1-\mu\xi_0)}}$$

$$\psi_1 = \sin^{-1} \sqrt{\frac{(1-\xi)(1-\mu\xi_0)}{(1-\xi_0)(1-\mu\xi)}}$$

and Λ_0 is the function (equation (16)) plotted in figure 6.

In the second case ($1 < \xi < \frac{1}{\mu}$)

$$\left(\frac{\Delta u}{V\alpha}\right)_{tip} = -\frac{\sigma_s}{\sqrt{m\lambda}} \frac{K_0}{\xi \sqrt{1-\mu\xi_0}} \left[\sqrt{\xi-\xi_0} - \sqrt{(\xi-1)\xi_0} \frac{Z(\psi_2, k)}{k \sin \psi_2} \right] \quad (73b)$$

where

$$\psi_2 = \sin^{-1} \sqrt{\frac{m_t(1-\mu\xi)}{m\xi}}$$

and Z is the function (equation (41)) plotted in figure 14.

In the third case ($\xi > \frac{1}{\mu}$),

$$\left(\frac{\Delta u}{V\alpha}\right)_{tip} = \frac{-\sigma_s}{\sqrt{m\lambda\xi}} \frac{\Lambda_0(\psi_3, k)}{\sin \psi_3} \quad (73c)$$

where

$$\psi_3 = \sin^{-1} \sqrt{\frac{\mu\xi-1}{\xi-1}}$$

Along the Mach line from the leading-edge tip all three equations reduce to the value

$$\frac{\Delta u^*}{V\alpha} = \frac{-\sigma_s}{\sqrt{m\lambda\xi}} \quad (74)$$

By the procedure just described, approximate cancellation of all pressure differences outboard of the tip has been effected, but the pressures induced by such cancellation now violate the condition of zero lift in the wake. Approximate cancellation of the induced pressure differences in the wake region can be accomplished, as before, by making use of the known value of the tip-induced velocity at the trailing edge of the wing, but assuming the entire error to originate at the leading edge of the tip. Equation (31) is directly applicable, with Δu^* given by equation (74) and $(\Delta u)_{tip}$ by

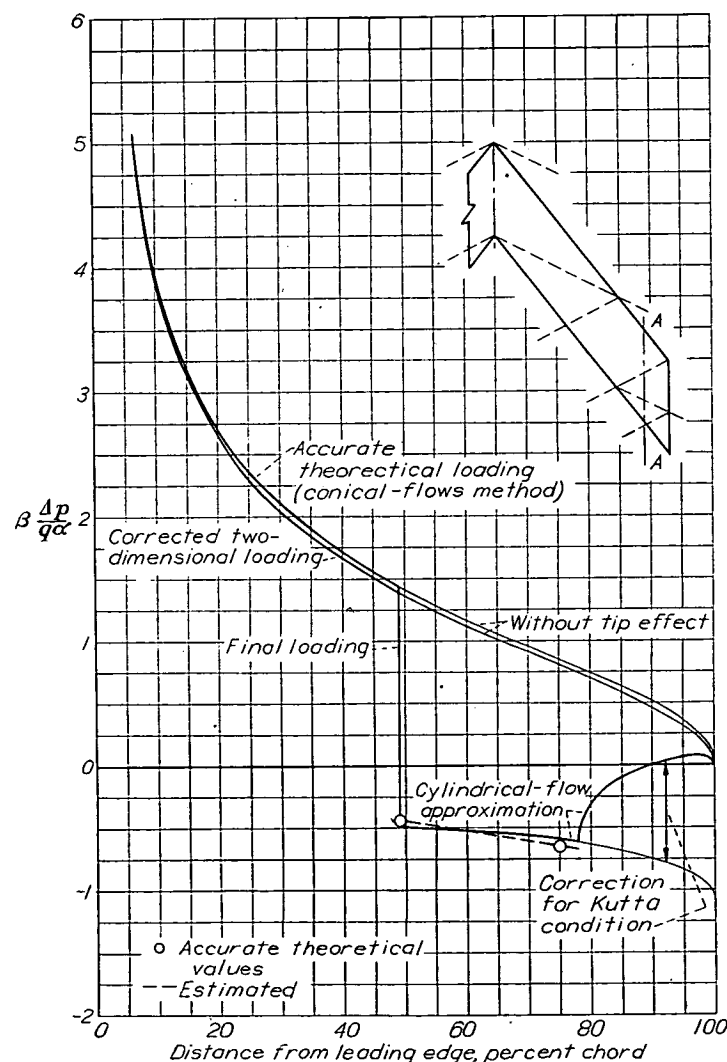
equation (73). On the trailing edge of an untapered wing, $\xi=1$ and

$$\left(\frac{\Delta u}{V\alpha}\right)_{\xi=1} = \frac{-\sigma_s}{\sqrt{m}} k' K_0 \quad (75)$$

There is no corresponding simplification for the tapered wing.

Numerical examples, tip effect.—Equations (73) and (31) have been used to calculate the tip effect in two cases, namely: $m=m_t=0.4$, $\beta s=0.94c_0$; and $m=0.4$, $m_t=0.6$, $\beta s=0.86c_0$. The tip effect has been calculated for each wing at $\beta y=0.8c_0$, where the loading was previously calculated (figs. (17b) and (18b)) assuming the wing to extend indefinitely. The tip locations were selected so that in each case only one reflection of the primary tip effect affected the section at $\beta y=0.8c_0$.

Figure 25 shows the results of the calculations. The heavy solid curve in each case was calculated entirely by the corrected two-dimensional theory—that is, by equations (60), (73), and (31). As a check on the accuracy of the cylindrical-flow approximation for the flow outboard of the tip location, the accurate theoretical loading was calculated



(a) Untapered wing; $m=0.4$; $\beta A=1.88$. Section at $\beta y=0.8c_0$, or 85 percent semispan

FIGURE 25.—Load distribution over streamwise section near tip as calculated by two-dimensional formulas, compared with more accurate theoretical values.

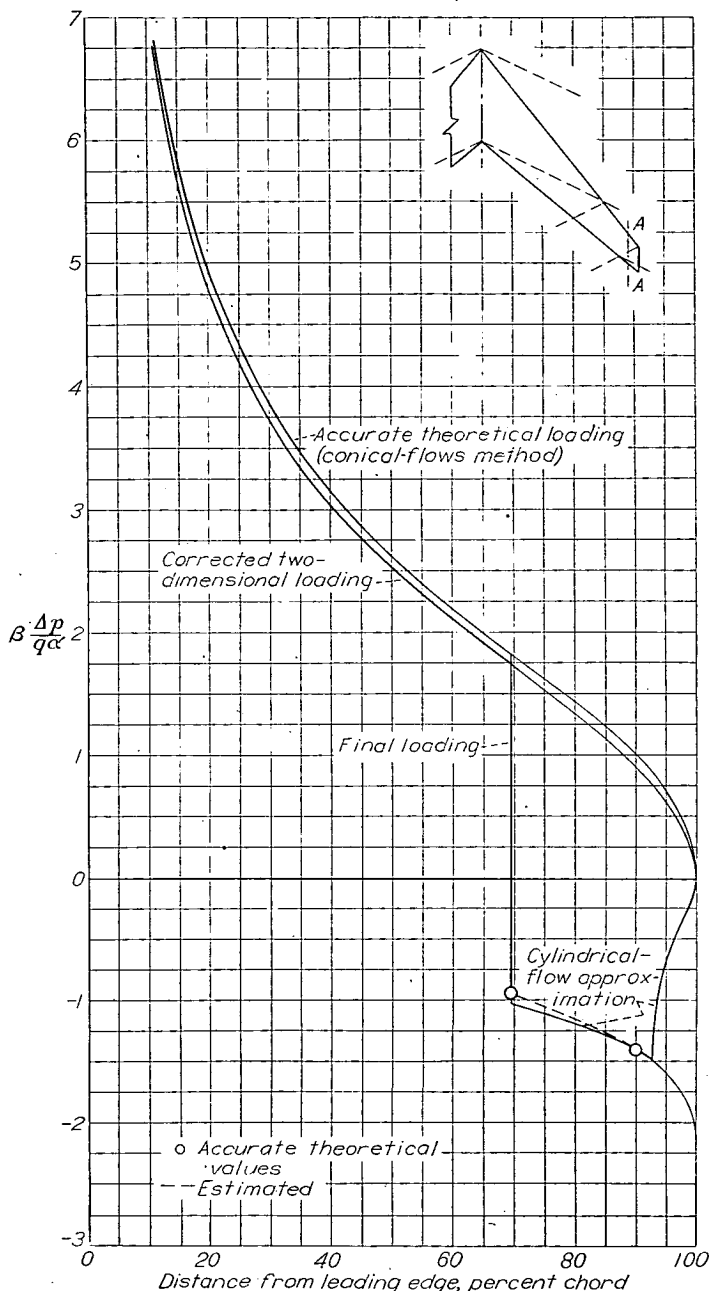
(b) Tapered wing; $m=0.4$; $m_t=0.6$. Section at $\beta y=0.8c_o$, or 93 percent semispan

FIGURE 25—Continued

for one point within the region of influence of the tip in each case. The procedure employed for the exact calculation was as follows:

The accurate loadings with no side-edge effects had already been calculated, as has been noted, by the conical-flow method. A primary tip correction was calculated for each case by equation (15). This correction is the effect of canceling the unmodified triangular-wing loading off the tip station. The remaining pressure differences to be canceled consisted of those introduced by the leading-edge and trailing-edge corrections. These pressures were computed by means of equations (26) and (46) of the present report and canceled by the method of reference 5.

The results are designated by the circled points on each figure. At the point at which the section enters the tip Mach cone in each case, a second circled point indicates the accurate theoretical loading. The value differs from that calculated by the approximate formulas only as the two loadings without tip effects differ.

It may be pointed out in concluding this section on load calculations that, while the formulas have been developed for plan forms with streamwise tips, the procedure may be adapted by obvious means to raked tips as well. However, in every case the deviation in the tip regions of the physical flow from the assumed potential flow must be borne in mind.

III—LIFT⁶

GENERAL PROCEDURE FOR CONICAL FLOWS

The total lift for any wing is, of course, the integral of the loading over the wing area. In general, however, it is difficult to obtain an analytic expression for the lift by a direct integration of the lift distribution. In the conical-flow method, advantage may be taken of the simplicity of the component fields by integrating the lift associated with each one and then combining the results in the same way as the pressure fields.

Conical elements of area are employed for the integrations. These are infinitesimal triangles bounded by two adjacent rays of the conical field and the intercepted boundary of the wing plan form. Over each of the infinitesimal triangles the velocity u of the conical field will be constant. Thus it remains only to perform a single integration, with respect to the conical variable of the field, to obtain the total lift associated with that field.

GENERAL FORMULA FOR THE LIFT INDUCED BY A SINGLE TIP ELEMENT

The lift $(\Delta L)_a$ induced on the wing by a single canceling tip element is obtained first. Although the notation of the solution (equation (11)) used to cancel the triangular-wing loading is employed, the derivation will hold generally for any canceling element bounded on one side by the tip of a swept-back wing, since no use is made of the fact that the other boundary of the element passes through the origin of the x, y axes. We write

$$(\Delta L)_a = 2\rho V \int_{-1}^0 (\Delta u)_a \frac{dS}{dt_a} dt_a \quad (76)$$

where $(\Delta u)_a$ (equation (11)) is the streamwise increment of velocity induced by the canceling field and $\frac{dS}{dt_a} dt_a$ (fig. 26) is the element of wing area S for integration. For simplicity it will be specified that the Mach cone from the apex of the element does not include the apex of the trailing edge nor any part of the opposite tip. Then (see fig. 26)

$$\frac{dS}{dt_a} = \frac{m_t^2}{2\beta} \left(\frac{x_t - x_a}{m_t - t_a} \right)^2 \quad (77)$$

⁶ It may be noted that, as a result of the reversibility property (reference 17), the formulas for the lift given herein for swept-back wings are equally applicable to the swept-forward wings having the same plan forms but reversed in heading.

When $m_t = m$, this reduces to

$$\frac{L_0}{q\alpha} = \frac{8s^2}{3ma_t^2} \frac{\beta u_0}{V\alpha} [(m+a_t)\sqrt{m^2-a_t^2} - (m-a_t)] \quad (85b)$$

It should be noted that, for a given plan form, $\frac{L_0}{q\alpha}$ varies with Mach number only as u_0 .

WING WITH SUPERSONIC TRAILING EDGE (TIP CORRECTION)

Proceeding to the calculation of the tip correction to the lift, we integrate the change in lift $(\Delta L)_a$ (equation (78)) induced by each element a over the range $a_t \leq a \leq m$. The quantity $u_\Delta(m)$ is substituted for u_a of the initial canceling element and $\frac{du_\Delta}{da} da$ for u_a for the remaining ones. As in calculating the tip-induced pressure correction, the difficulty is encountered that $u_\Delta(m)$ is infinite, and therefore the total lift correction must be written in terms of limiting values. Following the substitution

$$x_t - x_a = \beta s \left(\frac{1}{a_t} - \frac{1}{a} \right) \quad (86)$$

in equation (78), it is convenient to define the function

$$G(a) = \frac{(a-a_t)^2}{a_t^2 a^2} g(a)$$

Then the total induced lift may be written

$$\Delta L = 2\rho V m_t^2 \beta s^2 \lim_{a \rightarrow m} \left[-u_\Delta(a)G(a) + \int_{a_t}^a \frac{du_\Delta}{da} G(a) da \right] \quad (87)$$

Integrating by parts results in cancellation of the first term inside the brackets. Since $G(a_t)$ is zero, equation (87) reduces to

$$\left(\frac{\Delta L}{q\alpha} \right)_{tip} = -4m_t^2 \beta s^2 \int_{a_t}^m \frac{u_\Delta(a)}{V\alpha} G'(a) da \quad (88a)$$

$$\begin{aligned} \left(\frac{\Delta L}{q\alpha} \right)_{tip} = & -\frac{4s^2}{3ma_t^2} \frac{\beta u_0}{V\alpha} \left(2(m+a_t)\sqrt{m^2-a_t^2} + \sqrt{\frac{ma_t(m^2-a_t^2)}{(1+m)(1+a_t)}} \left\{ \frac{3}{m}(m-a_t) - 2m + \frac{(m-a_t)^2}{m(1+m)^2} \left[\frac{2}{a_t} - \frac{(1-m)^3}{2(m+a_t)} \right] - \right. \right. \\ & \left. \left. \frac{2m^2(1+a_t)^3}{a_t(1+m)^2(m+a_t)} - \frac{m+a_t}{2} \right\} + \sqrt{\frac{2}{1+m}} \left\{ \frac{(m-a_t)(m+2a_t)}{ma_t} F(\psi, k) - \left[\frac{2(m+a_t)}{a_t} - \frac{a_t}{m} - \frac{(m-a_t)^2}{2ma_t(1+m)} \right] E(\psi, k) \right\} \right) \end{aligned} \quad (88c)$$

where

$$\psi = \sin^{-1} \sqrt{\frac{m-a_t}{m(1+a_t)}} \text{ and } k = \sqrt{\frac{1-m}{2}}$$

The primary tip correction, however, is usually quite large. It may therefore be desirable to take into account the secondary correction resulting from its cancellation at the subsonic trailing edge. Rather than compute a single secondary correction to the lift, as an additional item, it is again found advantageous to treat each superposed field individually, that is, to cancel each conical tip field at the trailing edge and find the net effect on the lift, then integrate over all the tip elements for a combined primary and secondary tip correction.

⁷ Or see reference 10 for an approximate formula valid when m is close to one.

where

$$G'(a) = \frac{a-a_t}{a_t^2 a^2} \left[\left(\frac{a_t}{a} + \frac{m_t-a_t}{m_t-a} \right) g(a) - \frac{a-a_t}{2(m_t-a)\sqrt{(m_t+m_t^2)(a+a^2)}} \right]$$

is the derivative of $G(a)$.

Equation (88a) has been integrated (appendix C) in terms of an incomplete elliptic integral of the third kind. If the necessary tables are not available, it may be preferable to integrate numerically.⁷ In that case it is noted that

$$\frac{1}{\sqrt{m^2-a^2}} = \frac{d}{da} \sin^{-1} \frac{a}{m}$$

and $u_\Delta da$ is rewritten as

$$mu_0 d \left(\sin^{-1} \frac{a}{m} \right)$$

Equation (88a) then becomes

$$\left(\frac{\Delta L}{q\alpha} \right)_{tip} = -4m_t^2 m \beta s^2 \frac{u_0}{V\alpha} \int_{\sin^{-1} \frac{a_t}{m}}^{\frac{\pi}{2}} G'(a) d \left(\sin^{-1} \frac{a}{m} \right) \quad (88b)$$

In this way infinite values in the integrand are avoided.

WING WITH SUBSONIC TRAILING EDGE

The expressions (equations (85)) for the uncorrected lift apply regardless of whether the trailing edge is subsonic or supersonic. The formulas for the tip correction may serve as a first approximation when the trailing edge is subsonic if the accuracy of a second correction is not required. For that purpose the special value for the untapered wing will be of interest:

If the wing is untapered the elliptic integrals in equation (88a) (see appendix C) reduce to the first and second kind and the primary induced lift may be written in the following closed form:

Tip correction with subsonic trailing edge.—For the cancellation at the trailing edge of a pressure field originating at a point x_a, s on the wing tip, equation (82) is applied, with the parameter a , which defines one boundary of the oblique canceling field, replaced by t_a , referring to a ray from x_a, s . The velocity u_a is the gradient

$$\frac{d\Delta u}{dt_a} dt_a$$

of the field (equation (11)) to be canceled. The distance from the apex of the canceling field to the wing tip is expressible as

$$-\frac{x_t - x_a}{\beta} \frac{m_t t_a}{m_t - t_a}$$

Then the effect on the lift of canceling the single field from x_a, s is

$$(\Delta_2 L)_a = -\rho V m_i^2 (x_i - x_a)^2 \int_{-1}^0 \frac{t_a}{(m_i - a)^2} \frac{d\Delta u}{dt_a} \times \left[\sqrt{\frac{(m_i - t_a)(1 - t_a)}{m_i}} - \frac{m_i - t_a}{m_i} \right] dt_a \quad (89)$$

which may be integrated, after substitution for $\frac{d\Delta u}{dt_a}$, to give

$$(\Delta_2 L)_a = -\rho V m_i^2 u_a \frac{(x_i - x_a)^2}{\beta(m_i - a)} \left\{ \sqrt{\frac{a(1+a)}{m_i(1+m_i)}} [1 - E_0(k)] - \frac{a}{m_i} \left[1 - \frac{\Lambda_0(\psi, k)}{\sin \psi} \right] \right\} \quad (90)$$

in which

$$k = \sqrt{\frac{1 - m_i}{1 + m_i}}$$

and

$$\psi = \sin^{-1} \sqrt{\frac{m_i - a}{m_i(1 - a)}}$$

If the foregoing result is added to the lift associated with the original tip element, given by equations (78) and (79), it is found that the latter lift is exactly canceled by the algebraic terms in the reflected lift, leaving

$$(\Delta L)_a = \rho V m_i^2 u_a \frac{(x_i - x_a)^2}{\beta(m_i - a)} \left[\sqrt{\frac{a(1+a)}{m_i(1+m_i)}} E_0 - \frac{a}{m_i} \frac{\Lambda_0(\psi, k)}{\sin \psi} \right] \quad (91)$$

for the lift induced by one tip element and its cancellation at the trailing edge. It will generally be found that further steps in the cancellation process are unnecessary for engineering accuracy.

For the total tip-induced correction to the lift, it is necessary to write as before

$$(\Delta L)_{itp} = 2\rho V m_i^2 \beta s^2 \lim_{a \rightarrow m} \left[-u_\Delta(a) J(a) + \int_a^m \frac{du_\Delta}{da} J(a) da \right] \quad (92)$$

where $J(a)$ is

$$\frac{(a - a_i)^2}{a_i^2 a^2} \frac{1}{m_i - a} \left[\sqrt{\frac{a(1+a)}{m_i(1+m_i)}} E_0 - \frac{a}{m_i} \frac{\Lambda_0(\psi, k)}{\sin \psi} \right]$$

An integration by parts reduces equation (92) to

$$(\Delta L)_{itp} = -2\rho V m_i^2 \beta s^2 \int_{a_i}^m u_\Delta(a) J'(a) da \quad (93)$$

with

$$J'(a) = \frac{a - a_i}{a_i^2 a^2} \frac{1}{m_i - a} \left\{ \left[\left(\frac{a_i}{a} + \frac{m_i - a_i}{m_i - a} \right) - \frac{a - a_i}{2(1+a)} \left(\frac{1}{a} - \frac{1+m_i}{m_i - a} \right) \right] \sqrt{\frac{a(1+a)}{m_i(1+m_i)}} E_0(k) - \frac{a - a_i}{1 - a^2} \sqrt{\frac{a(1+a)}{m_i(1+m_i)}} K_0(k) - \frac{a}{m_i} \left[\left(\frac{a_i}{a} + \frac{m_i - a_i}{m_i - a} \right) + \frac{(1 - m_i)(a - a_i)}{2(1 - a)(m_i - a)} \right] \frac{\Lambda_0(\psi, k)}{\sin \psi} \right\} \quad (94)$$

If the wing is untapered, $J'(a)$ becomes indeterminate when $a = m$. The limiting value is

$$J'(m) = \frac{m - a_i}{m^3 a_i^2 (1 - m^2)} \left\{ \left[\frac{1 - 3m}{3} \left(\frac{a_i}{m} + \frac{1 - a_i}{1 - m} \right) - \frac{m - a_i}{10m} \left(3 - \frac{8m^2}{1 - m^2} \right) \right] E_0(k) + 2m \left[\frac{1}{3} \left(\frac{a_i}{m} + \frac{1 - a_i}{1 - m} \right) - \frac{m - a_i}{10m} \left(\frac{1 + 3m}{1 - m^2} \right) \right] K_0(k) \right\}$$

Further integration must be performed numerically. In order to avoid infinite values in the integrand, note again that

$$u_\Delta(a) = m u_0 \frac{d}{da} \sin^{-1} \frac{a}{m} \quad (95)$$

so that equation (93) may be rewritten

$$\left(\frac{\Delta L}{q\alpha} \right)_{itp} = -4m m_i^2 s^2 \frac{\beta u_0}{V\alpha} \int_{\sin^{-1} \frac{a_i}{m}}^{\pi/2} J'(a) d\left(\sin^{-1} \frac{a}{m} \right) \quad (96)$$

Trailing-edge corrections.—In deriving the trailing-edge corrections to the total lift, primary and secondary effects will again be combined. Further corrections will be omitted.

For the symmetrical wake correction, the element of area is obtained from equation (80) by setting y_a equal to zero, and substituting t_0 for t_a . Then the decrement in lift induced by the application of the symmetrical canceling element at the trailing edge is, from equation (20),

$$(\Delta_1 L)_0 = -2\rho V \beta s^2 \frac{u_0}{K'(m_i)} \int_{m_i}^1 F(\phi, \sqrt{1 - m_i^2}) \frac{dt_0}{t_0^2} \quad (97)$$

or

$$\frac{(\Delta_1 L)_0}{q\alpha} = \frac{-4s^2}{m_i} \frac{\beta u_0}{V\alpha} \left[1 - \frac{1}{K'_0(m_i)} \right] \quad (98)$$

where

$$K'_0(m_i) = \frac{2}{\pi} K(\sqrt{1 - m_i^2})$$

The effect of canceling the pressure field induced by the symmetrical wake correction at the wing tips is obtained with the aid of the previously derived formula (equation (78)) for the lift associated with a single tip element. The parameter defining the boundary of the canceling tip element is now t_0 instead of a , and the velocity on the canceling sector is

$$\frac{d(\Delta u)_0}{dt_0} dt_0 = \frac{u_0 dt_0}{K'(m_i) \sqrt{(1 - t_0^2)(t_0^2 - m_i^2)}} \quad (99)$$

The distance from the apex of the canceling sector to the trailing edge may be expressed as

$$\beta s \left(\frac{1}{m_i} - \frac{1}{t_0} \right) \quad (100)$$

so that the secondary effect of the symmetrical trailing-edge correction becomes

$$(\Delta_2 L)_0 = 2\rho V m_i^2 \beta s^2 \frac{u_0}{K'(m_i)} \int_{m_i}^1 \left(\frac{1}{m_i} - \frac{1}{t_0} \right)^2 \frac{g(t_0) dt_0}{\sqrt{(1-t_0^2)(t_0^2-m_i^2)}} \quad (101)$$

or

$$\frac{(\Delta_2 L)_0}{q\alpha} = \frac{4s^2 \beta u_0}{m_i V\alpha} \left[1 - \frac{1}{K_0'(m_i)} - \frac{2^{3/2}}{\sqrt{1+m_i}} \frac{K_0(k) - E_0(k)}{K_0'(m_i)} \right] \quad (102)$$

with $k = \sqrt{\frac{1-m_i}{2}}$.

Addition of this secondary correction to the primary effect given in equation (98) results in the single correction

$$\left(\frac{\Delta L}{q\alpha} \right)_0 = \frac{-8s^2 \beta u_0}{m_i V\alpha} \sqrt{\frac{2}{1+m_i}} \frac{K_0(k) - E_0(k)}{K_0'(m_i)} \quad (103)$$

By a similar procedure, the effect of canceling one of the oblique trailing-edge fields at the tip is readily obtained and added to the primary effect given by equation (82) to yield

$$(\Delta L)_a = \rho V \beta (s - y_a)^2 \frac{u_a}{a} \frac{1}{\sqrt{m_i}} \left[E_0(k) \sqrt{(m_i - a)(1 - a)} - \frac{m_i - a}{\sqrt{1 + m_i}} \sqrt{\frac{1 + a}{a}} \Lambda_0(\psi, k) \right] \quad (104)$$

with $k = \sqrt{\frac{1-m_i}{1+m_i}}$ and $\psi = \sin^{-1} \sqrt{\frac{(1+m_i)a}{m_i(1+a)}}$ as the combined primary and secondary correction to the lift due to a single oblique trailing-edge cancellation.

For the total correction to the lift due to cancellation of the gradient of the triangular-wing loading in the wake, equation (104) is integrated graphically or numerically across the span as follows:

$$\frac{\Delta L}{q\alpha} = \frac{-4m \beta u_0}{\sqrt{m_i} V\alpha} \int_0^{at} \frac{(s - y_a)^2}{(m^2 - a^2)^{3/2}} \left[E_0(k) \sqrt{(m_i - a)(1 - a)} - \frac{m_i - a}{\sqrt{1 + m_i}} \sqrt{\frac{1 + a}{a}} \Lambda_0(\psi, k) \right] da \quad (105)$$

Numerical examples to be presented will show this component of the lift to be very small, in general.

WING WITH INTERACTING LEADING AND TRAILING EDGES

In computing the load distribution it was found that, when interaction takes place between the flow fields of the leading and trailing edges, the wing plan form appears to comprise two principal regions separated (see fig. 16) by the Mach line arising at the point of intersection x_1, y_1 of the trailing-edge Mach line and the leading edge. Ahead of this line (region I) the flow is most readily described in terms of conical fields. Behind this line the flow is more nearly two-dimensional. On this basis, the total lift will be found in two parts, using for region I the conical-flow

expressions for the loading, and for the remainder of the wing the quasi-two-dimensional approximation.

LIFT ON INBOARD PORTION OF WING

The uncorrected triangular-wing loading will first be integrated over region I, shown shaded in figure 28. For

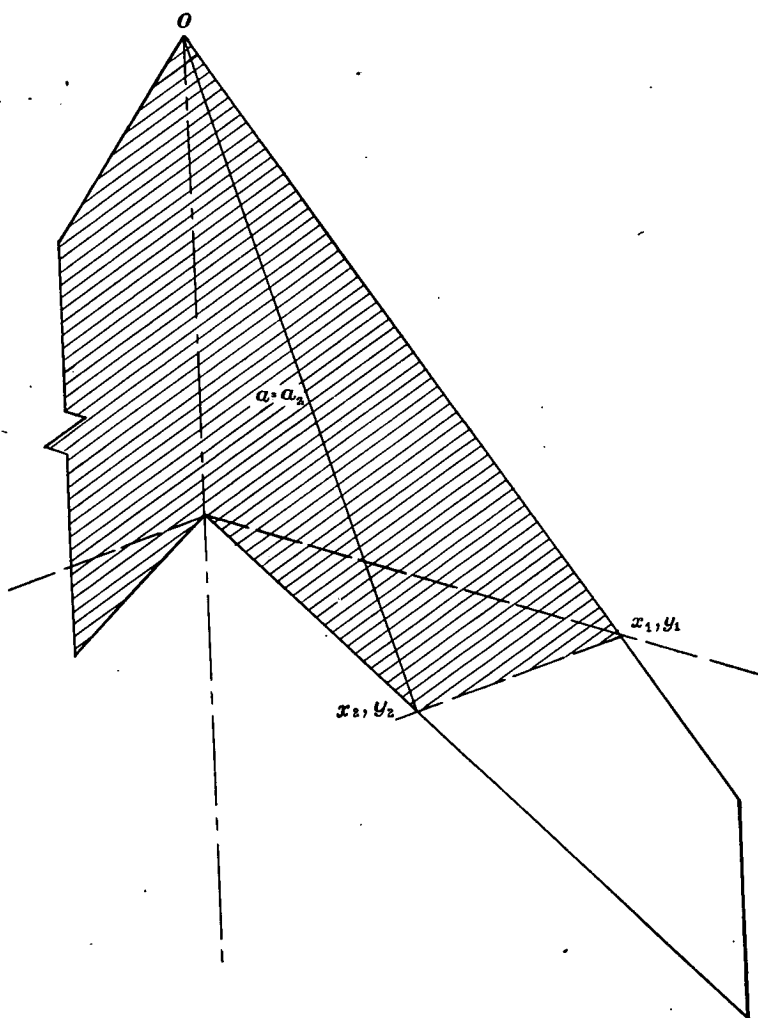


FIGURE 28.—Inboard portion (region I) of high-aspect-ratio wing.

this purpose the region is considered in two parts, separated by the ray a_2 from the wing apex to the point x_2, y_2 . When a is less than a_2 , the element of area is as before

$$\frac{m_i^2 c_0^2}{2\beta(m_i - a)^2} da$$

When $a > a_2$, the element of area is

$$\frac{(1+m)^2 c_0^2}{2\beta(1-m)^2(1+a)^2} da$$

Thus, the uncorrected lift in the entire shaded region is

$$L_0 = \frac{2\rho V c_0^2}{\beta} \left[\int_0^{a_2} \frac{m_i^2}{(m_i - a)^2} u_{\Delta} da + \int_{a_2}^m \frac{(1+m)^2}{(1-m)^2(1+a)^2} u_{\Delta} da \right] \quad (106)$$

or

$$\frac{L_0}{q\alpha} = \frac{4m}{\beta} \frac{u_0}{V\alpha} c_0^2 \left\{ \frac{1+m}{(1-m)^3} \left[\frac{1}{\sqrt{1-m^2}} \cos^{-1} \frac{m^2+a_2}{m(1+a_2)} - \frac{\sqrt{m^2-a_2^2}}{1+a_2} \right] + \frac{m_i^2}{m_i^2-m^2} \left(\frac{m}{m_i} - \frac{\sqrt{m^2-a_2^2}}{m_i-a_2} \right) + \left(\frac{m_i}{\sqrt{m_i^2-m^2}} \right)^3 \left[\cos^{-1} \frac{m^2-m_i a_2}{m(m_i-a_2)} - \cos^{-1} \frac{m}{m_i} \right] \right\} \quad (107)$$

with

$$a_2 = \frac{2mm_i}{1+m+m_i-mm_i} \quad (108)$$

When $m_i = m$ (untapered wing), the second part of equation (107) becomes indeterminate. In this case,

$$\frac{L_0}{q\alpha} = \frac{4m}{\beta} \frac{u_0}{V\alpha} c_0^2 \left\{ \frac{1+m}{(1-m)^3} \left[\frac{1}{\sqrt{1-m^2}} \cos^{-1} \frac{m^2+a_2}{m(1+a_2)} - \frac{\sqrt{m^2-a_2^2}}{1+a_2} \right] + \frac{1}{3} \left[\left(1 + \frac{m}{m-a_2} \right) \sqrt{\frac{m+a_2}{m-a_2}} - 2 \right] \right\} \quad (109)$$

The trailing-edge corrections to the loading are to be integrated over the part of the shaded region behind the trailing-edge Mach lines. Integration of the symmetrical wake correction (equation (20)) yields

$$\frac{(\Delta L)_0}{q\alpha} = \frac{-16m^2 c_0^2}{\beta(1+m_i)(1-m)^2} \frac{u_0}{V\alpha} \left\{ 1 - \frac{2}{1-m_i} \left[1 - \frac{E_0'(m_i)}{K_0'(m_i)} \right] \right\} \quad (110)$$

For each oblique element, the reduction in lift is given by

$$\frac{1}{q\alpha} \frac{d\Delta L}{da} da = -\frac{2u_a}{\beta V\alpha} \frac{1+m_i}{1+a} (m_i-a)(x_2-x_a)^2 \left[\sqrt{\frac{(1+m_i)(1-a)}{2(m_i-a)}} - 1 \right] \quad (111)$$

with $u_a = \frac{du_\Delta}{da} da$, $x_a = \frac{m_i c_0}{m_i - a}$ and

$$x_2 = \left(\frac{1+m}{1-m} + m_i \right) \frac{c_0}{1+m_i} \quad (112)$$

The total lift in region I is then given by

$$\left(\frac{L}{q\alpha} \right)_I = \frac{L_0}{q\alpha} + \frac{(\Delta L)_0}{q\alpha} + \frac{2}{q\alpha} \int_0^{a_2} \frac{d\Delta L}{da} da \quad (113)$$

The quantity $\frac{\beta^2}{m^2 c_0^2} \left(\frac{L}{q\alpha} \right)_I$ is plotted against m_i in figure 29 for several values of the ratio m/m_i .

LIFT ON OUTER PORTIONS OF WING

In order to find the total lift (except for tip losses) on the remainder of the wing (fig. 30), a double integration with respect to x and y is performed on equation (60). A first integration, with respect to y , yields for the indefinite integral

$$\int \frac{u}{V\alpha} dy = \frac{\sigma(x)}{\beta} \sqrt{c_0} \left[\sqrt{\frac{(mx-\beta y)(m_i c_0 - m_i x + \beta y)}{m_i c_0 - (m_i - m)x}} + \sqrt{m_i c_0 - (m_i - m)x} \tan^{-1} \sqrt{\frac{mx-\beta y}{m_i c_0 - m_i x + \beta y}} \right] \quad (114)$$

The values of βy to be substituted as limits in equation (114) are indicated in figure 30. Along the leading edge, the right-hand member of equation (114) reduces to zero; along the trailing edge it becomes

$$\frac{\pi}{2} \frac{\sigma(x)}{\beta} \sqrt{c_0} \sqrt{m_i c_0 - (m_i - m)x}$$

Then the total lift on the outboard region (both wing halves), except for tip losses, is

$$\left(\frac{L}{q\alpha} \right)_{II} = \frac{8\sqrt{c_0}}{\beta} \left[\int_{x_1}^{x_2} \sigma(x) \left(f_1 \tan^{-1} \frac{f_2}{f_3} + \frac{f_2 f_3}{f_1} \right) dx + \frac{\pi}{2} \int_{x_2}^{x_1} \sigma(x) f_1 dx - \int_{\frac{\beta s}{m}}^{x_1} \sigma(x) \left(f_1 \tan^{-1} \frac{f_4}{f_5} + \frac{f_4 f_5}{f_1} \right) dx \right] \quad (115)$$

where

$$f_1 = \sqrt{m_i c_0 - (m_i - m)x}$$

$$f_2 = \sqrt{(1+m)(x-x_1)}$$

$$f_3 = \sqrt{m(x-\beta s/m)}$$

$$f_4 = \sqrt{(1+m_i)(x_2-x)}$$

$$f_5 = \sqrt{m_i(x_i-x)}$$

The indicated integrations may be performed numerically or graphically, using values of $\sigma(x)$ taken from the charts of figure 23.

TIP-INDUCED CORRECTION TO THE LIFT

In deriving a tip correction to the lift, the same simplifying assumption of completely cylindrical flow will be adopted concerning the pressure field to be canceled as was used in obtaining a tip correction to the loading. As in the preceding section, a combined primary and secondary tip correction will be derived. All further corrections will be omitted.

If the notation of equation (63) is used the distance from the apex $x_{e,s}$ of a canceling element to the trailing-edge tip is $c_i(1-\xi_c)$, and the lift induced by the element and its cancellation at the trailing edge is, from equation (91),

$$(\Delta L)_c = \rho V m_i^2 u_c \frac{c_i^2(1-\xi_c)^2}{\beta(m_i-m)} \left[\sqrt{\frac{m(1+m)}{m_i(1+m_i)}} E_0(k) - \frac{m}{m_i} \frac{\Lambda_0(\psi, k)}{\sin \psi} \right] \quad (116)$$

where

$$k = \sqrt{\frac{1-m_i}{1+m_i}}$$

as before, and

$$\psi = \sin^{-1} \sqrt{\frac{m_i - m}{m_i(1-m)}}$$

since the outer boundary of each element now has the slope $\frac{m}{\beta}$.

It is seen that only u_c and $(1-\xi_c)^2$ in the coefficient of equation (116) vary with the element. For the first element ($\xi_c=0$), the velocity u_c is the initial value of the uncorrected velocity along the tip section given in equation (65), and, for the other elements, the differential of that velocity. Then

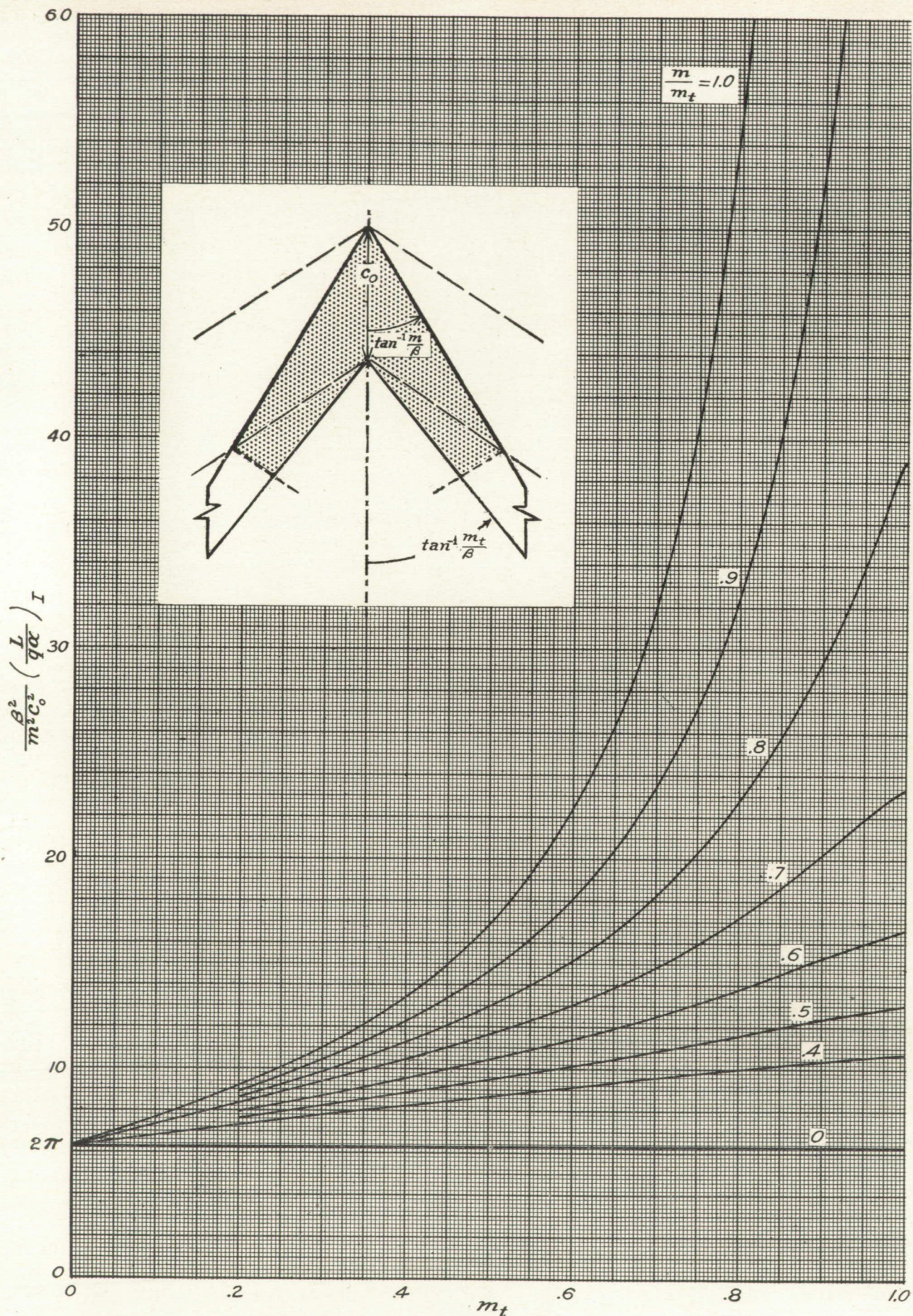


FIGURE 29.—Chart for the computation of lift in region I.

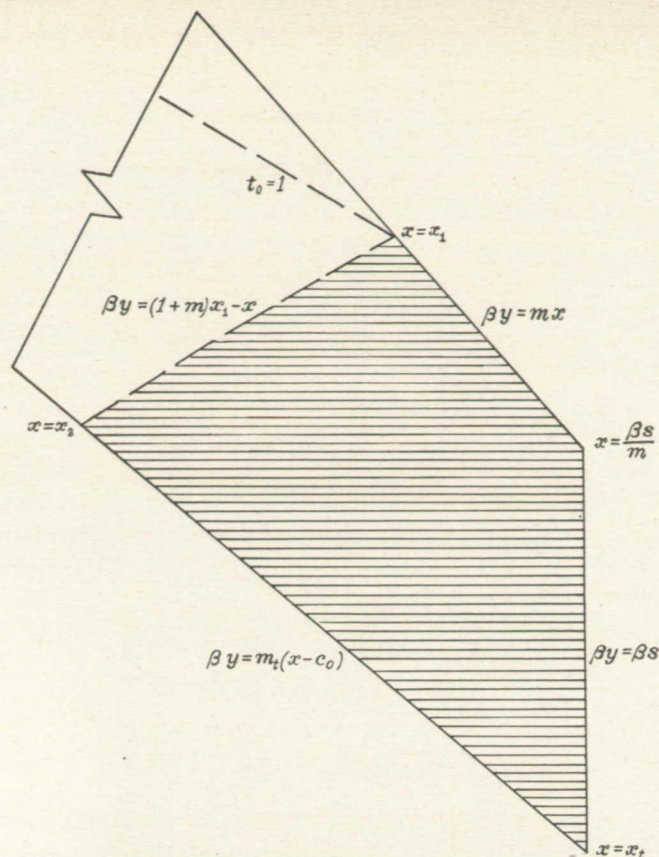


FIGURE 30.—Boundaries of outboard region of high-aspect-ratio wing, for use as limits of integration in equation (114).

the combined primary and secondary tip correction on both wing halves is

$$\frac{\Delta L}{q\alpha} = -\frac{4m_t^2 c_t^2 \sigma_s}{\beta(m_t - m)\sqrt{m\lambda}} \left[\sqrt{\frac{m(1+m)}{m_t(1+m_t)}} E_0(k) - \frac{m}{m_t} \frac{\Lambda_0(\psi, k)}{\sin \psi} \right] \lim_{\xi_c \rightarrow 0} \left[(1 - \xi_c)^2 \sqrt{\frac{1 - \xi_c}{\xi_c(1 - \mu \xi_c)}} + \int_{\xi_c}^1 (1 - \xi_c)^2 \frac{d}{d\xi_c} \sqrt{\frac{1 - \xi_c}{\xi_c(1 - \mu \xi_c)}} d\xi_c \right] \quad (117)$$

Integration by parts gives, finally,

$$\frac{\Delta L}{q\alpha} = -\frac{8\pi m_t^2 c_t^2 \sqrt{m}}{3\beta(m_t - m)^3 \sqrt{\lambda}} \sigma_s \left[\sqrt{\frac{m(1+m)}{m_t(1+m_t)}} E_0(k) - \frac{m}{m_t} \frac{\Lambda_0(\psi, k)}{\sin \psi} \right] \left[(3m - m_t) K_0(\sqrt{\mu}) - 2m_t \left(2 - \frac{m_t}{m} \right) E_0(\sqrt{\mu}) \right] \quad (118a)$$

If the wing is untapered, equation (118a) takes on the value

$$\frac{\Delta L}{q\alpha} = -\frac{\pi c_0^2 \sqrt{m}}{\beta(1 - m^2)} \sigma_s [2m K_0(k) + (1 - 3m) E_0(k)] \quad (118b)$$

Except for the occurrence of σ_s, c_t and λ in the coefficients, the tip correction obtained in the foregoing way is a function of m and m_t only, independent of the tip location. Values of $\frac{\beta \sqrt{\lambda}}{\sigma_s c_t^2} \left(\frac{\Delta L}{q\alpha} \right)_{tip}$ have been plotted in figure 31 in a form similar to the chart of $\left(\frac{L}{q\alpha} \right)_i$ (fig. 29).

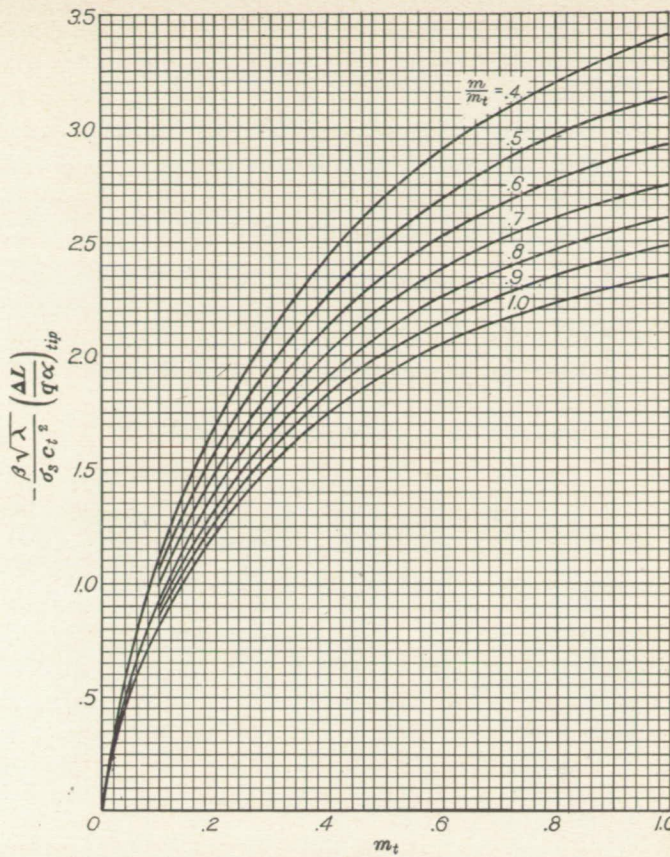


FIGURE 31.—Chart for the correction of the lift for tip effect, using two-dimensional formulas.

APPLICATION OF LIFT FORMULAS

CASES COMPUTED

The lift-curve slope $C_{L\alpha}$ has been calculated for two families of untapered wings with varying aspect ratios as follows:

$m=0.2$		$m=0.4$	
$\beta \frac{s}{c_0}$	βA	$\beta \frac{s}{c_0}$	βA
0.3	0.6	0.6	1.2
.4	.8	.8	1.6
.6	1.2	1.2	2.4
		1.6	3.2

and for two tapered wings:

$m=0.4, m_t=0.6$		
$\beta \frac{s}{c_0}$	βA	λ
0.6	1.6	$\frac{1}{2}$
.8	2.4	$\frac{1}{3}$

It should be noted that the untapered-wing cases (except for the last one under $m=0.4$) represent three wings of fixed geometry at two different Mach numbers such that β is doubled in going from the first to the second. No calculation was made for $m=0.2$ to correspond to the last case

under $m=0.4$ because at the lower Mach number it was not possible to calculate satisfactory values of σ out to the wing tip. The tapered wings were chosen to show, by comparison with the first two of the untapered $m=0.4$ wings, the effect of taper with the span held constant and, by comparison with the second and third untapered $m=0.4$ wings, the effect of taper with a given aspect ratio.

SUMMARY OF COMPUTATIONS

With $m=0.4$ and $\beta s/c_o=0.6$, the trailing-edge Mach lines do not intersect the leading edge, and the values of C_{L_α} were

obtained entirely by means of the conical-flow formulas, as follows:

Component of lift	Equation No.	Tapered wing, $\beta A=1.6$		Untapered, $\beta A=1.2$	
		$\beta^2 L/q\alpha c_o^2$	% total	$\beta^2 L/q\alpha c_o^2$	% total
Uncorrected triangular wing.....	(85)	2.093	121.1	2.595	143.0
Tip effect.....	(96)	-.190	-11.0	-.422	-23.3
Symmetrical trailing-edge correction.....	(103)	-.159	-9.2	-.340	-18.7
Oblique trailing-edge correction.....	(105)	-.015	-0.9	-.019	-1.0
Totals.....		1.729	100.0	1.814	100.0
$\beta C_{L_\alpha} = \beta L/q\alpha S$, per radian.....		1.920		1.512	

The calculations for the remaining values of βA are summarized in the following table:

Component of lift	Obtained from	Untapered wings						Tapered wing $m=0.4$; $m_t=0.6$; $\beta s=0.8$; $\beta A=2.4$
		$m=0.2$			$m=0.4$			
		$\beta A=0.6$	$\beta A=0.8$	$\beta A=1.2$	$\beta A=1.6$	$\beta A=2.4$	$\beta A=3.2$	
Lift on inboard portion.....	Fig. 29 or equation (113).....	0.366	0.366	0.366	2.128	2.128	2.128	1.981
Lift on outboard portion.....	Equation (115).....	.180	.386	.830	.849	2.593	4.459	.462
Tip correction.....	Fig. 31 or equation (118).....	-.085	-.090	-.098	-.363	-.392	-.415	-.092
Totals, $\beta^2 L/q\alpha c^2$		.461	.662	1.098	2.614	4.329	6.172	2.351
$\beta C_{L_\alpha} = \beta L/q\alpha S$, per radian.....		.77	.83	.92	1.63	1.80	1.93	2.20

DISCUSSION OF RESULTS

The results of the calculations are plotted against the reduced aspect ratio βA in figure 32. The curves for the untapered wings may be seen to be approaching, at the upper end, the value $2\pi m/\sqrt{1-m^2}$ given by simple sweep theory.

At the lower end, the curves should approach the origin along the line $C_{L_\alpha} = \frac{\pi}{2} A$ given by low-aspect-ratio theory (reference 13). The two points on the $m=0.2$ curve for $\beta A < 1$ are not entirely accurate because no account was taken of the interference between the flow fields from the tips. The points are included, however, because, with so much sweep, the wing areas affected are small and the interference effects should be negligible. The resulting curve appears consistent with the corresponding curve calculated

by the slender-wing theory of reference 3, although a disparity in plan form lessens the significance of the comparison.

The slender-wing-theory values are also plotted for $m=0.4$. In that case, however, the assumption of extreme slenderness is no longer justified and introduces an appreciable error. (It should be mentioned that the asymptote for the slender-wing-theory curves is *below* the value given by simple sweep theory by the factor $\sqrt{1-m^2}$.)

An estimate of the accuracy of the lift formulas of the present report, compared with results which would take into account all the successive reflections at the tips and trailing edge, may be made from the following observations:

The values obtained (in the first table) from equations (96) and (103), which combine primary and secondary corrections, differ from values obtainable for the primary corrections alone by only 1 percent of the total lift in the case of the tapered wing, and 4 percent of the total lift for the untapered wing. Third-order corrections would be only a fraction of those small corrections and would, in turn, be partly canceled out by a fourth-order correction.

The results in the second table, incorporating the two-dimensional approximations, agree within 2 or 3 percent with values calculated entirely by the conical-flow method.

IV—DRAG DUE TO LIFT

The drag due to lift of a wing with supersonic leading edge is simply the lift times the angle of attack. When the leading edge is subsonic, the drag is reduced by a suction force due to the upwash around the leading edge. In the linearized theory this force appears as the limit of the product of an infinite velocity across an infinitesimal frontal area.

The formula for the suction force on a subsonic leading edge has been derived (see, e. g., Hayes, reference 18) by

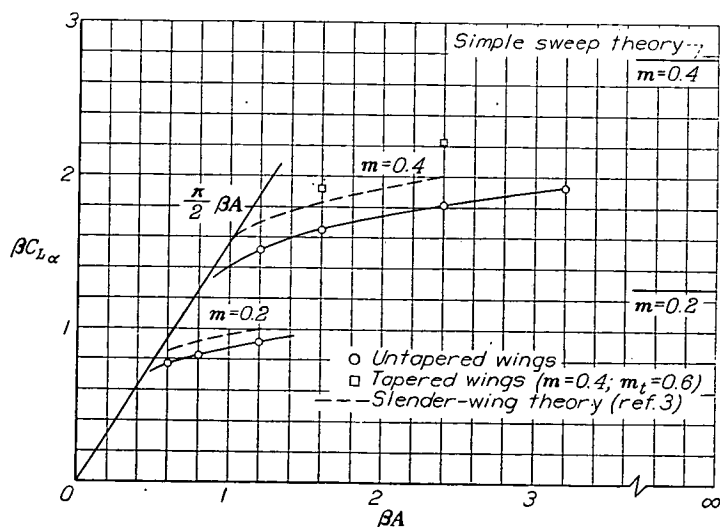


FIGURE 32.—Variation of lift-curve slope with aspect ratio.

assuming the flow near the leading edge to be essentially two-dimensional and applying the results of two-dimensional potential theory. The simple result obtained in that manner has been verified for the swept-back wing of finite span by application of the somewhat different approach of reference 19.

By the two-dimensional approach, the suction force is found to be proportional to the square of the strength of the leading-edge singularity in the perturbation velocity u . The latter is the quantity discussed earlier in connection with the adjustment of the two-dimensional loading to the loading on the swept-back wing. With the use of the previous terminology it is possible to write for the longitudinal component of the suction force per unit streamwise length of leading edge,

$$\frac{dT}{dx} = \frac{\rho\pi}{m} \sqrt{1-m^2} C_\Delta^2 \quad (119)$$

where C_Δ (equation (53)) is the value, at the leading edge, of the coefficient of $(mx-\beta y)^{-1/2}$ in u_Δ .

Then, if the trailing-edge Mach line does not intersect the leading edge, the thrust is merely

$$\left. \begin{aligned} T &= \rho\pi u_0^2 \sqrt{1-m^2} \int_0^{\beta s} x dx \\ &= \frac{\rho\pi u_0^2 \beta^2 s^2}{2m^2} \sqrt{1-m^2} \end{aligned} \right\} \quad (120)$$

The total drag due to lift is obtained by subtracting the thrust from the product of the lift and the angle of attack, or, in coefficient form,

$$C_D = \alpha C_L - C_T \quad (121)$$

where C_T is the thrust coefficient T/qS . Thus, in the foregoing case,

$$C_D = \frac{\alpha^2}{S} \left[\frac{L}{q\alpha} - \pi \left(\frac{\beta s}{m} \right)^2 \left(\frac{u_0}{V\alpha} \right)^2 \sqrt{1-m^2} \right] \quad (122)$$

When a portion of the leading edge is influenced by the trailing edge, the leading-edge singularity takes on, for that portion, the value given by expression (56), which then replaces C_Δ in equation (119) for the thrust. The total thrust is

$$\left. \begin{aligned} 2 \int_0^{\beta s} \frac{dT}{dx} dx &= 2 \frac{\rho\pi}{m} \sqrt{1-m^2} \left\{ \int_0^{x_1} C_\Delta^2 dx + \int_{x_1}^{\beta s} \left[C_\Delta + (\Delta C)_0 + \int_0^{a_0} \frac{d\Delta C}{da} da \right]^2 dx \right\} \end{aligned} \right\} \quad (123)$$

where

$$x_1 = \frac{c_0}{1-m}$$

locates the intersection of the trailing-edge Mach line with the leading edge. Integrating the first term and reducing so coefficient form gives

$$C_T = \frac{\pi\alpha^2 \sqrt{1-m^2}}{S} \left[\left(\frac{u_0}{V\alpha} \right)^2 x_1^2 + \frac{4c_0}{m} \int_{x_1}^{\beta s} \sigma^2 dx \right] \quad (124)$$

so that

$$C_D = \frac{\alpha^2}{S} \left\{ \frac{L}{q\alpha} - \pi \sqrt{1-m^2} \left[\left(\frac{u_0}{V\alpha} \right)^2 x_1^2 + \frac{4c_0}{m} \int_{x_1}^{\beta s} \sigma^2 dx \right] \right\} \quad (125)$$

In figure 33, $\frac{1}{\beta}$ times the drag-rise factor $\frac{C_D}{C_L^2}$ is plotted against the reduced aspect ratio βA for two combinations of sweep and Mach number, $m=0.2$ and $m=0.4$, for untapered wings. Comparison is made with a theoretical minimum for slender wings in supersonic flight obtained by R. T. Jones (reference 18) and assuming the wing to be narrow compared with the Mach cone, Jones has derived a minimum wave-drag coefficient

$$C_{D_w} = \frac{\beta^2}{2\pi A_x} C_L^2 \quad (126)$$

where A_x is the aspect ratio defined in the streamwise, instead of the spanwise, direction; that is, if l (numerically equal to x_l) is the over-all length of the wing in the stream direction,

$$A_x = l^2/S \quad (127)$$

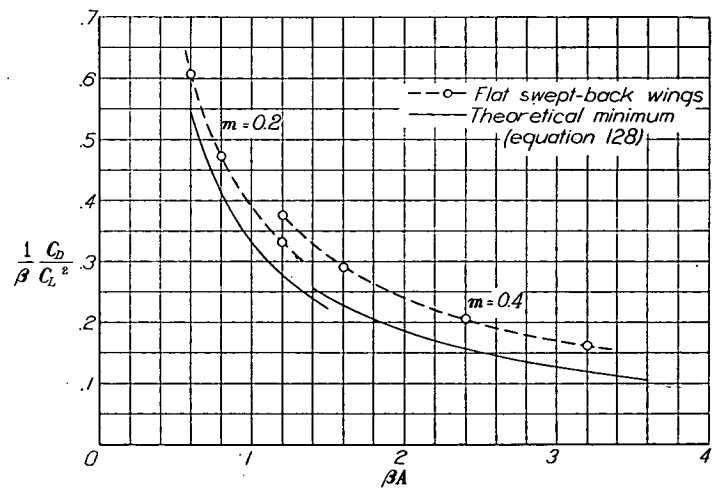


FIGURE 33.—Variation of drag-rise factor with aspect ratio for untapered wings.

The wave drag is to be added to the vortex drag, which is the induced drag of subsonic flow, calculated from the spanwise loading. Using the minimum induced drag obtained from lifting-line theory gives as the minimum supersonic drag-rise factor⁸

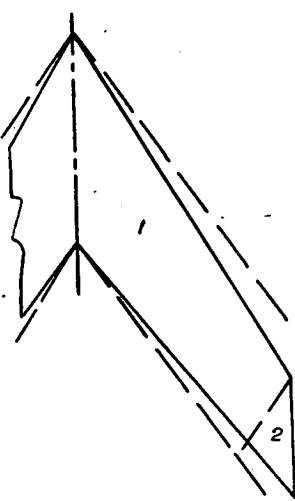
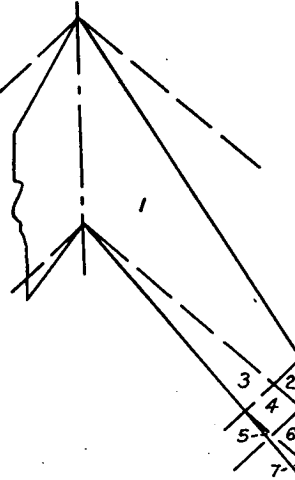
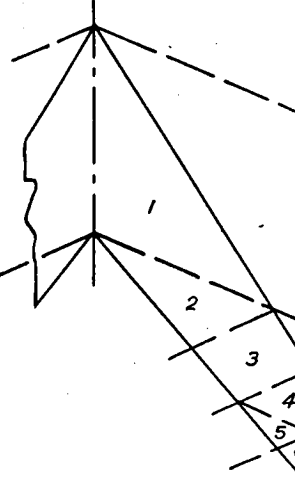
$$\frac{C_D}{C_L^2} = \frac{1}{\pi A} + \frac{\beta^2}{2\pi A_x} \quad (128)$$

It may be seen that the drag rise of the constant-chord swept-back wings is fairly close to this minimum, especially at the lower values of m for which equation (128) was derived.

V—SUMMARY OF FORMULAS

The formulas for the loading, lift, and drag coefficients are summarized in the following table, in which the equations are identified by number.

⁸ This result has since been published in The Journal of the Aeronautical Sciences, vol. 18, no. 2, Feb. 1951, pp. 75-81.

Case	$\frac{\Delta p}{q\alpha} \leftarrow \frac{4u}{V\alpha}$		$C_{L\alpha} = \frac{1}{S} \cdot \frac{L}{q\alpha}$	C_D
——— Mach lines	Region	Equations for u	Equa. for $\frac{L}{q\alpha}$	Equa. No.
	1 2	(6) (6) + (15) ^a	(85) + (88)	(122)
	1 2 3 4 5 6 7	(6) (6) + (15) ^a (6) + (26b) (6) + (15) ^a + (26b) (6) + (15) ^a + (26b) + (31) (6) + (15) ^a + (26b) + (32) (6) + (15) ^a + (26b) + (31) + (32)	(85) + (96) + (103) + (105) ^a	(122)
	1 2 3 4 5 6	(6) (6) + (26b) (60) ^b (60) ^b + (73) ^{a,b,c} (60) ^b + (73) ^{a,b,c} + (31) Not evaluated	(113) ^d + (115) ^b + (118) ^{b,e}	(125) ^b
^a In evaluating, use fig. 6. ^b In evaluating, use fig. 23. ^c In evaluating, use fig. 14. ^d or see fig. 29. ^e or see fig. 31.				

APPENDIX A

SYMBOLS

GENERAL			
V	free-stream velocity	ξ	streamwise distance of x, y back from leading edge, as a fraction of the tip chord (equation (69))
M	free-stream Mach number	ξ_0	distance of x_0, s behind leading-edge tip, as a fraction of the tip chord (equation (70))
β	$\sqrt{M^2 - 1}$	ξ_c	distance of x_c, s behind leading-edge tip, as a fraction of the tip chord (equation (64))
u, v, w	perturbation velocities in streamwise, cross-stream, and vertical directions, respectively	CONICAL COORDINATES	
ρ	density of air	In the following, all slopes are measured counterclockwise from a line extending downstream from the apex of the wing or of the pertinent canceling sector:	
q	dynamic pressure $(\frac{1}{2} \rho V^2)$	m	$\frac{\text{slope of leading edge}}{\text{slope of Mach lines}} = \beta \cot \Lambda$
Δp	pressure difference between upper and lower surfaces, or local lift	m_t	$\frac{\text{slope of trailing edge}}{\text{slope of Mach lines}}$
α	angle of attack, radians	a	$\frac{\text{slope of ray from the origin}}{\text{slope of Mach lines}} = \beta \frac{y}{x}$
L	lift	a_0	the value of a corresponding to a primary canceling element of which the apex lies on the Mach forecone of the point at which the load is being calculated (equation (13) for tip corrections, equation (25) for trailing-edge corrections)
T	leading-edge thrust, or component of leading-edge suction force in flight direction	a_0'	limiting value of a for leading-edge correction (equation (47))
C_L	lift coefficient $(\frac{L}{qS})$	a_2	$a(x_2, y_2)$ (equation (108))
C_{L_α}	lift-curve slope $(\frac{dC_L}{d\alpha})$	a_t	$a(x_t, s)$
C_D	drag coefficient $(\frac{D}{qS})$	t_a	$\frac{\text{slope of ray from apex of element } a}{\text{slope of Mach lines}} = \beta \frac{y - y_a}{x - x_a}$
C_T	thrust coefficient $(\frac{T}{qS})$	t_b	$\frac{\text{slope of ray from } x_b, y_b}{\text{slope of Mach lines}} = \beta \frac{y - y_b}{x - x_b}$
WING DIMENSIONS		t_c	$\frac{\text{slope of ray from } x_c, s}{\text{slope of Mach lines}} = \beta \frac{y - s}{x - x_c}$
c_0	root chord	t_m	$\frac{\text{slope of ray from leading-edge tip}}{\text{slope of Mach lines}} = \beta \frac{y - s}{x - (\beta s/m)}$
c_t	tip chord	t^*	$\frac{\text{slope of ray from } x^*, y^*}{\text{slope of Mach lines}} = \beta \frac{y - y^*}{x - x^*}$
s	semispan	τ_0	limiting value of t_0 for leading-edge correction (equation (39))
S	wing area	τ_a	limiting value of t_a for leading-edge correction (equation (44))
l	over-all length in the streamwise direction	COMPONENTS OF STREAMWISE PERTURBATION VELOCITY	
Λ	angle of sweep of the leading edge	u_Δ	basic (uncorrected) perturbation velocity as given by solution for triangular wing (equation (6) for subsonic leading edge)
λ	taper ratio (c_t/c_0)	u_0	value of u_Δ at $a=0$ (equation (7))
A	aspect ratio $(4s^2/S)$	Δu	correction to u_Δ induced by cancellation of pressure differences outside the wing plan form
A_x	streamwise aspect ratio (l^2/S)	u_a	constant perturbation velocity on sector used in canceling triangular-wing loading
RECTANGULAR COORDINATES		u_b	constant perturbation velocity on sector used in secondary cancellation
x, y	Cartesian coordinates in the stream direction and across the stream, in the plane of the wing	u_c	constant perturbation velocity on sector used outboard of tip in canceling assumed cylindrical field
x_a, y_a	coordinates of apex of conical field used to cancel triangular-wing loading (Equation (8) at tip, equations (21) and (22) at trailing edge)		
x_b, y_b	coordinates of apex of conical field used in secondary cancellations		
x_c, s	coordinates of point on tip; apex of conical field used to cancel assumed cylindrical load		
x_0, y_0	coordinates of intersection of Mach forecone from x, y with edge at which correction is being made		
x_1, y_1	coordinates of intersection of trailing-edge Mach cone with leading edge (x_1 given by equation (61))		
x_2, y_2	coordinates of intersection of Mach line from x_1, y_1 with trailing edge (x_2 given by equation (112))		
x^*, y^*	coordinates of intersection of tip Mach line with trailing edge		
x_t, s	coordinates of intersection of tip and trailing edge		

$(\Delta u)_0$	symmetrical trailing-edge correction to u_Δ (equation (20))
$(\Delta_2 u)_0$	correction induced by canceling $(\Delta u)_0$ at leading edge (equation (38))
$(\Delta u)_a$	correction to u_Δ due to single oblique trailing-edge element (equation (24))
Δu^*	value of tip correction to u_Δ at the point x^*, y^*

ARBITRARY MATHEMATICAL SYMBOLS

C_Δ	value of coefficient of $\frac{1}{\sqrt{mx-\beta y}}$ in u_Δ at the leading edge (equation (53))
$(\Delta C)_0$	decrement in C_Δ due to reflection of $(\Delta u)_0$ at leading edge (equation (54))
$\frac{d\Delta C}{da} da$	decrement in C_Δ due to reflection of $(\Delta u)_a$ at leading edge (equation (55))
σ	Non-dimensional expression for strength of the leading-edge singularity (equation (59))
σ_s	value of σ at leading-edge tip $\left[\sigma \left(\frac{\beta s}{m} \right) \right]$
μ	taper parameter $\left(\frac{m_t - m}{m_t} \right)$
g	function defined by equation (79)
C	inverse-cosine term of leading-edge correction function (equation (35))
R	radical term of leading-edge correction function (equation (36))
$r. p.$	real part

ELLIPTIC INTEGRALS AND FUNCTIONS

k	modulus of elliptic integral, defined where used (also with subscripts)
k'	complimentary modulus ($\sqrt{1-k^2}$)
ϕ or ψ	argument of elliptic integrals, defined where used (also with subscripts)
$F(\phi, k)$	incomplete elliptic integral of the first kind of modulus k and argument ϕ
$K, K(k)$	complete elliptic integral of the first kind; that is, $K = F\left(\frac{\pi}{2}, k\right)$
$E(\phi, k)$	incomplete elliptic integral of the second kind of modulus k and argument ϕ
$E, E(k)$	complete elliptic integral of the second kind; that is, $E = E\left(\frac{\pi}{2}, k\right)$
K_0	$\frac{2}{\pi} K$
E_0	$\frac{2}{\pi} E$
K'	$K(k')$
E'	$E(k')$
Z	zeta function (equation (41))
Λ_0	function used in evaluation of elliptic integral of the third kind, circular case (equation (16))
Ω	function used in evaluation of elliptic integral of the third kind, circular case (equation (B11))

APPENDIX B

EVALUATION OF THE INTEGRAL IN EQUATION (26)

It is first necessary to recall that t_a is a function (equation (23)) of x , y , and a . After substitution for t_a in equation (26), we may integrate by parts to obtain

$$\int_0^{a_0} \cos^{-1} \frac{(1-a)(t_a - m_t) - (m_t - a)(1-t_a)}{(1-m_t)(t_a - a)} \frac{du_\Delta(a)}{da} da =$$

$$-u_0 \left[\cos^{-1} \frac{(1+m_t)\beta y - 2m_t(x-c_0)}{(1-m_t)\beta y} \right.$$

$$\left. (x - \beta y) \sqrt{\frac{\beta y - m_t(x-c_0)}{x - \beta y - m_t c_0}} \times \right.$$

$$\left. \int_0^{a_0} \frac{da}{(\beta y - ax) \sqrt{(1-a)(a_0 - a)(m - a)(m + a)}} \right] \quad (B1)$$

The integral term on the right-hand side of equation (B1) is an elliptic integral of the third kind which may be evaluated through the substitution of

$$\omega = sn^{-1} \sqrt{\frac{2m(a_0 - a)}{(m + a_0)(m - a)}} \quad k = \sqrt{\frac{(1-m)(m + a_0)}{2m(1 - a_0)}}$$

If the value of ω at the lower limit is designated by ω_0 , this substitution gives

$$\int_0^{a_0} \frac{da}{(\beta y - ax) \sqrt{(1-a)(a_0 - a)(m - a)(m + a)}} =$$

$$\frac{1}{\beta y - a_0 x} \sqrt{\frac{2}{m(1 - a_0)}} \int_0^{\omega_0} \frac{1 - \frac{m + a_0}{2m} sn^2 \omega}{1 + n sn^2 \omega} d\omega \quad (B2)$$

where

$$n = \frac{(m + a_0)(mx - \beta y)}{2m(\beta y - a_0 x)} \quad (B3)$$

or

$$\int_0^{a_0} \frac{da}{(\beta y - ax) \sqrt{(1-a)(a_0 - a)(m - a)(m + a)}} =$$

$$\frac{-1}{mx - \beta y} \sqrt{\frac{2}{m(1 - a_0)}} \left[\omega_0 - \left(1 + \frac{mx - \beta y}{\beta y - a_0 x} \right) \Pi_3(\omega_0, k, n) \right] \quad (B4)$$

where

$$\Pi_3(\omega_0, k, n) = \int_0^{\omega_0} \frac{d\omega}{1 + n sn^2 \omega}$$

is the normal form of the elliptic integral of the third kind.

It is first noted that $n > 0$. For this case it can be shown that the substitution

$$\nu = tn^{-1} \left(\sqrt{\frac{n}{k^2}}, k' \right) \quad (\text{B5})$$

gives

$$\Pi_3(\omega_0, k, n) = \omega_0 cn^2(\nu, k') + \frac{sn(\nu, k') cn(\nu, k')}{dn(\nu, k')} \left[\frac{\omega_0}{K_0} \Lambda_0(k, \phi) + \Omega \right] \quad (\text{B6})$$

where

$$\phi = \tan^{-1} \sqrt{\frac{n}{k^2}} \quad (\text{B7})$$

is the amplitude of the elliptic function ν , Λ_0 is the function defined in equation (16), and Ω is an angular function of k, ν and ω_0 which will be discussed later.

If

$$\psi = \sin^{-1} \sqrt{\frac{2a_0}{m+a_0}} \quad (\text{B8})$$

then

$$\omega_0 = F(\psi, k) \quad (\text{B9})$$

$$\text{From equation (B5), } sn(\nu, k') = \sqrt{\frac{n}{n+k^2}}, \quad cn(\nu, k') = \sqrt{\frac{k^2}{n+k^2}}$$

and $dn(\nu, k') = \sqrt{\frac{k^2(1+n)}{n+k^2}}$ may be found, so that equation (B6) may be rewritten without recourse to the Jacobian elliptic functions as

$$\Pi_3(\omega_0, k, n) = \frac{k^2}{n+k^2} F(\psi, k) + \sqrt{\frac{n}{(1+n)(n+k^2)}} \left[\frac{F(\psi, k)}{K_0} \Lambda_0(k, \phi) + \Omega \right] \quad (\text{B10})$$

This expression is to be substituted in equation (B4) and the result used in equation (B1). As previously mentioned, the functions K_0 and Λ_0 are tabulated in reference 11 and Λ_0 is plotted in figure 6. The function Ω is given by⁹

$$\Omega = \tan^{-1} \frac{2 \sum_{j=1}^{\infty} (-1)^{j+1} q^{(j^2)} \sin 2j \frac{\omega_0}{K_0} \sinh 2j \frac{\nu}{K_0}}{1 - 2 \sum_{j=1}^{\infty} (-1)^{j+1} q^{(j^2)} \cos 2j \frac{\omega_0}{K_0} \cosh 2j \frac{\nu}{K_0}} \quad (\text{B11})$$

with

$$q = e^{-\frac{\pi K'}{K}}$$

(tabulated in reference 15).

⁹ The symbol q in equation (B11) is standard notation for the nome of the Jacobian theta function, and is not related to the dynamic pressure q of the text.

APPENDIX C

INTEGRATION FOR LOSS OF LIFT AT THE TIP OF WING WITH SUBSONIC LEADING EDGE

From equations (88a) and (6)

$$\left(\frac{\Delta L}{q\alpha} \right)_{tip} = -\frac{4m_i^2 m s^2 \beta u_0}{V\alpha} \int_{a_i}^m \frac{G'(a)}{\sqrt{m^2 - a^2}} da \quad (\text{C1})$$

where

$$G'(a) = \frac{a - a_i}{a_i^2 a^2 (m_i - a)} \left[\left(\frac{a_i}{a} + \frac{m_i - a_i}{m_i - a} \right) \left(\sqrt{\frac{a + a^2}{m_i + m_i^2}} - \frac{a}{m_i} \right) - \frac{a - a_i}{2\sqrt{m_i + m_i^2} \sqrt{a + a^2}} \right] \quad (\text{C2})$$

The terms in $G'(a)$ are of two types; namely, those that contain $\sqrt{a + a^2}$ and those that do not. The former combine with the radical $\sqrt{m^2 - a^2}$ in equation (C1) to form elliptic integrals of the first, second, and third kinds. The latter give rise to terms in equation (C1) which are integrable by elementary means. It is convenient, therefore, to consider the integral in two parts, writing

$$-\int_{a_i}^m \frac{G'(a)}{\sqrt{m^2 - a^2}} da = I_1 + I_2$$

where I_1 is that part of the integral not requiring elliptic integrals.

Then

$$I_1 = \int_{a_i}^m \frac{a - a_i}{a_i^2 m_i a (m_i - a)} \left(\frac{a_i}{a} + \frac{m_i - a_i}{m_i - a} \right) \frac{da}{\sqrt{m^2 - a^2}}$$

$$= \frac{1}{a_i^2 m_i (m_i^2 - m^2)} \left[\frac{(m_i - a_i)^2}{\sqrt{m_i^2 - m^2}} \cos^{-1} \frac{m_i a_i - m^2}{m(m_i - a_i)} - \frac{m_i a_i - m^2}{m^2 \sqrt{m^2 - a_i^2}} \right] \quad (\text{C3})$$

The remaining terms, involving $\sqrt{a + a^2}$ and $\sqrt{m^2 - a^2}$, are integrated by means of the substitution

$$sn \omega = \sqrt{\frac{m - a}{m(1 + a)}} \quad k = \sqrt{\frac{1 - m}{2}} \quad (\text{C4})$$

The result is

$$I_2 = \frac{1}{a_i^2 m_i \sqrt{2m m_i (1 + m_i)}} \left\{ \left[\frac{m_i - a_i}{m_i} \left(\frac{m_i - a_i}{1 + m_i} - \frac{a_i}{m} \right) - \frac{a_i (m - a_i)}{m^2} \right] F(\psi, k) + \frac{2a_i}{m} \left[\left(1 + \frac{m_i - a_i}{m_i} \right) E(\psi, k) + \left(a_i - \frac{m_i - a_i}{m_i} \right) \sqrt{\frac{m + a_i}{2a_i}} \sin \psi \right] - \frac{(1 + m)(m_i - a_i)^2}{(m_i - m)^2} \left[\left(1 + \frac{m}{m_i} + \frac{m_i - m}{1 + m_i} \right) \Pi_3(\omega_i, k, n) + \frac{2m(1 + m)}{m_i - m} \frac{\partial \Pi_3(\omega_i, k, n)}{\partial n} \right] \right\} \quad (\text{C5})$$

where $\omega_i = \omega(a_i)$, ψ is its amplitude, and

$$n = \frac{m(1+m_i)}{m_i - m} \quad (C6)$$

From equation C(4),

$$\psi = \sin^{-1} \sqrt{\frac{m - a_i}{m(1 + a_i)}} \quad (C7)$$

The elliptic integral

$$\Pi_3(\omega_i, k, n) = \int_0^{\omega_i} \frac{d\omega}{1 + n \operatorname{sn}^2 \omega}$$

is evaluated in equation (B10). Its derivative with respect to the parameter n may be obtained for this case ($n > 0$) in the form

$$\begin{aligned} \frac{\partial \Pi_3}{\partial n} = & \frac{1}{2n} \sqrt{\frac{n}{(n+k^2)(1+n)}} \left\{ \left(\frac{1}{1+n} - \frac{n}{n+k^2} \right) \left[\frac{\Delta_0(\phi, k)}{K_0} F(\psi, k) + \right. \right. \\ & \left. \Omega(\omega_i, n, k) \right] - \sqrt{\frac{n}{(n+k^2)(1+n)}} \left[k^2 \left(\frac{1}{k^2} + \frac{1+n}{n+k^2} \right) F(\psi, k) - \right. \\ & \left. \left. E(\psi, k) - \frac{nsn\omega_i cn\omega_i dn\omega_i}{1+n sn\omega_i} \right] \right\} \quad (C8) \end{aligned}$$

where ϕ and Ω are the angles defined in equations (B7) and (B11) and the elliptic functions $cn\omega_i$ and $dn\omega_i$, obtained from equation (C4), have the values

$$cn\omega_i = \sqrt{\frac{a_i(1+m)}{m(1+a_i)}} \quad dn\omega_i = \sqrt{\frac{(1+m)(m+a_i)}{2m(1+a_i)}} \quad (C9)$$

REFERENCES

1. Evvard, John C.: Use of Source Distributions for Evaluating Theoretical Aerodynamics of Thin Finite Wings at Supersonic Speeds. NACA Rep. 951, 1950.
2. Harmon, Sidney M., and Jeffreys, Isabella: Theoretical Lift and Damping in Roll of Thin Wings With Arbitrary Sweep and Taper at Supersonic Speeds. NACA TN 2114, 1950.

3. Lomax, Harvard, and Heaslet, Max. A.: Linearized Lifting-Surface Theory for Swept-Back Wings with Slender Plan Forms. NACA TN 1992, 1949.
4. Lagerstrom, P. A.: Linearized Supersonic Theory of Conical Wings. NACA TN 1685, 1948.
5. Mirels, Harold: Lift-Cancellation Technique in Linearized Supersonic Wing Theory. NACA TN 2145, 1950.
6. Busemann, Adolf: Infinitesimal Conical Supersonic Flow. NACA TM 1100, 1947.
7. Cohen, Doris: The Theoretical Lift of Flat Swept-Back Wings at Supersonic Speeds. NACA TN 1555, 1948.
8. Cohen, Doris: Theoretical Loading at Supersonic Speeds of Flat Swept-Back Wings with Interacting Trailing and Leading Edges. NACA TN 1991, 1949.
9. Cohen, Doris: Formulas and Charts for the Supersonic Lift and Drag of Flat Swept-Back Wings With Interacting Leading and Trailing Edges. NACA TN 2093, 1950.
10. Malvestuto, Frank S. Jr., Margolis, Kenneth, and Ribner, Herbert S.: Theoretical Lift and Damping in Roll of Thin Swept-Back Wings of Arbitrary Taper and Sweep at Supersonic Speeds. Subsonic Leading Edges and Supersonic Trailing Edges. NACA Rep. 970, 1949.
11. Heuman, Carl: Tables of Complete Elliptic Integrals. Jour. of Math. and Physics, V. 19-20, 1940-41, pp. 127-206.
12. Legendre, Adrien Marie: Tables of the Complete and Incomplete Elliptic Integrals. Biometrika Office, University College, Cambridge University Press, Cambridge, England, 1934.
13. Jones, R. T.: Properties of Low-Aspect-Ratio Pointed Wings at Speeds Below and Above the Speed of Sound. NACA Rep. 835, 1946.
14. Ribner, Herbert S.: Some Conical and Quasi-Conical Flows in Linearized Supersonic Wing Theory. NACA TN 2147, 1950.
15. Spenceley, G. W., and Spenceley, R. M.: Smithsonian Elliptic Functions Tables. Smithsonian Institution, Washington, D. C., 1947.
16. Adams, Edwin P., and Hippisley, R. L.: Smithsonian Mathematical Formulae and Tables of Elliptic Functions. Smithsonian Institution, Washington, D. C., 1939.
17. Brown, Clinton E.: The Reversibility Theorem for Thin Airfoils in Subsonic and Supersonic Flow. NACA TN 1944, 1949.
18. Hayes, W. D.: Linearized Supersonic Flow. North American Aviation, Inc., Rep. AL-222, June 18, 1947.
19. Jones, R. T.: Leading-Edge Singularities in Thin-Airfoil Theory. Jour. Aero. Sci., vol. 17, no. 5, May, 1950, pp. 307-310.

REPORT 1051

AN ANALYSIS OF BASE PRESSURE AT SUPERSONIC VELOCITIES AND COMPARISON WITH EXPERIMENT¹

By DEAN R. CHAPMAN

SUMMARY

In the first part of the investigation an analysis is made of base pressure in an inviscid fluid, both for two-dimensional and axially symmetric flow. It is shown that for two-dimensional flow, and also for the flow over a body of revolution with a cylindrical sting attached to the base, there are an infinite number of possible solutions satisfying all necessary boundary conditions at any given free-stream Mach number. For the particular case of a body having no sting attached only one solution is possible in an inviscid flow, but it corresponds to zero base drag. Accordingly, it is concluded that a strictly inviscid-fluid theory cannot be satisfactory for practical applications.

An approximate semi-empirical analysis for base pressure in a viscous fluid is developed in a second part of the investigation. The semi-empirical analysis is based partly on inviscid-flow calculations. In this theory an attempt is made to allow for the effects of Mach number, Reynolds number, profile shape, and type of boundary-layer flow. Some measurements of base pressure in two-dimensional and axially symmetric flow are presented for purposes of comparison. Experimental results also are presented concerning the support interference effect of a cylindrical sting, and the interference effect of a reflected bow wave on measurements of base pressure in a supersonic wind tunnel.

INTRODUCTION

The present investigation is concerned with the pressure acting on the base of an object moving at a supersonic velocity. This problem is of considerable practical importance since in certain cases the base drag can amount to as much as two-thirds of the total drag of a body of revolution, and over three-fourths of the total drag of an airfoil. In the past, numerous measurements of base pressure on bodies of revolution have been made both in supersonic wind tunnels and in free flight, but these experimental investigations have had no adequate theory to guide them. As a result, the present-day knowledge of base pressure is undesirably limited and some inconsistencies appear in the existing experimental data.

Various hypotheses as to the fundamental mechanism which determines the base pressure on bodies of revolution were advanced years ago by Lorenz, Gabeaud, and von Kármán. (See references 1, 2, and 3, respectively.) These

hypotheses, which neglect the influence of the boundary layer, do not appear to be adequate for predicting the base pressure or for correlating experiments.

A semi-empirical theory of base pressure for bodies of revolution has been advanced by Cope in reference 4. Cope's analysis and the semi-empirical analysis of the present report were developed independently and are similar in one significant respect; both consider the influence of the boundary layer on base pressure. The basic concepts and the details of the two analyses, though, are entirely different. Cope's equations are developed only for axially symmetric flow, and do not provide for the effects of variations in profile shape on base pressure. He evaluates the base pressure by equating the pressure in the wake, as calculated from the boundary-layer flow, to the corresponding pressure as calculated from the exterior flow. In calculating the pressure from the boundary-layer flow, however, several approximations and assumptions are necessarily made which, according to Cope, result in no more than a first approximation.

The primary purpose of the investigation described in the present report is to formulate a method which is of value for quantitative calculations of base pressure both on airfoils and bodies. The analysis is divided into two parts. Part I consists of a detailed study of the base pressure in two-dimensional and axially symmetric inviscid flow. The purpose of part I is to develop an understanding of the problem in its simplest form, and to study the effects on base pressure of variations in profile shape. In part II a semi-empirical theory is formulated since the results of part I indicate that an inviscid-flow theory cannot possibly be satisfactory for engineering estimates of base pressure. A comparison of the semi-empirical analysis with experimental results is also presented in part II of the report.

Much of the present material was developed as part of a thesis submitted to the California Institute of Technology in 1948. Acknowledgment is made to H. W. Liepmann of the California Institute of Technology for his helpful discussions regarding the theoretical considerations, and to A. C. Charters of the Ballistic Research Laboratories for making available numerous unpublished spark photographs which were taken in the free flight experiments of reference 5.

¹ Supersedes NACA TN 2137, "An Analysis of Base Pressure at Supersonic Velocities and Comparison with Experiment," by Dean R. Chapman, 1950. The present report includes reference to some experiments not discussed therein, and incorporates a more detailed analysis of the effects of variations in profile shape on base pressure in inviscid flow.

NOTATION

- C constant
 d rod or support diameter
 h base thickness (base diameter for axially symmetric flow, trailing-edge thickness for two-dimensional flow)
 L length upstream of base (body length for axially symmetric flow, airfoil chord for two-dimensional flow)
 M Mach number
 p pressure
 P pressure coefficient referred to free-stream conditions

$$\left(\frac{p - p_{\infty}}{\frac{1}{2} \rho_{\infty} U_{\infty}^2} \right)$$

- P_b' base pressure coefficient referred to conditions on a hypothetical extended afterprofile $\left(\frac{p_b - p'}{\frac{1}{2} \rho' U'^2} \right)$

- P_{b_i} base pressure coefficient for maximum drag in inviscid flow over a semi-infinite profile

- P_b^* value of P_b' obtained by extrapolating to zero boundary-layer thickness the curve of P_b' versus $\frac{\delta}{h}$

- q dynamic pressure $\left(\frac{1}{2} \rho U^2 \right)$

- R gas constant

- Re Reynolds number based on the length L

- r radial distance from axis of symmetry to point in the flow

- T temperature

- t thickness of wake near the trailing shock wave

- U velocity

- β angle of boattailing at base

- γ ratio of specific heats (1.4 for air)

- δ boundary-layer thickness

- ρ density

SUPERSCRIT

' conditions on hypothetical extended afterprofile averaged over a region occupying the same position relative to the base as the dead-air region

SUBSCRIPTS

- ∞ conditions in the free stream
 b conditions at base
 o stagnation conditions

I. BASE PRESSURE IN AN INVISCID FLUID

Throughout this part of the report the effects of viscosity are completely ignored and the flow field determined for an inviscid fluid wherein both the existence of a boundary layer and the mixing of dead air with the air outside a free streamline are excluded from consideration. It is assumed throughout that a dead-air region of constant pressure exists just behind the base and is terminated by a single trailing shock wave. As will be seen later, the assumption of zero viscosity oversimplifies the actual conditions; the results obtained with this assumption agree qualitatively with a number of experimental results, but provide quantitative information only on the effects of profile shape on base pressure.

TWO-DIMENSIONAL INVISCID FLOW OVER A SEMI-INFINITE PROFILE

In order to achieve the greatest possible simplicity at the outset, the case of a semi-infinite profile will be considered first. By this is meant a profile of constant thickness which extends from the base to an infinite distance upstream (fig. 1). The problem at hand is to determine the flow pattern in the neighborhood of the base. Since the effects of viscosity are at present ignored and only steady symmetrical flows are considered, the problem is simply that of determining the flow over a two-dimensional, flat, horizontal surface which has a step in it (fig. 2).

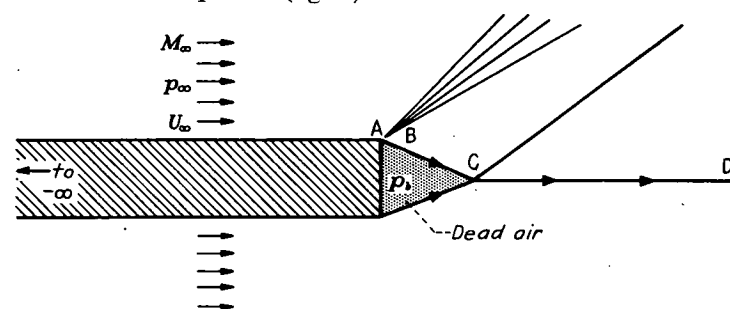


FIGURE 1.—Semi-infinite profile.

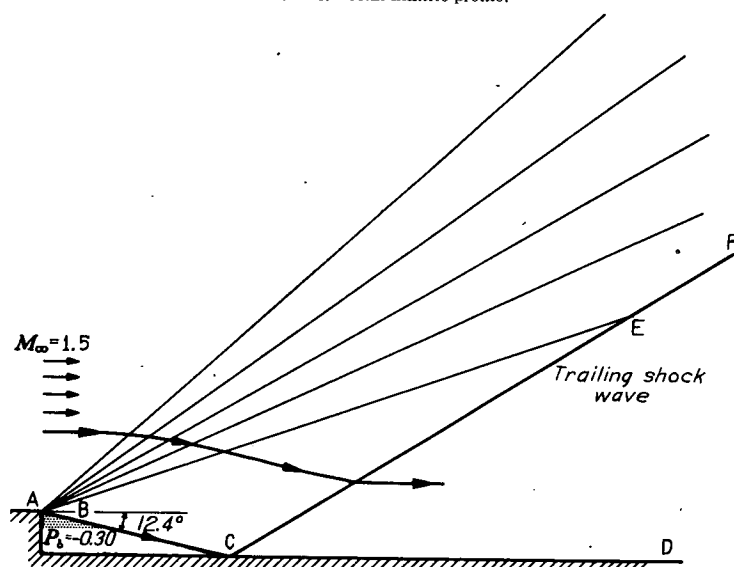


FIGURE 2.—Example of inviscid flow over a two-dimensional profile.

It is easy to construct a possible flow pattern which satisfies all necessary boundary conditions including the requirement of constant pressure in the dead-air region. For example, suppose the free-stream Mach number is 1.50 and some particular value of the base pressure coefficient, say $P_b = -0.30$ ($p_b/p_{\infty} = 0.53$), is arbitrarily chosen. Since the base pressure is prescribed, the initial angle of turning through the Prandtl-Meyer expansion (fig. 2) is uniquely determined, and in this particular case is equal to 12.4° at B. The pressure, and hence the velocity and Mach number, must be constant along the free streamline BC. For the example under consideration, the Mach number along the free streamline is calculated from the Prandtl-Meyer equations to be 1.92. For a uniform two-dimensional flow over a convex corner, the pressure depends only on the angle of inclination of a streamline, hence it follows that BC is a straight line. The triangle BCE therefore bounds a region of uniform flow having the same pressure as the dead-air

region. As the trailing shock wave (fig. 2) extends outward from E to infinity, interference from the expansion waves gradually decreases its strength until it eventually becomes a Mach wave. That part of the shock wave from C to E must deflect the flow through the same angle as the expansion waves originally turned it (12.4° for the particular example under consideration). This deflection certainly is possible since the Mach number in the triangle BCE is 1.92 which, according to the well-known shock-wave equations, is capable of undergoing any deflection smaller than 21.5° . As the flow proceeds downstream from the trailing shock wave CEF, the pressure approaches the free-stream static pressure, thus satisfying the boundary condition at infinity.

It is evident that a possible flow pattern has been constructed which satisfies all the prescribed requirements as well as the necessary boundary conditions. This flow, however, certainly is not the only possible one for the particular Mach number (1.50) under consideration, since any negative value of P_b algebraically greater than -0.30 also would have permitted a flow pattern to be constructed and still satisfy all boundary conditions. This is not necessarily true, though, if values of P_b algebraically less than -0.30 are chosen, as can be seen by picturing the conditions that would result if the base pressure were gradually decreased. The angle of turning through the Prandtl-Meyer expansion would increase and point C in figure 2 simultaneously would move toward the base. The base pressure can be decreased in this manner only until a condition is reached in which the shock wave at C turns the flow through the greatest angle possible for the particular local Mach number existing along the free streamline. The base pressure cannot be further reduced and still permit steady inviscid flow to exist. The flow pattern corresponding to this condition of a maximum-deflection shock wave can be considered as a "limiting" flow of all those possible. There are obviously an infinite number of possible flows for a given free-stream Mach Number, but only one limiting flow.

The limiting value of the base pressure coefficient can be calculated as a function of the free-stream Mach number by reversing the procedure described above for constructing possible flow patterns. Thus, for a given value of the local Mach number along the free streamline a limiting flow pattern can be constructed by requiring that the angle of turning be equal to the maximum-deflection angle possible for a shock wave at that particular local Mach number. By use of the Prandtl-Meyer relations the appropriate value of the free-stream Mach number is then directly calculated from the angle of turning and the local Mach number along the free streamline. This process can be repeated for different values of the local Mach number along the free streamline and a curve drawn of the limiting base pressure coefficient as a function of Mach number. Such a curve is presented in figure 3. The shaded area represents all the possible values of the base pressure coefficient for two-dimensional inviscid flow. The upper boundary of the shaded area corresponds to the limiting flow condition for various free-stream Mach numbers.

There is no reason a priori to say that for a given M_∞ the limiting flow pattern represents that particular one which most nearly approximates the flow of a real fluid.

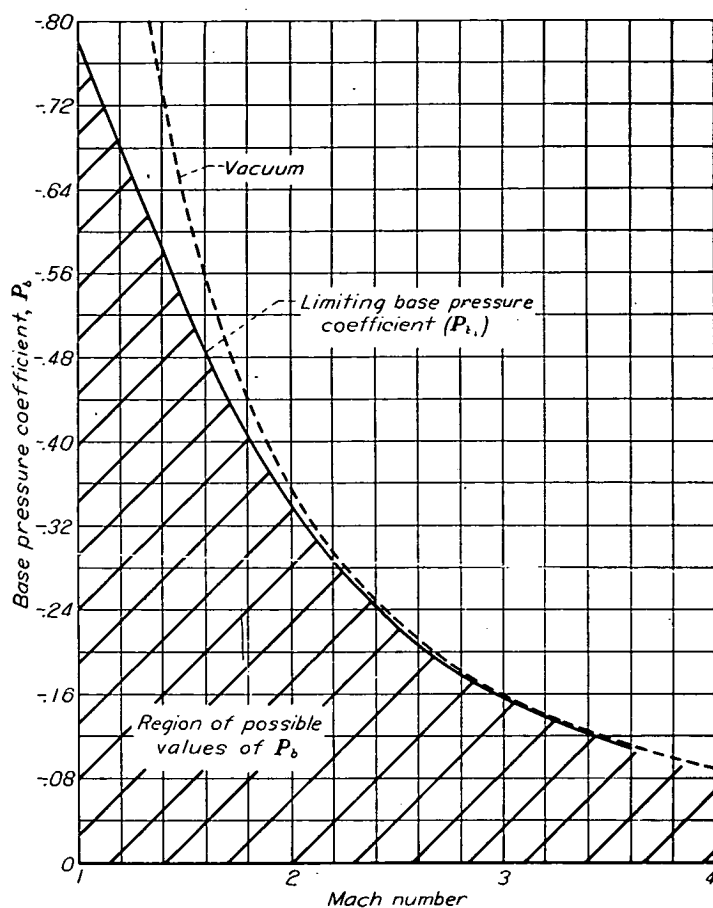


FIGURE 3.—Base pressure for two-dimensional inviscid flow.

The curve representing these limiting flow patterns can be considered simply as being the curve of maximum base drag (and hence maximum entropy increase) possible in an inviscid flow. This is the only interpretation that will be given to this curve for the time being. Since it is these limiting solutions which will be singled out later for further use, a special symbol P_{b_i} will be used to designate the base pressure coefficient of such flows. It is evident from figure 3 that in the Mach number region shown the values of P_{b_i} for two-dimensional flow correspond to very high base drags, being almost as high as if a vacuum existed at the base. At Mach numbers greater than or equal to 6.0, the values of P_{b_i} exactly correspond to a vacuum at the base.

AXIALLY SYMMETRIC INVISCID FLOW OVER A SEMI-INFINITE BODY

In principle the same method of procedure can be used for inviscid axially symmetric flow as was used for inviscid two-dimensional flow. The axially symmetric flows, however, are somewhat more involved than the corresponding two-dimensional flows. For example, in axially symmetric flow the expansion wavelets issuing from the corner of the base are not straight lines as they are in Prandtl-Meyer flow. Moreover, additional complications arise since the flow conditions upstream of the trailing shock wave do not depend solely on the inclination of the streamlines at a given point, but depend on the whole history of the flow upstream of the Mach lines passing through that point. As a consequence of these complications, the free streamline of constant pressure cannot be straight.

In order to construct possible flow patterns as was done in the two-dimensional case, the method of characteristics for axially symmetric flow must be used. The details of the particular characteristics method employed are described in reference 6. By employing the characteristics method the inviscid flow field corresponding to a given base pressure can be constructed step by step for any given value of the Mach number. The shape of the free streamline is, of course, determined by the condition that the pressure and the velocity must be constant along it. An example of such a construction for a free-stream Mach number of 1.5 is given in figure 4 (a). In this particular case, the base pressure

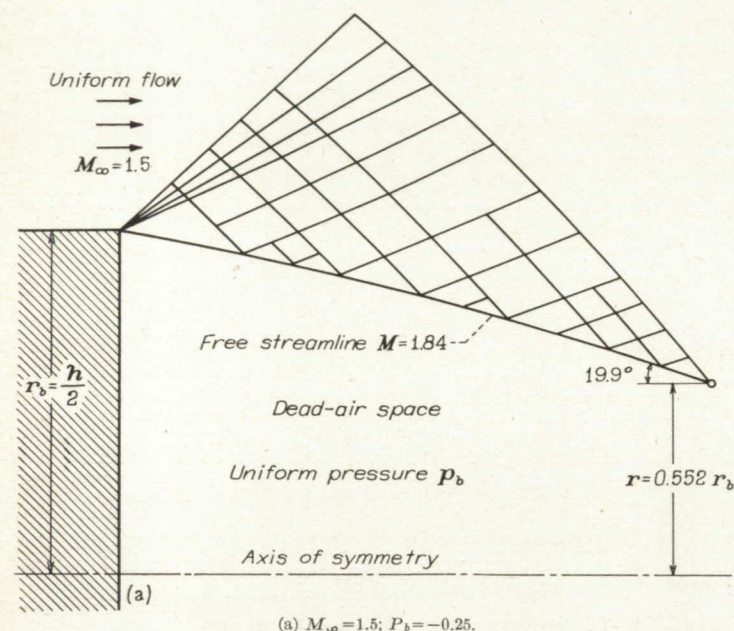


FIGURE 4.—Typical Mach nets for inviscid flow over the base of a semi-infinite body of revolution.

coefficient which has been chosen arbitrarily is -0.25 . It is to be noted that there is a striking difference between the axially symmetric flow (fig. 4 (a)) and the two-dimensional flow (fig. 2). The inviscid flow pattern for the axially symmetric case cannot be constructed all the way to the axis of symmetry and still satisfy the prescribed boundary conditions. This is a consequence of the curvature of the free streamline and the fact that the Mach number along the free streamline in the case under consideration is 1.84, which, at the most, is capable of deflecting a streamline only 19.9° by a single shock wave. As is illustrated in figure 4 (a), the angle of inclination of the free streamline for this example is already 19.9° at a value of $r/r_b = 0.552$, where r is the radial distance from the axis and $r_b = h/2$ is the radius of the base. Since the angle of inclination of the constant-pressure free streamline would continue to increase monotonically as the axis is approached, the flow pattern of figure 4 (a) cannot be constructed farther than the point shown ($r/r_b = 0.552$) and still permit the flow to be deflected through a single shock wave and become parallel to the axis of symmetry. This phenomenon is not attributable to the particular combination of Mach number and base pressure selected for figure 4 (a). In figures 4 (b), 4 (c), and 4 (d), other examples are presented which illustrate the flow for different values of Mach number and for different base pressures. In each case the free streamline has been ter-

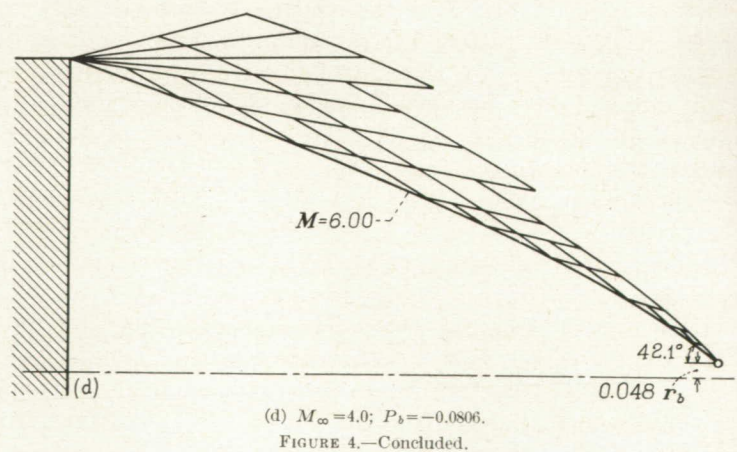
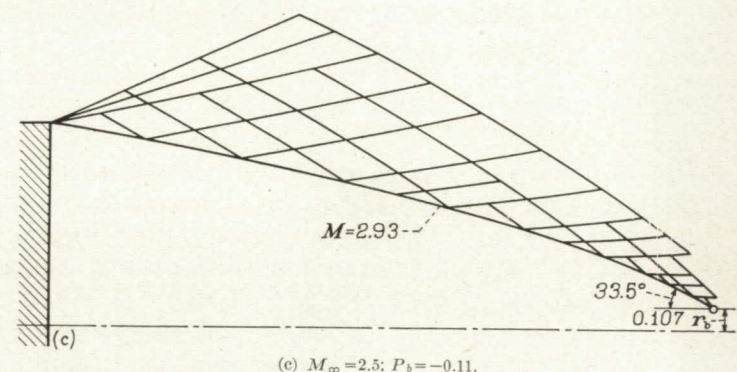
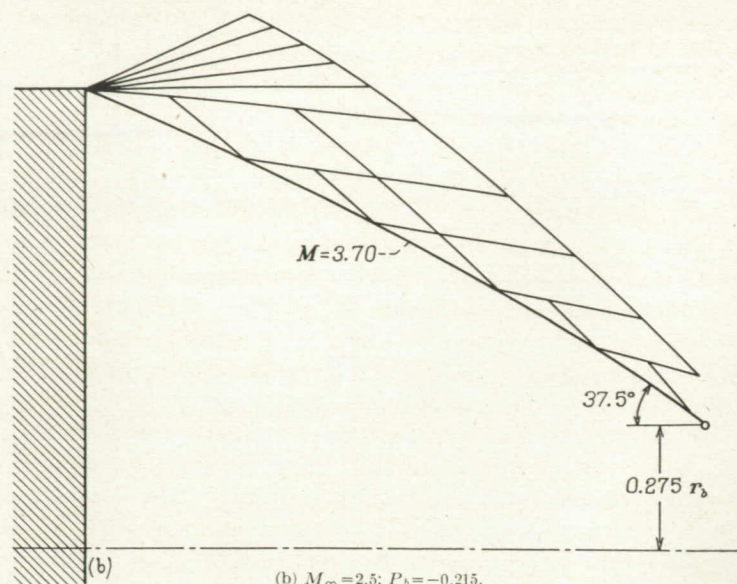


FIGURE 4.—Concluded.

minated at the point where the local angle of inclination is equal to the angle corresponding to the greatest possible deflection by a single shock wave. It is evident that none of these flow patterns could be constructed down to the axis of symmetry. Altogether, approximately 30 flow patterns were constructed by the characteristics method; in no case could the flow be constructed all the way to the axis.

The flow patterns built up by the method of characteristics should not be regarded as unrealistic simply because the flow cannot be constructed all the way to the axis. In a real fluid the flow outside the boundary layer is similar because the wake behind the body fills the region near the axis and prevents the outer flow from reaching the axis. This fact suggests that the axially symmetric inviscid-flow patterns

should be investigated further as they might bear some relation to actual flows if the displacement effect of the wake were considered.

The flow fields containing a free streamline not meeting the axis of symmetry can be considered as those that would exist in inviscid flow about a body of revolution which has an infinitely long cylindrical rod (or "sting") attached to the base. As an example, the flow of figure 4 (a) would correspond to a body having a rod of diameter $d=0.552h$ attached to the base. (See fig. 5.) With such a model the trailing shock wave turns the free streamline through the greatest deflection possible for the given local Mach number along the free streamline. The flow field is therefore the limiting flow field of all those possible for the given free-stream Mach number and the given ratio of d/h .

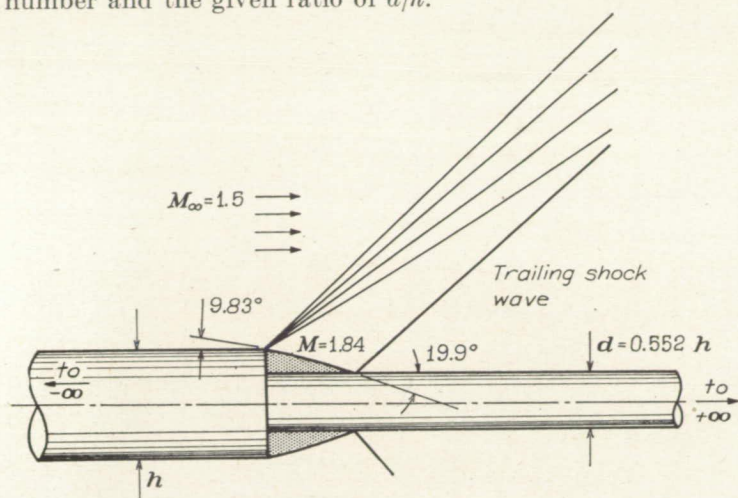


FIGURE 5.—Axially symmetric semi-infinite body with rod attached.

Just as in the case of the two-dimensional body, there are also an infinite number of possible flow patterns for the body of revolution with a rod attached. This is true because for a given configuration as many additional flow patterns as desired can be constructed by simply selecting the base pressure to be any pressure between the free-stream pressure and the pressure corresponding to the limiting flow. The limiting flow pattern is to be given the same physical significance for axially symmetric flow as for two-dimensional flow; that is, the corresponding base pressure coefficient P_{b_i} represents the maximum base drag possible for an inviscid flow with a single trailing shock wave and a given ratio of d/h .

By choosing different values of the base pressure coefficient for a fixed Mach number, the inviscid solutions determined by the method of characteristics enable a plot of P_{b_i} against d/h to be made. This procedure has been carried out for Mach numbers of 1.25, 1.5, 2.0, 2.5, 3.0, and 4.0. The results are shown in figure 6. Each point on the curves in this figure represents one flow pattern constructed by the characteristics method. The values for $d/h=0$ correspond to the semi-infinite body without a rod attached. It is to be noted that for each curve in figure 6 the value of P_{b_i} extrapolates to zero as d/h approaches zero. This means that the base pressure is equal to the free-stream static pressure, the free streamline is undeflected, and the base drag is zero. Hence, the limiting flow pattern

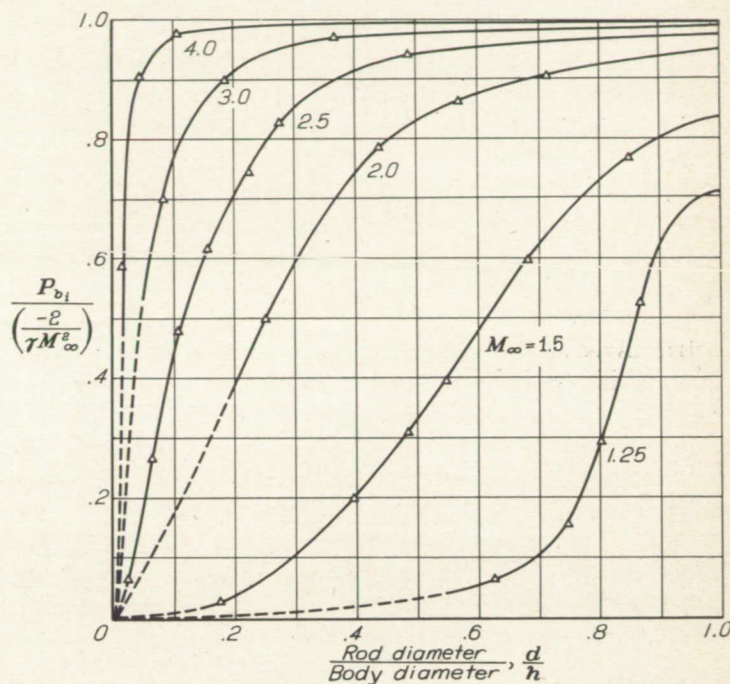
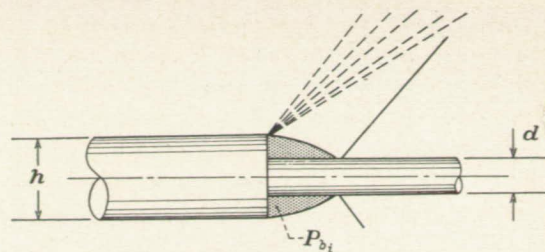


FIGURE 6.—Parameter proportional to the maximum base drag possible in an inviscid axially symmetric flow.

and the infinity of possible inviscid flows for $0 < d/h < 1$ degenerate into a single trivial solution corresponding to zero base drag for $d/h=0$. This behaviour appears anomalous on first thought, particularly when one remembers that the coefficient P_{b_i} represents the maximum possible base drag that can exist for an inviscid flow of the type being considered. An explanation can be obtained from a consideration of the equations of motion since they are the basis for the method of characteristics. This explanation, however, is not essential for an understanding of the main conclusions regarding base pressure, and hence is presented as Appendix A.

In figure 6 the limiting values as d/h approaches 1.0 correspond to the previously treated case of two-dimensional flow. It can be seen that this must be the case by visualizing the limiting process as taking place with both d and h approaching infinity, but with the difference $(h-d)$ held constant. The configuration approached in this manner would be a two-dimensional step of height $(h-d)/2$; hence the pressure coefficient approached would be the limiting base pressure coefficient for two-dimensional inviscid flow. On the other hand, if d/h is equal to unity (instead of approaching it from values always less than unity), then the corresponding configuration would be a semi-infinite body of revolution with a cylindrical rod of the same diameter attached to the base. Although no dead-air region exists

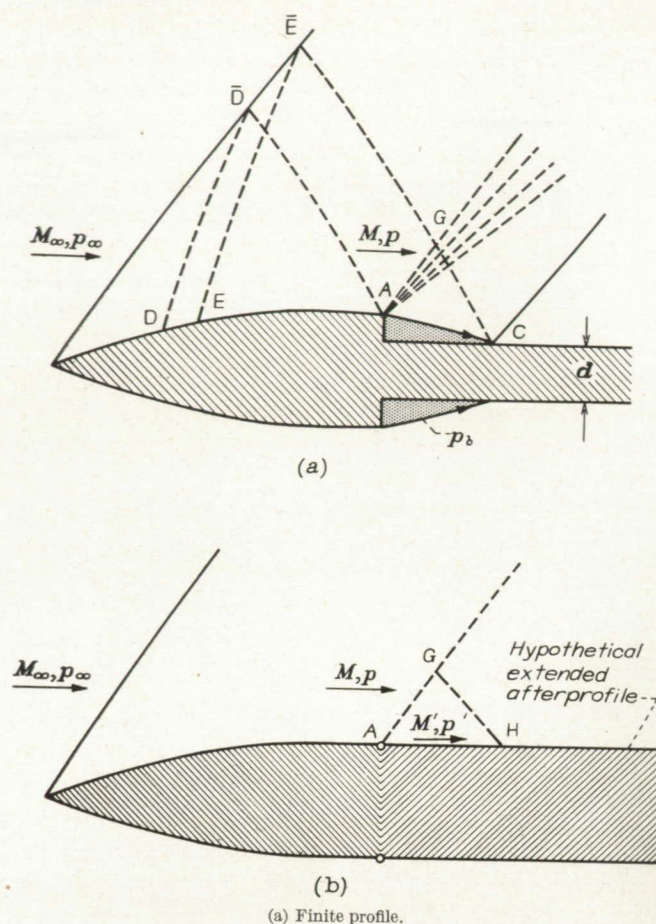
in this latter case since the flow is everywhere uniform, the base pressure in the physical sense would be the static pressure at the junction of body and rod, and hence P_{b_i} would be zero.

The occurrence of more than one possible solution in two-dimensional flow and also in axially symmetric flow with a rod attached does not represent a new occurrence in inviscid flow theory. A similar situation occurs, for example, in airfoil theory for an inviscid, incompressible fluid. As is well known, a satisfactory solution in this case has been found in the use of the so-called Kutta condition, which can be readily justified on the basis of qualitative consideration of viscous effects near the trailing edge. Apart from the effects of viscosity several other considerations, such as stability of the flow, also have been of importance in other unrelated problems when selecting a suitable inviscid flow solution from a possible choice of more than one. As an example of this, the inviscid channel flow studied in reference 7 may be cited. For the present problem, however, the preceding analysis of axially symmetric inviscid flows points toward viscous effects (rather than stability of inviscid flow) as being the essential mechanism determining the base pressure. Before considering viscous effects, however, the effect on base pressure of variations in profile shape will be analyzed in detail since experiments have indicated widely different results for various profiles. The method presented later for correlating base pressure data requires that the measurements first be corrected for the effect of profile shape. In the section which follows equations are developed for such a correction.

TWO-DIMENSIONAL AND AXIALLY SYMMETRIC INVISCID FLOW OVER FINITE PROFILES

In this section consideration is given to the flow over a finite two-dimensional profile concurrently with that of a finite body of revolution. For either type of flow, the presence of the profile causes the Mach number and pressure in the flow field ahead of the base (M, p) to be nonuniform and different from free-stream conditions (M_∞, p_∞). This is illustrated in figure 7 (a) for a profile without boattailing. As a result of the disturbance caused by the profile, the base pressure depends on profile shape even in an inviscid flow. In this section, a method is developed for calculating corrected free-stream conditions (M', p') to which the base pressure can be referred and be nearly independent of profile shape. This method does not depend on the magnitude of the base pressure or on the dimension d (fig. 7 (a)), and hence is useful in comparing experimental measurements made on various airfoils and bodies of revolution.

To fix ideas, the Mach lines shown as dotted lines in figure 7 (a) will be thought of as representing weak pressure waves; those with positive tangents (e. g., $D\bar{D}$) being members of the so-called first family, and those with negative tangents (e. g., $\bar{D}A$) being members of the so-called second family. Weak pressure waves issuing from the body can affect the base pressure in several ways. For example, waves of the first family starting between D and E not only affect conditions at A , but also affect conditions between A



(a) Finite profile.
(b) Finite profile with extended afterprofile.
FIGURE 7.—Sketch of inviscid flow over finite profile without boattailing.

and G . Such waves reflect from the bow shock wave between D and $\bar{E}$ and then become members of the second family of waves between $\bar{D}A$ and $\bar{E}G$ which directly interact with the dead-air region. Waves of the second family beyond $\bar{E}G$ would not affect the base pressure in an inviscid flow. The net effect of profile shape on the base pressure of a finite body, therefore, will be determined both by conditions at A and by the variation of conditions between A and G . If a hypothetical afterprofile were extended from the base, as illustrated in figure 7 (b), then such conditions would cause the average pressure (p') and Mach number (M') along AH of the extended afterprofile to differ from the corresponding free-stream conditions. These differences would represent the disturbance field induced near the base by the profile shape, and the base pressure referred to M' and p' (e. g., a curve of P_b' or p_b/p' versus M') could be regarded as corrected for the effects of profile shape in inviscid flow.² By applying the compatibility equations of the method of characteristics for either two-dimensional or axially symmetric flow to the triangle AGH in figure 7 (b), it can be deduced that the magnitude of the velocity averaged at points A and H is approximately equal to the magnitude of the velocity at point G . Thus, M' and p' can be evaluated either from conditions along a hypothetical extended afterprofile, or else from conditions at an appropriate point (G) in the flow over the given profile.

A second case to be considered is that of a profile having a

² It may be noted that M' and p' are analogous in some respects to the corrected free-stream Mach number and pressure used in subsonic wind-tunnel operation: the former represent the average Mach number and pressure induced in the vicinity of the base by the presence of the profile; whereas the latter represent the average Mach number and pressure induced in the vicinity of the test model by the presence of the tunnel walls. Both corrections are accurate only when the induced disturbance field is small and approximately uniform over the region in question.

negative boattail angle (β), as illustrated in figure 8 (a). This flow can be converted to an equivalent flow over a profile without boattailing having the same base pressure as the flow of figure 8 (a) and certain nonuniform conditions ahead of the base. This equivalent flow, illustrated in figure 8 (b), is identical to the type already considered and is such that the flow within $C'O'G'$ coincides with the flow within COG in figure 8 (a). Point G, therefore, is defined by the intersection of the Mach line passing through C, and the particular Mach line passing through O on which the velocity vector at O is parallel to the free-stream direction. Hence, for this second case also, M' and p' can be determined approximately either from conditions on a hypothetical extended afterprofile, or else from conditions in the original flow at point G.

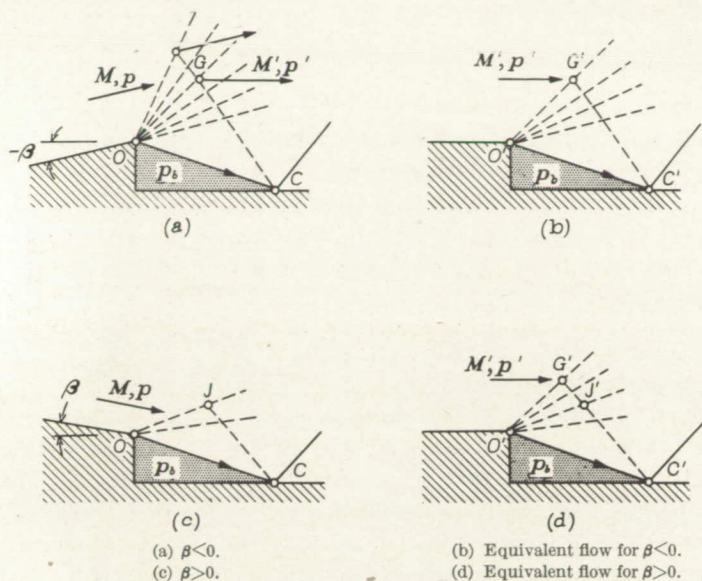


FIGURE 8.—Sketch of inviscid flow in vicinity of base for profiles with boattailing.

A third and last case to be considered is that of a profile having a positive boattail angle, as illustrated in figure 8 (c). This flow also can be converted to an equivalent flow over a profile without boattailing having the same base pressure as the original flow (fig. 8 (c)), and certain nonuniform conditions ahead of the base. As sketched in figure 8 (d), the equivalent flow ahead of the base is such that the conditions downstream of $O'J'$ are identical to conditions downstream of OJ in figure 8 (c).³ Thus for $\beta > 0$, M' and p' can be determined approximately from conditions at G' in the equivalent flow, or else from conditions along a hypothetical profile extended downstream from O' , but M' and p' do not necessarily exist at any easily determined point in the original flow.

For any profile the relationship between the base pressure coefficient $P_b' \equiv (p_b - p')/q'$ which corresponds to the Mach number M' , and the base pressure coefficient $P_b \equiv (p_b - p_\infty)/q_\infty$ which corresponds to the Mach number M_∞ and to the given profile, is given by the equation

$$P_b' = \frac{q_\infty}{q'} (P_b - P') \quad (1)$$

where

$$P' = (p' - p_\infty)/q_\infty \quad (2)$$

and, if the profile disturbance field is small,

$$\frac{q'}{q_\infty} = 1 + \left(\frac{M_\infty^2}{2} - 1 \right) P' - \frac{2}{\gamma M_\infty^2} \left(1 + \frac{\gamma - 1}{2} M_\infty^2 \right) \frac{\Delta p_o}{p_o} \quad (3)$$

In this last equation (derived in appendix C), $\Delta p_o/p_o$ is the fractional loss in total pressure on passing through the bow wave. If the ratio p_b/p_∞ is used instead of the coefficient P_b , the analogous relation between the ratio p_b/p' and p_b/p_∞ obviously is

$$\frac{p_b}{p'} = \frac{1}{(p'/p_\infty)} \frac{p_b}{p_\infty} \quad (4)$$

For a given profile, these equations enable a curve of P_b' (or p_b/p') versus M' to be plotted if a curve of P_b (or p_b/p_∞) versus M_∞ is known.

In order to further clarify the concept of the disturbance field induced by profile shape, and also to illustrate the magnitude of the variations in base pressure that might be expected between different profiles, some representative calculations of M' and p' have been prepared in tables I, II, and III. For simplicity in these calculations, M' and p' have been evaluated along the hypothetical extended afterprofile at a distance h behind the base position, rather than to use in each case a more involved average over the appropriate extent of dead air. Table I applies to two-dimensional flow over the particular profile shown. The computations for $M_\infty = 1$ are based on the pressure distributions calculated by Guderley and Yoshihara in reference 8; the computations for other Mach numbers in this table are based on shock-expansion theory. It is evident that the disturbance field near the base is significant at low supersonic Mach numbers where the bow wave is detached, and also at hypersonic Mach numbers where the bow wave is very strong. At moderate supersonic Mach numbers, however, the profile disturbance field in two-dimensional flow is negligible, and conditions on a thin airfoil depend solely on the local surface inclination. It follows that the base pressure under such circumstances is nearly independent of profile shape and boattail angle. (If the angle of attack is small the base pressure is also nearly independent of angle of attack under these conditions.)

Table II, which is based on the method of characteristics, applies to the cone-cylinder body of revolution shown, and illustrates that the correction for the profile disturbance field is not large if the afterbody comprises a cylinder several diameters long. For example, at a Mach number of 1.5 for which the value of p_b/p_∞ is about 0.7, the value $p'/p_\infty = 0.98$ corresponds to a correction of about 6.7 percent to the base drag (since the base drag is proportional to $(1 - p_b/p_\infty)$).

Table III applies to a cone ($\beta = -10^\circ$), and illustrates that the correction for such profiles can be sizable. At a Mach number of 1.5, for example, the induced pressure field in this case amounts to over one-fourth of the base drag. For larger apex angles, the corresponding correction for cones can be considerably larger. It is to be noted that the induced pressure field usually represents a much more important correction to base drag than the induced Mach number field.

³ Such an equivalent flow can readily be constructed if the Mach number on the surface just ahead of the base in the original flow is sufficiently large, or if β is sufficiently small, to insure supersonic velocities along $O'G'$ in the equivalent flow.

II. A SEMI-EMPIRICAL METHOD FOR CORRELATING BASE PRESSURE MEASUREMENTS AND COMPARISON WITH EXPERIMENTAL RESULTS

QUALITATIVE EFFECTS OF VISCOSITY ON THE BASE-PRESSURE FLOW

A sketch showing the qualitative flow characteristics for the viscous-fluid flow in the region of the base is given in figure 9. The flow along the first expansion wavelet starts with the nonuniform distribution of Mach number M , pressure p , and with a boundary-layer thickness δ . Because the base pressure is lower than the pressure p , a small fan of expansion wavelets originates at point A. The existence

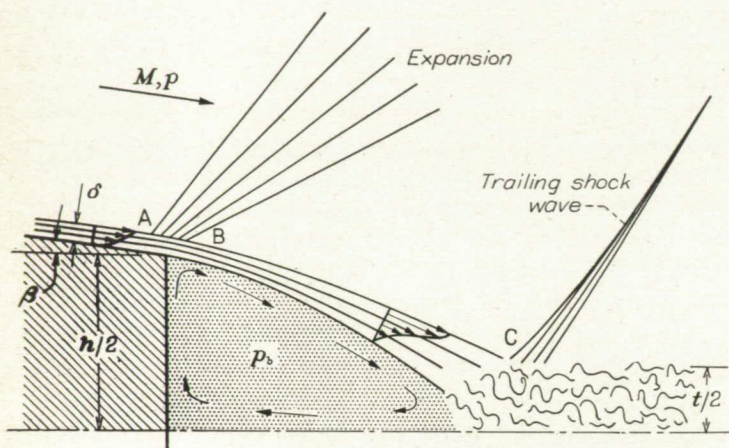


FIGURE 9.—Sketch of the viscous-fluid flow in the neighborhood of the base.

of dead air in a small volume immediately behind the base is a result of the separation at point B. As a consequence of the formation of a dead-air region it might be expected that the pressure along the streamline BC is approximately constant. The qualitative form of the boundary-layer profiles at stations between points B and C must take on the same nature as those existing at the boundary of a supersonic jet issuing into ambient air. Because of the viscosity of the fluid, the dead air is induced into a circulatory motion in the directions indicated by the small arrows in figure 9. The viscous mixing process causes the boundary layer to thicken as it approaches point C. In axially symmetric flow there is an additional reason for further spreading of the streamlines in the boundary layer as the trailing shock wave is approached. Since the mean radius of a streamtube in the boundary layer continually decreases as the trailing shock wave is approached, additional spreading is brought about in order to keep the annular cross-sectional area of the streamtubes approximately constant.

With this qualitative picture of the flow processes in mind, a brief description can be given as to how the base pressure arrives at its steady-state equilibrium value. To fix conditions in mind, suppose a jet of air is pumped from the body into the dead-air region and then is suddenly stopped. At the instant the jet is turned off, point C is far downstream of its equilibrium position. Due to the scavenging effect of the outside flow on the mass of dead air, some of this dead air is removed, thus causing the angle of turning at the corner to be increased and the pressure of the dead-air region to be decreased. The larger angle of turning increases the velocity outside the boundary layer, which in

turn increases the scavenging action, thereby again lowering the pressure and starting the cycle over again. Thus, point C moves rapidly to a position as close to the base as possible. There is, however, at least one important factor which prevents point C from going as far toward the base as that point which would roughly represent the limiting solution for inviscid flow. As C moves toward the base, the pressure ratio of the trailing shock wave increases, making it more difficult for the scavenged air and the low-velocity air in the boundary layer to overcome the pressure rise of the shock wave and flow downstream. The opposition of this effect to the ones mentioned previously would serve to establish equilibrium. It appears, therefore, that a satisfactory theory of base pressure would have to consider the mixing process in conjunction with the inviscid-fluid characteristics of the outer flow.

BASIS FOR CORRELATION OF EXPERIMENTAL DATA

It is assumed that the flow expands over the corner of the base as illustrated in figure 9. The base thickness h would be the trailing-edge thickness in the case of two-dimensional flow, and would be the base diameter in the case of axially symmetric flow. An attempt to correlate the various measurements of base pressure is made on the basis of the relationship

$$P_b' = f\left(M', \frac{\delta}{h}, \beta\right) \quad (5)$$

which assumes that the base pressure coefficient corrected for the profile disturbance field is affected by viscous effects only through the ratio of boundary-layer thickness to base thickness. Actually, even for a fixed value of δ/h the base pressure would be affected by anything that affects the distribution of fluid properties within the boundary layer or within the mixing layer downstream of the base. It will be seen subsequently, though, that in many cases the above relationship yields acceptable results.

If the boundary-layer flow is laminar, then from dimensional analysis and the classical considerations of the terms involved in the boundary-layer equations, it follows that

$$\delta \sqrt{\frac{U_\infty}{\nu_\infty L}} = f(M_\infty, \text{profile shape})$$

Rewriting this equation,

$$\frac{\delta}{h} = \frac{L/h}{\sqrt{\frac{U_\infty L}{\nu_\infty}}} f(M_\infty, \text{profile shape}) = \frac{C}{\sqrt{Re}} \frac{L}{h}$$

where C is a function of the Mach number and profile shape, but independent of viscosity. For a given L/h , variations in profile shape affect the boundary-layer thickness principally through the action of the pressure gradients set up by the particular profile contour. As a first approximation the effects of variations in pressure distribution on the thickness of the boundary layer just ahead of the base will be neglected since these effects in most cases should be small compared to the effects of Reynolds number and L/h ratio. Within the limits of this simplification, the above equation is

applicable to any profile shape or length. Hence in correlating the data for laminar-boundary-layer flow, the parameter $L/(h\sqrt{Re})$ is used in the absence of direct measurements of δ/h .

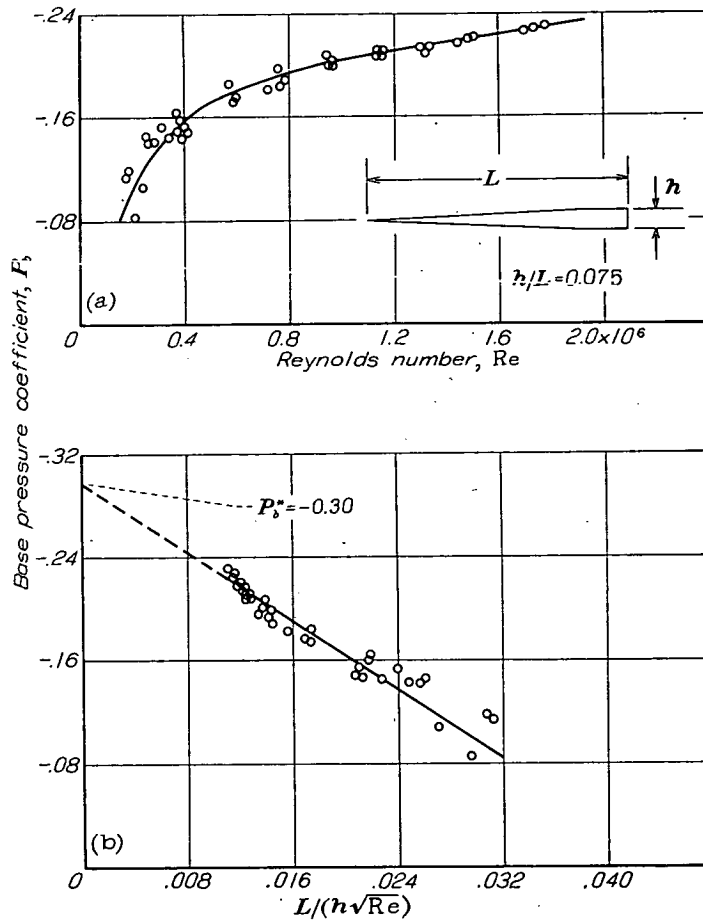
In the case of turbulent flow a similar parameter can be obtained. By approximating the turbulent boundary-layer profile with a $1/7$ -power law, the ratio δ/h for low-speed flow turns out to be inversely proportional to the $1/5$ power of the Reynolds number. (For example, see reference 9.) Using this result, the appropriate parameter in correlating base-pressure data for turbulent boundary-layer flow would be $L/[h(Re)^{1/5}]$.

EXPERIMENTAL DATA FOR TWO-DIMENSIONAL FLOW

At present the available experimental results on base pressure in two-dimensional flow are rather limited, but they are sufficient to provide a qualitative check on one particular result of the inviscid-flow calculations; this result concerns the essential difference, as indicated by the inviscid-flow calculations, between the base pressure in two-dimensional flow and in axially symmetric flow. The absolute magnitude of the base pressure coefficient for two-dimensional inviscid flow at a given Mach number is represented by the limit of the values for axially symmetric flow as d/h approaches unity in figure 6. For low and moderate supersonic Mach numbers this limiting value is several times the value for a body of revolution, which, as will be seen later, is represented in figure 6 by a d/h ratio somewhere between 0.5 and 0.8. For high supersonic Mach numbers the difference between the two types of flow, according to figure 6, is small. These considerations which indicate that, except at high supersonic Mach numbers, a pronounced difference should exist between the base pressure in two-dimensional and axially symmetric flow, are in agreement with existing data. In reference 10, the wind-tunnel measurements for two-dimensional flow over a wedge airfoil at a Mach number of 1.4 and a Reynolds number of 0.6 million indicate a value of -0.41 for the base pressure coefficient. Measurements presented later for axially symmetric flow at the same Mach number and Reynolds number, however, indicate values around -0.20 . This large difference is in accord qualitatively with the considerations based on the curves of figure 6.

In order to make a preliminary evaluation of the Reynolds number effect on base pressure in two-dimensional flow, some measurements have been made on a constant-chord wing of finite span having a thick trailing edge.⁴ Because the ambient air near the wing tips can flow laterally around the tip and into the low-pressure region behind the base, the data cannot be considered as strictly representing two-dimensional flow. Nevertheless, the ratio of span to base thickness (40) was sufficiently large on the wing employed so that tip effects should not affect conclusions concerning the qualitative influence of Reynolds number on base pressure in two-dimensional flow. The results of base-pressure measurements taken at a Mach number of 2.0 are shown in figure 10 (a). It is apparent that the base drag increases considerably as the Reynolds number increases. Since the surfaces of the wings were smooth, and the highest Reynolds

number attained was 1.8 million, the data are representative of the case of laminar flow in the boundary layer. A plot of these data against the parameter $L/(h\sqrt{Re})$ is shown in figure 10 (b). It is to be noted that in this form a straight line can be faired through the data in the region covered by the tests. For larger values of $L/(h\sqrt{Re})$ the line would be expected to curve and approach the line representing zero base drag.



(a) Base pressure as a function of Reynolds number.
(b) Base pressure as a function of $L/h\sqrt{Re}$.
FIGURE 10.—Measured base pressure on a finite-span wing; $M_\infty = 2.0$, ratio of wing span to base thickness = 40.

EXPERIMENTAL DATA FOR AXIALLY SYMMETRIC FLOW

Fortunately, there are sufficient experimental data available for axially symmetric flow to make a fairly extensive correlation of P_b' with the parameters $L/(h\sqrt{Re})$ and $L/[h(Re)^{1/5}]$, where h is now the base diameter. Most of these data have been obtained from wind-tunnel measurements on bodies of revolution mounted from the rear by a cylindrical support. Accordingly, a knowledge of the possible support and wall interference effects is necessary for a satisfactory interpretation of the wind-tunnel measurements. Some experimental data on support interference and reflected bow-wave interference are presented in appendix B. It will suffice for the present purposes to state that the wind-tunnel measurements were taken with a support sting of sufficient unobstructed length so that no interference effect of support length is present in the data. Likewise, no appreciable interference resulting from the

⁴ These data were taken in the Ames 1- by 3-foot supersonic wind tunnel No. 1 employing a wing of 9-inch span with a base-pressure orifice located 1 inch outboard of the plane of symmetry.

reflected bow wave is present in the data. As regards the effects of support diameter, it is known from a relatively complete set of interference measurements made by Perkins (reference 11), part of which is presented later, that the data taken at $M=1.5$ are essentially free of support interference. At the higher Mach numbers, however, a complete set of support-diameter interference measurements was not made. Consequently, some effect may be present in the data taken at $M=2.0$ and $M=2.9$. For consistency, these data which may be affected to a small extent by support-diameter interference have been taken with a fixed value of 0.4 for the ratio of support diameter to base diameter. By comparing the base pressure measured on various bodies tested with the same relative support diameter, the effects of body shape can be deduced if it is assumed that changes in nose shape do not produce significant changes in the support interference. This is believed to be a valid assumption for the body and support dimensions used.

In reducing the experimental data for correlation, the measurements are first corrected for the disturbance field induced by profile shape. All bodies of revolution used in the present experiments consisted of either a cone-cylinder (10° semiangle of cone) or an ogive-cylinder (10-caliber ogival radius) combination. In order to determine the body-shape correction (P') the pressure distribution over such combinations has been calculated using the method of characteristics. Two typical pressure distributions for a Mach

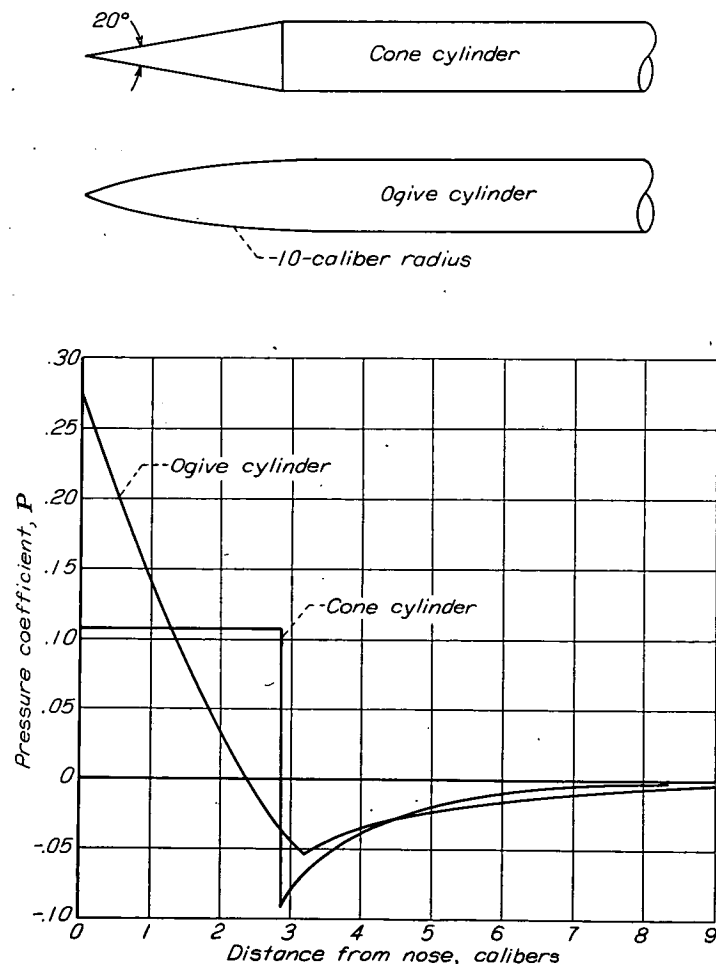


FIGURE 11.—Typical pressure distribution as determined by the method of characteristics; $M_\infty=2.0$.

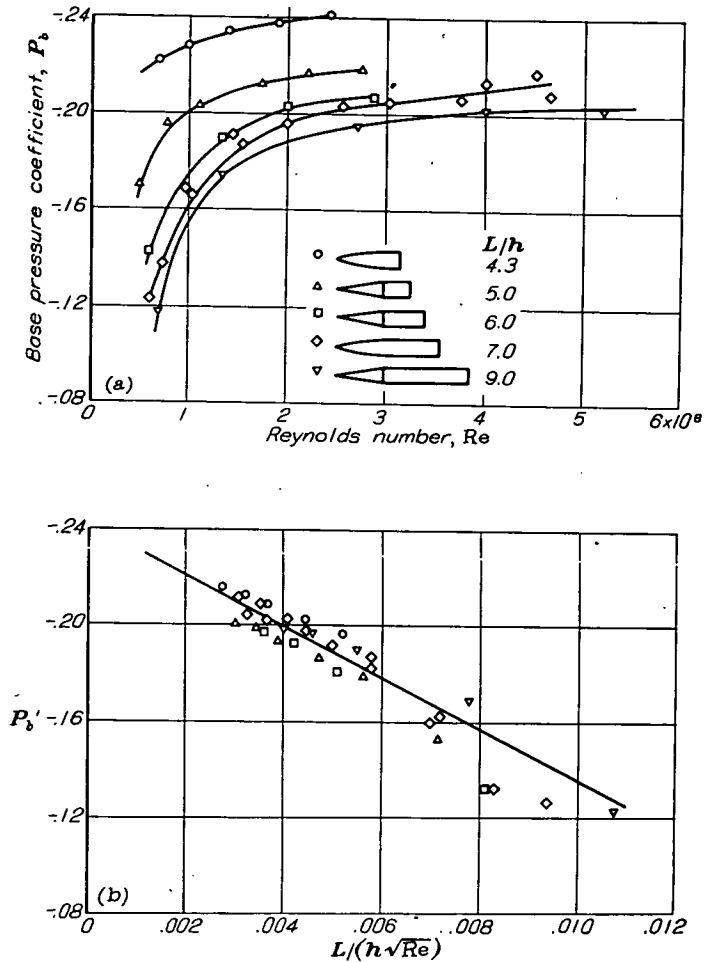
number of 2.0 are shown in figure 11. For the reasons explained earlier, the correction P' is determined by selecting the value of the pressure coefficient existing on an extension of the cylindrical afterbody at a location approximately one diameter downstream of the base position. The values of P' determined in this manner enable the corresponding values of P_b' to be determined from equations (1) and (3).

The quantity P_b' should not depend on the body shape for a given M' . For all but a few exceptional shapes, such as a simple cone without an afterbody, the Mach number M' in the present tests is sufficiently close to the free-stream Mach number to enable a direct comparison to be made between various body shapes after correcting for the pressure disturbance field only. For these exceptional cases, which represent small values of the length-diameter ratio, an additional correction $\frac{\partial P_b}{\partial M} (M' - M_\infty)$ is added to the right side of equation (1), so that the comparison of various bodies is made on the basis of a constant M' equal to M_∞ . Since even in an extreme case this latter correction is small compared to P' , the derivative $\frac{\partial P_b}{\partial M}$ can be roughly estimated

without affecting the final results appreciably. For the data to be presented subsequently, this correction was made only for those bodies with a length-diameter ratio of 4 or less, since it amounted to only 6 percent of the measured data in the most extreme case ($L/h=0.9$) and was negligible for the bodies with L/h greater than 4.

In attempting to correlate the available experiments it will be convenient to consider first the case of laminar flow in the boundary layer, and then the case of turbulent flow. The experiments representing the case of laminar boundary-layer flow were conducted on bodies of revolution with polished surfaces, and those representing turbulent flow were conducted on the same models with artificial roughness added in the form of a narrow transition strip. (See reference 12.) Although for simplicity the data are referred to simply as representing either laminar or turbulent flow, in a few cases the actual boundary layer may be in the transition state. It is to be noted that with smooth models transition (insofar as it affects base pressure) probably begins at Reynolds numbers of the order of 4 million. Likewise, with roughness added in order to obtain turbulent flow, the artificial roughness may not bring about complete transition ahead of the base at Reynolds numbers less than about 2 million.

Laminar boundary-layer flow approaching base.—Wind-tunnel measurements of the base pressure for various bodies of revolution at a Mach number of 1.53 are shown in figure 12 (a). These data, taken from reference 12, include the effect of variations in Reynolds number and body shape. The large effect of both Reynolds number and body shape is evident. Since the boundary-layer flow is laminar for these data, the extent to which correlation is achieved is most easily determined by plotting P_b' as a function of $L/(h\sqrt{Re})$. Figure 12 (b) shows the data of figure 12 (a) plotted in this form, from which it is evident that the experimental data correlate reasonably well to a single curve. The scatter of the various measurements about the mean line is attributed partly to the fact that the thickness and



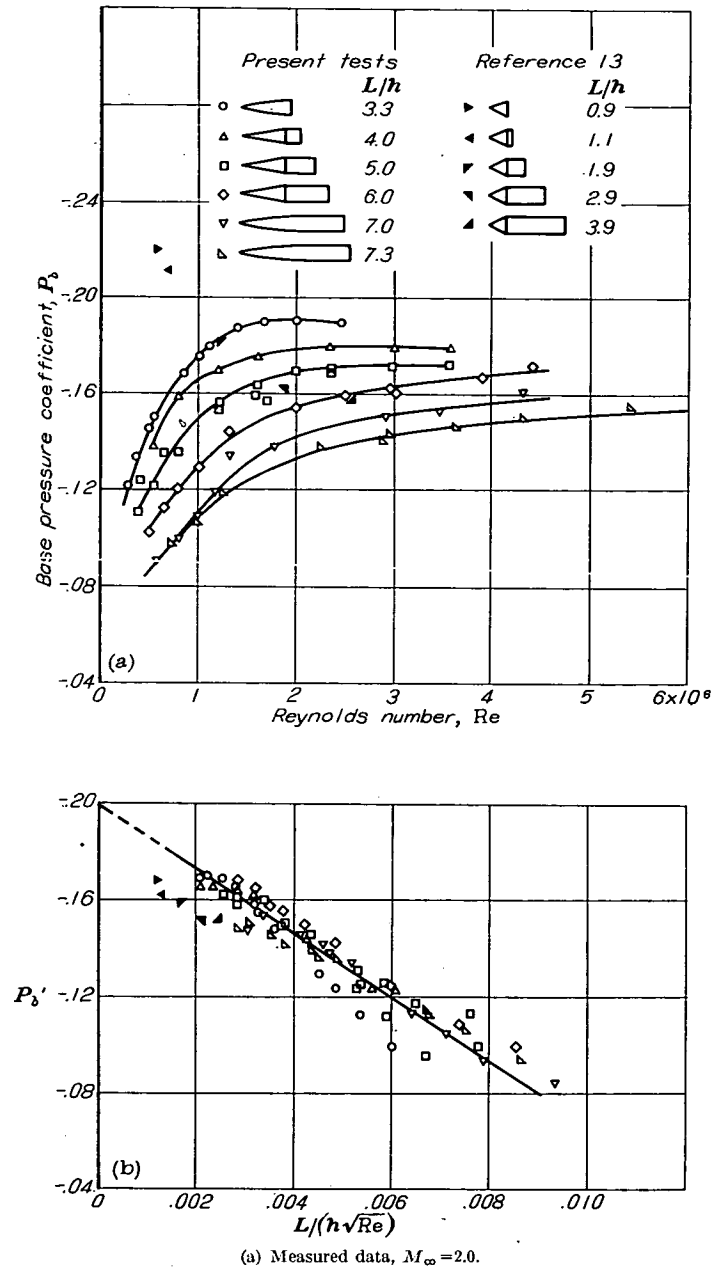
(a) Measured data, $M_\infty = 1.53$.
(b) Correlation of measured data, $M_\infty = 1.53$.

FIGURE 12.—Measured and correlated base pressure data; $M_\infty = 1.53$, laminar boundary-layer flow.

velocity profile of the boundary layer approaching the base, and hence the base pressure, are not strictly a function of the Reynolds number and length-diameter ratio alone.

The results of some measurements of the base pressure for various bodies with laminar boundary-layer flow at a Mach number of 2.0 are shown in figure 13 (a). The data through which curves are drawn were taken in the Ames 1- by 3-foot supersonic wind tunnel No. 1 under conditions similar to the tests at a Mach number of 1.53 reported in reference 12. The remaining data points were obtained from the experiments of Kurzweg (reference 13) by plotting his data for insulated smooth bodies as a function of Mach number, and reading the values of base pressure for $M_\infty = 2.0$ from the faired curves. The same qualitative effects of body shape and Reynolds number as were observed at a Mach number of 1.53 are evident from these data obtained at the higher Mach number. Figure 13 (b) shows the data of figure 13 (a) plotted in the form suitable for correlation according to the theoretical considerations. Considering the wide variety of body shapes tested, it can be seen that these data also correlate reasonably well to a single straight line. If the tests were extended to larger values of L/h , this line presumably would curve and approach the abscissae axis.

Turbulent boundary-layer flow approaching base.—The results of wind-tunnel measurements of base pressure on bodies of revolution at a Mach number of 1.5 with turbulent



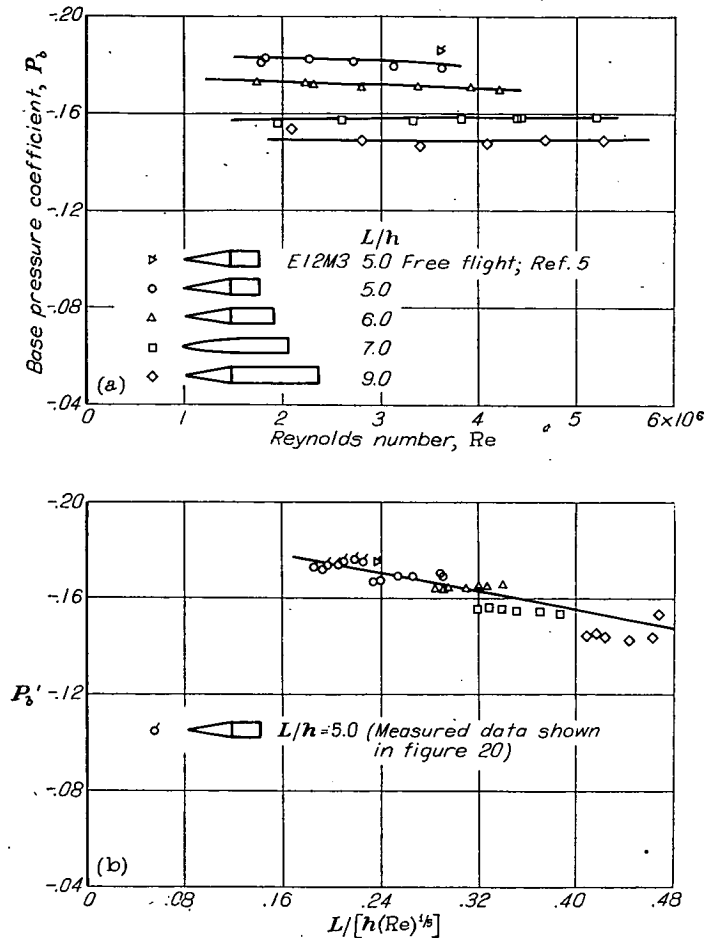
(a) Measured data, $M_\infty = 2.0$.
(b) Correlation of measured data, $M_\infty = 2.0$.

FIGURE 13.—Measured and correlated base pressure data; $M_\infty = 2.0$, laminar boundary-layer flow.

boundary-layer flow approaching the base are shown in figure 14 (a). Also shown in this figure are the results of free-flight measurements reported by Charters and Turetsky in reference 5. It is evident from this figure that the effect of Reynolds number on base pressure is small; whereas figure 12 (a) shows that it is large in the case of laminar boundary-layer flow.

The measured data of figure 14 (a) are shown in figure 14 (b) plotted in the form suitable for purposes of correlating experimental data. Since the body-shape correction (P_b') is independent of viscous effects, the same corrections have been used for the case of turbulent flow as were used for laminar flow. It may be seen from figure 14 (b) that the data correlate fairly well to a straight line.

Some experimental data for turbulent boundary-layer flow at a Mach number of 2.0 are shown in figure 15 (a) and the plot of P_b' against $L/[h(Re)^{1/5}]$ is shown in figure 15 (b). The curves in these figures show the same charac-

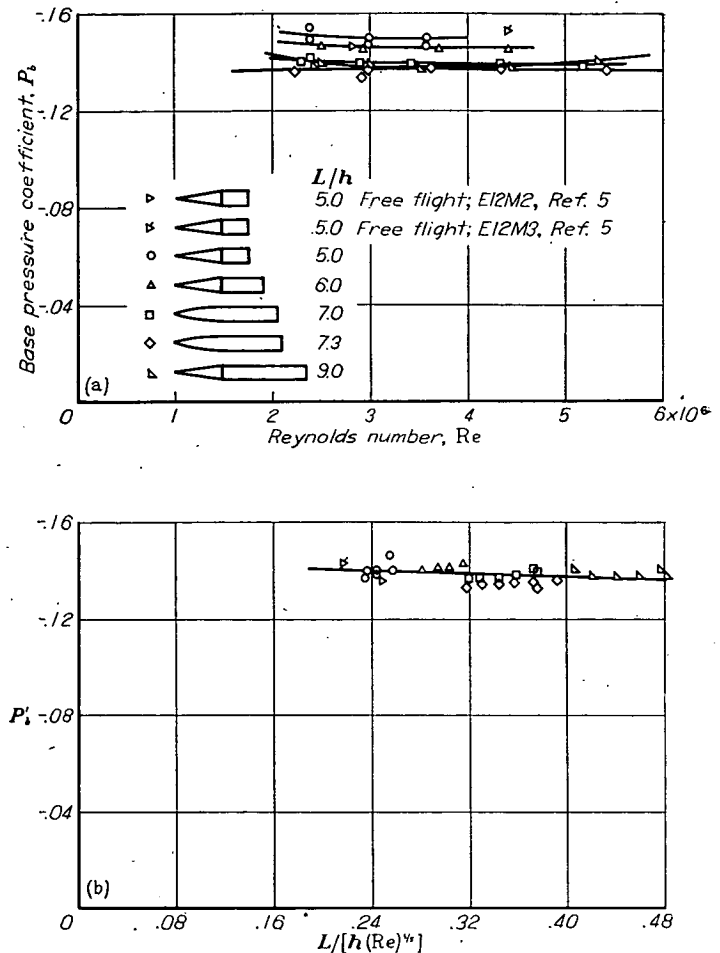


(a) Measured data, $M_\infty = 1.5$.
(b) Correlation of measured data, $M' = 1.5$.
FIGURE 14.—Measured and correlated base pressure data; $M_\infty = 1.5$, turbulent boundary-layer flow.

teristic of relatively constant base pressure as was noted above for turbulent boundary-layer flow at a Mach number of 1.5. Again, there is a reasonably good correlation of these data, as is evident from figure 15 (b).

COMPARISON OF EXPERIMENTAL RESULTS WITH THE INVISCID-FLOW CALCULATIONS

Since the intercept (P_b^*) of a curve of P_b' versus δ/h is independent of the Reynolds number, some correlation (possibly only qualitative) might be expected between the experimental values of P_b^* and the inviscid-flow calculations, provided allowance is made for the displacement effect of the wake near the trailing shock wave. As long as the wake thickness is well defined (reasonably steady wake) a simple and plausible method of estimating P_b^* would be to evaluate the base pressure coefficient for maximum drag in an inviscid flow wherein an equivalent solid object, such as illustrated in figure 5, replaced the wake. Such an object would have no effect in inviscid two-dimensional flow but would have a pronounced effect in axially symmetric flow. If in axially symmetric flow a rod of diameter d is considered to replace the wake of diameter t , the resulting maximum drag in inviscid flow would be the same as calculated in part I where the corresponding base pressure coefficient was designated by P_{b_i} . (See fig. 6.) Thus an estimate for the variation of P_b^* with Mach number in axially symmetric flow would be



(a) Measured data, $M_\infty = 2.0$.
(b) Correlation of measured data, $M' = 2.0$.
FIGURE 15.—Measured and correlated base pressure data; $M_\infty = 2.0$, turbulent boundary-layer flow.

$$P_b^* \approx P_{b_i} \text{ for } \frac{d}{h} = \frac{t}{h} \quad (6)$$

and in two-dimensional flow it would be

$$P_b^* \approx P_{b_i} \quad (7)$$

Since a fluctuating wake presumably cannot be replaced by a rod without essentially altering the flow conditions near the base, the above estimates cannot be expected under such conditions to yield anything more than the right order of magnitude.

Some information on the thickness and steadiness of the wake has been obtained from an examination of numerous spark photographs taken of projectiles in free flight.⁵ Typical spark photographs are shown in figure 16, and the results of measuring the wake thickness on a large number of similar photographs are shown in figure 17. Figure 16 (a) represents the case of laminar flow in the boundary layer at a free-stream Mach number of 1.73. Under these conditions the wake thickness appears to be reasonably well defined, although the trailing shock wave is not well defined near the wake. Figures 16 (b) and 16 (c) indicate that for turbulent boundary-layer flow on bodies of revolution the trailing shock wave and the wake are not very steady at Mach numbers below about 2. Thus it is not surprising

⁵ These shadowgraphs were made available through the courtesy of the Ballistic Research Laboratories, Aberdeen, Md.

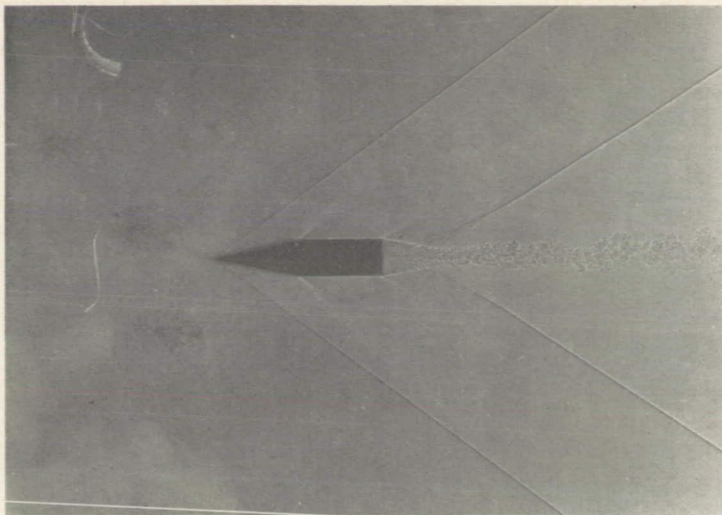
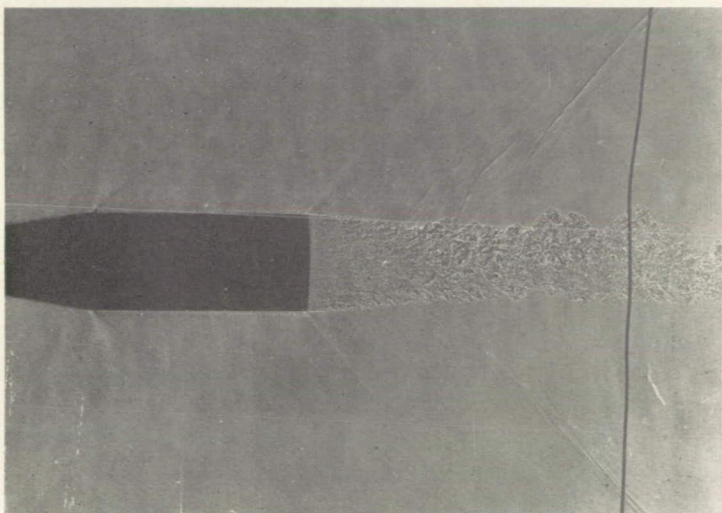
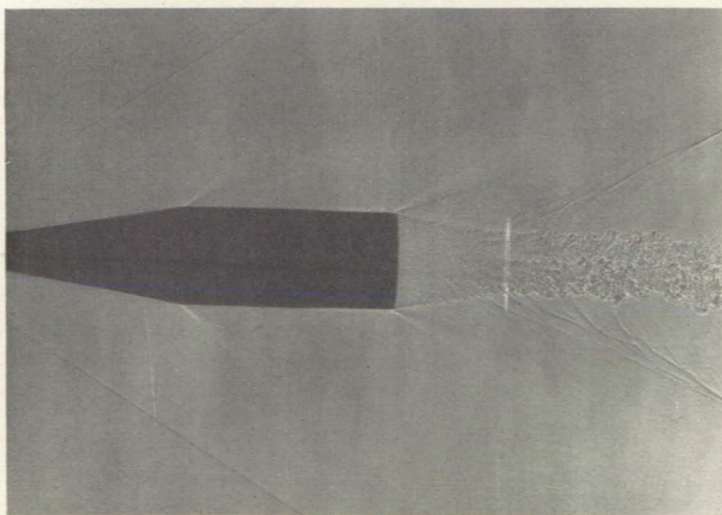
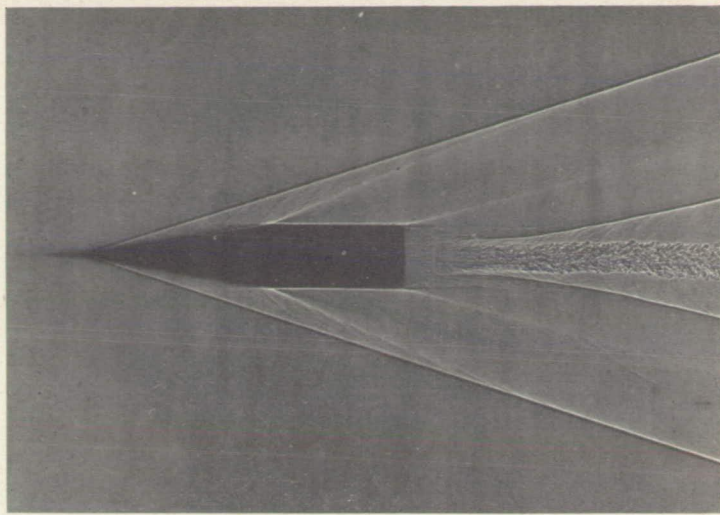
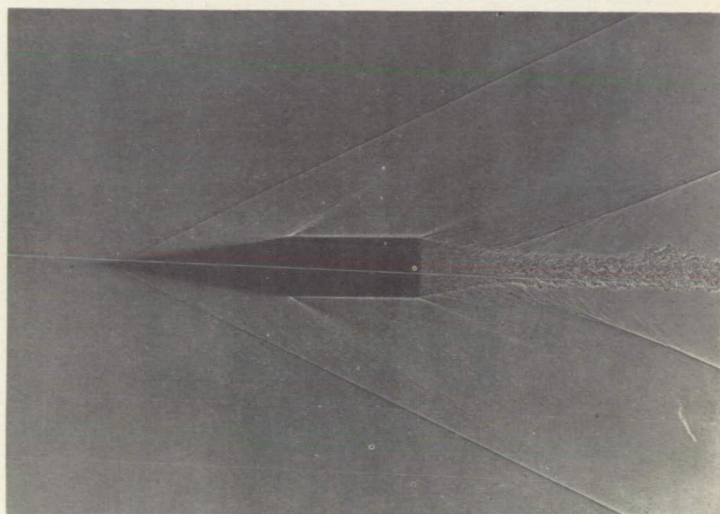
(a) $M_\infty = 1.73$, laminar.(b) $M_\infty = 1.28$, turbulent.(c) $M_\infty = 1.88$, turbulent.(d) $M_\infty = 2.33$, turbulent.(e) $M_\infty = 3.64$, turbulent.

FIGURE 16.—Concluded.

that, as will be seen later, equation (6) is in poor agreement with measurements for turbulent boundary-layer flow at Mach numbers below about 2. At higher Mach numbers the trailing shock wave and the wake become more clearly defined (figs. 16 (d) and 16 (e)), but the accuracy of equation (6) in this region cannot as yet be tested because of insufficient experimental data.

A comparison between inviscid-flow calculations and experimental values of P_b^* is more direct for airfoils than for bodies of revolution since the wake thickness presumably need not be accounted for in two-dimensional flow. The value of P_b^* as determined from the finite-span wing data in figure 10 (b) is -0.30 . This is fairly close to the limiting pressure coefficient (P_{bi}) for two-dimensional flow, which is -0.33 for a Mach number of 2.0. (See fig. 3.) Definite conclusions as to the significance of this agreement, however, will have to await the results of measurements on airfoils at other Mach numbers, and on airfoils with turbulent flow in the boundary layer.

FIGURE 16.—Shadowgraphs of projectiles in flight. (Courtesy Ballistic Research Laboratories, Aberdeen, Md.).

For laminar flow on bodies of revolution at Mach numbers of 1.5 and 2.0, the wake thickness (t/h) from figure 17 is 0.55 and 0.49, respectively. From figure 6, the corresponding values of P_{b_i} are -0.25 and -0.29, respectively. On the other hand, the values of $P_{b_i}^*$ determined from the intercepts of the extrapolated lines in figures 12 (b) and 13 (b) are -0.24 and -0.20, respectively. Hence, although the inviscid-flow calculations may provide a reasonable approximation for two-dimensional flow near $M=2.0$, and for axially symmetric flow near $M=1.5$, there is a serious discrepancy with the experimental results for axially symmetric flow at $M=2.0$. This large discrepancy indicates that the simple relation given by equation (6) which attempts to connect $P_{b_i}^*$ with the inviscid calculations is not always a satisfactory approximation. The good agreement obtained in two of the three cases may be entirely fortuitous. Additional experiments are needed to clarify this point.

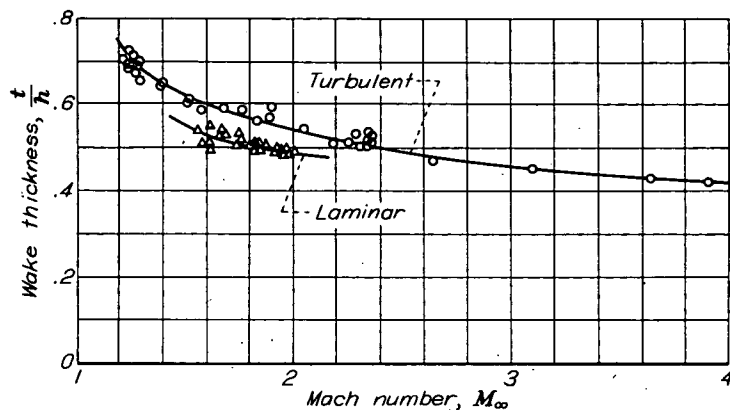


FIGURE 17. Wake thickness as a function of Mach number (determined from shadowgraphs of the Ballistic Research Laboratories, Aberdeen, Md.).

The fact that the inviscid-flow calculations agree qualitatively, though not quantitatively, with experimental results can be seen by a comparison with measurements of the base pressure at various Mach numbers but with an essentially constant Reynolds number. Figure 18 shows some experimental free-flight data of Charters (reference 5) together with the corresponding wind-tunnel data of Kurzweg (reference 13), and the present investigation.⁶ These experimental data are for turbulent flow in the boundary layer. In this figure the ordinate of the curve labeled "curve based on equation (6)" is proportional to the value of the limiting pressure coefficient P_{b_i} determined at each Mach number in the manner indicated by equation (6). It is apparent that the curve based on the calculations of P_{b_i} for inviscid flow gives the right order of magnitude for the base pressure coefficient, but does not give good quantitative agreement. As an incidental point, it may be noted that the various wind-tunnel and free-flight measurements shown in this figure agree quite well at all Mach numbers.

VARIATION OF BASE PRESSURE WITH REYNOLDS NUMBER FOR NATURAL TRANSITION

Since the base pressure is different for laminar and turbulent boundary-layer flow approaching the base, it is of interest to examine the results of measurements in the intermediate range of Reynolds number where the transition

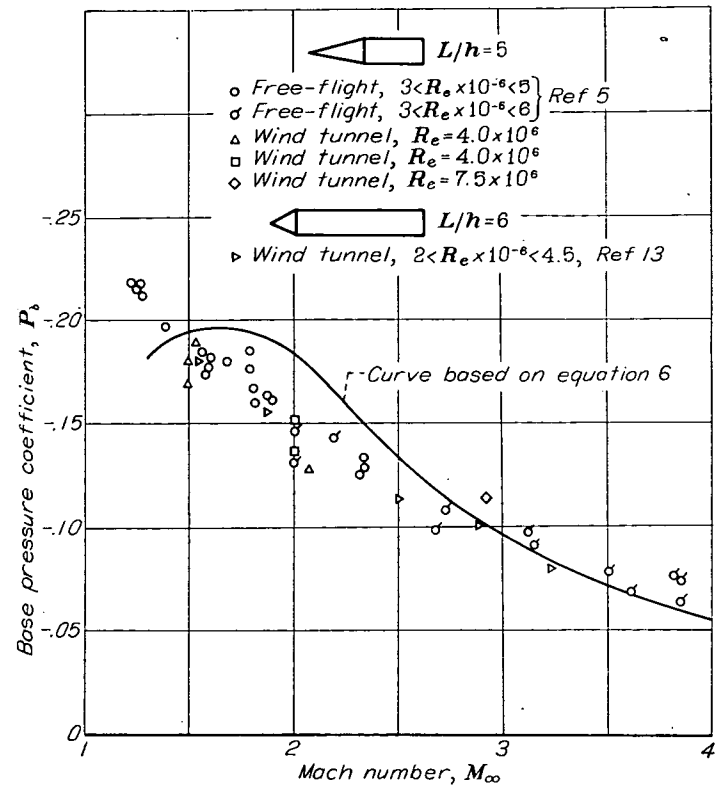


FIGURE 18.—Variation of base pressure coefficient with Mach number for turbulent boundary layer flow.

"point" moves from a position downstream of the base to a position upstream of the base. Figure 19 shows the results of some base-pressure measurements at a Mach number of 2.0 on a body of revolution in the Reynolds number range from 0.4 million to 10 million. At Reynolds numbers below about 2 million, where the boundary-layer flow is laminar, the base pressure coefficient depends to a great extent on the Reynolds number, as was noted earlier. In the Reynolds number range from 4 to 6 million, where the transition point moves ahead of the base, the base pressure again is sensitive to changes in the Reynolds number (and presumably also

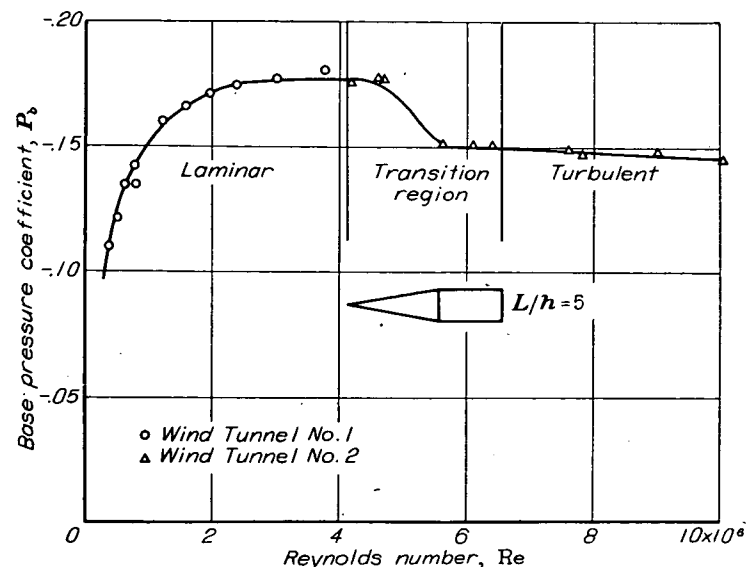


FIGURE 19.—Variation of base pressure coefficient with Reynolds number for natural transition; $M_\infty=2.0$.

⁶ In the present experiments measurements occasionally were made in more than one facility. For example, the three experimental points in figure 18 representing the wind-tunnel data at Mach numbers near 1.5 represent measurements with three different nozzles.

to other factors affecting transition such as surface roughness, free-stream turbulence, and rate of heat transfer). At the higher Reynolds numbers where a turbulent boundary layer exists for some distance ahead of the base, the base pressure is not sensitive to changes in the Reynolds number.

From the viewpoint of reliably extrapolating small-scale measurements, it is encouraging that the base pressure coefficient for turbulent boundary-layer flow is not sensitive to changes in the Reynolds number. At a Mach number of 2.0 this insensitivity is evident from a comparison of the data for the model with an L/h of 5 in figures 15 (a) and 19. At a Reynolds number of 2×10^6 , where turbulent flow is attained on the models by using artificial roughness, the base pressure coefficient does not differ by more than 3 or 4 percent from the value at a Reynolds number of 1×10^7 , where turbulent flow is attained without such an artifice. At a Mach number of 1.5 the measurements indicate this same characteristic, as can be seen from the data given in figure 20. These data at the somewhat lower Mach number do not show any appreciable dependence on Reynolds number within the range from 2×10^6 to 1.6×10^7 . It is interesting that the free-flight data of Hill and Alpher (reference 14) also show no significant effect of Reynolds number within

the range from 2×10^7 to 1×10^8 . These latter data, however, give a widely different value for the base pressure. It is evident from figure 20 that the base pressures measured in reference 14 differ from the values of references 5 and 13 and the present wind-tunnel tests because of some factor other than differences in Reynolds number. The possible effects of support interference in the present wind-tunnel tests would not appear to contribute any appreciable amount to this discrepancy for two reasons. First, good agreement is obtained at all Mach numbers between the present wind-tunnel tests and the free-flight firings of Charters; and second, the measurements of support interference as described in appendix B indicate that for the support dimensions used ($d/h=0.25$ and $d/h=0.40$ in fig. 20) these effects are an order of magnitude smaller than the observed discrepancies. Since the models of reference 14 were equipped with tail fins of sufficient size so that their presence at moderate supersonic Mach numbers might be expected to lower considerably the pressure in the vicinity of the dead air (algebraically lower the effective P'), it would appear that the observed discrepancy is attributable to the effect of tail fins on base pressure.⁷

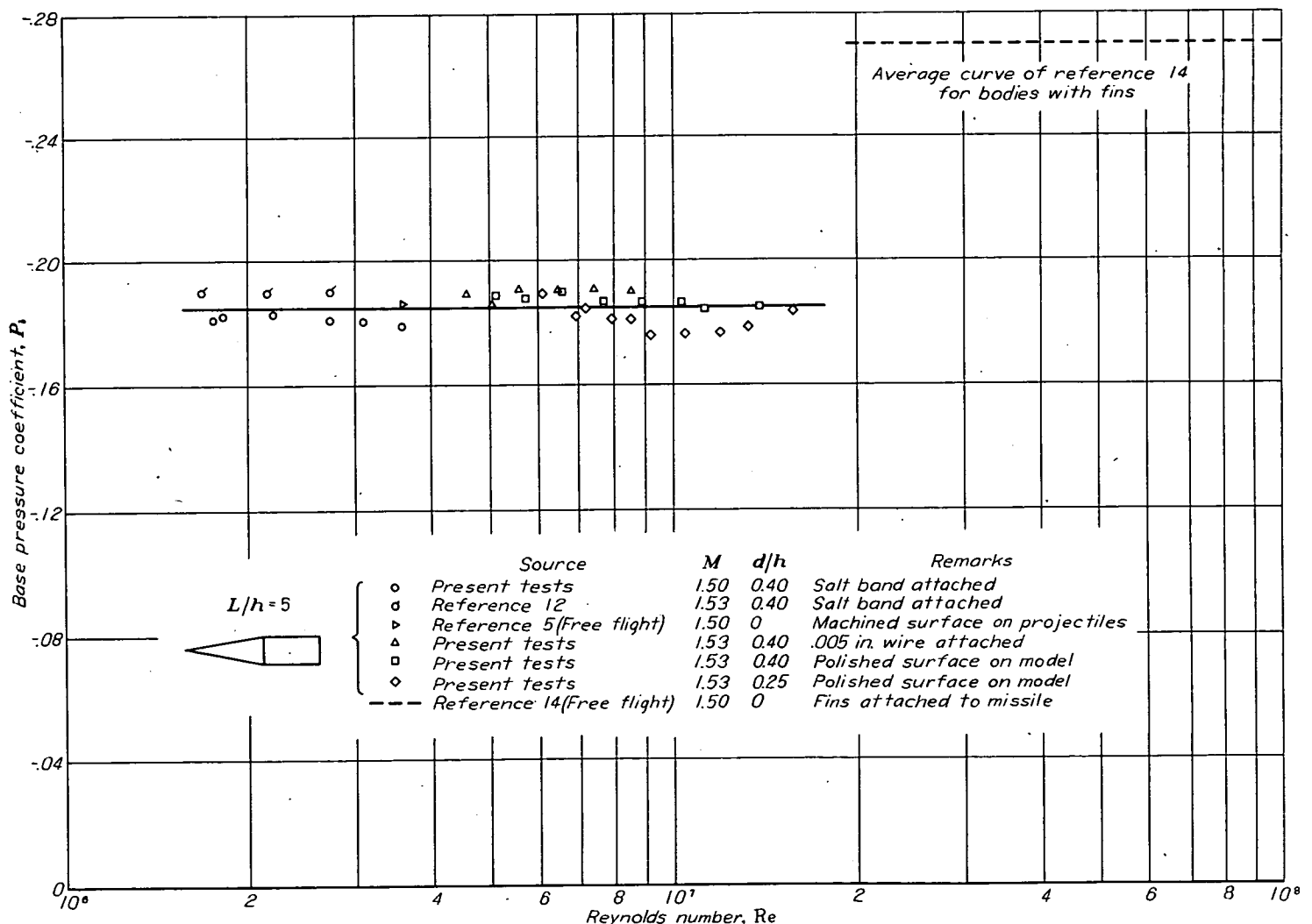


FIGURE 20.—Variation of base pressure coefficient with Reynolds number for turbulent boundary-layer flow; $M_\infty = 1.5$.

⁷ Subsequent experiments conducted at the Ames Laboratory by J. R. Spahr and R. R. Dickey have shown that this is the case.

CONCLUDING REMARKS

The simplest approach to an analysis of base pressure for supersonic flow is that of considering the flow of an inviscid fluid. Although such an approach has produced many useful theories when applied to other aerodynamic problems, it produces results of very limited value when applied to the present problem. The inviscid-fluid theory indicates that the only possible base pressure for a body of revolution without a rod attached to the base is the free-stream static pressure. Moreover, this simple theory also indicates that for two-dimensional flows, as well as axially symmetric flows with a rod attached to the base, there are an infinite number of possible solutions for a given body shape and Mach number.

The first of the above-mentioned shortcomings of inviscid theory can be remedied by allowing qualitatively for the existence of a wake, since by so doing the high-velocity streamlines are displaced from the axis of symmetry and a base drag other than zero can be obtained. The second shortcoming, of having an infinite number of possible solutions from which to choose, is not easily remedied. In particular, the comparison between the inviscid-flow calculations and experiment has shown that if the limiting flow pattern (maximum drag possible) at each Mach number is singled out from the infinity of possible inviscid-flow solutions, then the characteristics of base pressure observed thus far can be explained, but only qualitatively. Thus, the experimental finding that an increase in support diameter behind a body of revolution can considerably decrease the base pressure is explained by an interpretation of the behavior in an inviscid-fluid flow. Also, the experimental result of a much lower base pressure in two-dimensional flow (at low and moderate supersonic Mach numbers) than in axially symmetric flow is satisfactorily explained by the inviscid-flow

calculations. As regards quantitative results, though, the calculations based on the maximum drag possible in inviscid flow do not agree with the observed effects for turbulent boundary-layer flow, and agree only in certain cases with the observed effects for laminar boundary-layer flow.

In an attempt to formulate a more accurate quantitative analysis a semi-empirical analysis has been developed. The available experimental data correlate reasonably well when the base pressure coefficient, corrected for the effects of profile shape, is plotted as a function of a parameter which is approximately proportional to the ratio of boundary-layer thickness to base thickness. As a result of this correlation several general conclusions can be drawn. One such conclusion is that the variation of base pressure with Reynolds number is small at high Reynolds numbers where the boundary layer approaching the base is turbulent, but is large at low Reynolds numbers where the boundary layer is laminar. Another conclusion is that the effect on base pressure of the disturbance field induced by profile shape can be adequately explained on the basis of inviscid calculations.

In order to develop a thorough understanding of the behavior of base pressure in supersonic flow, further experimental and theoretical investigations are required. At present, experimental results are especially needed as regards the base pressure in two-dimensional flow, even at low supersonic Mach numbers. Experiments conducted at high supersonic Mach numbers are also needed, both for two-dimensional flow and for axially symmetric flow.

AMES AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., May 11, 1950.

APPENDIX A

AXIALLY SYMMETRIC FLOWS CONVERGING TOWARD THE AXIS

The rather anomalous result obtained when applying the method of characteristics to base-pressure flows can be clarified by examining the equations of motion on which the method of characteristics is based. The differential equation for the velocity potential ϕ of an inviscid axially symmetric compressible flow is (see reference 6, for example)

$$\left(1 - \frac{\phi_x^2}{a^2}\right) \phi_{xx} - 2 \frac{\phi_x \phi_r}{a^2} \phi_{xr} + \left(1 - \frac{\phi_r^2}{a^2}\right) \phi_{rr} + \frac{\phi_r}{r} = 0 \quad (A1)$$

where a is the local velocity of sound, x is the coordinate measured parallel to the direction of the undisturbed stream, and r is the radial coordinate. If a transformation is made to a new system (ξ, η) of curvilinear coordinates, where ξ and η are distances measured along the two Mach lines issuing from a point, then the equation of motion for the velocity potential becomes simply (the details of the algebra involved in making this transformation may be found in reference 6),

$$\frac{\partial^2 \phi}{\partial \xi \partial \eta} = \frac{\sin^2 \alpha}{r} \frac{\partial \phi}{\partial r} \quad (A2)$$

where α is the local Mach angle. It is to be noted that the new variables have the simple physical significance that lines of constant ξ and η are the Mach lines of the flow. The derivative of the velocity potential in any given direction is the projection of the velocity vector along that direction, and the order of differentiation in equation (A2) can be interchanged. With

$$\frac{\partial \phi}{\partial \xi} = p \quad \frac{\partial \phi}{\partial \eta} = q \quad (A3)$$

and

$$\frac{\partial \phi}{\partial r} = v = w \sin \theta$$

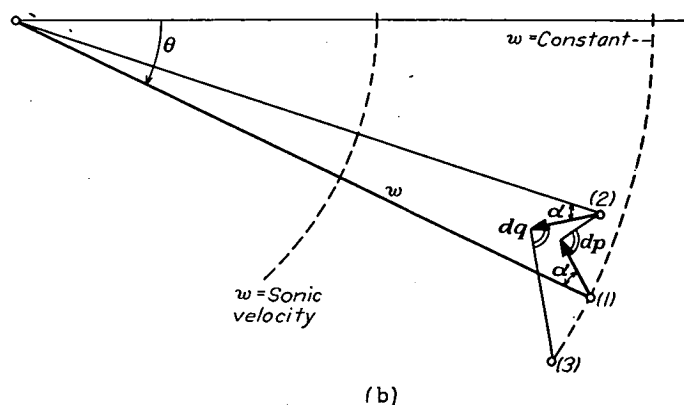
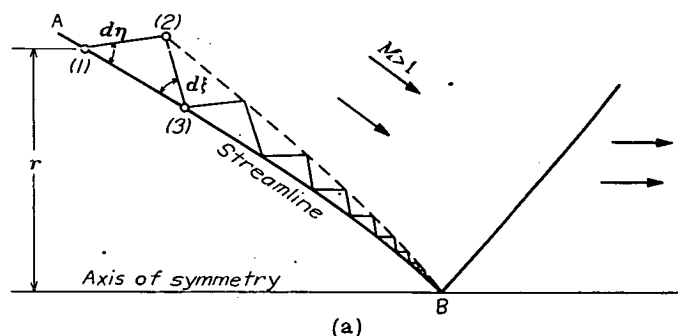
where w is the velocity vector inclined at an angle θ with respect to the axis, it follows from equation (A2) that along Mach lines

$$dp = \frac{\sin^2 \alpha}{r} v d\eta \quad dq = \frac{\sin^2 \alpha}{r} v d\xi \quad (A4)$$

Thus, dp is the increment in the projection of the velocity vector along the ξ direction when passing a distance $d\eta$ in the physical plane along the η direction, and dq is the increment in the projection of the velocity vector in the η direction when passing a distance $d\xi$ along the ξ direction. Equations (A4) are the fundamental equations used in the step-by-step construction of a supersonic flow by Sauer's or Frankl's method of characteristics.

The reasons for the singular behavior as the flow approaches the axis of symmetry can now be explained with the help of equations (A4). Suppose a series of steps were laid off in the physical plane in the manner indicated by the sketch

shown in figure 21 (a). The small increments ($d\xi$ and $d\eta$) along the Mach lines are laid off such that they are always small compared to the distance from the axis r and also such that for all steps $d\xi/r$ and $d\eta/r$ are always very nearly equal to a constant, say $\bar{C}$. It is to be noted that if such a flow converging to the axis is possible, then there would be an infinite number of such steps along the streamline AB in figure 21 (a).



(a) Assumed flow in the physical plane.

(b) Increments in hodograph plane corresponding to figure 21 (a).

FIGURE 21.—Characteristics construction for flows converging to the axis.

Now consider the increments in the hodograph plane corresponding to those laid off in the physical plane (fig. 21 (a)). Figure 21 (b) illustrates the way, according to equations (A3) and (A4), in which the increments must be laid off in the velocity plane. Points having the same number in figures 21 (a) and 21 (b) represent the same point in the flow. Let the smallest average Mach angle along the steps in the physical plane be α_m , and the smallest vertical-velocity component be v_m , then for all steps along AB

$$|dp| > |v_m \bar{C} \sin^2 \alpha_m| = \text{constant}$$

and

$$|dq| > |v_m \bar{C} \sin^2 \alpha_m| = \text{constant}$$

This means that every increment in the hodograph plane is greater than a constant value. This value cannot be zero unless points 1 and 3 are identical, which would represent the exceptional case of a "reversed" conical flow. On passing from point A to point B there are, however, an infinite number of such increments. They must be laid out along the arc of a circle in the hodograph plane since AB is a streamline of constant pressure. Hence, before reaching point B the inclination angle of the velocity vector must be greater than 46° (approximate maximum deflection angle through a single shock wave for $\gamma=1.4$). Because this situation obviously prevents a shock wave from being fitted into the flow, there results a contradiction to the assumption that the over-all flow is possible. It appears, therefore, that these flows are not always possible.

The preceding discussion, though not a mathematically rigorous exposition, points out the reason why the inclination angle θ of a free streamline can increase at an excessive rate as the axis is approached. The source of the trouble is inherently associated with the last term in the equation of motion (A1), since it has r in the denominator and a non-vanishing factor in the numerator. The appearance of r in the denominator of this equation stems entirely from the

continuity equation. This leads to a qualitative explanation of the observed behavior near the axis of the inviscid flows. Consider the changes that must occur on going from point 1 to point 3 in the physical plane (fig. 21 (a)). If the flow were two-dimensional, then the free streamline would be straight and θ_1 would equal θ_3 , thereby preserving the cross-sectional area between two adjacent streamlines on passing from 1 to 3. The term involving $1/r$ does not occur for plane flow and no difficulties arise. In the axially symmetric case, the fundamental condition is again that the cross-sectional area of an annular streamtube must be preserved, since w_1 is equal to w_3 . This means that for purely geometric reasons the streamlines bounding the annular streamtube must spread apart as the axis is approached. In order to have the pressure at point 3 equal to that at point 1, the free streamline curves toward the axis, permitting the bounding streamlines to spread, thereby allowing the continuity equation to be satisfied. Because of the $1/r$ term in the continuity equation, the curvature rapidly increases as the axis is approached. Hence, before the axis is reached, the inclination of the free streamline exceeds the largest value which any oblique shock wave can possibly overcome.

APPENDIX B

WIND-TUNNEL SUPPORT INTERFERENCE AND REFLECTED BOW-WAVE INTERFERENCE

When a body of revolution is tested in a wind tunnel it is usually supported from the rear by a cylindrical rod. As a result the measured values of base pressure may be considerably affected, for one thing, by the presence of the support. Support interference on base pressure is a complicated function of the diameter of support rod, the unobstructed length of support rod, the Mach number, and the Reynolds number. If, as is the case for the experiments referred to herein, the support length is much greater than the base diameter, then the only appreciable interference must arise from the "diameter effect" of the rod. From theoretical considerations certain inferences can be drawn regarding the resulting support-diameter interference on base pressure.

For a fixed Mach and Reynolds number, an increase in the support diameter brings about two different effects. First, the wake thickness is increased, thereby making it possible for lower base pressures to exist. (See fig. 6.) A second effect resulting from an increase in support diameter is that the appropriate dimensionless boundary-layer thickness $\delta/(h-d)$ is increased, thereby tending to increase the base pressure. The two effects, therefore, oppose each other. For values of d/h near unity the second effect must predominate; whereas for small values of d/h the first effect would (on the basis of fig. 6) be expected to predominate, especially at low supersonic Mach numbers.

Before comparing these theoretical considerations with experimental measurements of the effect of variations in d/h it will be advantageous to first consider the effects of having only a finite length of unobstructed support rod. To examine this effect, base-pressure measurements have been taken with a constant value of d/h , but with various lengths of

unobstructed support. In these experiments the model was located at a fixed position in the test section so as to eliminate possible effects of axial pressure gradients along the test section. The results from $M=2.0$ and 2.9 are illustrated by the curves in figure 22, which show, for $d/h=0.3$, no change

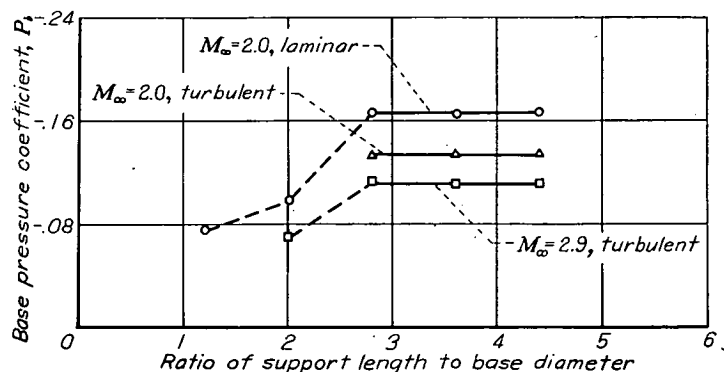


FIGURE 22.—Effect of support length on base pressure; $d/h=0.3$.

in base pressure if the support length is greater than about 3 base diameters. Since support lengths of over 4 body diameters have been used in all subsequent tests, it is concluded that any interference in the wind-tunnel measurements of base pressure at $M=2.0$ and 2.9 is not attributable to effects of support length.

The results of base-pressure measurements for various support diameters with laminar boundary-layer flow are shown in figure 23 (a). The data for a Mach number of 1.5 (reported by Perkins in reference 11) show the expected increase, and then eventual decrease in base drag as the support diameter is progressively increased. At a Mach

number of 2.9 the data show a monotonic decrease in base drag as the support diameter is increased. Schlieren photographs show that the wake thickness t/h varies from approximately 0.5 to 1.0 as d/h varies from 0 to 1.0. Consequently, it turns out that the behavior of the three curves in figure 23 (a) is qualitatively the same as would be indicated if equation (6) were used to estimate P_b^* . (It is to be remembered that t/h is the "effective" d/h of fig. 6.)

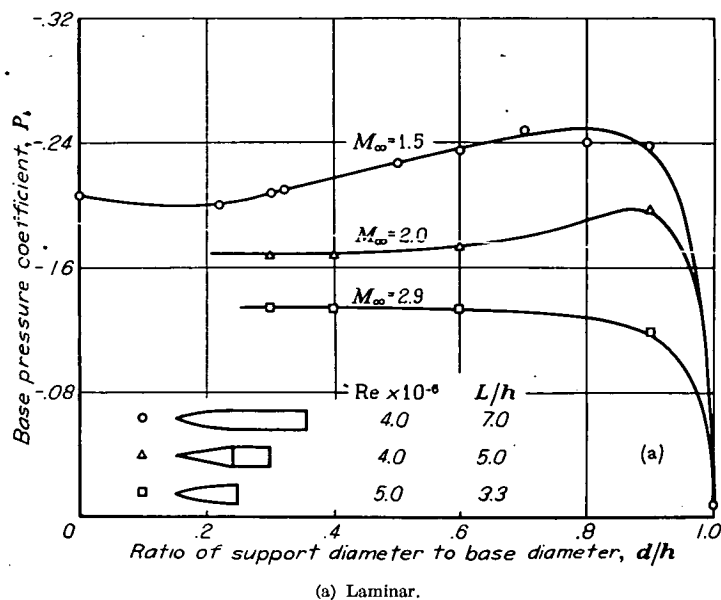


FIGURE 23.—Effect of support diameter on base pressure.

The corresponding results for turbulent boundary-layer flow are shown in figure 23 (b). At Mach numbers of 1.5 and 2.0 these data show the same trends as for laminar boundary-layer flow, but at a Mach number of 2.9 the trend is not the same. At Mach numbers near 3, and possibly higher, it appears that the relative importance of the two above-mentioned effects of increasing d/h depends on the condition of the boundary-layer flow.

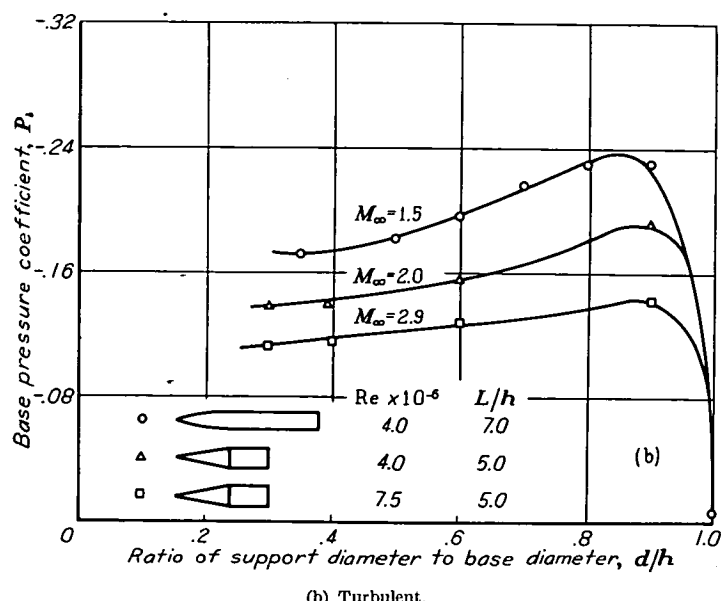


FIGURE 23.—Concluded.

It may be noted from figure 23 (a) that there is one point corresponding to $d/h=0$ on the curve representing laminar flow at a Mach number of 1.5. This point, which was determined from the measurements using a side support, gives the same value for the base pressure as exists for a support with a d/h ratio of about 0.3. At all the other Mach numbers, where special interference measurements were not made, the base pressure was measured with a constant value of 0.4 for the ratio d/h . From the curves in figure 23 (a) it may be inferred that, at least for Reynolds numbers of the order of 4 million, these base-pressure data for laminar flow are not significantly affected by support interference.

Unfortunately, an investigation of support interference for turbulent boundary-layer flow has not been made using a side support. Definite quantitative statements about the possible effects of support interference in the turbulent-flow data (figs. 14, 15, 18, 19, and 20) cannot be made at present. Evidence that the combined effects of support and wall interference are not large, however, is given by the good agreement obtained at all Mach numbers between the free-flight firings of reference 5 and the various wind-tunnel measurements (figs. 14, 15, 18, and 20).

A possible source of wall interference arises from the reflection of a bow wave from the side walls, and the eventual intersection and interaction with the wake at some downstream position. This interaction for $M=2.0$ and $M=2.9$ occurs at a position varying from 7 to 22 base diameters downstream of the base. Since the large disturbance caused by the balance housing has no measurable effect at distance of 3 base diameters from the base (see fig. 22), there is no reason to expect that the base-pressure measurements at $M=2.0$ and $M=2.9$ might be affected by reflections of bow waves from the tunnel side walls. At a Mach number of 1.5, however, the downstream position of interaction is closer; it varies from approximately 2.7 base diameters for the model with an L/h of 7, to 5.4 base diameters for the model with an L/h ratio of 4.3. In view of the possible interference from reflected bow waves at low supersonic Mach numbers, a special investigation was made in 1946 prior to the tests of reference 12 to determine the magnitude of this effect. The results, taken at a Mach number of 1.53,⁸ are presented here as they aid in evaluating the accuracy of the wind-tunnel measurements of base pressure, and they show that the conclusion of Faro (reference 15) regarding the magnitude of the bow-wave interference effect in the present experiments is incorrect.

Figure 24 illustrates the test setup employed in evaluating the effect of a reflected bow wave on base pressure. Because of symmetry the two outer dummy models caused two shock waves, similar to reflected bow waves, to interact with the wake behind the base of the center model (on which the base pressure was measured). By varying the distance between the dummy models of the test setup, the position of interaction was readily changed. The strength of the bow wave on the models employed (6-caliber ogival radius) in this special investigation varied from approximately two to eight times the strength of the bow wave on the various models for which base-pressure data are presented.

⁸ This Mach number differs somewhat from that of more recent tests (at $M=1.50$) since the earlier tests were conducted in 1946 at a time when the 1- by 3-foot supersonic wind tunnel was temporarily equipped with a set of fixed nozzle blocks instead of the flexible plates now employed.

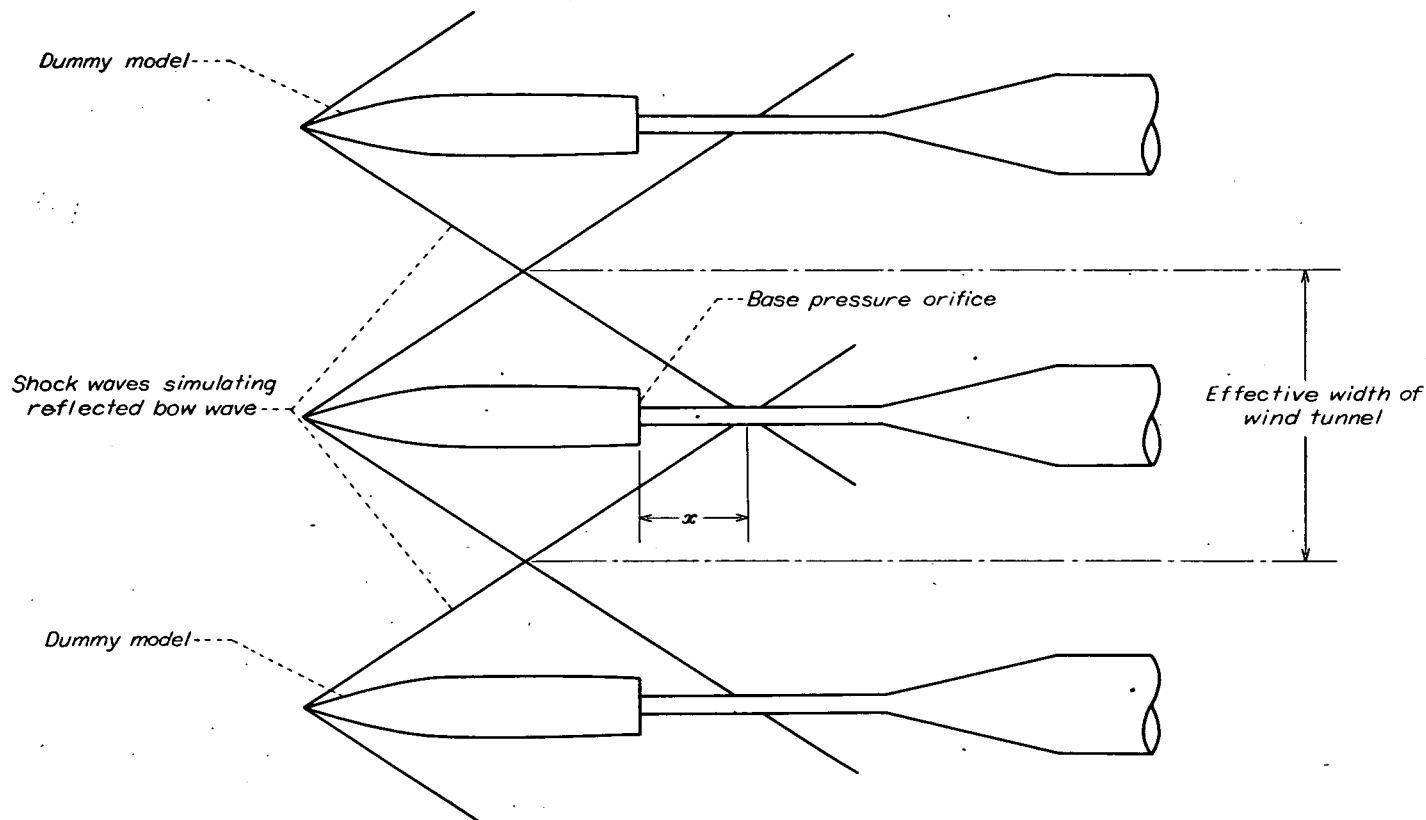
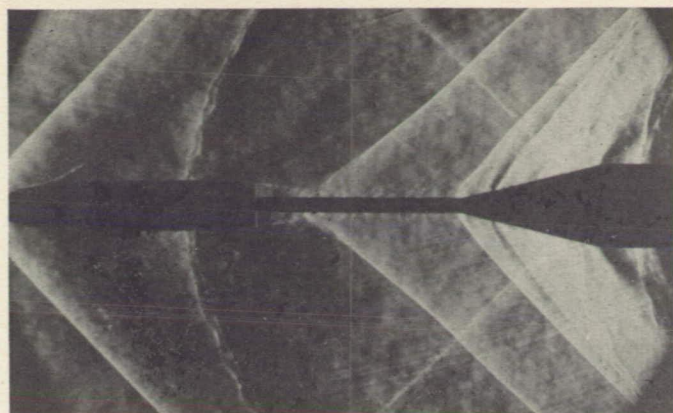


FIGURE 24.—Sketch of test setup used for determining the effect of a reflected bow wave on base pressure.

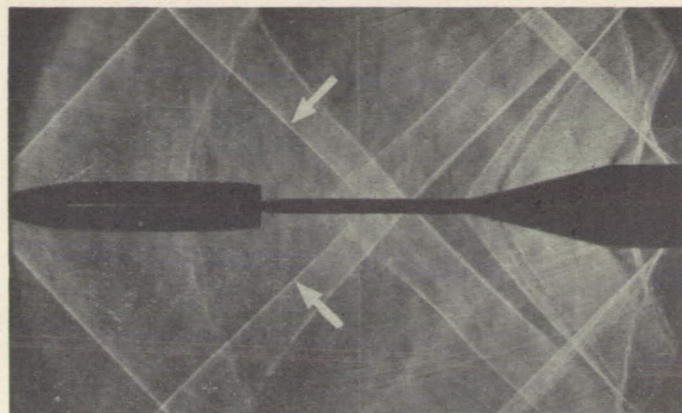
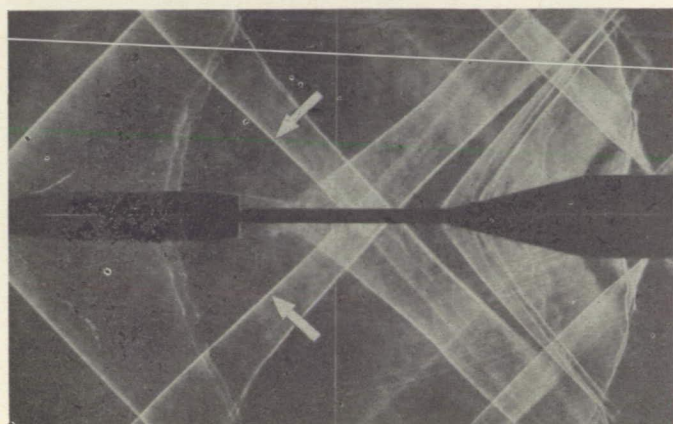
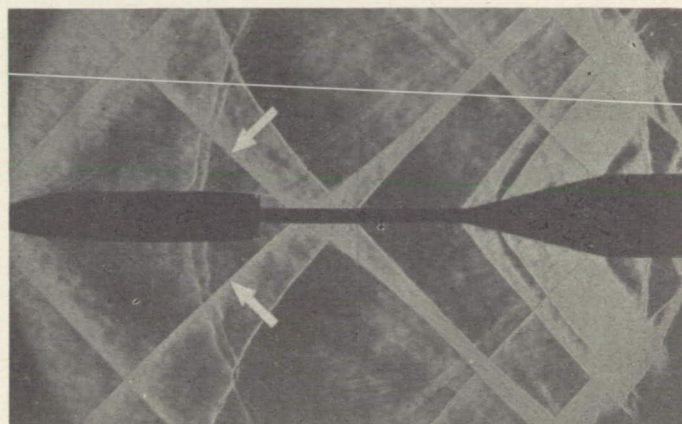
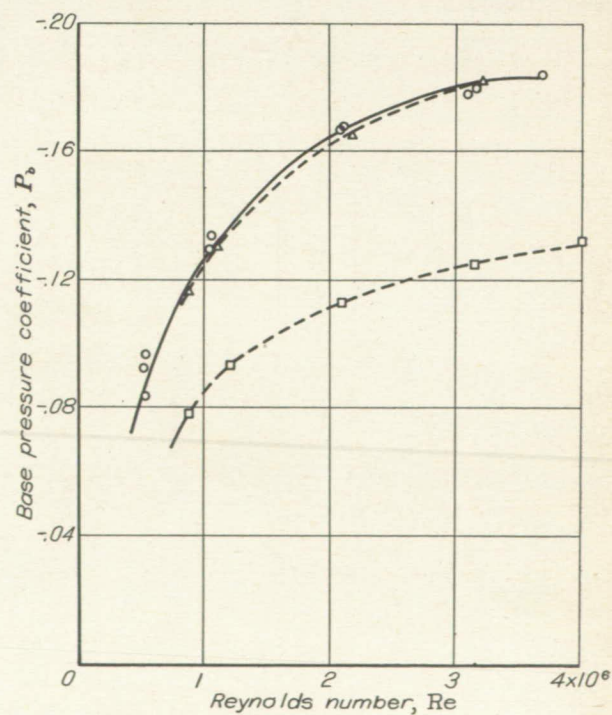
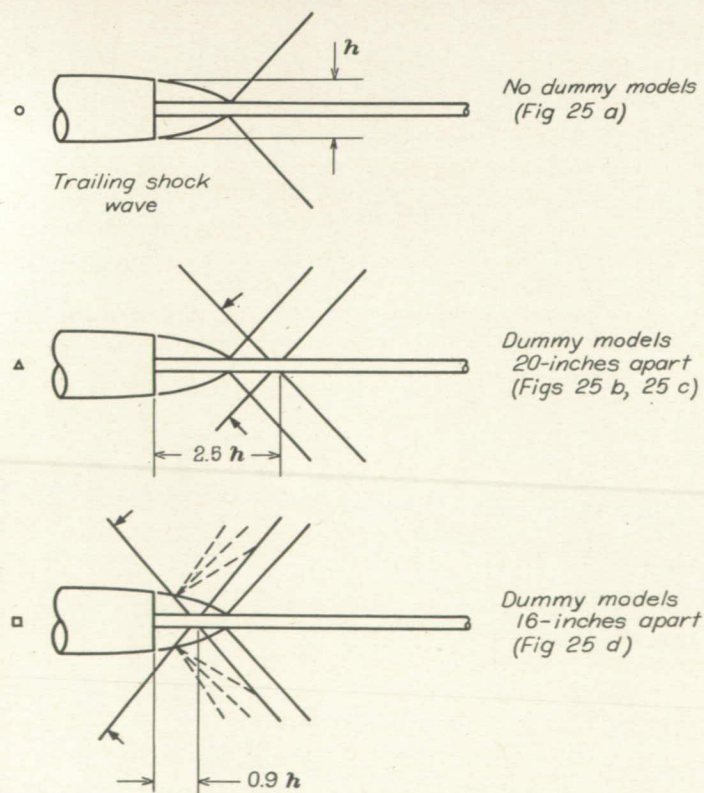
Schlieren photographs of the flow for two different positions of interaction, and two different Reynolds numbers, are given in figure 25. The distance x , from the base to the position of interaction, is equal to $2.5h$ in both figures 25 (b) and 25 (c). This particular position simulates the closest position to the base of the interaction of reflected waves in the present tests. The corresponding base-pressure measurements⁹ without and with the interference wave present are illustrated in figure 26 by the circle and triangle symbols, respectively. The data show no appreciable effect on base pressure of the shock wave which simulates a reflected bow wave. If a reflected bow wave comes

too close to the base, however, then large interference effects are possible, as illustrated by the square symbols in figure 26, and the corresponding schlieren photographs in figure 25 (d). Except for purposes of illustrating this effect, base-pressure measurements were, of course, not taken under these latter conditions of important interference from reflected waves. Since the simulated reflection waves of the models used in this special investigation were several times stronger than the bow waves on the models for which the base pressure was measured, it is clear that the wind-tunnel measurements presented are not appreciably affected by interference of a reflected bow wave.

⁹ These data fall slightly below other data presented herein because of the very small amount of boattailing on the models used in this special investigation.



(a) Flow without dummy models.

(b) $Re=0.9 \times 10^6$; $x=2.5h$.(c) $Re=2.7 \times 10^6$; $x=2.5h$.(d) $Re=2.7 \times 10^6$; $x=0.9h$.FIGURE 25.—Schlieren photographs for various positions of intersection of the shock waves simulating reflected bow waves; $M_\infty=1.53$.FIGURE 26.—Effect of reflected bow waves on base pressure; $M_\infty=1.53$.

APPENDIX C

DERIVATION OF APPROXIMATE EQUATION FOR q'/q_∞

The ratio q'/q_∞ can be written as

$$\frac{q'}{q_\infty} = \frac{\rho' U'^2}{\rho_\infty U_\infty^2} = \frac{\rho'}{\rho_\infty} \frac{\bar{p}_o}{\rho_o} \frac{\rho_o}{\rho_\infty} \left(1 + 2 \frac{\Delta U}{U_\infty} \right) \quad (C1)$$

In this and subsequent equations, powers higher than the first of quantities such as $\frac{\Delta U}{U_\infty} \equiv \frac{U' - U_\infty}{U_\infty}$ are assumed to be small in comparison to unity, and are therefore neglected. In equation (C1), ρ_o and $\bar{p}_o$ represent the stagnation densities corresponding to conditions in the free stream and to conditions just ahead of the base, respectively. Designating $\Delta M \equiv M' - M_\infty$ and again considering only first-order terms, it follows that

$$\begin{aligned} \frac{\rho'}{\rho_o} \frac{\bar{p}_o}{\rho_o} \frac{\rho_o}{\rho_\infty} &= \left[\frac{1 + \frac{\gamma-1}{2} M'^2}{1 + \frac{\gamma-1}{2} M_\infty^2} \right]^{\frac{-1}{\gamma-1}} \left(1 - \frac{\Delta p_o}{p_o} \right) = \\ &= 1 - \frac{M_\infty \Delta M}{1 + \frac{\gamma-1}{2} M_\infty^2} - \frac{\Delta p_o}{p_o} \end{aligned} \quad (C2)$$

where Δp_o is the loss in total pressure on passing through the nose shock wave, and may often be neglected. From the energy equation

$$\frac{\Delta U}{U_\infty} = \frac{U'^2 - U_\infty^2}{2U_\infty^2} = \frac{c_p(T_\infty - T')}{U_\infty^2} = \frac{c_p T_\infty}{U_\infty^2} \left(1 - \frac{T'}{T_\infty} \right)$$

or, using $c_p = \gamma R/(\gamma-1)$ and $M = U/\sqrt{\gamma R T}$

$$\frac{\Delta U}{U_\infty} = \frac{1}{(\gamma-1) M_\infty^2} \left[1 - \left[\frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M'^2} \right] \right] =$$

$$\frac{\Delta M}{M_\infty \left(1 + \frac{\gamma-1}{2} M_\infty^2 \right)} \quad (C3)$$

hence the combination of equations (C1), (C2), and (C3) gives

$$\frac{q'}{q_\infty} = 1 + \left(\frac{2}{M_\infty^2} - M_\infty \right) \frac{\Delta M}{1 + \frac{\gamma-1}{2} M_\infty^2} - \frac{\Delta p_o}{p_o} \quad (C4)$$

The pressure coefficient P' is related to ΔM and Δp_o by

$$\begin{aligned} P' &= \frac{p' - p_\infty}{\frac{\gamma}{2} \rho_\infty M_\infty^2} = \frac{2}{\gamma M_\infty^2} \left(\frac{p'}{\bar{p}_o} \frac{\bar{p}_o}{p_o} \frac{p_o}{p_\infty} - 1 \right) = \\ &= \frac{2}{\gamma M_\infty^2} \left[\left[\frac{1 + \frac{\gamma-1}{2} M_\infty^2}{1 + \frac{\gamma-1}{2} M'^2} \right]^{\frac{\gamma}{\gamma-1}} \left(1 - \frac{\Delta p_o}{p_o} \right) - 1 \right] = \\ &= - \frac{2 \Delta M}{M_\infty \left(1 + \frac{\gamma-1}{2} M_\infty^2 \right)} - \frac{2}{\gamma M_\infty^2} \frac{\Delta p_o}{p_o} \end{aligned} \quad (C5)$$

Substitution of equation (C5) into equation (C6) yields the relation

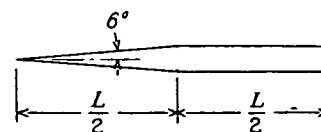
$$\frac{q'}{q_\infty} = 1 + \left(\frac{M_\infty^2}{2} - 1 \right) P' - \frac{2}{\gamma M_\infty^2} \left(1 + \frac{\gamma-1}{2} M_\infty^2 \right) \frac{\Delta p_o}{p_o} \quad (C6)$$

presented earlier as equation (3).

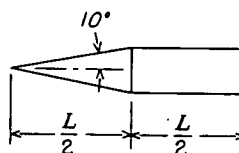
REFERENCES

1. Lorenz, H.: Der Geschosswiderstand. *Physikalische Zeitschrift*, vol. 18, 1917, p. 209; vol. 29, 1928, p. 437.
2. Gabeaud: Sur la resistance de l'air aux vitesses ballistiques. *Comptes Rendus de L'Academie des Sciences*, vol. 192, 1931, p. 1630.

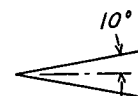
3. von Kármán, Theodore, and Moore, Norton B.: The Resistance of Slender Bodies Moving at Supersonic Velocities. A. S. M. E. Trans., vol. 54, 1932, pp. 303-310.
4. Cope, W. F.: The Effect of Reynolds Number on the Base Pressure of Projectiles, NPL Rep. Eng. Div. 63/44, Jan. 1945.
5. Charters, A. C., and Turetsky, R. A.: Determination of Base Pressure from Free-Flight Data. Aberdeen Ballistic Res. Lab., Rep. 653, Mar. 1948.
6. Sauer, Robert: Theoretische Einführung in die Gasdynamik. Julius Springer, Berlin, 1943. (Reprinted by J. W. Edwards Bros., Inc., 1947, Ann Arbor, Mich.)
7. Kantrowitz, Arthur: The Formation and Stability of Normal Shock Waves in Channel Flows. NACA TN 1225, 1947.
8. Guderley, Gottfried, and Yoshihara, Hideo: The Flow Over a Wedge Profile at Mach Number One. AF Tech. Rep. No. 5783, July, 1949.
9. Prandtl, L., and Tietjens, O. G.: Applied Hydro and Aeromechanics. McGraw-Hill Book Co., Inc., New York, 1934, p. 76.
10. Valensi, J., and Pruden, F. W.: Some Observations on Sharp Nosed Profiles at Supersonic Speed. British R. & M. No. 2482.
11. Perkins, Edward W.: Experimental Investigation of the Effects of Support Interference on the Drag of Bodies of Revolution at a Mach Number of 1.5. NACA TN 2292 (Issued as RM A8B05). May 7, 1948).
12. Chapman, Dean R., and Perkins, Edward W.: Experimental Investigation of the Effects of Viscosity on the Drag of Bodies of Revolution at a Mach Number of 1.5. NACA RM A7A31a, 1947.
13. Kurzweg, H. H.: New Experimental Investigations on Base Pressure in the NOL Supersonic Wind Tunnels at Mach Numbers 1.2 to 4.24. NOLM 10113, January 1950.
14. Hill, Freeman K., and Alpher, Ralph A.: Base Pressures at Supersonic Velocities. Jour. Aero. Sci., vol. 16, no. 3, Mar. 1949, pp. 153-160.
15. Faro, I. D. V.: Experimental Determination of Base Pressure at Supersonic Velocities. Johns Hopkins Univ., Bumblebee Report 106, November 1949.

TABLE I.—VALUES OF M' AND p' FOR A TWO-DIMENSIONAL AIRFOIL

M_∞	M'	p'/p_∞
1	1.25	0.73
1.5	1.50	1.00
2	2.00	1.00
3	2.99	1.01
8	7.85	1.14
∞	82	∞

TABLE II.—VALUES OF M' AND p' FOR A CONE-CYLINDER BODY OF REVOLUTION

M_∞	M'	p'/p_∞
1.5	1.51	0.98
2	2.02	.97
3	3.03	.95
7	7.02	.86

TABLE III.—VALUES OF M' AND p' FOR A CONE

M_∞	M'	p'/p_∞
1.5	1.58	0.88
2	2.09	.87
3	3.13	.82
7	7.16	.76

REPORT 1052

A SUMMARY OF LATERAL-STABILITY DERIVATIVES CALCULATED FOR WING PLAN FORMS IN SUPERSONIC FLOW

By ARTHUR L. JONES and ALBERTA ALKSNE

SUMMARY

A compilation of theoretical values of the lateral-stability derivatives for wings at supersonic speeds is presented in the form of design charts. The wing plan forms for which this compilation has been prepared include a rectangular, two trapezoidal, two triangular, a fully-tapered swept-back, a swept-back hexagonal, an unswept hexagonal, and a notched triangular plan form. A full set of results, that is, values for all nine of the lateral-stability derivatives for wings, was available for the first six of these plan forms only. The reasons for the incompleteness of the results available for other plan forms are discussed.

The values of the derivatives presented were obtained directly from tabulated results in the reports referenced or were calculated from expressions presented in these reports. The expressions for the derivatives were derived using linearized theory for compressible flow. The values presented, however, do not represent exact linear-theory solutions in every case due to the fact that approximations and simplifications were sometimes necessary in the derivation of the expressions or in the calculations of the numerical results. The effects of these approximations and simplifications on the accuracy and applicability of the results are considered.

INTRODUCTION

The calculation of the lateral-stability derivative coefficients (hereinafter referred to as the lateral-stability derivatives) for thin wings at supersonic speeds has been accomplished for a number of plan forms and the results are reported in references 1 through 18. These results are necessarily incomplete in view of the fact that there is an unlimited number of plan forms that could be investigated. There appears to be a sufficient quantity of results available, however, to warrant the preparation of a summary. In fact, some summaries, such as references 10 and 17, are already available but the results presented in these reports are restricted to a fairly small number of plan forms or to a small number of derivatives. It is the purpose of this report, therefore, to assemble and present a more extensive set of numerical results than was heretofore available. These results will be limited to the thin-airfoil, inviscid-flow solutions obtained from application of the linearized theory of compressible flow. They will be presented in the form of design charts showing the variations of the derivatives¹ with Mach number, aspect ratio, and other plan-form parameters. The derivatives have been evaluated in a manner that per-

mits their direct application in an analysis using the stability axes system.

A discussion of the limitations in the applicability and availability of the lateral stability derivative results is included. Some of the limitations are inherent in the linearized theory itself, but most of the limitations are due to the approximations and simplifications sometimes found necessary in the application of the theory.

The plan forms for which the results are presented are illustrated in figure 1. Included are a rectangular, two

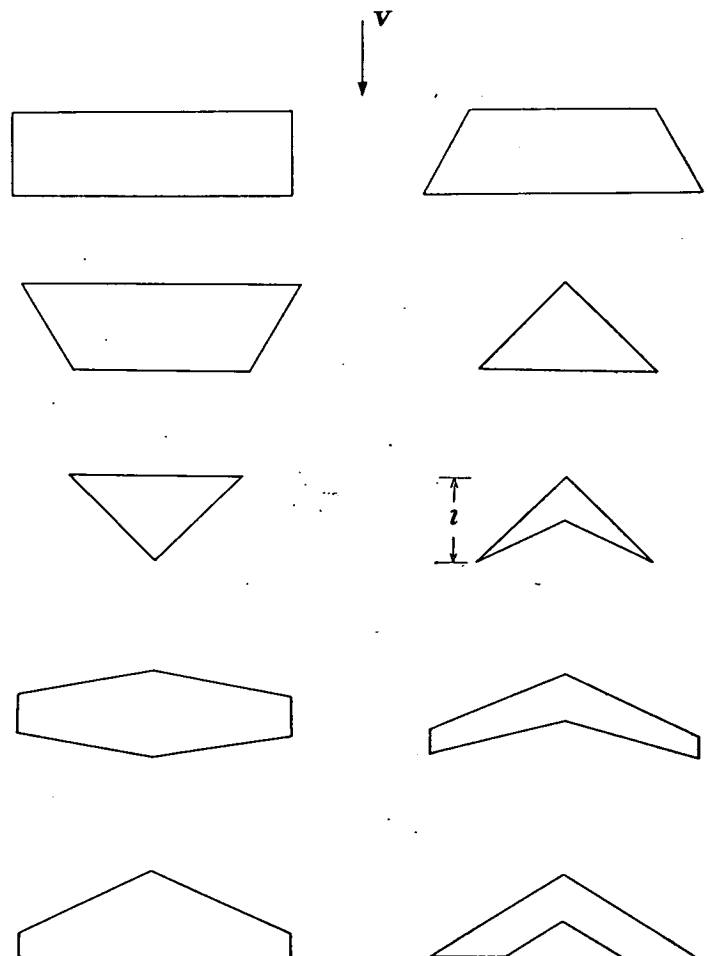


FIGURE 1.—Types of plan forms for which summarized results are presented.

¹ Although the scale labels show that some of the stability derivatives presented in the figures are divided by angle of attack or angle of attack squared, reference to the variation of this quotient with other variables will be simply referred to as the variation of the derivative. This usage is consistent with the terminology used in previous NACA reports.

triangular, two trapezoidal, a fully-tapered swept-back, a swept-back hexagonal, an unswept hexagonal, and a notched triangular plan form. This group may not cover all the plan forms for which at least partially complete results are available. It is, however, a fairly representative group. In table I the sources of the results for the lateral-stability derivatives of the plan forms surveyed are listed.

SYMBOLS AND COEFFICIENTS

A	aspect ratio
b	span of wing measured normal to plane of symmetry
B	$\sqrt{M^2-1}$
c_r	chord of wing root
$c. g.$	center of gravity
C_l	rolling-moment coefficient $\left(\frac{L}{qSb}\right)$
C_n	yawing-moment coefficient $\left(\frac{N}{qSb}\right)$
C_Y	side-force coefficient $\left(\frac{Y}{qS}\right)$
C_{l_p}	damping-in-roll derivative $\left[\frac{\partial C_l}{\partial(p\dot{b}/2V)}\right]$
C_{l_r}	rolling-moment-due-to-yawing derivative $\left[\frac{\partial C_l}{\partial(r\dot{b}/2V)}\right]$
C_{l_β}	rolling-moment-due-to-sideslip derivative $\left(\frac{\partial C_l}{\partial\beta}\right)$
C_{n_p}	yawing-moment-due-to-rolling derivative $\left[\frac{\partial C_n}{\partial(p\dot{b}/2V)}\right]$
C_{n_r}	yawing-moment-due-to-yawing derivative $\left[\frac{\partial C_n}{\partial(r\dot{b}/2V)}\right]$
C_{n_β}	yawing-moment-due-to-sideslip derivative $\left(\frac{\partial C_n}{\partial\beta}\right)$
C_{Y_p}	side-force-due-to-rolling derivative $\left[\frac{\partial C_Y}{\partial(p\dot{b}/2V)}\right]$
C_{Y_r}	side-force-due-to-yawing derivative $\left[\frac{\partial C_Y}{\partial(r\dot{b}/2V)}\right]$
C_{Y_β}	side-force-due-to-sideslip derivative $\left(\frac{\partial C_Y}{\partial\beta}\right)$
l	over-all longitudinal length of swept-back wing (See fig. 1.)
$l_{c. g.}$	longitudinal location of center of gravity aft of the leading edge of the root chord
L	rolling moment (See fig. 2(b).)
m	slope of right wing tip or leading edge relative to plane of symmetry (Positive for raked-out tip, negative for raked-in tip.)
M	free-stream Mach number
N	yawing moment (See fig. 2(b).)
p	rate of roll, radians per second
q	free-stream dynamic pressure
r	rate of yaw, radians per second
S	area of wing
V	free-stream velocity
x	longitudinal coordinate
$x_{c. g.}$	arbitrary longitudinal location of the center of gravity with respect to the location specified in this report (Positive for locations forward of those specified.)
y	lateral coordinate

Y	side force (See fig. 2 (b).)
z	vertical coordinate
α	angle of attack, radians
β	angle of sideslip (positive when sideslipping to right), degrees
λ	taper ratio

AXES

At least three systems of axes are associated with the development and application of stability derivatives. For instance, the theoretical expressions for the stability derivatives are most easily derived using a set of three orthogonal axes, known as the wind axes, that are oriented as shown in figure 2 (a). The origin of these axes is at the leading edge of the root chord. The x axis is an extension of the free-stream vector through the origin and is positive rearward. The y axis lies in the plane of the wing perpendicular to the x axis and is positive toward the right tip. The z axis stands perpendicular to the x and y axes and is positive upward. These axes are commonly used in wing-theory calculations.

Body-axes systems (see fig. 2 (b)) are sometimes used in the calculations of the motion of an aircraft. These calculations are commonly referred to as dynamic-stability calculations. The origin of the axes for such calculations is usually the center of gravity rather than the leading edge of the root chord. Thus the expressions initially derived for the derivatives should be corrected for the change in moment-center location accompanying the change in location of the origin of the body axes.

A third system of axes known as the stability axes (fig. 2 (c)) is often preferred for dynamic-stability calculations. The basic difference between the orientations of a stability-axes system and a body-axes system is the location of the x axis. In a stability-axes system the x axis is assumed to lie along the intersection of the plane of symmetry and a plane perpendicular to the plane of symmetry which contains the free-stream velocity vector (or the velocity vector of the center of gravity). Thus the stability axes, in general, correspond to the body axes rotated through an angle of $-\alpha$ and, in order to transform expressions for stability derivatives that are applicable to a body-axes system to expressions applicable to a stability-axes system, it is necessary to correct for the angle of attack. If the origins of the body and stability axes are coincident, this correction can be made by using the transformation formulas in reference 19. If the origins are not coincident, the correction must also include the effects of the chordwise distance between the origins. Most of the reports used as sources for the material presented, references 1 through 17, provide either formulas for both systems of axes or means for converting from one system to the other. It is interesting to note that the definitions of the positive directions for the velocities, forces, and moments of the wing conform to the right hand screw-rule for a set of body or stability axes.

For the calculation of the stability derivatives presented herein, the location of the center of gravity was selected as $c_r/2$ for the rectangular, trapezoidal, and unswept hexagonal wings, $(2/3) c_r$ for the triangular wings, and $(2/3) l$ for the swept-back wings. The values of the derivatives pertain to a stability-axes system so located. For arbitrary locations

of the center of gravity the derivatives that are affected by changes in the location can be evaluated using the following formulas:

$$(C_{l_r})_{x_{c.g.}} = C_{l_r} - \frac{2x_{c.g.}}{b} C_{l_\beta}$$

$$(C_{n_p})_{x_{c.g.}} = C_{n_p} - \frac{x_{c.g.}}{b} C_{Y_p}$$

$$(C_{n_\beta})_{x_{c.g.}} = C_{n_\beta} - \frac{x_{c.g.}}{b} C_{Y_\beta}$$

$$(C_{n_r})_{x_{c.g.}} = C_{n_r} - \frac{x_{c.g.}}{b} \left(C_{Y_r} + 2C_{n_\beta} - 2 \frac{x_{c.g.}}{b} C_{Y_\beta} \right)$$

$$(C_{Y_r})_{x_{c.g.}} = C_{Y_r} - \frac{2x_{c.g.}}{b} C_{Y_\beta}$$

DERIVATION OF RESULTS

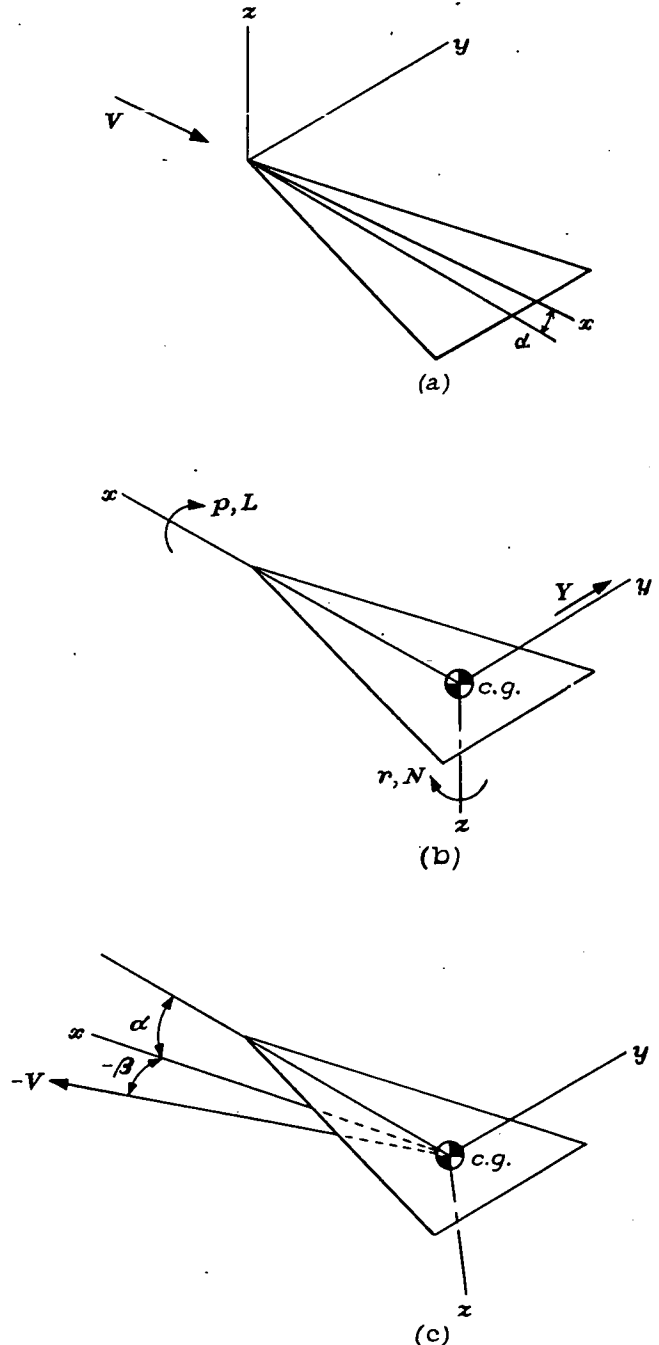
The derivation of theoretical results involves both the formulation of expressions for the derivatives and the calculation of the numerical values of the derivatives for specific plan forms and conditions. In some of the reports referenced, the analytical expressions for the derivatives were derived and presented but no numerical results were included. In other reports, both the analytical expressions and a few numerical results were given. The following discussion, therefore, will be concerned with the general method of analysis used in obtaining the analytical expressions and the way in which the numerical results presented were compiled.

GENERAL METHOD OF ANALYSIS

A stability derivative is an expression of the rate of change of a force or moment with respect to the motion producing it. Once the shape of the object for which a stability derivative is desired has been decided upon and its motion prescribed, an analysis to determine the resulting force or moment can be started. The first step in the analysis is the formulation of the boundary conditions and at this point an idealization is usually made. For a wing of finite dimensions it is convenient to assume that the effects of wing thickness and of the viscosity of the flow can be ignored or at least estimated independently. On the basis of this assumption, the wing and its flow field can be represented by a thin plate operating in an inviscid flow. This idealization permits the linearized theory for compressible flow to be used for the analysis, provided the angle of attack of the wing is kept small.

The first step in the calculation of the lateral-stability derivatives for a wing, therefore, is the specification of boundary conditions for a thin wing in sideslip, roll, or yaw. This specification can be made readily for two of these three lateral motions. For instance, a wing in sideslip at a small angle of attack can be represented by a flat plate in the same position and the plate can be simulated by use of a uniform distribution of the vertical perturbation velocity over the plan form. A rolling wing can be represented by a twisted thin plate and simulated by a linear spanwise variation of the downwash perturbation velocity over the plan-form area. This representation of the rolling wing is an illustration of the manner in which quasi-steady flow conditions

can be substituted for conditions that are, in fact, unsteady. Such a substitution permits the use of a simple steady-state analysis which yields results of sufficient accuracy for the motions considered in stability theory if not for the more rapid motions involved in flutter theory. A yawing wing, however, cannot be suitably represented by a flat or twisted thin plate in steady flow. In fact, no quasi-steady flow conditions have been developed at the present time which would permit the straightforward application of a steady-state analysis. Thus it appears that the most desirable procedure in the case of the yawing wing would be to undertake the analysis as a problem in unsteady flow. Such analyses are quite complex, however, and have not as



(a) Wind-axes system used in derivations of expressions for stability derivatives;
(b) Body-axes system and definition of positive moments and velocities;
(c) Stability-axes system usually preferred for dynamic-stability calculations.

FIGURE 2.—Three systems of axes associated with the development and application of stability derivatives.

yet been carried out extensively. Consequently, the results presented herein for the derivatives due to yawing are approximations based on strip theory. In the case of the rectangular plan form of infinite span or the trapezoidal plan form with raked-in supersonic tips, the validity of these results has been confirmed by an unsteady-flow solution presented in reference 20.

Having specified the boundary conditions, the next step in the analysis is to apply the linearized theory for compressible flow to the calculation of the pressure distributions. The details of this step may be found in a number of reports dealing with supersonic wing theory. (See, for example, references 21, 22, and 23.) Unfortunately, only pressures acting normal to the surface of the flat plate are calculated directly from the application of the linearized theory. These pressures are not a complete representation of the pressures acting on a wing of finite thickness, and in order to obtain a better representation the theory of edge suction must be applied and the pressures acting on the edges of the thin plate determined. Then by using both the normal and edge pressures, the moments and forces acting on the wing can be determined for the motion prescribed and the stability derivative can be calculated.

The foregoing general description of the method of calculating stability derivatives for wings, using the linearized theory of compressible flow, is merely a brief summarization of the procedure. For a more detailed description of the steps involved, the reports referenced should be consulted.

LIMITATIONS OF BASIC THEORY

The idealization of the flow field discussed above and the resulting simplifications in the theoretical analysis are reflected as limitations in the applicability of the results obtained in this manner. An obvious limitation, for instance, is that the results should be applicable only at the small angles of attack to which the wing was restricted for the analysis. Comparisons of experimental and theoretical results, however, have shown that this particular limitation usually can be exceeded in practical applications without incurring excessive inaccuracy. The less obvious limitations can be deduced from considerations of the effects of neglecting thickness and viscosity. For instance, the two basic reasons for differences between the experimental and theoretical results are: (1) the difference in pressure distributions caused by neglecting the effects of thickness, camber, and viscosity in the calculations; and (2) the existence of a skin-friction force caused by viscosity. In general, it does not seem likely that the pressure distributions due to thickness or camber will vary appreciably with any of these lateral motions. Consequently the linearized-theory results should not be greatly limited in application due to the effects of thickness or camber. On the other hand, the effect of viscosity on the pressure distribution, which is principally the effect of the boundary layer, is of consequence. It is very difficult, however, to determine the nature of the boundary-layer flow and its effect on the pressure distribution in steady straight flight and even more difficult in the case of flight involving the lateral motions. Thus the contribution of the pressure-distribution effects of viscosity to the values of the

lateral-stability derivatives is a relatively unknown factor and probably the greatest source of discrepancy between the theoretical and actual values of the lateral derivatives. The skin-friction force, which is also due to viscosity, will vary appreciably with yawing velocity but not significantly with rolling velocity or sideslip. The effects of this force can be estimated and added to the linearized-theory value of C_n to reduce the limitations of applicability of this derivative. The contribution of the skin-friction force to C_l , however, is not significant since it is mainly a drag force and has only a very small component in the direction that would produce a rolling moment.

Another assumption which is inherent in the analysis applied is that the wing be completely rigid. The applicability of the theoretical results obtained on the basis of complete rigidity is limited, of course, by the effects of the deflection or distortion of the actual wing. However, only wings that have exceptionally long and slender panel lengths are subject to much distortion or deflection. Furthermore, current studies of the effect of aeroelasticity on stability derivatives are directed toward the determination of deviations from rigid-body results and it can be anticipated that the possibility of making appropriate changes in the values of the derivatives given herein may result from such analyses.

COMPILATION OF NUMERICAL RESULTS

The numerical results presented were obtained directly from the results tabulated in references 1 through 17 or were calculated using expressions presented in these reports. Wherever possible, both the numerical values of the derivatives and the expressions from which they were obtained were checked against identical results available from duplicate analyses. The plotting and cross plotting of the results in the construction of the figures presented also provided an effective check against errors.

Use was made of the reversability theorem (see reference 24) in the calculation and checking of the results for the damping-in-roll derivative C_{l_p} . No evidence or proof exists, however, that would permit the reversability theorem to be applied to the calculation or checking of the results for the other lateral derivatives.

RESULTS AND DISCUSSION

PRESENTATION OF RESULTS

As mentioned previously, the plan forms covered in this summary include a rectangular, two triangular, two trapezoidal, a fully-tapered swept-back, a swept-back hexagonal, an unswept hexagonal, and a notched triangular plan form. Estimates of all the lateral stability derivatives are available for the first six of these plan forms. Only partially complete results are available for the other plan forms.

The values of the derivatives plotted against aspect ratio and against Mach number parameter B are shown in figures 3 through 19. The results are grouped by derivatives rather than by plan forms. The values of the damping-in-roll parameter BC_{l_p} for all the plan forms surveyed are presented in figures 3 and 4, which show the variation of this derivative with the aspect-ratio parameter BA . A similar presentation,

figure 7, was used for the side-force derivative C_{Y_p} . The other derivatives required separate plots to show the aspect-ratio and Mach number variations for each type of plan form. In table II, a convenient cross reference of the figure numbers, plan forms, and derivatives is presented.

For these general results, the aspect-ratio range investigated extended from 0 to 9 and values of the derivatives were calculated for constant values of B of 1, 2, and 4. The Mach number parameter range investigated extended from 0 to 4, corresponding to Mach numbers of 1 and 4.13, respectively, and the derivatives were calculated for aspect ratios of 2, 4, and 6. More extensive ranges and additional intermediate values of these variables could have been investigated, but the complexity of the calculations made it impractical to do so. It is felt, however, that included within these limits of aspect ratio and Mach number are most of the plan forms and speeds currently of interest. Furthermore, by interpolation and, in some instances, by careful extrapolation of the results presented, it is possible to obtain theoretical values of the derivatives for plan forms of the type covered, but not specifically investigated, that should be useful as preliminary design estimates.

For the trapezoidal plan forms, the tip rake was selected to correspond to an angle of sweep of approximately 63° ($m = \pm 0.5$) and the aspect ratio was varied by changing the span. Supplementary plots showing the variation of the derivatives with tip rake for a constant aspect ratio are presented in figures 20 through 28. For the fully-tapered swept-back plan forms, the value of the ratio of the root chord to over-all length c_r/l was set at 0.5 and the aspect ratio was varied by changing the sweep of the leading edge. Supplementary plots were used in this case to show the variation of the derivatives with the ratio of root chord to the over-all length for a constant value of leading-edge sweep. These variations are presented in figures 29 through 37.

In figures 38 through 67 the variations with taper ratio and with leading-edge slope of the derivatives available for the swept-back and unswept hexagonal and the notched triangular plan forms are presented.

ASPECT RATIO AND MACH NUMBER RANGES COVERED

In many instances, the results previously described were available only within small portions of the ranges of aspect ratio and Mach number mentioned. The reasons for these limitations are discussed in the following paragraphs of this section.

For the lateral motions of rolling and sideslip, the linearized theory is directly applicable and the results available are limited only by the complexity of the calculations involved in determining the load distributions for certain plan forms and certain conditions. For the yawing motions, however, the difficulty in specifying quasi-steady boundary conditions precludes the direct application of steady-state linearized theory. Approximate solutions based on strip theory can be used, however, wherever suitable solutions of this type can be developed. Rigorous unsteady-flow solutions would of course be preferable for the yawing derivatives but only one such solution has been reported at this time. Thus it is apparent that the derivatives due to yawing

are neither as available nor as generally applicable as the derivatives due to rolling or sideslip.

The cause of the calculation complexities that limit the determination of the load distribution to certain plan forms and certain conditions is the existence of regions on a wing which are affected by the interaction of flows past two or more subsonic edges² lying within the Mach forecones of points contained within the region. Such regions exist on any wing having interacting subsonic edges, that is, on any wing having one subsonic edge lying within the region of influence of another. These regions also exist on a wing having mutually interacting subsonic edges. The determination of the load distribution for such regions is extremely difficult using the basic integral-equation methods (references 21 and 22) developed for supersonic wing theory. This is especially true for the regions influenced by mutually interacting subsonic edges. As a consequence, special cancellation-of-load techniques have been used (references 11, 15, and 18) to handle plan forms having interacting or mutually interacting subsonic edges. The application of these cancellation techniques, however, usually involves rather lengthy calculations and, consequently, numerical results are available for only relatively few of the plan forms that require such methods. Thus most of the results presented herein are for plan forms that do not have interacting subsonic edges.

The actual ranges of the aspect-ratio parameter for which results are presented are given in table III. The limits of these ranges are explained in the discussion that follows.

Rectangular plan form.—For values of BA less than 1 the Mach lines from the tip of the rectangular plan form would intersect the opposite edges and thereby change them from noninteracting to interacting subsonic edges. Thus a lower limit is set on the aspect-ratio parameter BA for which C_{l_p} , C_{n_p} , C_{Y_p} , C_{l_β} , C_{n_β} , and C_{Y_β} are presented. The derivatives due to yawing, C_{l_r} , C_{n_r} , and C_{Y_r} , were calculated using the analysis presented in reference 6. This analysis indicated that reasonably accurate approximate values of C_{l_r} , C_{n_r} , and C_{Y_r} could be obtained as a fraction $\frac{\alpha(1-B^2)}{B^2}$

of the C_{l_p} , C_{n_p} , and C_{Y_p} results. Thus the limitations of the aspect-ratio parameter for C_{l_r} , C_{n_r} , and C_{Y_r} are the same as those for C_{l_p} , C_{n_p} , and C_{Y_p} .

Trapezoidal plan form.—The lower limits on the aspect-ratio parameter for both types of trapezoidal plan form (tips raked in and tips raked out) are to prevent the plan-form tips from becoming interacting subsonic edges. If the plan form has supersonic tips, however, the lower limit is the aspect ratio for which the trapezoid becomes a triangle.

The derivatives due to yawing were obtained from the application of the factor $\frac{\alpha(1-B^2)}{B^2}$ developed for the rectangular plan form. The justification of this step is given in a later section of the report wherein the sources and development of the results are discussed.

² In the terminology that has become associated with supersonic flow analysis, the term "subsonic edge" refers to an edge swept behind its Mach line (because the component of flow normal to the edge is subsonic) and the term "supersonic edge" refers to an edge swept ahead of its Mach line.

Triangular plan form with apex forward.—There is no aspect-ratio-parameter limitation necessary for the derivatives due to rolling or sideslip. The derivatives due to yawing, however, have been obtained only for plan forms having a BA of 4 or less because no solution, approximate or otherwise, has been developed for the yawing triangular wing with supersonic leading edges.

Triangular plan form with base forward.—For values of BA less than 4, the edges of this plan form become mutually interacting subsonic trailing edges and the reflections of the Mach lines establish an infinite number of regions which would have to be analyzed independently. This limit applies to the derivatives due to sideslip and yawing and would apply to the derivatives due to rolling if it were not for the fact that the reversability theorem (see reference 24) permits these derivatives to be calculated for the same plan-form range as obtained for the triangular plan forms with apex forward.

The derivatives due to yawing were obtained by applying the factor $\frac{\alpha(1-B^2)}{B^2}$ developed in the rectangular-wing analysis to the derivatives due to rolling. The justification of using this factor for a base-forward triangular plan form is given in a later section of this report. Inasmuch as the reversability theorem does not apply to the derivatives due to yawing, the lower limit for BA is 4 rather than zero.

Fully-tapered swept-back plan form.—The lower limit of BA , $4\frac{l}{c_r}\left(1-\frac{c_r}{l}\right)$, is common to all the derivatives in order to assure that the trailing edge of this plan form does not become subsonic. The upper limit of BA for the derivatives due to sideslip and yawing, $4l/c_r$, assures that the leading edge will not become supersonic. Values of the derivatives C_{l_p} , C_{n_p} , and C_{Y_p} , however, have been obtained from references 13 and 16 for plan forms having supersonic leading edges.

Swept-back hexagonal plan form.—The plan forms considered to fall in this classification have streamwise tips and range from the constant-chord swept-back, $\lambda=1$, to the swept-back plan form having a straight trailing edge, $A=4m(1-\lambda)/(1+\lambda)$. The aspect-ratio parameter range for this wing, given in table III, indicates that results are presented for plan forms having either subsonic or supersonic leading edges and having supersonic trailing edges. An added restriction provided by the limits on the aspect-ratio parameter range is that the Mach lines from the leading edges of the tips must not cross on the wing.

Unsweppt hexagonal plan form.—These plan forms have streamwise tips and range from the swept back with a straight trailing edge to the hexagonal plan form having fore-and-aft symmetry, which can be expressed as an aspect ratio range of

$$\frac{2m(1-\lambda)}{(1+\lambda)} \leq A \leq \frac{4m(1-\lambda)}{(1+\lambda)}$$

The results presented for the derivatives due to rolling and yawing are subject to the same aspect-ratio-parameter limitations that were given for the derivatives due to rolling for the swept-back hexagonal plan form.

Notched triangular plan form.—The derivatives due to rolling are presented for both subsonic and supersonic leading edges. It should be pointed out that for the damping-in-roll derivative the plan forms having subsonic leading edges differ from those having supersonic leading edges in that the trailing edges of the former always lie along the Mach line from the trailing end of the root chord; whereas the trailing edges of the latter are parallel to the leading edges. (See reference 2.) The derivatives due to sideslip are presented only for the plan form having supersonic leading edges.

SOURCES AND DEVELOPMENT OF RESULTS

The sources of the results for the lateral-stability derivatives of the various plan forms surveyed are listed in table I. An indication of the rigor which has been maintained in the individual developments of these results is given in the following discussion. Sources of some of the results available for plan forms other than those surveyed are also mentioned.

Derivative C_{l_p} . The rolling wing lends itself to readily specified boundary conditions and a rigorous application of the linearized-theory analysis. The damping moment obtained varies linearly with the rolling velocity p and, accordingly, the expression obtained for the derivative C_{l_p} is usually a relatively simple one. Furthermore, the numerical results are readily computed and can be presented in one figure showing both the variation with Mach number and aspect ratio. Consequently, there are more results available for the damping-in-roll derivative than for any of the other lateral derivatives, particularly in regard to the number of plan forms investigated.

In addition to the results presented in this summary, more extensive damping-in-roll results are available in the referenced reports for plan forms having arbitrary sweep and taper. In references 9 and 10, C_{l_p} is determined for such plan forms having subsonic leading edges and supersonic trailing edges. In reference 13, these results are extended to include such plan forms having supersonic leading and trailing edges. The effect of subsonic trailing edges has been investigated in references 11 and 15 using the cancellation-of-pressure technique, but the numerical results are limited to one or two plan forms. Values of C_{l_p} described as the upper limits for swept-back plan forms having subsonic trailing edges are presented in reference 9. These values were obtained by extending the expressions developed for C_{l_p} beyond the limits of applicability for which they were derived. The damping-in-roll derivative for swept-back tapered wings having raked-in or cross-stream tips and subsonic leading edges has been investigated and the results are presented in reference 12. Thus, it is possible to calculate the damping-in-roll derivative for a large number of plan forms of arbitrary sweep, taper, and tip rake. In some cases the numerical values of the derivative have been calculated and presented. In other cases the calculations must be carried out, a step which involves rather lengthy calculations when the trailing edges are subsonic and the cancellation techniques must be used.

It should be mentioned, however, that in many instances values for the damping-in-roll derivative can be determined

for plan forms and Mach angle configurations that would be extremely tedious if at all possible to calculate, by application of the plan form reversibility theorem presented in reference 24.

Derivative C_{n_p} : The contribution of the normal force to this derivative is $-\alpha C_{l_p}$ which can be evaluated from the damping-in-roll results previously discussed. The effects of edge suction on this derivative are also readily calculated. Thus it is possible to calculate C_{n_p} for the same extensive range of plan forms for which it was possible to calculate C_{l_p} . The results actually available for C_{n_p} , however, cover only the plan forms considered in this summary.

For the trapezoidal plan forms having raked-out tips, the expression for C_{n_p} applicable to the stability-axes system has not been published previously. For subsonic tips, and with the center of gravity located at $c_r/2$, the expression is,

$$C_{n_p} = \frac{-256\alpha\sqrt{1-B^2m^2}}{9\pi B^3 A^3(1+Bm)\left(1+\sqrt{1-\frac{4Bm}{BA}}\right)^3} \left\{ B \left[\frac{3(1+Bm)(m^2-1)-2m^2}{\left(1+\sqrt{1-\frac{4Bm}{BA}}\right)} \right] - (BA)[3B(m^2-1)+m] + \frac{9B^2A^2}{8} m \left(1+\sqrt{1-\frac{4Bm}{BA}}\right) - \frac{B}{4} \left[9BA - \frac{12Bm}{\left(1+\sqrt{1-\frac{4Bm}{BA}}\right)} - \frac{8}{\left(1+\sqrt{1-\frac{4Bm}{BA}}\right)} \right] \right\}$$

and for supersonic tips the expression is

$$C_{n_p} = -\alpha C_{l_p}$$

Derivative C_{Y_p} : The derivative C_{Y_p} exists only when edge suction forces are present. The results available for C_{Y_p} cover only the plan forms considered in this summary and some swept-back tapered wings with streamwise tips having either supersonic or subsonic leading edges. (See references 14 and 16, respectively.)

The expressions for C_{Y_p} for the trapezoidal plan forms having raked-out or raked-in tips have not been published previously. For the subsonic raked-out-tip plan forms the expression applicable to the stability axes system is

$$C_{Y_p} = \frac{64\alpha\sqrt{1-B^2m^2}}{\pi B^2 A^2(1+Bm)\left(1+\sqrt{1-\frac{4Bm}{BA}}\right)^2} \left[BA - \frac{4Bm}{3\left(1+\sqrt{1-\frac{4Bm}{BA}}\right)} - \frac{8}{9\left(1+\sqrt{1-\frac{4Bm}{BA}}\right)} \right]$$

For the supersonic raked-out-tip plan forms and for the raked-in-tip plan forms there is no suction effect and C_{Y_p} is, consequently, zero.

Derivative C_{l_p} : For wings in sideslip, the boundary conditions are easily specified and the linearized-theory analysis to obtain the normal force is easily carried out. Unfortun-

nately, however, the rolling moment cannot be generally expressed as a linear function of sideslip angle. (See reference 5.) To obtain the derivative C_{l_p} , therefore, it is necessary to plot C_l against the sideslip angle β and to measure the slope or, if the sideslip is restricted to very small angles (reference 4), it is possible to linearize the expression for C_l and obtain an explicit expression for C_{l_p} . Although no derivatives due to sideslip are presented for the unswept and swept-back hexagonal plan forms, these derivatives can be calculated for these plan forms using references 21, 22, or 23, or if a cancellation technique is needed, reference 18 gives a demonstration that is directly applicable to these cases.

It should be pointed out that for the trapezoidal plan forms with a tip slope m of $\pm \frac{1}{2}$ the Mach cones and tips coincide at

$B=2$. Double values of the derivatives due to sideslip, C_{l_p} , C_{n_p} , and C_{Y_p} , occur at this value of B and in order to avoid any ambiguity the limiting values of the derivatives obtained by approaching $B=2$ from the lower values of B (tips subsonic) were labeled $B=2(-)$ and the limiting values of the derivatives obtained by approaching from higher values of B (tips supersonic) were labeled $B=2(+)$.

For plan forms having streamwise tips, such as rectangular plan forms, there is some doubt as to the validity of the results obtained by applying the Kutta condition to the trailing tip at small angles of sideslip. In reference 6 it is assumed that the Kutta condition does not apply, whereas in reference 5 it is assumed that the Kutta condition applies to the trailing tip at small angles of sideslip as well as large. This difference in basic assumptions leads to two entirely different values for C_{l_p} . From physical considerations of what the flow must be past a sharp trailing edge, such as the tip of a thin airfoil, the theoretical analysis based on the latter assumption is the correct one. From physical considerations of the flow about a wing tip of finite thickness, the correctness of either assumption depends upon the ability of the boundary layer on the tip to resist separation. If the suction force caused by the high velocity flow of air from the bottom surface to the top surface is strong enough to cause the boundary layer to separate, then a flow corresponding to the Kutta condition will result and the edge suction force will no longer exist. The angle of sideslip at which separation of the flow around the tip will take place is, at present, undetermined. It will depend on the angle of attack and on the shape of the tip to a large extent. It is definitely possible, however, for separation to occur on tips raked at a slight angle into the stream, such as the advancing tip of the rectangular wing in sideslip, as well as on a trailing tip. If this were the case, the Kutta condition would have to be applied to both tips at small angles of sideslip rather than to one or neither as assumed in the previously mentioned analyses. Until experimental results are obtained that will provide definite quantitative evidence to the contrary, therefore, it seems most reasonable to assume that the solution based on the flow over a thin trailing edge is valid. In other words, it seems most reasonable to assume that the Kutta condition holds for all trailing edges and the edge suction exists on all leading edges no matter what the angle of inclination is

between the edge and the stream. Accordingly, the results presented in this survey correspond to those reported in reference 5.

Derivative $C_{n\beta}$: The contribution of the edge suction effects to this derivative is easily estimated but the resulting expression for the yawing moment is nonlinear with sideslip. The normal force contribution $-\alpha C_{l\beta}$ is also nonlinear with sideslip as described in the discussion of $C_{l\beta}$. Both of these nonlinearities can be handled, however, in the manner mentioned in that discussion.

For the plan forms having streamwise tips the application of the Kutta condition to the trailing tip at small angles of sideslip causes a jump in the yawing-moment curve at zero sideslip. (See references 8 and 17.) This jump makes it difficult to establish a rational value for $C_{n\beta}$ especially in view of the fact that in actuality the jump will, undoubtedly, be rounded off and the curve will have no discontinuity in slope at zero sideslip. The value of the derivative selected for the curve with the jump was based on the slope of the yawing-moment curve as it approached zero sideslip. This value is questionable in magnitude. It has the proper sign, however, and should be an underestimation of the actual value of $C_{n\beta}$.

Derivative $C_{Y\beta}$: Inasmuch as the edge-suction force is the only source of side force for wing plan forms, the derivative $C_{Y\beta}$ is zero (at zero sideslip) for plan forms having supersonic raked-out tips or raked-in tips. For trapezoidal plan forms having subsonic tips, the expression for the side force is

$$C_{Y\beta} = \frac{-32\alpha^2}{\pi A \left(1 + \sqrt{1 - \frac{4Bm}{BA}}\right)^2} \frac{(1+B^2)}{B^2(1+Bm)\sqrt{1-B^2m^2}}$$

The jump that appears in the yawing moment versus sideslip curve at zero sideslip for plan forms having streamwise tips occurs also in the side force versus sideslip curve. The evaluation of the derivatives under these conditions was handled in the manner described in the discussion of $C_{n\beta}$.

Derivative C_{l_r} : As mentioned in a preceding section of this report, the yawing wing cannot be fitted satisfactorily to quasi-steady boundary conditions that would permit the use of an exact linearized-theory analysis based on a steady-state flow. For rectangular and triangular wings, however, modified strip-theory analyses have been applied in references 6 and 4, respectively, and approximate values for the rolling-moment-due-to-yawing derivative have been obtained. The analysis for the triangular wing takes into account the spanwise variation of speed but not the spanwise variation of Mach number. A chordwise variation of the effective sideslip angle for a yawing wing is also included in the triangular-wing analysis. The contribution of this latter factor is predominant for triangular wings of low aspect ratio (that is, triangular wings having small vertex angles). Both the spanwise variations of Mach number and of speed were taken into account in a somewhat similar analysis for the rectangular wing. From this latter analysis a factor of

$\frac{\alpha(1-B^2)}{B^2}$ was found to exist between the loading due to rolling and the loading due to yawing over the portion of the rectangular wing where the flow was two dimensional, and it was assumed that this factor could also be applied in the vicinity of the tips. An analysis of a rectangular wing of infinite span based on unsteady flow, made in reference 20, provided verification of the existence of the factor $\frac{\alpha(1-B^2)}{B^2}$

for the region of two-dimensional flow. In view of this verification, it seems that the rectangular-wing values for C_{l_r} obtained from reference 6 should be a good approximation, at least for plan forms having high aspect ratios. Furthermore, application of the factor $\frac{\alpha(1-B^2)}{B^2}$ to the rolling-moment results for a trapezoidal plan form having supersonic raked-in tips should provide exact theoretical values for C_{l_r} (using the unsteady-flow-analysis results as a norm) because of the lack of tip effects. Inasmuch as any plan form having relatively large regions of two-dimensional flow should be suited to this approximate analysis, it was decided to use this factor to obtain approximate values of C_{l_r} for the trapezoidal plan forms, the base-forward triangular plan form with supersonic tips, and the unswept hexagonal plan form. By this step, theoretical estimates of C_{l_r} were made available for all but the apex-forward triangular plan form with supersonic leading edges, the base-forward triangular plan form with subsonic trailing edges, the swept-back hexagonal plan form, and the notched triangular plan form.

Derivative C_{n_r} : Application of the factor $\frac{\alpha(1-B^2)}{B^2}$ to the calculation of the edge-suction-force contribution to C_{n_r} from the edge-suction force due to rolling will yield an admittedly rougher approximation than the application of this factor to the determination of the normal force due to yawing. However, in view of the fact that it was the only method available for estimating the edge-suction force in yawing, it was used to obtain the C_{n_r} results for all the plan forms similarly analyzed for C_{l_r} .

Derivative C_{Y_r} : The application of the factor $\frac{\alpha(1-B^2)}{B^2}$ to calculate C_{l_r} and C_{n_r} from C_{l_p} and C_{n_p} , respectively, for certain plan forms was extended to the side-force calculations in order to obtain values for the derivative C_{Y_r} . The apex-forward triangular plan form having subsonic leading edges was excepted inasmuch as the derivative C_{Y_r} was available from reference 4.

CONCLUDING REMARKS

Values of the lateral-stability derivatives for wings at supersonic speeds, calculated using the linearized theory for compressible flow, have been presented in the form of design charts showing the variations of the derivatives with Mach number and aspect ratio for nine plan forms. These plan forms were: (1) rectangular; (2) trapezoidal with raked-out tips; (3) trapezoidal with raked-in tips; (4) triangular with apex forward; (5) triangular with base forward; (6) fully

apered swept-back; (7) swept-back hexagonal; (8) unswept hexagonal; and (9) notched triangular.

Limitations in the applicability and availability of the lateral-stability derivatives are discussed.

AMES AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., *June 26, 1951.*

REFERENCES

1. Ribner, Herbert S.: The Stability Derivatives of Low-Aspect-Ratio Triangular Wings at Subsonic and Supersonic Speeds. NACA TN 1423, 1947.
2. Jones, Arthur L., and Alksne, Alberta: The Damping Due to Roll of Triangular, Trapezoidal, and Related Plan Forms in Supersonic Flow. NACA TN 1548, 1948.
3. Brown, Clinton E., and Adams, Mac C.: Damping in Pitch and Roll of Triangular Wings at Supersonic Speeds. NACA Rep. 892, 1948.
4. Ribner, Herbert S., and Malvestuto, Frank S., Jr.: Stability Derivatives of Triangular Wings at Supersonic Speeds. NACA Rep. 908, 1948. (Formerly NACA TN 1572)
5. Jones, Arthur L., Spreiter, John R., and Alksne, Alberta: The Rolling Moment Due to Sideslip of Triangular, Trapezoidal, and Related Plan Forms in Supersonic Flow. NACA TN 1700, 1948.
6. Harmon, Sidney M.: Stability Derivatives at Supersonic Speeds of Thin Rectangular Wings with Diagonals Ahead of Tip Mach Lines. NACA Rep. 925, 1949. (Formerly NACA TN 1706)
7. Malvestuto, Frank S., Jr., and Margolis, Kenneth: Theoretical Stability Derivatives of Thin Sweptback Wings Tapered to a Point with Sweptback and Sweptforward Trailing Edges for a Limited Range of Supersonic Speeds. NACA TN 1761, 1949.
8. Jones, Arthur L., and Alksne, Alberta: The Yawing Moment Due to Sideslip of Triangular, Trapezoidal, and Related Plan Forms in Supersonic Flow. NACA TN 1850, 1949.
9. Malvestuto, Frank S., Jr., Margolis, Kenneth, and Ribner, Herbert S.: Theoretical Lift and Damping in Roll of Thin Sweptback Wings of Arbitrary Taper and Sweep at Supersonic Speeds. Subsonic Leading Edges and Supersonic Trailing Edges. NACA TN 1860, 1949.
10. Piland, Robert O.: Summary of the Theoretical Lift, Damping-in-Roll, and Center-of-Pressure Characteristics of Various Wing Plan Forms at Supersonic Speeds. NACA TN 1977, 1949.
11. Walker, Harold J., and Ballantyne, Mary B.: Pressure Distribution and Damping in Steady Roll at Supersonic Mach Numbers of Flat Swept-Back Wings with Subsonic Edges. NACA TN 2047, 1950.
12. Margolis, Kenneth: Theoretical Lift and Damping in Roll of Thin Sweptback Tapered Wings with Raked-In and Cross-Stream Wing Tips at Supersonic Speeds. Subsonic Leading Edges. NACA TN 2048, 1950.
13. Harmon, Sidney M., and Jeffreys, Isabella: Theoretical Lift and Damping in Roll of Thin Wings with Arbitrary Sweep and Taper at Supersonic Speeds. Supersonic Leading and Trailing Edges. NACA TN 2114, 1950.
14. Margolis, Kenneth: Theoretical Calculations of the Lateral Force and Yawing Moment Due to Rolling at Supersonic Speeds for Sweptback Tapered Wings with Streamwise Tips. Subsonic Leading Edges. NACA TN 2122, 1950.
15. Ribner, Herbert S.: On the Effect of Subsonic Trailing Edges on Damping in Roll and Pitch of Thin Sweptback Wings in a Supersonic Stream. NACA TN 2146, 1950.
16. Harmon, Sidney M., and Martin, John C.: Theoretical Calculations of the Lateral Force and Yawing Moment Due to Rolling at Supersonic Speeds for Sweptback Tapered Wings with Streamwise Tips. Supersonic Leading Edges. NACA TN 2156, 1950.
17. Jones, Arthur L.: The Theoretical Lateral-Stability Derivatives for Wings at Supersonic Speeds. Jour. Aero. Sci., vol. 17, no. 1, Jan. 1950, pp. 39-46.
18. Lampert, Seymour: Rolling and Yawing Moments for Swept-Back Wings in Sideslip at Supersonic Speeds. NACA TN 2262, 1951.
19. Glauert, H.: A Non-Dimensional Form of the Stability Equations of an Aeroplane. British A. R. C., R. & M. 1093, 1927.
20. Lomax, Harvard, Heaslet, Max. A., and Fuller, Franklyn B.: Three-Dimensional, Unsteady-Lift Problems in High-Speed Flight—Basic Concepts. NACA TN 2256, 1950.
21. Heaslet, Max. A., and Lomax, Harvard: The Use of Source-Sink and Doublet Distributions Extended to the Solution of Arbitrary Boundary Value Problems in Supersonic Flow. NACA Rep. 900, 1948. (Formerly NACA TN 1515)
22. Evvard, John C.: Theoretical Distribution of Lift on Thin Wings at Supersonic Speeds (an Extension). NACA TN 1585, 1948.
23. Pell, William H.: Lifting Surfaces in Supersonic Flow. Part II—Linearized Three-Dimensional Theory. AMC Technical Report 102-AC49/8-100, December 1949.
24. Brown, Clinton E.: The Reversibility Theorem for Thin Airfoils in Subsonic and Supersonic Flow. NACA TN 1944, 1949.

TABLE I.—REFERENCE NUMBERS OF THE SOURCES OF THE STABILITY—DERIVATIVE RESULTS SUMMARIZED

Deriv- ative \ Plan form	Rectangular	Trapezoidal with raked- out tips	Trapezoidal with raked- in tips	Triangular with apex forward	Triangular with base forward	Swept back	Swept-back hexagonal	Unswept hexagonal	Notched triangular
C_{l_p}	2, 6	2	2	2, 4	2, 4	7, 9	9, 10, 11, 13, 15	9, 10, 11, 13, 15	2
C_{n_p}	6	(¹)	2	4	2, 4	7, 14, 16	14, 16	14, 16	-----
C_{Y_p}	6	(¹)	(¹)	4	(¹)	7, 14, 16	14, 16	14, 16	-----
C_{l_β}	5, 6	5	5	4, 5	5	5, 7	-----	-----	5
C_{n_β}	6, 8	8	8	4, 8	8	7, 8	-----	-----	8
C_{Y_β}	6	(¹)	(¹)	4	(¹)	7	-----	-----	(¹)
C_{l_r}	6	² 6	² 6	4	² 6	7	-----	² 6	-----
C_{n_r}	6	² 6	² 6	4	² 6	7	-----	² 6	-----
C_{Y_r}	6	² 6	² 6	4	² 6	7	-----	² 6	-----

¹ Previously unpublished. ² By extension.TABLE II.—FIGURE NUMBERS FOR DERIVATIVES AND PLAN FORMS¹

Deriv- ative \ Plan form	Rectan- gular	Trapezoidal with raked-out tips	Trapezoidal with raked-in tips	Triangular with apex forward	Triangular with base forward	Swept back	Swept-back hexagonal	Unswept hexagonal	Notched triangular
C_{l_p}	3(a)	3(a), 4, 20	3(a), 4, 20	3(a)	3(a)	3(a), 29	3(b), 3(c), 3(d), 38, 41, 59, 56	3(e), 44, 3(f), 50, 62	3(g)
C_{n_p}	5(a), 6(a)	5(b), 6(b), 21	5(c), 6(c), 21	5(d), 6(d)	5(e), 6(e)	5(f), 6(f), 30	5(g), 5(h), 5(i), 6(g), 6(h), 6(i), 39, 42, 57, 60	5(j), 6(j), 5(k), 6(k), 45, 51, 63	-----
C_{Y_p}	7(a)	7(a), 22	7(a), 22	7(a)	7(a)	7(a), 31	7(b), 7(c), 40, 43, 58, 61	7(d), 7(e), 46, 52, 64	-----
C_{l_β}	8(a), 9(a)	8(b), 9(b), 23	8(c), 9(c), 23	8(d), 9(d)	8(e), 9(e)	8(f), 9(f), 32	-----	-----	8(g)
C_{n_β}	10(a), 11(a)	10(b), 11(b), 24	10(c), 11(c), 24	10(d), 11(d)	10(e), 11(e)	10(f), 11(f), 33	-----	-----	10(g), 11(g)
C_{Y_β}	12(a), 13(a)	12(b), 13(b), 25	12(c), 13(c), 25	12(d), 13(d)	12(e), 13(e)	12(f), 13(f), 34	-----	-----	12(g), 13(g)
C_{l_r}	14(a), 15(a)	14(b), 15(b), 26	14(c), 15(c), 26	14(d), 15(d)	14(e), 15(e)	14(f), 15(f), 35	-----	14(g), 15(g), 14(h), 15(h), 47, 53, 65	-----
C_{n_r}	16(a), 17(a)	16(b), 17(b), 27	16(c), 17(c), 27	16(d), 17(d)	16(e), 17(e)	16(f), 17(f), 36	-----	16(g), 17(g), 16(h), 17(h), 48, 54, 66	-----
C_{Y_r}	18(a), 19(a)	18(b), 19(b), 28	18(c), 19(c), 28	18(d), 19(d)	18(e), 19(e)	18(f), 19(f), 37	-----	18(g), 19(g), 18(h), 19(h), 49, 55, 67	-----

¹ In cases where more than one curve falls on a single line, the leaders identifying the curves are used to indicate the lower limit of each curve with respect to the variable plotted horizontally.

TABLE III.—RANGES OF ASPECT-RATIO PARAMETER FOR WHICH DERIVATES ARE PRESENTED

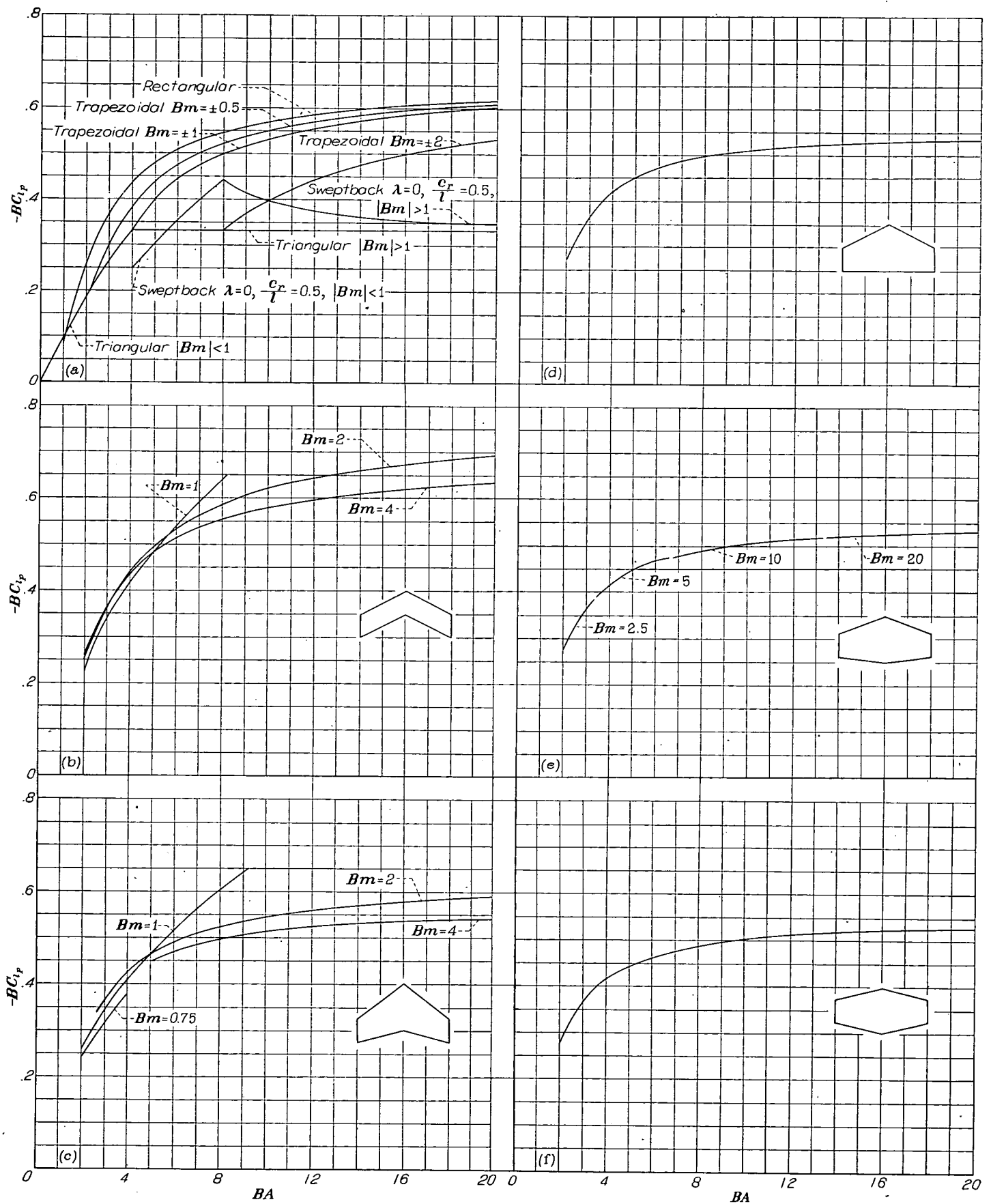
Deriv- ative \ Plan form	Rectangular	Trapezoidal ¹ with raked-out tips	Trapezoidal ¹ with raked-in tips	Triangular with apex forward	Triangular with base forward
C_{l_p}	$1 \leq BA \leq \infty$	$(1+Bm)^2 \leq BA \leq \infty$	$(1-Bm)^2 \leq BA \leq \infty$	$0 \leq BA \leq \infty$	$0 \leq BA \leq \infty$
C_{n_p}	do	do	do	do	Do.
C_{Y_p}	do	do	do	do	Do.
C_{l_β}	do	do	do	do	$4 \leq BA \leq \infty$
C_{n_β}	do	do	do	do	Do.
C_{Y_β}	do	do	do	do	Do.
C_{l_r}	do	do	do	$0 \leq BA \leq 4$	Do.
C_{n_r}	do	do	do	do	Do.
C_{Y_r}	do	do	do	do	Do.

¹ Limits given are for subsonic tips ($|Bm| < 1$); for supersonic tips ($|Bm| \geq 1$) the limits are $|4Bm| \leq BA \leq \infty$.

TABLE III.—CONCLUDED

Deriv- ative \ Plan form	Swept back	Swept-back ² hexagonal	Unswapt ² hexagonal	Notched triangular
C_{l_p}	$\frac{4l}{c_r} \left(1 - \frac{c_r}{l}\right) \leq BA \leq \infty$	$\frac{4Bm}{(Bm+1)(1+\lambda)} \leq BA$ $\leq \frac{4Bm(1-\lambda)}{(1-Bm)(1+\lambda)}$	$\frac{4Bm}{(Bm+1)(1+\lambda)} \leq BA$ $\leq \frac{4Bm(1-\lambda)}{(1-Bm)(1+\lambda)}$	$0 \leq BA \leq \frac{4l}{c_r \left(2 - \frac{c_r}{l}\right) \left(1 - \frac{c_r}{l}\right)}$
C_{n_p}	do	do	do	do
C_{Y_p}	do	do	do	do
C_{l_β}	$\frac{4l}{c_r} \left(1 - \frac{c_r}{l}\right) \leq BA \leq \frac{4l}{c_r}$	do	do	$\frac{4l}{c_r \left(2 - \frac{c_r}{l}\right)} \leq BA \leq \frac{4l}{c_r \left(2 - \frac{c_r}{l}\right) \left(1 - \frac{c_r}{l}\right)}$
C_{n_β}	do	do	do	Do.
C_{Y_β}	do	do	do	Do.
C_{l_r}	do	do	$\frac{4Bm}{(Bm+1)(1+\lambda)} \leq BA$ $\leq \frac{4Bm(1-\lambda)}{(1-Bm)(1+\lambda)}$	do
C_{n_r}	do	do	do	do
C_{Y_r}	do	do	do	do

² These limits apply for subsonic leading edges, $Bm < 1$, but for supersonic leading edges, $Bm \geq 1$, the upper limit is changed to ∞ .



(a) First six plan forms investigated.

(b) Swept-back hexagonal plan form; $\lambda = 1$.(c) Swept-back hexagonal plan form; $\lambda = 0.5$.(d) Swept-back hexagonal plan form; $\lambda = 0.5, Bm = 3BA/4$.(e) Unswept hexagonal plan form; $\lambda = 0.5$.(f) Unswept hexagonal plan form; $\lambda = 0.5, Bm = 3BA/2$.

FIGURE 3.—Variation of damping-in-roll parameter with aspect-ratio parameter.

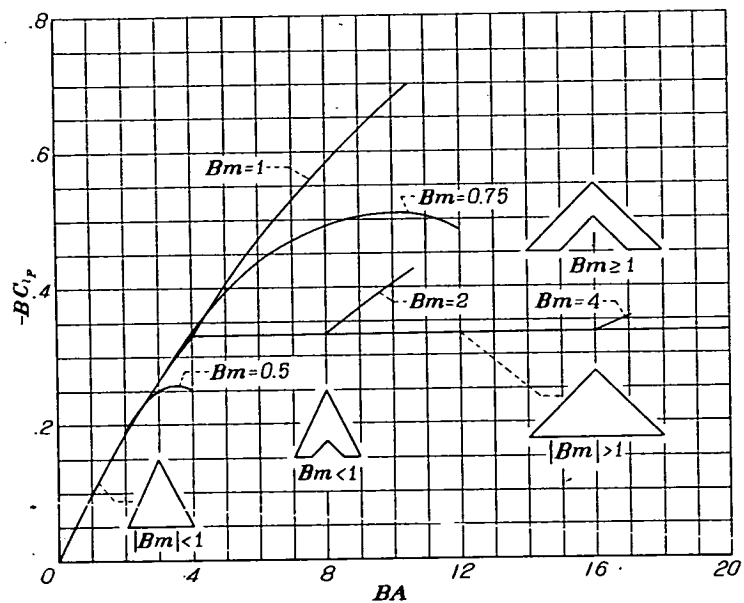


FIGURE 3.—Concluded. (g) Triangular and notched triangular plan forms.

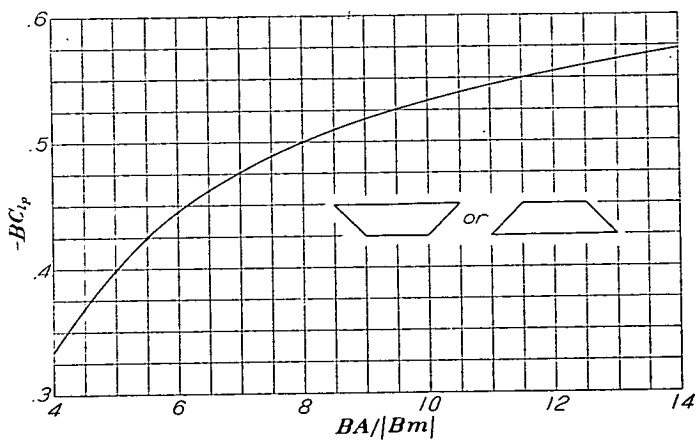
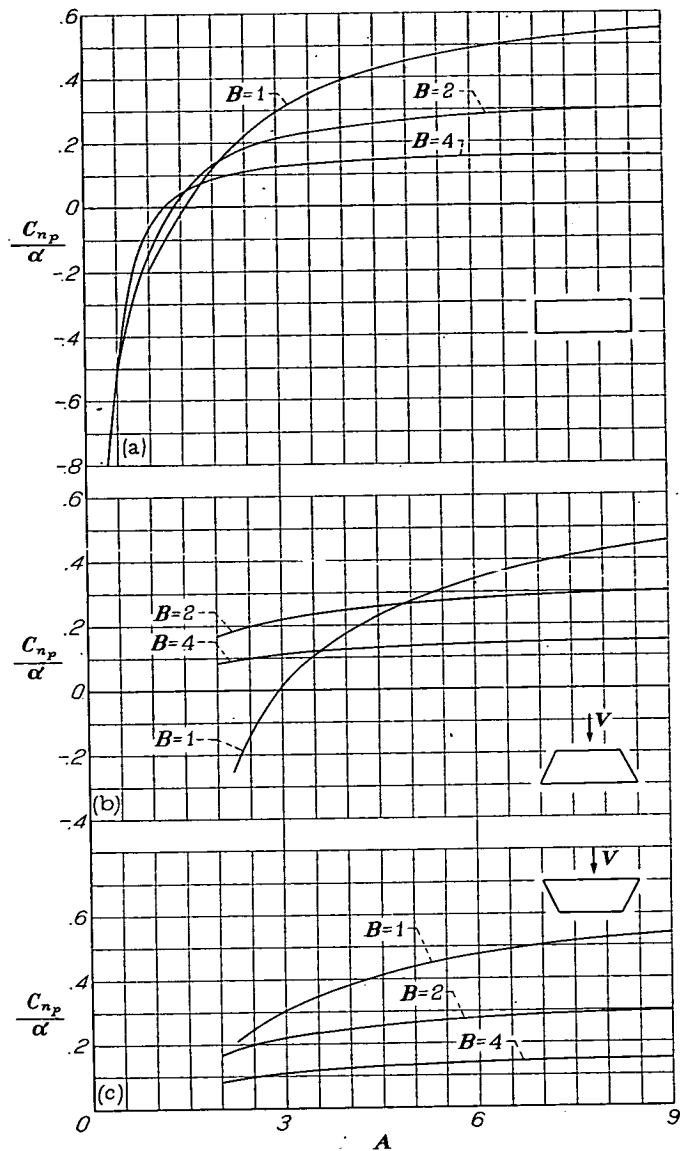
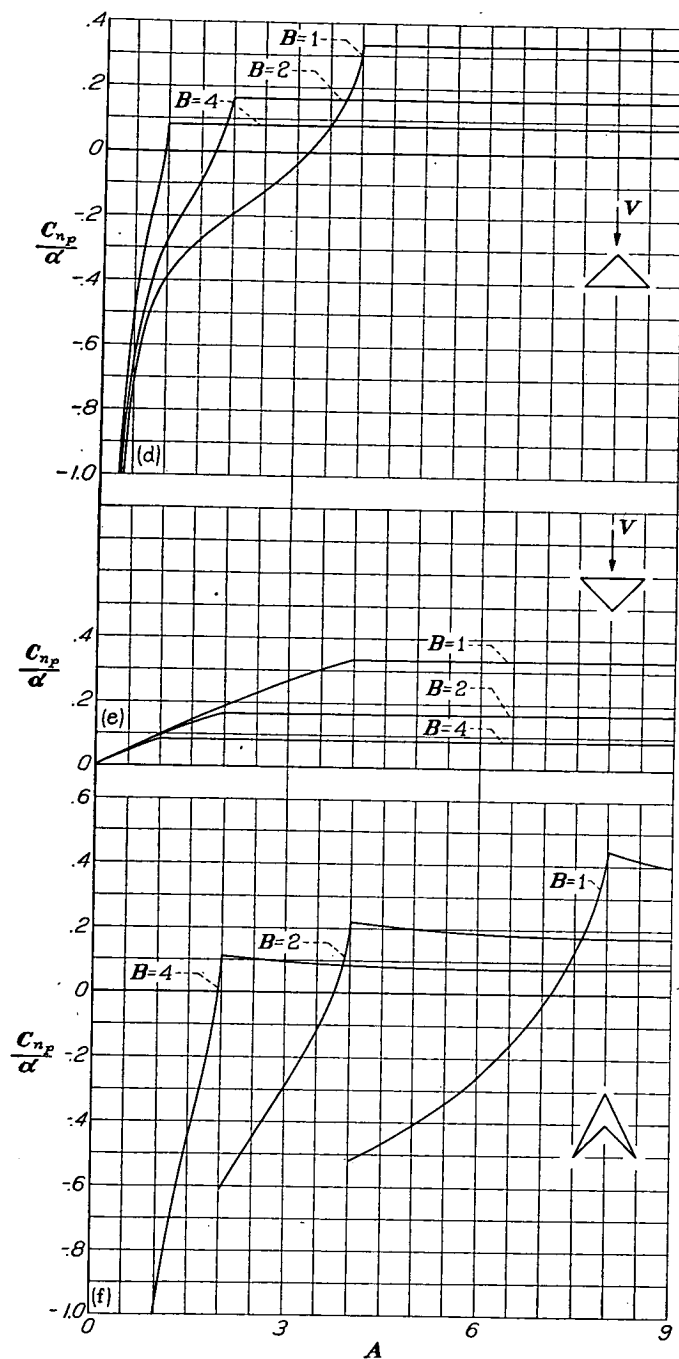


FIGURE 4.—Variation of damping-in-roll parameter with plan-form parameter for trapezoidal plan forms having supersonic tips; $|Bm| \geq 1$.



(a) Rectangular plan form.
 (b) Trapezoidal plan form; $m=0.5$.
 (c) Trapezoidal plan form; $m=-0.5$.

FIGURE 5.—Variation of yawing-moment-due-to-roll derivative with aspect ratio.



(d) Triangular plan form. Apex forward.

(e) Triangular plan form. Base forward.

(f) Swept-back plan form; $\lambda=0$; $\frac{c_r}{l}=0.5$.

FIGURE 5.—Continued.

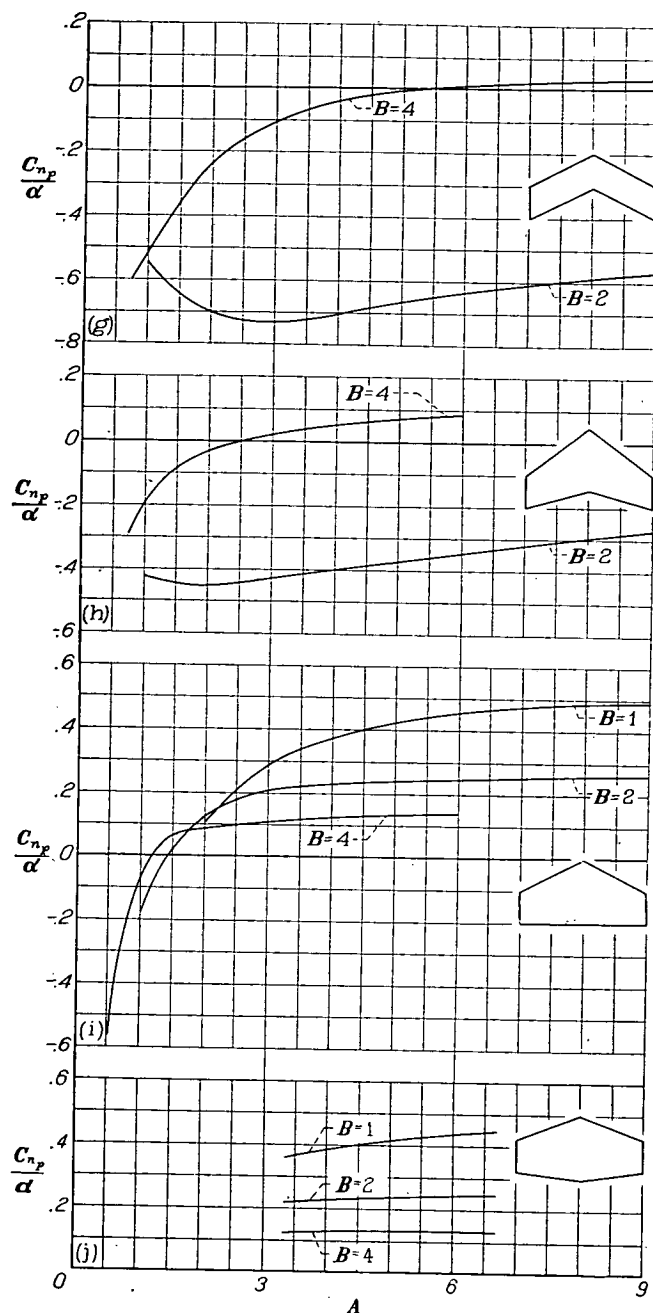
(g) Swept-back hexagonal plan form; $\lambda=1.0$; $m=0.5$.(h) Swept-back hexagonal plan form; $\lambda=0.5$; $m=0.5$.(i) Swept-back hexagonal plan-form; $\lambda=0.5$, $Bm=3BA/4$.(j) Unswept hexagonal plan form; $\lambda=0.5$; $m=5$.

FIGURE 5.—Continued.

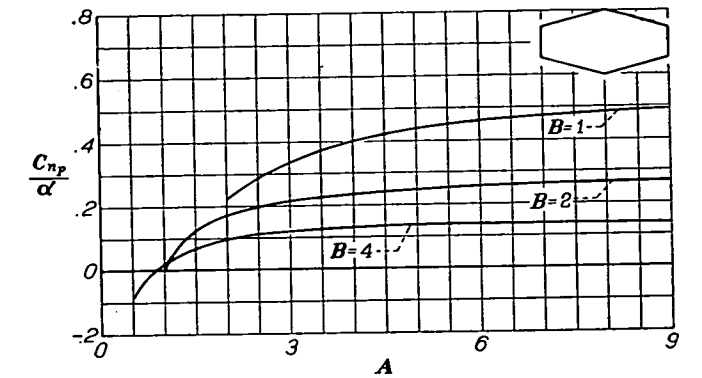
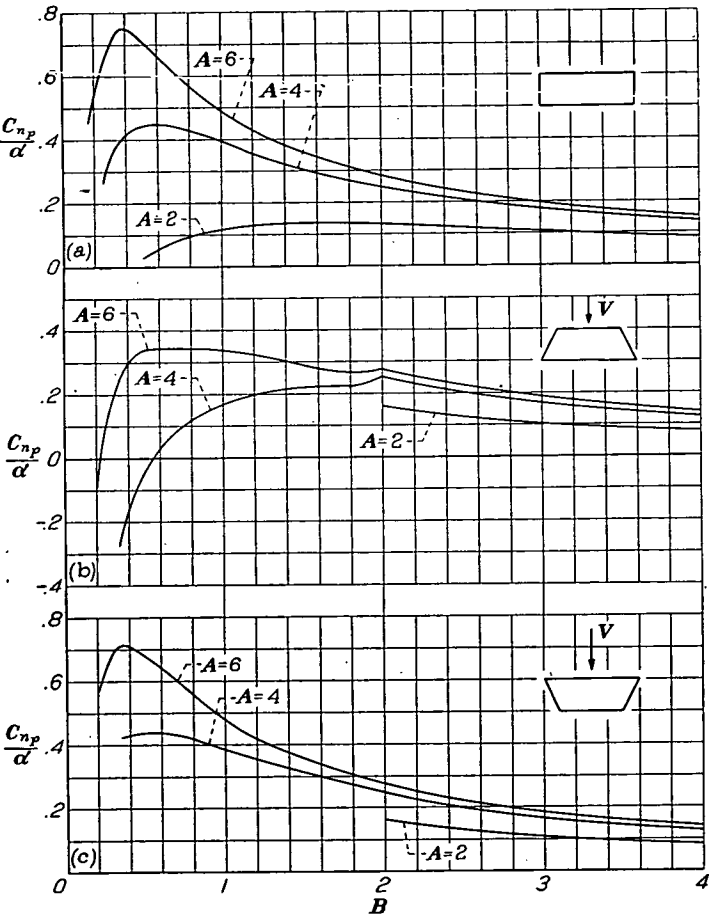
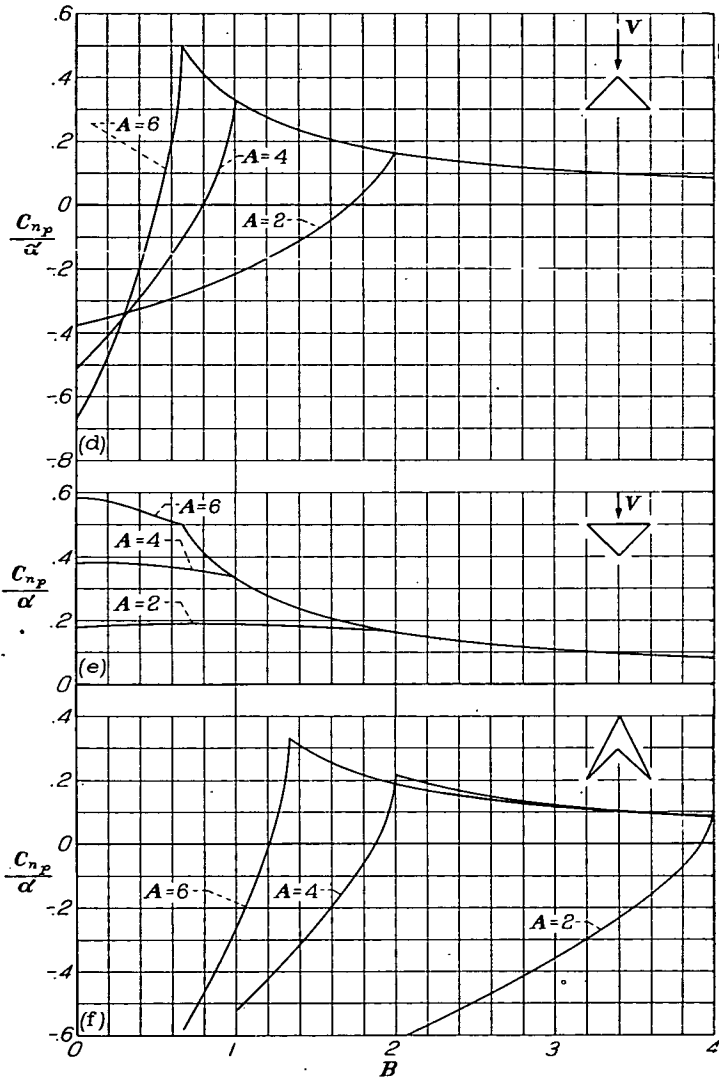


FIGURE 5.—Concluded. (k) Unswept hexagonal plan form; $\lambda=0.5$; $Bm=3BA/2$.



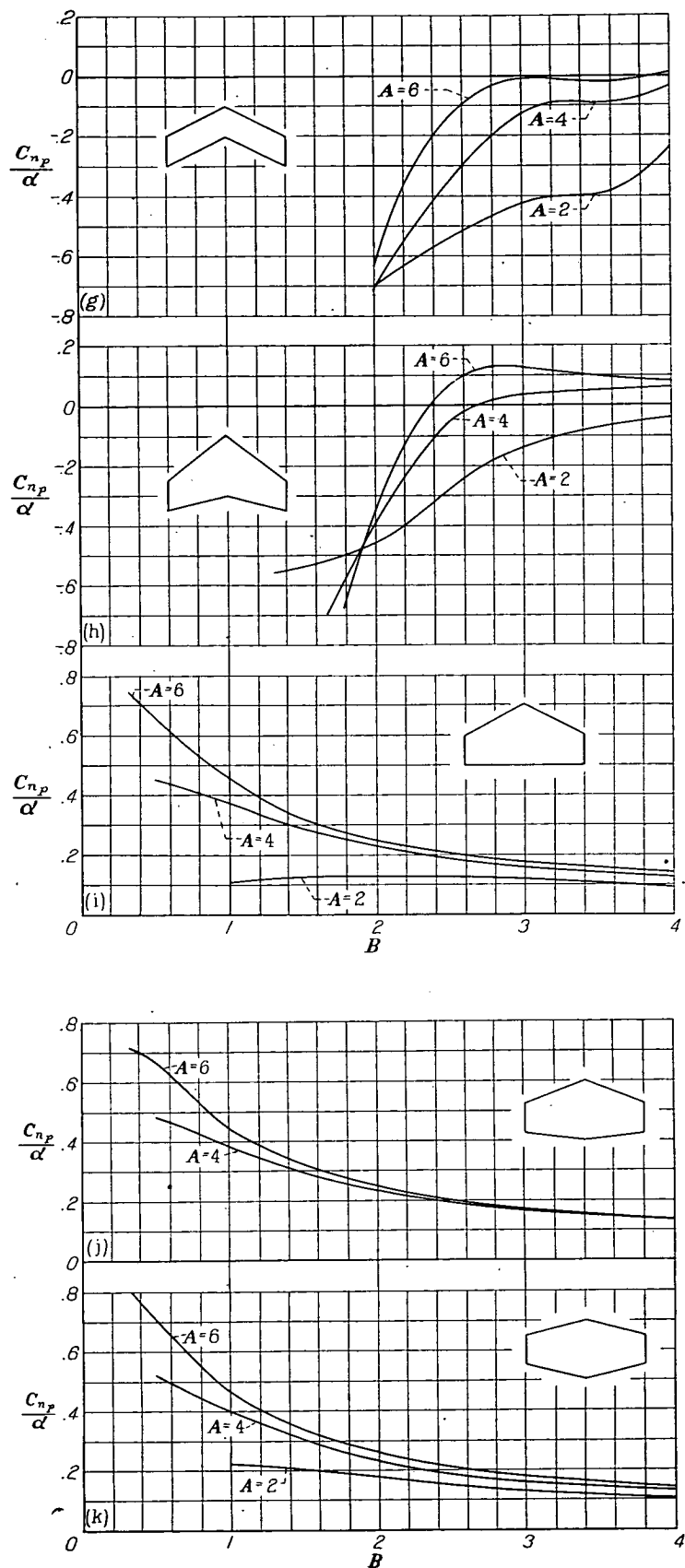
- (a) Rectangular plan form.
- (b) Trapezoidal plan form; $m=0.5$.
- (c) Trapezoidal plan form; $m=-0.5$.

FIGURE 6.—Variation of yawing-moment-due-to-roll derivative with Mach number parameter B .



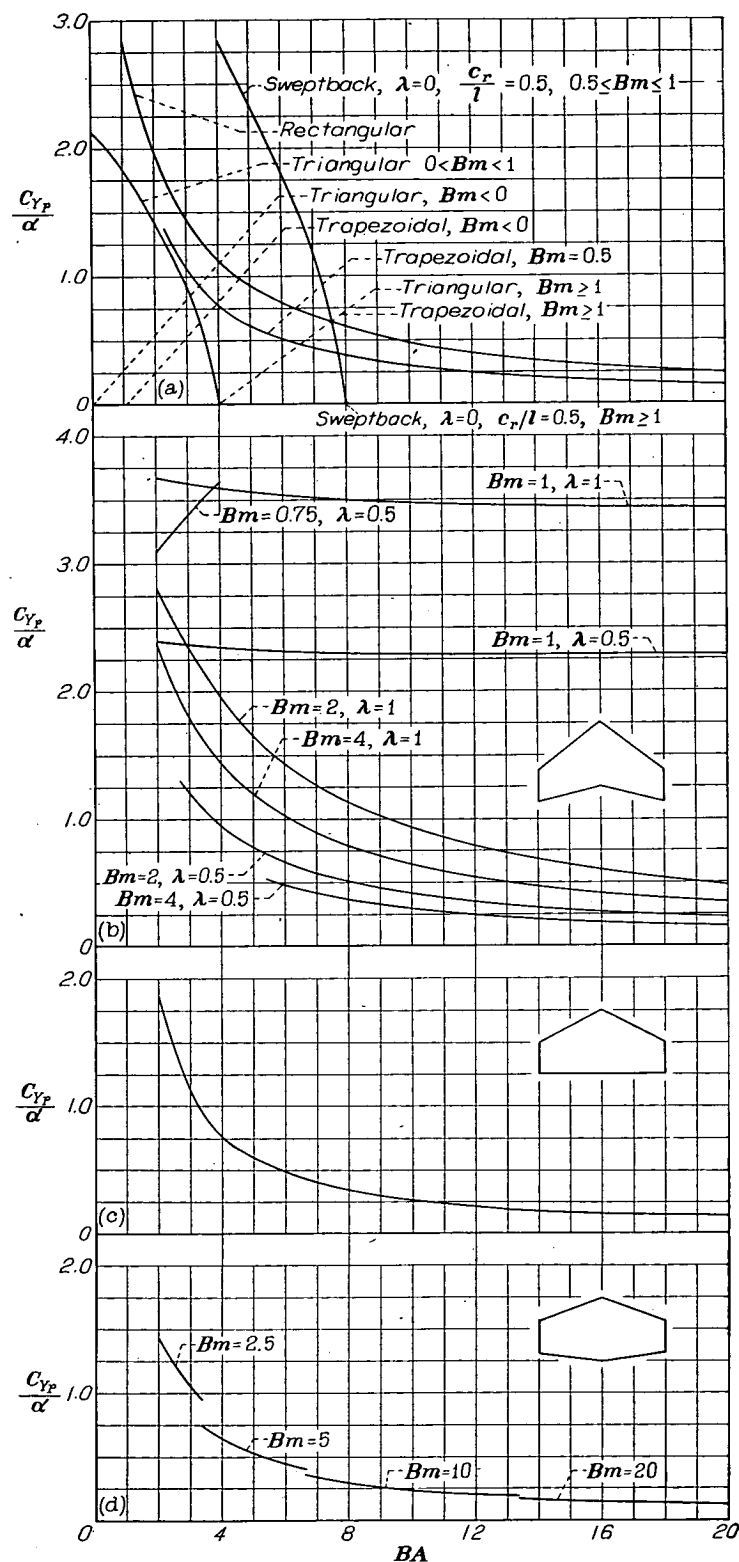
- (d) Triangular plan form. Apex forward.
- (e) Triangular plan form. Base forward.
- (f) Swept-back plan form; $\lambda=0$; $c_r=0.5$.

FIGURE 6.—Continued.



(g) Swept-back hexagonal plan form; $\lambda=1.0$; $m=0.5$.
 (h) Swept-back hexagonal plan form; $\lambda=0.5$; $m=0.5$.
 (i) Swept-back hexagonal form; $\lambda=0.5$, $Bm=3BA/4$.
 (j) Unswept hexagonal plan form; $\lambda=0.5$; $m=5$.
 (k) Unswept hexagonal plan form; $\lambda=0.5$; $Bm=3BA/4$.

FIGURE 6.—Concluded.



(a) First six plan forms investigated. (See footnote, table II.)
 (b) Swept-back hexagonal plan form.
 (c) Swept-back hexagonal plan form; $\lambda=0.5$, $Bm=3BA/4$.
 (d) Unswept hexagonal plan form; $\lambda=0.5$.

FIGURE 7.—Variation of side-force derivative with aspect ratio parameter.

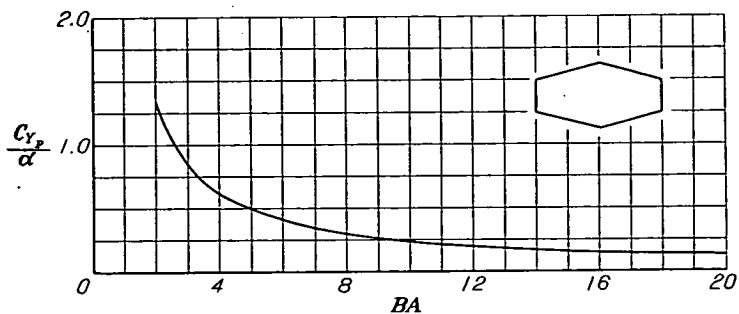
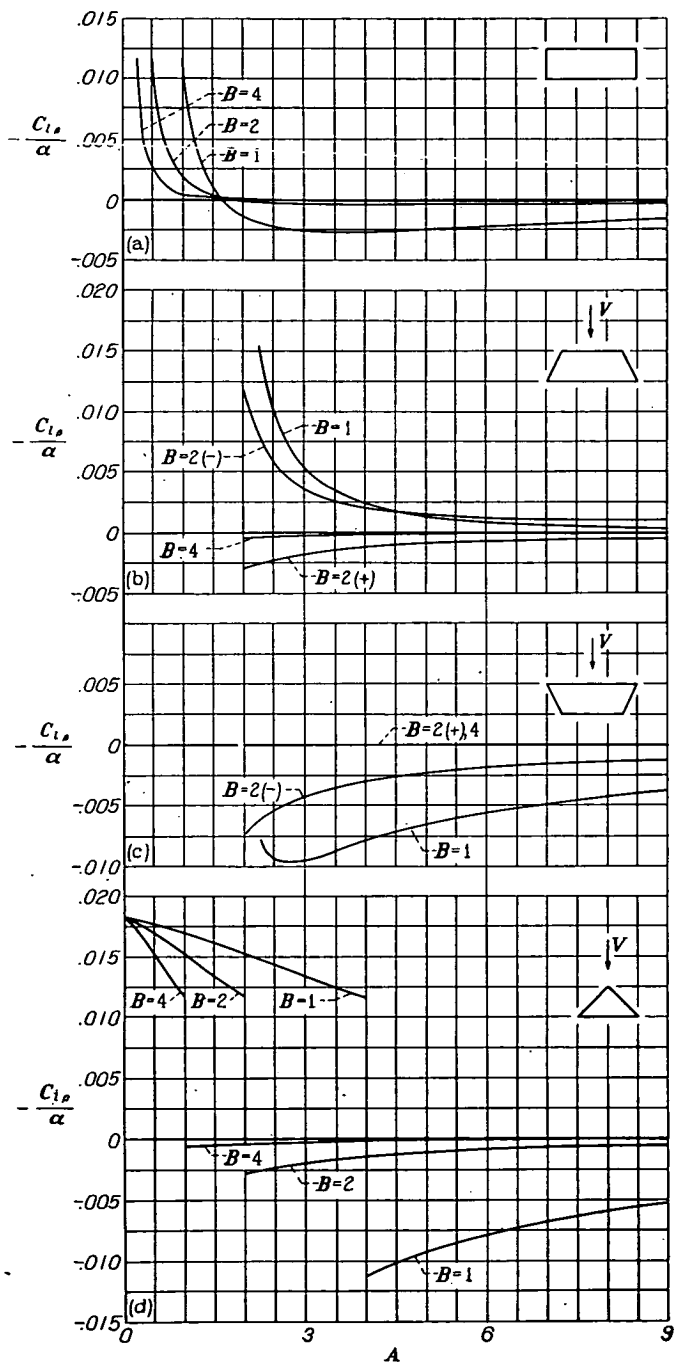
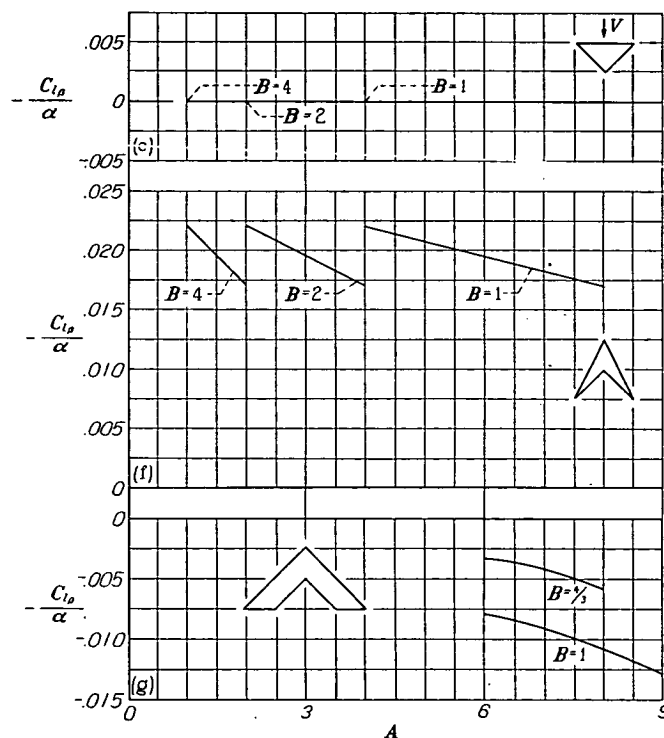


FIGURE 7.—Concluded. (e) Unswept hexagonal plan form; $\lambda=0.5$; $Bm=3BA/2$.



- (a) Rectangular plan form.
 (b) Trapezoidal plan form; $m=0.5$.
 (c) Trapezoidal plan form; $m=-0.5$.
 (d) Triangular plan form. Apex forward.

FIGURE 8.—Variation of rolling-moment-due-to-sideslip derivative with aspect ratio.

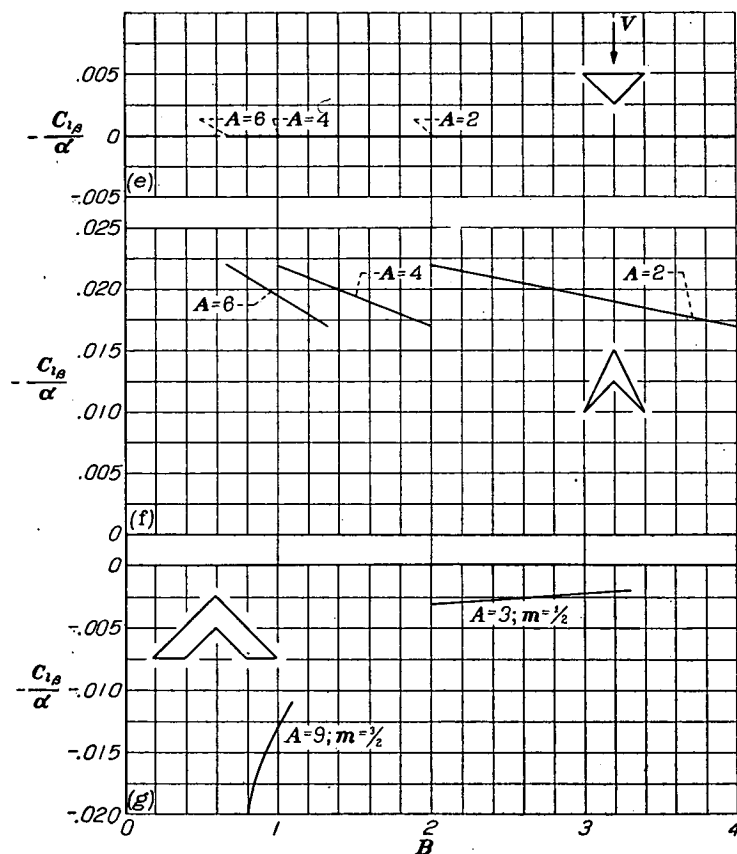
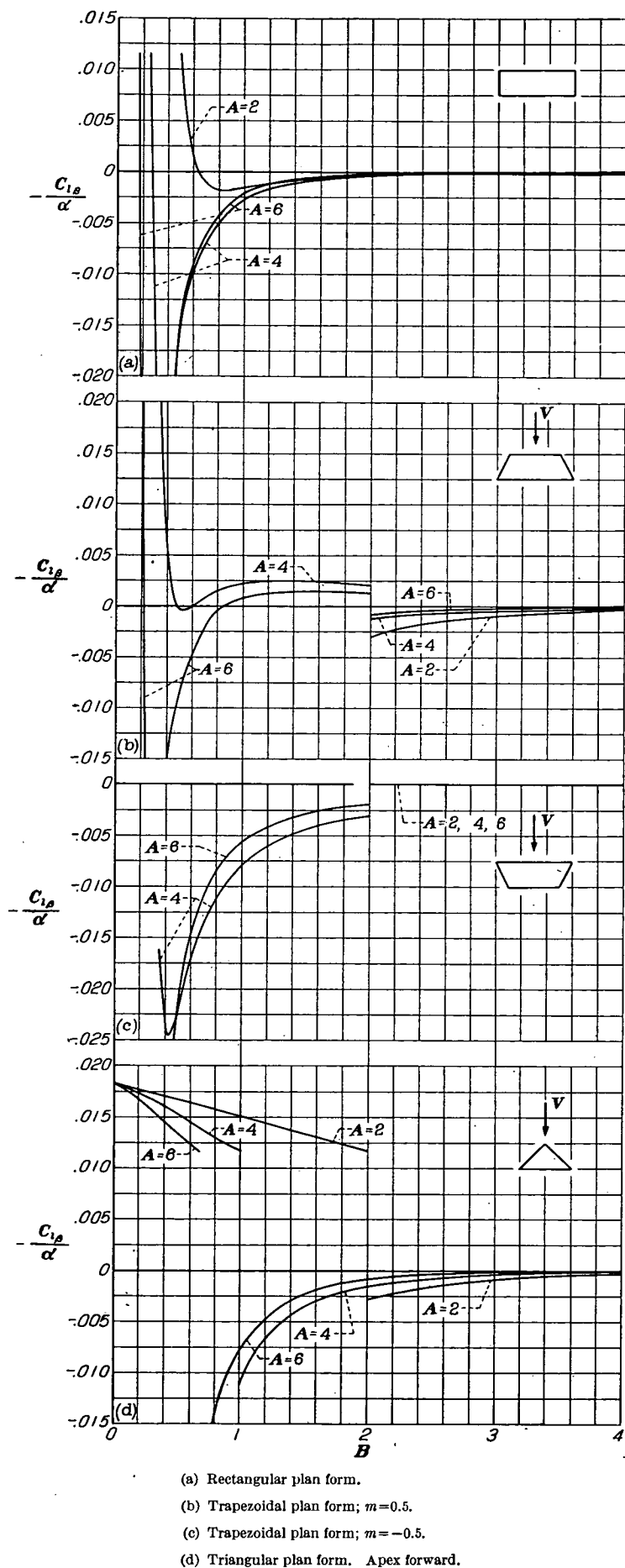


(e) Triangular plan form. Base forward. (See footnote, table II.)

(f) Swept-back plan form; $\lambda=0$; $c_r=0.5$.

(g) Notched triangular plan form; $m=3/4$.

FIGURE 8.—Concluded.



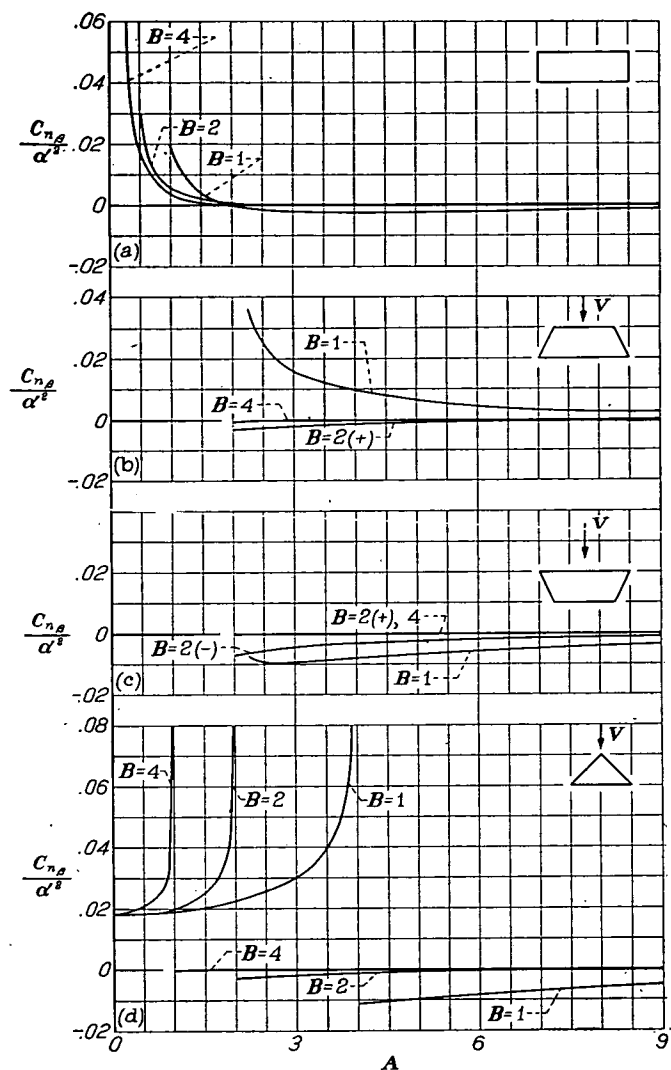
(e) Triangular plan form. Base forward. (See footnote, table II.)

(f) Swept-back plan form; $\lambda=0$; $\frac{c_r}{l}=0.5$.

(g) Notched triangular plan form.

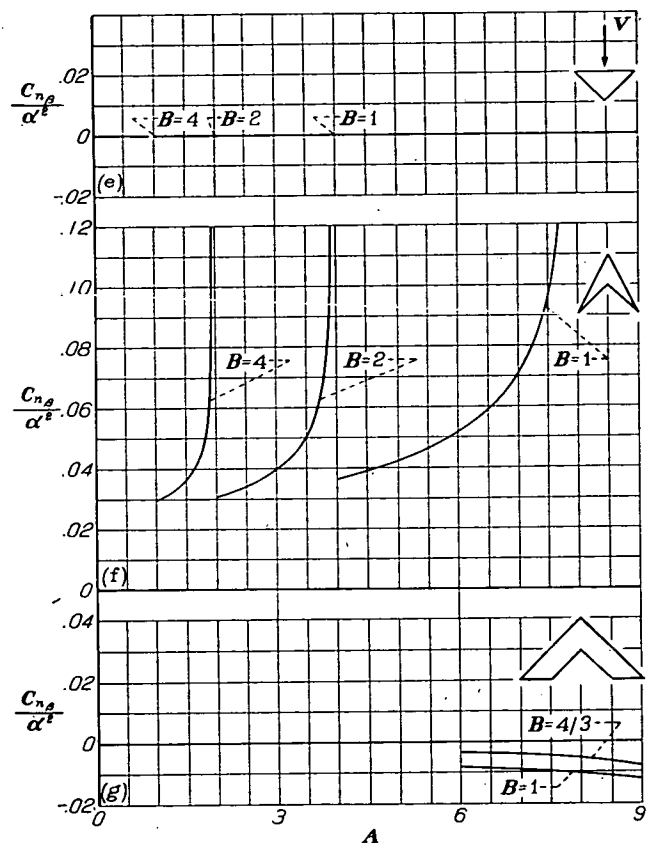
FIGURE 9.—Concluded.

FIGURE 9.—Variation of rolling-moment-due-to-sideslip derivative with Mach number parameter B .



- (a) Rectangular plan form.
 (b) Trapezoidal plan form; $m=0.5$.
 (c) Trapezoidal plan form; $m=-0.5$.
 (d) Triangular plan form. Apex forward.

FIGURE 10.—Variation of yawing-moment-due-to-sideslip derivative with aspect ratio.



- (e) Triangular plan form. Base forward. (See footnote, table II.)
 (f) Swept-back plan form; $\lambda=0$; $\frac{c}{l}=0.5$.
 (g) Notched triangular plan form; $m=3/2$.

FIGURE 10.—Concluded.

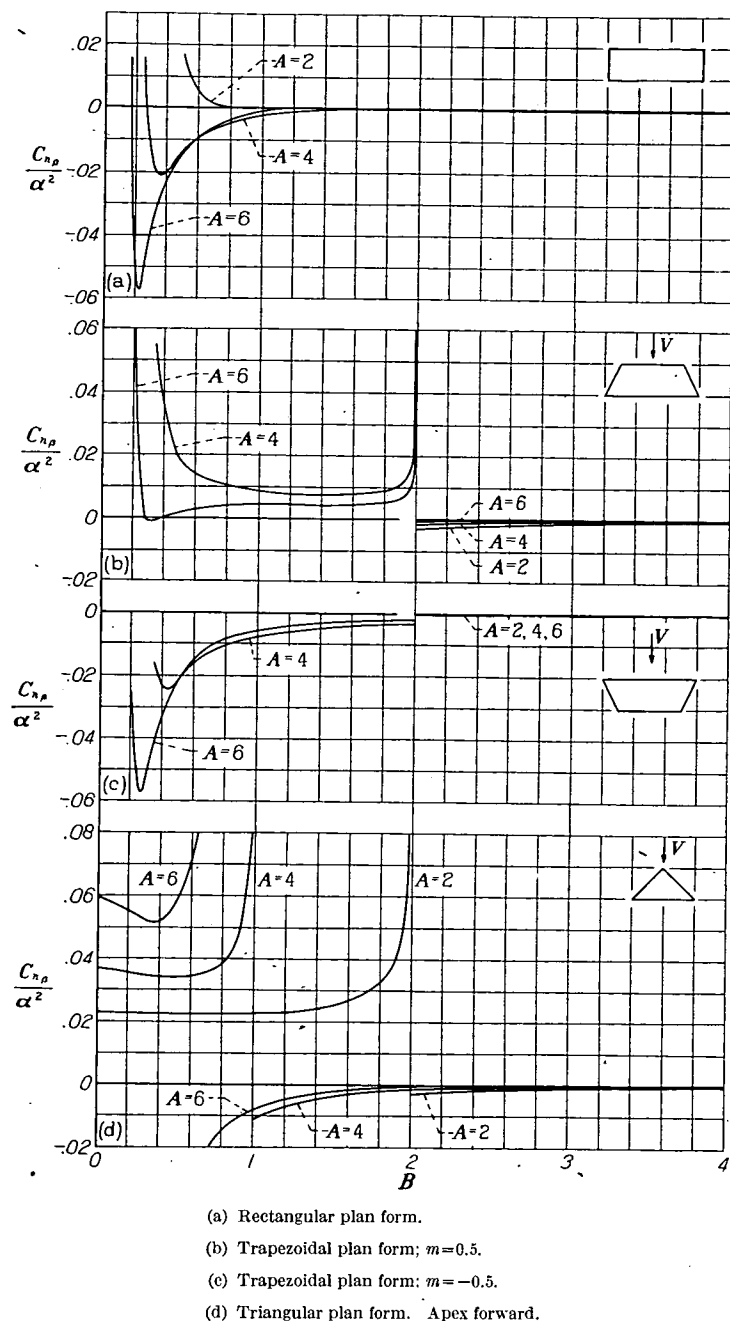


FIGURE 11—Variation of yawing-moment-due-to-sideslip derivative with Mach number parameter B .

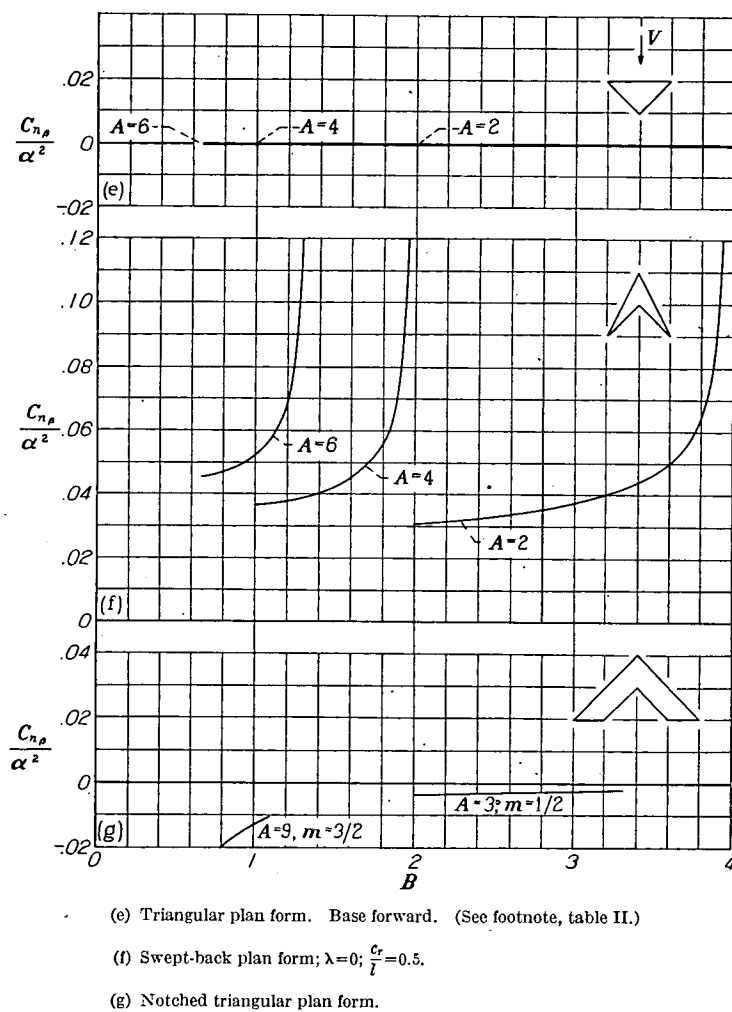
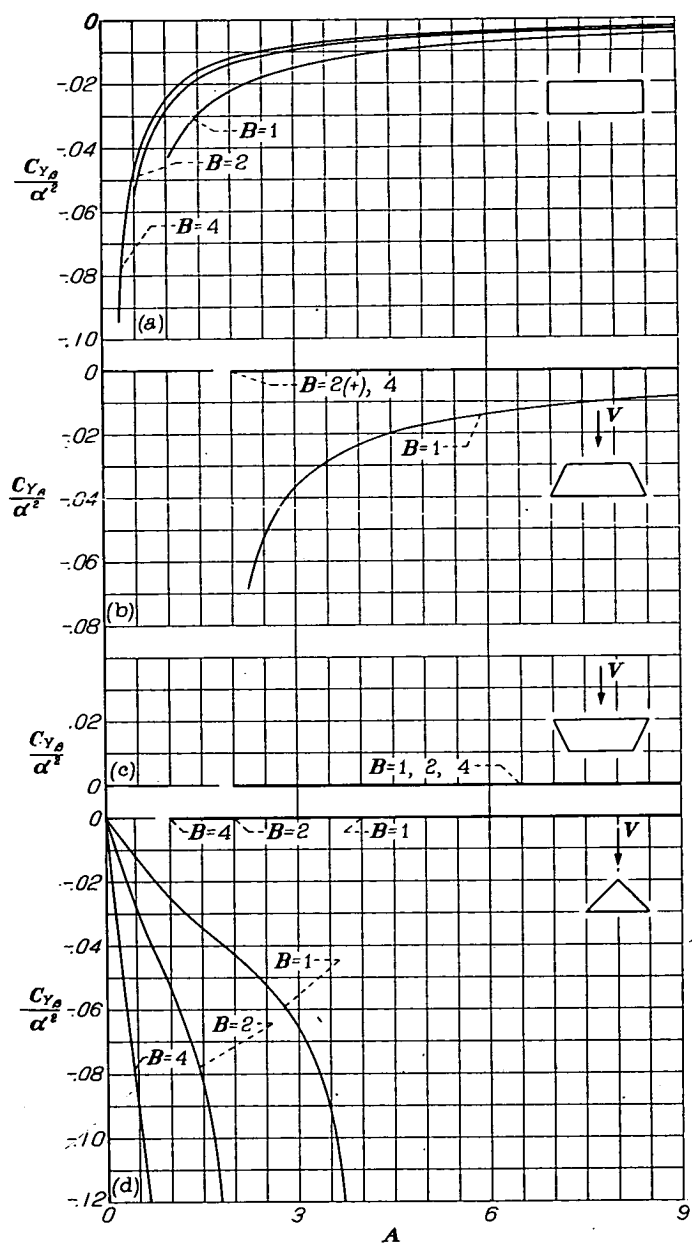
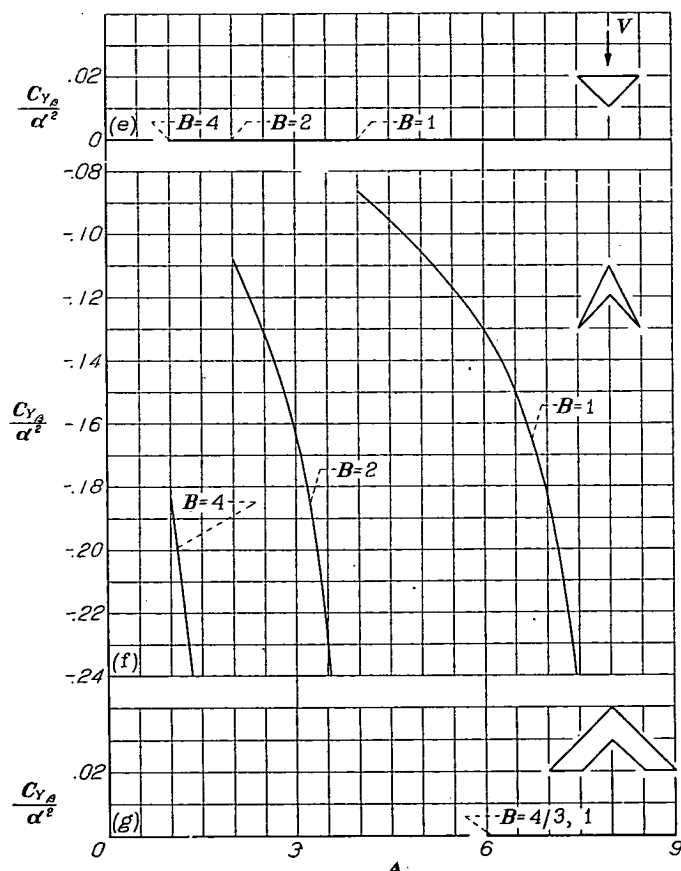


FIGURE 11.—Concluded.

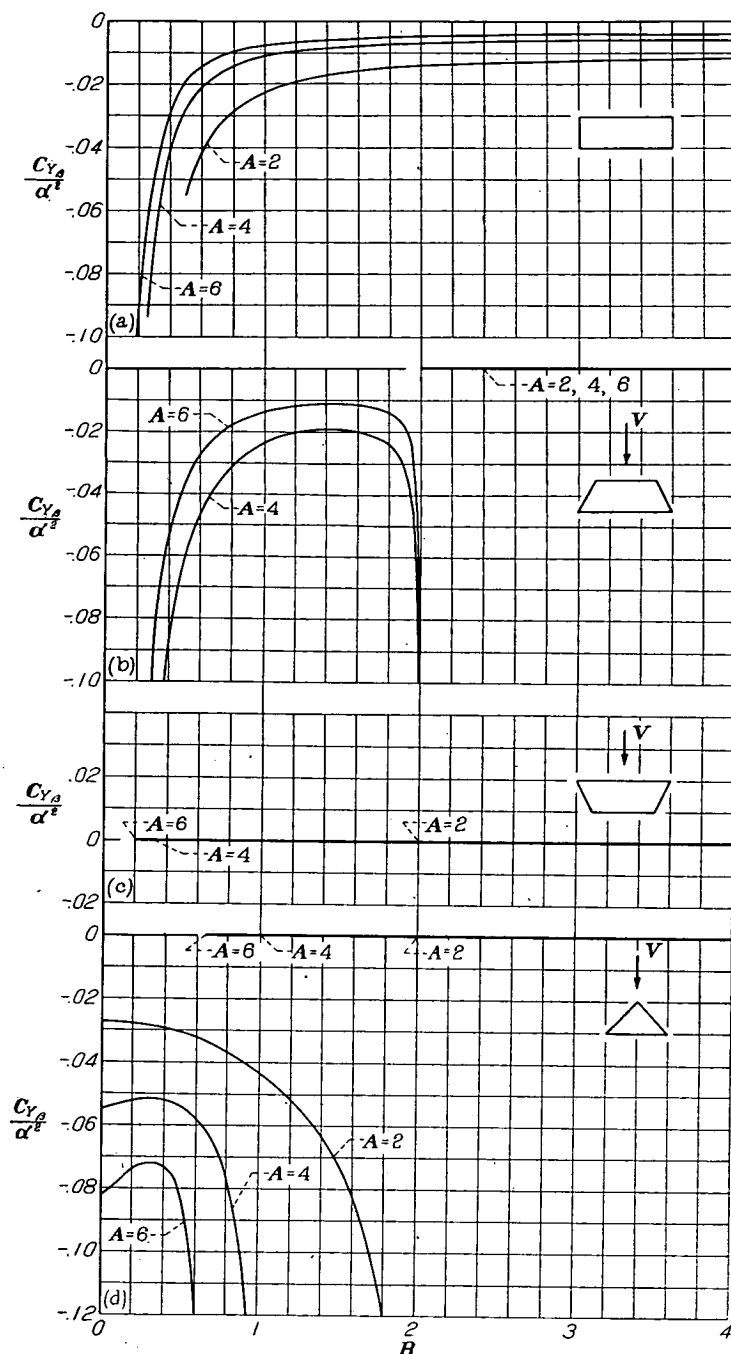


(a) Rectangular plan form.
(b) Trapezoidal plan form; $m=0.5$.
(c) Trapezoidal plan form; $m=-0.5$.
(d) Triangular plan form. Apex forward. (See footnote, table II.)
FIGURE 12.—Variation of side-force-due-to-sideslip derivative with aspect ratio.



(e) Triangular plan form. Base forward. (See footnote, table II.)
(f) Swept-back plan form; $\lambda=0$; $\frac{c_r}{l}=0.5$.
(g) Notched triangular plan form; $m=3/2$.

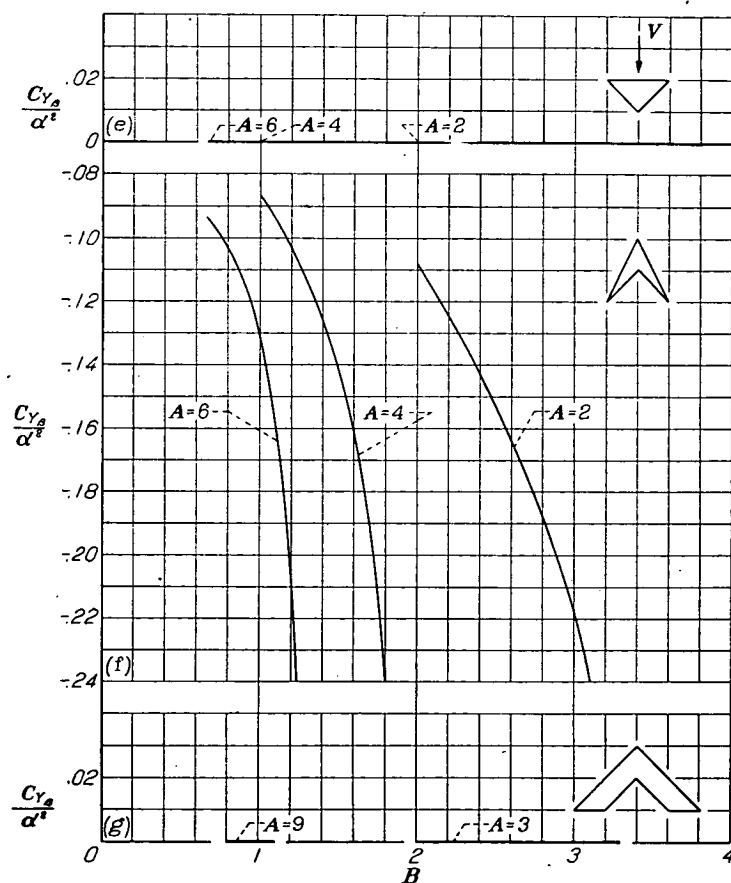
FIGURE 12.—Concluded.



(a) Rectangular plan form.

(b) Trapezoidal plan form; $m=0.5$.(c) Trapezoidal plan form; $m=-0.5$. (See footnote, table II.)

(d) Triangular plan form. Apex forward. (See footnote, table II.)

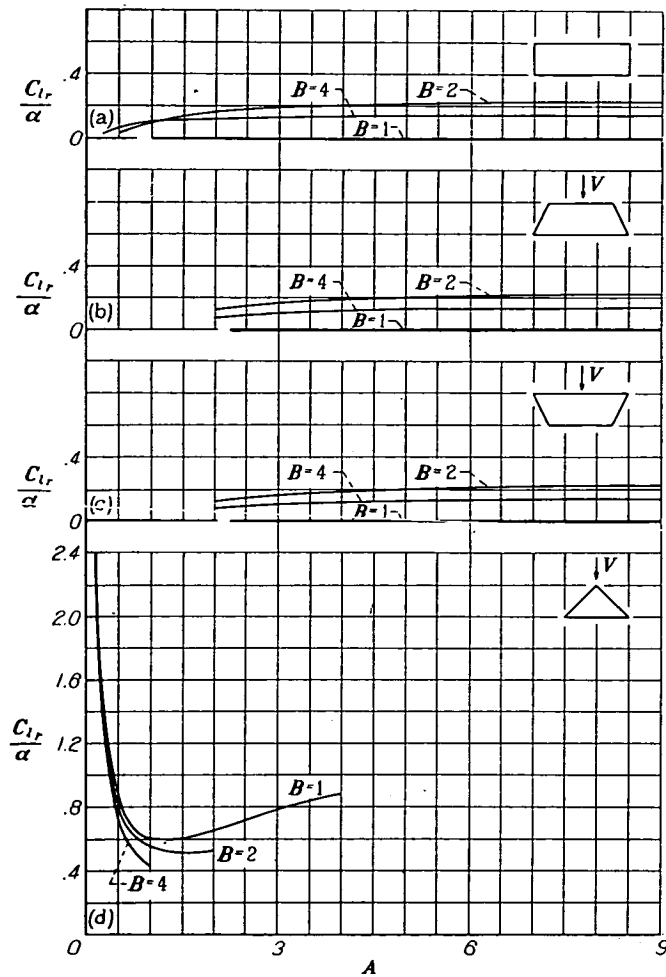
FIGURE 13.—Variation of side-force derivative with Mach number parameter B .

(e) Triangular plan form. Base forward. (See footnote, table II.)

(f) Swept-back plan form; $\lambda=0$; $\frac{C_r}{l}=0.5$.

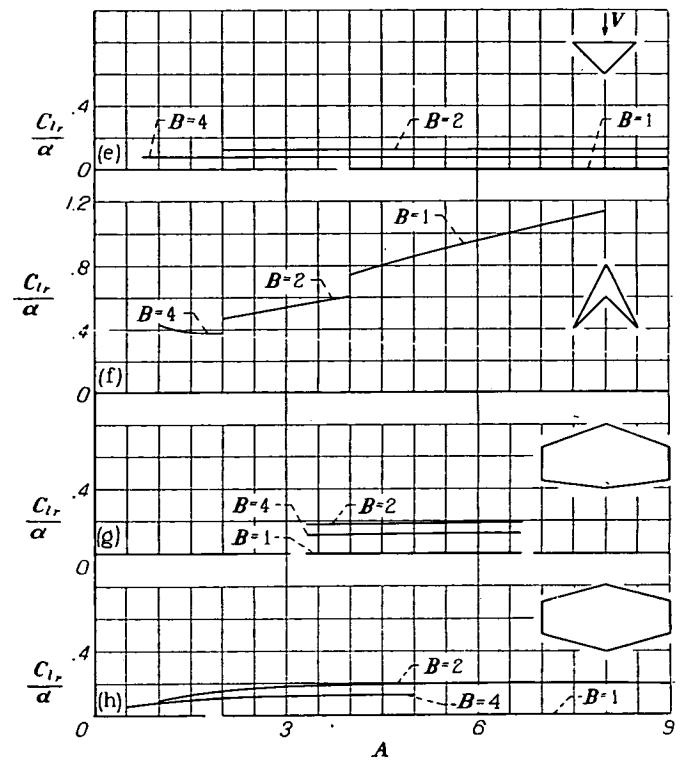
(g) Notched triangular plan form.

FIGURE 13.—Concluded.



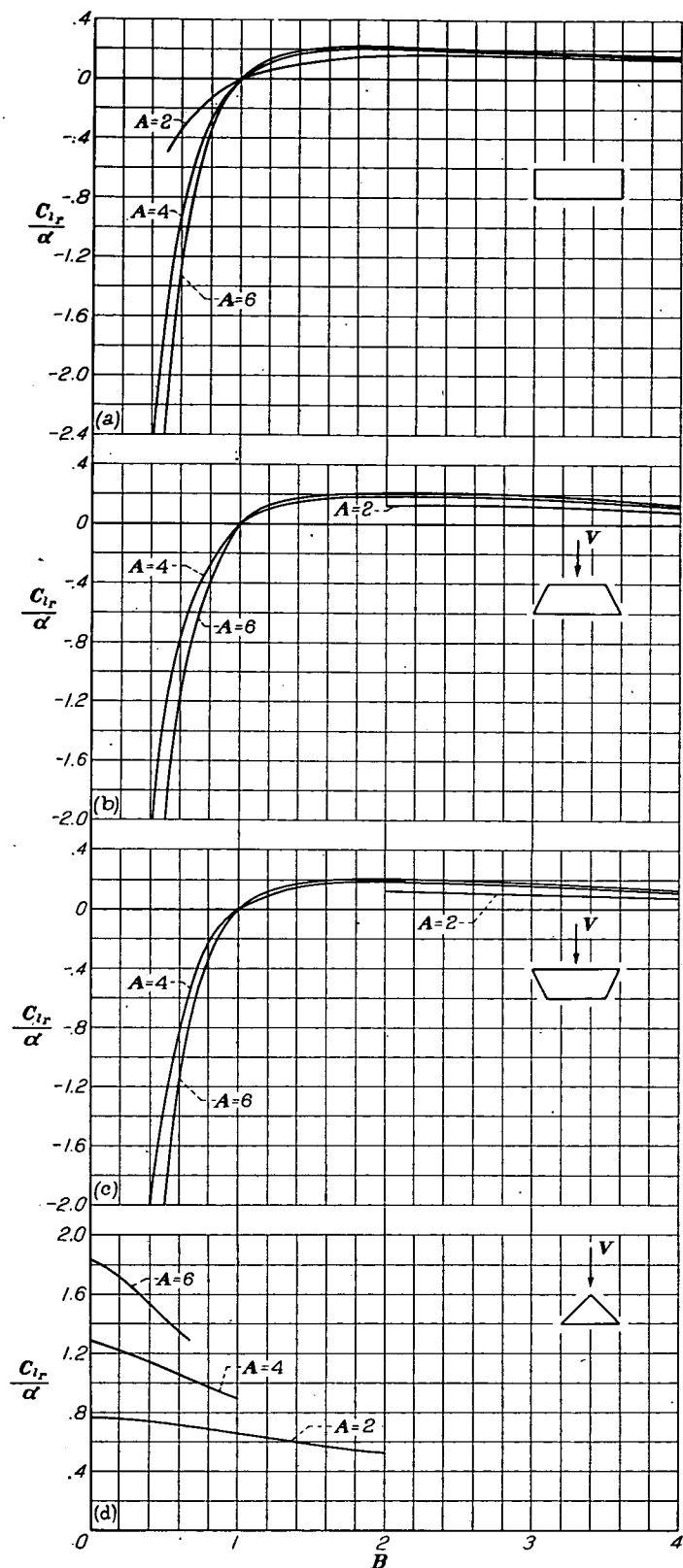
- (a) Rectangular plan form.
 (b) Trapezoidal plan form; $m=0.5$.
 (c) Trapezoidal plan form; $m=-0.5$.
 (d) Triangular plan form. Apex forward.

FIGURE 14.—Variation of rolling-moment-due-to-yawing derivative with aspect ratio.



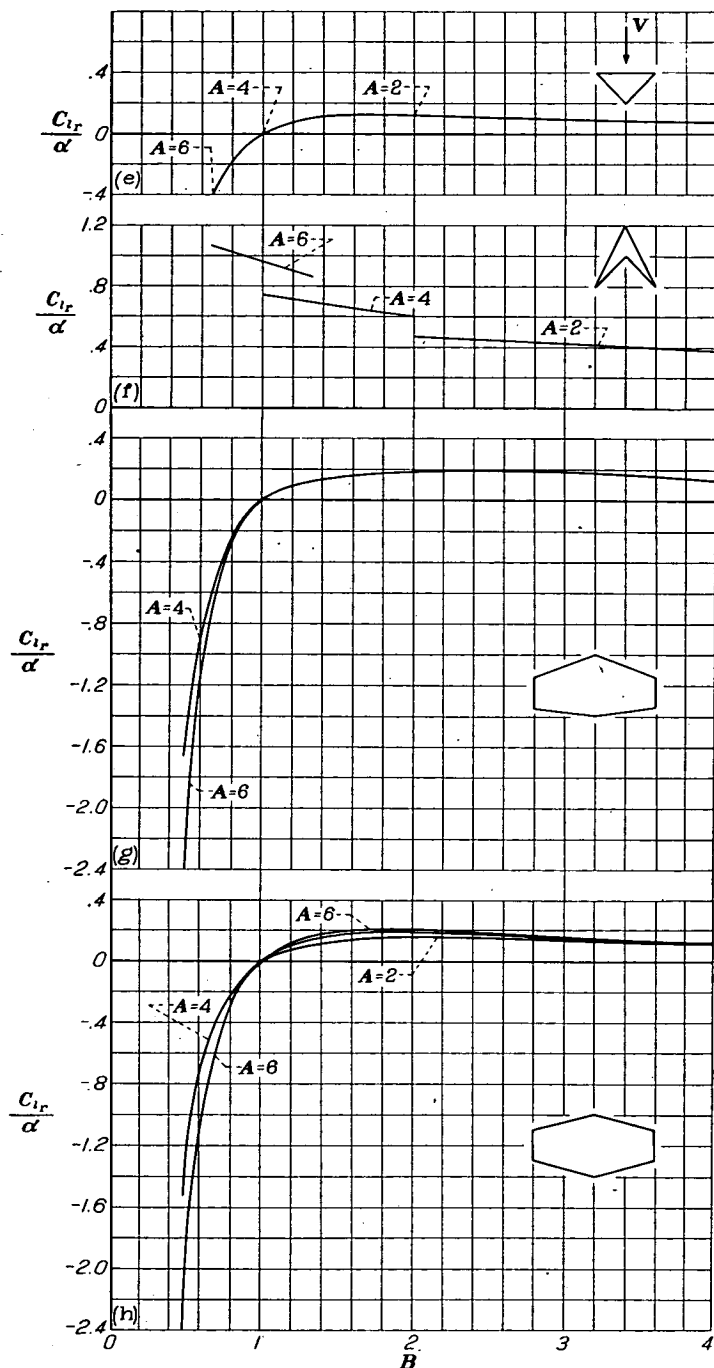
- (e) Triangular plan form. Base forward.
 (f) Swept-back plan form; $\lambda=0$; $\frac{C_r}{l}=0.5$.
 (g) Unswept hexagonal plan form; $\lambda=0.5$; $m=5$.
 (h) Unswept hexagonal plan form; $\lambda=0.5$; $Bm=3BA/2$.

FIGURE 14.—Concluded.



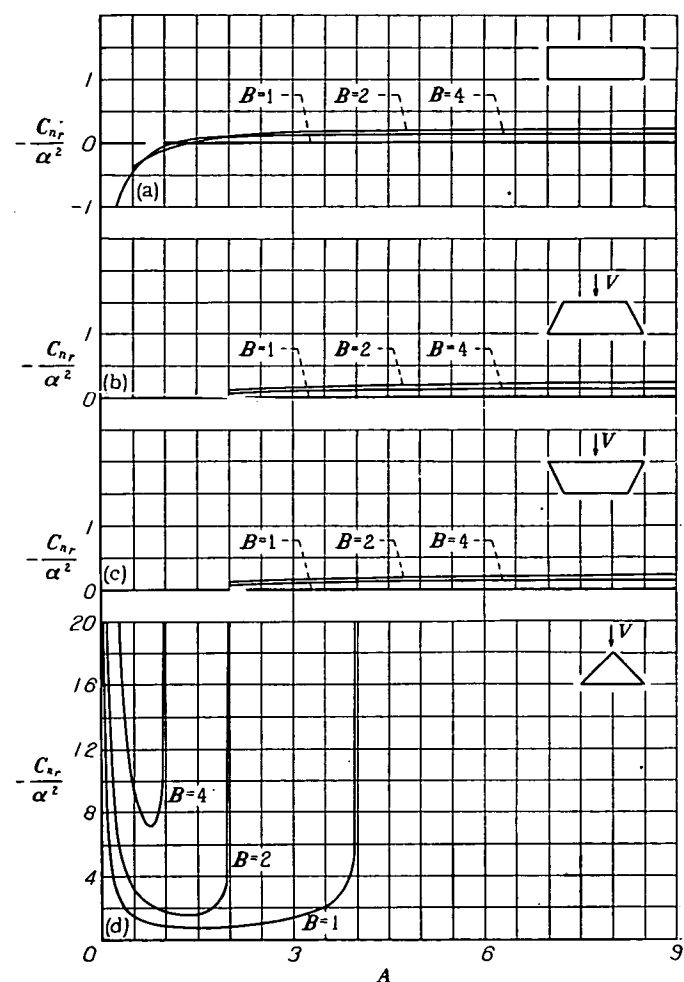
(a) Rectangular plan form.
 (b) Trapezoidal plan form; $m=0.5$.
 (c) Trapezoidal plan form; $m=-0.5$.
 (d) Triangular plan form. Apex forward.

FIGURE 15.—Variation of rolling-moment-due-to-yawing derivative with Mach number parameter B .



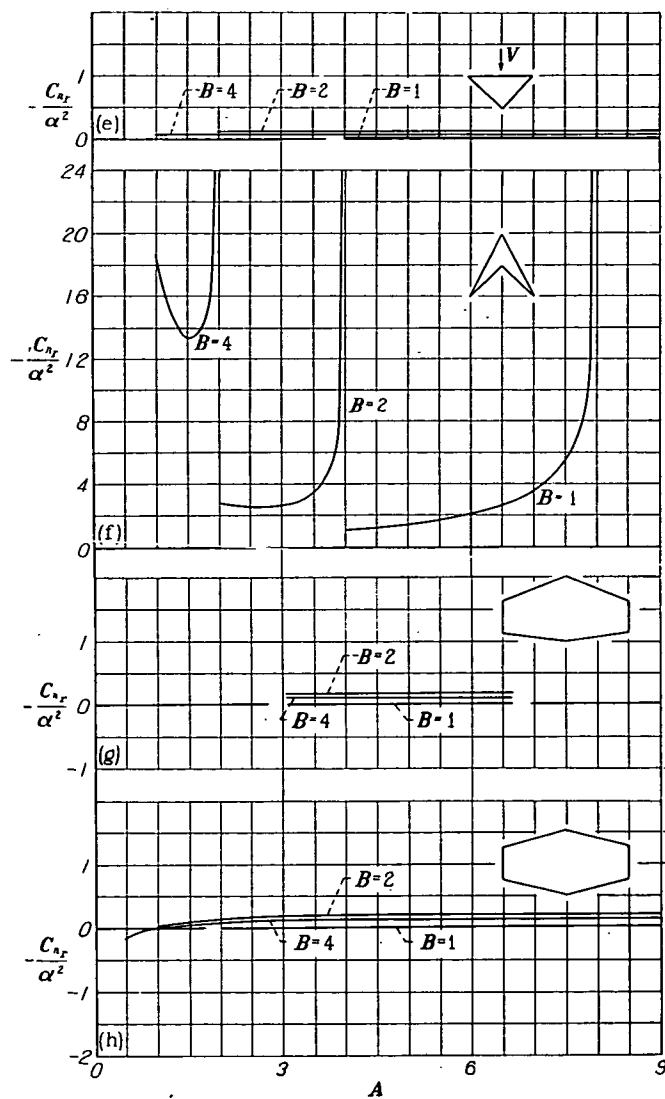
(e) Triangular plan form. Base forward. (See footnote, table II.)
 (f) Swept-back plan form; $\lambda=0$; $\frac{c_r}{l}=0.5$.
 (g) Unswept hexagonal plan form; $\lambda=0.5$; $m=5$.
 (h) Unswept hexagonal plan form; $\lambda=0.5$; $Bm=3BA/2$.

FIGURE 15.—Concluded.



- (a) Rectangular plan form.
 (b) Trapezoidal plan form; $m=0.5$.
 (c) Trapezoidal plan form; $m=-0.5$.
 (d) Triangular plan form. Apex forward.

FIGURE 16.—Variation of damping-in-yaw derivative with aspect ratio.



- (e) Triangular plan form. Base forward.
 (f) Swept-back plan form; $\lambda=0$; $\frac{C_r}{l}=0.5$.
 (g) Unswept hexagonal plan form; $\lambda=0.5$; $m=5$.
 (h) Unswept hexagonal plan form; $\lambda=0.5$; $Bm=3BA/2$.

FIGURE 16.—Concluded.

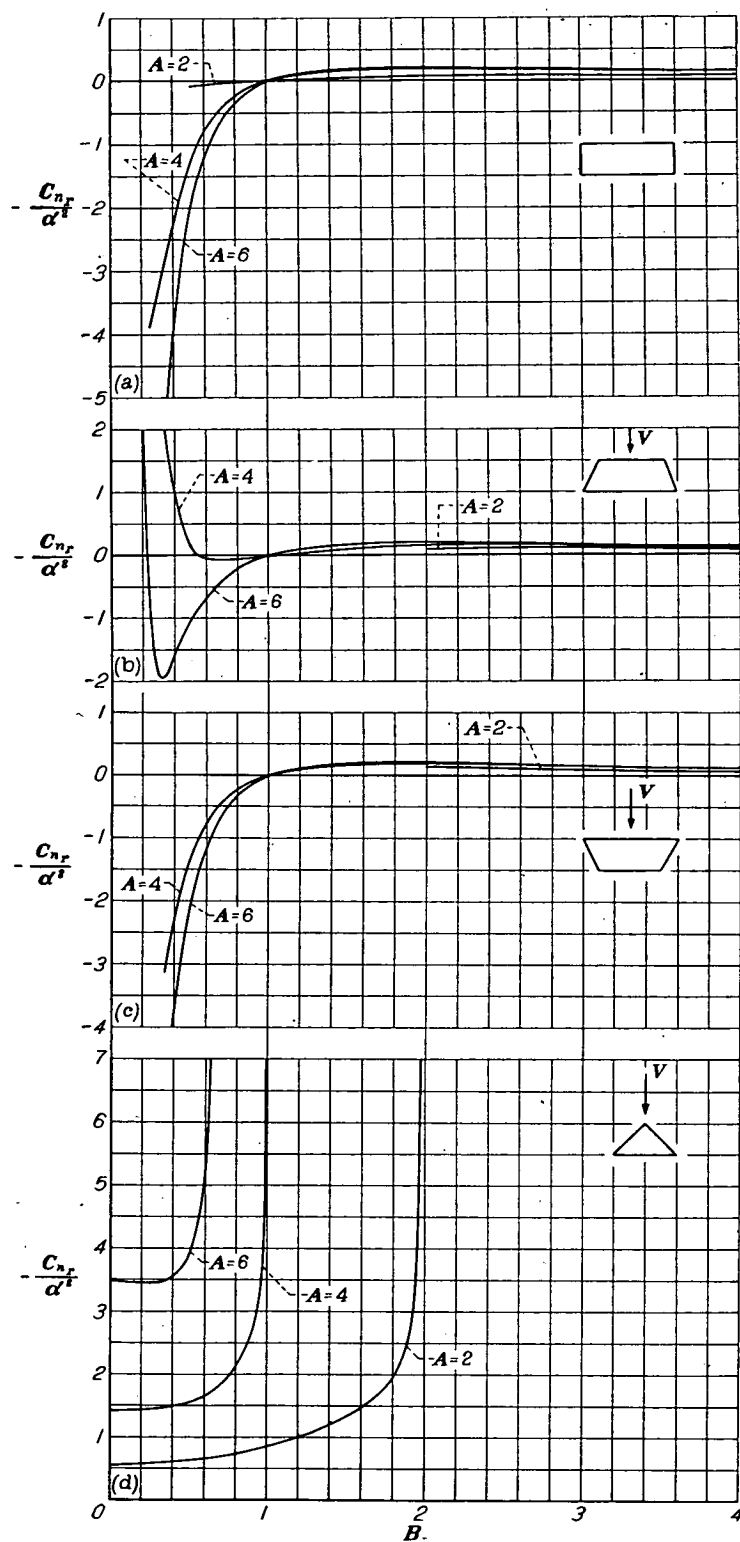


Figure 17.—Variation of damping-in-yaw derivative with Mach number parameter B .

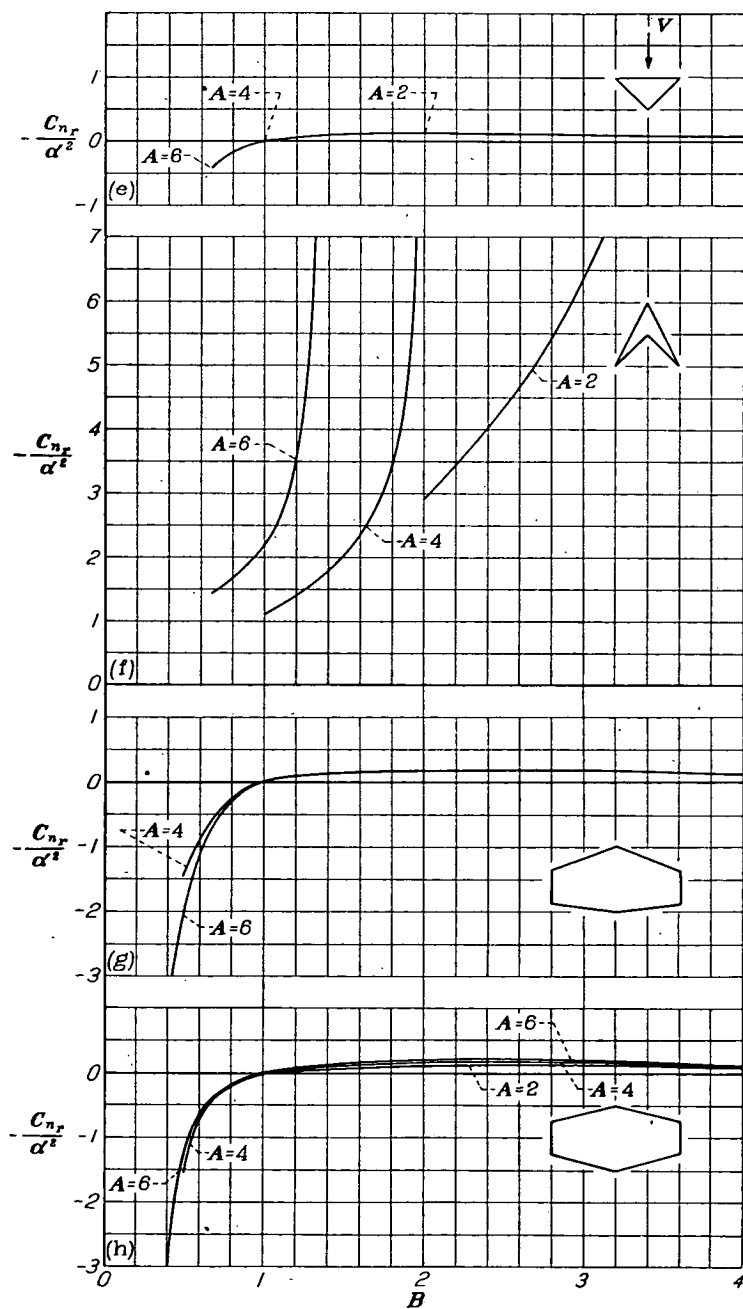
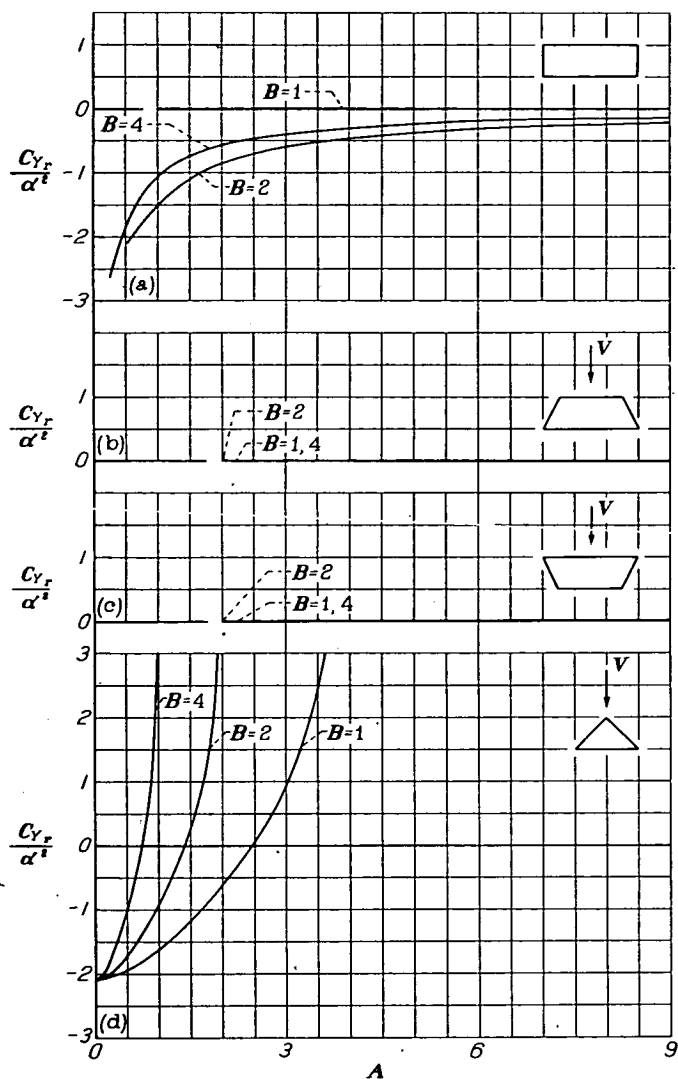


FIGURE 17.—Concluded.



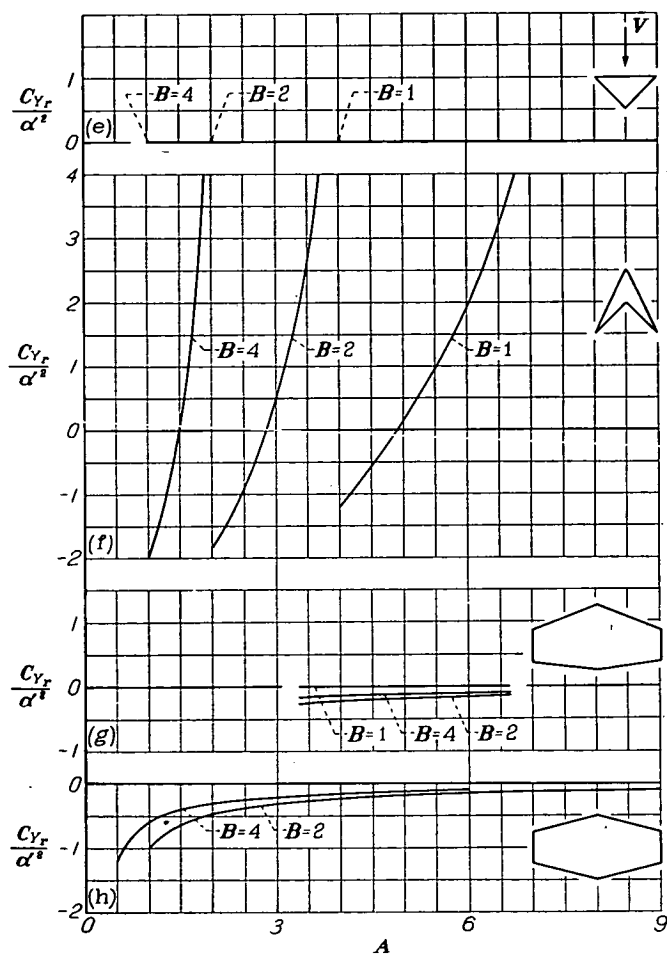
(a) Rectangular plan form.

(b) Trapezoidal plan form; $m=0.5$. (See footnote, table II.)

(c) Trapezoidal plan form; $m=-0.5$. (See footnote, table II.)

(d) Triangular plan form. Apex forward.

FIGURE 18.—Variation of side-force-due-to-yawing derivative with aspect ratio.



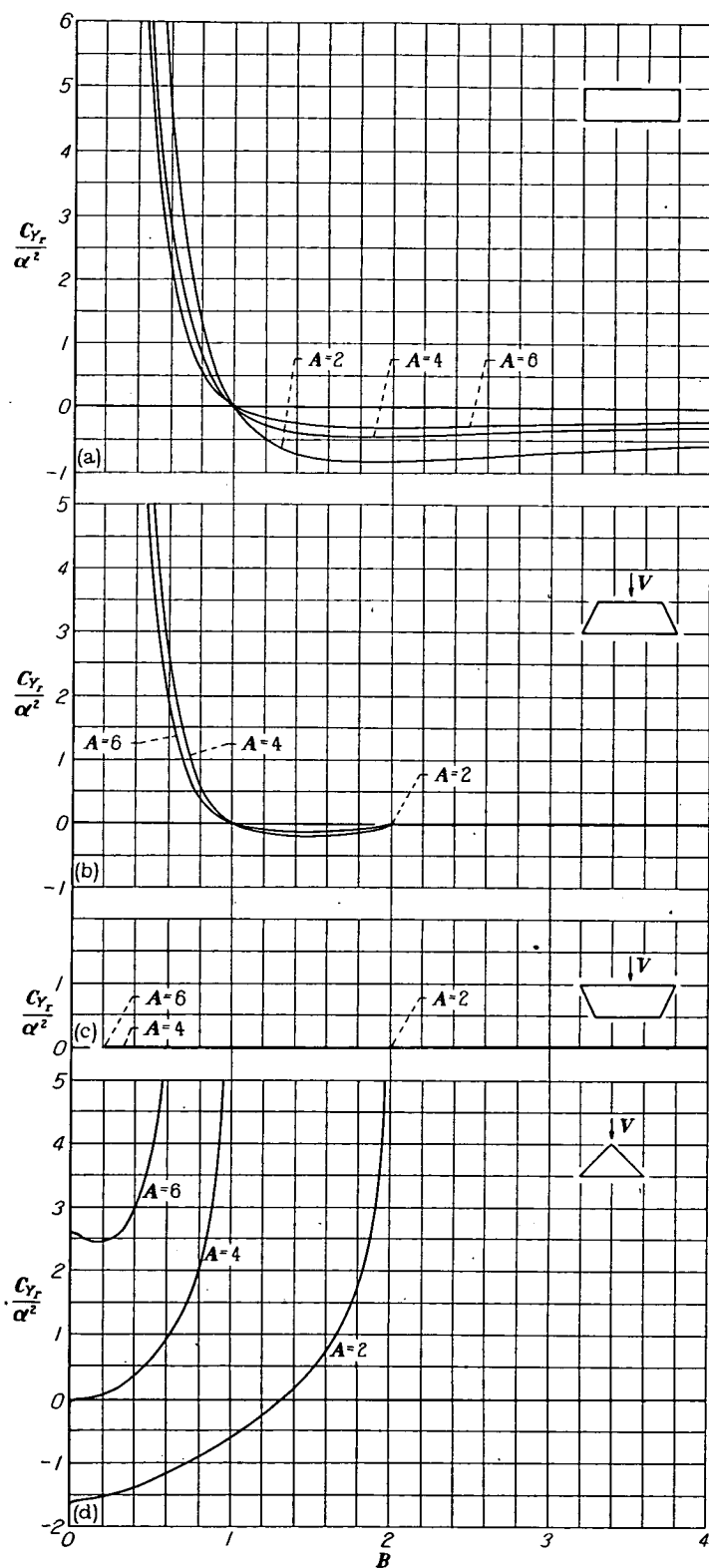
(e) Triangular plan form. Base forward. (See footnote, table II.)

(f) Swept-back plan form; $\lambda=0$; $\frac{c_r}{l}=0.5$.

(g) Unswept hexagonal plan form; $\lambda=0.5$; $m=5$.

(h) Unswept hexagonal plan form; $\lambda=0.5$; $Bm=3BA/2$.

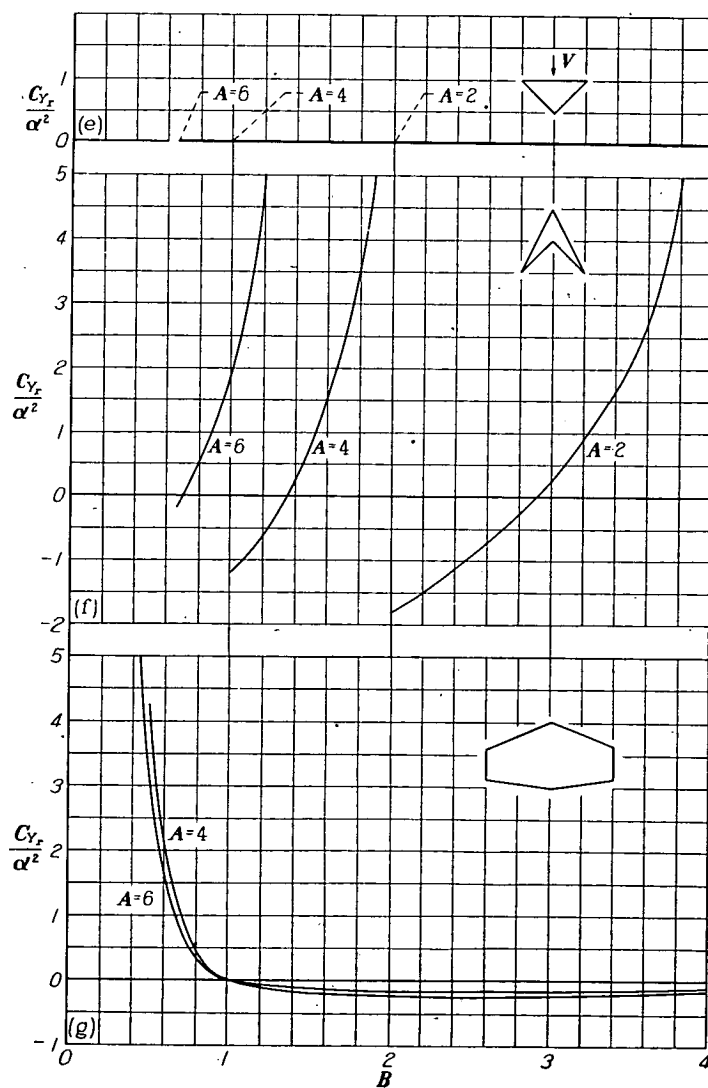
FIGURE 18.—Concluded.



(a) Rectangular plan form.

(b) Trapezoidal plan form; $m=0.5$.(c) Trapezoidal plan form; $m=-0.5$. (See footnote, table II.)

(d) Triangular plan form. Apex forward.

FIGURE 19.—Variation of side-force-due-to-yawing derivative with Mach number parameter B .

(e) Triangular plan form. Base forward. (See footnote, table II.)

(f) Swept-back plan form; $\lambda=0$; $\frac{c_r}{l}=0.5$.(g) Unswept hexagonal plan form; $\lambda=0.5$; $m=5$.

FIGURE 19.—Continued.

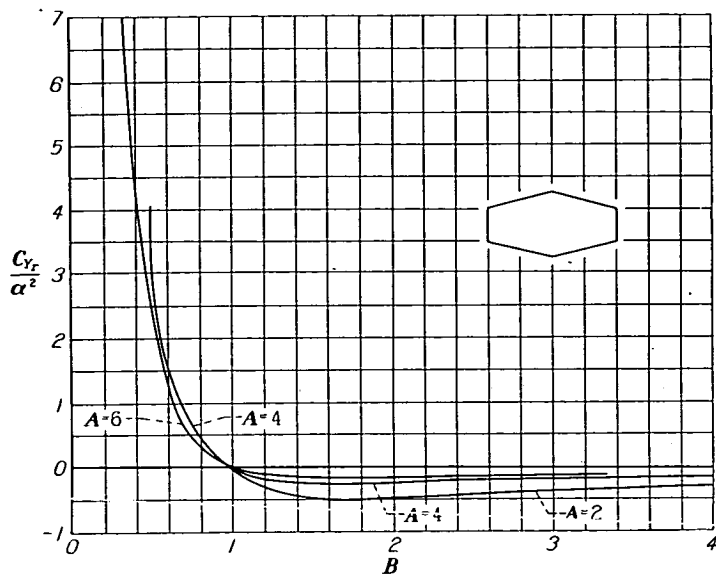


FIGURE 19.—Concluded. (h) Unswept hexagonal plan form; $\lambda=0.5$; $Bm=3BA/2$.

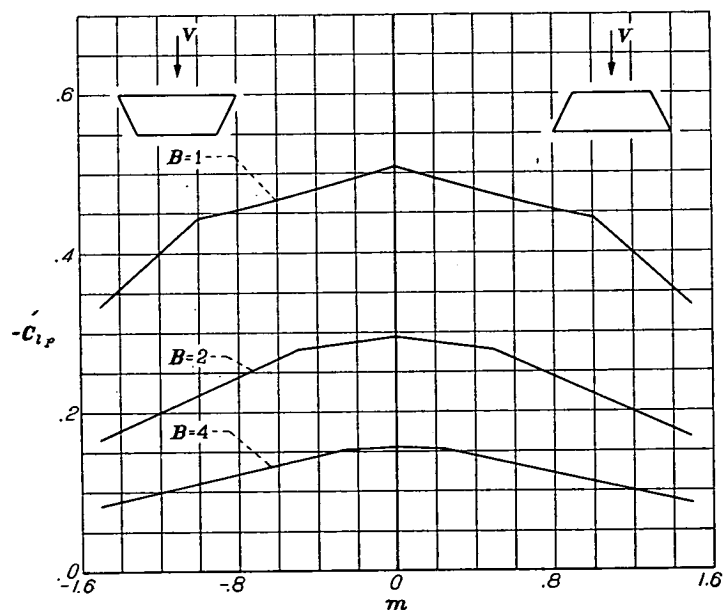


FIGURE 20.—Variation of damping-in-roll derivative with tip slope for trapezoidal plan forms; $A=6$.

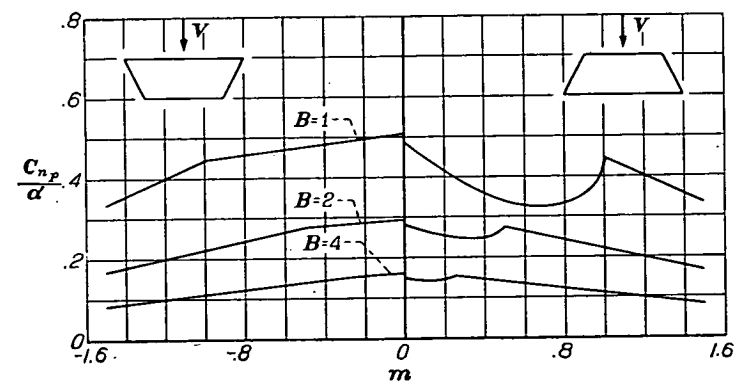


FIGURE 21.—Variation of yawing-moment-due-to-roll derivative with tip slope for trapezoidal plan forms; $A=6$.

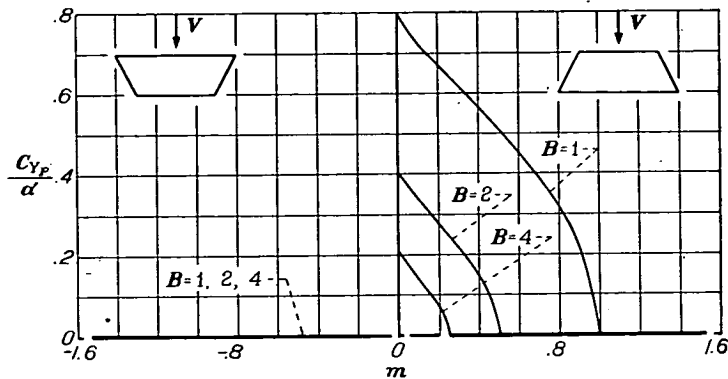


FIGURE 22.—Variation of side-force-due-to-rolling derivative with tip slope for trapezoidal plan forms; $A=6$.

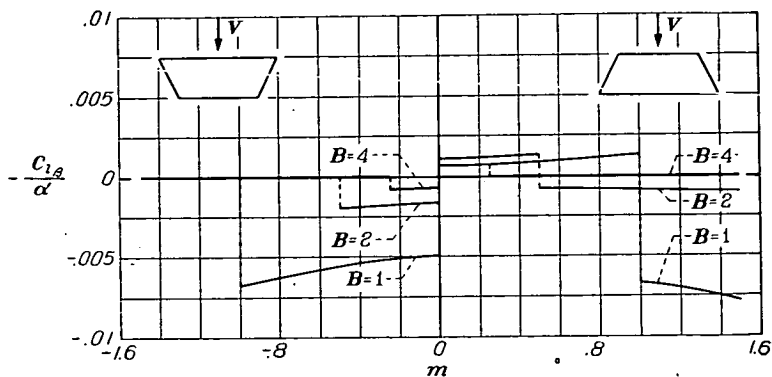


FIGURE 23.—Variation of rolling-moment-due-to-sideslip derivative with tip slope for trapezoidal plan forms; $A=6$.

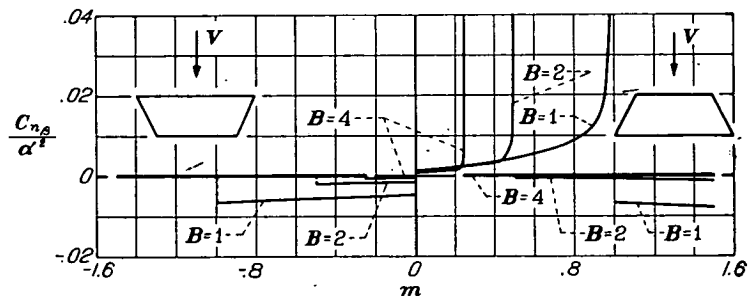


FIGURE 24.—Variation of yawing-moment-due-to-sideslip derivative with tip slope for trapezoidal plan forms; $A=6$.

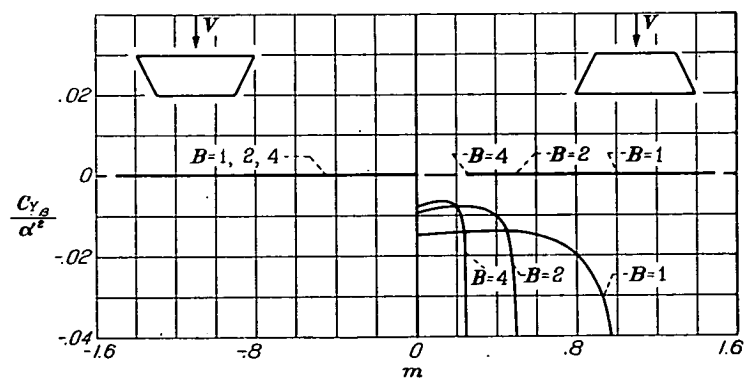
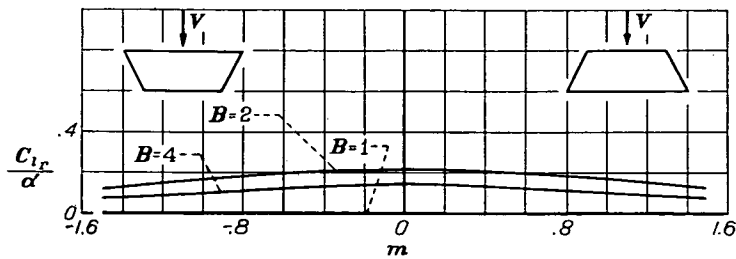
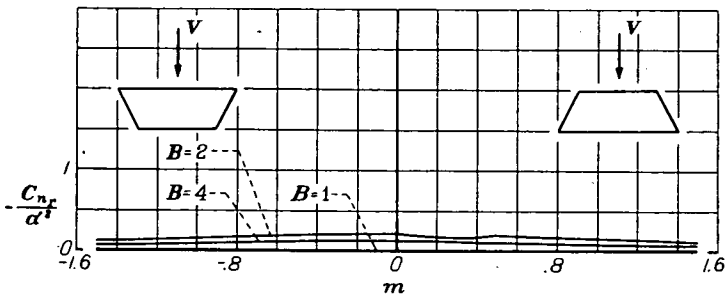
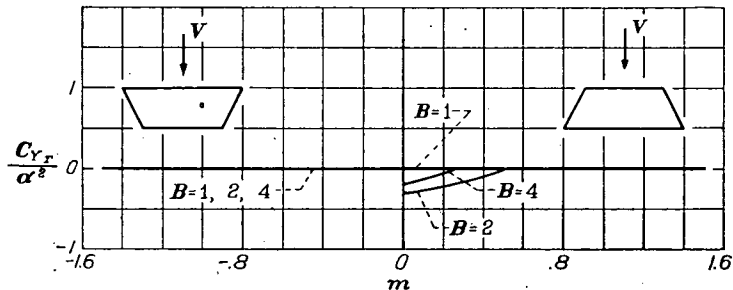
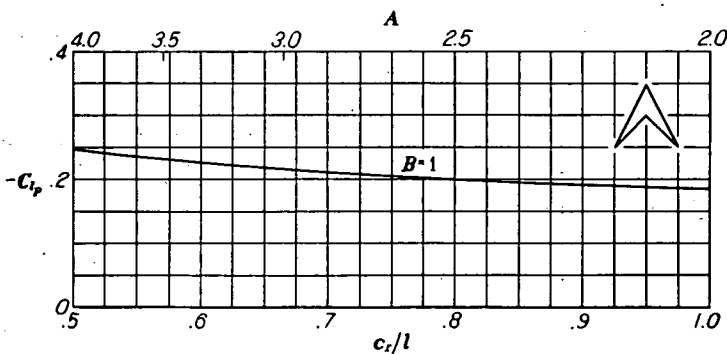
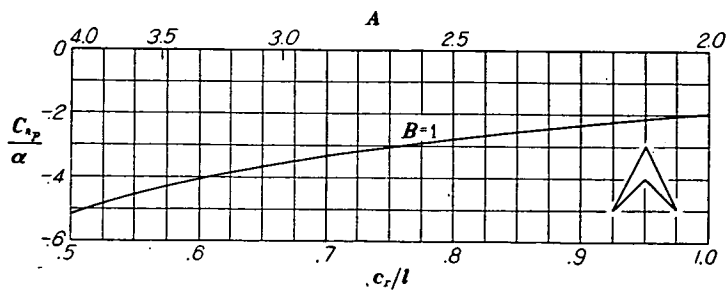
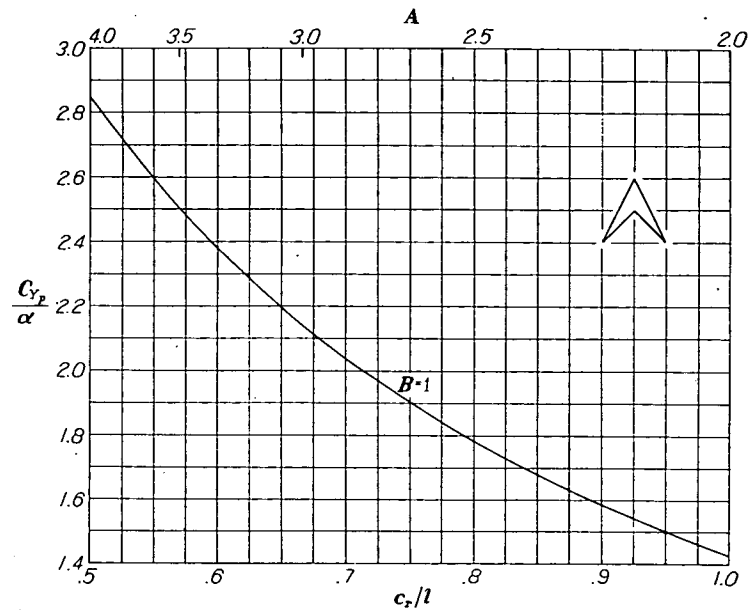
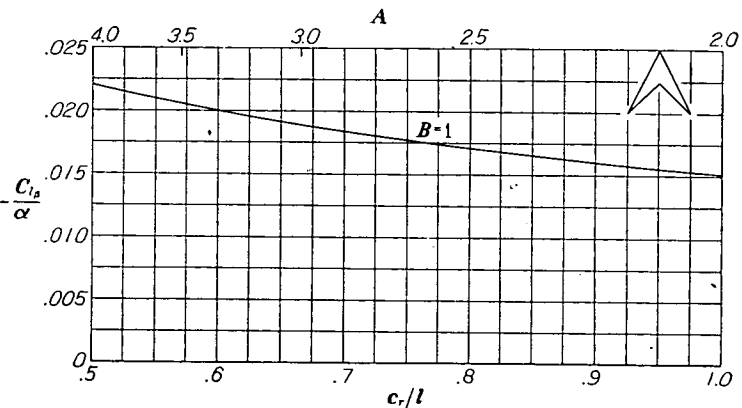
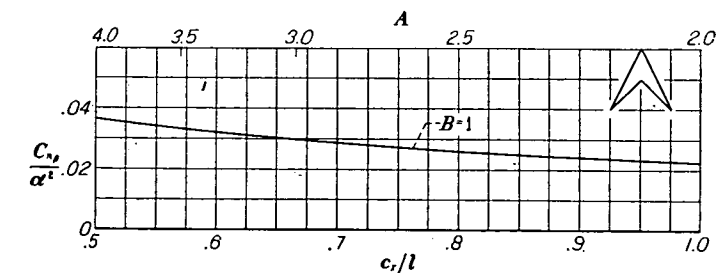
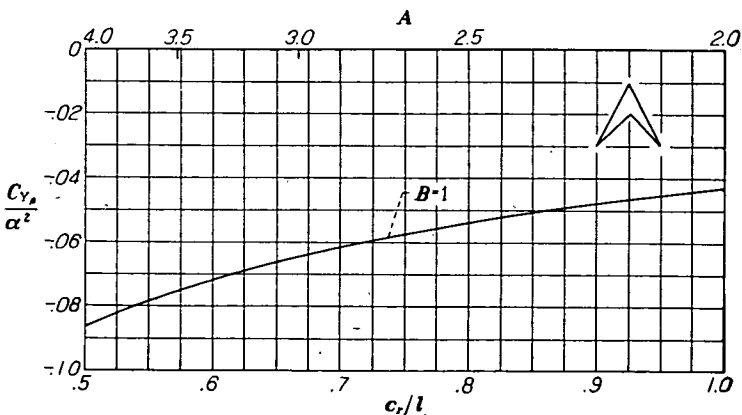


FIGURE 25.—Variation of side-force-due-to-sideslip derivative with tip slope for trapezoidal plan forms; $A=6$. (See footnote, table II.)

FIGURE 26.—Variation of rolling-moment-due-to-yawing derivative with tip slope for trapezoidal plan forms; $A=6$.FIGURE 27.—Variation of damping-in-yaw derivative with tip slope for trapezoidal plan forms; $A=6$.FIGURE 28.—Variation of side-force-due-to-yawing derivative with tip slope for trapezoidal plan forms; $A=6$.FIGURE 29.—Variation of damping-in-roll derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.FIGURE 30.—Variation of yawing-moment-due-to-roll derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.FIGURE 31.—Variation of side-force-due-to-roll derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.FIGURE 32.—Variation of rolling-moment-due-to-sideslip derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.FIGURE 33.—Variation of yawing-moment-due-to-sideslip derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.FIGURE 34.—Variation of side-force-due-to-sideslip derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.

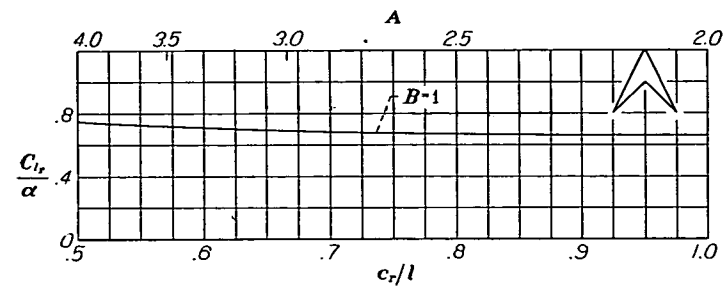


FIGURE 35.—Variation of rolling-moment-due-to-yawing derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.

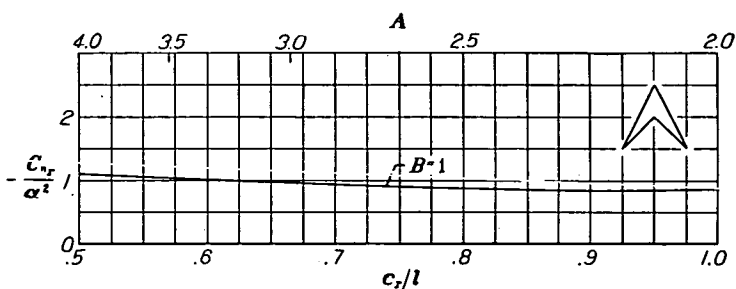


FIGURE 36.—Variation of damping-in-yaw derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.

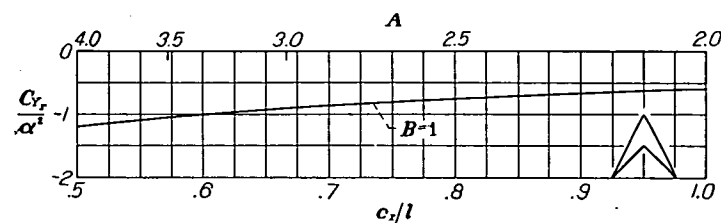


FIGURE 37.—Variation of side-force-due-to-yawing derivative with ratio of root chord to over-all length of swept-back plan form; $m=0.5$.

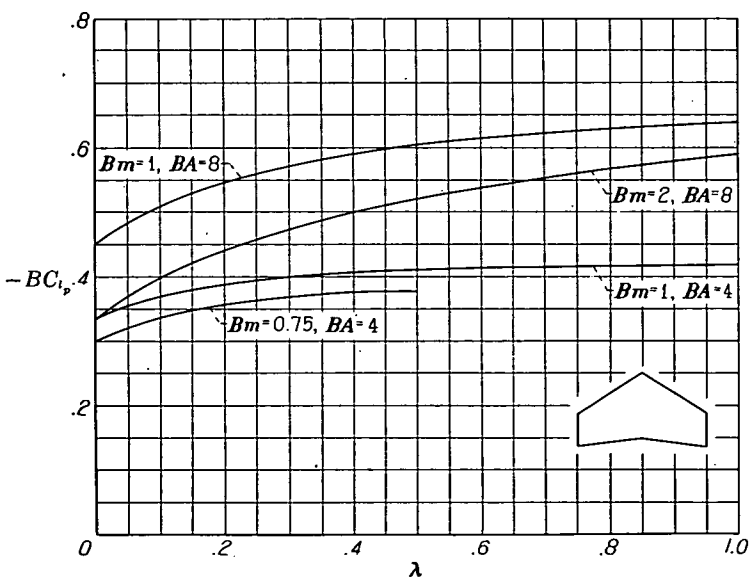


FIGURE 38.—Variation of damping-in-roll parameter with taper ratio for swept-back hexagonal plan form.

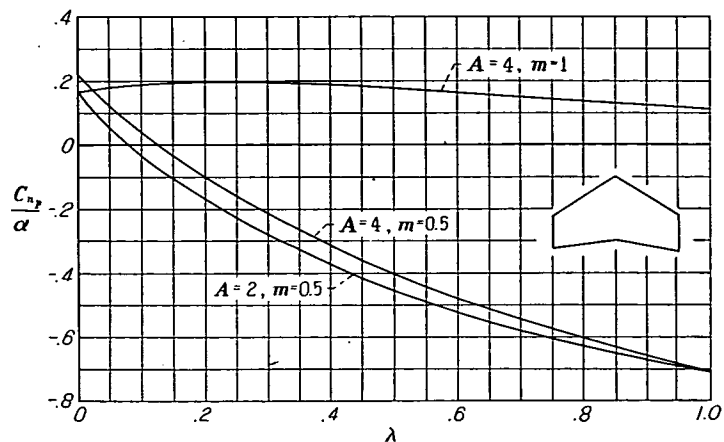


FIGURE 39.—Variation of yawing-moment-due-to-roll derivative with taper ratio for swept-back hexagonal plan form; $B=2$.

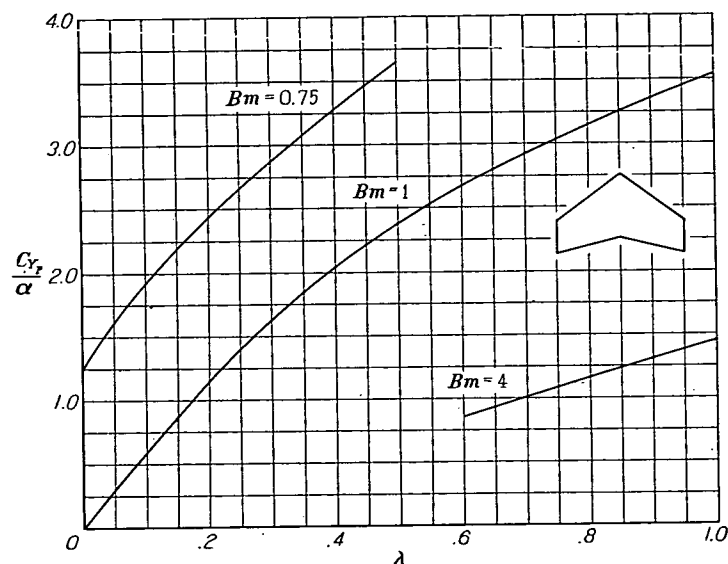


FIGURE 40.—Variation of side-force-due-to-roll derivative with taper ratio for swept-back hexagonal plan form; $BA=4$.

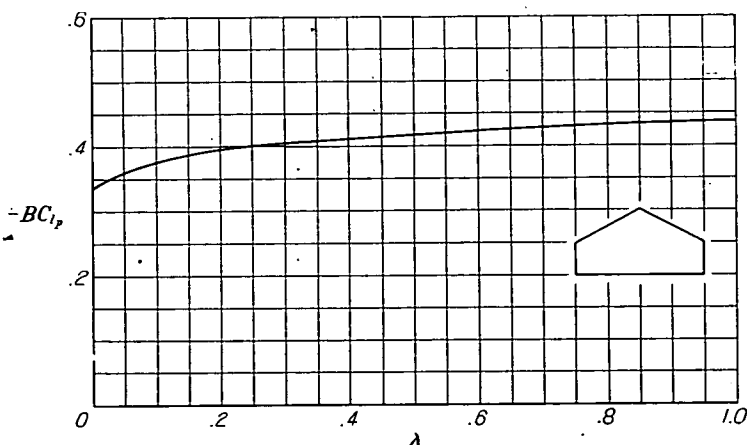


FIGURE 41.—Variation of damping-in-roll parameter with taper ratio for swept-back hexagonal plan form; $Bm=BA(1+\lambda)/4(1-\lambda)$; $BA=4$.

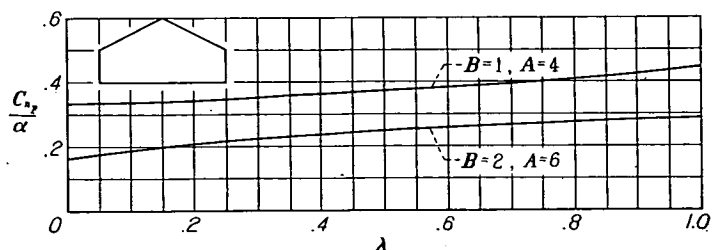


FIGURE 42.—Variation of yawing-moment-due-to-roll derivative with taper ratio for swept-back hexagonal plan form; $Bm = BA(1+\lambda)/4(1-\lambda)$.

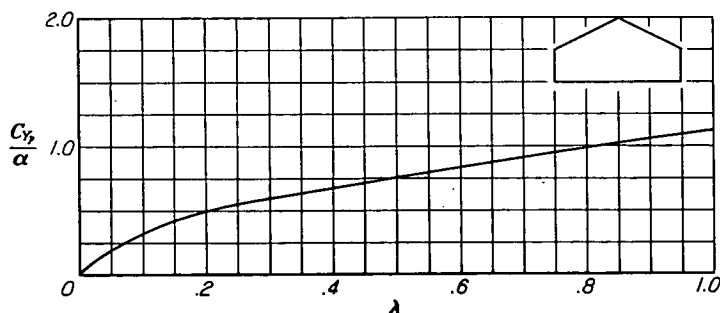


FIGURE 43.—Variation of side-force-due-to-roll derivative with taper ratio for swept-back hexagonal plan form. $Bm = BA(1+\lambda)/4(1-\lambda)$; $BA=4$.

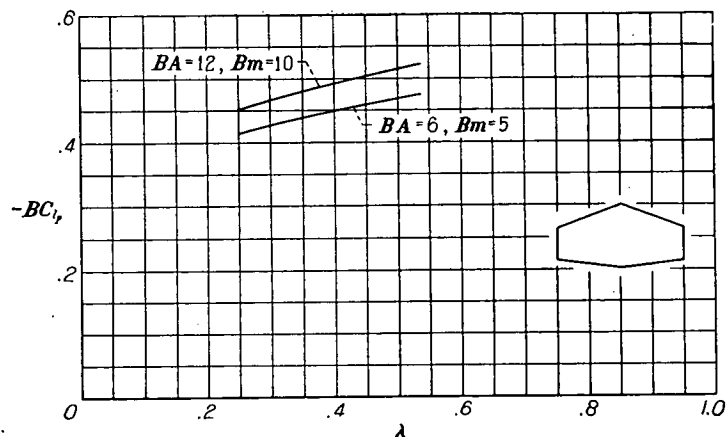


FIGURE 44.—Variation of damping-in-roll parameter with taper ratio for unswept hexagonal plan form.

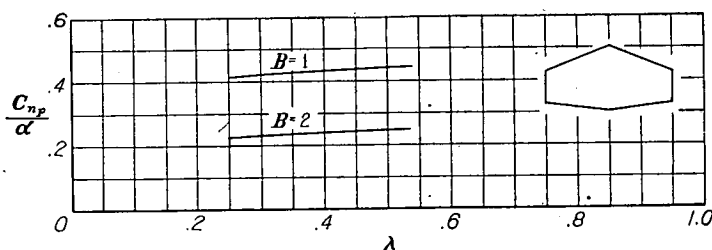


FIGURE 45.—Variation of yawing-moment-due-to-roll derivative with taper ratio for unswept hexagonal plan form; $A=6$; $m=5$.

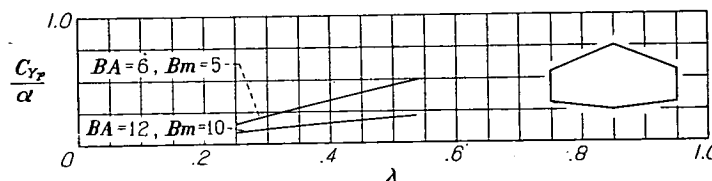


FIGURE 46.—Variation of side-force-due-to-roll derivative with taper ratio for unswept hexagonal plan form.

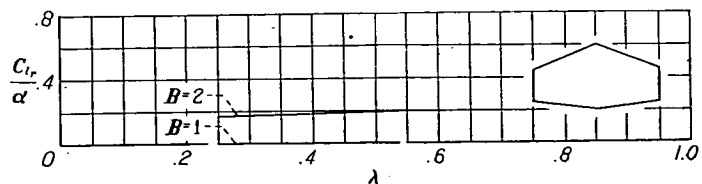


FIGURE 47.—Variation of rolling-moment-due-to-yawing derivative with taper ratio for unswept hexagonal plan form; $A=6$; $m=5$.

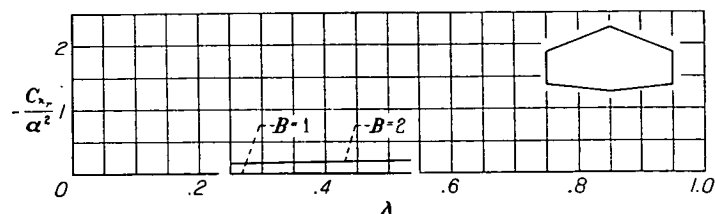


FIGURE 48.—Variation of damping-in-yaw derivative with taper ratio for unswept hexagonal plan form; $A=6$; $m=5$.

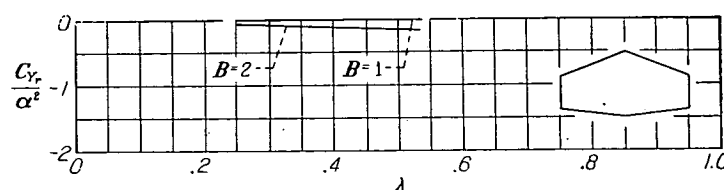


FIGURE 49.—Variation of side-force-due-to-yawing derivative with taper ratio for unswept hexagonal plan form; $A=6$; $m=5$.

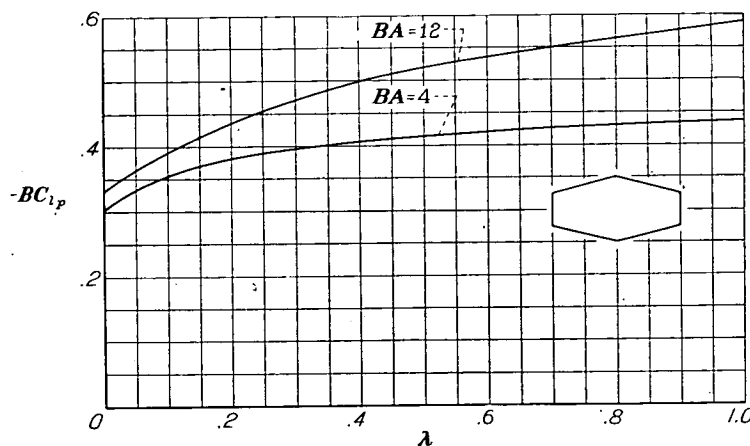


FIGURE 50.—Variation of damping-in-roll parameter with taper ratio for unswept hexagonal plan form; $Bm = BA(1+\lambda)/2(1-\lambda)$.

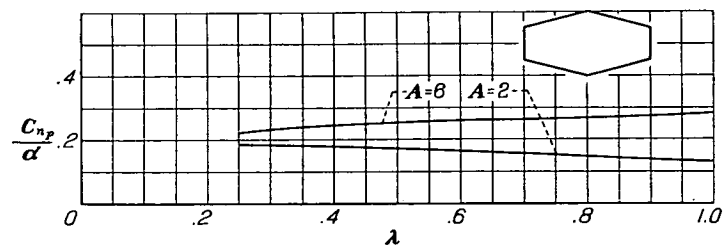


FIGURE 51.—Variation of yawing-moment-due-to-roll derivative with taper ratio for unswept hexagonal plan form; $B=2$; $Bm = BA(1+\lambda)/2(1-\lambda)$.

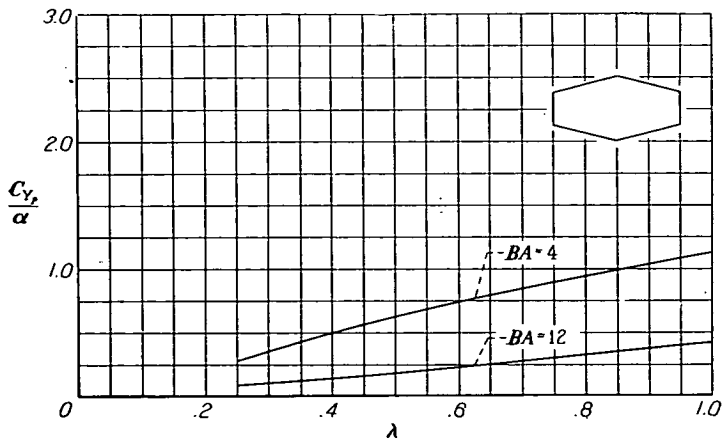


FIGURE 52.—Variation of side-force-due-to-roll derivative with taper ratio for unswept hexagonal plan form; $Bm = BA(1+\lambda)/2(1-\lambda)$.

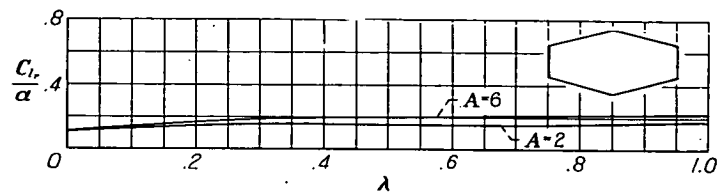


FIGURE 53.—Variation of rolling-moment-due-to-yawing derivative with taper ratio for unswept hexagonal plan form; $B=2$; $Bm = BA(1+\lambda)/2(1-\lambda)$.

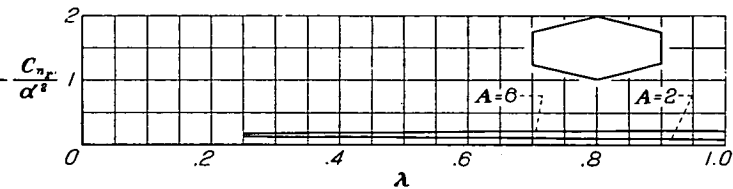


FIGURE 54.—Variation of damping-in-yaw derivative with taper ratio for unswept hexagonal plan form; $B=2$; $Bm = BA(1+\lambda)/2(1-\lambda)$.

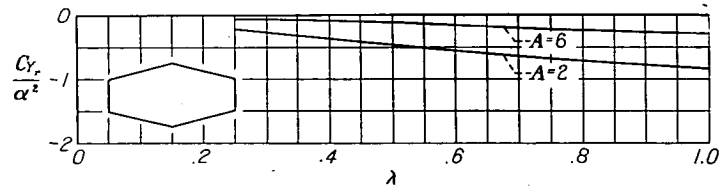


FIGURE 55.—Variation of side-force-due-to-yawing derivative with taper ratio for unswept hexagonal plan form; $B=2$; $Bm = BA(1+\lambda)/2(1-\lambda)$.

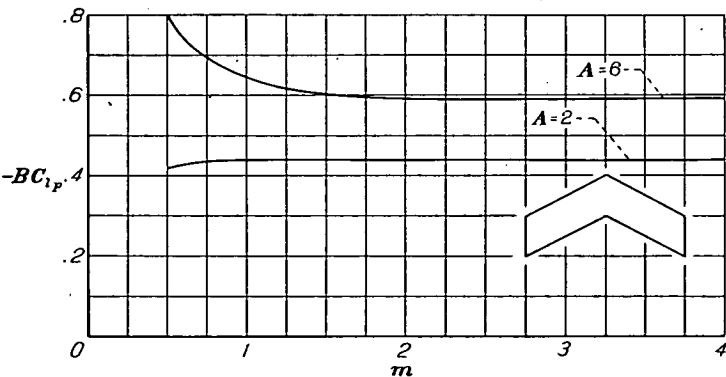


FIGURE 56.—Variation of damping-in-roll parameter with leading-edge slope for swept-back hexagonal plan form; $\lambda=1$; $B=2$.

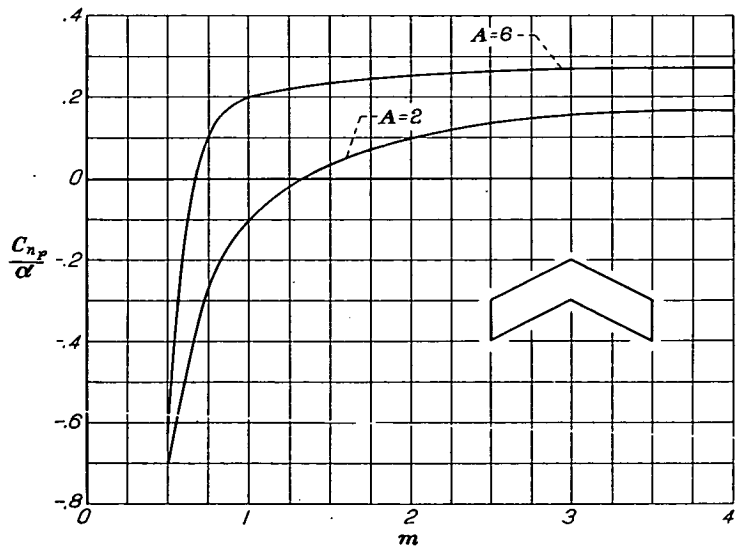


FIGURE 57.—Variation of yawing-moment-due-to-roll derivative with leading-edge slope for swept-back hexagonal plan form; $\lambda=1$; $B=2$.

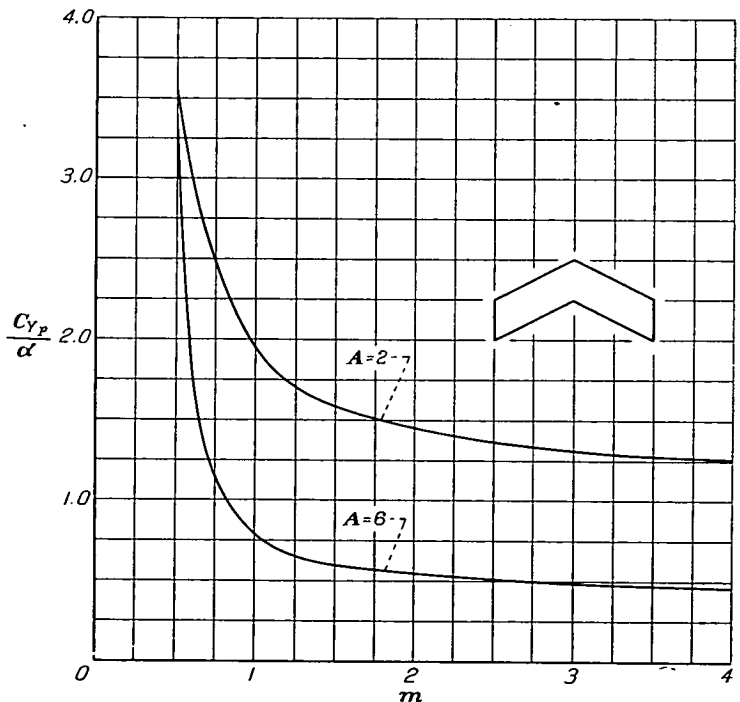


FIGURE 58.—Variation of side-force-due-to-roll derivative with leading-edge slope for swept-back hexagonal plan form; $\lambda=1$; $B=2$.

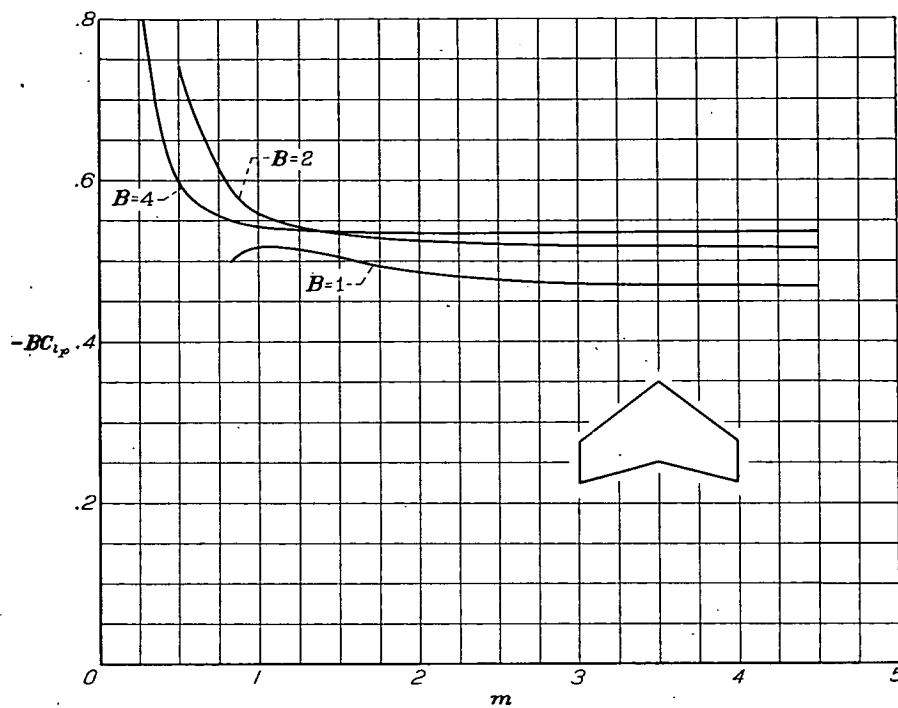


FIGURE 59.—Variation of damping-in-roll parameter with leading-edge slope for swept-back hexagonal plan form; $\lambda=0.5$; $A=6$.

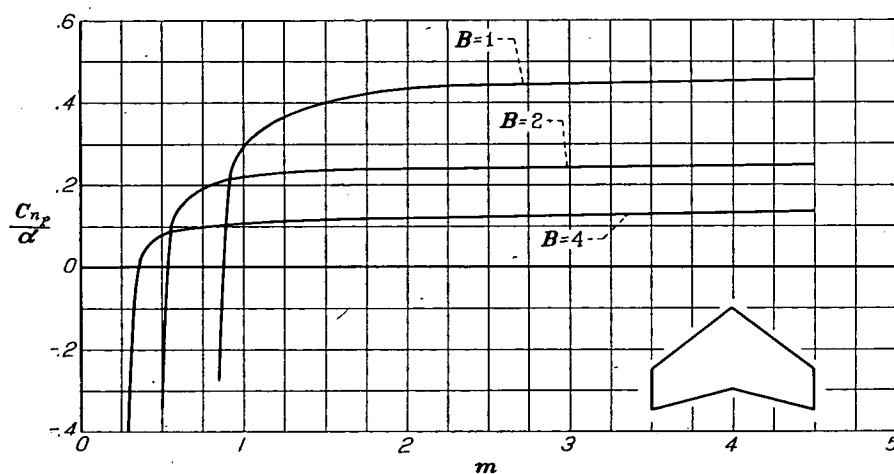


FIGURE 60.—Variation of yawing-moment-due-to-roll derivative with leading-edge slope for swept-back hexagonal plan form; $\lambda=0.5$; $A=6$.

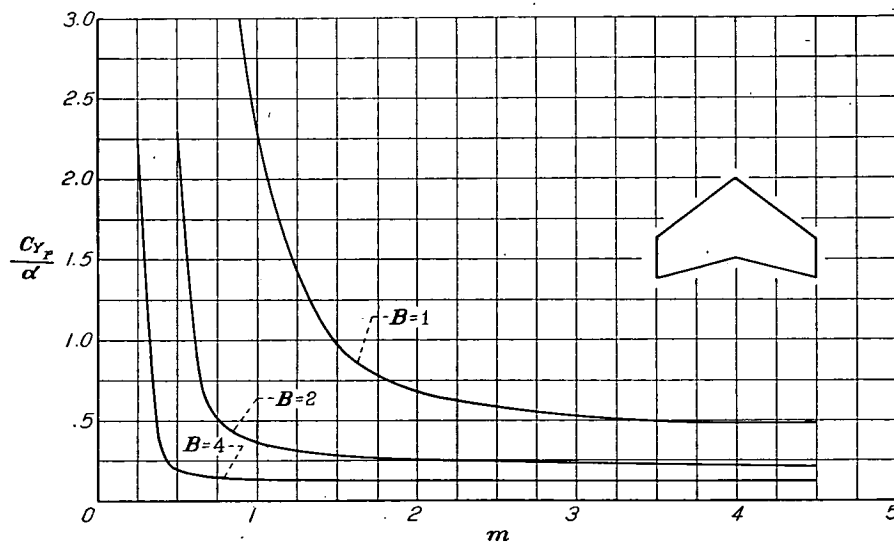


FIGURE 61.—Variation of side-force-due-to-roll derivative with leading-edge slope for swept-back hexagonal plan form; $\lambda=0.5$; $A=6$.

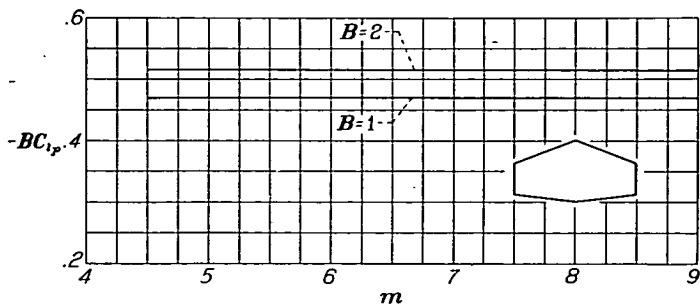


FIGURE 62.—Variation of damping-in-roll parameter with leading-edge slope for unswept hexagonal plan form; $\lambda=0.5$; $A=6$.

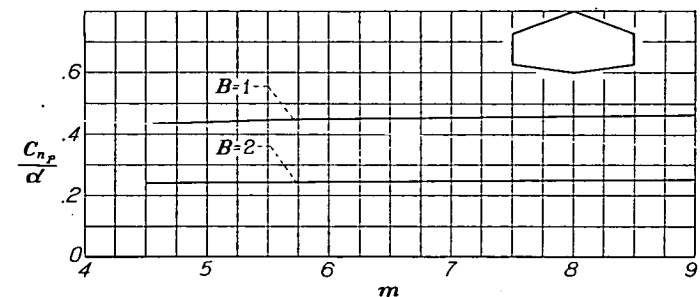


FIGURE 63.—Variation of yawing-moment-due-to-roll derivative with leading edge slope for unswept hexagonal plan form; $\lambda=0.5$; $A=6$.

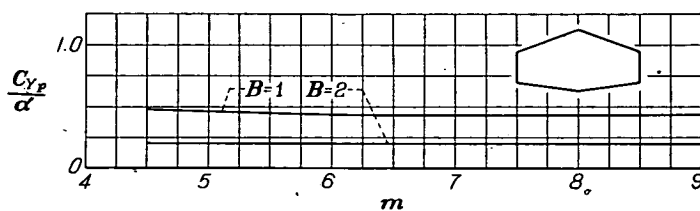


FIGURE 64.—Variation of side-force-due-to-roll derivative with taper ratio for unswept hexagonal plan form; $\lambda=0.5$; $A=6$.

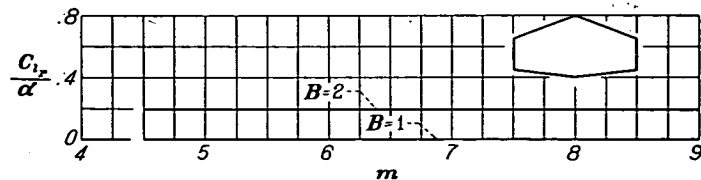


FIGURE 65.—Variation of rolling-moment-due-to-yawing derivative with leading-edge slope for unswept hexagonal plan form; $\lambda=0.5$; $A=6$.

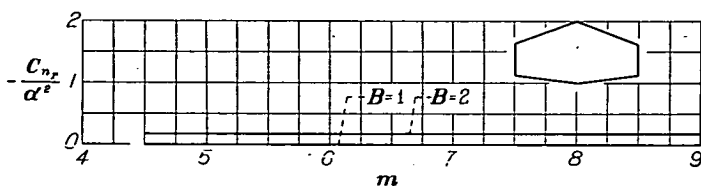


FIGURE 66.—Variation of damping-in-yaw derivative with leading-edge slope for unswept hexagonal plan form; $\lambda=0.5$; $A=6$.

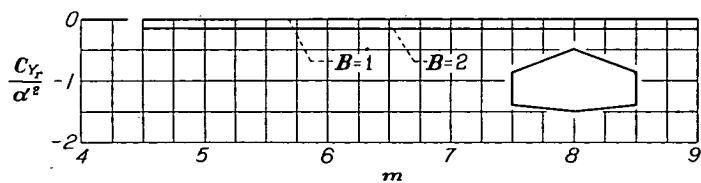


FIGURE 67.—Variation of side-force-due-to-yawing derivative with leading-edge slope for unswept hexagonal plan form; $\lambda=0.5$; $A=6$.

REPORT 1053

INVESTIGATION OF TURBULENT FLOW IN A TWO-DIMENSIONAL CHANNEL¹

By John Laufer

SUMMARY

A detailed exploration of the field of mean and fluctuating quantities in a two-dimensional turbulent channel flow is presented. The measurements were repeated at three Reynolds numbers, 12,300, 30,800, and 61,600, based on the half width of the channel and the maximum mean velocity. A channel of 5-inch width and 12:1 aspect ratio was used for the investigation.

Mean-speed and axial-fluctuation measurements were made well within the laminar sublayer. The semitheoretical predictions concerning the extent of the laminar sublayer were confirmed. The distribution of the velocity fluctuations in the direction of mean flow u' shows that the influence of the viscosity extends farther from the wall than indicated by the mean velocity profile, the region of influence being approximately four times as wide.

Fluctuations perpendicular to the flow in the lateral and vertical directions v' and w' , respectively, and the correlation coefficient $k = \frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}}$ were also measured. The turbulent

shearing stress was computed by three different methods. The results show satisfactory agreement for the two lower Reynolds numbers. In the case of the highest Reynolds number, however, the total shearing stress τ obtained from the fluctuation measurements was approximately 20 percent lower than that computed from the mean-velocity and mean-pressure measurements. All dimensionless mean fluctuating quantities were found to decrease with increasing Reynolds number. Measurements of the scales of turbulence L_v and L_z and microscales of turbulence λ_v and λ_z across the channel are presented and their variation with Reynolds number is discussed. Using a new technique, values for the microscale λ_z were obtained; a new method for estimating the scale L_z is also given.

The energy balance in the turbulent flow field was calculated from the measured quantities. From this calculation it is possible to give a descriptive picture of turbulent-energy diffusion in the center portion of the channel cross section.

For the flow corresponding to a Reynolds number of 30,800 the energy spectrum of the u'^2 -fluctuations at various points across the channel, including one in the laminar sublayer, was obtained.

At station $y/d=0.4$, where y is lateral distance and d is half width of the channel, the contribution to the turbulent shear stress from various frequency bands was measured and it was found that the contribution corresponding to frequencies above 1500 cycles per second is negligible. Since the spectrum of u'^2

at this point extends to about 5000 cycles per second it is evident that the high frequencies are nearly isotropic in agreement with Kolmogoroff's hypothesis.

INTRODUCTION

In recent years a considerable step forward was made in the theory of turbulence. Kolmogoroff (reference 1), Heisenberg (reference 2), and Onsager (reference 3) obtained independently an energy-spectrum law that holds in rather restricted types of turbulent flows. This progress gave a new impetus to both theoretical and experimental investigations. The experimental worker may follow two principal methods of approach to the problem. First, he may establish flow fields which satisfy sufficiently the assumptions of the new theory, namely, that the flow is isotropic and of high Reynolds number so that the influence of viscosity is a minimum and the effect of the turbulence-producing mechanism is small. Under these conditions he may measure quantities such as correlation functions, scales, and microscales that are defined exactly in the flow field and may compare his results with those predicted by the theory. The main difficulty with this method is to predict how closely one has to approximate in the actual flow the conditions assumed in the theory. In other words, the sensitivity of the theory to deviations from true isotropic conditions is not known and in case the measurements do not agree with the theory one does not know whether to attribute the disagreement to faulty assumptions in the theory or to the incomplete isotropic conditions of the flow. Furthermore, the imposition of the condition of isotropy to fluctuating fields may be a very strong restriction and the study of such fields may not yield the complete picture of the turbulence mechanism.

The second method of approach is to establish a simple nonisotropic turbulent flow field with well-defined boundary conditions and, in the light of the existing theories, to try to obtain information on the mechanism of energy transfer from the low frequencies of the energy spectrum to the high ones. The present investigation of a turbulent channel flow has this purpose in mind in its long-range program.

The difficulties of this method are immediately realized. Because of the nonisotropic nature of the flow, characteristic quantities such as scale and microscale are no longer well-defined and it is very possible that in the first stages of the investigation certain quantities will be measured that later will prove to be trivial. On the other hand, the main advantage of a fully developed channel flow is the fact that, in contrast with the flow behind grids, flow conditions are

¹Supersedes NACA TN 2123 "Investigation of Turbulent Flow in a Two-Dimensional Channel" by John Laufer, 1950.

steady; no decay of mean or fluctuating quantities exists in the direction of the flow. Consequently, the turbulent energy goes through all of its stages of transformation across the channel section—turbulent-energy production from the mean flow, energy diffusion, and turbulent and laminar dissipation—and one may study these transformations in detail.

It has become clear that the phenomenological theories of turbulence, such as the mixing-length theories, have lost most of their importance. These theories, developed in the late twenties and early thirties, were aimed specifically at an evaluation of the mean-velocity distribution in turbulent flow. The existing experimental evidence shows clearly that the mean-velocity distribution is very insensitive to the essential assumptions introduced into the phenomenological theories. In fact, purely dimensional arguments generally suffice to give the shape of the mean velocity profile with sufficient accuracy. For the further development of an understanding of turbulence, detailed measurements of the field of fluctuating rather than of mean velocities are necessary. The program of experimental research of which this work is one part is based on this reasoning. It is quite apparent and natural that the same conclusions have been drawn by workers in the turbulent field elsewhere and that, in general, the main emphasis of current experimental investigation is the exploration of the field of the fluctuating-velocity components.

The present set of experiments deals with flow in a two-dimensional channel, that is, with pressure flow between two flat walls. Channel flow of this type is the simplest type of turbulent flow near solid boundaries which can be produced experimentally. The simpler Couette type flow requires that one wall move with constant velocity, a condition which is difficult to realize experimentally. It can be approximated by the flow between concentric cylinders, but complications due to centrifugal forces arise here. The simple geometry of a two-dimensional channel allows an integration of the Reynolds equations, and the turbulent shearing stress can then be related directly to the shearing stress on the surfaces which, in turn, can be determined from the mean-pressure gradient or the slope of mean velocity profile at the wall.

The relation between the apparent stresses and the wall shearing stress can be used to advantage in two fashions. It is here possible to obtain the magnitude of the correlation coefficient responsible for the apparent shear by measuring only the intensities of the turbulent fluctuations. The turbulent shearing stress can also be measured directly by means of the hot-wire anemometer. A comparison with the shear distribution obtained from mean-pressure-gradient or mean-velocity-profile measurements serves then as a very useful check on the underlying assumptions, on the one hand, and specifically as a check on the reliability and accuracy of the direct measurements, on the other.

Measurements of channel flow have previously been made by Dönch (reference 4), Nikuradse (reference 5), Watten-

dorf and Kuethe (references 6 and 7), and Reichardt (references 8 and 9). A few unpublished measurements have been made recently at the Polytechnic Institute of Brooklyn.

Dönch's and Nikuradse's measurements were concerned only with the mean-velocity distribution and are thus of not too much interest for a comparison with the present set of measurements. Wattendorf measured the intensity of the fluctuating-velocity components and then deduced the correlation coefficient from the mean-pressure measurements. The technique for measuring the axial component of the velocity fluctuation was well-developed at the time, but the cross component was only tentatively measured.

The most complete set of measurements is due to the work of Reichardt, who measured velocity fluctuations in the direction of the flow and normal to the wall as well as the turbulent shear directly. Reichardt found very good agreement between the shearing stress determined in these two ways; his paper comes closest to the present investigation and his results will be used for comparison. The Reynolds number in Reichardt's measurements was 8000, which is lower than the range covered by the present measurements. A criticism which can be made of Reichardt's investigation is his use of a tunnel of only 1:4 aspect ratio. The two-dimensional character of the flow is thus somewhat doubtful. Wattendorf's experiments were made in a channel of very large aspect ratio (18:1) and are thus free of this criticism.

In a preliminary investigation a channel 1 inch wide and 60 inches high was chosen. The measurements, however, have shown that in this case the scale of turbulence is so small that great care must be taken to correct the hot-wire readings for the effect of wire length. Measurements of the microscales were, in fact, impossible in the 1-inch channel. The microscales were about 0.1 centimeter and thus smaller than the length of the wire. The corrections for the case of measurements of v' and w' were about 30 percent. Since the method of correction becomes very inaccurate for such large ratios of wire length to microscale, the measurements were repeated in a 5-inch channel with a 12:1 aspect ratio. This ratio was still large enough to insure two-dimensional flow and the length corrections were greatly reduced.

This investigation was conducted at the Guggenheim Aeronautical Laboratory, California Institute of Technology, under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

The author wishes to acknowledge the constant advice and help in both experimental methods and interpretation of results of Dr. H. W. Liepmann during this investigation. He also wishes to thank Dr. C. B. Millikan for his continuous interest in this research. The cooperation of Dr. F. E. Marble and Mr. F. K. Chuang is much appreciated.

SYMBOLS

x longitudinal coordinate in direction of flow;
 $x=0$ corresponds to channel exit

y	lateral coordinate; $y=0$ corresponds to channel wall
z	vertical coordinate
d	half width of channel (2.5 in.)
δ	thickness of laminar sublayer
u	mean velocity at any point in channel
U_o	maximum value of mean velocity
u'	instantaneous value of velocity fluctuations in direction of mean flow x
v', w'	instantaneous values of velocity fluctuations normal to mean flow in directions y and z , respectively
$\tilde{u}'$	root-mean-square value of velocity fluctuations in direction of mean flow x
$\tilde{v}', \tilde{w}'$	root-mean-square values of velocity fluctuations normal to mean flow in directions y and z , respectively
p	pressure at any point in channel
ρ	air density
τ	total shearing stress
τ_o	shearing stress at wall
U_τ	friction velocity ($\sqrt{\tau_o/\rho}$)
μ	absolute viscosity of air
ν	kinematic viscosity
R	Reynolds number based on half width of channel and maximum mean velocity
k	correlation coefficient responsible for apparent shear $\left(\frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}}\right)$
R_x, R_y, R_z	correlation coefficients as functions of X, Y , and Z , respectively
$\lambda_x, \lambda_y, \lambda_z$	microscales of turbulence $\left(\frac{\overline{u'^2}}{\lambda_x^2} = \overline{\left(\frac{\partial u'}{\partial x}\right)^2}, \frac{\overline{u'^2}}{\lambda_y^2} = \frac{1}{2} \overline{\left(\frac{\partial u'}{\partial y}\right)^2}, \frac{\overline{u'^2}}{\lambda_z^2} = \frac{1}{2} \overline{\left(\frac{\partial u'}{\partial z}\right)^2}\right)$

L_x, L_y, L_z	scales of turbulence $\left(L_x \equiv \int_0^\infty R_x dX, L_y \equiv \int_0^\infty R_y dY, L_z \equiv \int_0^\infty R_z dZ\right)$
X, Y, Z	distances (in the x -, y -, and z -direction, respectively) between points at which correlation fluctuations are measured
σ	ratio of square of compensated and uncompensated fluctuations
M	time constant expressing thermal lag of hot-wire
n	frequency, cycles per second
$F_{u'^2}(n)$	fraction of turbulent energy $\overline{u'^2}$ associated with band width dn
$F_{u'v'}(n)$	fraction of turbulent shear $\overline{u'v'}$ associated with band width dn
W	dissipation of turbulent energy
t	time
e_1, e_2	voltage fluctuations

ANALYTICAL CONSIDERATIONS

EQUATIONS OF MOTION FOR TWO-DIMENSIONAL CHANNEL FLOW

The mean- and fluctuating-velocity components are denoted by u_i and u'_i , respectively; Cartesian coordinates are designated by x_i ; and p_{ik} denotes the components of the stress tensor which includes both the viscous and apparent (Reynolds) stresses. The Reynolds equation and the continuity equation for steady flow can thus be written, using Cartesian tensor notation,

$$\rho u_k \frac{\partial u_i}{\partial x_k} = \frac{\partial p_{ik}}{\partial x_k} \quad (1a)$$

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (1b)$$

where

$$p_{ik} = \begin{vmatrix} -p + 2\mu \frac{\partial u_1}{\partial x_1} - \rho \overline{u_1'^2} & \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \rho \overline{u_1' u_2'} & \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) - \rho \overline{u_1' u_3'} \\ \mu \left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) - \rho \overline{u_1' u_2'} & -p + 2\mu \frac{\partial u_2}{\partial x_2} - \rho \overline{u_2'^2} & \mu \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) - \rho \overline{u_2' u_3'} \\ \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) - \rho \overline{u_1' u_3'} & \mu \left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) - \rho \overline{u_2' u_3'} & -p + 2\mu \frac{\partial u_3}{\partial x_3} - \rho \overline{u_3'^2} \end{vmatrix}$$

For channel flow between two parallel flat walls,

$$\begin{matrix} u_1 = u(y) \\ u_2 = 0 \\ u_3 = 0 \end{matrix} \quad p_{ik} = \begin{vmatrix} -p - \rho \overline{u'^2} & \mu \frac{du}{dy} - \rho \overline{u'v'} & 0 \\ \mu \frac{du}{dy} - \rho \overline{u'v'} & -p - \rho \overline{v'^2} & 0 \\ 0 & 0 & -p - \rho \overline{w'^2} \end{vmatrix}$$

where x , y , and z replace x_i , and u' , v' , and w' replace u_i' for convenience in writing equations (1a) and (1b) in simpler form. Equation (1b) is automatically satisfied and equation (1a) becomes:

$$\left. \begin{aligned} 0 &= \frac{\partial}{\partial x} (-p - \rho \overline{u'^2}) + \frac{\partial}{\partial y} \left(\mu \frac{du}{dy} - \rho \overline{u'v'} \right) \\ 0 &= \frac{\partial}{\partial y} (-p - \rho \overline{v'^2}) + \frac{\partial}{\partial x} \left(\mu \frac{du}{dy} - \rho \overline{u'v'} \right) \end{aligned} \right\} \quad (2)$$

In fully developed turbulent flow the variation of mean values of the fluctuating quantities with x should be zero; that is, the flow pattern is independent of the streamwise direction. Hence, equations (2) become:

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \nu \frac{d^2 u}{dy^2} - \frac{d \overline{u'v'}}{dy}$$

or

$$\frac{1}{\rho} \frac{\partial p}{\partial x} = \frac{1}{\rho} \frac{d\tau}{dy} \quad (3a)$$

$$\frac{1}{\rho} \frac{\partial p}{\partial y} = - \frac{\partial \overline{v'^2}}{\partial y} \quad (3b)$$

Differentiating equation (3b) with respect to x gives $\frac{1}{\rho} \frac{\partial^2 p}{\partial x \partial y} = 0$; consequently $\partial p / \partial x$ is independent of y and equation (3a) is immediately integrable. Thus

$$\frac{y}{\rho} \frac{\partial p}{\partial x} = \nu \frac{du}{dy} - \overline{u'v'} + \text{Constant}$$

In the center of the channel the shear vanishes; hence if the channel has the width $2d$

$$\text{Constant} = \frac{d}{\rho} \frac{\partial p}{\partial x}$$

and hence

$$\frac{1}{\rho} \frac{\partial p}{\partial x} (y-d) = \nu \frac{du}{dy} - \overline{u'v'}$$

Or in nondimensional form:

$$\frac{y-d}{\rho U_o^2} \frac{\partial p}{\partial x} = \frac{\nu}{U_o^2} \frac{du}{dy} - \frac{\overline{u'v'}}{U_o^2} \quad (4)$$

In terms of the shearing stress at the wall,

$$- \frac{\tau_o}{\rho U_o^2} \frac{y-d}{d} = \frac{\nu}{U_o^2} \frac{du}{dy} - \frac{\overline{u'v'}}{U_o^2} \quad (5)$$

It is evident from equations (4) and (5) that τ_o can be determined in three different ways:

(a) From the mean-pressure gradient

$$\tau_o = -d \frac{\partial p}{\partial x}$$

(b) From the slope of the mean velocity profile near the wall

$$\tau_o = \mu \left(\frac{du}{dy} \right)_{y=0}$$

(c) From a direct measurement of $\overline{u'v'}$

$$\tau_o = \left(\rho \overline{u'v'} - \mu \frac{du}{dy} \right)_{y=d}$$

If y/d is not too small,

$$\mu \frac{du}{dy} < \rho \overline{u'v'}$$

and then

$$\tau_o = \rho \overline{u'v'} \frac{d}{y-d}$$

The technically simplest way to determine τ_o , and thus in fact the complete shear distribution, is a measurement of $\partial p / \partial x$. This is the method applied in most investigations.

To determine the slope of the velocity profile near the wall, the profile has to be known up to points very close to the wall, that is, at least to distances $y/d = 10^{-2}$. This in general requires the use of the hot-wire anemometer and very precise measurements.

The third method requires a direct measurement of the correlation between the axial and lateral velocity fluctuations. The technique of this type of measurement is known and was first applied by Reichardt and by Skramstad and, in somewhat different form, in recent investigations at the National Bureau of Standards, Polytechnic Institute of Brooklyn, and California Institute of Technology.

In the present investigation all three methods have been applied and the results compared, with the exception of the flow at the lowest Reynolds number. In this case the pressure gradient is extremely small (approx. 0.0003 mm of alcohol/cm) and reasonably accurate measurements were not possible. This comparison of the three methods has the advantage that it gives a good indication of the absolute accuracy of measurements of the fluctuating quantities and the correlation coefficient k .

ENERGY EQUATION

Writing equation (1a) in the form

$$\rho u_k \frac{\partial u_i}{\partial x_k} = - \frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i \quad (6)$$

and multiplying it by u_i , the following relation is obtained:

$$\frac{1}{2} \rho \frac{\partial u_i u_i u_k}{\partial x_k} = - \frac{\partial p u_i}{\partial x_i} + \mu u_i \frac{\partial^2 u_j}{\partial x_j \partial x_j}$$

Transforming the last term by partial integration the energy equation of the mean flow becomes

$$\frac{1}{2} \rho \frac{\partial u_i u_i u_k}{\partial x_k} = -\frac{\partial p u_i}{\partial x_i} + \frac{1}{2} \mu \frac{\partial^2 u_i u_i}{\partial x_j \partial x_j} - \mu \left(\frac{\partial u_i}{\partial x_j} \right) \left(\frac{\partial u_i}{\partial x_j} \right) \quad (7)$$

The velocity perturbations can now be introduced:

$$u_1 = u + u'$$

$$u_2 = v'$$

$$u_3 = w'$$

Since the velocities are independent of the coordinates x and z the following equation is obtained after averaging:

$$\rho \overline{u'v'} \frac{du}{dy} + \rho u \frac{d\overline{u'v'}}{dy} + \frac{1}{2} \rho \frac{d}{dy} \overline{v'(u'^2 + v'^2 + w'^2)} = -u \frac{\partial p}{\partial x} - \frac{\partial \overline{v'p}}{\partial y} + \mu \frac{d^2 u}{dy^2} + \frac{1}{2} \mu \frac{d^2}{dy^2} \overline{(u'^2 + v'^2 + w'^2)} - \mu \left(\frac{\partial u_i'}{\partial x_j} \right) \left(\frac{\partial u_i'}{\partial x_j} \right)$$

Making use of equation (3a) this simplifies to

$$\tau \frac{du}{dy} = \frac{d}{dy} \left[\rho v' \left(\frac{u'^2 + v'^2 + w'^2}{2} \right) + v' p \right] + \mu \left(\frac{du}{dy} \right)^2 - \frac{1}{2} \mu \frac{d^2}{dy^2} \overline{(u'^2 + v'^2 + w'^2)} + \mu \left(\frac{\partial u_i'}{\partial x_j} \right) \left(\frac{\partial u_i'}{\partial x_j} \right)$$

This form was obtained by Von Kármán (reference 10) while discussing a nonisotropic flow in terms of the statistical theory of turbulence. In order to see the relative orders of magnitudes of the different terms in the equation, Von Kármán expresses them in terms of a single velocity $q = \sqrt{u'^2 + v'^2 + w'^2}$ and a characteristic length D corresponding to the width of the channel. Since

$$\frac{\tau}{\rho} \approx -\overline{u'v'} = 0(q^2)$$

and

$$\frac{du}{dy} = 0 \left(\frac{\sqrt{\frac{\tau}{\rho}}}{D} \right) = 0 \left(\frac{q}{D} \right)$$

the above equation may be written as

$$A_1 \frac{q^3}{D} = A_2 \frac{q^3}{D} + A_3 \frac{\nu q^2}{D^2} + A_4 \frac{\nu q^2}{D^2} + A_5 \frac{\nu q^2}{\lambda^2}$$

where the coefficients have the characteristics of correlation functions and λ is the microscale of turbulence. For a high Reynolds number flow $qD/\nu \gg 1$, it follows that $\nu q^2/D^2 \ll q^3/D$. Thus the second and third terms on the right side of the equation may be neglected. Since the pertinent quantities have been measured during the present work, the order of magnitude of these terms can be directly evaluated and the omission of the terms is found to be justified. The above equation thus contains only terms of the forms q^3/D and $\nu q^2/\lambda^2$; these terms should be of the same order of magnitude and therefore $q\lambda^2/\nu D = 0(1)$. It is of considerable interest to

see whether experimental results confirm this relation. Taking as an example results from the measurements at $R=30,800$ and $y/d=0.5$ (where $q^3/D \approx 11 \times 10^3$ and $\nu q^2/\lambda^2 \approx 1.7 \times 10^3$)

$$q = 52 \text{ centimeters per second}$$

$$\lambda_y = 0.5 \text{ centimeter}$$

therefore

$$\frac{q\lambda^2}{\nu D} = \frac{52 \times 0.25}{0.155 \times 12.7} = 6.6$$

This ratio seems to be fairly constant for different Reynolds numbers and across the channel cross section.

In view of the above dimensional considerations, the energy equation may be written

$$\tau \frac{du}{dy} = \frac{d}{dy} \left(\rho v' \frac{u'^2 + v'^2 + w'^2}{2} + v' p \right) + \mu \left(\frac{\partial u_i'}{\partial x_j} \right) \left(\frac{\partial u_i'}{\partial x_j} \right) \quad (8)$$

The equation expresses the fact that the energy produced by the turbulent shear forces at a certain point is partly diffused and partly dissipated. This equation has been used in estimating the energy diffusion across the channel, since from the measured quantities the production term and dissipation can be calculated.

EQUIPMENT AND PROCEDURE

WIND TUNNEL

The investigation was carried out in the wind tunnel shown in figure 1. The turbulence level is controlled by a honeycomb and seamless precision screens, followed by an approximately 29:1 contraction. The screens have 18 meshes per inch and a wire diameter of 0.018 inch. The honeycomb consists of paper mailing tubes, 6 inches long and 1 inch in diameter.

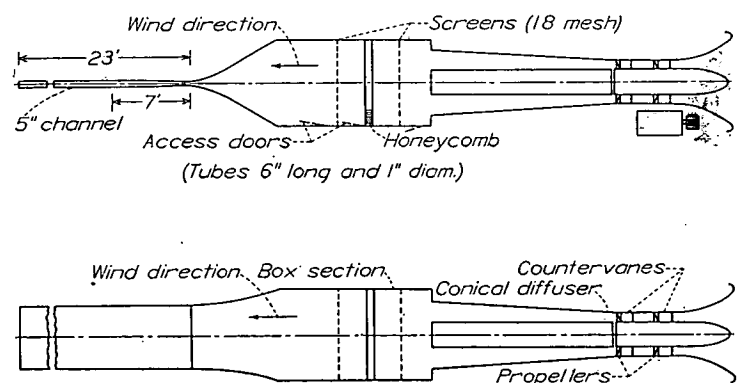


FIGURE 1.—Diagram of two-dimensional tunnel.

The over-all length of the channel is 23 feet. At the entrance section it is 3 inches wide and has an aspect ratio of 20:1. In a distance of 7 feet it expands to a width of 5 inches and the aspect ratio is reduced to 12:1. The walls of the exit portion of the channel (about 6 ft) are made of 3/4-inch-

thick plywood with a $\frac{1}{4}$ -inch birch inside cover. They were specially treated in order to acquire a smooth finish and were reinforced in order to avoid warping (fig. 2); in spite of this a few percent of width variation existed.

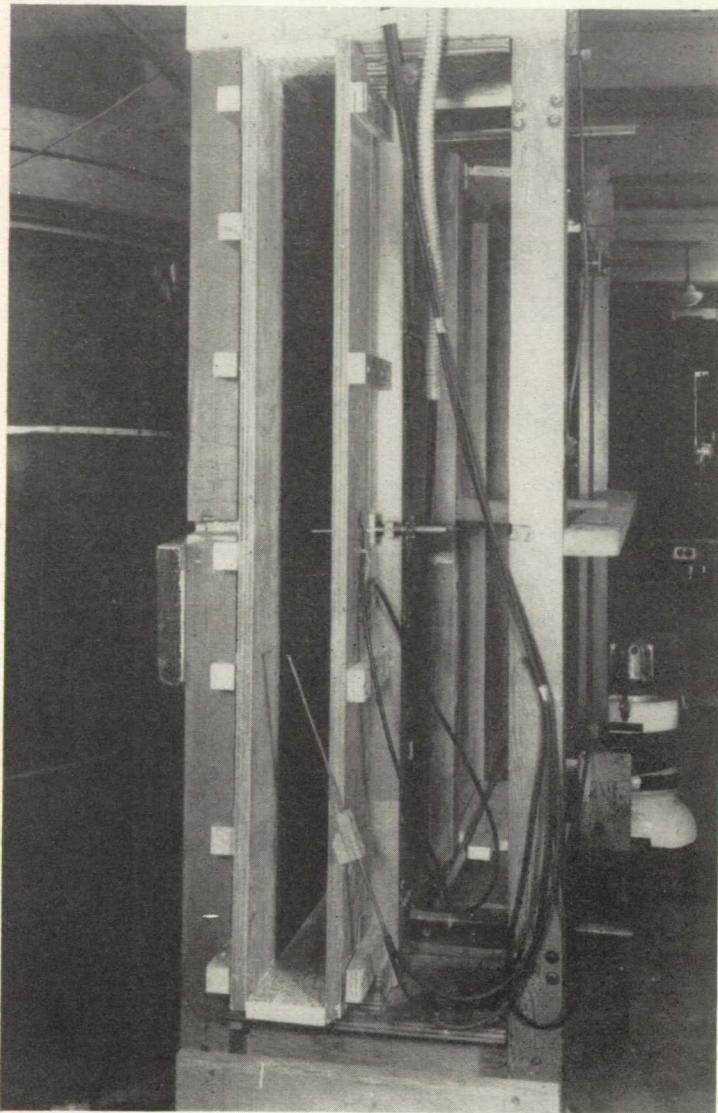


FIGURE 2.—Exit of channel.

The tunnel is operated by a 62-horsepower stationary natural-gas engine—normally operating at a fraction of its rating—which drives two eight-bladed fans. The speed is remotely controlled by means of a small electric motor which drives the throttle through a gear and lead-screw system.

The experiments were carried out at speeds of 3, 7.5, and 15 meters per second.

TRAVERSING MECHANISM

Figure 3 shows the type of traversing mechanism used during the experiments. It consists simply of a micrometer screw on which the hot-wire support is fastened. The support can rotate freely in a plane perpendicular to the air flow; thus, the hot-wire may be adjusted exactly parallel to the wall.

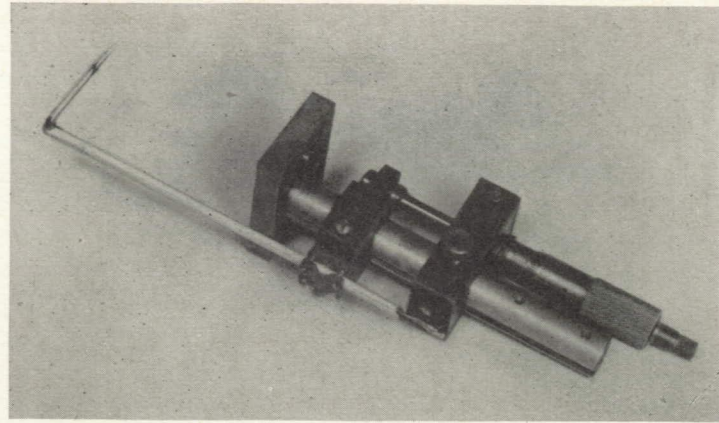


FIGURE 3.—Traversing mechanism.

The zero reading of the traversing mechanism ($y=0$) was carefully found using the following method: The hot-wire was placed close to the wall (approx. 0.025 cm away); the distance between the wire and its image in the polished wall was measured by means of an ocular micrometer. The position of the wire is, of course, one-half of the observed distance and could be determined with an accuracy of ± 0.0005 centimeter.

In order to obtain the pressure distribution along the middle of the channel a 6-inch-long thin-walled tube $\frac{3}{8}$ -inch in diameter was used with a small static-pressure hole drilled close to its end. The tube was free to slide in a support at the entrance of the channel so that the position of the pressure hole relative to the channel exit could be changed. The tube was kept straight and under tension by means of weights.

HOT-WIRE EQUIPMENT

All velocity measurements were made with hot-wire anemometers. The frequency response of the amplifier-compensator unit of the hot-wire equipment, using a 0.00024-inch wire at standard operating conditions, is flat from approximately 2 to 10,000 cycles per second.

The compensation of the hot-wire for thermal lag is accomplished by a capacity network. The range of time constants was chosen from 0 to 1 millisecond corresponding to the characteristics of 0.00005- to 0.00024-inch wires at the operating conditions employed in general at GALCIT. No attempt was made to extend the range of compensation to larger values of M , since in this case the noise level soon becomes appreciable.

The correct setting of the compensating unit was found using the square-wave method described by Kovátszay (reference 11). The time constants of the wires used for turbulence measurements fell between 0.1 and 0.9 millisecond.

The output readings were taken with a thermocouple and precision potentiometer.

Mean-speed measurements.—A 1/2-mil (0.0005-in.) platinum wire of 1-centimeter length was used to measure the mean speed. The measurements were made by the constant-resistance method. This method has the advantage of keeping the wire temperature constant through the velocity field.

Turbulence measurements.—For the investigation of turbulent fluctuations, 0.00024-inch Wollaston wire was used. The wire was soft-soldered to the tips of fine sewing needles after the silver coating had been etched off. The two lateral components v' and w' and the correlation coefficient $R_y = \frac{\overline{u'v'}}{\sqrt{\overline{u'^2}}\sqrt{\overline{v'^2}}}$ were measured using the X-wire technique. The

method was essentially the same as described in detail in previous reports from this laboratory. (See, for example, reference 12.)

The X-meters for shear measurements had angles between the wires of approximately 90° . The angles of the v', w' -meter were of the order of 30° . The wire length of the u' -meter was about 1.5 millimeters and that of the v' -meter was 3 millimeters.

The parallel-wire technique used recently at the Polytechnic Institute of Brooklyn was also tried in order to obtain the lateral component of the velocity fluctuation and the shear close to the wall. The parallel-wire instrument is superior to the X-meter since it allows an exploration of the turbulent field up to distances of a few thousandths of an inch from the wall. However, it was found impossible to obtain reliable values with this instrument. The corrections due to u' -fluctuations and to unequal heating of the wires were pronounced and not easily accountable. The method was therefore temporarily abandoned after considerable time and labor had been spent on it.

Measurements of correlations between two points.—The correlation functions between values of u' at points along the y - and z -axis, that is,

$$R_y = \frac{\overline{u'(0)u'(Y)}}{\sqrt{\overline{u'(0)^2}}\sqrt{\overline{u'(Y)^2}}}$$

$$R_z = \frac{\overline{u'(0)u'(Z)}}{\overline{u'^2}}$$

were measured using the standard technique (reference 12). The scales of turbulence L_y and L_z and the microscales λ_y and λ_z at different points across the channel were obtained from these measurements.

Measurements of λ_z .—A method suggested by Townsend (reference 13) was applied to measure λ_z . Using an electronic differentiation circuit the amplifier output signal was differentiated and λ_z computed from the following relation:

$$\left(\frac{\partial u'}{\partial t}\right)^2 = u^2 \left(\frac{\partial u'}{\partial x}\right)^2 = u^2 \frac{\overline{u'^2}}{\lambda_z^2}$$

The error involved in the approximation $\frac{\partial}{\partial t} \rightarrow u \frac{\partial}{\partial x}$ can be estimated and it may be seen that the above relation holds with reasonable accuracy over the large center portion of the channel.

A new technique for the measurement of λ_z has also been applied. The method is described in detail in reference 14 and consists in counting the zeros of an oscillograph trace

of the u' -fluctuations. From these counts λ_z may be calculated directly by assuming a normal and independent distribution for both u' and $\partial u'/\partial x$:

$$\frac{1}{\lambda_z} = \frac{\pi}{U_o} \times \text{Average number of zeros of } u' \text{ per second}$$

It is known that the distribution of u' is closely a Gaussian one even in nonisotropic turbulence (see, for instance, reference 15); however, for the case of $\partial u'/\partial x$ a small deviation from the normal distribution was found (reference 13). For the preliminary measurements of λ_z reported presently, no corrections were applied as yet for this effect. Figure 4 shows an oscillograph trace of the u' -fluctuation in the middle of the channel at $R=30,800$. The trace represents an interval of approximately 1/20 second.

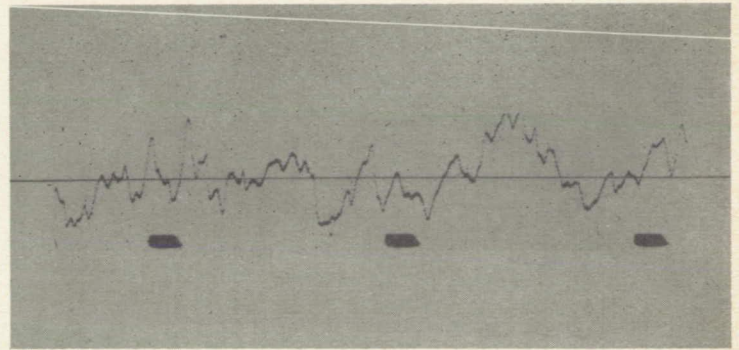


FIGURE 4.—Typical oscillogram of u' -fluctuation at $R=30,800$ and $y/d=1.0$. Horizontal line corresponds to $u'(t)=0$; interval is approximately 1/20 second.

Measurement of L_z .—The following simple procedure² was applied to obtain a rough estimate of scale of turbulence corresponding to correlations between points along the x -axis: Denote by $F_{u'^2}(n)$ the fraction of turbulent intensity which is contributed by frequencies between n and $n+dn$; that is,

$$\overline{u'^2}(n) dn = \overline{u'^2} F_{u'^2}(n) dn$$

and thus

$$\int_0^\infty F_{u'^2}(n) dn = 1$$

Consider now an uncompensated hot-wire. If the time constant of this uncompensated wire is M , the response will be

$$[u'(n)^2]_{uncomp.}$$

where

$$[u'(n)^2]_{uncomp.} = \frac{\overline{u'(n)^2}}{1 + M^2 n^2}$$

The total intensity for the uncompensated wire will then be given by³

$$(\overline{u'^2})_{uncomp.} = \overline{u'^2} \int_0^\infty \frac{F_{u'^2}(n)}{1 + M^2 n^2} dn$$

² This method was suggested by Dr. H. W. Liepmann.

³ Formulas of this general type have been proposed by Kampé de Fériet and by Frenkiel for determining the spectrum of turbulence from uncompensated hot-wire measurements by varying M (reference 16).

Thus, if the ratio of the response of the same wire compensated and uncompensated is denoted by σ ,

$$\frac{1}{\sigma} = \int_0^\infty \frac{F_{u'^2}(n)}{1 + M^2 n^2} dn \quad (9)$$

In order to estimate L_x , $F_{u'^2}(n)$ is assumed to have the simple form

$$F_{u'^2}(n) = \frac{2}{\pi} \frac{L_x/U_o}{1 + n^2 \frac{L_x^2}{U_o^2}}$$

which was given by Dryden (reference 17) and corresponds to an exponential correlation curve.

Then

$$\frac{1}{\sigma} = \frac{2}{\pi} \int_0^\infty \frac{L_x/U_o}{\left(1 + n^2 \frac{L_x^2}{U_o^2}\right)(1 + M^2 n^2)} dn$$

or with

$$\eta = L_x n / U_o$$

$$\alpha = M U_o / L_x$$

then

$$\frac{1}{\sigma} = \frac{2}{\pi} \int_0^\infty \frac{d\eta}{(1 + \eta^2)(1 + \alpha^2 \eta^2)} = \frac{1}{1 + \alpha}$$

Hence

$$L_x = \frac{M U_o}{\sigma - 1}$$

Thus by measuring the ratio of the mean squares of the fluctuating velocities with and without compensation, L_x can be estimated.

The procedure was checked in the flow behind a $\frac{1}{2}$ -inch grid where the lateral scale was measured and where because of isotropy the relation $L_x = 2L_y$ should hold fairly well.

The results were:

$$\sigma = 1.85$$

$$M = 7 \times 10^{-4} \text{ second}$$

$$U_o = 1.5 \times 10^3 \text{ centimeters per second}$$

therefore

$$L_x = 1.24 \text{ centimeters}$$

Direct measurements gave $L_y = 0.553$ centimeter. Hence $L_x = 1.11$ centimeters which shows that this method of estimating the value of L_x is satisfactory.

Measurement of turbulent-energy spectrum.—In order to be able to measure accurately the energy distribution over a wide frequency band, it was found necessary to use extremely thin wires with small time constants and thus to increase the signal-to-noise ratio. Wires of 0.00005-inch diameter, operating with time constants of approximately 2×10^{-4} second, were used. In this way it was possible to detect

energy values in the high-frequency bands that were 10^{-7} times the value corresponding to zero frequency.

The hot-wire signal was fed into a Hewlett-Packard wave analyzer, the output circuit of which was somewhat altered in order to be able to feed the output signal into a root-mean-square voltmeter. The averaging characteristics of the root-mean-square meter were checked and, for a narrow-band-width signal, were found to give the same mean values as those calculated from thermocouple readings.

With this method the energy of the $\overline{u'^2}$ -fluctuation was measured at different points across the channel. At $y/d = 0.4$ the spectrum of the turbulent shear $\overline{u'v'}$ was also obtained.⁴ An X-type meter was used, both wires making an angle of 45° with the free-stream direction. If $\overline{e_1^2}$ and $\overline{e_2^2}$ are the mean-square voltage fluctuations (corrected for time lag) across the two wires it may be shown (reference 12) that

$$\overline{e_1^2} - \overline{e_2^2} = \text{Constant} \times \overline{u'v'}$$

If e_1 and e_2 are fed into the wave analyzer then the above relation holds for all band widths dn ; that is,

$$\overline{e_1(n)^2} - \overline{e_2(n)^2} = \text{Constant} \times \overline{u'v'(n)}$$

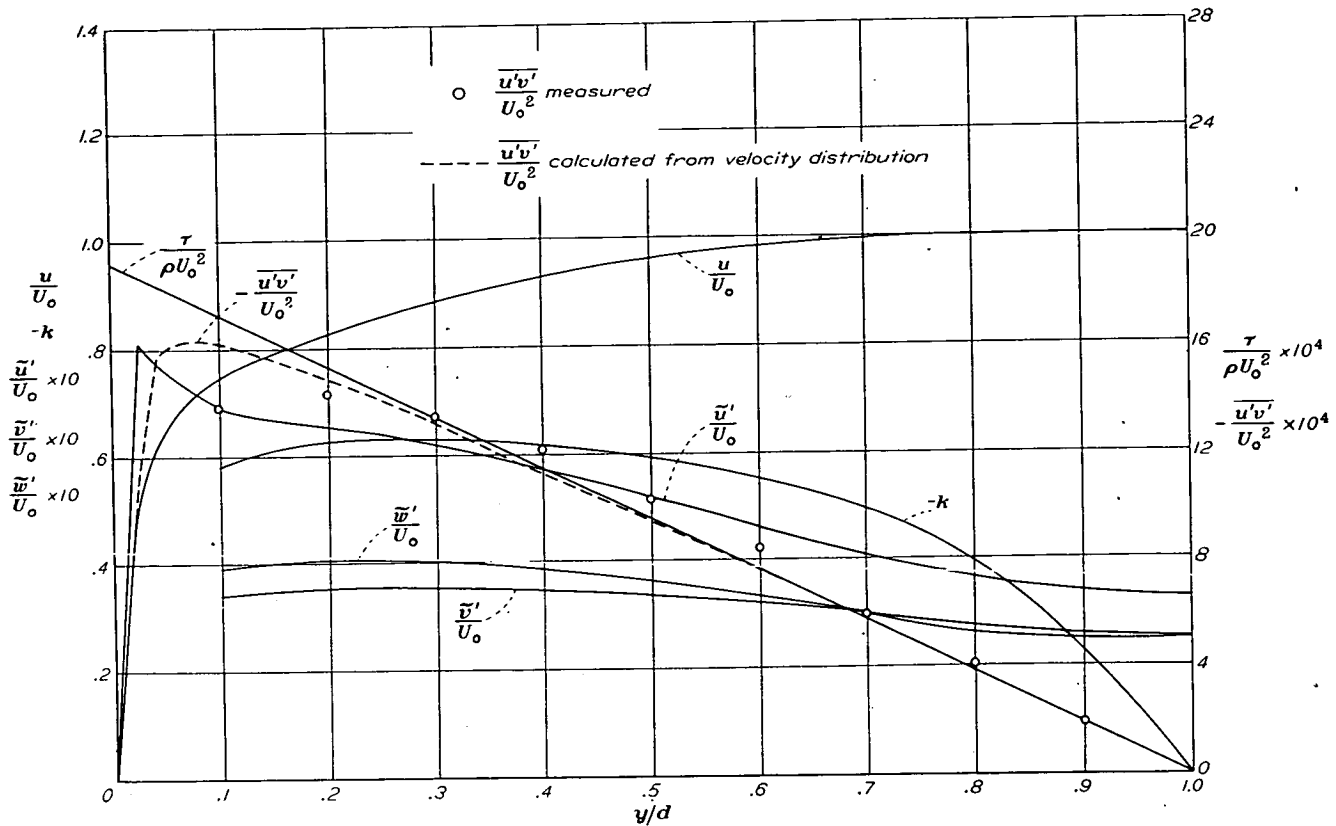
PRELIMINARY MEASUREMENTS IN A 1-INCH CHANNEL

In the initial stages of the turbulent-channel-flow investigation a two-dimensional channel of 1-inch width was used. Measurements of mean and fluctuating velocities and of the correlation coefficient k were completed. The scales L_y and L_z were also measured at the channel center and were found to be about 0.2 to 0.3 centimeter. This small-scale turbulence existing in the channel imposed a definite limitation on the accuracy of the fluctuating measurements. Wire-length corrections as high as 30 percent had to be applied to the measurements of v' and w' . Furthermore, no microscales could be measured accurately; thus one of the objectives of the investigation, the calculation of the energy dissipation across the channel, could not be obtained. Nevertheless, results show good consistency; the three independent measurements of the turbulent shear indicate satisfactory agreement.

It is of interest to present these preliminary measurements, the Reynolds number of which was 12,200, and to compare them with those obtained in the 5-inch channel. Figure 5 shows the distribution of all the measured quantities in the 1-inch channel and it may be directly compared with figure 16 which shows the quantities measured in the 5-inch channel at nearly the same Reynolds number.

Although the slopes of the mean velocities at the wall are almost the same in each case (for the 1-in. channel $\tau_o/\rho U_o^2 = 0.0019$; for the 5-in. channel $\tau_o/\rho U_o^2 = 0.0018$), the velocity ratios u/U_o in the 5-inch channel seem to be lower across most of the channel. This indicates that a more intense

⁴ This method appears to have been first used by Dr. Stanley Corrsin in a turbulent jet (private communication).


 FIGURE 5.—Distribution of measured quantities across two-dimensional channel of 1-inch width. $R=12,200$.

turbulent-energy production $\tau \frac{du}{dy}$ takes place in the 5-inch channel. Indeed, the turbulent-velocity fluctuations $\tilde{u}'/U_0$, $\tilde{v}'/U_0$, and $\tilde{w}'/U_0$ (especially $\tilde{u}'/U_0$) increase at a faster rate toward the wall in the wider channel, although their values at the channel center check very well in the two cases. The shear distribution $\tau/\rho U_0^2$ is almost the same for the two channels; it follows, therefore, from the relation

$$\frac{\overline{u'v'}}{U_0^2} = k \frac{\tilde{u}'}{U_0} \frac{\tilde{v}'}{U_0}$$

that k must have a higher absolute value in the 1-inch channel since both $\tilde{u}'/U_0$ and $\tilde{v}'/U_0$ are smaller there. Indeed the maximum value of k in this case is -0.6 , while in the 5-inch channel it is -0.5 .

As a matter of interest it could be mentioned that if the basis of comparison is not Reynolds number but maximum mean speed (for the 1-in. channel $U_0=15$ m/sec) then figure 5 shows a distribution of $\tilde{u}'$, $\tilde{v}'$, and $\tilde{w}'$ very similar to that of figure 18, the absolute values being somewhat lower in the 5-inch channel.

RESULTS AND DISCUSSION

MEAN-VELOCITY DISTRIBUTION

A careful study of the two-dimensional nature of the channel flow was first made. Mean-velocity measurements were carried out at approximately $x=-2$ inches at different

heights in the channel: At positions 6 inches from the bottom, 6 inches from the top, and at the middle. Agreement among the three sets of measurements confirmed the two-dimensionality. A further check was made on possible end effects that might influence the results. The length of the channel was extended by 6 inches and the mean-velocity measurements were repeated at $x=-8$ inches. No change in the profile was noticed. Figure 6 shows the mean-velocity distributions at three Reynolds numbers, $R=61,600$, $R=30,800$, and $R=12,300$. The distributions follow Von Kármán's logarithmic law very well, except, of course, near the wall and at the center of the channel (fig. 7). The values of U_τ were obtained from the velocity gradients at the wall.

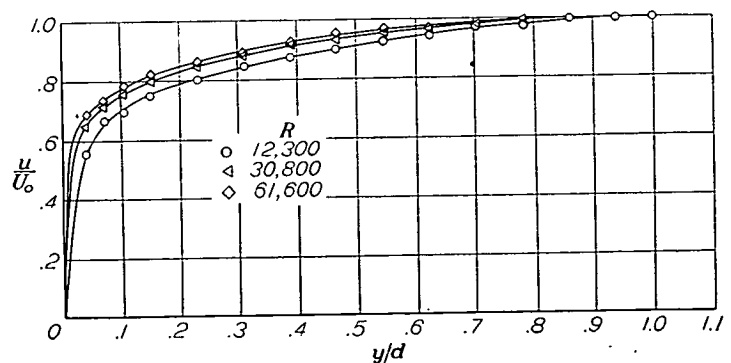


FIGURE 6.—Mean-velocity distribution. Measured points close to wall are not shown.

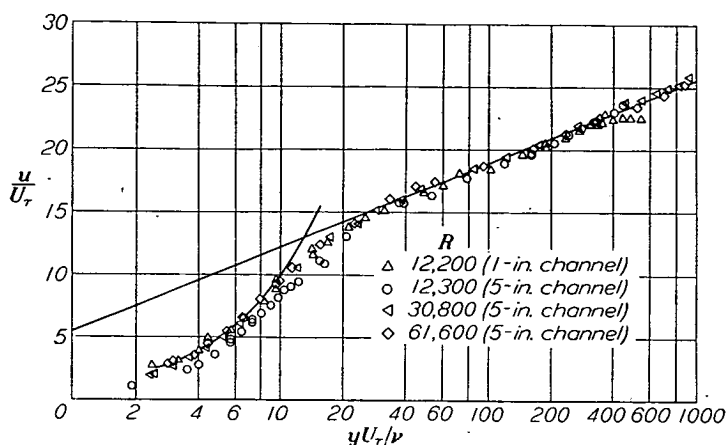


FIGURE 7.—Logarithmic representation of mean-velocity distribution.

Measurements were made with special care close to the wall. Velocities at a large number of points were recorded within the laminar sublayer in order to establish with reasonable accuracy the shape of the velocity profile at the wall (fig. 8). The thickness of the laminar sublayer (the point where the velocity distribution deviates from the logarithmic law) was found to be $\delta \approx 30\nu/U_\tau$ (fig. 7).

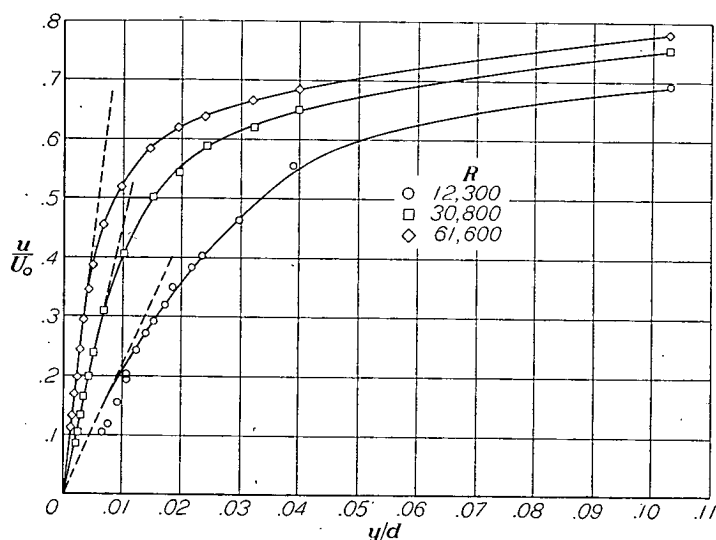


FIGURE 8.—Mean-velocity distribution near wall.

In figure 8 it can be seen that for the case of the lowest Reynolds number a few points near the wall indicate very low velocities. Since the point at which $y/d=0$ has been determined with great accuracy and since the equations of motion of the channel flow require a negative curvature for the velocity profile, it was concluded that the hot-wire indicated too low velocities near the wall for the lowest Reynolds number. (The dashed curve for this case in fig. 8 shows the interpolated velocity distribution for $0 < y/d < 0.01$.) It can be shown that very high local-velocity fluctuations cause the hot-wire to read velocities lower than the true

mean value. Since the mean current of the wire varies with the fourth root of the velocity,

$$\sqrt{u} = \sqrt{\bar{u}} \left[\left(1 + \frac{u'}{\bar{u}} \right)^2 + \left(\frac{v'}{\bar{u}} \right)^2 + \left(\frac{w'}{\bar{u}} \right)^2 \right]^{1/8}$$

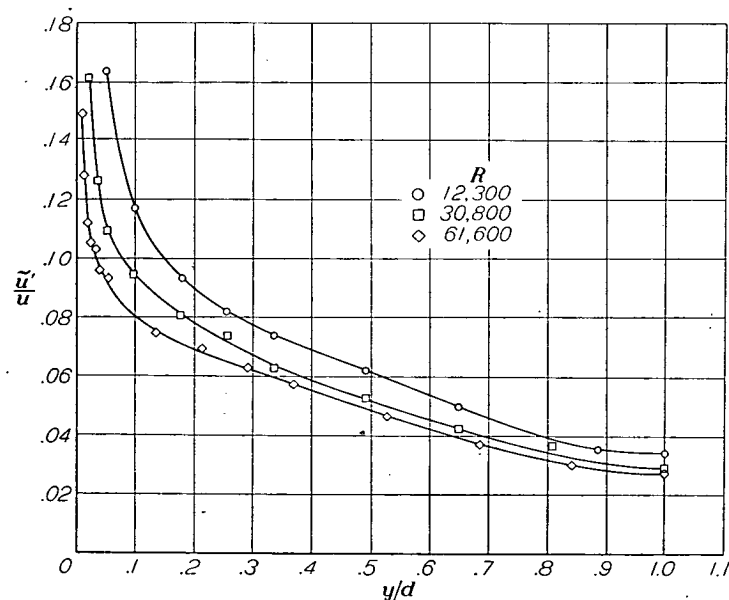
where u is the velocity to which the hot-wire responds and $\bar{u}$ is the true mean velocity. Near the wall $\left(\frac{v'}{\bar{u}} \right)^2 \ll 1$ and can therefore be neglected. Because of the cosine-type directional characteristics of the hot-wire the effect of the w' -fluctuation can also be neglected. The velocity measured by the wire can be written, after expanding the above expression, as

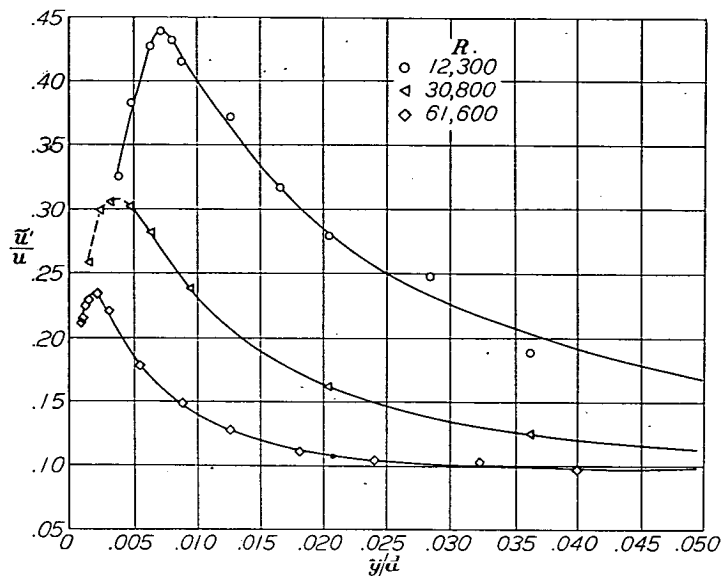
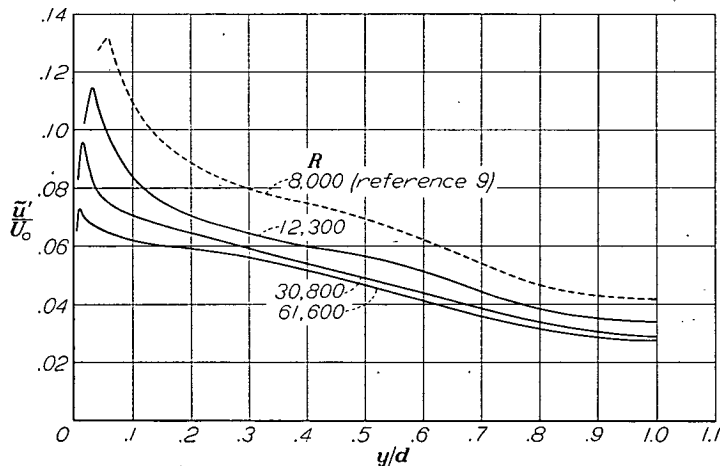
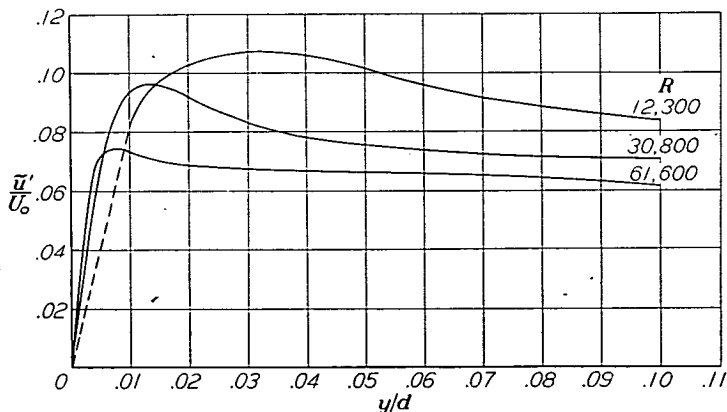
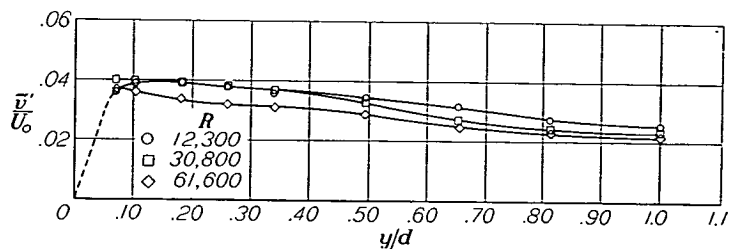
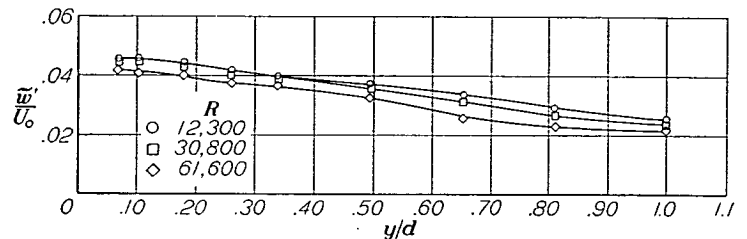
$$u = \bar{u} \left[1 - \frac{3}{16} \left(\frac{u'}{\bar{u}} \right)^2 + \dots \right]$$

Since the average of the odd terms is zero the series converges rapidly. In the region in question, the velocity fluctuations obtained were very high indeed ($\bar{u}'/\bar{u} > 0.30$). Their absolute values, however, are probably even larger, since the nonlinearity effect of the large fluctuations on the hot-wire response still further amplified by the very low mean velocities is not taken into account because the wire-length corrections are neglected. The mean-velocity correction therefore given by the above relation is too small to give the required negative curvature of the profile.

TURBULENCE LEVELS

Figures 9 to 14 represent the results of measurements of the three components $\tilde{u}'$, $\tilde{v}'$, and $\tilde{w}'$ of the turbulent-velocity-fluctuation distribution in the channel for the three Reynolds numbers.

FIGURE 9.—Velocity fluctuations u' relative to local speeds.


 FIGURE 10.—Velocity fluctuations u' relative to local speeds near wall.

 FIGURE 11.—Velocity fluctuations $\tilde{u}'$.

 FIGURE 12.—Velocity fluctuations $\tilde{u}'$ near wall.

 FIGURE 13.—Distribution of velocity fluctuations $\tilde{v}'$.

 FIGURE 14.—Distribution of velocity fluctuations $\tilde{w}'$.

The velocity fluctuations $\tilde{u}'$ relative to local speeds increase very rapidly near the wall as is shown in figures 9 and 10. Measurements very close to the wall indicate that $\tilde{u}'/u$ reaches a maximum within the laminar boundary layer ($yU_0/\nu \approx 17$) and it tends toward a constant value at the wall which is independent of the Reynolds number. This point will be discussed in detail under "Reynolds Number Effect."

The absolute values of the distribution of $\tilde{u}'$ show the same general shape as that obtained by Reichardt (reference 9), having the characteristic maximum near the wall and thus showing the strong action of viscosity even for values of $y \approx 4\delta$.

Using the X-type hot-wire technique for obtaining the velocity-fluctuation components v' and w' , no measurements could be obtained near the wall. Figures 13 and 14 show that, while in the center of the channel the magnitudes of $\tilde{v}'$ and $\tilde{w}'$ are the same, $\tilde{w}'$ increases faster toward the wall. This agrees with the ultramicroscope measurements in a pipe by Fage and Townend (reference 18).

No length corrections were necessary to the measurements of u' except near the wall; however, no corrections were applied in this region since no measurements of λ_z could be made. In this region, furthermore, the fluctuations are very large and the values given in figure 10 must be accepted with reserve. The hot-wire response for large velocity fluctuations is not well-understood yet and no correction was attempted. Length corrections were applied to the measurements of v' and w' .

CORRELATION COEFFICIENT AND SHEAR DISTRIBUTION

The correlation coefficient is fairly constant across most of the channel (fig. 15) as indicated already by Wattendorf

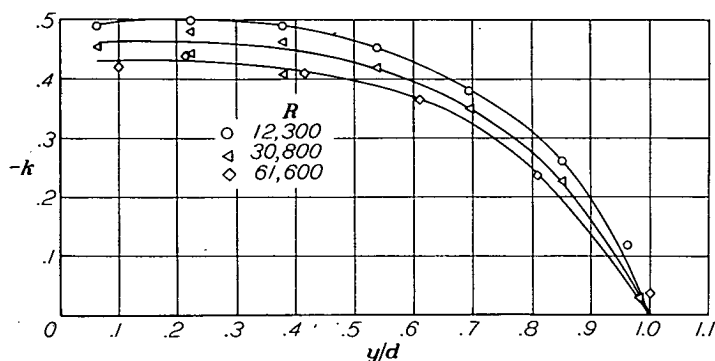


FIGURE 15.—Correlation-coefficient distribution.

and Kueth (reference 6). The maximum values of k obtained decrease slowly with increasing Reynolds number and thus show a definite Reynolds number dependence. Existing results indicate, however, that parameters other than Reynolds numbers also influence the value of k . The following table shows magnitudes of the maximum correlation coefficient as obtained by different investigators working with channels of various widths and with various Reynolds numbers:

Experiments by—	Channel width (cm)	R	k_{max}
Reichardt (reference 9) ----	24.6	8,000	-.45
Laufer (present paper) ----	2.5 (1 in.)	12,200	-.63
Laufer (present paper) ----	12.7 (5 in.)	12,300	-.50
Wattendorf (reference 7) ---	5.0	15,500	-.52
Laufer (present paper) ----	12.7 (5 in.)	30,800	-.45
Laufer (present paper) ----	12.7 (5 in.)	61,600	-.40

Wattendorf obtained his value of k from his measurements of the quantities $\tilde{u}'/U_o$, $\tilde{v}'/U_o$, and $\overline{u'v'}/U_o^2$. Unfortunately his preliminary measurements of $\tilde{v}'$ are incorrect; their magnitude is larger than that of $\tilde{u}'$ at all values of y across the channel contrary to results obtained by Reichardt and by the present writer. Wattendorf himself points out the inaccuracy of the values of $\tilde{v}'$ in his paper. Because of the large values of $\tilde{v}'$ his computed correlation coefficients are too low. The value $k_{max} = -0.52$ listed in the above table was obtained by using Wattendorf's values $\overline{u'v'}/U_o^2 = 0.001$ and $\tilde{u}/U_o = 0.055$ and the value $\tilde{v}'/U_o = 0.035$ ($y/d = 0.5$) obtained by the author for both the 1-inch (2.5-cm) and 5-inch (12.7-cm) channel at $R = 12,200$ and $R = 12,300$, respectively.

The variation of k indicates that for the same Reynolds number the correlation coefficient tends to increase with increasing maximum mean velocity (i. e., in a channel of decreasing width).

Figures 16 to 18 show the distribution of all the measured quantities including the shear distribution. For the case of the two higher Reynolds numbers, three independent measurements were made for the determination of the shear distribution by methods indicated in "Equations of Motion for Two-Dimensional Channel Flow." Figure 19 indicates that consistent results are obtained for the value of τ whether calculated from the pressure gradient along the channel or from the velocity gradient at the wall. The turbulent-shear distributions obtained from hot-wire measurements and those calculated from figure 17 show satisfactory agreement. However, for the flow at the highest Reynolds number, the

hot-wire measurements gave a 20-percent-lower value for the shear coefficient, as may be seen in figure 18.⁵ At present, the writer can give no satisfactory explanation for this discrepancy. In the flow at the lowest Reynolds number, no pressure measurements were made because the very low pressure gradient (approx. 0.0003 mm of alcohol/cm) did not permit accurate measurements. However, in this case the wall shear computed from the measured mean fluctuating quantities $\tilde{u}'$, $\tilde{v}'$, and k and the mean-velocity measurements may be compared with that obtained in the 1-inch-channel flow having nearly the same Reynolds number. The comparison shows satisfactory agreement: For the 1-inch channel $\tau_o/\rho U_o^2 = 0.0019$; for the 5-inch channel $\tau_o/\rho U_o^2 = 0.0018$.

SCALE AND MICROSCALE MEASUREMENTS

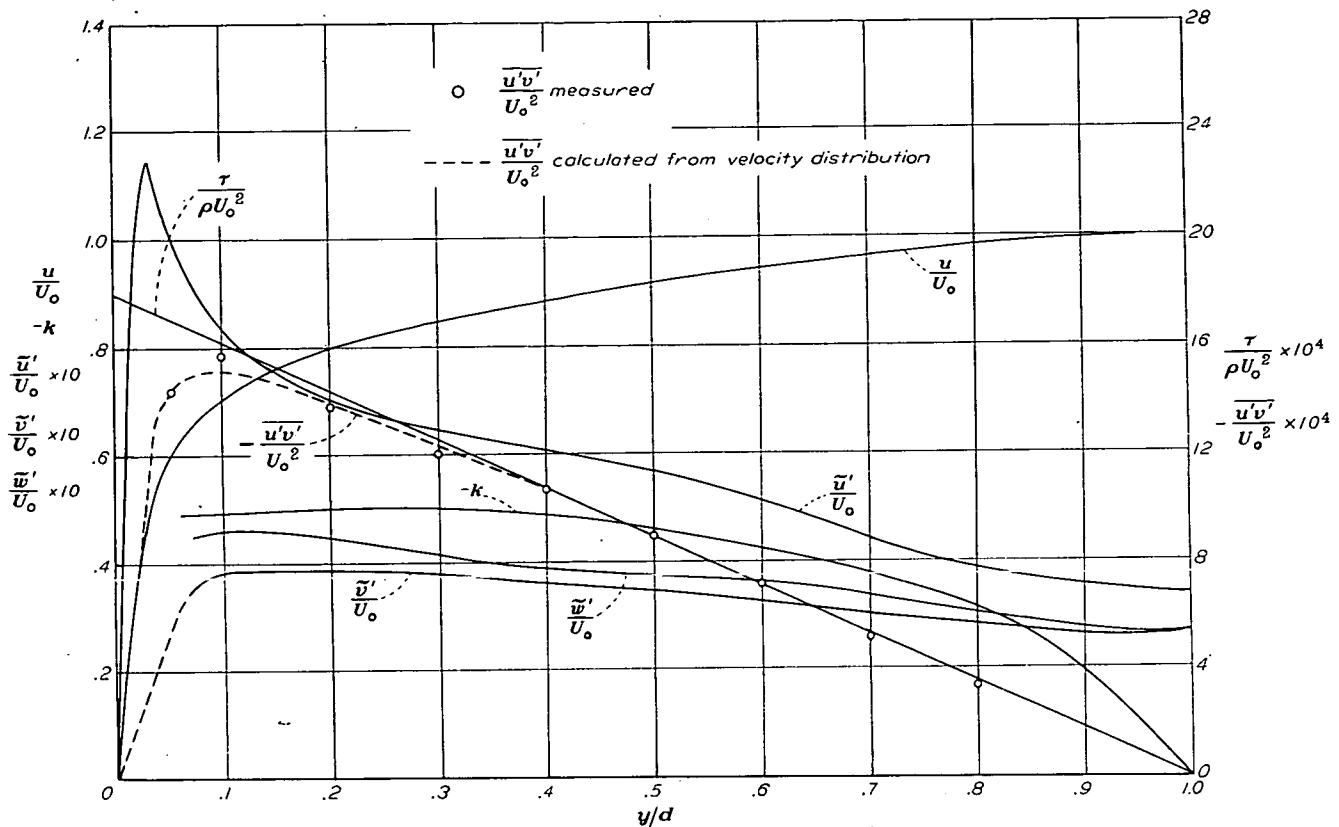
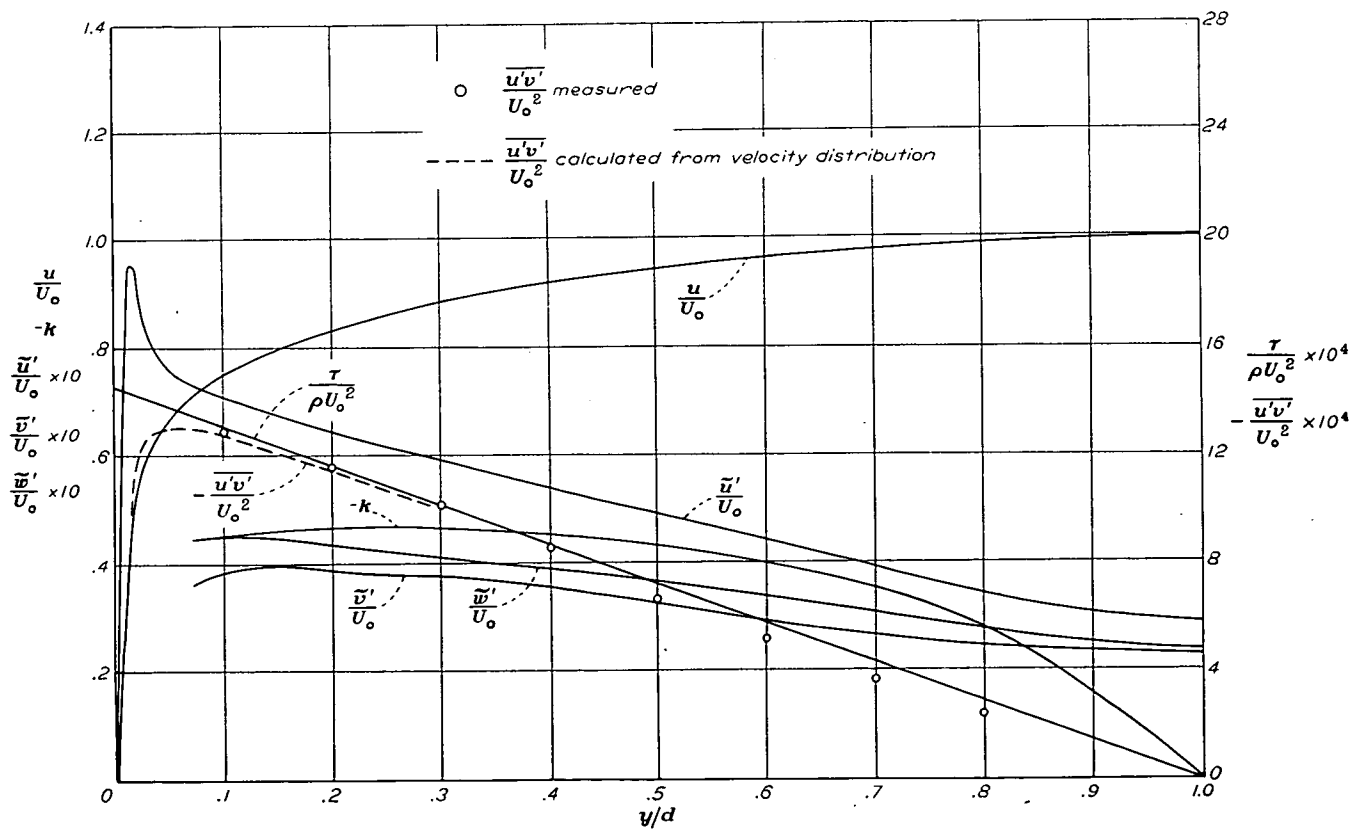
For a further understanding of the structure of the turbulent field, correlations of u' -fluctuations at two different points were carried out. Since the field is not isotropic, the scale and microscale measurements were repeated both in the y - and z -direction for different values of y/d .

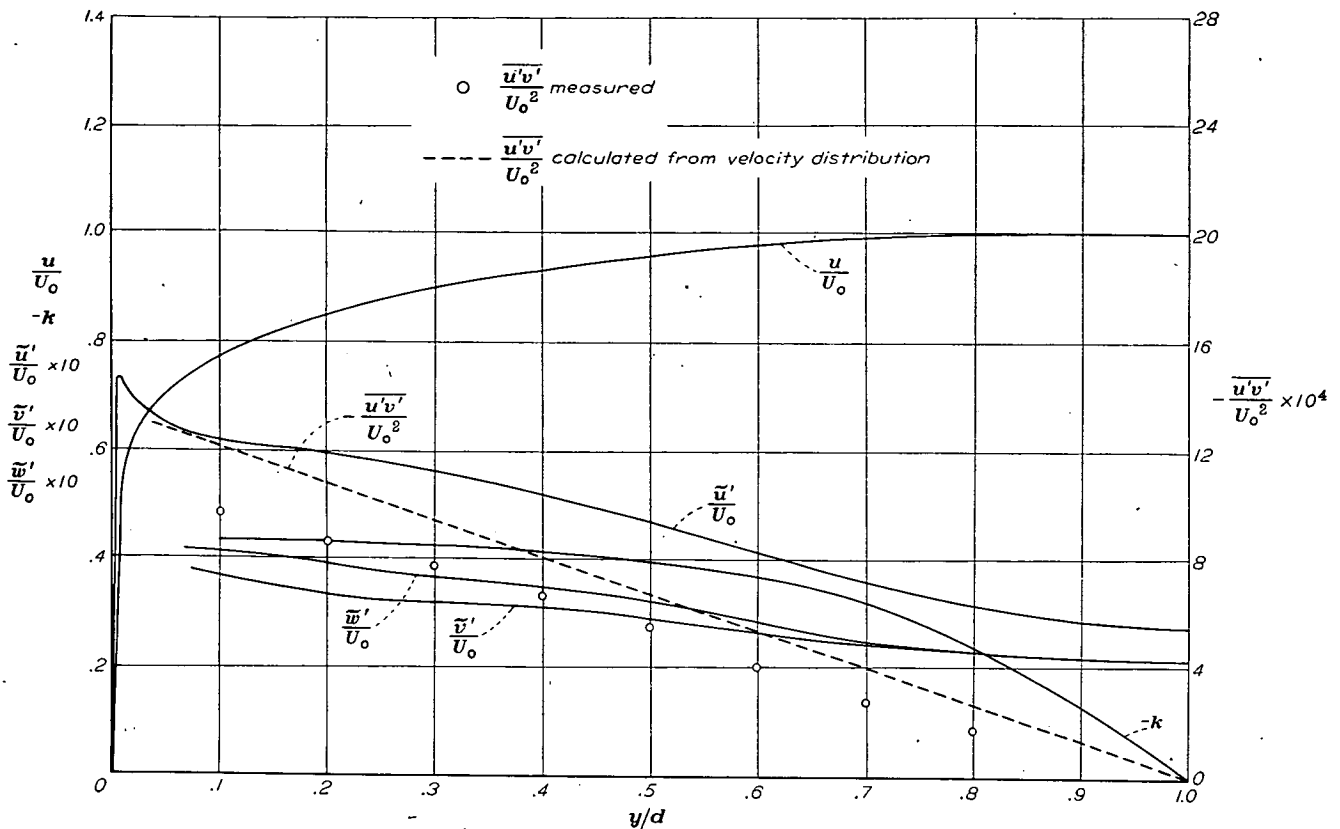
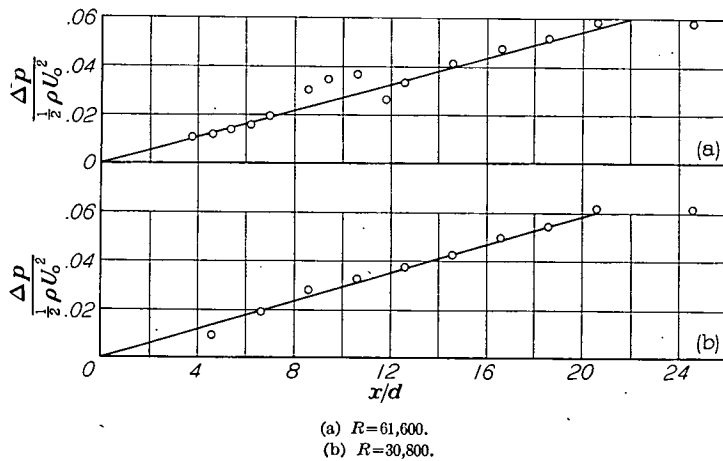
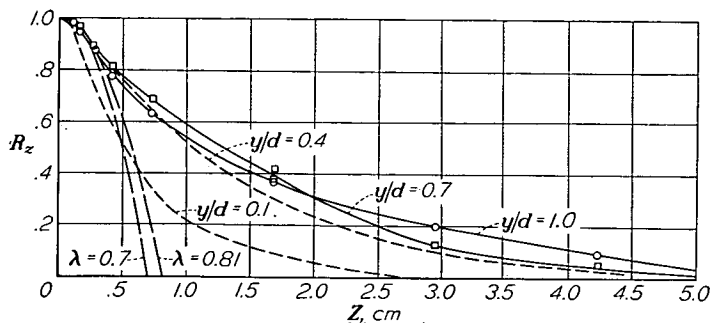
R_z -correlation.—Figure 20 shows typical R_z -correlation distributions at different values of y/d corresponding to $R = 30,800$. For larger values of z , inaccuracies in the measurements did not permit the exploration of the negative region of the correlation distribution. From the measured distributions of R_z at four stations across the channel and for different values of R , the values of L_z and λ_z were calculated (figs. 21 and 22). In these figures L_z is seen to be decreasing uniformly toward the wall, while λ_z reaches a definite maximum at about $y/d = 0.7$ and then decreases with decreasing y/d .

R_y -correlation.—Distributions of the R_y -correlation at a given value of y/d were obtained by fixing the stationary hot-wire at the given value of y/d and traversing with the moving hot-wire away from the fixed one toward the channel center. Values of L_y and λ_y were then calculated from these distributions of R_y . In the region $1.0 > y/d > 0.1$ the gradients of the various quantities are slight; no significant asymmetry in the function R_y is therefore expected. Measurements of the top part of the R_y -correlation distribution ($1.0 > R_y > 0.8$) by Prandtl and Reichardt (reference 19) justify this belief fairly well.

By comparing figure 20 with figure 23, the strong effect of the existing nonisotropy can immediately be seen. Although both R_y and R_z correspond to Von Kármán's g -function (reference 10) they exhibit different behaviors across the channel. The R_z -correlation function falls more and more rapidly to zero with decreasing y/d . Traversing from $y/d = 1.0$ to $y/d = 0.70$ the distribution of R_y is found to behave similarly; however, the dashed curve measured at $y/d = 0.4$ (fig. 23) indicates considerably increased correlations for larger values of Y . It is thus seen that around $y/d = 0.7$ there is a definite decrease in energy content of the fluctuations having low frequencies. Further consideration of this fact is given later in the discussion of the energy balance in the channel.

⁵ This result was accepted after the absolute values were carefully checked; the v' -wire response was compared with u' -fluctuations in an isotropic field and the two-dimensional character of the flow was checked.


 FIGURE 16.—Distribution of mean and fluctuating quantities. $R=12,300$.

 FIGURE 17.—Distribution of mean and fluctuating quantities. $R=30,800$.

FIGURE 18.—Distribution of mean and fluctuating quantities. $R=61,600$.FIGURE 19.—Pressure distribution along channel. Solid lines indicate pressure gradient obtained from measured du/dy .FIGURE 20.— R_z -correlation. $R=30,800$. At $y/d=1.0$, $L_z=1.65$ centimeters, $\lambda_z=0.70$ centimeter; at $y/d=0.7$, $L_z=1.55$ centimeters, $\lambda_z=0.81$ centimeter.

The values of $L_y = \int_0^\infty R_y dY$ (fig. 21) are not so reliable as those of L_z since R_y could not be measured at sufficiently large values of Y because of the limitation of the traversing mechanism used.

The distribution of λ_y across the channel is similar to that of λ_z (fig. 22). From $y/d=1.0$ to $y/d \approx 0.7$ both λ_y and λ_z increase almost proportionally to u' , indicating that the turbulent-energy dissipation $W \propto u'^2/\lambda^2$ is approximately constant in this region. It should be noticed, however, that λ_z is consistently larger than λ_y throughout the channel cross section. Some measurements of λ_y close to the wall are also indicated in figure 23. No length correction was found to be necessary for the measured values of λ_y and λ_z .

A rough estimate of L_x by the method already described gave a value approximately twice that of L_z at the center of the channel (fig. 21). Its value increases to a maximum at $y/d \approx 0.5$ and then decreases rapidly. No values for L_x are given for the lowest Reynolds number; in this case the value of σ is very close to unity and, since L_x is proportional to $\frac{1}{\sigma-1}$, σ should be known within an accuracy of 1 percent to give consistent results for L_x . Unfortunately measurements of L_x cannot be made with this accuracy, particularly when the u' -fluctuations are of rather low frequencies as is the case for $R=12,300$. It should be mentioned that the accuracy of the determination of L_x is more limited by the inaccurately measured value of σ than by the fact that an approximate spectrum function $F_{u'^2}(n)$ is used in equation (9).

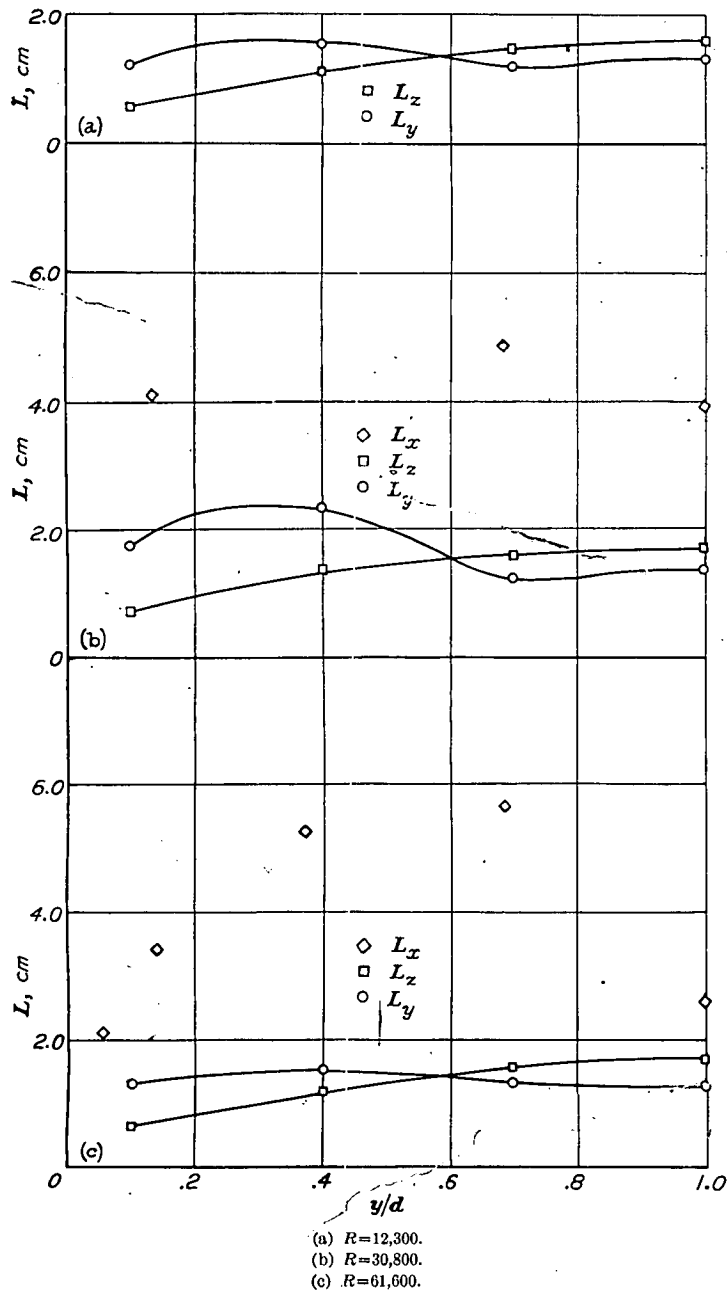


FIGURE 21.—Scale of turbulence distribution.

The distribution of λ_z obtained by the differentiation method shows the same behavior as that of λ_y and λ_x (fig. 22). It has the characteristic maximum at $y/d \approx 0.7$. No measurements were made for $R=12,300$ because of the inaccuracy due to the large noise-to-signal ratios.

As a matter of interest the results of some measurements of λ_z obtained with the zero-counting method are compared with those described above. (See fig. 22.) Since the validity of the assumptions involved in this method is not clarified yet, these measurements should be regarded as preliminary.

SPECTRUM MEASUREMENTS

The spectra of the velocity fluctuations $\overline{u'^2}$ have been obtained at various values of y/d across the channel. With

improved instrumentation it was possible to reduce experimental scatter by a considerable amount. The accuracy of the present measurements is believed to be within ± 10 percent with the exception of values corresponding to low frequencies ($n < 100$ cps) because of the large-amplitude fluctuations and with the exception of values corresponding to high frequencies ($n > 4000$ cps) because of noise and of possible wire-length effect. A typical spectrum distribution taken at $y/d=1.00$ is shown in figure 24. (The measured spectra at various positions of y/d are given in table I.) No attempt is made to compare the measured distributions with existing theories since the restrictive assumptions of these theories (isotropy and flows at high Reynolds numbers) are not satisfied in the present experiments. The only frequency range where comparison is possibly justified is the viscous region, that is, the high-frequency part of the spectrum where viscosity plays a dominant role. Unfortunately here the accuracy of the measurements is not good enough to afford any definite conclusions. Therefore, the rather close agreement of the measured spectrum (for all values of y/d except in the laminar sublayer) with the n^{-7} -law predicted by Heisenberg (reference 2) should be accepted with reserve (fig. 24).

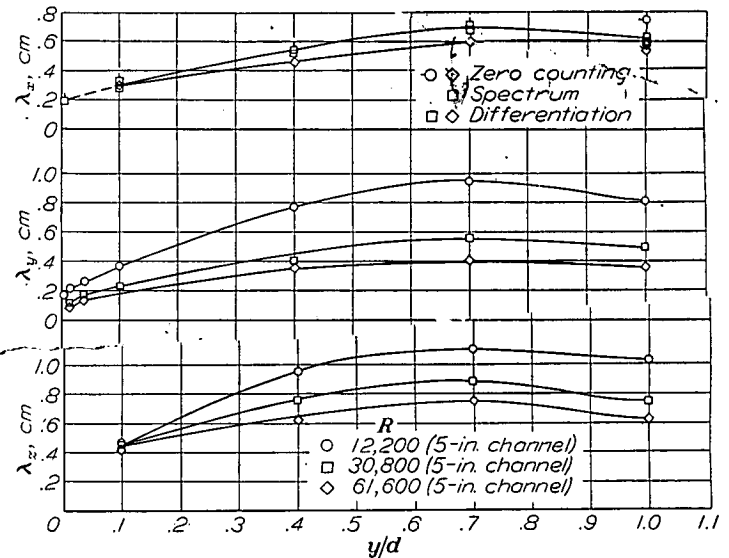
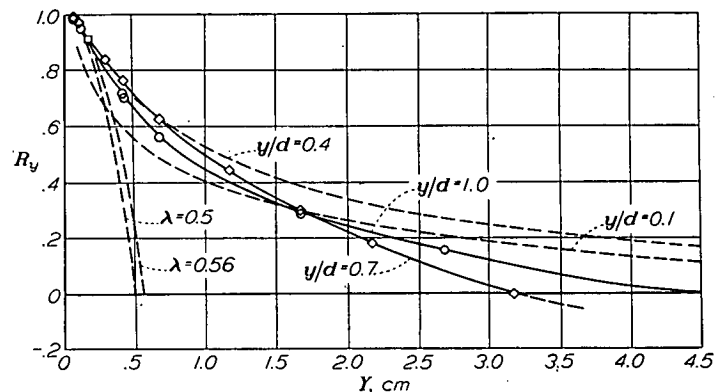


FIGURE 22.—Microscale measurements.


 FIGURE 23.— R_y -correlation. $R=30,800$. At $y/d=1.0$, $L_y=1.3$ centimeters, $\lambda_y=0.50$ centimeter; at $y/d=0.7$, $L_y=1.2$ centimeters, $\lambda_y=0.56$ centimeter.

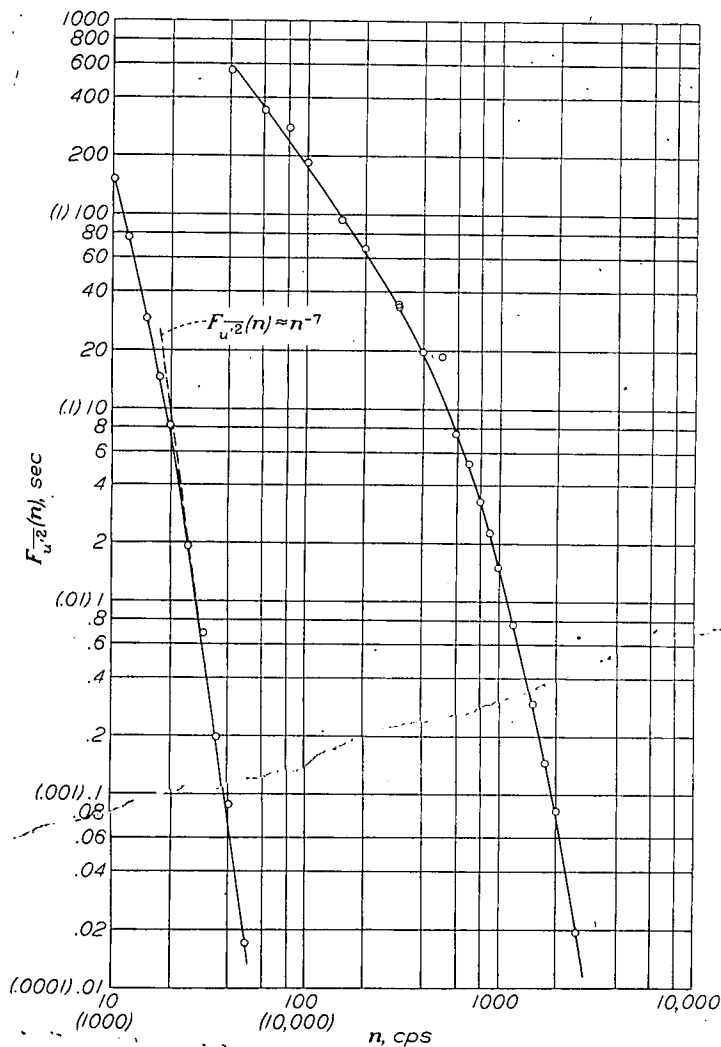


FIGURE 24.—Frequency spectrum of $\bar{u}'^2$ at channel center. Scale values within parentheses correspond to curve on left.

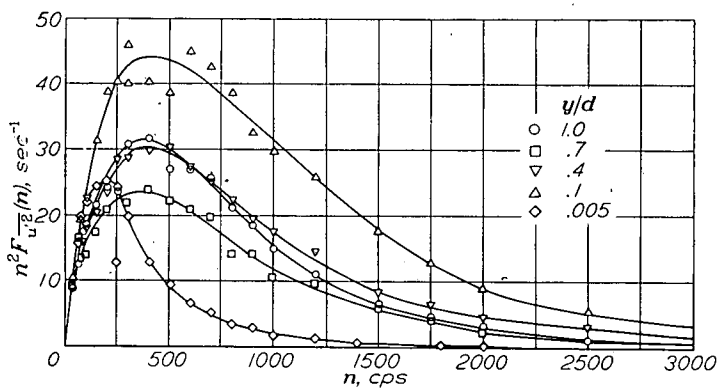


FIGURE 25.—Spectrum distributions across channel. $R=30,800$.

For a comparison of the energy spectra at different points in the channel, figure 25 shows the distribution of $n^2 F_{u'^2}(n)$ instead of that of $F_{u'^2}(n)$ as a function of the frequency. This method of presentation emphasizes energies corresponding to the higher frequencies but gives a clearer over-all comparison of the different sets of measurements. The characteristic maximum of the distribution of λ_x immediately becomes apparent in this figure. From the fact that the

TABLE I.—MEASURED VALUES OF THE SPECTRUM OF $\bar{u}'^2$

Frequency (cps)	$F_{u'^2}(n)$ (sec)				
	$y/d=1.0$	$y/d=0.7$	$y/d=0.4$	$y/d=0.1$	$y/d=0.005$
30					772×10^{-5}
40	561×10^{-5}	585×10^{-5}	571×10^{-5}	664×10^{-5}	786
50					618
60	349	464			604
70					519
80	273	211	252	308	436
90					373
100	186	141	182.5	226	345
125					309
150	95.4	77.3	91.6	140	254
175					272
200	68.2	52.6	56.8	97.2	222
250	23.1	37.9	45.8	64.9	157
300	33.9	23.5	32.2	44.8	150
400	34.6	25.6	34.7	51.3	109
500	19.8	15.0	19.17	25.2	114
600	18.9	8.09	12.25	15.5	86.5
700	7.57	5.86	7.6	12.5	74.9
800	5.27	4.0	5.14	8.73	63.6
900	3.31	2.67	3.53	6.07	63.6
1000	2.29	1.77	2.51	4.04	38.6
1200	1.50	1.08	1.76	2.99	38.0
1500	.764	.677	1.01	1.80	21.5
1750	.295	.257	.379	.787	8.03
2000	.147	.16	.221	.422	3.74
2500	.0832	.0584	.109	.224	1.83
3000	.0192	.0186	.0487	.0873	1.06
3500	.00682	.00668	.0141	.0313	.531
4000	.00196	.00272	.00492	.00121	.352
4500	.000855	.00074	.0021	.00435	.185
5000		.000608	.00157	.0029	.090
5500	.000170	0	.00052	.00117	.016
6000	0		.000252	.00062	.008
7000				.000313	.0039
				.000109	.00102

correlation coefficient is a Fourier transform of the spectrum function it follows that

$$-\left(\frac{\partial^2 R_x}{\partial X^2}\right)_{x=0} = \frac{1}{\lambda_x} = \frac{4\pi^2}{u^2} \int_0^\infty n^2 F_{u'^2}(n) dn$$

Thus λ_x can be computed from the distributions of $n^2 F_{u'^2}(n)$. The calculated values check within a few per cent with direct measurements (fig. 22).

REYNOLDS NUMBER EFFECT

Figure 7 shows the distribution of mean velocities plotted in the form of "friction velocity" u/U_τ against "friction-distance parameter" yU_τ/ν . In this form the profiles are independent of the Reynolds number $U_\tau d/\nu$ and follow Von Kármán's logarithmic law:

$$\frac{u}{U_\tau} = A \log \frac{yU_\tau}{\nu} + B$$

The constants A and B are 6.9 and 5.5, respectively. Comparing the present measurements with previous ones it is to be noted that because of the very smooth wall surfaces used, the value B is larger than that of other investigators. Making allowances for this effect, measurements are in fairly good agreement with those of Dönch (reference 4) and Reichardt (reference 9). Nikuradse's channel results (reference 5) differ from all other existing experiments. It is pertinent to mention that for the Reynolds number range in question ($R < 100,000$) the slope of the logarithmic mean-

velocity distribution (the constant A) both in pipes and channels is larger than that established by Nikuradse's pipe measurements ($A=5.75$).

The Reynolds number has a definite influence on the velocity fluctuations also. Along most of the cross section where the influence of the viscosity is negligible the fluctuations $\tilde{u}'$, $\tilde{v}'$, and $\tilde{w}'$ decrease slowly with increasing Reynolds number.

Very close to the wall, according to Prandtl's hypothesis, all velocities of the form $\text{Velocity}/U_\tau$ must be a function of the friction-distance parameter only. It has already been pointed out that the mean velocity obeys this similarity law. Figure 26 shows the distribution of $\tilde{u}'/U_\tau$ as a function

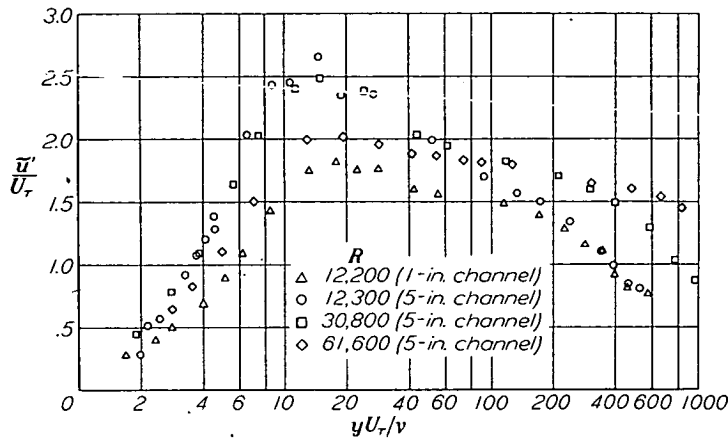


FIGURE 26.—Logarithmic representation of velocity fluctuations u' .

of yU_τ/ν for the different flow conditions investigated. The following remarks can be made with reference to this figure:

(1) For values of $yU_\tau/\nu < 100$, values of $\tilde{u}'/U_\tau$ corresponding to the various cases indicate similar behavior. They reach a maximum at about $yU_\tau/\nu \approx 17$.

(2) It is believed that the similarity is actually more complete than indicated in figure 26. Because the microscale is not known in regions very close to the wall no length corrections could be applied. These corrections would of course be appreciably higher for the two high-velocity flows ($R=61,600$, $2d=5$ in. and $R=12,200$, $2d=1$ in.) and would therefore bring the various distributions of $\tilde{u}'/U_\tau$ closer together in the region in question.

(3) The effect of viscosity is more pronounced on the fluctuating quantities than on the mean velocity.

(4) Taylor pointed out in 1932 (supporting his arguments by Fage and Townend's ultramicroscope measurements) that $\tilde{u}'/u$ and $\tilde{w}'/u$ approach a finite value at the wall (reference 18). It follows from the similarity law that this value should be an absolute constant independent of the Reynolds number. Figure 10 indicates this to be true, the constant being $(\tilde{u}'/u)_{y=0} \approx 0.18$.

It is of considerable interest to discuss the variation of the scale and microscale with Reynolds number. For flows behind grids where the turbulence is isotropic the scale is independent of the mean velocity and depends on the mesh size of the grid. Similar behavior was found for the channel flow. Figure 21 shows the distributions of L_y and L_z for different velocities. These distributions indicate no con-

sistent variation with velocity. Furthermore, measurements in a 1-inch channel give a value for L_z five times lower and a ratio for L_y somewhat larger than the values obtained in the present investigation.

The variation of λ depends, of course, on the velocity and channel width. The values of λ decrease with increasing velocity; however, the variation of λ with channel width is less than that of L .

FULLY DEVELOPED CHARACTER OF TURBULENCE

The flow in the channel is called fully developed if the variations of the mean values of the velocity and the mean squares of the velocity fluctuations with x are very small. That the mean velocity profile does not vary downstream is evident from the pressure-gradient measurements (fig. 19). The gradient in x of $\overline{u'^2}$ was measured on the axis of the channel. It was found that $\overline{u'^2}$ was indeed decreasing with x . The gradient, however, was very small as compared with $\frac{1}{\rho} \frac{\partial p}{\partial x}$:

$$\frac{\partial \overline{u'^2}}{\partial x} \approx 0.01 \frac{1}{\rho} \frac{\partial p}{\partial x}$$

Hence for all practical purposes $\partial \overline{u'^2}/\partial x$ can be neglected. No measurements have been made concerning $\partial \overline{u'v'}/\partial x$, since the scatter in the values would cover any effect. However, there is little doubt that $\partial \overline{u'v'}/\partial x$ is of the same order as $\partial \overline{u'^2}/\partial x$ and that the use of equations (3a) and (3b) is therefore justified here.

ENERGY BALANCE IN FLUCTUATING FIELD

The energy equation for a two-dimensional channel has the form given by equation (8) and is valid throughout the cross section of the channel with the exception of a small region near the wall. The term $\tau \frac{du}{dy}$ on the left side of equation (8) corresponds to the energy produced by the shearing stresses and it can be obtained directly from the measurements of τ and from the mean velocity profile. The second term on the right $\mu \left(\frac{\partial u_i'}{\partial x_j} \right) \left(\frac{\partial u_i'}{\partial x_j} \right)$ expresses the amount of energy that is being dissipated because of the breaking down of the larger eddies to smaller ones. The term may be written explicitly

$$W = \mu \left[\left(\frac{\partial u'}{\partial x} \right)^2 + \left(\frac{\partial u'}{\partial y} \right)^2 + \left(\frac{\partial u'}{\partial z} \right)^2 + \left(\frac{\partial v'}{\partial x} \right)^2 + \left(\frac{\partial v'}{\partial y} \right)^2 + \left(\frac{\partial v'}{\partial z} \right)^2 + \left(\frac{\partial w'}{\partial x} \right)^2 + \left(\frac{\partial w'}{\partial y} \right)^2 + \left(\frac{\partial w'}{\partial z} \right)^2 \right]$$

The problem is to express these functions in terms of easily measurable quantities. In the case of isotropic turbulence Taylor solved the problem by introducing the microscale of turbulence λ and obtained for the dissipation

$$W = 15\mu \frac{\overline{u'^2}}{\lambda^2} \quad (10)$$

It is attempted here to carry over his analysis to the case of channel flow. The following assumptions have to be made in this connection:

(a) The gradient of the velocity fluctuations is small in all three directions. With the exception of the region near the wall this is justifiable from the measurements.

(b) Since only correlations of u' have been measured, it is assumed that the correlations of w' as functions of Z (Von Kármán's f -function) have the same curvature at the origin ($Y=Z=0$) as the correlation of u' has as a function of X ; that is,

$$\left(\frac{\partial^2 R_z^{w'}}{\partial Z^2}\right)_0 = \left(\frac{\partial^2 R_x^{u'}}{\partial X^2}\right)_0 = -\frac{1}{\lambda_x^2}$$

Furthermore the correlations of v' as functions of X and Z and the correlations of w' as functions of X (Von Kármán's g -functions) have the same curvature at the origin as the correlation of u' has as a function of Z ; that is,

$$\left(\frac{\partial^2 R_x^{v'}}{\partial X^2}\right)_0 = \left(\frac{\partial^2 R_z^{v'}}{\partial Z^2}\right)_0 = \left(\frac{\partial^2 R_x^{w'}}{\partial X^2}\right)_0 = \left(\frac{\partial^2 R_z^{u'}}{\partial Z^2}\right)_0 = -\frac{2}{\lambda_z^2}$$

Finally,

$$\left(\frac{\partial^2 R_y^{v'}}{\partial Y^2}\right)_0 = \left(\frac{\partial^2 R_y^{w'}}{\partial Y^2}\right)_0 = \left(\frac{\partial^2 R_y^{u'}}{\partial Y^2}\right)_0 = -\frac{2}{\lambda_y^2}$$

With these assumptions the derivatives of the fluctuations could be expressed in terms of the measured values of λ_x , λ_y , and λ_z :

$$W = \mu \left[\frac{\overline{u'^2}}{\lambda_x^2} + \frac{2\overline{u'^2}}{\lambda_y^2} + \frac{2\overline{u'^2}}{\lambda_z^2} + \frac{2\overline{v'^2}}{\lambda_x^2} + \frac{2\overline{v'^2}}{\lambda_y^2} + \frac{2\overline{v'^2}}{\lambda_z^2} + \frac{2\overline{w'^2}}{\lambda_x^2} + \frac{2\overline{w'^2}}{\lambda_y^2} + \frac{\overline{w'^2}}{\lambda_z^2} \right]$$

At the middle of the channel W turned out to be 2.62 ergs per cubic centimeter per second for $R=30,800$. Using the isotropic relation

$$W = 15\mu \frac{\overline{u'^2}}{\lambda_y^2} = 15\mu \frac{\overline{u'^2}}{\lambda_x^2} = 15\mu \frac{\overline{u'^2}}{\lambda_z^2}$$

the values 2.88, 4.38, and 2.24 ergs per cubic centimeter per second were obtained depending upon whether λ_x , λ_y , or λ_z was used. (For the value of $\overline{u'^2}$ the algebraic mean of the squares of the fluctuations was used.)

Figure 27 shows the distribution of W in the center region

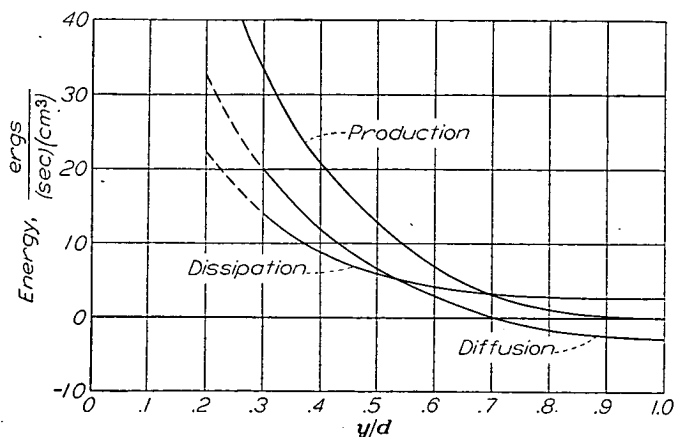


FIGURE 27.—Turbulent-energy balance in center region of channel. $R=30,800$.

of the channel. Taylor obtained a similar distribution of the dissipated energy across the channel (reference 20); however, his numerical magnitudes are too high since he substituted in the isotropic relation, in equation (10), the values of λ_y , which turn out to be less than those of λ_x and λ_z .

From the known distributions of the energy production and dissipation the diffusion of energy is easily calculated from equation (8). It should be pointed out that because of the approximations involved in the calculation of the dissipation and diffusion terms, the conclusions derived from them are more or less of a qualitative nature. It is seen from figure 27 that at $y/d \approx 0.7$ the diffusion term is zero. It was also pointed out earlier that in this region the R_y -correlations show a considerable change in shape indicating a shift in energy from the lower to higher frequencies of the velocity fluctuations. These two facts suggest that the energy diffusion is associated mainly with the low frequencies of the fluctuations.

The equation expressing the balance of the three forms of energy furnishes the following picture of the turbulent flow field in the channel where viscous dissipation is still negligible: Two planes passing through points where the diffusion of energy vanishes divide the channel flow into three parts. The middle region, the width of which is of the order of L_x , receives most of its energy by diffusive action and this energy is dissipated here at a constant rate. In the two outside regions all three energy terms increase rapidly, the production term being the dominant one, and their interaction is more involved.

This picture of the flow field is only of a descriptive nature. The purpose of further investigations should be to obtain information on the development and mechanism of such an energy balance.

LOCALLY ISOTROPIC CHARACTER OF TURBULENT CHANNEL FLOW

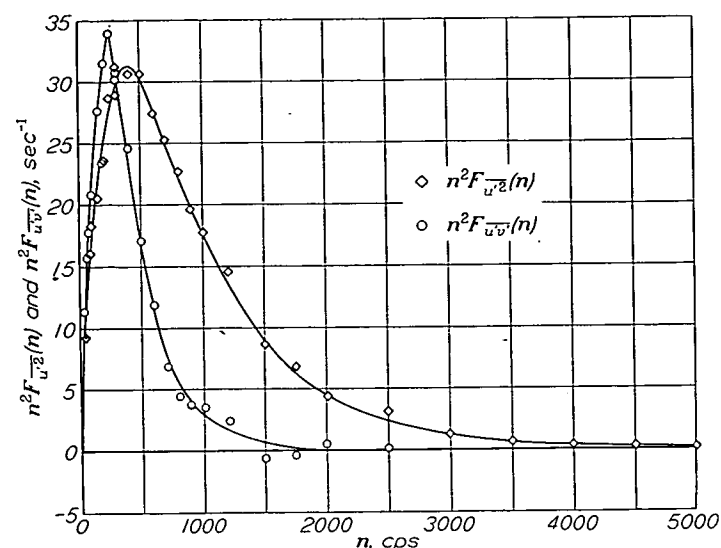
The concept of locally isotropic turbulence obtained by Kolmogoroff requires the smaller eddies in turbulent flow to approach isotropy. Smaller eddies are the ones with length dimension l small compared with the scale of turbulence L . The smallest characteristic length in turbulent motion is Kolmogoroff's η defined as

$$\eta = \left(\frac{\nu^3}{\epsilon} \right)^{1/4}$$

where ϵ is the total dissipated energy. Clearly, to approach locally isotropic conditions it is necessary that

$$L \gg \eta$$

It follows from this hypothesis that beyond sufficiently high frequencies (of the order of $\tilde{u}'/\eta$, say) no correlations exist between the components of the velocity fluctuations. Figure 28 shows the measured spectrum of $\overline{u'^2}$ as compared with the spectrum of $\overline{u'v'}$ at $y/d=0.4$. It is seen that the shear spectrum tends to zero at a frequency of about 1500 cycles per second while $\overline{u'^2}(n)$ still has a detectable value at 5000 cycles per second. This result verifies Kolmogoroff's assumption. It should be mentioned that the


 FIGURE 28.—Comparison of spectra of $\overline{u'^2}$ and $\overline{u'v'}$ at $y/d=0.4$.

absolute values of $F_{u'v'}(n)$ are not so accurate as those of $F_{u'^2}(n)$ since they represent the small difference between two large values of hot-wire signal.

It is to be noted that even though local isotropy was shown to exist in the channel flow, the values of the various vorticity terms $(\overline{u_i^2}/\lambda_j^2)$ differ appreciably, indicating that the relative magnitudes of these terms do not constitute a sensitive test for the existence of local isotropy. This is evident from the comparison of the distributions $n^2 F_{u'^2}(n)$ and $n^2 F_{u'v'}(n)$ (fig. 28). It is apparent from the figures that a large part of the contribution to $\int_0^\infty n^2 F_{u'^2}(n) dn$ comes from a frequency well below 1500 cycles per second. Local isotropy, according to the shear spectrum, exists only above a frequency of 1500 cycles per second.

CONCLUDING REMARKS

The measurements presented here confirm the general conceptions concerning the mean velocity profile in a turbulent channel. The extent of the laminar sublayer, the velocity profile in the sublayer, and the over-all velocity

distribution as measured here are in good agreement with general theoretical expectations.

Measurements of the turbulent field show that the hot-wire technique is well enough developed to give consistent results for the intensities and correlation functions.

Detailed measurements of the velocity fluctuations in the direction of the flow u' were carried out well within the laminar sublayer. It was found that the similarity law for $\tilde{u}'/U_\tau$, where U_τ is friction velocity, holds fairly well in the vicinity of the wall and as a consequence the magnitude of $\tilde{u}'/u$, where u is local mean velocity, approaches an absolute constant at the wall (approx. 0.18), the value being independent of the Reynolds number. The magnitudes of the velocity fluctuations normal to the flow in the lateral and vertical directions $\tilde{v}'$ and $\tilde{w}'$ are nearly the same in the middle region of the channel, $\tilde{w}'$ increasing more rapidly toward the wall.

The measured microscales of turbulence λ_x , λ_y , and λ_z consistently show a maximum at $y/d \approx 0.7$, where y is lateral distance and d is half width of the channel; they increase proportionally with $\tilde{u}'$, indicating a constant rate of energy dissipation W in the center portion of the channel since $W \propto \overline{u'^2}/\lambda^2$.

The scales of turbulence L_y and L_z in the center region are independent of Reynolds number and depend only on the channel width. The microscales, however, show a dependency on the Reynolds number.

From the calculated magnitudes of the energy produced by the shearing stresses and of the dissipated energy a descriptive picture of the energy diffusion in the center region of the channel is obtained.

The spectrum measurements of the $\overline{u'^2}$ -fluctuations tend to indicate that $F_{u'^2}(n)$ behaves as n^{-7} over a large band in the high-frequency region (where $F_{u'^2}(n)$ is fraction of turbulent energy associated with band width dn and n is frequency). From the comparison of the spectra of $\overline{u'^2}$ and of the turbulent shear the existence of local isotropy in the channel flow is verified.

GUGGENHEIM AERONAUTICAL LABORATORY,
CALIFORNIA INSTITUTE OF TECHNOLOGY,
PASADENA, CALIF., September 1, 1949.

REFERENCES

1. Kolmogoroff, A.: The Local Structure of Turbulence in Incompressible Viscous Fluid for Very Large Reynolds' Numbers. *Comp. rend., acad. sci. URSS*, vol. 30, no. 4, Feb. 10, 1941, pp. 301-305.
2. Heisenberg, W.: Zur statistischen Theorie der Turbulenz. *Zeitschr. Phys.*, Bd. 124, Heft 7/12, 1948, pp. 628-657.
3. Onsager, Lars: The Distribution of Energy in Turbulence. Abstract, *Phys. Rev.*, vol. 68, nos. 11 and 12, second ser., Dec. 1 and 15, 1945, p. 286.
4. Dönch, Fritz: Divergente und konvergente turbulente Strömungen mit kleinen Öffnungswinkeln. *Forsch.-Arb. Geb. Ing.-Wes.*, Heft 282, 1926.
5. Nikuradse, Johann: Untersuchungen über die Strömungen des Wassers in konvergenten und divergenten Kanälen. *Forsch.-Arb. Geb. Ing.-Wes.*, Heft 289, 1929.
6. Wattendorf, F. L., and Kueth, A. M.: Investigations of Turbulent Flow by Means of the Hot-Wire Anemometer. *Physics*, vol. 5, no. 6, June 1934, pp. 153-164.
7. Wattendorf, F. L.: Investigations of Velocity Fluctuations in a Turbulent Flow. *Jour. Aero. Sci.*, vol. 3, no. 6, April 1936, pp. 200-202.
8. Reichardt, H.: Die quadratischen Mittelwerte der Längsschwankungen in der turbulenten Kanalströmung. *Z. f. a. M. M.*, Bd. 13, Heft 3, June 1933, pp. 177-180.
9. Reichardt, H.: Messungen turbulenter Schwankungen. *Die Naturwissenschaften*, Jahrg. 26, Heft 24/25, June 17, 1938, pp. 404-408.
10. Von Kármán, Th.: The Fundamentals of the Statistical Theory of Turbulence. *Jour. Aero. Sci.*, vol. 4, no. 4, Feb. 1937, pp. 131-138.
11. Kovasznay, L.: Calibration and Measurement in Turbulence Research by the Hot-Wire Method. *NACA TM 1130*, 1947.
12. Liepmann, Hans Wolfgang, and Laufer, John: Investigations of Free Turbulent Mixing. *NACA TN 1257*, 1947.
13. Townsend, A. A.: Measurement of Double and Triple Correlation Derivatives in Isotropic Turbulence. *Proc. Cambridge Phil. Soc.*, vol. 43, pt. 4, Oct. 1947, pp. 560-570.
14. Liepmann, H. W., Laufer, J., and Liepmann, Kate: On the Spectrum of Isotropic Turbulence. *NACA TN 2473*, 1951.
15. Townsend, A. A.: Measurements in the Turbulent Wake of a Cylinder. *Proc. Roy. Soc. (London)*, ser. A, vol. 190, no. 1023, Sept. 9, 1947, pp. 551-561.
16. Frenkiel, F. N.: Étude statistique de la turbulence theorie de la correlation avec deux fils chauds non compenses. *Comp. rend., acad. sci. (Paris)*, t. 222, 1946, pp. 1377-1378 and 1474-1476.
17. Dryden, Hugh L.: Turbulence Investigations at the National Bureau of Standards. *Proc. Fifth Int. Cong. Appl. Mech. (Sept. 1938, Cambridge, Mass.)*, John Wiley & Sons, Inc., 1939, pp. 362-368.
18. Fage, A., and Townend, H. C. H.: An Examination of Turbulent Flow with an Ultramicroscope. *Proc. Roy. Soc. (London)*, ser. A, vol. 135, no. 828, April 1, 1932, pp. 656-677.
19. Prandtl, L., and Reichardt, H.: Einfluss von Wärmeschichtung auf die Eigenschaften einer turbulenten Strömung. *Deutsche Forschung*, Heft 21, 1934, pp. 110-121.
20. Taylor, G. I.: Statistical Theory of Turbulence. III—Distribution of Dissipation of Energy in a Pipe over Its Cross-Section. *Proc. Roy. Soc. (London)*, ser. A, vol. 151, no. 873, Sept. 2, 1935, pp. 455-464.

REPORT 1054

INTEGRALS AND INTEGRAL EQUATIONS IN LINEARIZED WING THEORY¹

By HARVARD LOMAX, MAX. A. HEASLET, and FRANKLYN B. FULLER

SUMMARY

The formulas of subsonic and supersonic wing theory for source, doublet, and vortex distributions are reviewed, and a systematic presentation is provided which relates these distributions to the pressure and to the vertical induced velocity in the plane of the wing. It is shown that care must be used in treating the singularities involved in the analysis and that the order of integration is not always reversible. Concepts suggested by the irreversibility of order of integration are shown to be useful in the inversion of singular integral equations when operational techniques are used. A number of examples are given to illustrate the methods presented, attention being directed to supersonic flight speeds.

INTRODUCTION

One of the most fundamental approaches to the analytical investigation of linearized wing theory, throughout the subsonic and supersonic ranges, stems from the use of certain elementary mathematical expressions that are identified with the physical properties of sources, doublets, and elementary horseshoe vortices. By means of these expressions boundary-value problems involving wings with thickness, camber, and angle of attack can be solved. These problems naturally fall into two categories: one, involving bodies with symmetrical thickness and no lift, is analyzed by means of source distributions; and the other, involving lifting plates without thickness, is analyzed by means of doublet and vortex distributions.

All these distributions require the treatment of singularities in the mathematical analysis. Thus; for subsonic Mach numbers, the concept of the generalized principal part plays an important role in the calculation of the induced velocities in the plane of a vortex sheet. In supersonic wing theory, the generalized principal part is again used in the analysis of vortex distributions, and it has further application in the treatment of conical-flow problems. However, the existence in supersonic flow of pressure discontinuities (due to Mach, or linearized shock, waves) brings about another type of singularity the mathematical analysis of which leads to the introduction of the finite-part concept. The integrals in both subsonic and supersonic wing theory thus require careful attention to the discontinuities in the integrand and, as an illustration, indiscriminate use of such standard devices as inversion of the order of integration can lead to incorrect results.

When direct problems are involved, that is, when prescribed functions are to be integrated (as in the problem of finding the pressure on a wing with symmetrical thickness), a guide to the proper method of calculation is often furnished by physical intuition. However, when inverse problems arise, that is, when integral equations are to be inverted (as for the flat plate of arbitrary plan form), the mathematical methods are more abstract. Nevertheless, the solutions to several types of inverse aerodynamic problems have been obtained by reasoning that required an understanding of the physical nature of the flow field. This method of solution may be sufficient for the particular problem involved but it is difficult to generalize. By using the aerodynamic data to construct mathematical boundary-value problems requiring the inversion of singular integral equations and by obtaining these inversions from a purely mathematical (operational) basis, a technique evolves whereby the existing solutions for two-dimensional subsonic, and three-dimensional supersonic wing problems (e. g., thin airfoil, conical flow, and Esvard solutions) are synthesized. Furthermore, the solution to the general supersonic wing problem is suggested.

The purpose of the present report is: First, to review the formulas of linearized wing theory in which source, doublet, and elementary horseshoe vortex distributions are introduced and to relate these distributions to the pressure and vertical induced velocity in the plane of the wing; second, to present an operational technique that can be used to invert the singular integral equations appearing in the application of the above formulas; and finally, to present certain special examples which will illustrate the basic concepts.

LIST OF IMPORTANT SYMBOLS

B_n	$\frac{1}{k_n^2} (E_n - k_n'^2 K_n)$
c	chord of a wing
C_D	drag coefficient $\left(\frac{\text{drag}}{\frac{1}{2} \rho_\infty V_\infty^2 S} \right)$
C_p	pressure coefficient $\left(-2 \frac{u}{V_\infty} \right)$
E	complete elliptic integral of second kind, modulus k
K	complete elliptic integral of first kind, modulus k
K_n, E_n	complete elliptic integrals of first and second kinds, respectively, with moduli k_n
k, k_n	moduli of elliptic integrals

¹ Supersedes NACA TN 2252, "Formulas for Source, Doublet, and Vortex Distributions in Supersonic Wing Theory" by Harvard Lomax, Max. A. Heaslet, and Franklyn B. Fuller, 1950.

k', k'_n	Complementary moduli ($\sqrt{1-k^2}, \sqrt{1-k_n^2}$)
L	lift
M_o	Mach number in free stream
m	slope of wing leading edge
m_1	cotangent of angle between the η and x axes
m_2	cotangent of angle between the ξ and x axes
$\frac{\Delta p}{q}$	loading coefficient (pressure on the lower surface minus pressure on the upper surface, divided by free-stream dynamic pressure)
r, s, z	characteristic coordinates
r_c	$\{(x-x_1)^2 + (1-M_o^2)[(y-y_1)^2 + z^2]\}^{1/2}$
r_o	$[(x-x_1)^2 + (1-M_o^2)(y-y_1)^2]^{1/2}$
S	wing area
t	maximum thickness of a wing
u, v, w	perturbation velocities in x, y, z directions, respectively
V_o	free-stream velocity
x, y, z	Cartesian coordinates
α	angle of attack of wing ($-\frac{w}{V_o}$)
β	$\sqrt{ 1-M_o^2 }$
Δ	jump in value of the quantity considered across the $z=0$ plane
λ	streamwise slope of surface ($\frac{w}{V_o}$)
μ_1	$\sqrt{1+m_1^2}$
μ_2	$\sqrt{1+m_2^2}$
μ	$\frac{1+m\beta}{1-m\beta}$
ν_o	$\frac{x-x_1}{(y-y_1)^2 + z^2}$
ξ, η, z	oblique coordinates
ρ_o	density of free stream
τ	area of integration
φ	perturbation velocity potential
ω_c	$\frac{x-x_1}{\beta\sqrt{(y-y_1)^2 + z^2}}$

SUBSCRIPTS

l	value of a quantity on the lower surface of a wing ($z=0$ plane)
u	value of a quantity on the upper surface of a wing ($z=0$ plane)

PART I—THE THREE FUNDAMENTAL FORMULAS

[SOME BASIC MATHEMATICAL FORMULAS]

[FIELD EQUATION FOR SUBSONIC FLOW]

The basic linearized partial differential equation governing a subsonic flow field is derived under the assumption that perturbation velocity components are small relative to the free-stream velocity V_o . Written in terms of the perturbation velocity potential $\varphi(x, y, z)$ the equation is

$$(1-M_o^2)\varphi_{xx} + \varphi_{yy} + \varphi_{zz} = 0 \quad (1)$$

where M_o is the free-stream Mach number and the x axis is parallel to the free-stream direction. Equation (1) is, in its

normalized form, Laplace's equation in three dimensions. If a sufficiently thin wing at a small angle of attack is situated on or in the immediate vicinity of the xy plane, the boundary conditions in the resulting linearized theory may be assumed specified at $z=0$ and, by means of Green's theorem (see, e. g., reference 1), a solution to equation (1) can be written in the form

$$\varphi(x, y, z) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\Delta w(x_1, y_1) \frac{1}{r_c} - \Delta \varphi(x_1, y_1) \frac{\beta^2 z}{r_c^3} \right] dx_1 dy_1 \quad (2)$$

where

$$\beta^2 = 1 - M_o^2$$

and

$$r_c = \{(x-x_1)^2 + (1-M_o^2)[(y-y_1)^2 + z^2]\}^{1/2}$$

Equation (2) relates the perturbation velocity potential at a point (x, y, z) in space to the discontinuities in the potential and vertical induced velocity at the "plane of the wing." Thus, $\Delta \varphi = \varphi_u - \varphi_l$ and $\Delta w = w_u - w_l$, where the subscripts u and l denote conditions on the upper and lower side of the xy plane.

In a later section, equation (2) will be used to obtain expressions for source, vortex, and doublet distributions in subsonic flow.

FIELD EQUATION FOR SUPERSONIC FLOW

The form of the basic linearized partial differential equation governing supersonic flow fields can be written in terms of the perturbation velocity potential as

$$(M_o^2 - 1)\varphi_{xx} - \varphi_{yy} - \varphi_{zz} = 0 \quad (3)$$

Since M_o is now greater than one, equation (3) is, in its normalized form, the wave equation. A solution to equation (3) that relates the potential in space to its jump $\Delta \varphi$ and the jump of the vertical velocity Δw across the $z=0$ plane has also been derived by means of Green's theorem. A form of such a solution, due to Volterra (reference 2), can be written

$$\varphi(x, y, z) = -\frac{1}{2\pi} \frac{\partial}{\partial x} \int_{\tau} \int \left\{ \Delta w(x_1, y_1) \operatorname{arc} \cosh \frac{x-x_1}{\beta\sqrt{(y-y_1)^2 + z^2}} - \Delta \varphi(x_1, y_1) \frac{z(x-x_1)}{[y-y_1)^2 + z^2]r_c} \right\} dx_1 dy_1 \quad (4)$$

where

$$\beta^2 = M_o^2 - 1$$

and

$$r_c = \{(x-x_1)^2 - (M_o^2 - 1)[(y-y_1)^2 + z^2]\}^{1/2}$$

The area τ is that part of the $z=0$ plane contained within the Mach forecone from the point (x, y, z) ; that is, the area bounded by the line $x_1 = -\infty$ and the hyperbola $(x-x_1)^2 - \beta^2(y-y_1)^2 - \beta^2 z^2 = 0$.

Obviously equations (2) and (4) are not similar although the basic equations to which they apply are. A formal similarity can be obtained, however, through the introduction of an integral operator, originated by Hadamard (reference 3), and referred to as the "finite part." The use of the finite part and certain other techniques necessary to reduce equation (4) to a form similar to equation (2) requires some attention and a discussion of the mathematical

difficulties involved will be given in the following section. Then an application of these techniques in subsequent sections will make it possible to obtain expressions for source, vortex, and doublet distributions in supersonic flow.

THE FINITE PART OF AN INTEGRAL

In the study of linearized supersonic flow problems, one is continually confronted with expressions of the form

$$\frac{\partial}{\partial x} \int_a^x \frac{f(x, y) dy}{\sqrt{x-y}}, \quad a < x \quad (5)$$

The integrals are of this form in the sense that the integrand is infinite at one (or both) of the limits and this limit is a function of the variable by which the partial derivative is to be taken. Such an expression is annoying because the derivative cannot be "moved" through the integral sign according to the usual rule, namely,

$$\frac{\partial}{\partial x} \int_a^x F(x, y) dy = F(x, x) + \int_a^x \frac{\partial F(x, y)}{\partial x} dy \quad (6)$$

Direct application of equation (6) to equation (5) obviously yields an unacceptable indeterminate form since the term corresponding to $F(x, x)$ is infinite. One way of avoiding the difficulty is to integrate equation (5) by parts so that the radical appears in the numerator of the integral and then to apply equation (6) to the resulting expression. Such a procedure can be carried out without the introduction of any new mathematical symbol or concept. However, this involves unnecessary restrictions on the integrand and often leads to unwieldy forms since the derivative of the function $f(x, y)$ with respect to y can be cumbersome.

Definition.—A more direct way of taking the derivative through the integral sign in equation (5) is accomplished by using the integral operator known as the finite part. Consider the simple equality

$$\frac{\partial}{\partial x} \int_a^x \frac{dy}{\sqrt{x-y}} = \frac{1}{\sqrt{x-a}}$$

The finite-part sign can be introduced by the definition

$$\int_a^x \frac{\partial}{\partial x} \frac{dy}{\sqrt{x-y}} = \frac{\partial}{\partial x} \int_a^x \frac{dy}{\sqrt{x-y}} \quad (7)$$

from which it follows

$$\int_a^x \frac{dy}{(x-y)^{3/2}} = \frac{-2}{\sqrt{x-a}} \quad (8)$$

The natural extension of this idea is to consider

$$J = \int_a^x \frac{\partial}{\partial x} \frac{A(y) dy}{\sqrt{x-y}} = \frac{\partial}{\partial x} \int_a^x \frac{A(y) dy}{\sqrt{x-y}} \quad (9)$$

where $A(y)$ is continuous at $x=y$ and is integrable elsewhere in the range of integration. The evaluation of J can be reduced to a form that requires only the definition introduced by equation (7). Thus, by adding and subtracting the same term, J can be written

$$J = \frac{\partial}{\partial x} \left[\int_a^x \frac{A(y) - A(x)}{\sqrt{x-y}} dy + A(x) \int_a^x \frac{dy}{\sqrt{x-y}} \right]$$

and it follows that

$$\int_a^x \frac{A(y) dy}{(x-y)^{3/2}} = \int_a^x \frac{A(x) - A(y)}{(x-y)^{3/2}} dy + A(x) \int_a^x \frac{dy}{(x-y)^{3/2}} \quad (10)$$

The final generalization of the definitions given by equations (7) and (9) is accomplished by considering the n^{th} derivative of the integrals and, furthermore, by allowing a functional dependence on x of the integrand A . First consider the definition

$$\begin{aligned} \int_a^x \left(\frac{\partial}{\partial x} \right)^n \frac{A(y) dy}{\sqrt{x-y}} &= (-1)^n \frac{1 \cdot 3 \cdots (2n-1)}{2^n} \int_a^x \frac{A(y) dy}{(x-y)^{n+1/2}} \\ &= \left(\frac{\partial}{\partial x} \right)^n \int_a^x \frac{A(y) dy}{\sqrt{x-y}} \end{aligned} \quad (11)$$

The second integral can be expressed in the form

$$\int_a^x \frac{A(y) dy}{(x-y)^{n+1/2}} = \int_a^x \frac{A(y) - B(x, y)}{(x-y)^{n+1/2}} dy + \int_a^x \frac{B(x, y) dy}{(x-y)^{n+1/2}} \quad (12)$$

where

$$B(x, y) = A(x) - A'(x)(x-y) + \dots + \frac{(-1)^{n-1} A^{(n-1)}(x)}{(n-1)!} (x-y)^{n-1}$$

and

$$\begin{aligned} \int_a^x \frac{dy}{(x-y)^{i+1/2}} &= \frac{(-1)^i 2^i}{1 \cdot 3 \cdots (2i-1)} \left(\frac{\partial}{\partial x} \right)^i \int_a^x \frac{dy}{\sqrt{x-y}} \\ &= \frac{-2}{(2i-1)(x-a)^{i-1/2}} \end{aligned}$$

Finally replacing $A(y)$ by $A(x, y)$, equation (11) again defines uniquely a finite-part integral provided that

$$\lim_{y \rightarrow x} \left[\sqrt{x-y} \left(\frac{\partial}{\partial x} \right)^n A(x, y) \right] = 0$$

Methods of evaluation.—If expressions of the type presented in equation (5) appear in an analytical development, it is now possible, by using the finite-part symbol, to take the partial derivative operation through the integral sign. Such a process needs no further amplification. In applying the results of such an analysis to the solution of some specific problem, however, one is confronted with the inverse operation, that is, the problem of evaluating the finite-part integrals. This can always be done, of course, by means of the definitions already given. Often, though, such evaluations can be simplified by using one of the two following processes.

The first process is readily outlined. Rewrite equations (8) and (10) in the form

$$\int_a^x \frac{f(x, y) dy}{(x-y)^{3/2}} = \lim_{\epsilon \rightarrow x} \int_a^\epsilon \frac{f(x, y) - f(x, x)}{(x-y)^{3/2}} dy - \frac{2f(x, x)}{\sqrt{x-a}}$$

and set the indefinite integral of

$$\int \frac{f(x, y) dy}{(x-y)^{3/2}}$$

equal to $F(x, y) + C$. It follows that

$$\int_a^x \frac{f(x, y) dy}{(x-y)^{3/2}} = -[F(x, a) + C] \quad (13)$$

where

$$C = \lim_{y \rightarrow x} \left[\frac{2f(x, y)}{\sqrt{x-y}} - F(x, y) \right] \quad (14)$$

The second technique avoids the necessity of evaluating the constant C . It depends on the use of the complex variable and is valid only when $f(x, y)$ is real in the interval $a \leq y \leq b$ where b is some number greater than x . Again set the indefinite integral of

$$\int \frac{f(x, y) dy}{(x-y)^{3/2}}$$

equal to $F(x, y) + C$. Now if r. p. stands for the real part of a function, the evaluation of the finite-part integral is provided by the equality

$$\int_a^x \frac{f(x, y) dy}{(x-y)^{3/2}} = \text{r. p.} \int_a^b \frac{f(x, y) dy}{(x-y)^{3/2}} = \text{r. p.} [F(x, b) - F(x, a)], \quad a < x < b \quad (15)$$

As an example of the second technique, consider the integral

$$I = \int_0^x \frac{y^2 dy}{(x^2 - y^2)^{3/2}} = \text{r. p.} \int_0^b \frac{y^2 dy}{(x^2 - y^2)^{3/2}}$$

where $x < b$. From the relation

$$\int \frac{y^2 dy}{(x^2 - y^2)^{3/2}} = \frac{y}{\sqrt{x^2 - y^2}} - \arcsin \frac{y}{x}$$

together with equation (15), it follows that

$$I = \text{r. p.} \left(\frac{b}{\sqrt{x^2 - b^2}} - \arcsin \frac{b}{x} \right) = -\frac{\pi}{2}$$

A simple extension of this result yields

$$\int_{-x}^x \frac{y^2 dy}{(x^2 - y^2)^{3/2}} = \text{r. p.} \int_{-b}^b \frac{y^2 dy}{(x^2 - y^2)^{3/2}} = -\pi$$

An application.—The above methods can be applied to give the following simple but useful result. Let $Y = a + by + cy^2 = (\lambda_2 - y)(y - \lambda_1)(-c)$ and $q = 4ac - b^2$, then

$$\int_t^{\lambda_2} \frac{(c_0 + c_1 y) dy}{Y^{3/2}} = - \int_{\lambda_1}^t \frac{(c_0 + c_1 y) dy}{Y^{3/2}} = \frac{-2}{q \sqrt{a + bt + ct^2}} [c_0(2ct + b) - c_1(bt + 2a)] \quad (16)$$

Included in this result is the equality

$$\int_{\lambda_1}^{\lambda_2} \frac{c_0 + c_1 y}{Y^{3/2}} dy = 0 \quad (17)$$

which contains the very important identity

$$\int_{\lambda_1}^{\lambda_2} \frac{dy}{(\lambda_2 - y)^{3/2} \sqrt{y - \lambda_1}} = \int_{\lambda_1}^{\lambda_2} \frac{dy}{(y - \lambda_1)^{3/2} \sqrt{\lambda_2 - y}} = 0 \quad (18)$$

The case of multiple integrals.—When applied to the analysis of single integrals, the above definitions of the finite part coincide with, or are a re-expression of, those given originally by Hadamard (reference 3). Hence,

$$\int_a^x \frac{f(x, y) dy}{(x-y)^{3/2}} = \int_a^x \frac{f(x, y) dy}{(x-y)^{3/2}}$$

where $\int$ was the symbol used by Hadamard to denote evaluation by finite-part methods. When applied to double integrals, however, the signs $\int$ and $\int$ are no longer equivalent. Hadamard, as well as A. Robinson (reference 4), maintains the convention that the order of integration in the operation $\int$ is reversible; that is,²

$$\int dy \int dx f(x, y) = \int dx \int dy f(x, y)$$

Such a convention requires that all singularities for which the order of integration is irreversible must be excluded from the area of integration. These singular regions are then treated separately. This convention has the disadvantage that, in evaluating multiple integrals, the value of a given integral is not independent of succeeding integrals. The operator $\int$ avoids difficulties of this kind and each definite integral is independent of succeeding operations. For example,

$$\int_0^x \frac{d\eta}{(\xi - \eta)^{3/2} \sqrt{\eta}} = \int_0^x d\xi \int_0^x \frac{d\eta}{(\xi - \eta)^{3/2} \sqrt{\eta}} = 0$$

but, according to reference 3,

$$\int_0^x \frac{d\eta}{(\xi - \eta)^{3/2} \sqrt{\eta}} = 0$$

whereas, according to the same reference,

$$\int_0^x d\xi \int_0^x \frac{d\eta}{(\xi - \eta)^{3/2} \sqrt{\eta}} = -2\pi$$

Although the use of the symbol $\int$ makes each integration independent of subsequent integrations, the order of integration for operations involving the sign $\int$ cannot be reversed. Hence,

$$\int dy \int dx f(x, y) \neq \int dx \int dy f(x, y)$$

For example, the relation

$$\int_0^x d\xi \int_0^x \frac{d\eta}{(\xi - \eta)^{3/2} \sqrt{\eta}} = 0$$

holds while the same integral taken over the same area but in reversed order is

²Since the order of integration plays an important role in the following development, integration first with respect to x and then with respect to y will be denoted $\int dy \int dx f(x, y)$ while integration first with respect to y and then with respect to x will be denoted $\int dx \int dy f(x, y)$. When the notation $\int \int f(x, y) dy dx$ is used, the order of integration is immaterial.

$$\int_0^x d\eta \int_{\eta}^x \frac{d\xi}{(\xi-\eta)^{3/2}\sqrt{\eta}} = -2\pi$$

The operation defined by the symbol $\oint$ will be used consistently throughout the present report. Hence attention must always be paid to the order but not to the multiplicity of integrations. It can be seen that the finite part of a conventional-type integral coincides with the value found by standard methods.

THE GENERALIZED PRINCIPAL PART OF AN INTEGRAL

Another type of important operation appearing in the development of both subsonic and supersonic wing theory appears implicitly in the expression

$$I_0 = \lim_{z \rightarrow 0+} \int_a^b \frac{\partial}{\partial z} \left[\frac{zf(y_1, z)}{z^2 + (y - y_1)^2} \right] dy_1 \quad (19)$$

where $f(y, z)$ and its derivatives are bounded and continuous in the interval $a \leq y \leq b$. In the attempt to simplify I_0 by letting z approach zero before performing the integration, a second special integral operator will be introduced.

To simplify I_0 , first integrate by parts

$$I_0 = \lim_{z \rightarrow 0+} \frac{\partial}{\partial z} \left[f(a, z) \arctan \frac{y-a}{z} - f(b, z) \arctan \frac{y-b}{z} + \int_a^b \frac{\partial f(y_1, z)}{\partial y_1} \arctan \frac{y-y_1}{z} dy_1 \right]$$

Then since

$$\lim_{z \rightarrow 0+} \int_a^b \frac{\partial f_z(y_1, z)}{\partial y_1} \arctan \frac{y-y_1}{z} dy_1 = \pi f_z(y, 0) - \frac{\pi}{2} [f_z(a, 0) + f_z(b, 0)]$$

I_0 becomes

$$I_0 = \frac{f(b, 0)}{y-b} - \frac{f(a, 0)}{y-a} - \int_a^b \frac{f_{y_1}(y_1, 0) dy_1}{y-y_1} + \pi f_z(y, 0) \quad (20)$$

Definition of the generalized principal part.—The expression of I_0 given in equation (20) contains the integral of the function $f'(y_1)/(y-y_1)$. Such an integration is not, in general, convergent; however, when the integral is so written without further qualification it is generally accepted that the singularity occurring in the integrand is to be treated using Cauchy's principal part. Evaluation by such a method is often indicated by the symbol $\oint$ and is defined by the equation

$$\oint_a^b \frac{A(y_1) dy_1}{y_1 - y} = \lim_{\epsilon \rightarrow 0} \left[\int_a^{y-\epsilon} \frac{A(y_1) dy_1}{y_1 - y} + \int_{y+\epsilon}^b \frac{A(y_1) dy_1}{y_1 - y} \right] \quad (21)$$

or, alternatively,

$$\oint_a^b \frac{A(y_1) dy_1}{y_1 - y} = -\frac{\partial}{\partial y} \int_a^b A(y_1) \ln|y_1 - y| dy_1 \quad (22)$$

To assure the convergence of the right-hand side of equations (21) and (22) it is sufficient but not necessary to assume that $A(y_1)$ is differentiable at the point $y_1 = y$ and that

elsewhere within the region of integration $A(y_1)$ is either continuous or possesses integrable singularities. The concept of the Cauchy principal part is so well known that the symbol on this integral is often omitted, as shall be done here.

The differential operator in equation (22) lends itself readily to a generalization of the principal-part concept. Thus, for the next higher order, the definition (see also, in this connection, reference 5)

$$\oint_a^b \frac{A(y_1) dy_1}{(y_1 - y)^2} = -\frac{\partial^2}{\partial y^2} \int_a^b A(y_1) \ln|y - y_1| dy_1 = \frac{\partial}{\partial y} \int_a^b \frac{A(y_1) dy_1}{y_1 - y} \quad (23)$$

applies and, in general,

$$n! \oint_a^b \frac{A(y_1) dy_1}{(y_1 - y)^{n+1}} = -\frac{\partial^{n+1}}{\partial y^{n+1}} \int_a^b A(y_1) \ln|y - y_1| dy_1 = \frac{\partial^n}{\partial y^n} \int_a^b \frac{A(y_1) dy_1}{y_1 - y} \quad (24)$$

The generalized principal part can also be expressed in the form

$$\oint_a^b \frac{A(y_1) dy_1}{(y_1 - y)^{n+1}} = \int_a^b \frac{A(y_1) - B(y, y_1)}{(y_1 - y)^{n+1}} dy_1 + \oint_a^b \frac{B(y, y_1) dy_1}{(y_1 - y)^{n+1}}$$

where

$$B(y, y_1) = A(y) + \frac{A'(y)}{1} (y_1 - y) + \dots + \frac{A^{(n-1)}(y)}{(n-1)!} (y_1 - y)^{n-1}$$

and

$$\oint_a^b \frac{dy_1}{(y_1 - y)^{i+1}} = \frac{1}{i!} \left(\frac{\partial}{\partial y} \right)^i \int_a^b \frac{dy_1}{y_1 - y} = \frac{-1}{i} \left[\frac{1}{(b-y)^i} - \frac{1}{(a-y)^i} \right], \quad i \geq 1.$$

The first n derivatives of $A(y_1)$ are assumed to exist and be single valued at $y_1 = y$ while elsewhere in the range of integration $A(y_1)$ may possess integrable singularities. This definition is in a form that involves no extension beyond the concept of Cauchy's principal part.

It is possible to extend the definition contained in equation (24) to include a functional dependency on y in the numerator of the integrands. Thus, replacing $A(y_1)$ by $A(y, y_1)$, equation (24) again defines uniquely a principal-part integral provided the first n derivatives of $A(y, y_1)$ with respect to y and y_1 exist at $y_1 = y$.

Method of evaluation.—Operations involving the symbol $\oint$ can always be performed by means of the definitions just given. However, another method can be used, which is often simpler to apply. If the indefinite integral of $A(y_1)/(y_1 - y)^{n+1}$ exists such that

$$\int \frac{A(y_1) dy_1}{(y_1 - y)^{n+1}} = G(y_1, y) + C \quad (25)$$

then the value of the generalized principal part can be found by following the conventional rules for substitution of limits; thus

$$\frac{f_a^b A(y_1) dy_1}{(y_1 - y)^{n+1}} = G(b, y) - G(a, y), \quad y \neq a, b \quad (26)$$

The proof of this result can be obtained by mathematical induction.

An application.—Returning to equation (20), since an integration by parts yields the relation

$$\frac{f_a^b f(y_1, 0) dy_1}{(y_1 - y)^2} = \frac{f(b, 0)}{y - b} - \frac{f(a, 0)}{y - a} - \int_a^b \frac{f_{y_1}(y_1, 0) dy_1}{y - y_1}$$

the limiting process symbolized by I_0 can be expressed as an integral that contains only the function evaluated at $z=0$ and not its derivative. Thus, finally

$$\lim_{z \rightarrow 0+} \int_a^b \frac{\partial}{\partial z} \left[\frac{zf(y_1, z)}{z_2 + (y - y_1)^2} \right] dy_1 = \frac{f_a^b f(y_1, 0) dy_1}{(y_1 - y)^2} + \pi f_z(y, 0) \quad (27)$$

THE OBLIQUE COORDINATE SYSTEM

Equations (1) through (4) gave the basic partial differential equations of wing theory, together with their solutions, in terms of the usual Cartesian coordinate system, the x axis extending in the direction of the undisturbed flow and the y and z axes oriented normal to this direction in such a way that boundary conditions for the wing can be specified in the $z=0$ plane. In the study of supersonic flow fields it is at times mathematically convenient to introduce in the $z=0$ plane new coordinate axes making arbitrary angles with the x and y axes.

The general case. Consider the ξ, η, z coordinate system (fig. 1) such that the z axis is normal to the plane supporting

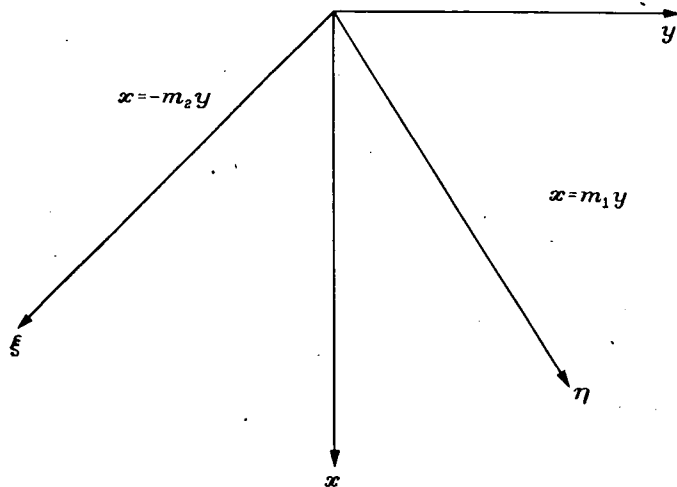


FIGURE 1.—Coordinate systems.

the boundary conditions while ξ and η are normal to z and coincident with the lines $x = -m_2 y$ and $x = m_1 y$. If

$$\mu_1 = \sqrt{1 + m_1^2}, \quad \mu_2 = \sqrt{1 + m_2^2} \quad (28)$$

the equations relating the two systems of coordinates are

$$\left. \begin{aligned} x &= \frac{\eta m_1}{\mu_1} + \frac{\xi m_2}{\mu_2}, & \xi &= \frac{\mu_2(x - m_1 y)}{m_1 + m_2} \\ y &= \frac{\eta}{\mu_1} - \frac{\xi}{\mu_2}, & \eta &= \frac{\mu_1(x + m_2 y)}{m_1 + m_2} \\ z &= z, & z &= z \end{aligned} \right\} \quad (29)$$

while the relation between the differential areas, as determined from the Jacobian of the transformations, is

$$dx_1 dy_1 = \frac{m_1 + m_2}{\mu_1 \mu_2} d\xi_1 d\eta_1 \quad (30)$$

The value of r_c , as defined under equation (4), becomes

$$r_c = \left[\frac{(\eta - \eta_1)^2 (m_1^2 - \beta^2)}{\mu_1^2} + \frac{2(\eta - \eta_1)(\xi - \xi_1)(m_1 m_2 + \beta^2)}{\mu_1 \mu_2} + \frac{(\xi - \xi_1)^2 (m_2^2 - \beta^2)}{\mu_2^2} - \beta^2 z^2 \right]^{1/2} \quad (31a)$$

where $\beta^2 = M_0^2 - 1$. If the variable ν_c is introduced such that

$$\nu_c = \frac{x - x_1}{(y - y_1)^2 + z^2} \quad (31b)$$

it follows that in the transformed coordinates this variable becomes

$$\nu_c = \frac{(\eta - \eta_1) \frac{m_1}{\mu_1} + (\xi - \xi_1) \frac{m_2}{\mu_2}}{\left[(\eta - \eta_1) \frac{1}{\mu_1} - (\xi - \xi_1) \frac{1}{\mu_2} \right]^2 + z^2} \quad (31c)$$

Finally, the differential operator $\partial/\partial x$ transforms to

$$\frac{\partial}{\partial x} = \frac{1}{m_1 + m_2} \left(\mu_2 \frac{\partial}{\partial \xi} + \mu_1 \frac{\partial}{\partial \eta} \right) \quad (31d)$$

The area τ over which the integration of equation (4) is to be taken is still, of course, bounded by the hyperbola $r_c^2 = 0$ and the line $x_1 = -\infty$. The asymptotes to the hyperbola become, however,

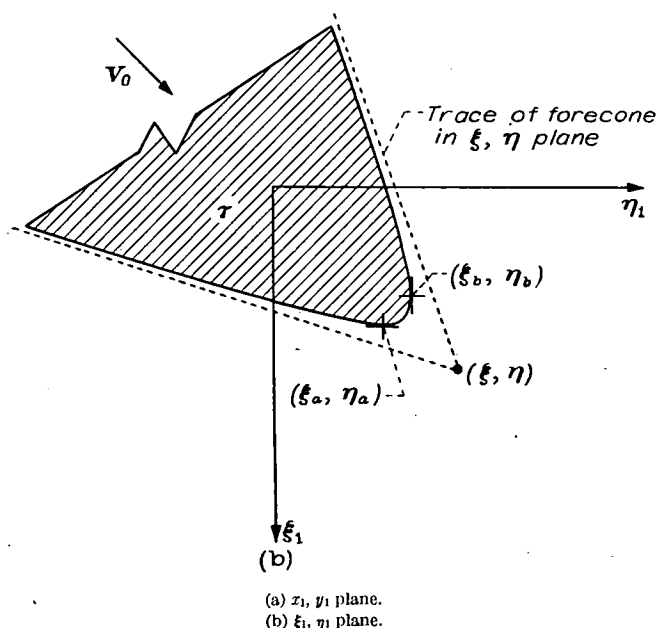
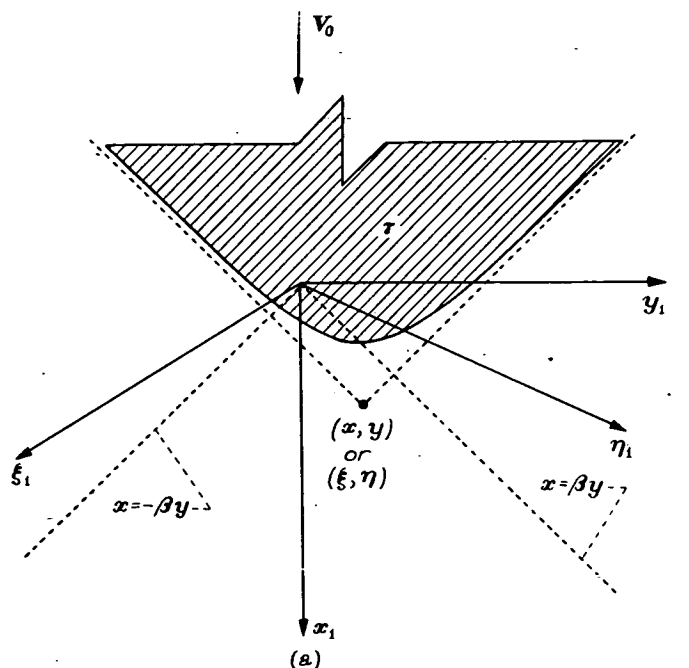
$$\left. \begin{aligned} \eta - \eta_1 &= -\frac{\mu_1}{\mu_2} (\xi - \xi_1) \frac{m_2 + \beta}{m_1 - \beta} \\ \eta - \eta_1 &= -\frac{\mu_1}{\mu_2} (\xi - \xi_1) \frac{m_2 - \beta}{m_1 + \beta} \end{aligned} \right\} \quad (32)$$

Figure 2 shows how the area τ in the xy plane transforms to the $\xi\eta$ plane.

(Although the ξ and η axes are oblique with respect to the original axial system, no inconsistency results if the equations in the transformed variables are plotted relative to orthogonal axes.) In case both m_1 and m_2 are less than β (the case for which the sketch was drawn), the asymptotes are straight lines with positive slopes, and the area is bounded in both the ξ and η directions by the maximum points ξ_a, η_a and ξ_b, η_b , respectively. These maximum points are shown in the figure and their analytical expressions are

$$\left. \begin{aligned} \xi_a &= \xi - \frac{z \mu_2 \sqrt{\beta^2 - m_1^2}}{m_1 + m_2}, & \eta_a &= \eta - \frac{z \mu_1 (m_1 m_2 + \beta^2)}{(m_1 + m_2) \sqrt{\beta^2 - m_1^2}} \\ \xi_b &= \xi - \frac{z \mu_2 (m_1 m_2 + \beta^2)}{(m_1 + m_2) \sqrt{\beta^2 - m_2^2}}, & \eta_b &= \eta - \frac{z \mu_1 \sqrt{\beta^2 - m_2^2}}{m_1 + m_2} \end{aligned} \right\} \quad (33)$$

If the oblique coordinate system is chosen such that m_1 is less than and m_2 is greater than β , one of the asymptotes in equation (32) has a slope of opposite sign and the area τ is unbounded in the η_1 direction although ξ_a, η_a is still the point which determines the farthest extent of τ in the ξ_1 direction.


 FIGURE 2.—Area of integration τ .

On the other hand, if m_2 is less than and m_1 is greater than β , τ extends infinitely in the ξ_1 direction and ξ_b, η_b represents its upper bound in terms of η_1 . Finally, if both m_1 and m_2 are greater than β , the area τ is not bounded for either negative or positive ξ_1 or η_1 .

The characteristic coordinate system.—Another special case of the ξ, η, z system (the x, y, z system is, of course, a special case obtained when $m_1=0$ and $m_2=\infty$) that is important enough to receive a particular notation is the one obtained when the ξ and η axes are parallel to the Mach lines (i. e., the traces of the characteristic cones) in the $z=0$ plane. These axes are shown in figure 3, will be designated r and s , and are given by equations (29) when $m_1=m_2=\beta$ and $\mu_1=\mu_2=M_0$; thus

$$\left. \begin{aligned} x &= \frac{\beta}{M_0} (s+r), & r &= \frac{M_0}{2\beta} (x-\beta y) \\ y &= \frac{1}{M_0} (s-r), & s &= \frac{M_0}{2\beta} (x+\beta y) \\ z &= z, & z &= z \end{aligned} \right\} \quad (34)$$

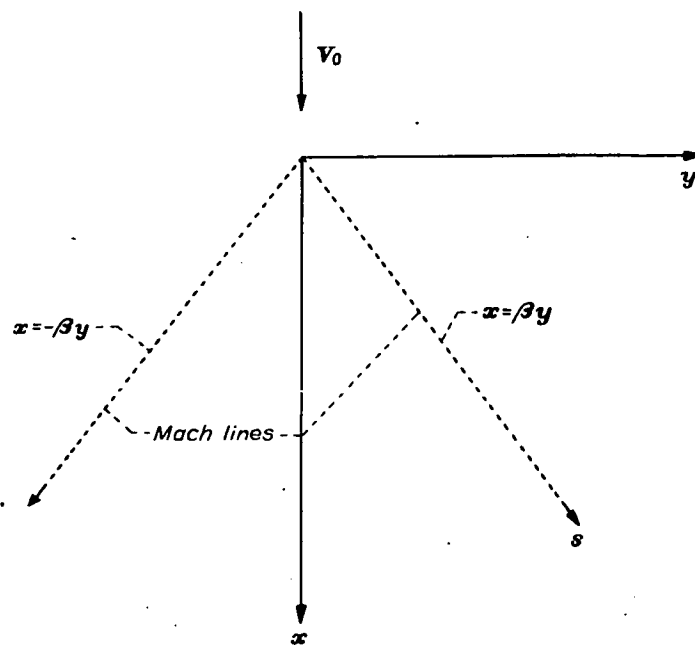
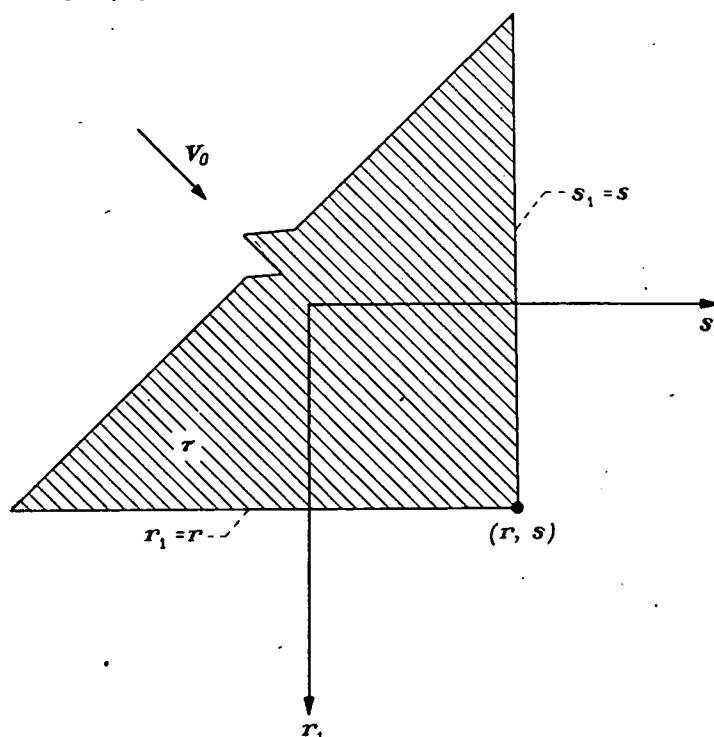


FIGURE 3.—Characteristic coordinate system.

When z is set equal to zero, the equation for r_c and the form of the area τ become especially simple. Thus

$$(r_c)_{z=0} = r_0 = \frac{2\beta}{M_0} \sqrt{(r-r_1)(s-s_1)} \quad (35)$$

and the area τ , shown in figure 4, is bounded by the straight lines $r_1=r$, $s_1=s$ and $r_1=s_1=-\infty$.


 FIGURE 4.—Integration area in characteristic system for $z=0$.

THE THREE FUNDAMENTAL FORMULAS IN SUBSONIC FLOW

The parallelism between the basic formulas in subsonic and supersonic wing theory is so obvious that it is advantageous to present first the somewhat more classical results applying to the purely subsonic regime. The immediate objective is therefore to present as briefly as possible the expressions for the perturbation velocity potential due to a distribution of sources, doublets, or elementary horseshoe vortices and then, by means of these expressions, to relate the pressure and vertical induced velocity in the $z=0$ plane to the weight of these distributions.

THE PERTURBATION POTENTIAL AT A POINT IN SPACE

The linearized form of the perturbation velocity potential due to a unit source, elementary horseshoe vortex, or doublet situated in a free stream moving at a uniform subsonic velocity V_∞ is given as follows:

$$\begin{aligned} \text{Unit source} & \dots\dots\dots \varphi = -1/4\pi r_c \\ \text{Unit elementary horseshoe vortex} & \dots\dots\dots \varphi = -z v_c / 4\pi r_c \\ \text{Unit doublet} & \dots\dots\dots \varphi = -z\beta^2 / 4\pi r_c^3 \end{aligned}$$

where r_c is defined in equation (31b) and $\beta^2 = 1 - M_\infty^2$.

It is well known that a distribution of sources in the $z=0$ plane splits the streamlines and forms a field symmetrical in u , v , and φ above and below the source plane. Hence, the strength of the sources is related to the term Δw (which, in turn, is related to the gradient of thickness of the simulated body) while the variables u , v , and φ are continuous. On the other hand, a distribution of elementary vortices or doublets causes a discontinuity in the streamwise induced velocity (or, what amounts to the same thing, the perturbation potential) across the reference plane but, at the same time, causes no division of the streamlines. The strengths of the vortices and doublets are therefore related to the terms Δu and $\Delta\varphi$, respectively, (which, in turn, are related to the wing loading), and produce no discontinuities in w . The exact analytical form of these distributions can be obtained readily from equation (2).

The source distribution.—The velocity potential induced by a distribution of sources over the $z=0$ plane follows immediately from equation (2) since, by symmetry, $\Delta\varphi$ must be zero. In practice, the area over which the sources are distributed is limited to the area S defined by the plan form. Hence,

$$\varphi(x, y, z) = \frac{-1}{4\pi} \iint_S \frac{\Delta w}{r_c} dx_1 dy_1 \quad (36)$$

The elementary-vortex distribution.—Equation (1) was written in terms of the perturbation potential φ . It could, however, after differentiation have been expressed in terms of any one of the induced velocity components and the solution in equation (2) would then also be expressed in terms of the particular velocity component chosen. Consider such a case, taking for the dependent variable the streamwise perturbation velocity instead of φ . Equation (2) then becomes

$$u(x, y, z) = \frac{-1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{\partial \Delta u}{\partial z} \frac{1}{r_c} - \Delta u \frac{\beta^2 z}{r_c^3} \right) dx_1 dy_1$$

If the field is to be without sources, both Δw and $\partial \Delta w / \partial z$ vanish. But for an irrotational field the equality $\partial \Delta w / \partial x = \partial \Delta u / \partial z$ holds and the first term in the integrand of the above equation is zero. By definition

$$\int_{-\infty}^x u(x_1, y, z) dx_1 = \varphi(x, y, z)$$

from which it follows that if the operator $\int_{-\infty}^x dx$ is applied to both sides of the resulting equation, the relation

$$\varphi(x, y, z) = \frac{-1}{4\pi} \iint_S \frac{z(x-x_1)\Delta u}{[(y-y_1)^2 + z^2]^{3/2}} dx_1 dy_1 \quad (37)$$

follows where the area of integration is limited to the wing plan form. This result expresses the perturbation velocity potential due to a distribution of elementary horseshoe vortices over a wing plan form in the $z=0$ plane.

The doublet distribution.—The solution for a doublet distribution, just as in the case of the sources, follows immediately from equation (2). Since the streamlines are not divided by the doublets the term containing Δw vanishes. The doublet distribution exists, however, not only over the wing area but also over the vortex wake streaming downstream behind the wing since the discontinuity in the potential persists in this region. Designating the wake area by W , the final expression for the perturbation potential associated with the doublet sheet becomes

$$\varphi(x, y, z) = \frac{\beta^2 z}{4\pi} \iint_{S+W} \frac{\Delta\varphi}{r_c^3} dx_1 dy_1 \quad (38)$$

REDUCTION TO THE PLANE OF THE WING

The aerodynamicist is usually interested in the forces on the surface of the wing itself and, as a consequence, it is pertinent to consider each of the above formulas in the limiting case as z approaches zero. An explicit expression of these results is given below.

The source distribution.—The limiting value of equation (36) as z approaches zero is obtained immediately by simply setting z equal to zero. The resulting expression is

$$\varphi(x, y, 0) = \frac{-1}{4\pi} \iint_S \frac{\Delta w}{r_0} dx_1 dy_1 \quad (39)$$

where

$$r_0 = [(x-x_1)^2 + \beta^2(y-y_1)^2]^{1/2}$$

Practical interest is usually concentrated on the relation between the pressure on the wing surface and the wing shape. Since in linearized theory pressure coefficient C_p and wing slope λ_u of the upper surface are known to be

$$C_p = \frac{-2u}{V_0}, \quad \lambda_u = \frac{w_u}{V_0} = \frac{1}{2} \frac{\Delta w}{V_0} \quad (40)$$

it follows, after differentiation of equation (39) with respect to x , that pressure coefficient is

$$C_p = \frac{1}{\pi} \frac{\partial}{\partial x} \iint_S \frac{\lambda_u}{r_0} dx_1 dy_1 \quad (41)$$

The elementary-vortex distribution.—When the strength of the elementary horseshoe vortices is known over a wing plan form, there is no difficulty in finding the potential in the $z=0$ plane since it follows from a direct integration of the vortex strength. The pertinent question is, rather, to determine the vertical induced velocity in the plane of the wing from the given vortex strength. If load coefficient is defined in the usual way

$$\frac{\Delta p}{q} = \frac{2\Delta u}{V_0} = \frac{4u_u}{V_0} \quad (42)$$

the answer to this question requires the evaluation of the following limiting process:

$$\frac{w}{V_0} = \lim_{z \rightarrow 0} \frac{-1}{8\pi} \iint_S \frac{\partial}{\partial z} \frac{z(x-x_1)}{[(y-y_1)^2 + z^2] r_c} \frac{\Delta p}{q} dx_1 dy_1$$

If, as in equation (27), the generalized principal part is introduced, the required expression becomes

$$\frac{w}{V_0} = \frac{-1}{8\pi} \iint_S \frac{(x-x_1)}{(y-y_1)^2 r_0} \frac{\Delta p}{q} dx_1 dy_1 \quad (43)$$

The doublet distribution.—In the case of doublet distributions, the relevant problem requires the expression of vertical induced velocity in the plane of the wing as a function of the doublet strength $\Delta\varphi$. If equation (38) is differentiated with respect to z and z is then set equal to zero, one finds without difficulty the final formula

$$\frac{w}{V_0} = \frac{\beta^2}{4\pi V_0} \iint_{S+W} \frac{\Delta\varphi}{r_0^3} dx_1 dy_1 \quad (44)$$

THE THREE FUNDAMENTAL FORMULAS IN SUPERSONIC FLOW

The purpose of this section is to repeat for supersonic wing theory the developments presented in the preceding section for subsonic theory. In order to maintain the formal analogy, it is necessary to introduce the concept of the finite part. This latter concept, in turn, introduces into the analysis integral expressions containing certain *inherent singularities*. Such singularities are, by definition, points across which the order of integration cannot be reversed.³ The study of these singularities and their effect on the fundamental formulas suggests the introduction of an oblique coordinate system defined in the first section of this report. Hence, the following analysis will be presented in the ξ, η, z system, while transformation to the Cartesian and characteristic systems, it will be remembered, can be made by considering the special cases

$$\left. \begin{array}{ll} \text{For} & \text{Let} \\ x, y, z & m_1=0, m_2=\infty \\ r, s, z & m_1=m_2=\beta \end{array} \right\} \quad (45)$$

Following the development of the basic formulas, a summary of results will be given in terms of the x, y and r, s coordinates.

³ More precisely, if an inherent singularity exists in the area over which a double integral is to be evaluated, the difference between the values of the integration made in one order and then over the same area but with reversed order is not zero.

THE PERTURBATION POTENTIAL AT A POINT IN SPACE

The velocity potential due to a unit source, a unit elementary horseshoe vortex, and a unit doublet is given as follows:

$$\left. \begin{array}{ll} \text{Unit source} & \varphi = -1/2\pi r_c \\ \text{Unit elementary horseshoe} & \\ \text{vortex} & \varphi = -z\nu_c/2\pi r_c \\ \text{Unit doublet} & \varphi = z\beta^2/2\pi r_c^3 \end{array} \right\} \quad (46)$$

where r_c and ν_c are defined in equations (31) and $\beta^2 = M_0^2 - 1$. In this notation, the only difference between these expressions and the corresponding ones for subsonic flow is the factor 2. A much more important difference exists, however, in the effect of the change of sign of $1 - M_0^2$ on r_c and ν_c .

The first task is the expression of the perturbation velocity potential at a point in space in terms of distributions of the elementary solutions and this can be accomplished by an appropriate analysis of equation (4). Just as in the subsonic case, the source strength is given by Δw , the vortex strength by Δu , and the doublet strength by $\Delta\varphi$.

The source distribution.—The potential induced by a source distribution can be obtained from equation (4) by setting the term containing $\Delta\varphi$ equal to zero (i. e., by removing all the circulation from the flow field). By means of the notation

$$\omega_c = \frac{x-x_1}{\beta \sqrt{(y-y_1)^2 + z^2}} = \frac{\frac{m_1}{\mu_1}(\eta-\eta_1) + \frac{m_2}{\mu_2}(\xi-\xi_1)}{\beta \sqrt{\left[\frac{1}{\mu_1}(\eta-\eta_1) - \frac{1}{\mu_2}(\xi-\xi_1)\right]^2 + z^2}} \quad (47)$$

The remaining expression can be written in the ξ, η, z coordinate system simply as

$$\varphi(\xi, \eta, z) = \frac{-1}{2\pi\mu_1\mu_2} \left(\mu_2 \frac{\partial}{\partial \xi} + \mu_1 \frac{\partial}{\partial \eta} \right) \iint_\tau \Delta w(\xi_1, \eta_1) \text{arc cosh } \omega_c d\xi_1 d\eta_1 \quad (48)$$

It is possible to "move" the partial derivative operations through the integral signs if the equality

$$\frac{\partial}{\partial x} \int_{b(x)}^{a(x)} f(x, y) dy = f(a, y) \frac{\partial a}{\partial x} - f(b, y) \frac{\partial b}{\partial x} + \int_b^a \frac{\partial f(x, y)}{\partial x} dy \quad (49)$$

is used. Obviously, if the operation indicated by equation (49) is to be applied to equation (48), the order of integration and the limits on the latter equation must be specified.

Consider the case when the ξ and η axes are chosen so that both m_1 and m_2 are less than β . The area τ is then the one shown in figure 2. Further consider the case when the first integration is made with respect to η_1 . Then, equation (48) becomes

$$\varphi(\xi, \eta, z) = \frac{-1}{2\pi\mu_1\mu_2} \left(\mu_2 \frac{\partial}{\partial \xi} + \mu_1 \frac{\partial}{\partial \eta} \right) \int_{-\infty}^{\xi_a} d\xi_1 \int_{\lambda_1}^{\lambda_2} d\eta_1 \Delta w(\xi_1, \eta_1) \text{arc cosh } \omega_c \quad (50)$$

where λ_1 and λ_2 are roots of the quadratic expression $r_c^2(\eta_1) = 0$ and ξ_a, η_a are defined in equation (33). It can be shown that when $\xi_1 = \xi_a$, the equality $\lambda_1 = \lambda_2$ applies, and, further, that the integrand of equation (50) does not vanish. Hence,

the two partial derivatives can be moved through the first integral sign without the appearance of the additional term involving the derivative of the upper limit. It is also not difficult to show the value of ω_c at η_1 equal to either λ_1 or λ_2 is just 1. Hence, the two partial derivatives can also be moved through the second integral sign without the appearance of additional terms. Finally, since

$$\frac{1}{m_1+m_2} \left(\mu_2 \frac{\partial}{\partial \xi} + \mu_1 \frac{\partial}{\partial \eta} \right) \text{arc cosh } \omega_c = \frac{\partial}{\partial x} \text{arc cosh } \omega_c = \frac{1}{r_c}$$

the velocity potential at a point (ξ, η, z) due to a distribution of sources in the $z=0$ plane can be written

$$\varphi(\xi, \eta, z) = -\frac{m_1+m_2}{2\pi\mu_1\mu_2} \iint_{\tau} \frac{\Delta w(\xi_1, \eta_1)}{r_c} d\xi_1 d\eta_1 \quad (51)$$

The elementary-vortex distribution.—The potential induced by a distribution of elementary horseshoe vortices can be derived in a manner analogous to the derivation for the subsonic case. Thus, the solution given by equation (4) is written for the induced velocity u rather than the velocity potential.⁴ Since the flow field contains no sources, $\partial \Delta u / \partial z$ is zero (by the same argument presented for the subsonic derivation) and the solution can be expressed in the form

$$u = \frac{1}{2\pi} \frac{\partial}{\partial x} \iint_{\tau} \frac{z \nu_c}{r_c} \Delta u(x_1, y_1) dx_1 dy_1$$

However, by definition

$$\varphi = \int_{-\infty}^x u dx$$

so the relation becomes

$$\varphi = \frac{1}{2\pi} \iint_{\tau} \frac{z \nu_c}{r_c} \Delta u(x_1, y_1) dx_1 dy_1$$

Finally, in terms of the ξ, η, z coordinate system the equation for the velocity potential due to a distribution of elementary horseshoe vortices in the $z=0$ plane can be written

$$\varphi = \frac{(m_1+m_2)z}{2\pi\mu_1\mu_2} \iint_{\tau} \frac{\nu_c}{r_c} \Delta u(\xi_1, \eta_1) d\xi_1 d\eta_1 \quad (52)$$

The doublet distribution.—The potential induced by a sheet of doublets can be obtained from equation (4) by setting the term involving Δw equal to zero (i. e., by removing all sources from the flow field). Expressing the result in the ξ, η, z coordinates, one has

$$\varphi = \frac{z}{2\pi} \left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \iint_{\tau} \frac{\nu_c}{r_c} \Delta \varphi(\xi_1, \eta_1) d\xi_1 d\eta_1 \quad (53)$$

Again it is necessary to carry the partial derivative operations through the two integral signs. And again, just as was the case for the sources, this requires that the order of integration and the limits on the integrals be specified.

⁴ The fact that equation (4) can be written for u as well as φ does not follow immediately in the case of supersonic flow because of the presence of the discontinuities in u along Mach waves emanating from supersonic edges. Proof of the validity for cases of interest herein is given in reference 6.

Once more consider the case when m_1 and m_2 are both less than β , the area τ is the same as that shown in figure 2, and the first integration is made with respect to η_1 . Then equation (53) becomes

$$\varphi = \frac{z}{2\pi} \left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \int_{-\infty}^{\xi_a} d\xi_1 \int_{\lambda_1}^{\lambda_2} d\eta_1 \frac{\nu_c}{r_c} \Delta \varphi(\xi_1, \eta_1) \quad (54)$$

More caution is necessary in moving the derivatives through the integrals than was required in the study of the source case, since at $\xi_1 = \xi_a$ the η_1 integral is *indeterminate*. It is true that the interval of integration λ_2, λ_1 is zero, but it is also true that the integrand is infinite. The value of such an indeterminate form must be obtained by some process such as the following.

The upper limit to the ξ_1 integral, ξ_a , is a function of the two variables z and ξ (see equation (33)); that is, in functional notation $\xi_a = \xi_a(\xi, z)$. Replace z in ξ_a by $\sqrt{z^2 + \epsilon^2}$, then when ϵ is zero ξ_a has not changed. But if $\xi_a(\xi, z)$ is replaced by $\xi_a(\xi, \sqrt{z^2 + \epsilon^2})$ in equation (54) and the *limit taken as ϵ approaches zero*, the indeterminate form mentioned above can be evaluated. Hence, consider

$$\varphi(\xi, \eta, z) = \lim_{\epsilon \rightarrow 0} \frac{z}{2\pi} \left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \int_{-\infty}^{\xi_a(\xi, \sqrt{z^2 + \epsilon^2})} d\xi_1 \int_{\lambda_1}^{\lambda_2} d\eta_1 \frac{\nu_c}{r_c} \Delta \varphi(\xi_1, \eta_1) \quad (55)$$

By applying the operation described by equation (49) and letting a primed function symbolize its value at $\xi_1 = \xi_a(\xi, \sqrt{z^2 + \epsilon^2})$, the expression

$$\begin{aligned} \varphi(\xi, \eta, z) = & \lim_{\epsilon \rightarrow 0} \frac{z}{2\pi\mu_1} \int_{\lambda_1'}^{\lambda_2'} d\eta_1 \frac{\nu_c'}{r_c'} \Delta \varphi'(\xi_a, \eta_1) + \\ & \frac{z}{2\pi} \int_{-\infty}^{\xi_a} d\xi_1 \left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \int_{\lambda_1}^{\lambda_2} d\eta_1 \frac{\nu_c}{r_c} \Delta \varphi(\xi_1, \eta_1) \end{aligned} \quad (56)$$

is obtained.

The first term in equation (56) can be greatly simplified by use of the mean value theorem. In the limit as ϵ goes to zero both λ_1' and λ_2' approach the common value of η_a . Thus, for ϵ very small the variation of $\Delta \varphi'$ and ν_c' in the range $\lambda_1' < \eta_1 < \lambda_2'$ is slight. The same cannot, of course, be said of $1/r_c'$, since λ_1' and λ_2' are the roots of $r_c' = 0$. Using the functional notation $\nu_c' = \nu_c'(\eta_1)$ and applying the mean value theorem, one can write for the first term in equation (56)

$$\lim_{\epsilon \rightarrow 0} \frac{z}{2\pi\mu_1} \nu_c'(\theta) \Delta \varphi'(\xi_a, \theta) \int_{\lambda_1'}^{\lambda_2'} \frac{d\eta_1}{r_c'}, \quad \lambda_1' < \theta < \lambda_2' \quad (57)$$

Now from the definition of r_c given by equation (31a)

$$\int_{\lambda_1'}^{\lambda_2'} \frac{d\eta_1}{r_c'} = \frac{\mu_1}{\sqrt{\beta^2 - m_1^2}} \int_{\lambda_1'}^{\lambda_2'} \frac{d\eta_1}{\sqrt{(\lambda_2' - \eta_1)(\eta_1 - \lambda_1')}} = \frac{\pi\mu_1}{\sqrt{\beta^2 - m_1^2}}$$

which is independent of ϵ . Hence, the expression given in equation (57) reduces to

$$\frac{1}{2} \Delta \varphi(\xi_a, \eta_a)$$

Moving the partial derivatives through the second integral sign in the last term of equation (56) can be accomplished by introducing the finite-part sign defined previously. Since

$$\frac{\mu_1\mu_2}{m_1+m_2} \left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \frac{v_c}{r_c} = -\frac{\beta^2}{r_c^3}$$

it follows that

$$\left(\frac{1}{\mu_2} \frac{\partial}{\partial \eta} + \frac{1}{\mu_1} \frac{\partial}{\partial \xi} \right) \int_{\lambda_1}^{\lambda_2} d\eta_1 \frac{v_c}{r_c} \Delta\varphi(\xi_1, \eta_1) = - \int_{\lambda_1}^{\lambda_2} \frac{\Delta\varphi(\xi_1, \eta_1) d\eta_1}{r_c^3} \left(\frac{m_1+m_2}{\mu_1\mu_2} \right)$$

By means of these equalities, equation (54) can finally be written in the form

$$\varphi(\xi, \eta, z) = \frac{i}{2} \Delta\varphi(\xi_a, \eta_a) - \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_{\tau} d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \quad (58)$$

If the area of integration is not changed but if the order of integration is reversed, it can be shown by a process identical to the one just described that

$$\varphi(\xi, \eta, z) = \frac{i}{2} \Delta\varphi(\xi_b, \eta_b) - \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\eta_1 \int_{\tau} d\xi_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \quad (59)$$

The area τ used in equations (58) and (59) has been defined as that shown in figure 2, the axes ξ and η both chosen so as to lie outside the Mach cone from the origin of the Cartesian system. As the ξ and η axes approach the Mach lines in the x, y plane, that is, as m_2 and m_1 approach β , the residual terms in equations (58) and (59) approach $(\frac{1}{2})\Delta\varphi(\xi_a, -\infty)$ and $(\frac{1}{2})\Delta\varphi(-\infty, \eta_b)$, respectively, which represent the jump in potential infinitely far ahead of and to one side of the point (ξ, η, z) at which the potential is being measured. Hence, in aerodynamic applications, the $(\frac{1}{2})\Delta\varphi(\xi_a, -\infty)$ and $(\frac{1}{2})\Delta\varphi(-\infty, \eta_b)$ can be taken as zero. Thus, when the ξ, η axes lie along the Mach lines, thereby becoming the r, s axes of equations (34), the expressions for φ are without the residue terms and the order of integration is immaterial. When m_1 and m_2 are greater than β the same is true (i. e., the terms $(\frac{1}{2})\Delta\varphi(\xi_a, \eta_a)$ and $(\frac{1}{2})\Delta\varphi(\xi_b, \eta_b)$ are missing from equations (58) and (59), respectively) so that the effect of a distribution of doublets on the velocity potential can be summarized as follows:

For $0 \leq m_1 < \beta, 0 \leq m_2 < \beta$

$$\varphi = \frac{1}{2} \Delta\varphi(\xi_a, \eta_a) - \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_{\tau} d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \quad (60a)$$

$$= \frac{1}{2} \Delta\varphi(\xi_b, \eta_b) - \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\eta_1 \int_{\tau} d\xi_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \quad (60b)$$

For $0 \leq m_1 < \beta, \beta \leq m_2 \leq \infty$

$$\varphi = \frac{1}{2} \Delta\varphi(\xi_a, \eta_a) - \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_{\tau} d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \quad (60c)$$

$$= -\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\eta_1 \int_{\tau} d\xi_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \quad (60d)$$

For $\beta \leq m_1 \leq \infty, 0 \leq m_2 < \beta$

$$\varphi = \frac{1}{2} \Delta\varphi(\xi_b, \eta_b) - \frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\eta_1 \int_{\tau} d\xi_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \quad (60e)$$

$$= -\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \int d\xi_1 \int_{\tau} d\eta_1 \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} \quad (60f)$$

For $\beta \leq m_1 \leq \infty, \beta \leq m_2 \leq \infty$

$$\varphi = -\frac{z\beta^2(m_1+m_2)}{2\pi\mu_1\mu_2} \iint_{\tau} \frac{\Delta\varphi(\xi_1, \eta_1)}{r_c^3} d\xi_1 d\eta_1 \quad (60g)$$

There exists an interesting corollary obtained by subtracting equation (60a) from (60b); namely, that the difference between an integration of supersonic doublets made first in one order and then in the reverse order is equal to the difference in the magnitude of the distribution at two points in the plane.

REDUCTION TO THE PLANE OF THE WING

The next problem is to consider the above formulas in the limiting case as z approaches zero.

The source distribution.—The potential in the plane of the disturbing source sheets follows immediately from equation (51) by simply setting z equal to zero; in this way

$$\varphi(\xi, \eta, 0) = -\frac{(m_1+m_2)}{2\pi\mu_1\mu_2} \iint_{\tau} \frac{\Delta w(\xi_1, \eta_1)}{r_0} d\xi_1 d\eta_1 \quad (61)$$

In order to relate the pressure coefficient C_p to the slope λ_u of the upper surface of the wing (where both C_p and λ_u are defined in equation (40)) the operator

$$-\frac{2}{V_0} \frac{\partial}{\partial x} = -\frac{2\mu_1\mu_2}{V_0(m_1+m_2)} \left(\frac{1}{\mu_1} \frac{\partial}{\partial \xi} + \frac{1}{\mu_2} \frac{\partial}{\partial \eta} \right)$$

must be applied to both sides of equation (61). Hence,

$$C_p = \frac{2}{\pi} \left(\frac{1}{\mu_1} \frac{\partial}{\partial \xi} + \frac{1}{\mu_2} \frac{\partial}{\partial \eta} \right) \iint_{\tau} \frac{\lambda_u(\xi_1, \eta_1)}{r_0} d\xi_1 d\eta_1 \quad (62)$$

The task of moving the partial derivatives through the two integral signs presents a problem identical to the one studied in reducing equation (53) to equations (60). Two inherent singularities in the area occur at the points ξ_a, η_a and ξ_b, η_b and form, just as in equations (60), certain residuals there. In the present discussion, interest is confined to the case when $z=0$. Equation (33) shows immediately

that the values of ξ_a , η_a and ξ_b , η_b for z equal to zero can be written

$$\left. \begin{aligned} (\xi_a)_{z=0} &= (\xi_b)_{z=0} = \xi \\ (\eta_a)_{z=0} &= (\eta_b)_{z=0} = \eta \end{aligned} \right\} \quad (63)$$

It can be shown, therefore (the details being omitted since they are precisely the same as those described in the reduction of equation (53)), that the following relations hold:

For $0 \leq m_1 < \beta$, $0 \leq m_2 < \beta$

$$C_p = \frac{2}{\sqrt{\beta^2 - m_1^2}} \lambda_u(\xi, \eta) - \frac{2(m_1 + m_2)}{\pi \mu_1 \mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\frac{m_1}{\mu_1}(\eta - \eta_1) + \frac{m_2}{\mu_2}(\xi - \xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \quad (64a)$$

$$= \frac{2}{\sqrt{\beta^2 - m_2^2}} \lambda_u(\xi, \eta) - \frac{2(m_1 + m_2)}{\pi \mu_1 \mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{\frac{m_1}{\mu_1}(\eta - \eta_1) + \frac{m_2}{\mu_2}(\xi - \xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \quad (64b)$$

For $0 \leq m_1 < \beta$, $\beta \leq m_2 \leq \infty$

$$C_p = \frac{2}{\sqrt{\beta^2 - m_1^2}} \lambda_u(\xi, \eta) - \frac{2(m_1 + m_2)}{\pi \mu_1 \mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\frac{m_1}{\mu_1}(\eta - \eta_1) + \frac{m_2}{\mu_2}(\xi - \xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \quad (64c)$$

$$= -\frac{2(m_1 + m_2)}{\pi \mu_1 \mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{\frac{m_1}{\mu_1}(\eta - \eta_1) + \frac{m_2}{\mu_2}(\xi - \xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \quad (64d)$$

For $\beta \leq m_1 \leq \infty$, $0 \leq m_2 < \beta$

$$C_p = \frac{2}{\sqrt{\beta^2 - m_2^2}} \lambda_u(\xi, \eta) - \frac{2(m_1 + m_2)}{\pi \mu_1 \mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{\frac{m_1}{\mu_1}(\eta - \eta_1) + \frac{m_2}{\mu_2}(\xi - \xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \quad (64e)$$

$$= -\frac{2(m_1 + m_2)}{\pi \mu_1 \mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\frac{m_1}{\mu_1}(\eta - \eta_1) + \frac{m_2}{\mu_2}(\xi - \xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) \quad (64f)$$

For $\beta \leq m_1 \leq \infty$, $\beta \leq m_2 \leq \infty$

$$C_p = -\frac{2(m_1 + m_2)}{\pi \mu_1 \mu_2} \iint_\tau \frac{\frac{m_1}{\mu_1}(\eta - \eta_1) + \frac{m_2}{\mu_2}(\xi - \xi_1)}{r_0^3} \lambda_u(\xi_1, \eta_1) d\xi_1 d\eta_1 \quad (64g)$$

The elementary-vortex distribution.—The reduction of equation (52), the formula expressing the potential due to a sheet of elementary horseshoe vortices, to the plane of the wing is of little interest since if Δu is known $\varphi_{z=0}$ can be obtained by the simple relation $\varphi_{z=0} = \frac{1}{2} \int_{-\infty}^x \Delta u \, dx$. However,

if the derivative with respect to z is determined on both sides of equation (52) and the limit is found as z approaches zero, the vertical induced velocity in the plane of the wing will be related to the vortex strength there. In a more physical sense, this will relate the slope of the lifting surface to the load distribution it supports. Such an equation is of basic importance.

The mathematical expression to be studied is

$$w = \lim_{z \rightarrow 0} \frac{\partial}{\partial z} \frac{(m_1 + m_2) V_0}{4 \pi \mu_1 \mu_2} \iint_\tau \frac{z \nu_c}{r_c} \frac{\Delta p(\xi_1, \eta_1)}{q} d\xi_1 d\eta_1 \quad (65)$$

The evaluation of w can be divided into two steps: First, the procedure necessary in order to carry the derivative through the first integral and, second, the calculation of I where

$$I = \lim_{z \rightarrow 0} \frac{(m_1 + m_2) V_0}{4 \pi \mu_1 \mu_2} \int \frac{\partial}{\partial z} \int_\tau \frac{z \nu_c}{r_c} \frac{\Delta p(\xi_1, \eta_1)}{q} d\xi_1 d\eta_1 \quad (66)$$

Again the order of integration is important. As in the preceding discussion of the doublet sheet (equations (53) through (60)) and the source sheet (equations (61) through (64)), assume first that the area of integration is the one given in figure 2. Thus, m_1 is less than β and the η_1 integration is performed first.

If ϵ is introduced in order to evaluate the indeterminate form, equation (65) is expressible in the form

$$w = \lim_{\substack{z \rightarrow 0 \\ \epsilon \rightarrow 0}} \frac{(m_1 + m_2) V_0}{4 \pi \mu_1 \mu_2} \frac{\partial}{\partial z} \int_{-\infty}^{\xi_a(\xi, \sqrt{z^2 + \epsilon^2})} d\xi_1 \int_{\lambda_1}^{\lambda_2} d\eta_1 \frac{z \nu_c}{r_c} \frac{\Delta p}{q} \quad (67)$$

where the terms λ_1 and λ_2 are the roots of the quadratic $r_c^2 = 0$ and where the limiting process for ϵ must be performed before that for z . As before, the primed expression denotes values for the particular case $\xi_1 = \xi_a(\xi, \sqrt{z^2 + \epsilon^2})$, and equation (67) reduces to

$$w = \lim_{\substack{z \rightarrow 0 \\ \epsilon \rightarrow 0}} \left(\frac{\partial \xi_a}{\partial z} \right) \frac{(m_1 + m_2) V_0}{4 \pi \mu_1 \mu_2} \int_{\lambda_1'}^{\lambda_2'} \frac{z \nu_c'}{r_c'} \left(\frac{\Delta p}{q} \right)' d\eta_1 + I \quad (68)$$

By means of the mean value theorem, the first term on the right-hand side of the equation can be simplified. The procedure involved in such an analysis was outlined in the derivation of equation (57). The process used here is identical and equation (68) becomes

$$w = \frac{V_0 \sqrt{\beta^2 - m_1^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + I \quad (69)$$

The term I , defined by equation (66), can be expressed in a simple form by introducing the notation for the generalized principal part (see equation (27)). This term becomes

$$I = \frac{(m_1 + m_2) V_0}{4 \pi \mu_1 \mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\nu_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \quad (70)$$

If the integration of the above expressions had been taken in the opposite order or if the range of m_1 and m_2 had been different, the residual term would change. This phenomena has been presented in connection with both doublet and

source distributions and is by now familiar. Finally, therefore, expressions relating the slope of the wing surface to its loading can be written:

For $0 \leq m_1 < \beta$, $0 \leq m_2 < \beta$

$$\frac{w}{V_0} = -\frac{\sqrt{\beta^2 - m_1^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{v_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \quad (71a)$$

$$= -\frac{\sqrt{\beta^2 - m_2^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{v_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \quad (71b)$$

For $0 \leq m_1 < \beta$, $\beta \leq m_2 \leq \infty$

$$\frac{w}{V_0} = -\frac{\sqrt{\beta^2 - m_1^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{v_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \quad (71c)$$

$$= \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{v_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \quad (71d)$$

For $\beta \leq m_1 \leq \infty$, $0 \leq m_2 < \beta$

$$\frac{w}{V_0} = -\frac{\sqrt{\beta^2 - m_2^2}}{4} \frac{\Delta p(\xi, \eta)}{q} + \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{v_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \quad (71e)$$

$$= \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{v_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} \quad (71f)$$

For $\beta \leq m_1 \leq \infty$, $\beta \leq m_2 \leq \infty$

$$\frac{w}{V_0} = \frac{m_1 + m_2}{4\pi\mu_1\mu_2} \oint_\tau \oint_\tau \frac{v_0}{r_0} \frac{\Delta p(\xi_1, \eta_1)}{q} d\xi_1 d\eta_1 \quad (71g)$$

The doublet distribution.—The pertinent problem in this case is to find the vertical induced velocity in the plane of the wing as a function of the jump in potential across the plane.

Consider equations (60) and take the partial derivative of both sides with respect to z ; then find the limit of the resulting expression as z approaches zero. If equation (60a) is used, for example, there results for the first term

$$\lim_{z \rightarrow 0} \frac{\partial}{\partial z} \frac{1}{2} \Delta \varphi(\xi_a, \eta_a) = \frac{1}{2} \left(\frac{\partial \Delta \varphi}{\partial \xi_a} \right)_{z=0} \frac{\partial \xi_a}{\partial z} + \frac{1}{2} \left(\frac{\partial \Delta \varphi}{\partial \eta_a} \right)_{z=0} \frac{\partial \eta_a}{\partial z}$$

which becomes

$$\frac{-1}{2(m_1 + m_2)\sqrt{\beta^2 - m_1^2}} \left[\mu_2(\beta^2 - m_1^2) \frac{\partial}{\partial \xi} \Delta \varphi(\xi, \eta) + \mu_1(m_1 m_2 + \beta^2) \frac{\partial}{\partial \eta} \Delta \varphi(\xi, \eta) \right] \quad (72)$$

The second term can be written

$$-\lim_{z \rightarrow 0} \frac{\partial}{\partial z} \frac{z \beta^2 (m_1 + m_2)}{2\pi\mu_1\mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_c^3}$$

and this reduces to

$$-\frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_0^3} - \frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \lim_{z \rightarrow 0} z \frac{\partial}{\partial z} \int d\xi_1 \oint_\tau d\eta_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_c^3} \quad (73)$$

Since, by equation (17),

$$\oint_{\lambda_1}^{\lambda_2} \frac{d\eta_1}{[(\lambda_2 - \eta_1)(\eta_1 - \lambda_1)]^{3/2}} = 0$$

the second term in expression (73) vanishes. Finally, therefore, the vertical induced velocity in the plane of the wing becomes:

For $0 \leq m_1 < \beta$, $0 \leq m_2 < \beta$

$$w = -\frac{1}{2(m_1 + m_2)\sqrt{\beta^2 - m_1^2}} \left[\mu_2(\beta^2 - m_1^2) \frac{\partial}{\partial \xi} \Delta \varphi(\xi, \eta) + \mu_1(m_1 m_2 + \beta^2) \frac{\partial}{\partial \eta} \Delta \varphi(\xi, \eta) \right] - \frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_0^3} \quad (74a)$$

$$= -\frac{1}{2(m_1 + m_2)\sqrt{\beta^2 - m_2^2}} \left[\mu_2(m_1 m_2 + \beta^2) \frac{\partial}{\partial \xi} \Delta \varphi(\xi, \eta) + \mu_1(\beta^2 - m_2^2) \frac{\partial}{\partial \eta} \Delta \varphi(\xi, \eta) \right] - \frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_0^3} \quad (74b)$$

For $0 \leq m_1 < \beta$, $\beta \leq m_2 \leq \infty$

$$w = -\frac{1}{2(m_1 + m_2)\sqrt{\beta^2 - m_1^2}} \left[\mu_2(\beta^2 - m_1^2) \frac{\partial}{\partial \xi} \Delta \varphi(\xi, \eta) + \mu_1(m_1 m_2 + \beta^2) \frac{\partial}{\partial \eta} \Delta \varphi(\xi, \eta) \right] - \frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_0^3} \quad (74c)$$

$$= -\frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_0^3} \quad (74d)$$

For $\beta \leq m_1 \leq \infty$, $0 \leq m_2 < \beta$

$$w = -\frac{1}{2(m_1 + m_2)\sqrt{\beta^2 - m_2^2}} \left[\mu_2(m_1 m_2 + \beta^2) \frac{\partial}{\partial \xi} \Delta \varphi(\xi, \eta) + \mu_1(\beta^2 - m_2^2) \frac{\partial}{\partial \eta} \Delta \varphi(\xi, \eta) \right] - \frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \oint d\eta_1 \oint_\tau d\xi_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_0^3} \quad (74e)$$

$$= -\frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \oint d\xi_1 \oint_\tau d\eta_1 \frac{\Delta \varphi(\xi_1, \eta_1)}{r_0^3} \quad (74f)$$

For $\beta \leq m_1 \leq \infty$, $\beta \leq m_2 \leq \infty$

$$w = -\frac{\beta^2(m_1 + m_2)}{2\pi\mu_1\mu_2} \oint_\tau \oint_\tau \frac{\Delta \varphi(\xi_1, \eta_1)}{r_0^3} d\xi_1 d\eta_1 \quad (74g)$$

SUMMARY OF RESULTS FOR φ , C_p , AND w IN THE PLANE OF THE WING;
CARTESIAN-COORDINATE SYSTEM

The special forms of equations (64), (71), and (74) when the ξ, η axes become the Cartesian axes are given in the following sections. In these cases, $\xi \rightarrow x$, $\eta \rightarrow y$, $m_1 \rightarrow 0$, and $m_2 \rightarrow \infty$.

Potential in terms of vertical velocity, nonlifting case.—

$$\varphi(x, y) = -\frac{1}{\pi} \int_{\tau} \int \frac{w_u(x_1, y_1)}{\sqrt{(x-x_1)^2 - \beta^2(y-y_1)^2}} dx_1 dy_1 \quad (75)$$

Pressure coefficient in terms of surface slope, nonlifting case.—

$$C_p = \frac{2}{\beta} \lambda_u(x, y) - \frac{2}{\pi} \int_{\tau} dx_1 \int_{\tau} dy_1 \frac{(x-x_1) \lambda_u(x_1, y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2]^{3/2}} \quad (76a)$$

$$C_p = -\frac{2}{\pi} \int_{\tau} dy_1 \int_{\tau} dx_1 \frac{(x-x_1) \lambda_u(x_1, y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2]^{3/2}} \quad (76b)$$

Vertical velocity in terms of loading coefficient, lifting case.—

$$\frac{w_u}{V_0} = -\frac{\beta}{4} \frac{\Delta p(x, y)}{q} + \frac{1}{4\pi} \int_{\tau} dx_1 \int_{\tau} dy_1 \frac{(x-x_1) \frac{\Delta p(x_1, y_1)}{q}}{(y-y_1)^2 \sqrt{(x-x_1)^2 - \beta^2(y-y_1)^2}} \quad (77a)$$

$$\frac{w_u}{V_0} = \frac{1}{4\pi} \int_{\tau} dy_1 \int_{\tau} dx_1 \frac{(x-x_1) \frac{\Delta p(x_1, y_1)}{q}}{(y-y_1)^2 \sqrt{(x-x_1)^2 - \beta^2(y-y_1)^2}} \quad (77b)$$

Vertical velocity in terms of surface potential, lifting case.—

$$w_u = -\frac{\beta}{2} \Delta u(x, y) - \frac{\beta^2}{\pi} \int_{\tau} dx_1 \int_{\tau} dy_1 \frac{\varphi_u(x_1, y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2]^{3/2}} \quad (78a)$$

$$w_u = -\frac{\beta^2}{\pi} \int_{\tau} dy_1 \int_{\tau} dx_1 \frac{\varphi_u(x_1, y_1)}{[(x-x_1)^2 - \beta^2(y-y_1)^2]^{3/2}} \quad (78b)$$

SUMMARY OF RESULTS FOR φ , C_p , AND w IN THE PLANE OF THE WING;
CHARACTERISTIC COORDINATE SYSTEM

The special forms of equations (64), (71), and (74) when the ξ, η axes become the characteristic axes are given in the following sections. In these cases, $\xi \rightarrow r$, $\eta \rightarrow s$, $m_1 \rightarrow \beta$, and $m_2 \rightarrow \beta$.

Potential in terms of vertical velocity, nonlifting case.—

$$\varphi = -\frac{1}{\pi M_0} \int_{\tau} \int \frac{w_u(r_1, s_1)}{\sqrt{(r-r_1)(s-s_1)}} dr_1 ds_1 \quad (79)$$

Pressure coefficient in terms of surface slope, nonlifting case.—

$$C_p = -\frac{1}{2\pi\beta} \int_{\tau} \int \frac{(r-r_1) + (s-s_1)}{[(r-r_1)(s-s_1)]^{3/2}} \lambda_u(r_1, s_1) dr_1 ds_1 \quad (80)$$

Vertical velocity in terms of loading coefficient, lifting case.—

$$\frac{w_u}{V_0} = \frac{\beta}{4\pi} \int_{\tau} \int \frac{(r-r_1) + (s-s_1)}{[(s-s_1) - (r-r_1)]^2 \sqrt{(r-r_1)(s-s_1)}} \frac{\Delta p(r_1, s_1)}{q} dr_1 ds_1 \quad (81)$$

Vertical velocity in terms of surface potential, lifting case.—

$$w_u = -\frac{M_0}{4\pi} \int_{\tau} \int \frac{\varphi_u(r_1, s_1)}{[(r-r_1)(s-s_1)]^{3/2}} dr_1 ds_1 \quad (82)$$

PART II—THE DIRECT PROBLEM

DISCUSSION

The term "direct problem" shall be defined herein as a problem requiring for its solution the evaluation of integrals with known integrands. The three fundamental formulas presented in part I, equations (64), (71), and (74), apply, respectively, to source, vortex, and doublet distributions, and a consideration of them shows that there are essentially only two different boundary-value problems of wing theory that lead to the direct classification. In the first of these problems (see equation (64)), the pressure coefficient is given by an integral involving the shape of a wing having thickness, but no angle of attack, twist, or camber. The other direct problem is represented by equations (71) and (74), where the angle of attack, twist, or camber of a wing having no thickness is given in terms of an integration involving, respectively, the wing loading or its streamwise integral, the discontinuity in velocity potential. The circumstances of the particular problem will determine which of the two alternative formulas is to be used.

THE AERODYNAMIC PROBLEM

The statement of the two problems can be given from a physical viewpoint as follows.

The thickness case.—The thickness of a wing that is symmetrical above and below a horizontal plane is given and the pressure distribution over the wing is to be determined. Such problems are of special interest in the study of wings in a supersonic flow since their evaluation is necessary for the calculation of the wave drag.

The lifting case.—The load distribution on a lifting plane, a surface without thickness, is given and the slope of the surface that will support such a loading is to be determined.

THE MATHEMATICAL PROBLEM

The mathematical statement of the two problems can be made by referring to the equations expressing the three fundamental formulas in the plane of the wing. For the thickness case, equations (41) or (64) apply, λ_u is given over the wing plan form and C_p is to be determined. For the lifting case, equations (43), (44), (71) or (74) apply, $\Delta p/q$ or $\Delta \varphi$ is given over the wing plan form and w_u is to be determined.

The solution of problems in the direct classification depends only on the analyst's facility in evaluating integrals. Although the integrations may be quite difficult to perform, this nevertheless must be regarded as a question of technique and, in a mathematical sense, a direct problem is solved.

PART III—THE INVERSE PROBLEM

INTRODUCTION

The term "inverse problem" shall be defined herein as a problem requiring for its solution the inversion of an integral equation. In application to the study of problems in aerodynamic wing theory, two different boundary-value problems appear in the inverse classification. These are provided, as was discussed in the presentation of the direct problem, by the two basic relationships that exist in the three fundamental formulas. In those equations the two basic relationships are: First, the pressure is given in terms of an integration involving the shape of a wing having thickness but with no angle of attack, twist, or camber; second, the angle of attack, twist, and camber of a wing having no thickness is given in terms of an integration involving either the wing loading or the discontinuity in the velocity potential.

THE AERODYNAMIC PROBLEM

The physical interpretation of the two types of inverse boundary-value problems is made as follows.

The thickness case.—The pressure distribution over a wing that is symmetrical above and below a horizontal plane is given and the shape of the wing is to be determined. To this bare statement of the problem, however, must be added certain auxiliary considerations. For example, it is physically evident that solutions yielding wings with negative volumes must be excluded. Consideration must also be given to the question of wing closure. It is apparent that these two conditions will serve to restrict the arbitrariness of the pressure distributions which can be prescribed. Finally, the question of the uniqueness of the wing shape arises. For example, it is known that the thin-airfoil-theory solution in the two-dimensional case is unique, provided the prescribed pressure distribution is one leading to a real and closed wing section. In the supersonic three-dimensional case, however, these conditions are no longer sufficient to guarantee a unique shape from a given pressure distribution (although the reverse is always true, i. e., a given shape produces a unique pressure). This fact will be illustrated later (Part IV) in connection with quasi-conical flow problems.

The lifting case.—The slope of a lifting plate, a surface without thickness, is given and the resulting load distribution is to be determined. To insure uniqueness in problems of this type it is sometimes necessary to impose an additional condition. For example, it is necessary to assume that the Kutta condition applies to all trailing edges for which the normal component of the free-stream velocity is subsonic.

THE MATHEMATICAL PROBLEM

The mathematical statement of the two problems can be made at once. Thus, for the thickness case, equations (41) or (64) apply, C_p is given over the area occupied by the wing plan form and λ_u is to be determined. For the lifting case, equations (43), (44), (71), or (74) apply, w_u is given over the wing plan form and $\Delta p/q$ is to be determined.

Of course, by definition, the solutions of both of these problems require the inversion of an integral equation. Further, these particular equations are known as singular

integral equations. Complete inversions to all the cases considered have not as yet been obtained. Some progress has been made, however, and the following section outlines one method by means of which certain singular integral equations can be inverted.

ON THE INVERSION OF SINGULAR INTEGRAL EQUATIONS

DEFINITIONS

An integral equation.—Consider the equation

$$\int_{L_1} g(x) K(x, y) dx = w(y) \quad (83)$$

If $w(y)$ and $K(x, y)$ are given functions and $g(x)$ is unknown, equation (83) is known as integral equation, and more specifically as an integral equation of the first kind. The path of integration L_1 lies along the x axis (in this report only real variables are considered although the methods and results can be generalized to include complex variables) and, in general, can depend on y . The term $K(x, y)$ is known as the kernel of the integral equation.

A singular integral equation.—An integral equation is referred to as singular either when the path of integration, L_1 , has infinite extent, or when the kernel, $K(x, y)$, is infinite at points of the interval L_1 . In other words, equation (83) is a singular integral equation if $K(x, y)$ is unbounded somewhere on L_1 .

An integral transform.—Again consider equation (83). If both sides of this equation are multiplied by the function $H(\lambda, y)$ and integrated with respect to y along the interval L_2 (which is, of course, independent of y but can be a function of λ), the equation is said to have been transformed and the operator

$$\int_{L_2} H(\lambda, y) dy \quad (84)$$

is referred to as an integral transform. The resulting expression

$$\int_{L_2} dy \int_{L_1} dx g(x) H(\lambda, y) K(x, y) = \int_{L_2} w(y) H(\lambda, y) dy \quad (85)$$

is obviously a function only of λ , both x and y being dummy variables of integration.

INHERENT SINGULARITIES

An inherent singularity can be defined first in terms of a function of two variables. Consider the function $f(x, y)$ and let the point a, b lie somewhere in the x, y plane. Then an inherent singularity will be said to exist at the point a, b if

$$\lim_{\epsilon \rightarrow 0} \epsilon^2 f(\epsilon \xi + a, \epsilon \eta + b) \neq 0$$

where

$$x - a = \epsilon \xi, \quad y - b = \epsilon \eta$$

In other words, a square of width 2ϵ is first placed on the x, y plane with a, b at its center, the function $f(x, y)$ is then evaluated at any point on the boundary of the square and multiplied by one-fourth the area of the square. Finally, in the limit as the width of the square vanishes, if this product is not

zero, the function $f(x, y)$ contains an inherent singularity at the point a, b .

Such a concept can obviously be generalized to include functions of three and more variables. For example, the function $f(x_1, x_2, \dots, x_n)$ contains an inherent singularity at the point $a_1, a_2, \dots, a_n$ if

$$\lim_{\epsilon \rightarrow 0} \epsilon^n f(\epsilon \xi_1 + a_1, \epsilon \xi_2 + a_2, \dots, \epsilon \xi_n + a_n) \neq 0$$

where

$$x_i - a_i = \epsilon \xi_i$$

RESIDUALS

Consider a double integration with respect to x and y of the function $f(x, y)$ over the area S in the x, y plane. Perform the integration of the same function over the same area but with the order of integrations reversed. The difference between the results of these two operations will be defined as the residual. Thus

$$R = \int dy \int_S dx f(x, y) - \int dx \int_S dy f(x, y) \quad (86)$$

Ordinarily the residual R is zero, since the order of integration for a double integral is usually immaterial. In the manipulation of singular integrals and singular integral transforms, however, a nonvanishing residual often exists. The evaluation of the residual can be accomplished in the following manner. Let the point a_1, b_1 be an inherent singularity in the area S . Then the residual from such a point is

$$R_1 = \lim_{\epsilon \rightarrow 0} \left[\int_{b_1-\epsilon}^{b_1+\epsilon} dy \int_{a_1-\epsilon}^{a_1+\epsilon} dx f(x, y) - \int_{a_1-\epsilon}^{a_1+\epsilon} dx \int_{b_1-\epsilon}^{b_1+\epsilon} dy f(x, y) \right]$$

Setting $\epsilon \xi + a_1 = x$ and $\epsilon \eta + b_1 = y$, one can write

$$R_1 = \int_{-1}^1 d\eta \int_{-1}^1 d\xi \lim_{\epsilon \rightarrow 0} \epsilon^2 f(\epsilon \xi + a_1, \epsilon \eta + b_1) - \int_{-1}^1 d\xi \int_{-1}^1 d\eta \lim_{\epsilon \rightarrow 0} \epsilon^2 f(\epsilon \xi + a_1, \epsilon \eta + b_1) \quad (87)$$

Hence the necessary condition for the existence of a residual is the occurrence of an inherent singularity in the area of integration over which the double integration is performed. The total residual is the sum of the residuals from each inherent singularity in the area involved.

THE NULL TRANSFORM

Definition.—The integral operator $\int_{L_3} H(\lambda, y) dy$ is said to be a null transform of order n to the function $K(x, y)$ in the interval L_3 if

$$\left(\frac{\partial}{\partial x} \right)^n \int_{L_3} H(\lambda, y) K(x, y) dy = 0 \quad (88)$$

where x, λ and, of course, y are on L_3 .

Examples.—The operator

$$\int_{L_3} H(\lambda, y) dy = \int_x^\lambda \frac{dy}{(\lambda - y)^{3/2}}$$

is a null transform of order zero to the function

$$K(x, y) = 1/\sqrt{y - x}$$

in the interval $x \leq y \leq \lambda$. Thus, (see equation (18)),

$$\int_x^\lambda \frac{dy}{\sqrt{(y-x)(\lambda-y)^3}} = 0 \quad (89)$$

The operator

$$\int_{L_3} H(\lambda, y) dy = \int_\lambda^1 \frac{dy}{\sqrt{(1-y)(y-\lambda)}}$$

is a null transform of order zero to the function $K(x, y) = 1/(y-x)$ in the interval $\lambda \leq y \leq 1$. Thus

$$\int_\lambda^1 \frac{dy}{(y-x)\sqrt{(1-y)(\lambda-y)}} = 0, \lambda < x < 1 \quad (90)$$

Finally, it can be shown from equation (90) that

$$\int_{L_3} H(\lambda, y) dy = \int_a^b \frac{\sqrt{(b-y)(y-a)}}{\lambda - y} dy, a < b$$

where a and b are constants, is a null transform of order one to the function $K(x, y) = 1/(y-x)$ in the interval $a < y < b$. Thus

$$\frac{\partial}{\partial x} \int_a^b \frac{\sqrt{(b-y)(y-a)}}{(y-x)(\lambda-y)} dy = 0, \begin{cases} a < x < b \\ a < \lambda < b \end{cases} \quad (91)$$

THE INVERSION OF SINGULAR INTEGRAL EQUATIONS BY MEANS OF NULL TRANSFORMS

Consider an integral equation of the first kind

$$w(y) = \int_{L_1} g(x) K(x, y) dx \quad (92)$$

such that the kernel $K(x, y)$ tends to infinity as x approaches y and let the point $x=y$ lie in L_1 . Equation (92) is, by definition, a singular integral equation of the first kind. Apply to both sides of this equation the integral transform

$\int_{L_2} H(\lambda, y) dy$ so that

$$\int_{L_2} w(y) H(\lambda, y) dy = \int_{L_2} dy \int_{L_1} dx g(x) H(\lambda, y) K(x, y) \quad (93)$$

Suppose that the area $L_1 + L_2$ of the double integral is bounded by a simple closed curve having the property that any line parallel to the x or y axis crosses its boundary at most twice. For such an area, it is always possible to write the reversed form of the double integral in equation (93) as

$$\int_{L_4} dx \int_{L_3} dy g(x) H(\lambda, y) K(x, y) \quad (94)$$

where L_3 can be a function of x and λ , and L_4 can be a function only of λ .

Subtracting expression (94) from the double integral in equation (93), one finds

$$\int_{L_2} dy \int_{L_1} dx g(x) H(\lambda, y) K(x, y) = \int_{L_4} dx \int_{L_3} dy g(x) H(\lambda, y) K(x, y) + R(\lambda) \quad (95)$$

where $R(\lambda)$ is, by definition, the residual. Hence, equation (93) can be rewritten in the form

$$\int_{L_2} w(y) H(\lambda, y) dy = \int_{L_4} dx \int_{L_3} dy g(x) H(\lambda, y) K(x, y) + R(\lambda) \quad (96)$$

One can now show that if $H(\lambda, y)K(x, y)$ contains an inherent singularity at the point $x=y=\lambda$, equation (96) is the inversion to equation (92) when $H(\lambda, y)$ is a null transform of order one or zero to the kernel $K(x, y)$ in the interval L_3 .

First, it is necessary to relate $R(\lambda)$ to $g(\lambda)$. By the definition given as equation (87), $R(\lambda)$ can be written

$$R(\lambda) = R_0(\lambda) + \int_{-1}^1 d\eta \int_{-1}^1 d\xi \lim_{\epsilon \rightarrow 0} g(\epsilon\xi + \lambda) K(\epsilon\xi + \lambda, \epsilon\eta + \lambda) H(\lambda, \epsilon\eta + \lambda) - \int_{-1}^1 d\xi \int_{-1}^1 d\eta \lim_{\epsilon \rightarrow 0} g(\epsilon\xi + \lambda) K(\epsilon\xi + \lambda, \epsilon\eta + \lambda) H(\lambda, \epsilon\eta + \lambda) \quad (97)$$

where $R_0(\lambda)$ is the sum of the remaining residuals (if there are any) from the other inherent singularities that might exist in the area $L_1 + L_2$. This reduces to

$$R(\lambda) = R_0(\lambda) + g(\lambda) R^*(\lambda) \quad (98)$$

where

$$R^*(\lambda) = \int_{-1}^1 d\eta \int_{-1}^1 d\xi \lim_{\epsilon \rightarrow 0} K(\epsilon\xi + \lambda, \epsilon\eta + \lambda) H(\lambda, \epsilon\eta + \lambda) - \int_{-1}^1 d\xi \int_{-1}^1 d\eta \lim_{\epsilon \rightarrow 0} K(\epsilon\xi + \lambda, \epsilon\eta + \lambda) H(\lambda, \epsilon\eta + \lambda) \quad (99)$$

If inherent singularities other than the one at $x=y=\lambda$ exist in $L_1 + L_2$, they must be at a point on the line $x=y$. The residual at such a point say $x=y=a$, would be the product of $g(a)$ and $R_0^*(a)$, the difference between the two appropriate double integrals. Hence, $R_0(\lambda)$ cannot contain $g(\lambda)$.

Now, if $\int_{L_3} H(\lambda, y) dy$ is a null transform of zero order, equation (96) becomes

$$\int_{L_2} w(y) H(\lambda, y) dy = R_0(\lambda) + g(\lambda) R^*(\lambda)$$

or

$$g(\lambda) = \frac{1}{R^*(\lambda)} \left[-R_0(\lambda) + \int_{L_2} w(y) H(\lambda, y) dy \right] \quad (100)$$

which is an inversion of equation (92). Further, if

$$\int_{L_3} H(\lambda, y) dy$$

is a null transform of the first order, so that

$$\int_{L_3} H(\lambda, y) K(x, y) dy = C(\lambda)$$

equation (96) becomes

$$\int_{L_2} w(y) H(\lambda, y) dy = C(\lambda) \int_{L_4} g(x) dx + R_0(\lambda) + g(\lambda) R^*(\lambda) \quad (101)$$

If L_4 is independent of λ this already is an inversion of equation (92). However, if L_4 contains λ then, after dividing through by $C(\lambda)$ and taking the derivative with respect to λ , a first-order differential equation in $g(\lambda)$ results. This is considered to be an inversion.

THE INVERSION OF SOME PARTICULAR SINGULAR INTEGRAL EQUATIONS

ABEL'S INTEGRAL EQUATION

Consider the special form of Abel's integral equation

$$w(y) = \int_a^y \frac{g(x) dx}{\sqrt{y-x}}, \quad a < y \quad (102)$$

It has been shown (see equation (89)) that

$$\int_x^\lambda \frac{dy}{(\lambda-y)^{3/2}}$$

is a null transform of order zero to the kernel $1/\sqrt{y-x}$ in the interval $x \leq y \leq \lambda$. Hence, applying the transform

$$\int_a^\lambda \frac{dy}{(\lambda-y)^{3/2}}$$

to both sides of equation (102), reversing the order of integration, and noting that

$$\int_a^\lambda dy \int_a^y dx \frac{g(x)}{\sqrt{y-x} (\lambda-y)^{3/2}} = \int_a^\lambda g(x) dx \int_x^\lambda dy \frac{1}{\sqrt{y-x} (\lambda-y)^{3/2}} + R(\lambda) = R(\lambda) \quad (103)$$

leads one to the result

$$\int_a^\lambda \frac{w(y) dy}{(\lambda-y)^{3/2}} = R(\lambda) \quad (104)$$

The only inherent singularity that appears in the area of integration occurs at the point $x=y=\lambda$. The residual is obtained by integrating over the shaded area in figure 5 and finding the limit as ϵ goes to zero.

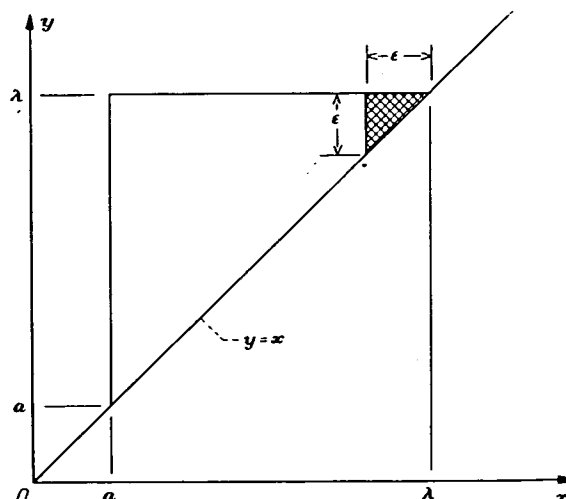


FIGURE 5.—Region of integration for equation (105a).

Thus

$$R(\lambda) = \lim_{\epsilon \rightarrow 0} \left[\int_{\lambda-\epsilon}^{\lambda} dy \int_{\lambda-\epsilon}^y dx \frac{g(x)}{\sqrt{y-x}(\lambda-y)^{3/2}} - \int_{\lambda-\epsilon}^{\lambda} g(x) dx \int_x^{\lambda} dy \frac{1}{\sqrt{y-x}(\lambda-y)^{3/2}} \right] \quad (105a)$$

and, since the second double integral is zero, the transformations $\epsilon\xi = \lambda - x$ and $\epsilon\eta = \lambda - y$ reduce this to

$$R(\lambda) = \lim_{\epsilon \rightarrow 0} \int_1^0 d\eta \int_1^{\eta} d\xi \frac{g(-\epsilon\xi + \lambda)}{\sqrt{\xi - \eta} \eta^{3/2}}$$

Finally, in the limit as ϵ goes to zero,

$$R(\lambda) = g(\lambda) \int_1^0 d\eta \int_1^{\eta} \frac{d\xi}{\eta^{3/2} \sqrt{\xi - \eta}} = -2\pi g(\lambda) \quad (105b)$$

Substitute equation (105b) into equation (104) and the inversion of the integral equation (102) can be written

$$g(\lambda) = \frac{-1}{2\pi} \int_a^{\lambda} \frac{w(y) dy}{(\lambda - y)^{3/2}} \quad (106)$$

By applying the definition of the finite part, one can rewrite equation (106) in the alternative form

$$g(\lambda) = \frac{1}{\pi} \frac{d}{d\lambda} \int_a^{\lambda} \frac{w(y) dy}{\sqrt{\lambda - y}} \quad (107)$$

which is the form of the inversion usually presented.

THE AIRFOIL EQUATION

The study of the singular integral equation known as the airfoil equation is closely associated with the study of boundary-value problems related to Laplace's equation in two dimensions. These boundary conditions are sometimes given along a straight line as is the case, for example, in the linearized study of two-dimensional subsonic wings. If the boundary conditions are given along a suitably prescribed curve, the curve can, by application of complex variable concepts, be mapped onto a straight line. For example, the Joukowski transformation maps a circle in one plane onto a straight line in another and in both planes the governing formula is the two-dimensional form of Laplace's equation. The solution to such boundary-value problems can be reduced to the inversion of the following singular integral equation:

$$w(y) = \int_a^b \frac{g(x) dx}{y - x}, \quad a < y < b \quad (108)$$

where a and b are constants.

It has been shown (see equation (91)) that

$$\int_a^b \frac{\sqrt{(b-y)(y-a)}}{\lambda - y} dy$$

is a null transform of order one to the kernel $1/(y-x)$ in the interval $a \leq y \leq b$. Applying this transform to equation (108), and using the definition of the residual, one obtains

$$\int_a^b \frac{w(y) \sqrt{(b-y)(y-a)}}{\lambda - y} dy = \int_a^b g(x) dx \int_a^b \frac{\sqrt{(b-y)(y-a)}}{(\lambda - y)(y - x)} dy + R(\lambda)$$

which reduces to the form

$$\int_a^b \frac{w(y) \sqrt{(b-y)(y-a)}}{\lambda - y} dy = \pi \int_a^b g(x) dx + R(\lambda) \quad (109)$$

Again the only inherent singularity in the area of integration is at the point $x=y=\lambda$. Evaluating the residual according to equation (87), one finds

$$R(\lambda) = g(\lambda) \sqrt{(b-\lambda)(\lambda-a)} \left[\int_{-1}^1 d\eta \int_{-1}^1 \frac{d\xi}{\eta(\xi-\eta)} - \int_{-1}^1 d\xi \int_{-1}^1 \frac{d\eta}{\eta(\xi-\eta)} \right]$$

and since

$$\int_{-1}^1 \frac{d\eta}{\eta(\xi-\eta)} = \frac{-1}{\xi} \ln \frac{1-\xi}{1+\xi}$$

this becomes

$$R(\lambda) = 2g(\lambda) \sqrt{(b-\lambda)(\lambda-a)} \int_{-1}^1 \frac{d\eta}{\eta} \ln \frac{1-\eta}{1+\eta} = -\pi^2 g(\lambda) \sqrt{(b-\lambda)(\lambda-a)} \quad (110)$$

By the combination of equation (110) with (109), the inversion to the airfoil integral equation (108) thus becomes

$$g(\lambda) = \frac{1}{\pi^2 \sqrt{(b-\lambda)(\lambda-a)}} \left[\pi \int_a^b g(x) dx - \int_a^b \frac{w(y) \sqrt{(b-y)(y-a)} dy}{\lambda - y} \right] \quad (111)$$

It is apparent that the inversion to equation (108) provided by equation (111) is not unique (because of the existence of the term $\int_a^b g(x) dx$ which can be thought of as an arbitrary constant). Hence, in the application of equation (108) to physical boundary-value problems it is not sufficient to specify the value of w along the y axis; some additional condition must also be supplied. Examples of such additional conditions in the study of aerodynamic problems are the specification of closure in the study of two-dimensional sections and the assumption of the Kutta condition along the trailing edge of two-dimensional lifting surfaces.

THE SUPERSONIC DOUBLET EQUATION

The general concepts of the method just applied to the solution of single-integral equations with a singular kernel can also be used to invert double-integral equations with singular kernels. Success in solving these more complicated forms depends again on the discovery of an appropriate integral transform—in this case a double integral transform—and the usefulness of these operators depends, in turn, on both the structure of their integrand and, what is just as important, the space (now four-dimensional) of integration. As it turns out, however, inversions can be obtained in many cases that are of importance in the study of supersonic aerodynamic problems.

Case 1—Supersonic leading edge.—Consider the equation that gives the vertical induced velocity in the plane of the wing in terms of the perturbation velocity potential on the upper surface of a lifting surface in a supersonic free stream, equation (82),

$$w_u(r_2, s_2) = \frac{-M_o}{4\pi} \iint_{\tau(r_2, s_2)} \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{[(r_2 - r_1)(s_2 - s_1)]^{3/2}} \quad (112)$$

The area $\tau(r_2, s_2)$ represents the area within the forecone from the point r_2, s_2 at which the vertical induced velocity is being measured. The details of the solution to equation (112) will now be presented for two different types of boundaries to the area $\tau(r_2, s_2)$, that is, from an aerodynamic standpoint, for two different types of wing plan forms.

First consider the case when $\tau = \tau_0$ is an area such as the one shown in figure 6. The two lines $r_1 = r_2$ and $s_1 = s_2$, which represent the traces of the Mach forecones from the point r_2, s_2 , form two bounds of the area while the third is given by the wing leading edge, the equation for which may be written as either $s_1 = f_0(r_1)$ or $r_1 = f_0^*(s_1)$.

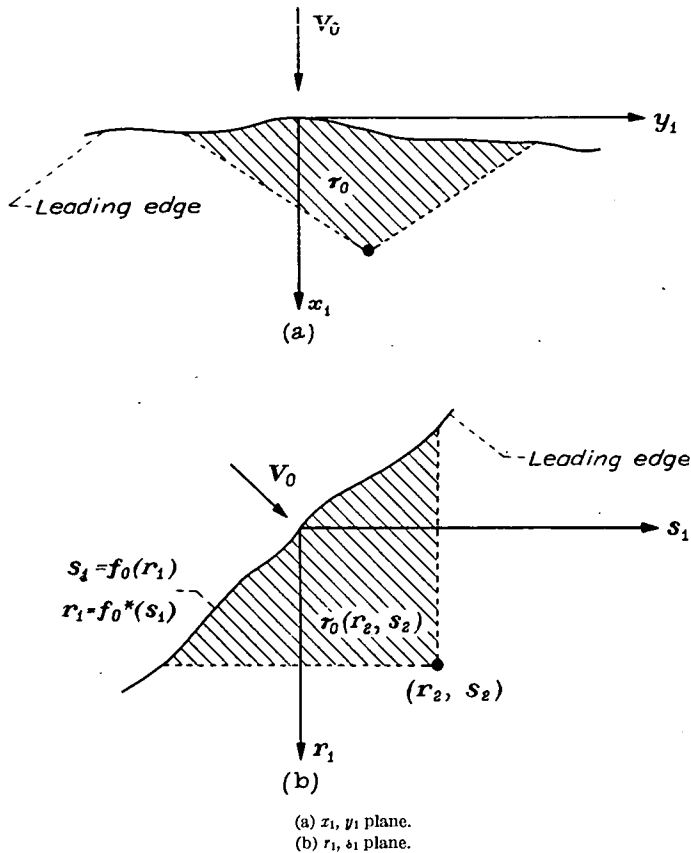


FIGURE 6.—Region of integration for wing with supersonic leading edge.

In the present case, it is assumed, furthermore, that $s_1 = f_0(r_1)$ is a continuous monotonic function with a negative slope, or, physically, that the wing has a supersonic type leading edge.⁵

The kernel of equation (112) is formed by the product of two functions, one independent of r_1 and r_2 and the other independent of s_1 and s_2 . It follows, therefore, that if an integral operator is used that contains the product of two linear functions of s_2 and r_2 to the $-1/2$ power, there exists the possibility of obtaining a null transform of order zero according to the equality given as equation (18). One can show, in fact, that the operator

$$\iint_{\tau_0(r, s)} \frac{dr_2 ds_2}{\sqrt{(r - r_2)(s - s_2)}} = \frac{2\beta}{M_o} \iint_{\tau_0(r, s)} \frac{dr_2 ds_2}{r_o(r, s; r_2, s_2)} \quad (113)$$

is a null transform of order zero to the right-hand side of equation (112) where $\tau_o(r, s)$ is the same area that appears in the integral equation.

If the transform given by equation (113) is applied to equation (112) the resulting expression is

$$\iint_{\tau_0(r, s)} \frac{w_u dr_2 ds_2}{r_o} = \frac{-2\beta^3}{\pi M_o^2} \iint_{\tau_0(r, s)} dr_2 ds_2 \iint_{\tau_0(r_2, s_2)} \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{r_o(r, s; r_2, s_2) r_o^3(r_2, s_2; r_1, s_1)} \quad (114)$$

Consider next the right-hand side of this equation but with the r_2, s_2 integrals taken first. Then by definition of the residual

$$\begin{aligned} & \frac{-2\beta^3}{\pi M_o^2} \iint_{\tau_0(r, s)} dr_2 ds_2 \iint_{\tau_0(r_2, s_2)} \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{r_o(r, s; r_2, s_2) r_o^3(r_2, s_2; r_1, s_1)} = \\ & \frac{-2\beta^3}{\pi M_o^2} \iint_{[\tau_0(r, s) + \tau_0(r_2, s_2)]} \varphi_u(r_1, s_1) dr_1 ds_1 \iint_{\tau_0(r, s)} \frac{dr_2 ds_2}{r_o(r, s; r_2, s_2) r_o^3(r_2, s_2; r_1, s_1)} + \\ & R(r, s) \end{aligned} \quad (115)$$

When the integration is made first with respect to r_2 and s_2 , the area of integration for these two variables can be visualized with the aid of figure 7. In such a case the points r_1, s_1 and r, s are fixed and the area is simply the one which lies in the forecone from the point r, s and in the aftercone from the point r_1, s_1 . This is represented by the shaded area in the figure. It is apparent from a study of figure 7 that, when the edge $s_2 = f_0(r_2)$ is a monotonic function as previously defined, the r_2 and s_2 integrals are always taken between the limits r_1 and r and s_1 and s , respectively. Hence, according to equation (18) the inte-

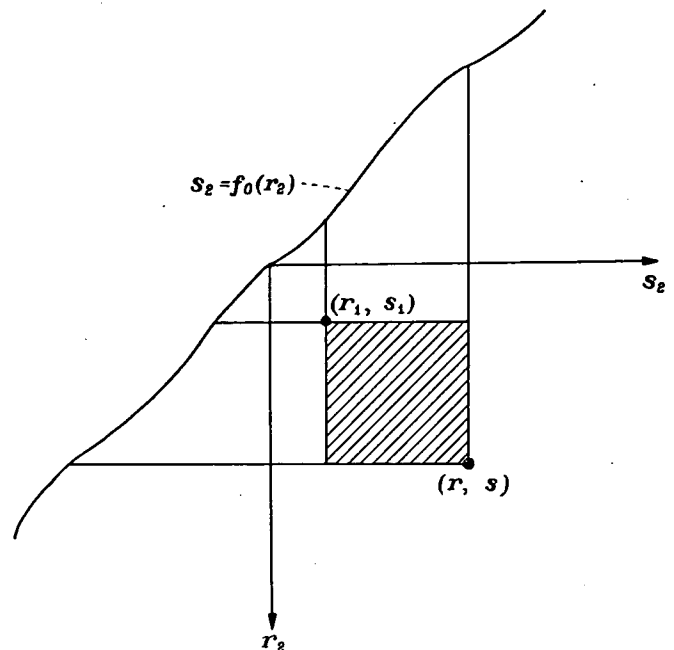


FIGURE 7.—Area of integration for fixed r_1, s_1, r, s ; equation (115).

⁵ Supersonic and subsonic edges have the property that the normal component of the free-stream velocity is supersonic and subsonic, respectively.

gral term on the right-hand side of equation (115) vanishes and equation (114) becomes

$$\int_{\tau_0(r,s)} \int \frac{w_u dr_2 ds_2}{r_0} = R(r,s) \quad (116)$$

The complete inversion of the integral equation can now be realized if an expression for $R(r,s)$ can be obtained in terms of $\varphi_u(r,s)$. The evaluation of this residual term follows along lines quite similar to those used in calculating the residual for the special form of Abel's integral equation given previously. Now, however, there is no longer a single inherent singularity; rather, the lines $r_1=r_2=r$ and $s_1=s_2=s$ are densely covered with them. First consider an integration made over the region close to the line $s_1=s_2=s$ (i. e., the sum of the areas a and b shown in fig. 8).

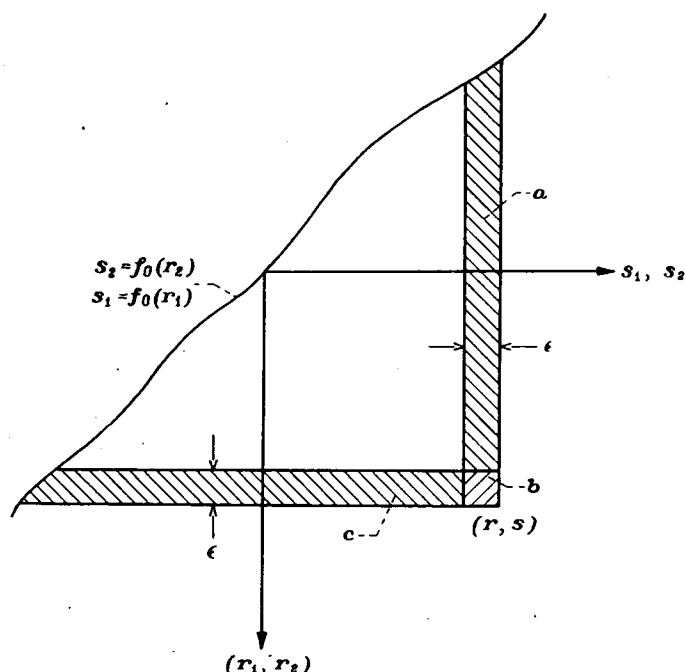


FIGURE 8.—Region of integration for evaluation of residual, equation (116).

Make the substitutions $s_1=s-\epsilon\sigma_1$ and $s_2=s-\epsilon\sigma_2$ and take the limit as ϵ goes to zero. There results for the portion of the residual due to the inherent singularities $R_{a+b}(r,s)$ along the line $s_1=s_2=s$ the equation⁶

$R_{a+b}(r,s)=$

$$\frac{-M_0^2}{8\pi\beta} \int_{f_0^*(s)}^r dr_2 \int_1^0 d\sigma_2 \int_{f_0^*(s)}^{r_2} dr_1 \int_1^{\sigma_2} d\sigma_1 \frac{\varphi_u(r_1,s)}{(\sigma_1-\sigma_2)^{3/2}(r_2-r_1)^{3/2}\sqrt{\sigma_2(r-r_2)}} \quad (117)$$

This becomes, after integrating with respect to σ_1 and σ_2 (which can be done immediately since the limits of the r_1 integral do not now contain σ_2 and, further, since the r_1, σ_2 plane contains no inherent singularity)

⁶ The first term on the right-hand side of equation (115) contributes nothing to the residual since it always contains an integral equivalent to that in equation (18) and, therefore, is identically zero. This same phenomenon also appeared in the study of the residual appearing in the inversion of Abel's integral equation. (See the development of equation (105).)

$$R_{a+b}(r,s) = \frac{M_0^2}{4\beta} \int_{f_0^*(s)}^r dr_2 \int_{f_0^*(s)}^{r_2} dr_1 \frac{\varphi_u(r_1,s)}{(r_2-r_1)^{3/2}\sqrt{r-r_2}} \quad (118)$$

It is now proposed to reverse the order of integration of the double integral in equation (118). But the r_1, r_2 plane contains an inherent singularity at the point $r_2=r_1=r$. By definition of the residual from this singularity as $R'(r,s)$, it follows that

$$R_{a+b}(r,s) = \frac{M_0^2}{4\beta} \int_{f_0^*(s)}^r dr_1 \varphi_u(r_1,s) \int_{r_1}^r \frac{dr_2}{(r_2-r_1)^{3/2}\sqrt{r-r_2}} + R'(r,s) = R'(r,s)$$

since the integral term is zero by the equality that has been used repeatedly. The evaluation of $R'(r,s)$ follows the identical line of argument used in obtaining equation (105b) from (105a). Hence, setting $r_2=r-\epsilon\rho_2$ and $r_1=r-\epsilon\rho_1$, and letting ϵ go to zero, one finds

$$R_{a+b}(r,s) = R'(r,s) = \frac{M_0^2}{4\beta} \varphi_u(r,s) \int_1^0 d\rho_2 \int_1^{\rho_2} \frac{d\rho_1}{(\rho_1-\rho_2)^{3/2}\sqrt{\rho_2}} = -\frac{\pi M_0^2}{2\beta} \varphi_u(r,s) \quad (119)$$

The part of the residual in equation (116) contributed by the inherent singularities in the areas a and b in figure 8 is, therefore, $-\frac{\pi M_0^2}{2\beta} \varphi_u(r,s)$. A similar calculation shows that the singularities in the areas $c+b$ give the same result; and, finally, a calculation for the area b itself also yields $-\frac{\pi M_0^2}{2\beta} \varphi_u(r,s)$. The value of R in equation (116) is obtained by combining the results for the various areas. Hence,

$$R = R_{a+b} + R_{b+c} - R_b = -\frac{\pi M_0^2}{2\beta} \varphi_u(r,s) \quad (120)$$

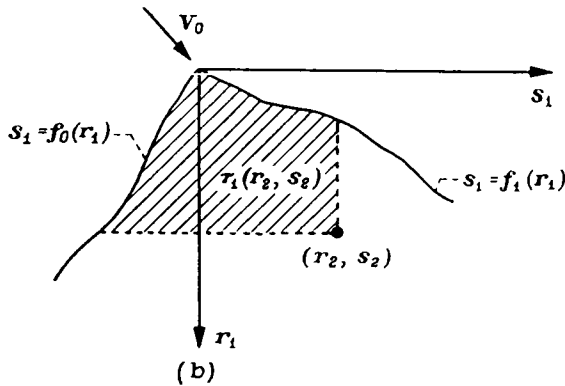
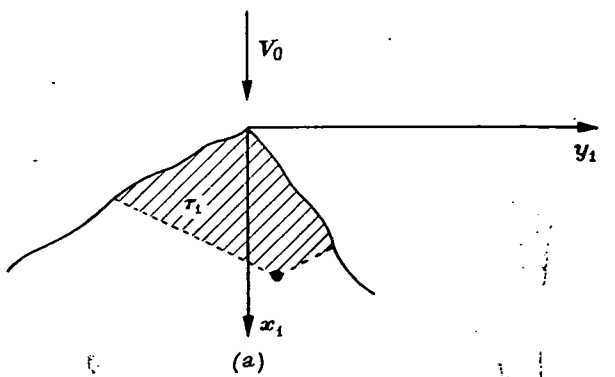
Combining the result expressed by the last equation with equation (116), one can finally write for the inversion of the integral equation (112), when $\tau=\tau_0$, the expression

$$\varphi_u(r,s) = \frac{-1}{\pi M_0} \iint_{\tau_0(r,s)} \frac{w_u dr_2 ds_2}{\sqrt{(r-r_2)(s-s_2)}} \quad (121)$$

Case 2—Combined subsonic and supersonic leading edges.—Consider the equation

$$w_u = \frac{-M_0}{4\pi} \iint_{\tau_1(r_2,s_2)} \frac{\varphi_u(r_1,s_1) dr_1 ds_1}{[(r_2-r_1)(s_2-s_1)]^{3/2}} \quad (122)$$

where $\tau_1(r_2,s_2)$ is the more complicated area shown in figure 9. Again the lines $r_1=r_2$ and $s_1=s_2$, which are the traces of the forecone from the point r_2, s_2 , form two bounds of the area. The remaining two boundaries are formed by the curves $s_1=f_0(r_1)$ (or in the inverse sense $r_1=f_0^*(s_1)$) and $s_1=f_1(r_1)$ (or $r_1=f_1^*(s_1)$) where f_0 has the same definition it had in the study of Case 1, that is, a monotonic curve with a negative slope. The curve $s_1=f_1(r_1)$ is also a monotonic function, but with a positive slope. For convenience the origin is placed at the point of intersection of the f_0 and f_1 curves.



(a) x_1, y_1 plane.
(b) (r_1, s_1) plane.

FIGURE 9.—Region of integration for wing with combined subsonic and supersonic edges.

A physical interpretation of the area τ_1 with regard to problems in supersonic wing theory is simple. The lines $r_1=r_2$ and $s_1=s_2$, as has been mentioned, are the traces of the Mach forecone from the point r_2, s_2 . The f_0 and f_1 curves are the edges of the wing plan form, the line $s_1=f_0(r_1)$ representing a supersonic leading edge and the line $s_1=f_1(r_1)$ representing a subsonic leading edge.

Proceeding exactly as in Case 1, consider the operator

$$\iint_{\tau_1(r,s)} \frac{dr_2 ds_2}{\sqrt{(r-r_2)(s-s_2)}}$$

and find whether or not it is a null transform of zero order to the kernel of equation (122), that is, whether or not the equality

$$\iint_{\tau_1(s,r)+\tau_1(r_2,s_2)} \varphi_u(r_1,s_1) dr_1 ds_1 \oint \oint \frac{dr_2 ds_2}{[(r_2-r_1)(s_2-s_1)]^{3/2} \sqrt{(r-r_2)(s-s_2)}} = 0 \quad (123)$$

is satisfied. A study of figure (10) shows that it is not, since for a point r_1, s_1 located above the line $r_2=f_1^*(s)$, a portion of the s_2 integral becomes

$$\int_{s_1}^{f_1(r_2)} \frac{ds_2}{(s_2-s_1)^{3/2} \sqrt{s-s_2}}$$

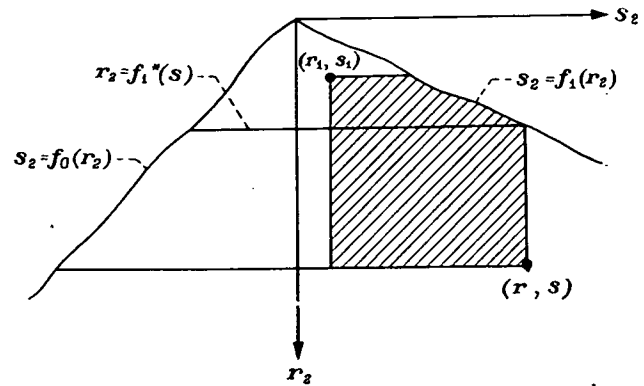
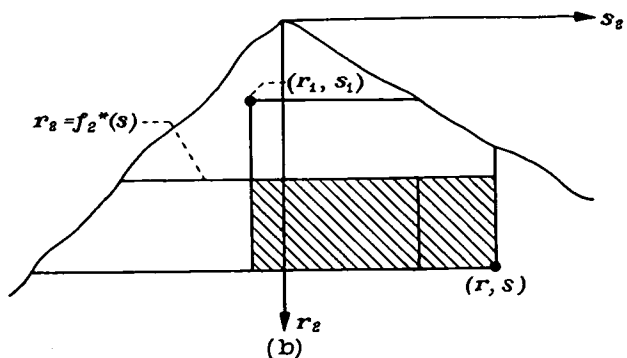
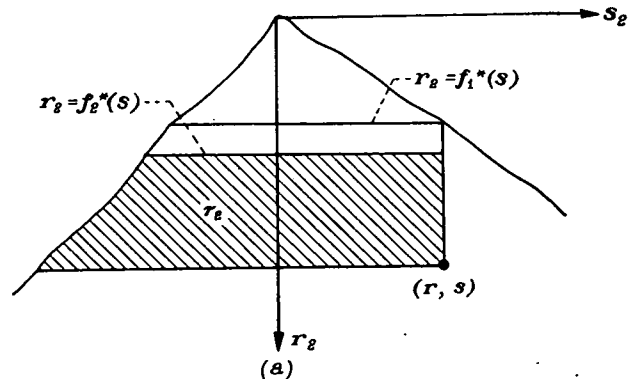


FIGURE 10.—Area of integration for fixed r_1, s_1, r, s ; equation (123).

which is not zero unless $f_1(r_2)$ is identically equal to s . For the same reason, portions of the r_2 integral will not vanish for the point r_1, s_1 so located.

However, the construction of a null transform of zero order can be accomplished by studying the above failure. Thus, if the s_2 integration were carried between the limits s_1 and s , for every location of the points r_1, s_1 and r, s , the operator $\oint h(r, s, r_2) dr_2 \oint (s-s_2)^{-1/2} ds_2$ would be a null transform of zero order regardless of the choice of h . The area of integration producing such a transform is simply the one shown in figure 11(a), an area bounded by the lines $s_2=s, r_2=r, s_2=f_0(r_2)$ and $r_2=f_2^*(s)$ where $r_2=f_2^*(s)$ is any line such that $f_2^*(s) \geq f_1^*(s)$. This area shall be designated as $\tau_2(r, s)$. When such a transform is applied and the s_2 integration performed first, it is apparent from figure 11(b) that the limits of the s_2 integral



(a) Area of integration for transform.
(b) Area of integration for fixed r_1, s_1, r, s .

FIGURE 11.—Integration regions for equations (124).

are always s_1 and s , even when the point r_1, s_1 is located above the line $r_2=f_1^*(s)$. Hence, if the residual $R(r, s)$ is defined by the relation

$$-\frac{M_0}{4\pi} \int h(r, s, r_2) dr_2 \int_{\tau_2(r, s)} \frac{ds_2}{\sqrt{s-s_2}} \iint_{\tau_1(r_2, s_2)} \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{[(r_2-r_1)(s_2-s_1)]^{3/2}} +$$

$$\frac{M_0}{4\pi} \int h(r, s, r_2) dr_2 \iint_{\tau_2(r, s)+\tau_1(r_2, s_2)} \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{(r_2-r_1)^{3/2}} \int_{s_1}^s \frac{ds_2}{\sqrt{s-s_2}(s_2-s_1)^{3/2}} = R(r, s) \quad (124a)$$

the second term on the left-hand side is zero, and equation (122) transforms into

$$\int_{f_2^*(s)}^r h(r, s, r_2) dr_2 \int_{f_0(r_2)}^s \frac{w_u(r_2, s_2)}{\sqrt{s-s_2}} ds_2 = R(r, s) \quad (124b)$$

The evaluation of the residual in equation (124b) follows the same pattern used in Case 1. Thus, after isolating the line of inherent singularities, $R(r, s)$ is given by the expression

$$R(r, s) = -\frac{M_0}{4\pi} \int_{f_2^*(s)}^r h dr_2 \int_{s-\epsilon}^s \frac{ds_2}{\sqrt{s-s_2}} \int_{f_1^*(s_2)}^{r_2} ds_1 \int_{s-\epsilon}^{s_2} dr_1 \frac{\varphi_u(r_1, s_1)}{[(r_2-r_1)(s_2-s_1)]^{3/2}}$$

which becomes (after setting $s_2=s-\epsilon\sigma_2$ and $s_1=s-\epsilon\sigma_1$ and letting ϵ go to zero)

$$R = \frac{M_0}{2} \int_{f_2^*(s)}^r h dr_2 \int_{f_1^*(s)}^{r_2} \frac{\varphi_u(r_1, s)}{(r_2-r_1)^{3/2}} dr_1 \quad (125)$$

It is now apparent that after setting $f_2^*(s)=f_1^*(s)$ and $h=1/\sqrt{r-r_2}$, equation (125) can be evaluated exactly as equation (118) was evaluated, so that for these values of f_2^* and h , $R(r, s) = -M_0\pi\varphi_u(r, s)$. Substituting into equation (124b), one finds for the inversion of equation (122) the result

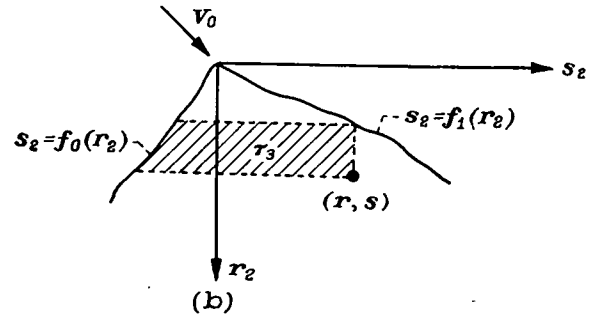
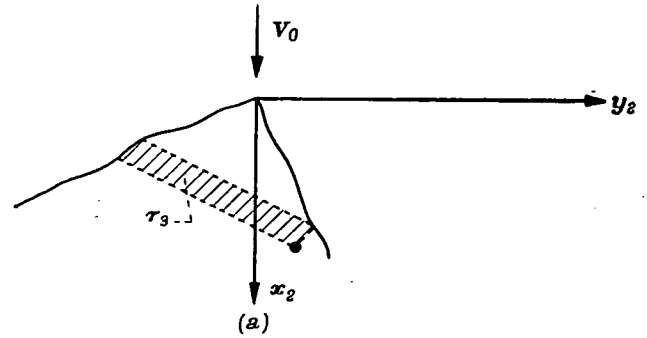
$$\varphi_u(r, s) = \frac{-1}{\pi M_0} \iint_{\tau_3(r, s)} \frac{w_u(r_2, s_2) dr_2 ds_2}{\sqrt{(r-r_2)(s-s_2)}} \quad (126)$$

where τ_3 is the shaded area in figure 12.

Case 3—Mixed boundary conditions.—Another very important kind of integral equation which can be solved directly by the proper choice of h in equation (125) is the "mixed" type problem, the boundary values of which are illustrated in figure 13. In this particular problem w_u is known over the portion of the r, s plane bounded by the curves $s=f_0(r)$ and $s=f_2(r)$ (the curve $s=f_2(r)$ is a monotonic function with a positive slope just like $s=f_1(r)$), while over another portion, bounded by the curves $s=f_2(r)$ and $s=f_1(r)$ the quantity $u_u(r, s) = \frac{M_0}{2\beta} \left(\frac{\partial}{\partial r} + \frac{\partial}{\partial s} \right) \varphi_u(r, s)$ is given. (This corresponds in aerodynamic applications to the specification of the vertical induced velocity over the former region and the loading over the latter.) It is also assumed that $u_u(r, s)$ is continuous across the line $s=f_2(r)$ and that $\varphi(r, s)$ vanishes along the line $s=f_1(r)$, that is, $\varphi(f_1^*(r), s)=0$.

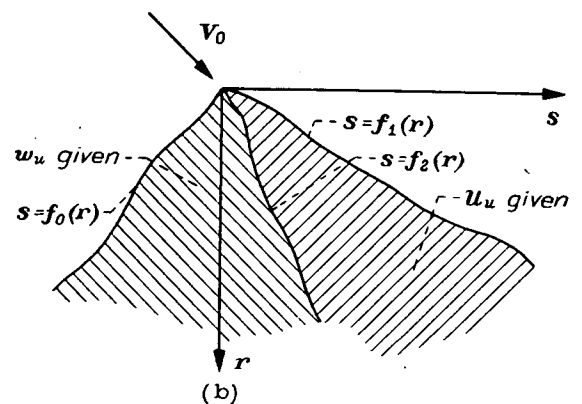
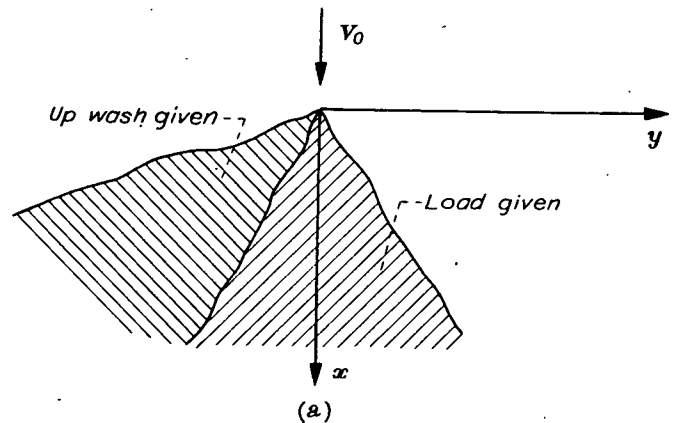
Taking for the value of h the expression

$$h = \frac{1}{\sqrt{r-r_2}} \left(\frac{\partial}{\partial r_2} + \frac{\partial}{\partial s} \right)$$



(a) x_2, y_2 plane.
(b) r_2, s_2 plane.

FIGURE 12.—Integration area τ_3 (equation (126)).



(a) x, y plane.
(b) r, s plane.

FIGURE 13.—Mixed boundary conditions.

and for the area of integration the region $\tau_2(r, s)$ shown in figure 11(a), one finds from equation (125) that the residual can be written

$$R = \frac{M_0}{2} \int_{f_2^*(s)}^r \frac{dr_2}{\sqrt{r-r_2}} \left(\frac{\partial}{\partial r_2} + \frac{\partial}{\partial s} \right) \int_{f_1^*(s)}^{r_2} \frac{\varphi_u(r_1, s) dr_1}{(r_2 - r_1)^{3/2}}$$

The partial derivative can be taken through the second integral sign (first integrating by parts in the case of $\partial/\partial r_2$) since $\varphi(f_1^*(s), s) = 0$, so that

$$R = \frac{M_0}{2} \int_{f_2^*(s)}^r dr_2 \int_{f_1^*(s)}^{r_2} dr_1 \frac{\left(\frac{\partial}{\partial r_1} + \frac{\partial}{\partial s} \right) \varphi_u(r_1, s)}{\sqrt{r-r_2}(r_2-r_1)^{3/2}}$$

Part of the boundary condition is that $u_u(r, s)$ is given over the region indicated in figure 13. Hence, the residual can be written

$$R = \beta \int_{f_2^*(s)}^r dr_2 \int_{f_1^*(s)}^{f_2^*(s)} dr_1 \frac{u_u(r_1, s)}{\sqrt{r-r_2}(r_2-r_1)^{3/2}} + \beta \int_{f_2^*(s)}^r dr_2 \int_{f_2^*(s)}^{r_2} dr_1 \frac{u_u(r_1, s)}{\sqrt{r-r_2}(r_2-r_1)^{3/2}}$$

The latter of these two terms is again just like the one given in equation (118) so its evaluation is immediate. The first term may be simplified by reversing the order of integration. (There is no residual since the point $r=r_1=r_2$ is not included in the area of integration.) Finally, R becomes

$$R = 2\beta \sqrt{r-f_2^*(s)} \int_{f_1^*(s)}^{f_2^*(s)} \frac{u_u(r_1, s) dr_1}{(r-r_1)\sqrt{f_2^*(s)-r_1}} - 2\pi\beta u_u(r, s)$$

and the inversion of this mixed type problem can be written

$$u_u(r, s) = \frac{\sqrt{r-f_2^*(s)}}{\pi} \int_{f_1^*(s)}^{f_2^*(s)} \frac{u_u(r_1, s) dr_1}{(r-r_1)\sqrt{f_2^*(s)-r_1}} - \frac{1}{2\beta\pi} \int_{f_2^*(s)}^r \frac{dr_2}{\sqrt{r-r_2}} \left(\frac{\partial}{\partial r_2} + \frac{\partial}{\partial s} \right) \int_{f_0(r_2)}^s \frac{w_u(r_2, s_2) ds_2}{\sqrt{s-s_2}} \quad (127)$$

An equation which does not impose the condition that $u_u(r, s)$ be continuous across the line $s=f_2(r)$ can readily be developed by using the operator $h=1/\sqrt{r-r_2}$. This leads directly to the result

$$\varphi(r, s) = \frac{\sqrt{r-f_2^*(s)}}{\pi} \int_{f_1^*(s)}^{f_2^*(s)} \frac{\varphi_u(r_1, s) dr_1}{(r-r_1)\sqrt{f_2^*(s)-r_1}} - \frac{1}{\pi M_0} \int_{f_2^*(s)}^r dr_2 \int_{f_0(r_2)}^s ds_2 \frac{w_u(r_2, s_2)}{\sqrt{(s-s_2)(r-r_2)}} \quad (128)$$

THE SUPERSONIC SOURCE EQUATION

The null transforms to the supersonic doublet equation were constructed by applying an integral operator having an integrand which, when combined with the kernel, would produce the integral

$$\int \frac{dx}{(x-b)^{3/2} \sqrt{a-x}}$$

and having an area which, when combined with that of the integral equation and traversed in reverse order, would pro-

duce the limits a and b on the integral. If one now considers the supersonic source equation (see equation (79))

$$\varphi_u(r_2, s_2) = \frac{-1}{\pi M_0} \iint_{\tau(r_2, s_2)} \frac{w_u(r_1, s_1) dr_1 ds_1}{\sqrt{(r_2-r_1)(s_2-s_1)}} \quad (129)$$

it is apparent that the null transform will differ from the transform used for the supersonic doublet equation only in the exponent of the integrand. Thus, for the area $\tau_0(r_2, s_2)$ (see fig. 6) the operator

$$\iint_{\tau_0(r, s)} \frac{dr_2 ds_2}{[(r-r_2)(s-s_2)]^{3/2}}$$

is a null transform of order zero to equation (129), and its application yields the solution

$$w_u(r, s) = \frac{-M_0}{4\pi} \iint_{\tau_0(r, s)} \frac{\varphi_u(r_2, s_2) dr_2 ds_2}{[(r-r_2)(s-s_2)]^{3/2}} \quad (130)$$

For the area $\tau_1(r_2, s_2)$ (see fig. 9), the operator

$$\iint_{\tau_1(r, s)} \frac{dr_2 ds_2}{[(r-r_2)(s-s_2)]^{3/2}}$$

is a null transform of order zero. Its application, under conditions like those specified in the development of equation (126), yields the inversion

$$w_u(r, s) = \frac{-M_0}{4\pi} \iint_{\tau_1(r, s)} \frac{\varphi_u(r_2, s_2) dr_2 ds_2}{[(r-r_2)(s-s_2)]^{3/2}} \quad (131)$$

Equation (130) gives vertical velocity in terms of a prescribed surface potential for a wing with a supersonic leading edge, while equation (131) does the same for a wing with a leading edge which is partly subsonic and partly supersonic. The transform with the area τ_3 (see fig. 12) can also be used to obtain inversions to equation (129) under conditions such as those imposed in the development of equations (127) and (128).

DISCUSSION OF INVERSION OF SUPERSONIC SOURCE AND DOUBLET EQUATIONS FROM A PHYSICAL BASIS

Each of the above examples used to illustrate the application of the null transform method to the inversion of double integral equations represents the solution of a class of supersonic problems. However, most of these solutions are well known and were originally obtained by reasoning that was suggested by knowledge of the physical structure of the problem. For example, the reciprocal relation between the source and doublet integral equations

$$\left. \begin{aligned} \varphi_u(r, s) &= \frac{-1}{\pi M_0} \int_{\tau_0} \int \frac{w_u(r_1, s_1) dr_1 ds_1}{\sqrt{(r-r_1)(s-s_1)}} \\ w_u(r, s) &= \frac{-M_0}{4\pi} \int_{\tau_0} \int \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{[(r-r_1)(s-s_1)]^{3/2}} \end{aligned} \right\} \quad (132)$$

when considered with respect to the area τ_0 , has a simple physical meaning. By definition τ_0 is an area bounded by the Mach forecone and a supersonic leading-edge. From a

physical standpoint it is clear that the flow field at a point, affected only by a supersonic edge, on the upper surface of a wing cannot be influenced by the shape of any part of the lower surface of the wing. In other words, the upper and lower surfaces are noninteracting. Hence, the upper surface of a wing does not "know" whether it is the upper surface of a lifting plate that is supporting loading and has no thickness, or the upper surface of a wing section that is symmetrical above and below the $z=0$ plane. Thus the source equation must be the inversion to the doublet equation and vice versa, and from a physical point of view the reciprocal relation given by equation (132) is obvious.

The solution to wing problems involving one supersonic and one subsonic edge, giving an area of integration for the integral equations corresponding to the area τ_1 in the previous discussion, was originally obtained by Evvard (see reference 7). It should be noted here that the inversions to the source and doublet equations (equations (126) and (131)) considered with respect to the area τ_1 , no longer form reciprocal relations.

Finally, the examples presented herein with regard to the mixed type of problem have also been derived (see references 8 and 9) using more or less physical arguments. The solutions to these mixed type problems form the basis of a lift cancellation technique that provides a very useful extension of Evvard's original discovery.

ITERATIVE METHODS OF SOLUTION

Other types of plan forms.—The question that naturally arises from a practical viewpoint is how the source and doublet equations can be inverted when the area τ_1 is not of the two special kinds discussed, or, in other words, when the wing plan forms are complicated by having more than one monotonic (in the r, s plane) subsonic edge. The answer must be that, unless null transforms with respect to these new integration areas can be discovered, the methods discussed here will not give the direct inversion to the problem. Several possibilities remain, however, so that even when the null transform cannot be found the concepts of the residual, inherent singularity, etc., can be used to simplify, if not solve, the supersonic source and doublet integral equations.

With respect to the lift cancellation techniques already mentioned, references 8 and 9 outline these methods in considerable detail and show how they can be applied to find the loading on wings in regions affected by two or more subsonic edges.

Regions influenced by multiple reflections of Mach waves.—A more direct example of how some of the concepts presented heretofore can be applied, even when the null transform is not available, is given by considering the following problem: Find the loading at the point (x, y) on the flat wing tip shown in figure 14. If the r, s coordinate system is used and the operator

$$\int_{f_2(s)}^r \frac{dr_2}{\sqrt{r-r_2}} \left(\frac{\partial}{\partial r_2} + \frac{\partial}{\partial s} \right) \int_{f_3(r)}^s \frac{ds_2}{\sqrt{s-s_2}}$$

(where the area of integration is region 4 in fig. 14 (b)) is applied to the doublet equation (equation (82)), two results

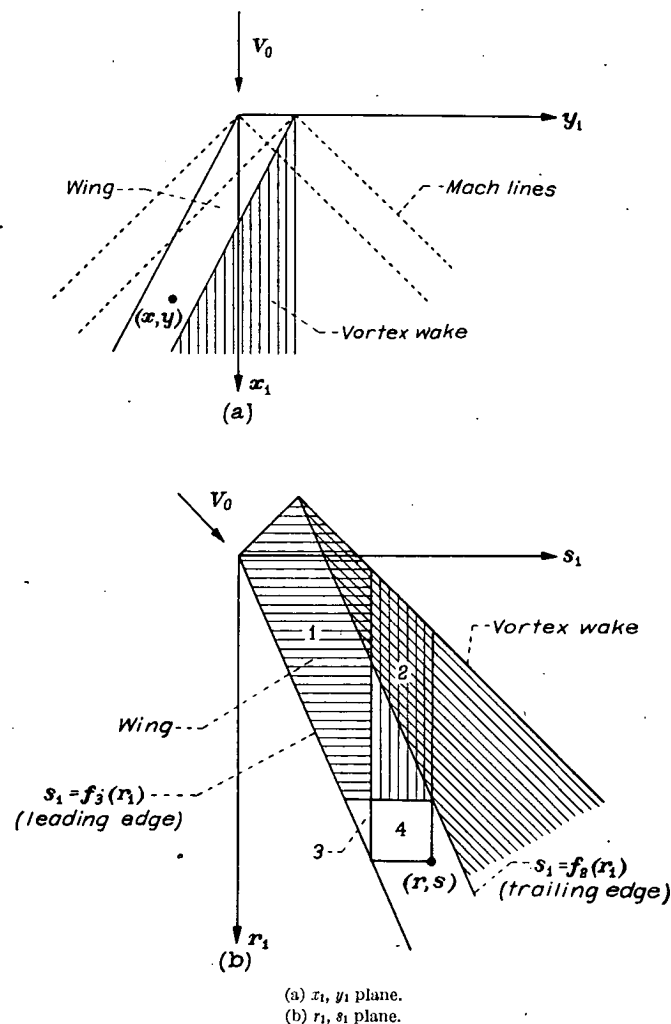


FIGURE 14.—Tip of swept forward wing and regions of integration.

can be anticipated: One, since the loading off the wing is zero, the residual will be $-2\pi\beta u_a(r, s)$; and the other, the transformed integral with the s_2 (in the notation of equation (124a)) integration performed first, is zero for all points r_1, s_1 lying in regions 2 and 4 in figure 14. These results follow directly from the discussion presented above in case 3 of the similar integral transform applied to the doublet equation (equation (122)) with the τ_1 area. Without proceeding further, therefore, it is apparent that the original integral equation has been reduced to: (1), an integral of the known function w_a over region 4 in figure 14; and (2), an integral of the unknown function φ_a over the regions 1 and 3 in figure 14. (It can be shown that the integration over region 3 will also vanish.) But regions 1 and 3 are ahead of the Mach forecone from (r, s) . Hence, by repeating the above process for regions farther and farther up (toward the origin) the wing, the problem must eventually be reduced to one of finding the solution for an area such as τ_1 ; in other words, to a problem involving only two edges, one subsonic and one supersonic. Since the latter problem is solved, the one considered in this section is also (theoretically at least) solved.

Triangular plan form with subsonic leading edges.—As a final exemplification of the preceding concepts, consider the problem of finding the loading on a flat triangular wing fly-

ing at a supersonic speed but with both leading edges subsonic (see fig. 15). To the supersonic doublet equation (equation (82)) apply the integral transform

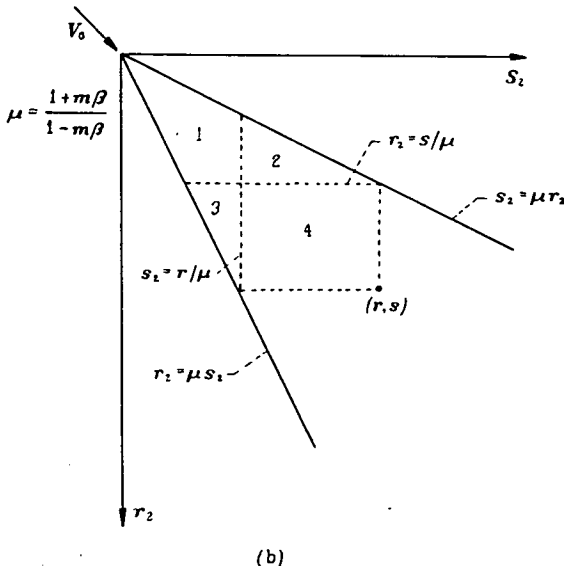
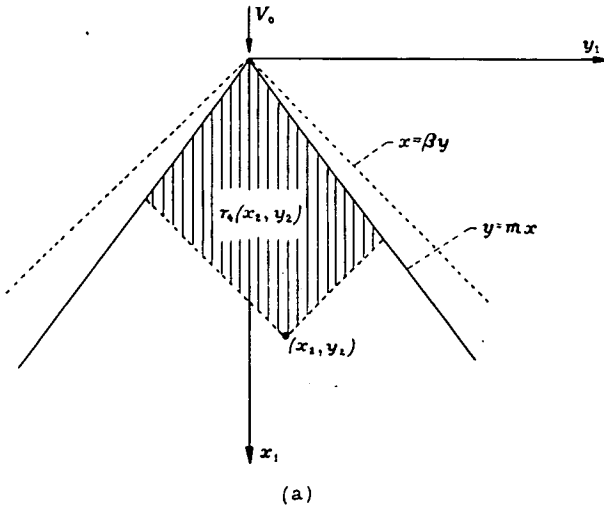
$$\frac{-1}{\pi M_0} \int_{s/\mu}^r dr_2 \int_{r/\mu}^s ds_2 \frac{1}{\sqrt{(r-r_2)(s-s_2)}}$$

where the area of integration is region 4 in figure 15(b). There results

$$\begin{aligned} \frac{-1}{\pi M_0} \int_{s/\mu}^r dr_2 \int_{r/\mu}^s ds_2 \frac{w_u(r_2, s_2)}{\sqrt{(r-r_2)(s-s_2)}} = \\ \frac{1}{4\pi^2} \int_{s/\mu}^r dr_2 \int_{r/\mu}^s ds_2 \oint_{\tau_1(r_2, s_2)} \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{\sqrt{(r-r_2)(s-s_2)[(r_2-r_1)(s_2-s_1)]^{3/2}}} \end{aligned} \quad (133)$$

where $\tau_1(r_2, s_2)$ is the original area of integration as shown in figure 15(a). Since w_u is a constant equal to $-V_0\alpha$, the left-hand side of equation (133) becomes

$$\frac{4V_0\alpha}{\pi M_0} \sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)} \quad (134)$$



(a) x_1, y_1 plane.
(b) r_2, s_2 plane.

FIGURE 15.—Integration areas for triangular wing.

where $\mu = (1+m\beta)/(1-m\beta)$, m being the slope of the leading edge in the x, y plane. Inverting the order of integration and evaluating the residual, one finds that equation (133) can be written

$$\begin{aligned} \varphi_u(r, s) = \frac{4V_0\alpha \sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)}}{\pi M_0} \\ \frac{\sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)}}{\pi^2} \int_{\tau_5} \int \frac{\varphi_u(r_1, s_1) dr_1 ds_1}{(r-r_1)(s-s_1) \sqrt{\left(\frac{r}{\mu}-s_1\right)\left(\frac{s}{\mu}-r_1\right)}} \end{aligned} \quad (135)$$

where τ_5 is the area shown as region 1 in figure 15(b).

Equation (135) is not, of course, the inversion of the supersonic doublet equation for the triangular flat plate. In fact, it simply represents the transformation of the doublet equation, which is a singular integral equation of the first kind, to a singular integral equation of the second kind. However, this latter form has the advantage that it is readily susceptible to the process of iteration. Thus, in the particular case of equation (135), it is possible to take as a first approximation to φ_u the value

$$\varphi_1(r, s) = \frac{4V_0\alpha}{\pi M_0} \sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)} \quad (136)$$

and, as the second, the value

$$\begin{aligned} \varphi_2(r, s) = \frac{4V_0\alpha}{\pi M_0} \sqrt{\left(r-\frac{s}{\mu}\right)\left(s-\frac{r}{\mu}\right)} \\ \left[1 - \frac{1}{\pi^2} \int_{\tau_5} \int \frac{\sqrt{\left(r_1-\frac{s_1}{\mu}\right)\left(s_1-\frac{r_1}{\mu}\right)} dr_1 ds_1}{(r-r_1)(s-s_1) \sqrt{\left(\frac{r}{\mu}-s_1\right)\left(\frac{s}{\mu}-r_1\right)}} \right] \end{aligned} \quad (137)$$

and so on. By means of the substitutions

$$\left. \begin{aligned} \mu r_1 &= \frac{s+r\mu\rho_1+s\sigma_1}{1+\rho_1+\sigma_1} \\ \mu s_1 &= \frac{r+r\rho_1+s\mu\sigma_1}{1+\rho_1+\sigma_1} \end{aligned} \right\} \quad (138)$$

the double integral in the equation for φ_2 can be evaluated. Thus, after some manipulation, one finds⁷

$$\begin{aligned} \int_{\tau_5} \int \frac{\sqrt{\left(r_1-\frac{s_1}{\mu}\right)\left(s_1-\frac{r_1}{\mu}\right)} dr_1 ds_1}{(r-r_1)(s-s_1) \sqrt{\left(\frac{r}{\mu}-s_1\right)\left(\frac{s}{\mu}-r_1\right)}} \\ = 2\pi[2E_1 - (1-k_1^2)K_1] - \pi^2 \end{aligned} \quad (139)$$

where K_1 and E_1 are complete elliptic integrals of the first and second kinds, respectively, with moduli $k_1 = 1/\mu$. Using the identity

⁷ Notice that the area of integration, i. e., the area τ_5 , is bounded by the lines which represent the four roots to the radicals in the integrand. Notice also that the value of this double integral is independent of r and s and depends only on the parameter μ , the slope of the leading edges in the r, s plane.

$$2E_1 - (1 - k^2)K_1 = \frac{2E}{1 + m\beta} \quad (140)$$

where E has the modulus $k = \sqrt{1 - m^2\beta^2}$, and returning to the Cartesian coordinate system, one finds for the values of φ_1 and φ_2 ,

$$\varphi_1 = \frac{4V_0\alpha}{\pi(1+m\beta)} \sqrt{m^2x^2 - y^2} \quad (141)$$

$$\varphi_2 = \frac{4V_0\alpha}{\pi(1+m\beta)} \sqrt{m^2x^2 - y^2} \left[2 - \frac{4E}{\pi(1+m\beta)} \right] \quad (142)$$

The process of iteration could be continued, and φ_u would be expressed as an infinite series of terms containing the parameter $m\beta$. However, since the terms in the series expansion are all independent of x and y , or, in the characteristic system, of r and s , it is more efficient to write φ in the form

$$\varphi_u(r, s) = A \sqrt{\left(s - \frac{r}{\mu}\right) \left(r - \frac{s}{\mu}\right)} = \frac{M_0 A \sqrt{m^2x^2 - y^2}}{1 + m\beta} \quad (143)$$

and determine the magnitude of A by substituting this expression into equation (135). There results the equality

$$A = \frac{4V_0\alpha}{\pi M_0} \frac{A}{\pi^2} \left(\frac{4\pi E}{1 + m\beta} - \pi^2 \right)$$

from which it can be shown that

$$A = \frac{V_0\alpha}{M_0 E} (1 + m\beta) \quad (144)$$

Finally, therefore, the velocity potential on the upper surface of a triangular wing with subsonic leading edges can be written

$$\varphi_u(x, y) = \frac{V_0\alpha}{E} \sqrt{m^2x^2 - y^2} \quad (145)$$

and the familiar expression for the loading coefficient (see, e. g., reference 10) follows immediately

$$\frac{\Delta p}{q} = \frac{4\alpha m^2 x}{E \sqrt{m^2x^2 - y^2}} \quad (146)$$

The purpose of examining this particular problem was not, of course, to obtain the solution presented as equation (145) or (146), since that solution is by now quite well known. Rather, the purpose was to show how the supersonic doublet equation could be transformed to a singular-integral equation of the second kind and how this equation could, in turn, be solved by applying an iteration process. Such a method has far more general applications than are given here and is by no means limited to problems in which the flow is conical or quasi-conical.

PART IV—APPLICATIONS

DIRECT PROBLEMS

The following three examples will serve to illustrate how the formulas derived in Part I can be used to solve the direct problems outlined in Part II. The applications will be limited to supersonic flow problems.

RECTANGULAR WING WITH BICONVEX SECTION

Consider a rectangular, nonlifting wing with a chordwise section given everywhere by the equation

$$\lambda_u(x, y) = \frac{2t}{c^2} (c - 2x) \quad (147)$$

where λ_u is the slope of the upper surface, c is the chord, and t is the maximum thickness. The equation for the pressure on the surface of such a wing will change form in each of the four regions indicated in figure 16. In region 1 the pressure is the same as for a two-dimensional wing with the same section, and the remaining regions contain the three-dimensional or tip effects.

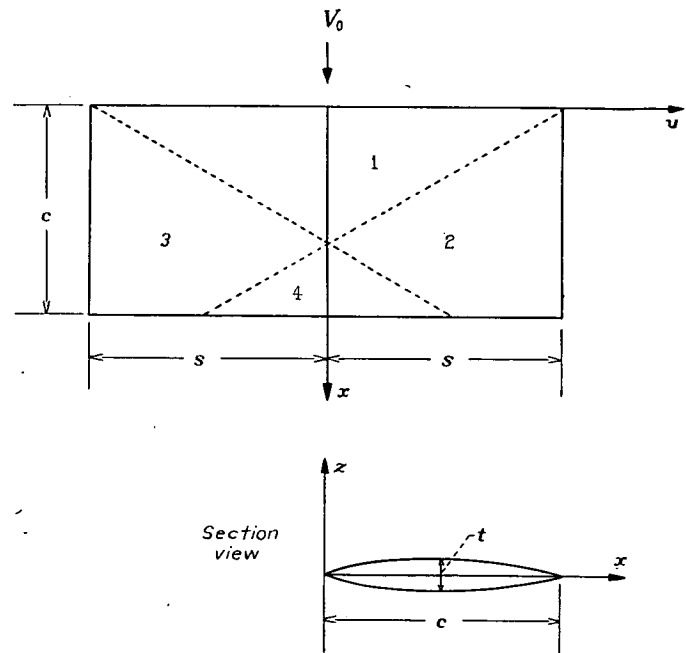


FIGURE 16.—Rectangular nonlifting wing with biconvex section.

The equation for pressure coefficient on the wing can be determined from either equation (76a) or (76b). Equation (76a), for example, becomes for region 1

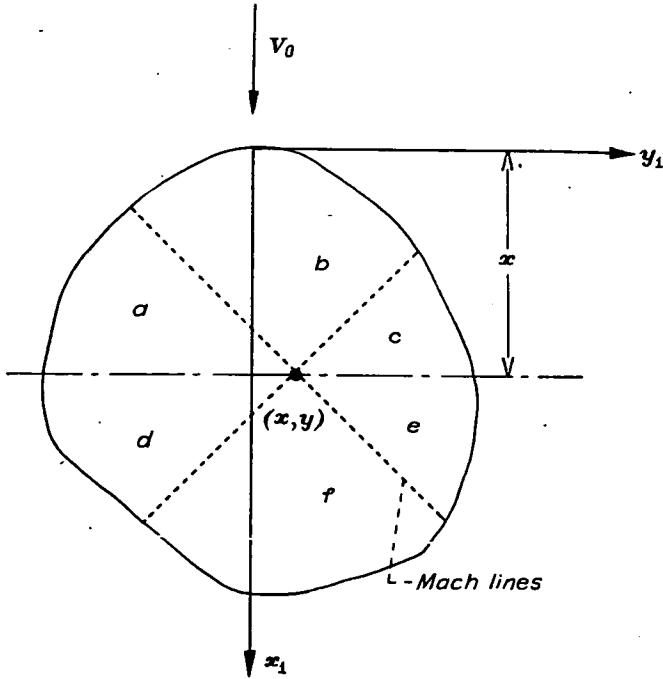
$$C_p = \frac{4t}{\beta c^2} (c - 2x) - \frac{4t}{\pi c^2} \int_0^x (c - 2x_1)(x - x_1) dx_1 \int_{y - \frac{x-x_1}{\beta}}^{y + \frac{x-x_1}{\beta}} \frac{dy_1}{r_0^3}$$

From the result given as equation (17), it is apparent that the integral term is zero and the pressure coefficient in region 1 of figure 16 is simply

$$C_p = \frac{4t}{\beta c^2} (c - 2x) \quad (148)$$

One can easily show that equation (76b) yields the same result.

The evaluation of pressure coefficient in regions 2, 3, and 4 can be carried out in a similar fashion. A slightly different approach can be used, however, that is useful in obtaining results in this and similar problems. Consider a wing with regions as shown in figure 17. Regions a, b, and c include all the area ahead of the line $x_1 = x$. It is obvious that r_0^3 is a pure real number for all x_1, y_1 inside region b (i. e., inside


 FIGURE 17.—Regions used in discussion of τ_i .

the forecone from x, y and a pure imaginary number for all x_1, y_1 inside regions a and c. Further, it is clear from physical considerations that λ_u is always real on the wing plan form. Hence, if the symbol τ_i is introduced to represent the combined area in regions a, b, and c, the area τ in equations (76)

can be replaced by the area τ_i and the real part of the result will be the correct answer for the pressure coefficient.⁸

By means of this concept, pressure coefficient for all points on the wing can be written in the form

$$C_p(x, y) = \frac{4t}{\beta c^2} (c - 2x) - \frac{4t}{\pi c^2} \text{R.P.} \int_0^x dx_1 \int_{-s}^s dy_1 \frac{(c - 2x_1)(x - x_1)}{r_0^3} \quad (149)$$

or, alternatively,

$$C_p(x, y) = \frac{-4t}{\pi c^2} \text{R.P.} \int_{-s}^s dy_1 \int_0^x dx_1 \frac{(c - 2x_1)(x - x_1)}{r_0^3} \quad (150)$$

where the letters R. P. indicate that the real part of the integral is to be taken. Evaluating, for example, the former equation one finds

$$C_p = \frac{4t}{\beta c^2} (c - 2x) + \frac{4t}{\pi c^2} \text{R.P.} \left[-\frac{c - 2x}{\beta} \arccos \frac{\beta(s - y)}{x} + 2(y - s) \operatorname{arcosh} \frac{x}{\beta(s - y)} - \frac{c - 2x}{\beta} \arccos \frac{\beta(s + y)}{x} - 2(y + s) \operatorname{arcosh} \frac{x}{\beta(y + s)} \right] \quad (151)$$

The real and imaginary parts of the arc cosine and arc cosh terms in this equation are given in the following table for all real values of the argument.

⁸ The areas d and e could also be included in the definition of τ_i since r_0^3 is imaginary within these regions also. However, region f must not be included since there the term r_0^3 is again real. The definition of τ_i adopted is usually the most convenient.

Range of x	$-1 > x$	$-1 < x < 1$	$1 < x$
$\operatorname{arcosh} x = \int_1^x \frac{dt}{\sqrt{t^2 - 1}}$	$i\pi + \operatorname{arcosh}(-x)$	$i \arccos x$	$\operatorname{arcosh} x$
$\arccos x = \int_x^1 \frac{dt}{\sqrt{1 - t^2}}$	$\pi - i \operatorname{arcosh}(-x)$	$\arccos x$	$i \operatorname{arcosh} x$
$\arcsin x = \int_0^x \frac{dt}{\sqrt{1 - t^2}}$	$-\frac{\pi}{2} + i \operatorname{arcosh}(-x)$	$\arcsin x$	$\frac{\pi}{2} - i \operatorname{arcosh} x$

Equation (151) contains, at once, the entire solution to the wing shown in figure 16. Thus, in region 1 none of the terms in the braces has a real part. Hence, equation (148) follows immediately. In region 3 the solution can be written

$$C_p = \frac{4t}{\pi \beta c^2} \left\{ (c - 2x) \left[\pi - \arccos \frac{\beta(y + s)}{x} \right] - 2\beta(y + s) \operatorname{arcosh} \frac{x}{\beta(y + s)} \right\}$$

The solution in region 2 follows from the one given in region 3 by symmetry; and, finally, the expression for the pressure coefficient in region 4 is given by equation (151) wherein every term is real.

DRAW REVERSIBILITY THEOREM

The well-known theorem that the drag of a symmetrical sharp-edged, nonlifting body is the same in forward and

reversed flight at the same speed (see references 11 and 12) can be derived in another way using the methods described in the above sections.

By definition, drag coefficient is

$$C_D = \frac{1}{S} \iint_S 2\lambda_u(x, y) C_p(x, y) dx dy \quad (152)$$

where S is the area of the wing. Using the real part concept outlined in the discussion of the preceding example, one can write

$$C_D = \frac{-4}{\pi S} \iint_S \lambda_u(x, y) dx dy \text{R.P.} \int_{\tau_i} dy_1 \int_{-s}^s dx_1 \frac{(x - x_1)\lambda_u(x_1, y_1)}{r_0^3} \quad (153)$$

The equation for the drag coefficient in reversed flight can be obtained by:

1. Replacing the area τ_i by τ_{ii} where $\tau_i + \tau_{ii} = S$

2. Rotating the axial system in the xy plane through 180°

3. Reversing the signs of $\lambda_u(x, y)$ and $\lambda_u(x_1, y_1)$

There results

$$C_{D_r} = \frac{-4}{\pi S} \int_S \lambda_u(x, y) dx dy \text{ R.P. } \int_{\tau_i} dy_1 \oint dx_1 \frac{(x_1 - x) \lambda_u(x_1, y_1)}{r_0^3} \quad (154)$$

Subtracting equation (154) from (153) gives

$$C_D - C_{D_r} = \frac{-4}{\pi S} \text{ R.P. } \int dy \int_S dx \int dy_1 \oint dx_1 \frac{x \lambda_u(x_1, y_1) \lambda_u(x, y)}{r_0^3} + \frac{4}{\pi S} \text{ R.P. } \int dy \int_S dx \int dy_1 \oint dx_1 \frac{x_1 \lambda_u(x_1, y_1) \lambda_u(x, y)}{r_0^3} \quad (155)$$

Since the symbols x_1, y_1, x, y are dummy variables of integration, the last term in equation (155) can be written

$$\frac{4}{\pi S} \text{ R.P. } \int dy_1 \int_S dx_1 \int dy \oint dx \frac{x \lambda_u(x, y) \lambda_u(x_1, y_1)}{r_0^3}$$

But reversing the operators $\oint dy_1 \oint dx_1$ and $\oint dy \oint dx$ (always preserving the same order within the operation) and subtracting gives

$$\int dy_1 \int_S dx_1 \int dy \oint dx \frac{x \lambda_u(x, y) \lambda_u(x_1, y_1)}{r_0^3} = \int dy \int_S dx \int dy_1 \oint dx_1 \frac{x \lambda_u(x, y) \lambda_u(x_1, y_1)}{r_0^3}$$

since the residual is zero. Hence, the second term in equation (155) is the same expression as the first except for the sign and

$$C_D - C_{D_r} = 0$$

or

$$C_D = C_{D_r} \quad (156)$$

as was to be shown.

LIFT ON WINGS WITH SUPERSONIC EDGES

The lift on any wing can be written

$$\frac{L}{q} = \iint_S \frac{\Delta p}{q} dy dx \quad (157)$$

Moreover,

$$\int_{\text{L.E.}}^{\text{T.E.}} \frac{\Delta p}{q} dx = \frac{4 \varphi_{\text{T.E.}}}{V_0}$$

where T. E. and L. E. denote the trailing and leading edges, respectively, and $\varphi_{\text{T.E.}}$ is the value of the velocity potential on the upper surface of the wing at the trailing edge.

Consider now a wing with all edges supersonic and a straight trailing edge not necessarily at right angles to the free-stream direction (see fig. 18). Let the wing be a plate having arbitrary twist and camber. Then, for a point on the wing, the velocity potential can be written in the x, y coordinate system on the basis of equation (121) as

$$\varphi_u = \frac{-1}{\pi} \text{ R.P. } \int_{\tau_i} \int \frac{w_u(x_1, y_1) dx_1 dy_1}{\sqrt{(x - x_1)^2 - \beta^2(y - y_1)^2}} \quad (158)$$

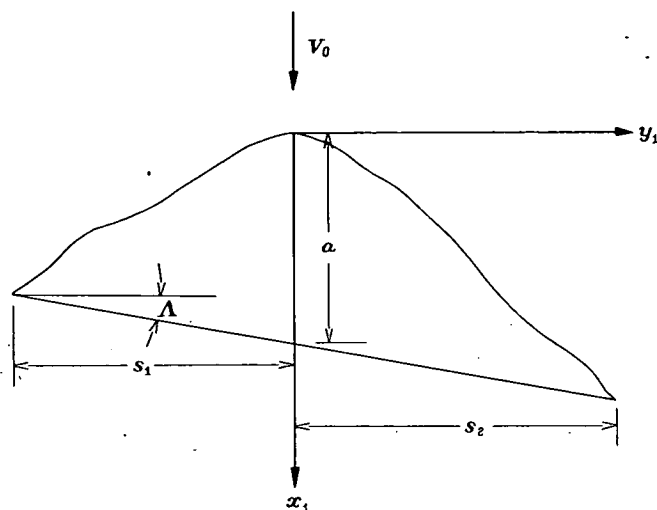


FIGURE 18.—Wing with supersonic edges.

where τ_i , as in the previous examples, is the area on the wing ahead of the line $x_1 = x$. If the equation of the trailing edge is

$$x_1 = a + y_1 \tan \Lambda$$

where a is some constant, the value of the potential at the trailing edge can be written

$$\varphi_{\text{T.E.}} = \frac{-1}{\pi} \text{ R.P. } \iint_S \frac{w_u(x_1, y_1) dx_1 dy_1}{\sqrt{(a + y \tan \Lambda - x_1)^2 - \beta^2(y - y_1)^2}} \quad (159)$$

where the area of integration is the whole wing plan form since the trailing edge of the wing is supersonic and the aftercone from the point at which $\varphi_{\text{T.E.}}$ is being evaluated cannot intersect the wing. The total lift L on the wing can, therefore, be written in the form

$$\frac{L}{q} = \frac{-4}{\pi V_0} \text{ R.P. } \int_{s_1}^{s_2} dy \iint_S \frac{w_u(x_1, y_1) dx_1 dy_1}{\sqrt{(a + y \tan \Lambda - x_1)^2 - \beta^2(y - y_1)^2}} \quad (160)$$

The area S does not depend on y , so the y integration can be made first and, since the edges are all supersonic, the interval $s_1 \leq y \leq s_2$ must always contain the roots λ_1 and λ_2 of the expression under the radical. Hence

$$\text{R.P. } \int_{s_1}^{s_2} \frac{dy}{\sqrt{(\beta^2 - \tan^2 \Lambda)(\lambda_1 - y)(y - \lambda_2)}} = \int_{\lambda_1}^{\lambda_2} \frac{dy}{\sqrt{(\beta^2 - \tan^2 \Lambda)(\lambda_1 - y)(y - \lambda_2)}}$$

and since

$$\int_{\lambda_1}^{\lambda_2} \frac{dy}{\sqrt{(\lambda_1 - y)(y - \lambda_2)}} = \pi$$

then

$$\frac{L}{q} = \frac{-4}{\sqrt{\beta^2 - \tan^2 \Lambda}} \iint_S \frac{w_u(x_1, y_1)}{V_0} dx_1 dy_1 \quad (161)$$

Defining the average angle of attack $\bar{\alpha}$ by the expression

$$\bar{\alpha} = \frac{1}{S} \iint_S \frac{-w_u(x_1, y_1) dx_1 dy_1}{V_0} \quad (162)$$

one can write equation (161) in the alternative form

$$C_L = \frac{4\bar{\alpha}}{\sqrt{\beta^2 - \tan^2 A}} \quad (163)$$

It is interesting to notice that the lift coefficient for the wing just studied is the same as that for a two-dimensional flat plate flying at an angle of attack $\bar{\alpha}$ into a free stream, the speed of which is given by the component of velocity normal to the trailing edge of the three-dimensional wing. This result has been derived previously in reference 13.

INVERSE PROBLEMS

LOW-ASPECT-RATIO RECTANGULAR WING

It will be noted in the summary of results for the fundamental formulas applicable to supersonic flow (equations (75) through (82)) that the results are presented for both orders of integration in the x, y coordinate system. While the analysis of direct problems can be carried out in all cases if, say, the x_1 integration is always performed first, it may sometimes be more convenient to perform the y_1 integration first. In the analysis of inverse problems, however, it is much more important that freedom exists in the choice of the first variable of integration. A good example of this is provided by the following approximate derivation of the loading on a slender (in the streamwise sense) rectangular flat plate.

By considering the special case when the wing chord is long compared to the span, one can obtain an approximate solution for the slender flat plate by assuming the loading coefficient has the form

$$\frac{\Delta p}{q} = 4\alpha f\left(\frac{x}{s}\right) \sqrt{1 - \left(\frac{y}{s}\right)^2} \quad (164)$$

where s is the semispan (see fig. 19) and $f(x/s)$ is an unknown function. The function f is to be determined by the condition that w_u is constant along the center line of the wing. If the solution to such a problem is to be determined by use of the doublet or vortex equation, it is obviously important that the first integration be made with respect to y_1 since the variation of $\Delta p/q$ with y_1 is known.

Since $\Delta p/q$ is given, let the vortex equation be used and let equation (164) be placed into (77a). For $y=0$ (and for added simplicity for $\beta=1$) the area τ is shown by the shaded area in figure 19 and the resulting equation can be written for $x > s$

$$\begin{aligned} \frac{w_u}{V_0} = & -\alpha f\left(\frac{x}{s}\right) + \frac{\alpha}{\pi} \int_0^{x-s} (x-x_1) f\left(\frac{x_1}{s}\right) dx_1 \\ & + \frac{\alpha}{\pi} \int_{-s}^s \frac{\sqrt{1-(y_1/s)^2}}{y_1^2 \sqrt{(x-x_1)^2 - y_1^2}} dy_1 + \frac{\alpha}{\pi} \int_{x-s}^x (x-x_1) f\left(\frac{x_1}{s}\right) dx_1 \\ & + \frac{\alpha}{\pi} \int_{-(x-x_1)}^{x-x_1} \frac{\sqrt{1-(y_1/s)^2}}{y_1^2 \sqrt{(x-x_1)^2 - y_1^2}} dy_1 \end{aligned}$$

for $0 < x < s$

$$\begin{aligned} \frac{w_u}{V_0} = & -\alpha f\left(\frac{x}{s}\right) + \frac{\alpha}{\pi} \int_0^x (x-x_1) f\left(\frac{x_1}{s}\right) dx_1 \\ & + \frac{\alpha}{\pi} \int_{-(x-x_1)}^{x-x_1} \frac{\sqrt{1-(y_1/s)^2}}{y_1^2 \sqrt{(x-x_1)^2 - y_1^2}} dy_1 \end{aligned}$$

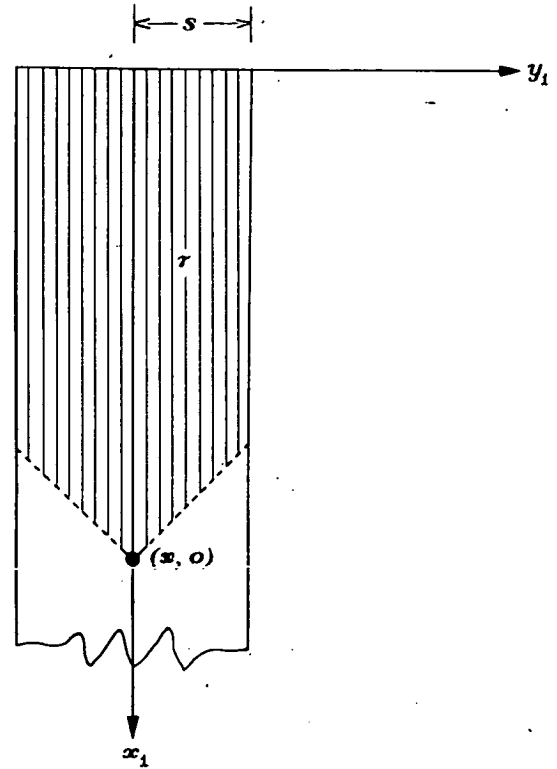


FIGURE 19.—Integration area for slender rectangular wing.

Introduce the notation

$$\theta_1 = \frac{x_1}{s}, \quad \theta = \frac{x}{s}, \quad k_1 = \frac{1}{\theta - \theta_1}, \quad k_2 = \theta - \theta_1$$

and these equations become,⁹ since $\alpha = -w_u/V_0$, for $0 < \theta < 1$

$$1 = f(\theta) + \frac{2}{\pi} \int_0^\theta k_2 B_2 f(\theta_1) d\theta_1 \quad (165a)$$

for $1 < \theta$

$$1 = f(\theta) + \frac{2}{\pi} \int_{\theta-1}^\theta k_2 B_2 f(\theta_1) d\theta_1 + \frac{2}{\pi} \int_0^{\theta-1} E_1 f(\theta_1) d\theta_1 \quad (165b)$$

Equations (165) are integral equations of the second kind (more specifically, Volterra's integral equations of the second kind) and the kernels are regular and bounded everywhere in the interval of integration. Hence, their solution can be determined readily by numerical processes. This has been done and the result in terms of the loading coefficient on the center line $\left(\frac{\Delta p}{q\alpha}\right)_0$ (from equation (164) $\left(\frac{\Delta p}{q\alpha}\right)_0 = 4f\left(\frac{x}{s}\right)$) is shown in figure 20.

For the purpose of comparison, the exact linearized value also is shown in the interval where it is known, together with another approximate solution obtained by Stewartson (reference 14) using a different approach. Near the leading edge, where the comparison with the exact results can be made, the agreement between the exact and approximate solutions obtained herein will be poorest because in this region the spanwise variation deviates most radically from the value assumed in the construction of the integral equation.

⁹ The symbols B and E indicate elliptic integrals. See the table of symbols.

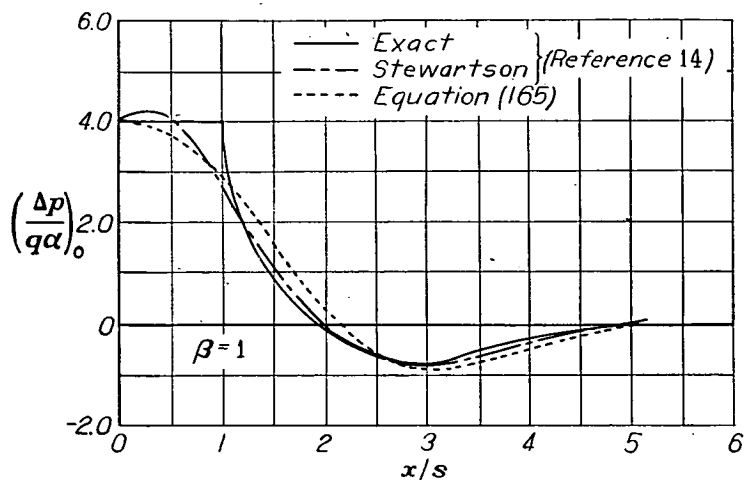


FIGURE 20.—Load coefficient for unit angle of attack along center line of rectangular wing.

CONICAL AND QUASI-CONICAL FLOW

Equation (108) and its solution, equation (111), occur repeatedly in the study of aerodynamics. In fact, equation (108) is often referred to as the airfoil equation since it plays a dominant role in the development of linearized, two-dimensional, subsonic wing theory. It appears also in the study of slender wings (reference 15) or wings flying at near sonic speeds (reference 16) since the boundary conditions lead again to the required inversion of the same type of integral relationship. In the present section, problems arising in supersonic conical or quasi-conical flow fields will be reduced also to this basic equation.

Several methods exist whereby the solution to conical flow problems can be determined. The one to be studied here is based on the construction of conical elements extending radially from the apex of the field and inclined at an angle arc $\tan m$ to the x axis (see fig. 21(a)). In order to obtain such an element it is sufficient to subtract two plan forms of prescribed loading or thickness, each plan form having one side directed along the x axis while the other sides are inclined at angles that differ only infinitesimally.

Consider first the construction of a quasi-conical,¹⁰ radial, lifting element that carries a load given by the expression

$$\frac{\Delta p}{q} = C y^{\kappa} \quad (166)$$

where C is a constant. The upwash field of a triangular plan form such as the one shown in figure 21(b) can be found by integrating elementary horseshoe vortices over the appropriate area τ . Thus, using equation (77b), performing an integration, and making the substitutions

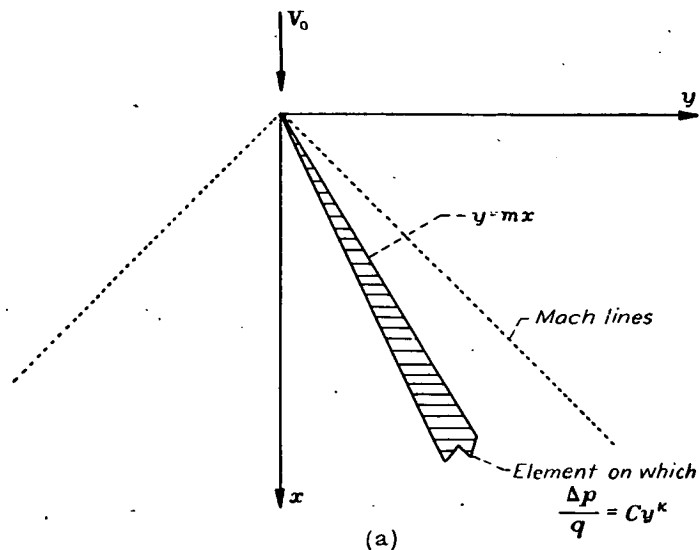
$$\theta = m\beta, \quad \eta = \beta y/x, \quad \eta_1 = \beta y_1/x \quad (167)$$

one finds the result

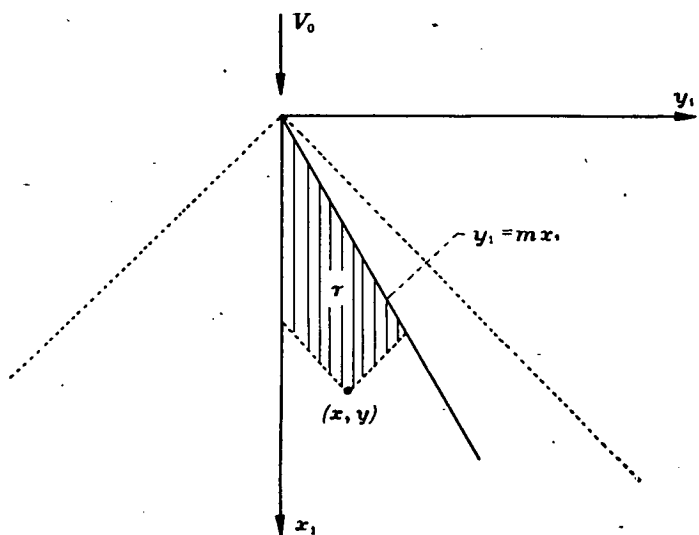
$$\left(\frac{\beta}{x}\right)^{\kappa} \frac{w_u}{V_0} = \frac{\beta C}{4\pi} \int_0^{\theta} \frac{\eta_1^{\kappa} d\eta_1}{(\eta - \eta_1)^2} \sqrt{\left(1 - \frac{\eta_1}{\theta}\right)^2 - (\eta - \eta_1)^2}, \quad 0 < \theta, \eta < \theta \quad (168a)$$

$$= \frac{\beta C}{4\pi} \int_0^{\theta} \frac{\eta_1^{\kappa} d\eta_1}{(\eta - \eta_1)^2} \sqrt{\left(1 - \frac{\eta_1}{\theta}\right)^2 - (\eta - \eta_1)^2}, \quad 0 < \theta, \theta < \eta \quad (168b)$$

¹⁰ A conical field is defined as one in which the induced velocities are constant along rays through a point. In the subsequent analysis this corresponds to the case when $\kappa = 0$. Quasi-conical fields are those in which κ is greater than zero.



(a)



(b)

(a) Lifting element.
(b) Triangular wing.

FIGURE 21.—Construction of conical element.

Equations (168) can be written in the functional notation

$$w_u = f(\theta, \eta)$$

It follows that if the analysis were repeated for a wing with a slightly larger apex angle, there would have resulted

$$w_u = f(\theta + \Delta\theta, \eta)$$

Subtracting these two expressions for w_u gives the increment in vertical induced velocity due to a quasi-conical element

$$dw_u = \lim_{\Delta\theta \rightarrow 0} \frac{f(\theta + \Delta\theta, \eta) - f(\theta, \eta)}{\Delta\theta} \Delta\theta = \frac{\partial f}{\partial \theta} d\theta \quad (169)$$

Carrying out the operation indicated by equation (169), making the substitution

$$\eta_1 = \theta \frac{\eta - t}{\theta - t}, \quad t = \theta \frac{\eta - \eta_1}{\theta - \eta_1}$$

and distributing these elements between θ_1 and θ_0 with weight $C(\theta)$, one can finally show that the equation

$$\left(\frac{\beta}{x}\right)^\kappa \frac{w_u}{V_0} = \frac{-\beta(\kappa+1)}{4\pi} \int_{\theta_1}^{\theta_0} \theta^\kappa C(\theta) H(\theta, \eta) d\theta \quad (170)$$

applies where

$$H(\theta, \eta) = \int_1^\eta \frac{(\eta-t)^\kappa \sqrt{1-t^2}}{(\theta-t)^{\kappa+2} t} dt, \quad \theta_1 \leq \theta < \eta$$

$$= \int_{-1}^\eta \frac{(\eta-t)^\kappa \sqrt{1-t^2}}{(\theta-t)^{\kappa+2} t} dt, \quad \eta < \theta \leq \theta_0$$

The function $H(\theta, \eta)$ has a simple pole at $\theta = \eta$ and the integral expression for w_u is therefore evaluated as a Cauchy principal part.

The boundary condition to be satisfied by equation (170) is that $\beta^* w_u / x^\kappa V_0$ is a given polynomial of degree κ in the variable η . Hence, equation (170) is a singular integral equation with a pole of the same order as that in the airfoil equation. In its present form the equation appears somewhat formidable, but it can be simplified considerably by a simple operation. Since $\beta^* w_u / x^\kappa V_0$ is a polynomial of degree κ , it follows that the $(\kappa+1)^{\text{st}}$ derivative of the right-hand member of equation (170) must vanish. Thus, using the concept of the generalized principal part, one has

$$0 = \frac{\sqrt{1-\eta^2}}{\eta} \int_{\theta_1}^{\theta_0} \frac{\theta^\kappa C(\theta) d\theta}{(\theta-\eta)^{\kappa+2}} \quad (171)$$

which, by definition, can be put in the form

$$\left(\frac{\partial}{\partial \eta}\right)^{\kappa+1} \int_{\theta_1}^{\theta_0} \frac{\theta^\kappa C(\theta) d\theta}{\theta-\eta}$$

The function $\theta^\kappa C(\theta)$ is therefore to be found through the inversion of the integral equation

$$\int_{\theta_1}^{\theta_0} \frac{\theta^\kappa C(\theta) d\theta}{\theta-\eta} = \sum_{i=0}^{\kappa} a_i \eta^i \quad (172)$$

But equation (172) is precisely the airfoil equation and its solution is given by equation (111). Hence,

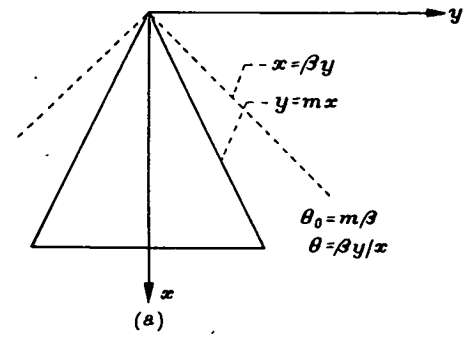
$$\theta^\kappa C(\theta) = \frac{1}{\pi^2 \sqrt{(\theta_0-\theta)(\theta-\theta_1)}} \left[A - \int_{\theta_1}^{\theta_0} \frac{\sum_{i=0}^{\kappa} a_i \eta^i \sqrt{(\theta_0-\eta)(\eta-\theta_1)}}{\theta-\eta} d\eta \right] \quad (173)$$

so that, finally,

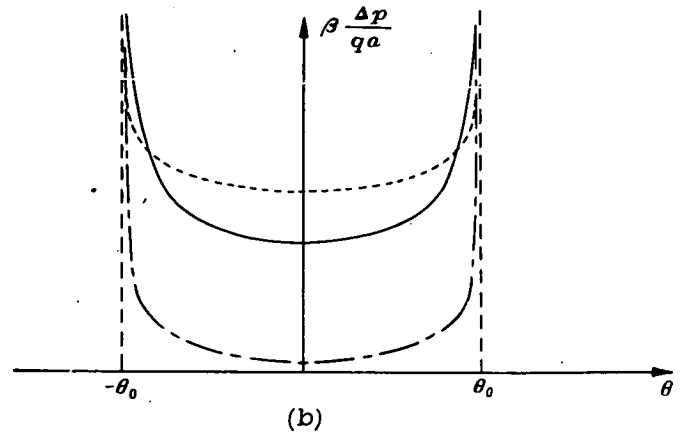
$$\frac{\Delta p}{q} = \left(\frac{x}{\beta}\right)^\kappa \sum_{i=0}^{\kappa+1} \frac{b_i \theta^i}{\sqrt{(\theta_0-\theta)(\theta-\theta_1)}} \quad (174)$$

where the coefficients b_i are functions of the constants θ_0 and θ_1 but not of θ . These coefficients must be determined from known conditions about the surface geometry.

Consider the unyawed (i. e., $\theta_1 = -\theta_0$) triangular wing shown in figure 22. If the loading is to be determined on a flat plate with such a plan form $\beta^* w_u / x^\kappa V_0$ becomes $-\alpha$, and equation (174) reduces to



$$\begin{aligned} \text{---} & w_u/V_0 = -\alpha \\ \text{---} & w_u/V_0 = -\alpha x \\ \text{---} & w_u/V_0 = -\alpha y^2/\beta^2 \end{aligned}$$



(a) Wing plan form.

(b) Load distribution at $x=1$.

FIGURE 22.—Load distribution on triangular wings with specified twist.

$$\frac{\Delta p}{q} = \frac{b_0 + b_1 \theta}{\sqrt{\theta_0^2 - \theta^2}} \quad (175)$$

By symmetry, the coefficient b_1 in equation (175) must be zero and b_0 can be evaluated by placing equation (175) into (170) and integrating. There results the solution for the pressure coefficient shown in figure 22 where $\alpha = a$ and already presented as equation (146).

One can go on to show that if, as for a wing pitching about the apex at a rate Q , (see reference 18),

$$w_u = -Qx, \quad \frac{\beta^* w_u}{x^\kappa V_0} = \frac{-\beta Q}{V_0}, \quad \kappa=1 \quad (176)$$

then

$$\frac{\Delta p}{q} = \frac{4xQ}{V_0 \beta \sqrt{\theta_0^2 - \theta^2}} \left[\frac{2\theta_0^2 - \theta^2}{\frac{\theta_0^2 K}{1-\theta_0^2} + \frac{1-2\theta_0^2}{1-\theta_0^2} E} \right] \quad (177)$$

where the complete elliptic integrals K and E have a modulus equal to $k = \sqrt{1-\theta_0^2}$. Further, if

$$\frac{w_u}{V_0} = h y^2, \quad \frac{\beta^* w_u}{x^\kappa V_0} = h \eta^2, \quad \kappa=2 \quad (178)$$

then

$$\frac{\Delta p}{q} = \frac{8h}{\beta} \left(\frac{x}{\beta}\right)^2 \theta_0^4 \left\{ \frac{[2\theta_0^2 K - (1+\theta_0^2)E] + [-(3-\theta_0^2)K + (4-2\theta_0^2)E]\theta^2}{[-5\theta_0^4 K^2 + 8\theta_0^2(1+\theta_0^2)KE + (4\theta_0^4 - 19\theta_0^2 + 4)E^2] \sqrt{\theta_0^2 - \theta^2}} \right\} \quad (179)$$

where $k = \sqrt{1 - \theta_0^2}$ ($= \sqrt{1 - m^2 \beta^2}$) is the modulus of the complete elliptic integrals. These solutions are all shown in figure 22, where in one case $a = Q/V_0$ and in the other $a = -h/\beta^2$.

If the function $\beta^* w_u/x^* V_0$ is discontinuous but a polynomial in each interval of continuity, the solution given as equation (173) still applies. For example, consider the case when $\kappa = 0$ and w_u/V_0 is a constant that changes sign in crossing the x axis. (See fig. 23.) For simplicity let $\theta_1 = -\theta_0$, then by equation (173)

$$\frac{w_u}{V_0} = -\delta \frac{\eta}{|\eta|}, \quad \kappa = 0 \quad (180)$$

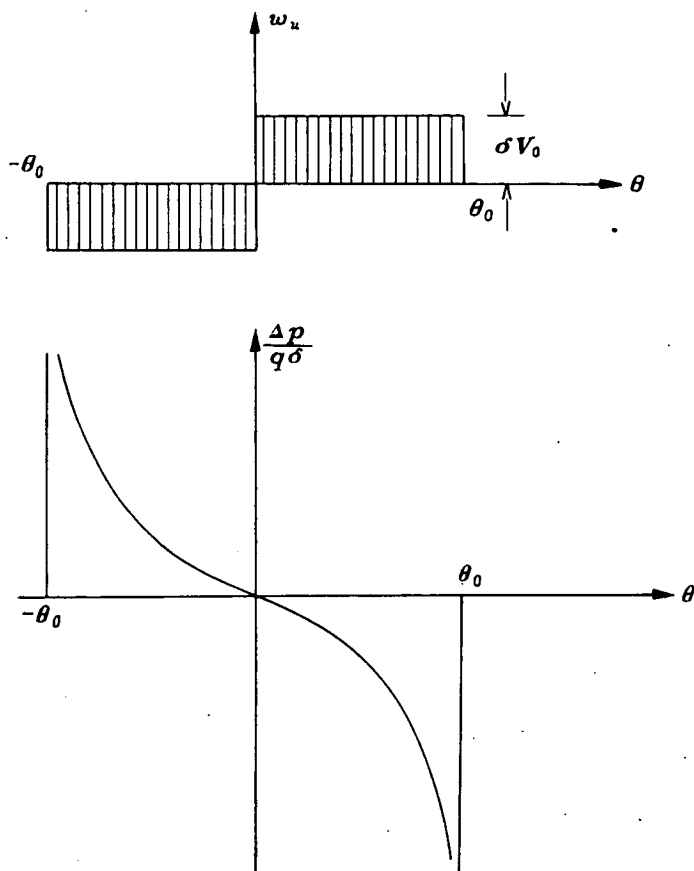


FIGURE 23.—Loading on differentially deflected triangular plate.

Again, as in the development of equation (175),

$$\frac{\Delta p}{q} = \frac{b_0 + b_1 \theta}{\sqrt{\theta_0^2 - \theta^2}} \quad (181)$$

Now, however, the solution must exhibit odd symmetry and the constant b_0 is zero. The constant b_1 can be evaluated by substituting equation (181) into equation (170). There results, finally,

$$\frac{\Delta p}{q} = \frac{8 \delta \theta_0 \theta}{\pi \beta \sqrt{\theta_0^2 - \theta^2}} \quad (182)$$

This solution is shown graphically in figure 23.

The methods described above can also be applied to problems involving wings with thickness and without lift. In these cases one constructs a radial element emanating from the origin and possessing a quasi-conical thickness distribution

$$\frac{dz_u}{dx} = \lambda_u = C y^* \quad (183)$$

The derivation of the induced pressure field associated with the element follows closely the analysis in the lifting case. First a triangular plan form is considered with one side having the slope m and the other parallel to the free-stream direction. The thickness is assumed to have the form given by equation (183) so that the pressure coefficient can be obtained from the equation

$$C_p = \frac{-2}{\pi} \int \lambda_u dy \int \frac{(x-x_1) dx_1}{[(x-x_1)^2 - \beta^2 (y-y_1)^2]^{3/2}} \quad (184)$$

Make the same notational changes as in the analysis of the lifting case; construct the element by taking the partial derivative with respect to θ ; and, finally, distribute weighted elements over the wing plan form by making C a function of θ and integrating with respect to θ between the limits θ_1 and θ_0 . There results

$$\left(\frac{\beta}{x}\right)^* C_p = \frac{-2(\kappa+1)}{\pi \beta} \int_{\theta_1}^{\theta_0} \theta^* C(\theta) H_1(\theta, \eta) d\theta \quad (185)$$

where

$$H_1(\theta, \eta) = \int_1^\eta \frac{t(\eta-t)^* dt}{(\theta-t)^{\kappa+2} \sqrt{1-t^2}}, \quad \theta_1 \leq \theta \leq \eta$$

$$= \int_{-1}^\eta \frac{t(\eta-t)^* dt}{(\theta-t)^{\kappa+2} \sqrt{1-t^2}}, \quad \eta < \theta \leq \theta_0$$

The boundary conditions require $(\beta/x)^* C_p$ to be a polynomial of degree κ in η . If the $(\kappa+1)^{\text{st}}$ derivative of equation (185) is set equal to zero, the relation

$$0 = \frac{\partial^{\kappa+1}}{\partial \eta^{\kappa+1}} \int_{\theta_1}^{\theta_0} \frac{\theta^* C(\theta) d\theta}{(\theta-\eta)^{\kappa+2}} = \left(\frac{\partial}{\partial \eta}\right)^{\kappa+1} \frac{1}{(\kappa+1)!} \int_{\theta_1}^{\theta_0} \frac{\theta^* C(\theta) d\theta}{\theta - \eta} \quad (186)$$

results. The function $C(\theta)$ satisfies the same integral equation that arose in the lifting case, and the solution can therefore be written immediately in the form

$$\lambda_u = y^* C(\theta) = \left(\frac{x}{\beta}\right)^* \sum_{i=0}^{\kappa+1} \frac{b_i \theta^i}{\sqrt{(\theta_0 - \theta)(\theta - \theta_1)}} \quad (187)$$

Again the coefficients b_i must be determined from known conditions about the surface geometry.

Consider first the case when the pressure is constant over an unyawed triangular plan form as in figure 24. Thus

$$C_p = C_{p_0}, \quad \kappa = 0 \quad (188)$$

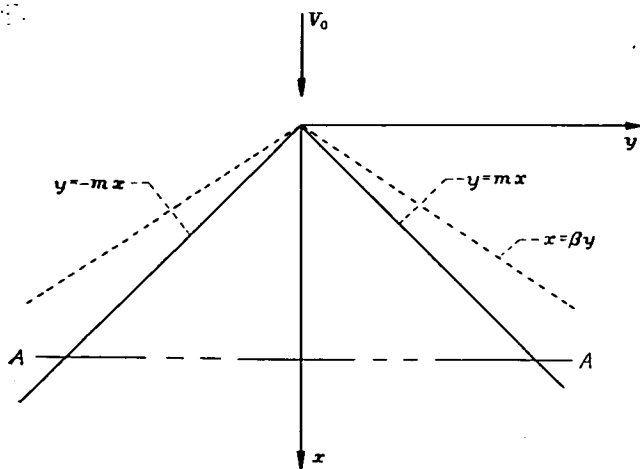
and (b_1 being zero by symmetry)

$$\lambda_u = \frac{dz_u}{dx} = \frac{b_0}{\sqrt{\theta_0^2 - \theta^2}} \quad (189)$$

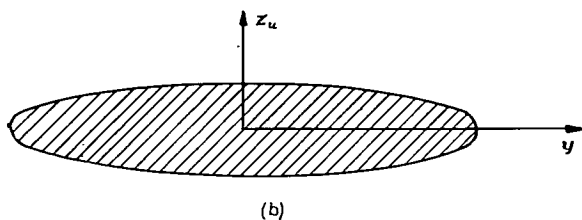
Evaluating the constant b_0 by substituting equation (189) into (185), it can be shown that the surface ordinate is

$$z_u = \frac{(1 - \theta_0^2) C_{p_0}}{2m^2(K - E)} \sqrt{m^2 x^2 - y^2} \quad (190)$$

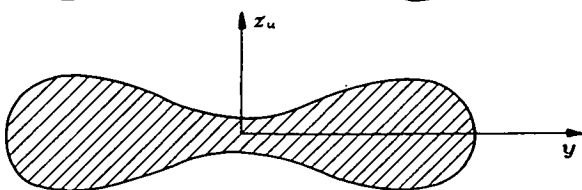
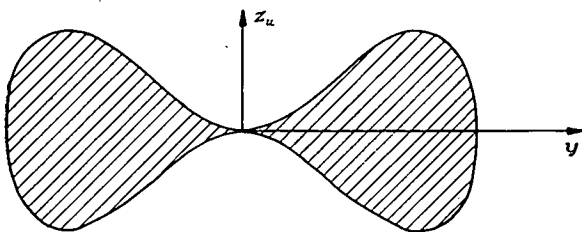
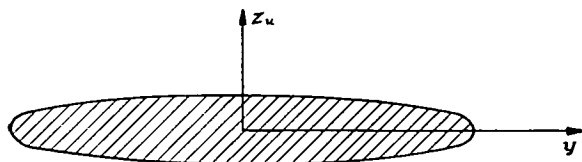
where the modulus of K and E is $k = \sqrt{1 - \theta_0^2}$. This result, which is the equation of an elliptical section, is shown in figure 24.



(a)



(b)



(c)

- (a) Triangular wing with subsonic edges.
 (b) Section AA for C_p constant.
 (c) Section AA for C_p varying linearly with x .

FIGURE 24.—Symmetrical triangular wing with specified pressure distributions.

Finally, consider an unyawed triangular wing for which the pressure varies linearly in the x direction. For such a case

$$C_p = (C_{p_0})x, \quad \kappa = 1 \quad (191)$$

and (b_1 again being zero by symmetry)

$$\lambda_u = \frac{dz_u}{dx} = \frac{x}{\beta} \frac{b_0 + b_2 \theta^2}{\sqrt{\theta_0^2 - \theta^2}} \quad (192)$$

from which it immediately follows by integration

$$z_u = \frac{b_0 x}{2\beta^2 m^2} \sqrt{m^2 x^2 - y^2} + \frac{y^2}{2\beta^2 m^2} (b_0 + 2\beta^2 m^2 b_2) \operatorname{arcosh} \frac{mx}{y} \quad (193)$$

Placing equation (192) into (185) and integrating, one can eventually show

$$C_{p_0} = \frac{2x}{\beta^2 (1 - \theta_0^2)^2} \{ b_0 [2K - E(3 - \theta_0^2)] + b_2 [(\theta_0^4 + \theta_0^2)K - 2\theta_0^2 E] \} \quad (194)$$

It is immediately apparent that the wing shape required to support a linear pressure gradient in the x direction is not unique, that there are, in fact, an infinite number of shapes that will induce the same pressure distribution. (The converse, however, is not true. That is, a given shape has only one possible distribution of pressure.)

Squire (reference 17) considered the thickness distribution that is obtained by neglecting the arc hyperbolic function in equation (193). His result corresponds to the case when b_0 is $-2\beta^2 m^2 b_2$ and can be written specifically

$$z_u = \frac{b_0 x}{2\beta^2 m^2} \sqrt{m^2 x^2 - y^2} \quad (195a)$$

$$C_{p_0} = \frac{b_0 x}{\beta^2 (1 - \theta_0^2)^2} [(3 - \theta_0^2)K - E(4 - 2\theta_0^2)] \quad (195b)$$

where b_0/β^2 can be related to the thickness chord ratio of the wing and the apex angle. Figure 24 shows how equation (194) can be used to obtain several triangular wing shapes all having the same linear pressure distribution. (It is interesting to note that no combination of b_0 and b_2 exists that will give a real wing shape with zero pressure coefficient since the resulting negative ordinates would require the surface to cross itself.)

AMES AERONAUTICAL LABORATORY,
 NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
 MOFFETT FIELD, CALIF., Oct. 16, 1950.

REFERENCES

1. Lomax, Harvard, and Heaslet, Max. A.: The Application of Green's Theorem to the Solution of Boundary-Value Problems in Linearized Supersonic Wing Theory. NACA Rep. 961, 1950. (Formerly NACA TN 1767.)
2. Heaslet, Max A., Lomax, Harvard, and Jones, Arthur L.: Volterra's Solution of the Wave Equation as Applied to Three-Dimensional Supersonic Airfoil Problems. NACA Rep. 889, 1947. (Formerly NACA TN 1412.)
3. Hadamard, Jacques: Lectures on Cauchy's Problem in Linear Partial Differential Equations. Yale University Press, New Haven, Conn., 1923.
4. Robinson, A.: On Source and Vortex Distributions in a Linearized Theory of Steady Supersonic Flow. College of Aeronautics, Cranfield, England. Rep. 9, Oct. 1947.
5. Ribner, Herbert S.: Some Conical and Quasi-Conical Flows in Linearized Supersonic-Wing Theory. NACA TN 2147, 1950.
6. Heaslet, Max. A., and Lomax, Harvard: The Use of Source-Sink and Doublet Distributions Extended to the Solution of Boundary-Value Problems in Supersonic Flow. NACA Rep. 900, 1948. (Formerly NACA TN 1515.)
7. Eppard, John C.: Use of Source Distributions For Evaluating Theoretical Aerodynamics of Thin Finite Wings at Supersonic Speeds. NACA Rep. 951, 1950.
8. Goodman, Theodore R.: The Lift Distribution on Conical and Non-Conical Flow Regions on Thin Finite Wings in a Supersonic Stream. Jour. Aero. Sci., vol. 16, no. 6, June 1949, pp. 365-374.
9. Mirels, Harold: Lift-Cancellation Technique in Linearized Supersonic-Wing Theory. NACA TN 2145, 1950.
10. Stewart, H. J.: The Lift of a Delta Wing at Supersonic Speeds. Quarterly of Applied Mathematics, vol. IV, no. 3, Oct. 1946, pp. 2462-54.
11. von Kármán, Theodore: Supersonic Aerodynamics—Principles and Applications. Jour. Inst. Aero. Sci., vol. 14, no. 7, July 1947, pp. 373-409.
12. Hayes, Wallace D.: Linearized Supersonic Flow. North American Aviation, Inc., Rep. AL-222, June 18, 1947.
13. Lagerstrom, Paco A., and Van Dyke, M. D.: General Considerations About Planar and Non-Planar Lifting Systems. Douglas Aircraft Company, Inc., Rep. No. SM-13432, June 1949.
14. Stewartson, K.: Supersonic Flow Over an Inclined Wing of Zero Aspect Ratio. Proc. Camb. Phil. Soc., vol. 46, part 2, April 1950, pp. 307-315.
15. Jones, Robert T.: Properties of Low-Aspect-Ratio Pointed Wings at Speeds Below and Above the Speed of Sound. NACA Rep. 835, 1946. (Formerly NACA TN 1032.)
16. Heaslet, Max. A., Lomax, Harvard, and Spreiter, John R.: Linearized Compressible-Flow Theory For Sonic Flight Speeds. NACA Rep. 956, 1950. (Formerly NACA TN 1824.)
17. Squire, H. B.: A Theory of the Flow Over a Particular Wing in a Supersonic Stream. R. A. E. Rep. No. Aero. 2184, British, Feb. 1947.
18. Brown, Clinton E. and Adams, Mac C.: Damping in Pitch and Roll of Triangular Wings at Supersonic Speeds. NACA Rep. 892, 1948. (Formerly NACA TN 1566.)

REPORT 1055

COMPARISON OF THEORETICAL AND EXPERIMENTAL HEAT-TRANSFER CHARACTERISTICS OF BODIES OF REVOLUTION AT SUPERSONIC SPEEDS ¹

By RICHARD SCHERRER

SUMMARY

An investigation of the three important factors that determine convective heat-transfer characteristics at supersonic speeds, location of boundary-layer transition, recovery factor, and heat-transfer parameter has been performed at Mach numbers from 1.49 to 2.18. The bodies of revolution that were tested had, in most cases, laminar boundary layers, and the test results have been compared with available theory. Boundary-layer transition was found to be affected by heat transfer. Adding heat to a laminar boundary layer caused transition to move forward on the test body, while removing heat caused transition to move rearward. These experimental results and the implications of boundary-layer-stability theory are in qualitative agreement. Theoretical and experimental values of the recovery factor, based on the local Mach number just outside the boundary layer, were found to be in good agreement for both laminar and turbulent boundary layers on both of the body shapes that were investigated. In general, values of the heat-transfer parameter (Nusselt number divided by the square root of the Reynolds number) as determined for both heated and cooled cones with uniform and nonuniform surface temperatures, were in good agreement with theory. It was also found that the theory for cones could be used to predict the values of heat-transfer parameter for a pointed body of revolution with large negative pressure gradients with good accuracy.

INTRODUCTION

Aerodynamic heating of a body flying at high speeds results from the viscous dissipation of kinetic energy in the boundary layer. Although the viscosity of air is quite low, the increase in surface temperature due to aerodynamic heating becomes large at supersonic speeds. At speeds as low as 1,000 feet per second under conditions where the ambient temperature is above 40° F, the temperature rise is sufficient to make cabin cooling a necessity for man-carrying aircraft. At higher speeds the temperature rise can become large

enough to cause serious structural problems. Although some natural cooling by radiation can be expected, continuous flight of airplanes and missiles at supersonic speeds requires some form of surface or internal temperature control.

In order to design cooling systems or adequate insulation for high-speed aircraft, the heat-transfer coefficients and the reference surface temperature (recovery temperature) must be known. The basic variables, the recovery temperature, and the Nusselt number are primarily functions of Mach number and Reynolds number. In addition to being influenced by the Mach number and Reynolds number, the Nusselt number is affected by a variety of other factors, the most important of which are surface-temperature gradients, longitudinal surface-pressure gradients, and body shape.

The recovery temperature and heat-transfer characteristics of bodies with laminar boundary layers are markedly different from those with turbulent boundary layers and for this reason the extent of the laminar-flow region in the boundary layer must be known. Because the over-all rates of heat transfer are, in general, less for laminar than for turbulent boundary layers, a body with a laminar boundary layer would be heated less rapidly than one with a turbulent boundary layer. This fact is of particular importance in the design of very high-speed missiles, and information on conditions which control the extent of the laminar-flow region in the boundary layer is needed. The theoretical work by Lees, reported in reference 1, indicates that heat transfer has a marked effect on the stability of laminar boundary layers on smooth flat plates; as a result, it is expected that there is a corresponding marked effect of heat transfer on the location of transition. Lees' theory indicates that the effect of transferring heat from a surface to a laminar boundary layer is to decrease the damping of small disturbances in the boundary layer, whereas extracting heat tends to increase the damping of small disturbances. The theory also indicates that at each supersonic Mach number a rate of heat extraction from a laminar boundary layer greater than a certain value results in complete damping of any small disturbances. This is of importance in the design of supersonic aircraft because of the possibility of extending the laminar region of the boundary layer.

The laminar boundary layer in compressible flow has proven to be much more amenable to theoretical treatment than the turbulent boundary layer. For this reason, a large

¹ Supersedes NACA RM A8L28, "Heat-Transfer and Boundary-Layer Transition on a Heated 20° Cone at a Mach Number of 1.53" by Richard Scherrer, William R. Wimbrow, and Forrest E. Gowen, 1949; NACA TN 1975, "Experimental Investigation of Temperature Recovery Factors on Bodies of Revolution at Supersonic Speeds" by William R. Wimbrow, 1949; NACA TN 2087, "Comparison of Theoretical and Experimental Heat Transfer on a Cooled 20° Cone with a Laminar Boundary Layer at a Mach Number of 2.02" by Richard Scherrer and Forrest E. Gowen, 1950; NACA TN 2131, "Boundary-Layer Transition on a Cooled 20° Cone at Mach Numbers of 1.5 and 2.0" by Richard Scherrer, 1950; and NACA TN 2148, "Laminar-Boundary-Layer Heat-Transfer Characteristics of a Body of Revolution with a Pressure Gradient at Supersonic Speeds" by William R. Wimbrow and Richard Scherrer, 1950.

body of theory exists for the laminar case while only a few investigators have developed heat-transfer theories for the turbulent case. For an example of the latter, see reference 2.

The heat-transfer literature devoted to temperature-recovery factors, reviewed in reference 3, indicates that theoretically the recovery factor for laminar boundary layers is equal to the square root of the Prandtl number, and, for turbulent boundary layers, is approximately equal to the cube root of the Prandtl number. Because in theory the recovery factors at moderate Mach numbers (1.0 to 3.0) are largely determined by the type of boundary layer, little variation would be expected between the recovery factors measured on flat plates, cones, and bodies with longitudinal pressure gradients.

The theory for determining heat transfer in the laminar boundary layer on a smooth, flat plate with a uniform surface temperature has been developed by both Crocco, and Hantsche and Wendt (reference 4).² The work of Crocco has been extended by Chapman and Rubesin (reference 5) to include the effects of nonuniform surface temperatures. The effects of body shape and longitudinal surface pressure gradients on the heat-transfer characteristics of laminar boundary layers have been investigated theoretically in references 6 and 7. Because of the additional variables considered, the results of these two investigations are complex and therefore are difficult to apply. In reference 8, Mangler has presented relations whereby the characteristics of laminar boundary layers on any body of revolution can be determined if the characteristics of the laminar boundary layer on a related two-dimensional body are known. These relations can be used to determine the heat-transfer characteristics of a cone from those of a flat plate, but beyond this application the relations are somewhat restricted because of the difficulty in determining laminar boundary-layer characteristics on two-dimensional bodies with pressure gradients. For a body of revolution with negative pressure gradients (ogive, parabolic arc, etc.) one can infer that the boundary layer, and hence the heat transfer, is affected by two factors not affecting the boundary layer on cones. One is the decreasing rate of change of circumference with length, and the other is the pressure gradient. Since the effects of these two factors on the heat transfer tend to be compensating, it is possible that the theory for laminar boundary layers on cones may be used to predict the heat transfer in the negative-pressure-gradient regions of other bodies of revolution.

The present investigation was undertaken to determine the applicability of the available theoretical results of boundary-layer transition, recovery factor, and heat-transfer parameter. The purposes of the experimental phase of the investigation are:

1. To determine if heat transfer, both to and from a surface, affects boundary-layer transition.
2. To measure recovery factor on cones and on a body with a negative longitudinal pressure gradient for both laminar and turbulent boundary layers.
3. To measure the heat transfer in laminar boundary layers on cones with uniform and nonuniform surface temperatures.
4. To measure the heat transfer in a laminar boundary layer on a body with a negative longitudinal pressure gradient

Tests were conducted at several Mach numbers from 1.49 to 2.18 on three bodies of revolution under the following conditions:

1. Uniform surface pressure and a uniform temperature, heated surface (20° cone).
2. Uniform surface pressure and a uniform temperature, cooled surface (20° cone).
3. Uniform surface pressure and a nonuniform temperature, heated surface (20° cone, modified).
4. Nonuniform surface pressure and a uniform temperature, heated surface (parabolic-arc body).

Although the theory for laminar boundary layers indicates no basic change in heat-transfer processes between heated and cooled surfaces, it was considered desirable to obtain data under both conditions of heat flow in order to have the comparison of theory and experiment cover a broad range of surface temperature.

NOTATION

The following symbols have been used in the presentation of the theoretical and experimental data:

A	surface area of an increment of body length, square feet
C_d	skin-friction drag coefficient based on surface area, dimensionless
c_p	specific heat at constant pressure, Btu per pound, °F
C_r	temperature recovery factor, $\left(\frac{T_r - T_v}{T_0 - T_v}\right)$, dimensionless
g	gravitational acceleration, 32.2 feet per second squared
H	wind-tunnel total pressure, pounds per square inch absolute
h	local heat-transfer coefficient $\left[\frac{Q}{A(T_r - T_s)}\right]$, Btu per second, square foot, °F
k	thermal conductivity, Btu per second, square foot, °F per foot
l	total body length, feet
M	Mach number, dimensionless
Nu	local Nusselt number $\left(\frac{h_s}{k_v}\right)$, dimensionless
p	static pressure, pounds per square inch absolute
Pr	Prandtl number $\left(\frac{c_p \mu_v}{k_v}\right)$, dimensionless
Q	total rate of heat transfer for an increment of body length, Btu per second
q	local rate of heat transfer, Btu per second, square foot
Re	local Reynolds number $\left(\frac{\rho_v V s}{\mu_v}\right)$, dimensionless
Re_l	free-stream Reynolds number based on body length $\left(\frac{\rho_a U l}{\mu_a}\right)$
r	body radius, feet
s	distance from nose along body surface, feet
T	temperature, °F
ΔT	temperature increment, °F
T_r	recovery temperature (surface temperature at condition of zero heat transfer), °F

² The work of Crocco, Hantsche and Wendt, and others has been conveniently summarized by Johnson and Rubesin in reference 4.

$$T_r^* = \frac{T_r + 460}{T_e + 460}$$

$$T_s^* = \frac{T_s + 460}{T_r + 460}$$

$T_{s\infty}$	theoretical surface temperature for infinite boundary-layer stability, °F
U	free-stream velocity, feet per second
V	velocity just outside the boundary layer, feet per second
W	flow rate of coolant, pounds per second
x	distance from the nose along the body axis, feet
x^*	length ratio $\left(\frac{x}{l}\right)$, dimensionless
μ	viscosity, pounds per foot, second
ρ	air density, pounds per cubic foot

SUBSCRIPTS

In addition, the following subscripts have been used:

a	fluid conditions in free stream
c	coolant
0	stagnation conditions in the free stream
s	conditions at the body surface
v	fluid conditions just outside the boundary layer
w	wind-tunnel wall

APPARATUS AND TESTS

Heat-transfer tests were conducted with three bodies: a heated 20° cone, a cooled 20° cone, and a heated parabolic-arc body. The two conical bodies were also used to investigate the effect of heat transfer on boundary-layer transition. The surface of each of the three models was ground and polished until the maximum roughness was less than 20 micro-inches in an attempt to eliminate surface finish as a variable in the investigation. All of the heat-transfer tests were similar in that measurements of the test conditions, local surface temperature T_s , and the incremental rates of heat transfer Q were made with each body. The recovery temperature, T_r , was measured in each set of experiments with the test-body heating or cooling system shut off and after both the wind tunnel and surface temperatures had reached equilibrium. The dimensionless heat-transfer parameter ($Nu/Re^{1/2}$, for laminar boundary layers) was determined from the experimental data by use of the following equation:

$$\frac{Nu}{Re^{1/2}} = \frac{Qs}{Ak_s(T_s - T_r)} \left(\frac{\mu_s}{\rho_s V s} \right)^{1/2} \quad (1)$$

WIND TUNNEL

The tests with all of the models were performed in the Ames 1- by 3-foot supersonic wind tunnel No. 1. This closed-circuit, continuous-operation wind tunnel is equipped with a flexible-plate nozzle that can be adjusted to give test-section Mach numbers from 1.2 to 2.4. Reynolds number variation is accomplished by changing the absolute pressure in the tunnel from one-fifth of an atmosphere to approximately three atmospheres—the upper limit on pressure being a function of Mach number and ambient temperature. The water content of the air in the wind tunnel is maintained at less than 0.0001 pound of water per pound of dry air in order to eliminate

humidity effects in the nozzle. All of the bodies tested were mounted in the center of the test section on sting-type supports. The total temperature of the air stream was measured by nine thermocouples in the wind-tunnel settling chamber.

HEATED CONE TESTS

Test body.—The details of the heated cone are shown in figure 1, and a photograph of the cone in the wind tunnel is shown in figure 2. The shell was machined from stainless steel because of its high electrical resistance, and all other parts were made of copper. Several longitudinal wall-thickness distributions, obtained by remachining the inside surface of the cone, were used to obtain different longitudinal surface-temperature distributions. The cone was heated by passing a high-amperage alternating electrical current (800 amperes maximum) at low voltage (0.45 volts maximum) longitudinally through the cone shell. Because the electrical resistance of the solid nose of the cone was very small, the forward 25 percent of the cone was, in effect, unheated.

Instrumentation.—The wiring diagram of the cone, showing the electrical heating circuit and the temperature- and power-measurement apparatus, is shown in figure 3. Nine iron-constantan thermocouples were installed at equal length increments along the cone to determine the surface-temperature distributions. The thermocouples were installed in holes drilled completely through the shell and were soldered in place. The ten voltage leads were installed in a similar manner and were arranged on the selector switch so as to measure the voltage drop of each successive increment along the cone. These measurements, with the current measurements, determined the local heat input to the cone.

Test procedure.—Data were obtained at length Reynolds numbers of approximately 2.5 and 5.0 million at Mach numbers of 1.50 and 1.99. Cone temperatures of 30° F and 60° F above the wind-tunnel total temperature were selected as nominal values at which to obtain data. The surface temperatures were set to the desired values by adjusting the input voltage, and, with the cone at the desired temperature, the following data were recorded: air stream total temperature and pressure, test-section static pressure, air stream humidity, current, incremental voltages, and surface temperatures.

Accuracy.—The accuracy of the experimental data was determined by estimating the uncertainty of the individual measurements which entered into the determination of the final results. The estimated uncertainties of the basic measurements are as follows:

Total temperature, T_0	$\pm 1.5^\circ \text{ F}$
Surface temperature, T_s	$\pm 0.5^\circ \text{ F}$
Total pressure, H	$\pm 0.01 \text{ psia}$
Static pressure, p	$\pm 0.01 \text{ psia}$
Current	$\pm 2 \text{ percent}$
Incremental voltages	$\pm 2 \text{ percent}$
Cone dimensions	$\pm 0.005 \text{ inch}$

The maximum probable error of any given parameter was obtained from the component uncertainties by the method described in reference 9. The maximum probable error of the heat-transfer parameter $Nu/Re^{1/2}$ is ± 6.8 percent and

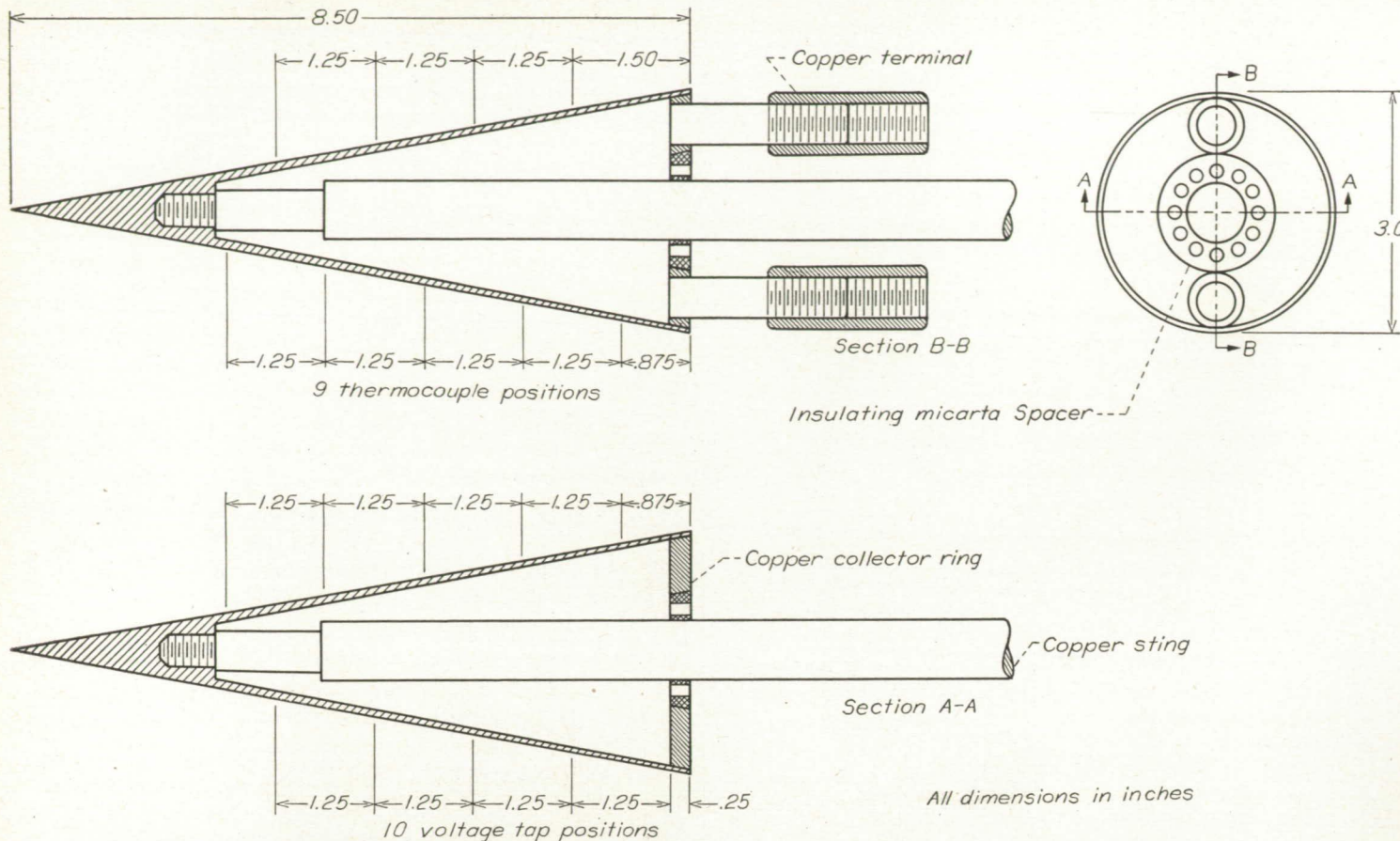


FIGURE 1.—Details of the electrically heated 20° cone.

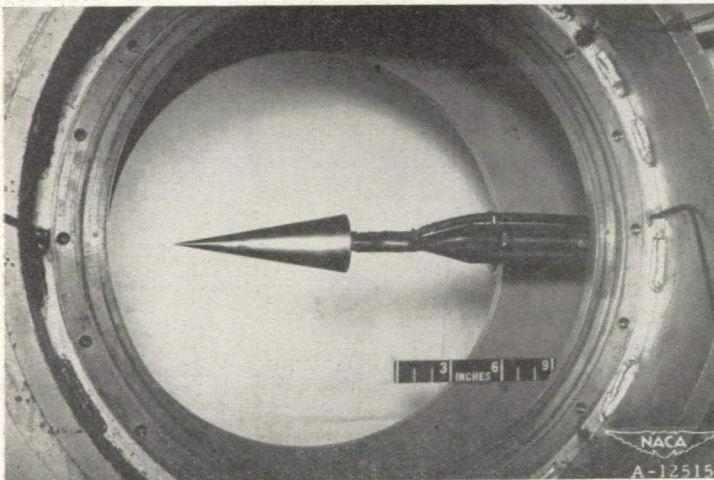


FIGURE 2.—Heated 20° cone installed in the test section of the 1- by 3-foot supersonic wind tunnel No. 1.

that of the recovery factor is ± 1.5 percent. Because the over-all accuracy is largely controlled by the accuracy of the total-temperature measurements, the experimental scatter of data points obtained at a fixed value of the total temperature is much less than that indicated by the values of over-all accuracy. Although the small pressure gradient (± 1 percent of the average dynamic pressure) in the test section of the wind tunnel was neglected in the reduction of the test data, its effect has been included in the calculation of the maximum probable error.

An effort was made to determine the magnitude of the error due to heat radiation from the cone to the tunnel walls by experimental means. The radiant-heat loss was evaluated by the extrapolation to zero pressure of data obtained at several low values of total pressure with the wind tunnel in operation³ and was found to be negligible.

COOLED-CONE TESTS

Air-cooled cone.—The construction details and the path of the cooling air through the cone are shown in figure 4. The outer shell was made of stainless steel and had a wall thickness of only 0.028 inch in order to minimize the heat conduction along the shell in the instrumented area. The inner cone also was made of stainless steel and had a wall thickness of 0.070 inch which was thin enough to make all conduction effects in the inner cone negligible. The annular gap between the cones was designed to give a uniform surface temperature in the instrumented region ($x^*=0.4$ to 0.8). Turbulent flow in this annular gap was necessary to obtain the most uniform distribution of coolant temperature across the gap; therefore, the internal cooling system was designed to operate at a Reynolds number sufficiently high to insure turbulent flow. In addition, the outer surface of the inner cone was roughened with $\frac{1}{32}$ -inch-wide circumferential grooves spaced at $\frac{1}{4}$ -inch intervals. During the investigation

³ An attempt was made to obtain the radiation calibration with the tunnel inoperative, but the cone surface temperatures were found to be very erratic because of free-convection currents. For this reason, the method was abandoned.

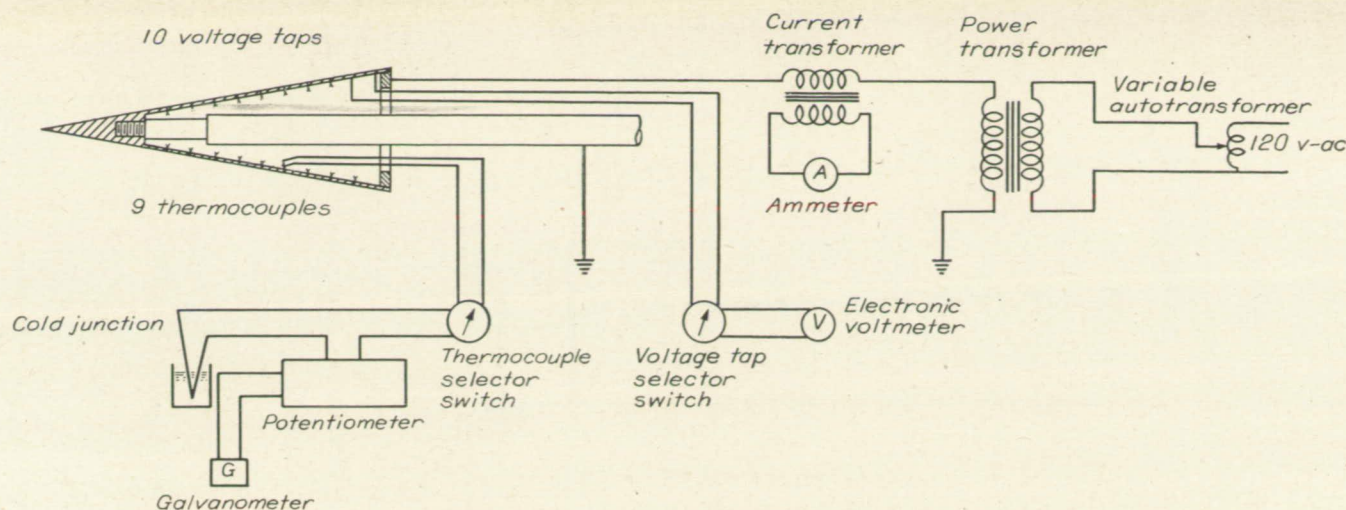


FIGURE 3.—Wiring diagram for the electrically-heated 20° cone.

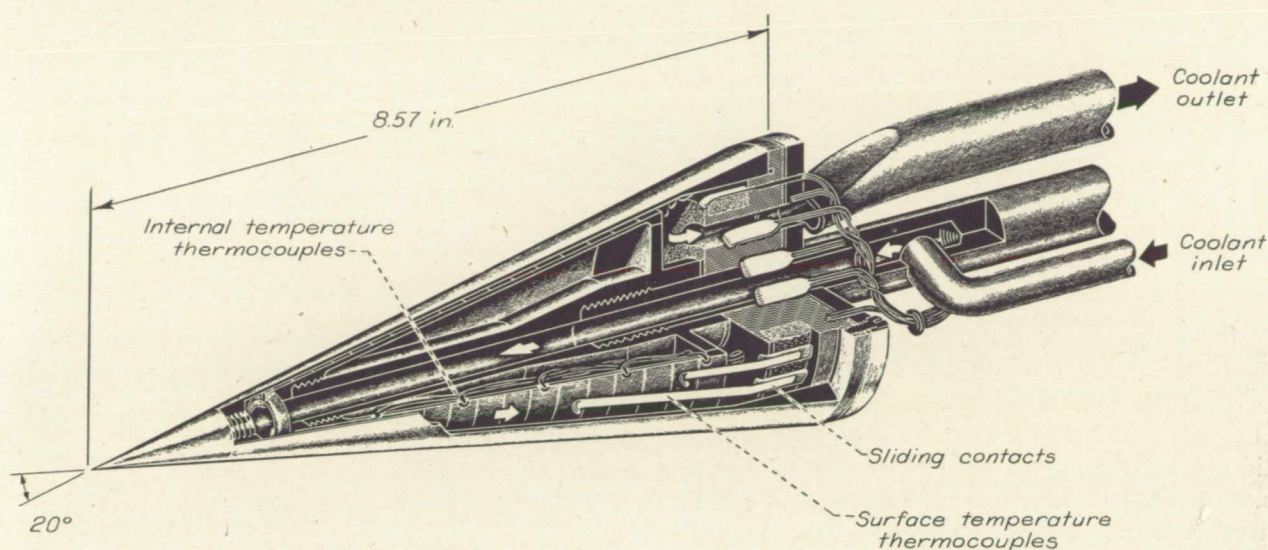


FIGURE 4.—Air-cooled cone.

on, tests were made with coolant flow rates of 70 percent and 40 percent of the design value to determine the effect of changes in the internal-flow Reynolds number on the internal-temperature measurements. The results indicated that this effect was negligible.

Cooling system.—The primary requirement of the cooling system was that it provide an adjustable and stable outlet temperature at a constant coolant flow rate. This requirement was satisfied by the cooling system shown in figure 5. The clean, dry air which was used as the coolant in the cone was obtained from the make-up air system of the wind tunnel and could be cooled to a minimum temperature of -90°F . In the development of the cooling system it was found that the use of the recovery heat exchanger provided a marked increase in thermal stability. The temperature drift of the cooling system was extremely slow and amounted to less than $\pm 0.1^{\circ}\text{F}$ per minute.

Instrumentation.—Local rates of heat transfer were obtained by measuring the incremental temperature changes of a known weight-flow of coolant flowing along the annular gap between the inner and outer cones. The incremental coolant temperatures were measured by iron-constantan thermo-

couples spaced 1 inch apart along the annular gap. Three thermocouple junctions were located 120° apart midway between the walls of the gap at each of the five longitudinal stations. The surface temperature was measured with stainless-steel-constantan thermocouples consisting of the cone shell and thin (0.002 inch thick) constantan ribbons. The tips of the ribbons were soldered into holes in the shell. The thin strips were cemented to the inner surface of the shell. This installation minimized the interference of the constantan wires with the flow of air in the coolant passage. The flow rate of cooling air was measured with a rotameter located at the outlet of the cooling system.

Test procedure.—The tests were performed at three values of the wind-tunnel total pressure which gave length Reynolds numbers of 2.2, 3.6, and 5.0 million. Data were taken at the recovery temperature and at three nominal surface temperatures at each Reynolds number. The minimum surface temperature obtainable with the cooling system at each pressure, and temperatures approximately 20°F and 40°F above the minimum temperatures, were selected. The investigation was performed only at one Mach number (2.02) because the theory for heat transfer

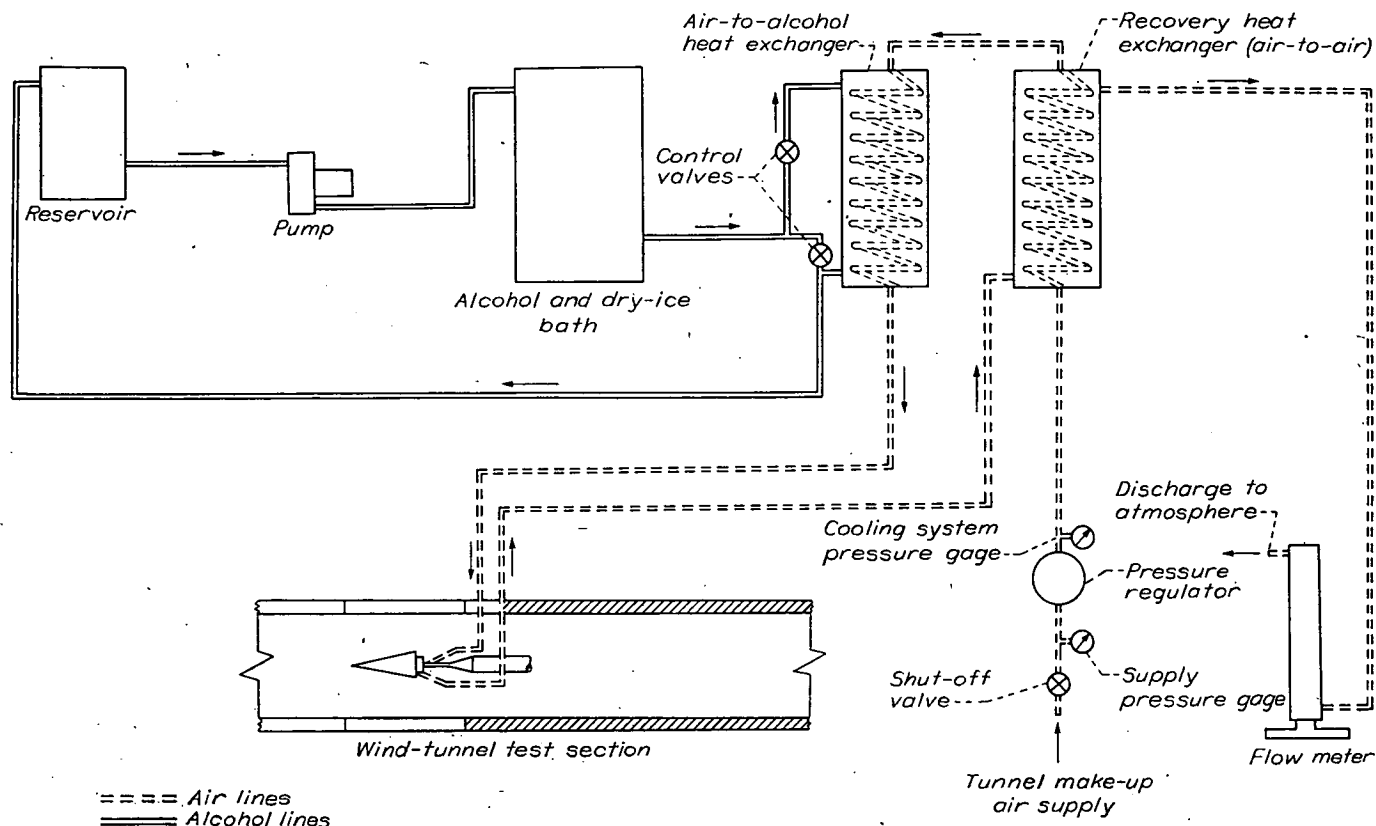


FIGURE 5.—Diagram of the air cooling system for the cooled 20° cone.

with uniform surface temperature and previous tests with the heated cone both indicated that the effect of Mach number on heat-transfer parameter is small within the range available in the wind tunnel (1.2 to 2.4). Schlieren observations, liquid-film tests, and the absence of discontinuities in the temperature-distribution measurements that would denote transition indicated that the boundary layer remained laminar for all test conditions.

In order to measure the effects of cooling on boundary-layer transition, it was necessary to have the transition point near the center of the instrumented area of the cone. However, it was known from the results of tests of the unheated cone that transition would occur downstream from the instrumented area at the condition of zero heat transfer at the maximum available Reynolds number. The required forward movement of transition was accomplished by the use of three small grooves around the cone at approximately the 8-, 10-, and 12-percent-length stations. All the grooves were 0.010 inch wide and 0.015 inch deep, and extreme care was taken to make them of uniform depth and with uniformly sharp edges in an attempt to obtain the same longitudinal position of transition on all rays of the cone. Data were obtained at several surface temperatures below the recovery temperature in successive decrements of about 10° F. The minimum surface temperatures investigated were the temperatures at which transition moved out of the instrumented area of the cone. The boundary-layer transition tests were conducted at a nominal length Reynolds number of 5.0 million at Mach numbers of 1.5 and 2.0.

Accuracy.—The accuracy of the final results is based on the accuracy of the individual measurements involved and on the probable uncertainty of some of the measurements

due to peculiarities of the test apparatus. The estimated accuracy of the individual measurements in this test is as follows:

Total temperature, T_0	$\pm 1.5^\circ$ F
Surface temperature, T_s	
At maximum T_s	$\pm 1.5^\circ$ F
At minimum T_s	$\pm 4.5^\circ$ F
Coolant-temperature increment, ΔT_c	
At maximum T_s	$\pm 0.07^\circ$ F
At minimum T_s	$\pm 0.05^\circ$ F
Internal air-flow rate, W (at design flow rate).....	$\pm 1.4\%$
Cone dimensions.....	± 0.002 in.
Cone-segment surface areas, A	$\pm 1.4\%$
Total pressure, H	± 0.01 psia
Static pressure, p	± 0.01 psia

Although the surface-temperature thermocouples were calibrated and were accurate to within $\pm 0.5^\circ$ F, the experimental variations from curves faired through the sets of surface-temperature data were found to exceed this value. These variations are believed to have resulted from a non-uniform coolant distribution in the annular gap. The accuracy of the surface-temperature measurements at each set of test conditions was taken as the local difference between the faired curves and the data point farthest from the curves. The local values of surface temperature used in the reduction of the data were obtained from the faired curves.

Because the range of temperature differences between the cone and wind-tunnel-wall temperatures is similar to that in the heated-cone tests, the effect of radiant heat transfer on the experimental accuracy again has been neglected. The effects of longitudinal and circumferential heat conduction in the thin stainless-steel inner cone aft of the 50-percent-length station have been neglected. An approximate calculation of

the actual conduction along the inner cone and through the inner-cone shell revealed that these effects were only about three-tenths of 1 percent of the total heat transfer. As in the case of the heated cone, the effect of the small static pressure gradient in the wind-tunnel test section has been neglected.

As in the heated-cone tests, the over-all accuracy of the final values of the heat-transfer parameter was calculated from the accuracy or uncertainty of each of the individual measurements for all test conditions by the method of reference 9. Because of the variation in the uncertainty of the surface temperature and internal-temperature increments, the final accuracy of the heat-transfer parameter varies from ± 12 percent at the most forward section of the cone to ± 8 percent at the rear of the cone.

PRESSURE-GRADIENT BODY TESTS

Body and instrumentation.—The body was axially symmetric and its radius at any longitudinal station is given by the relation

$$\frac{r}{L} = \frac{1}{3} \left[\left(\frac{x}{L} \right) - \left(\frac{x}{L} \right)^2 \right] \quad (2)$$

Since the forward portion of the body was adequate for the purpose of the present investigation, the length L in equation (2) was assigned a value of 18 inches, but only the first 8½ inches of the total length were employed. The vertex angle of the test body was 37° and its fineness ratio was 2.83. Like the cones, the exterior shell of the body was machined from stainless steel. The heating circuit, instrumentation and general design of the parabolic-arc body were similar to those of the heated cone; details of the body are shown in figure 6. The shell thickness was designed to provide a uni-

form surface temperature at a Mach number of 1.5 by assuming that the theory for cones was applicable. In addition to the heated body, another body identical in contour was employed to determine the pressure distribution and, consequently, the Mach number distribution just outside the boundary layer along the body.

Procedure and accuracy.—The test procedure and accuracy with the parabolic-arc body were the same as those with the electrically-heated 20° cone. Data were obtained at nominal length Reynolds numbers of 2.5, 3.75, and 5.0 million at Mach numbers of 1.49 and 2.18. Surface temperatures of 120°F , 160°F , and 200°F were arbitrarily chosen as values at which to obtain data. Pressure-distribution measurements also were made at the same Mach numbers and total pressures as for the heat-transfer measurements. Because the local static pressures on the parabolic-arc body were measured, the maximum probable error of the recovery factor is ± 1 percent rather than ± 1.5 percent as in the case of the conical bodies.

RESULTS AND DISCUSSION

EFFECT OF HEAT TRANSFER ON BOUNDARY-LAYER TRANSITION

The effect on transition of adding heat to a laminar boundary layer at a Mach number of 1.99 is shown by the longitudinal surface-temperature distributions in figure 7. The effect of removing heat from a laminar boundary layer at Mach numbers of 1.50 and 2.02 is shown in figure 8. The start of transition is indicated by an abrupt decrease in surface temperature for the case of heat addition and by a rise in surface temperature with no heat transfer or when heat is removed from the boundary layer. These changes in surface temperature result from the difference in heat-

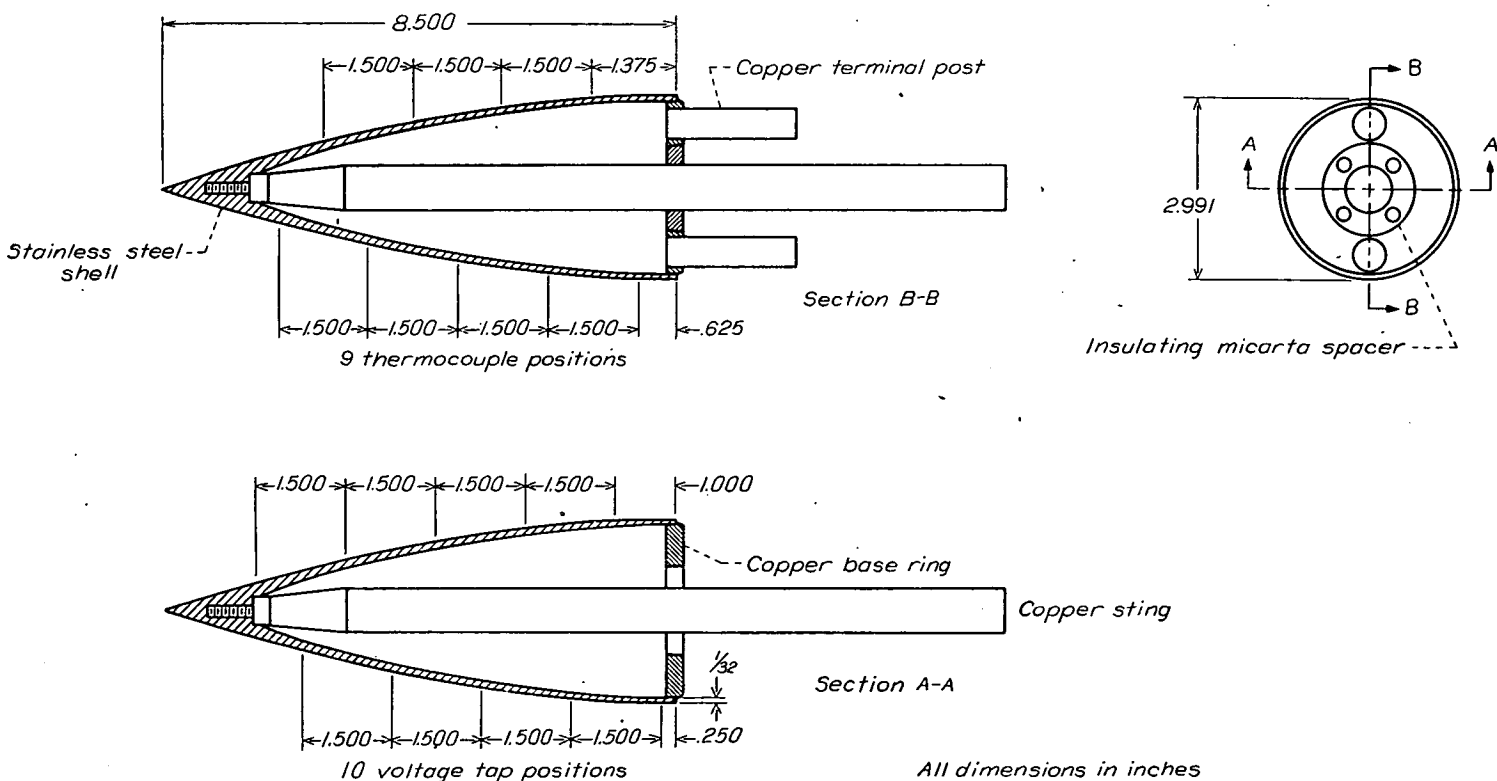


FIGURE 6.—Details of the electrically-heated parabolic body.

transfer coefficients between the laminar and turbulent boundary-layer regions. Only the data from one side of the cooled cone are presented in figure 8 because transition occurred in a similar manner on the opposite side, but at a position approximately 5 percent of the cone length aft of that shown at all test conditions. It should be noted that the lower curve in figure 7 and the upper curves in figure 8 were obtained at the condition of zero heat transfer ($T_s = T_r$).

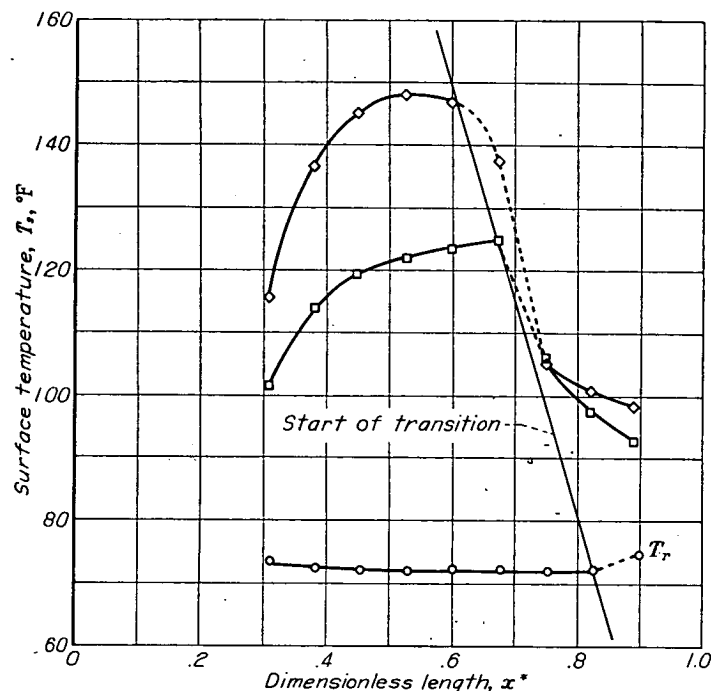


FIGURE 7.—The effect on transition of adding heat to the laminar boundary layer on a 20° cone; $M=1.99$, $Re_l=5.0 \times 10^6$.

In the case of the cooled cone (fig. 8), the fact that the rise in surface temperature is indicative of transition has been verified in four ways:

1. Liquid-film tests at the same length Reynolds number, at recovery temperature, have indicated transition in the region where the rise in surface temperature occurs.

2. The surface-temperature distributions ahead of the rise in surface temperature are almost identical in shape with those obtained in tests with the same body for which the measured heat transfer indicated laminar flow.

3. The region in which the rise in surface temperature occurs moved forward with an increase in the number of grooves.

4. The recovery factor increased by 0.04 in the region where the rise in surface temperature occurs,⁴ which is in agreement with the change between laminar and turbulent flow obtained both in theory and in other experiments.

The variation of the extent of the laminar boundary layer along the heated and cooled cones indicates that heat addition promotes early transition while removal delays transition. The theory of reference 1 and these results are in

⁴ The recovery factor on this model, without grooves, was 0.85, which is in agreement with the results for a laminar boundary layer obtained with the other models. The absolute values of the recovery factors ahead of and behind the rise in surface temperature were found to decrease continuously with an increase in number of grooves, but their difference was always 0.04 ± 0.005 . With the three grooves that were used, the laminar boundary-layer recovery factor was 0.81 and the turbulent boundary-layer value was 0.85. The reason for this change in recovery factor with number of grooves is unknown.

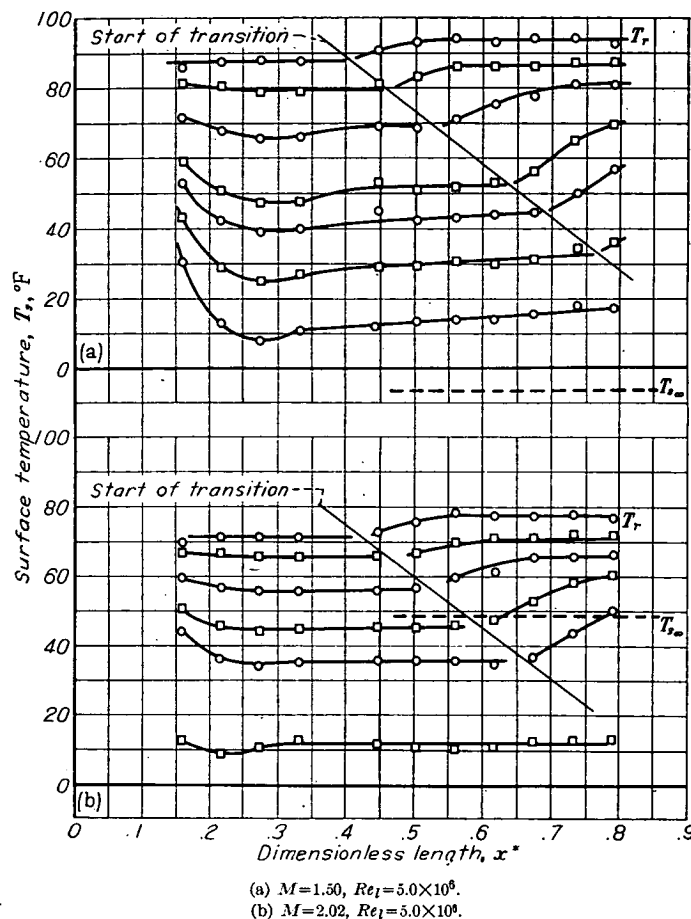


FIGURE 8.—Effect on transition of removing heat from a laminar boundary layer on a 20° cone with a small initial roughness.

qualitative agreement. For the present test conditions with the cooled cone, the theory indicates that the boundary layer should be infinitely stable at surface temperatures ($T_{s\infty}$) of -6.6°F and 49.0°F at Mach numbers of 1.5 and 2.0 respectively. The data shown in figure 8(b) show that transition occurred at surface temperatures below the theoretical infinite-stability limit at a Mach number of 2.02, and the data shown in figure 8(a) indicate that a similar result is possible at a Mach number of 1.50.

The limited range of the present experiments makes it impossible to draw any definite conclusions regarding the quantitative effect of heat transfer or Mach number on transition. The data indicate, however, that the movement of transition with surface temperature for the cooled cone is at least similar at both Mach numbers. The constant slopes of the lines (in figs. 7 and 8) through the points at the beginning of transition for both cones indicate that, for the present experiments, the increase in the length of laminar run is directly proportional to the difference between the surface and recovery temperatures, and therefore is a linear function of the rate of heat transfer. Although the present experiments have proven qualitatively that heat transfer affects transition, additional experiments at higher Reynolds numbers (without roughness) and in a low-turbulence air stream are required to obtain quantitative data on the effect of heat transfer on boundary-layer transition.⁵

⁵ A recent quantitative study of the effect of heat addition on transition has been made by Higgins and Pappas and published as NACA TN 2351.

RECOVERY FACTOR,*

Typical values of the recovery factor obtained at several Reynolds numbers at nominal Mach numbers of 1.5 and 2.0 with the heated-cone model at the condition of zero heat transfer are shown in figure 9. Data obtained with the cone at a Mach number of 2.0 with various means of causing transition are also shown. Data obtained with the parabolic-arc body are shown in figure 10. The theoretical values indicated in both figures are based on a Prandtl number of 0.719 which is the value for dry air at 70° F. The theoretical recovery factors are therefore 0.848, $Pr^{1/2}$, for laminar boundary layers and 0.896, $Pr^{1/3}$, for turbulent boundary layers.

The experimental values of the recovery factor for turbulent boundary layers on the heated cone were obtained by resorting to artificial means of causing transition. Three methods were used—a ring of 0.005-inch diameter wire, a coat of lampblack-lacquer mixture, and a band of salt crystals. The wire and salt crystals were located at about 1 inch from the cone tip, and the lampblack covered the cone forward of the 1-inch position.

The data shown in figure 9 for laminar boundary layers are in fairly good agreement with theory. The difference between the values for the two Mach numbers in the region $x^*=0.3$ to 0.6 is believed to be partly due to the fact that the average Mach number was used in the reduction of the data. The most rearward data points obtained with a laminar boundary layer at the maximum Reynolds numbers (fig. 9) indicate the beginning of natural transition. The data obtained with turbulent boundary layers are somewhat lower than the theoretical value, but the agreement is considered to be good enough to justify use of the theoretical value for design purposes. The differences between the results obtained with the various roughnesses decrease to small values toward the rear of the cone.

* A detailed discussion of the physical concept of the temperature recovery factor for laminar boundary layers is given by Wimbro in NACA TN 1975.

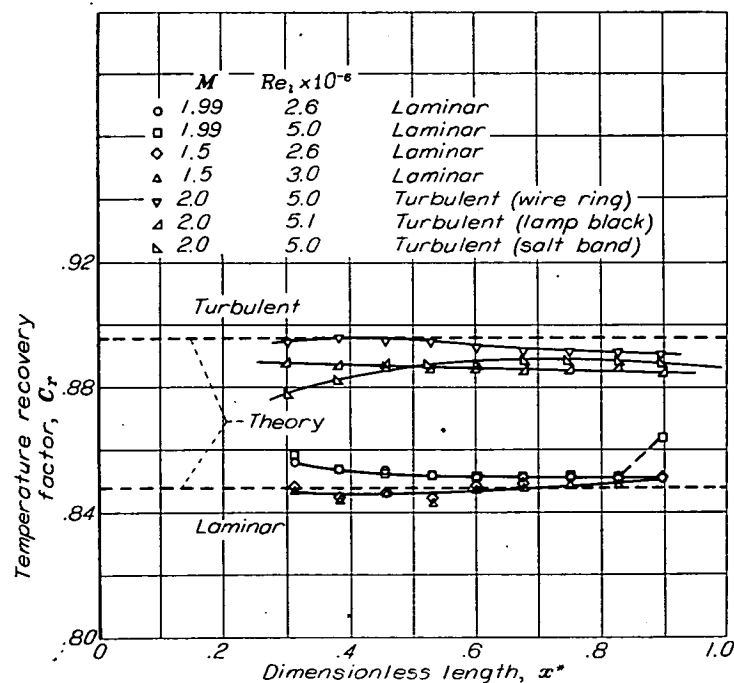


FIGURE 9.—Typical values of the temperature recovery factor for the 20° cone.

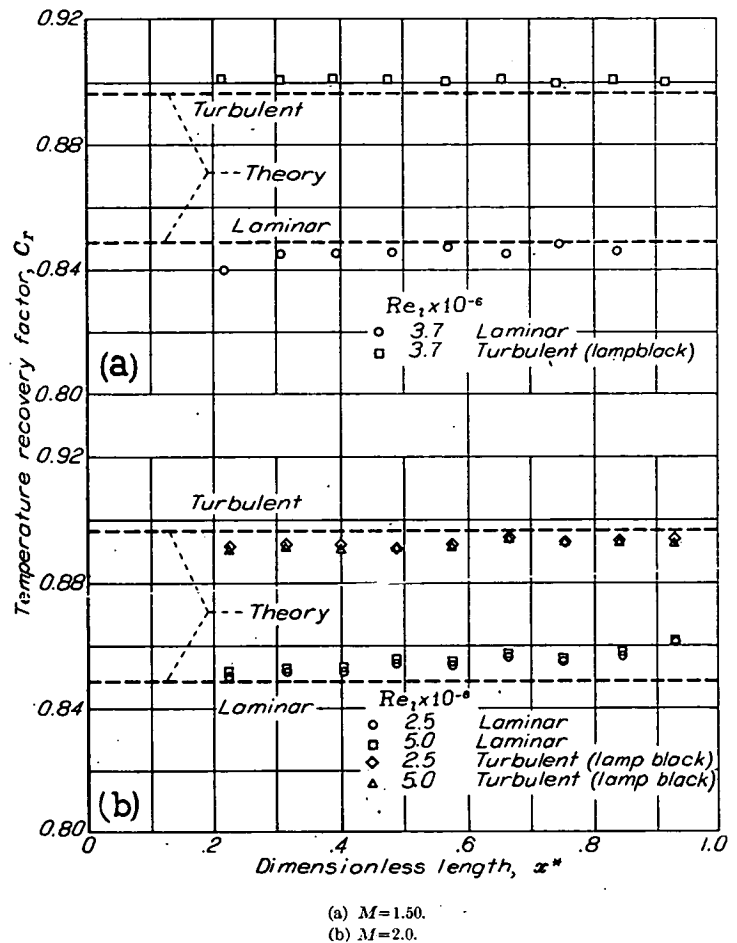


FIGURE 10.—Typical values of the temperature-recovery factor for the parabolic-arc body.

Since the external geometry and surface finish of the heated and cooled cones were identical, there were no essential differences in their recovery factors. For this reason, the recovery factors for the cooled cone are not presented.

The values of recovery factor for the parabolic-arc body with both laminar and turbulent boundary layers were based on the local Mach number just outside the boundary layer and these recovery factors are substantially constant along the body (fig. 10). The values of recovery factor for turbulent boundary layers, obtained by the use of lampblack, are somewhat greater than the values obtained with the cone (fig. 9). The data obtained with a laminar boundary layer on the parabolic-arc body at a Mach number of 2.0 show a gradual increase in recovery factor along the length of the body. This trend in the data is believed to have been caused by a gradual deterioration in the surface finish of the model due to testing and handling. Data obtained at a Mach number of 2.18 with this body when it was new and had a mirror-like finish indicated a constant recovery factor of 0.850. It is possible that surface roughness may cause some increase in the laminar boundary-layer recovery factor.

In general, the agreement between theory and experiment indicates that the recovery temperature on cones and in the negative-pressure-gradient region on bodies of revolution can be computed with satisfactory accuracy by use of the theoretical recovery factors and the local Mach number just outside the boundary layer.

HEAT TRANSFER

Results.—The theoretical values of the heat-transfer parameter for cones with laminar boundary layers and uniform surface temperatures are given by the following equation from reference 4:

$$\frac{Nu}{Re^{1/2}} = \frac{C_d \cdot Re^{1/2}}{2} Pr^{1/3} \sqrt{3} \quad (3)$$

in which the physical properties of the fluid are based on the temperature just outside the boundary layer. Crocco's theoretical calculations give values of the factor $C_d \cdot Re^{1/2}$ of 0.66 and 0.63 for small temperature differences ($T_s - T_r \approx 100^\circ \text{F}$) at Mach numbers of 1.5 and 2.0, respectively. If the Prandtl number is assumed to be 0.72, then the theoretical values of heat-transfer parameter at the two Mach numbers are 0.51 and 0.49.

The surface-temperature and heat-transfer-parameter data for the heated cone with an approximately uniform surface temperature at a Mach number of 1.50 are shown in figure 11 together with the theoretical value of the heat-transfer parameter for a uniform surface temperature (0.51). Similar data obtained at a Mach number of 1.99 for both the approximately uniform and the nonuniform surface temperatures are shown in figure 12, and the theoretical value of heat-transfer parameter for a uniform surface temperature (0.49) is shown for comparison. The data for the least uniform surface temperature shown in figure 12 is replotted in figure 13 and compared with a curve obtained by use of the theory of reference 5. The surface temperature distribution, converted according to the method of reference 8, is also shown in figure 13 and will be discussed later. The values of heat-transfer parameter for the nonuniform surface temperatures have been corrected for the effect of heat conduction in the shell due to the local surface temperature

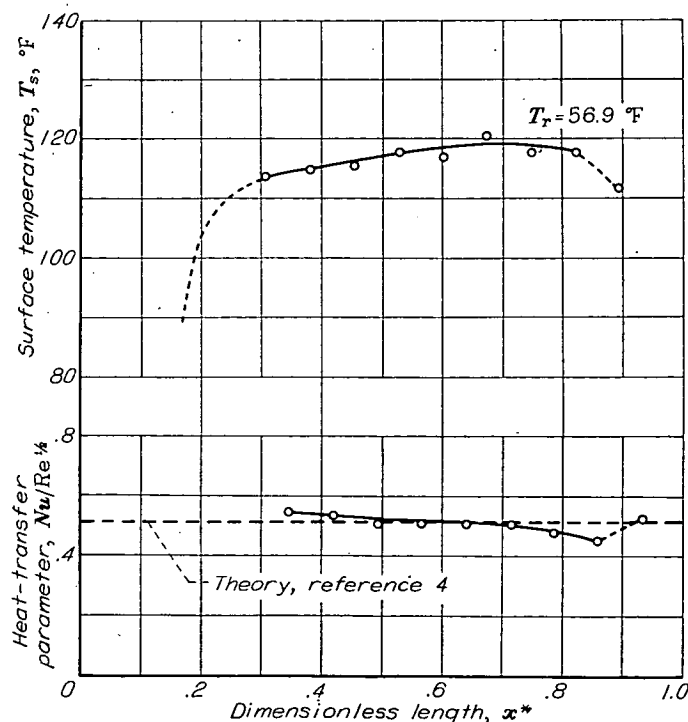


FIGURE 11.—Heat-transfer characteristics of the heated 20° cone with the most uniform surface temperature; $M=1.50$, $Re_1=2.6 \times 10^6$.

gradients. This correction for the approximately uniform surface temperatures was found to be negligible.

The heat-transfer data obtained with the cooled cone are

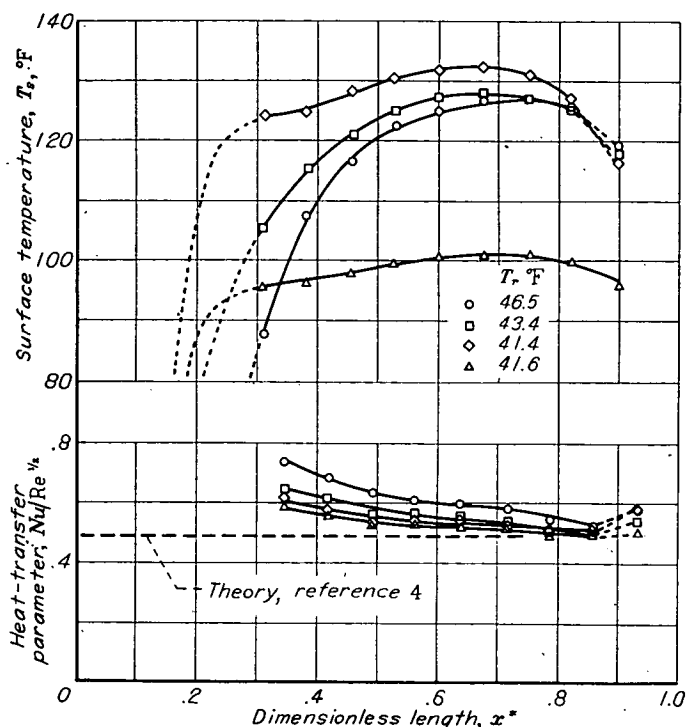


FIGURE 12.—Heat-transfer characteristics of the heated 20° cone with various surface-temperature distributions; $M=1.99$, $Re_1=2.6 \times 10^6$.

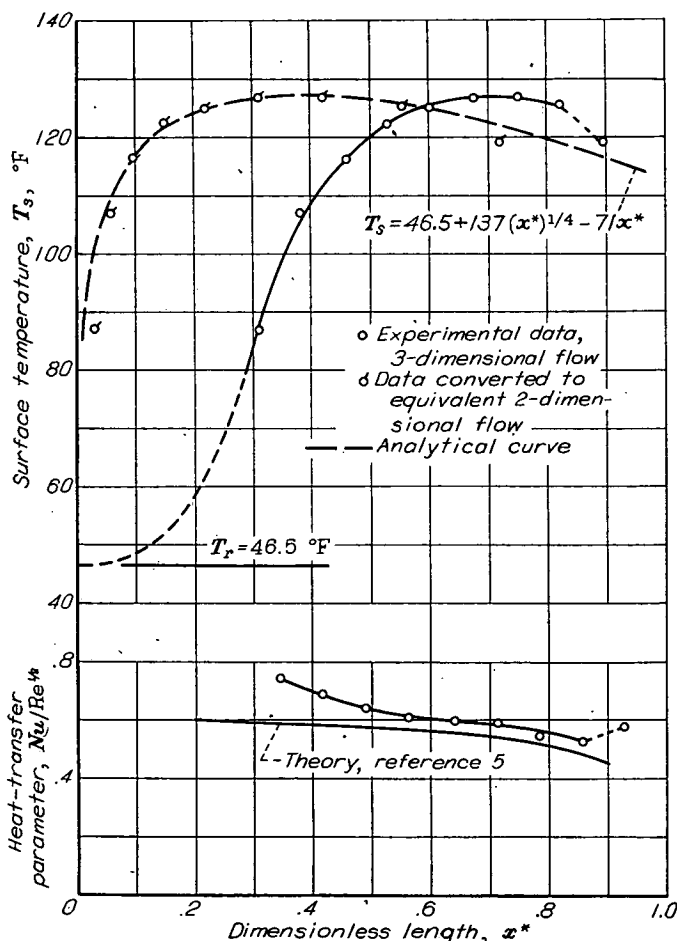


FIGURE 13.—Comparison of the theory for nonuniform surface temperatures with data from tests of the heated 20° cone; $M=1.99$, $Re_1=2.6 \times 10^6$.

presented in figure 14. All of the longitudinal surface-temperature distributions obtained during the tests were similar in shape, and therefore only three typical distributions, those for a Reynolds number of 2.2 million, are shown. Local values of heat-transfer parameter for the three surface temperatures at each of the three length Reynolds numbers are also presented. The surface-temperature gradient that existed over part of the segment nearest the tip resulted in a conduction loss of $3\frac{1}{2}$ to 4 percent. The heat-transfer data has been corrected for this effect. Downstream of the first element there was no longitudinal conduction correction because the surface temperature in this region was nearly constant. The scatter in the experimental data (fig. 14) is about equal to the estimated maximum probable error which resulted from the inaccuracy of the measurements of the incremental changes in coolant temperature along the heat-transfer passage and from the scatter of the surface-temperature data. The theoretical value of the heat-transfer parameter, 0.49, is also shown in figure 14. The data nearest the nose deviate quite markedly from theory, a fact which is attributed to the effects of surface-temperature gradients on the laminar boundary layer. The values of heat-transfer parameter obtained with the heated cone with the approximately uniform surface temperature at a nominal surface temperature of 100°F (fig. 12) have been plotted in figure 14 for comparison.

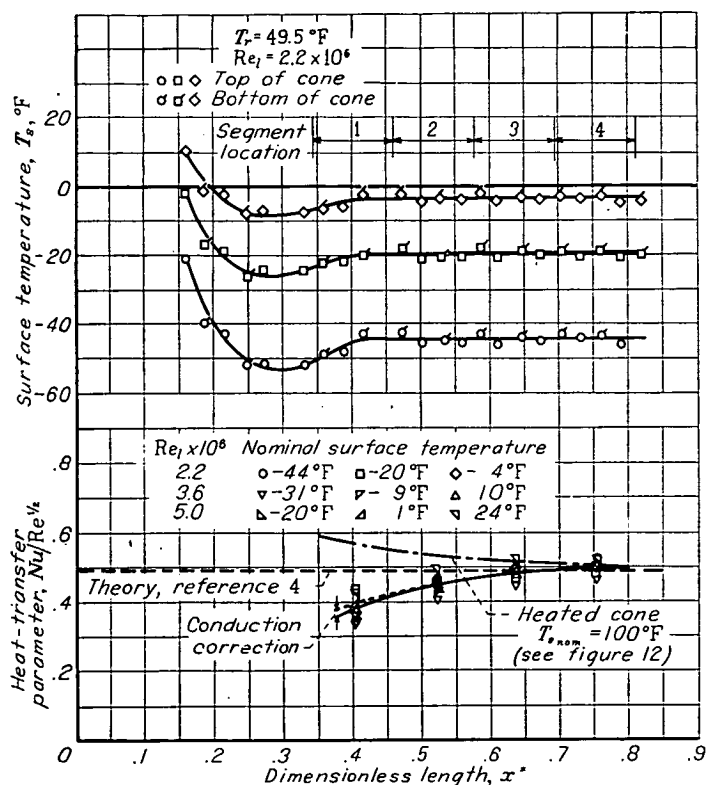


FIGURE 14.—Heat-transfer characteristics of the cooled 20° cone; $M=2.02$.

The measured pressure distributions required for the reduction of the heat-transfer data on the parabolic-arc body were reduced to the more convenient form of Mach number distributions and are shown in figure 15. The values of local Mach number were used to determine the local temperature at the outer edge of the boundary layer.

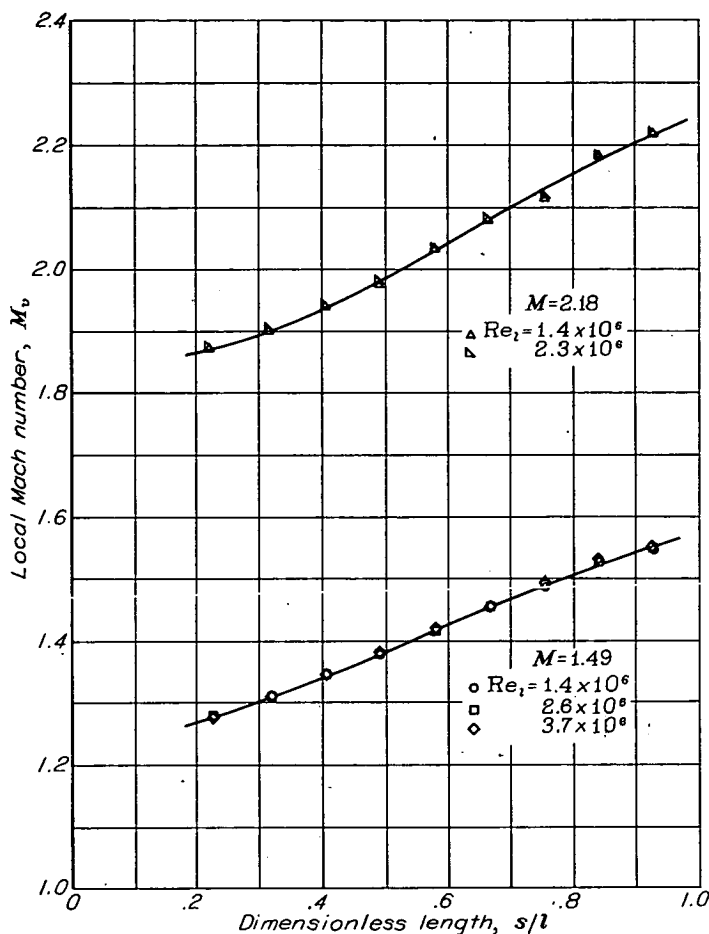
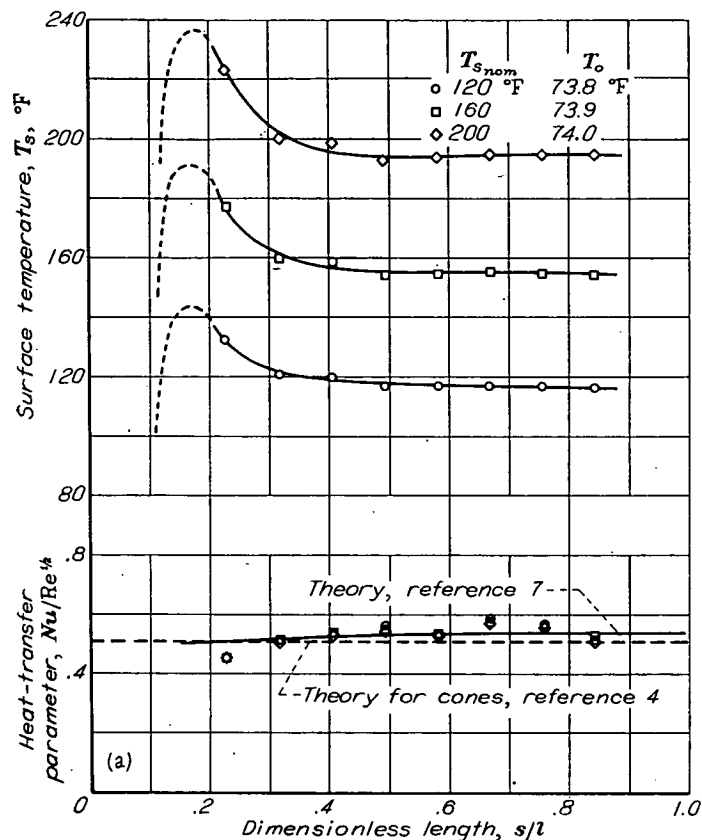
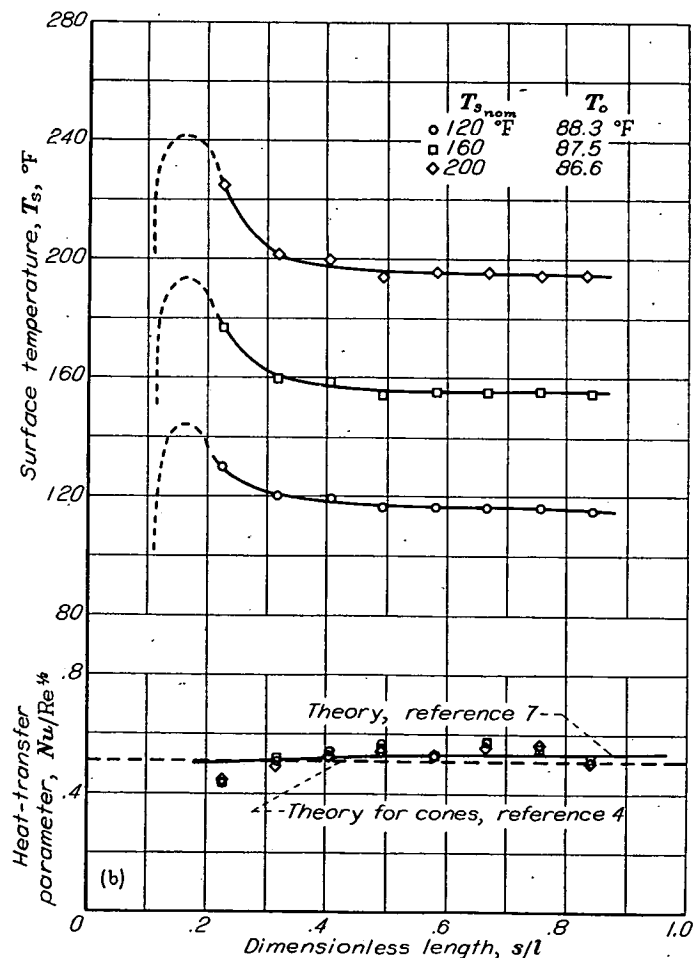
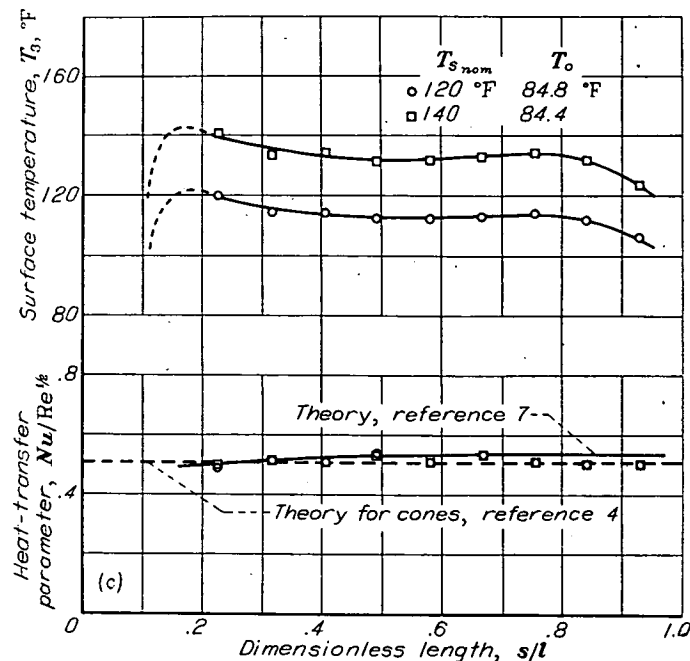


FIGURE 15.—Variation of local Mach number with length as determined by pressure measurements on the parabolic-arc body.

The heat-transfer data for the parabolic-arc body at a free-stream Mach number of 1.49 are shown in figure 16 and data obtained at a Mach number of 2.18 are shown in figure 17. Although heat-transfer data were obtained at three Reynolds numbers, 2.5, 3.75, and 5.0 million, at each Mach number, only representative data are presented in figures 16 and 17. The theoretical values of the heat-transfer parameter for cones are shown in figures 16 and 17 for comparison with the experimental data. In addition, local values of the heat-transfer parameter for a Mach number of 1.5 were computed for the test body by the method of reference 7. These values are shown in figure 16.

Inspection of the surface-temperature curves in figures 16(a), 16(b), and 17 reveals that the temperature at the first measuring station was considerably higher than the temperature at subsequent stations. After the tests were completed it was found that this local hot region was partially caused by poor electrical contact between the copper stinging and the stainless-steel shell. This connection was improved and the body was tested again at a Mach number of 1.49. As shown in figure 16 (c), a much more uniform surface temperature was obtained.

Heat transfer with uniform surface temperature.—All three of the bodies tested had severe surface temperature gradients near the noses because the noses were solid and essentially unheated or uncooled. In the case of the heated bodies, the data obtained with approximately uniform

(a) Original surface-temperature distribution $Re_l = 2.6 \times 10^6$.FIGURE 16.—Heat-transfer characteristics of the parabolic-arc body; $M=1.49$.FIGURE 16.—Continued. (b) Original surface-temperature distribution $Re_l = 3.7 \times 10^6$.FIGURE 16.—Continued. (c) Modified surface-temperature distribution $Re_l = 2.6 \times 10^6$.

surface temperatures (figs. 11, 12, and 16 (c)) indicate that when the large initial surface-temperature gradient is confined to the nose region on a body the effect on the local values of the heat-transfer parameter is small downstream of the initial gradient. In the case of the parabolic-arc body which had the most confined initial gradient region ($x^* = 0$ to 0.16), the data (fig. 16 (c)) indicate that the effect at the most forward data point $x^* = 0.23$ was almost negligible.

The fact that the effect of a surface-temperature gradient is dependent on its lengthwise location on the body can also be shown by the theory of reference 5. In this theory the surface-temperature distribution is expressed in a polynomial form such as the following:

$$T_s^* = a_0 + a_1 x^* + a_2 x^{*2} \dots a_n x^{*n} \quad (4)$$

The heat-transfer parameter for cones is then given by the following equation:

$$\frac{Nu}{Re^{1/2}} = -\frac{1}{2} \sqrt{3} \left[\sqrt{\frac{(T_s + 460)}{(T_v + 460)}} \frac{(T_v + 460) + 216}{(T_s + 460) + 216} \right] \sum_{n=0}^{\infty} \frac{a_n x^{*n} Y_n'(0)}{T_s^* - T_v^*} \quad (5)$$

where a and n are coefficients and exponents, respectively, as shown in equation (4) and the values of $Y_n'(0)$, taken from reference 5, are given in the following table:

n	$Y_n'(0)$
0	-0.5915
1	-0.9775
2	-1.1949
3	-1.3680
4	-1.4886
5	-1.5975
10	-2.0121

Although only the values of $Y_n'(0)$ for positive integer values of n are tabulated, it has been found that forms of equation (4) employing fractional exponents are particularly

useful in reducing the required number of terms in polynomials to give certain surface-temperature distributions. Since polynomials with fractional exponents satisfy the basic differential equations of reference 5, the use of fractional-exponent terms in equation 5 is acceptable. The appropriate values of $Y_n'(0)$ can be obtained for any fractional value of n by interpolation between the values tabulated above.

To illustrate that the effect of a surface-temperature gradient depends upon its position along the body, assume a number of separate surface-temperature distributions all given by the equation

$$T_s^* = a + b(x^*)^n \quad (6)$$

where, for a given distribution, $n = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, or zero. In going from the larger to the smaller values of n , the region of the large initial temperature gradients for each curve will be confined to a smaller and smaller region near the origin. It is obvious that at $n=0$ the surface temperature will be uniform. Now, turning to the values of $Y_n'(0)$, it can be seen that the values decrease as n decreases. Therefore, the heat-transfer parameter curves (equation (5)) for each successively smaller value of n approach the constant value for a uniform surface temperature ($n=0$). This is the effect that was observed in the experiments. (See fig. 12.)

In the case of the cooled cone, the region of nonuniform surface temperature extended to the 40-percent length station, and the surface-temperature gradient reversed in sign between the 20- and 40-percent length stations. (See fig. 14.) It might be expected from the theory of reference 5 that the lag in the boundary-layer temperature profile could result in a local decrement in the heat-transfer parameter when the surface-temperature gradient reversed. From the previous discussion, where it was shown that the effects of a surface-temperature gradient decrease downstream of the gradient, it would be expected that the agreement between the experimental and theoretical values of heat-transfer parameter would improve toward the rear of the body. These effects are shown by the data of figure 14. It is interesting to note that the local hot region ($x^*=0.2$) on the parabolic-arc body caused surface-temperature distributions similar to those of the cooled cone. (See figs. 14, 16(a), 16(b), and 17.) The effect of the reversal in surface-temperature gradients again caused the local values of heat-transfer parameter to be less than the "uniform-temperature" value.

The curve representing the heat-transfer data from tests of the heated cone with the most uniform surface temperature (see fig. 12) has been replotted in figure 14. As can be seen, in the regions where the effects of the initial surface-temperature gradient are small, the results from tests of the heated and cooled cones are in good agreement. It can be concluded that for the case of laminar boundary layers on bodies with uniform pressures and uniform surface temperatures, the available theory is satisfactory for engineering purposes.

Heat transfer with nonuniform surface temperatures.—The data obtained from tests of the heated cone with the least uniform surface-temperature distribution are replotted in

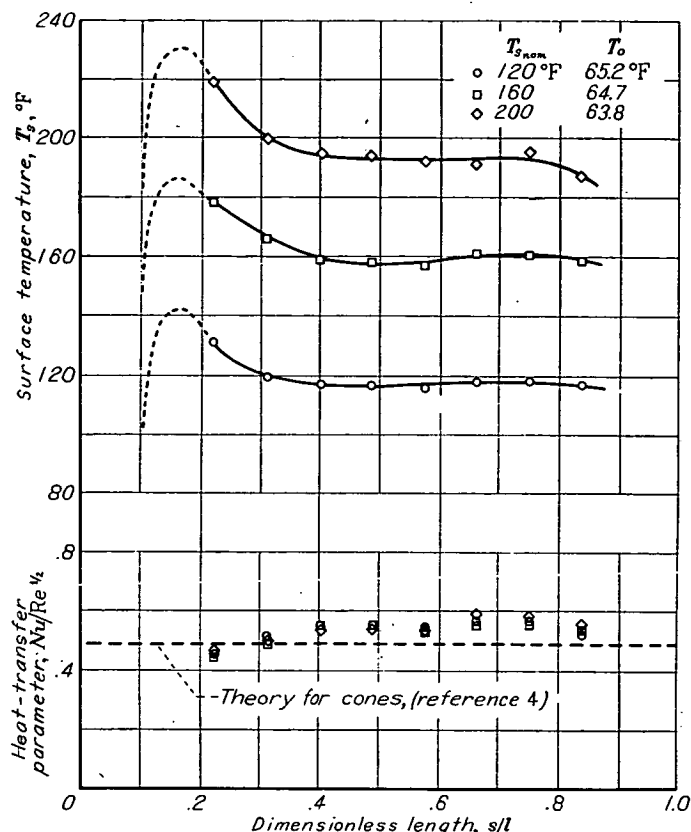


FIGURE 17.—Heat-transfer characteristics of the parabolic-arc body with the original surface-temperature distribution; $M=2.18$, $Re_1=2.6 \times 10^6$.

figure 13 for comparison with the theory of reference 5. (See equation (5) of the present report.) The first step in this comparison was the conversion of the experimental surface-temperature distribution from that of a cone to the equivalent distribution on a flat plate. Mangler, reference 8, has shown that the conversion is:

$$(x^*_{\text{cone}})^3 = x^*_{\text{plate}} \quad (7)$$

The converted distribution is shown by the "flagged" symbols in figure 13. The second step in the comparison was to determine the values of the coefficients $a_0, a_1, a_2, \dots, a_n$ and values of the exponents of equation (4) so that the equation would fit the converted surface-temperature distribution. It has been assumed from the shape of the curve through the experimental surface-temperature data shown in figure 13 that the dashed extension of the curve in the region $x^* = 0$ to 0.3 is a satisfactory approximation to the actual surface-temperature distribution. Although this assumed portion of the curve and the analytical curve may both be in error in the region $x^* = 0$ to 0.1 on the converted scale, it has already been shown that initial temperature gradients are not critical in determining heat transfer a short distance downstream from the gradients. For this reason, and others that will be discussed later, an expression was derived that fits the data quite accurately only from $x^* = 0.1$ to 0.7.

This equation in dimensionless form is

$$T_s^* - T_\infty^* = 0.425 (x^*)^{1/4} - 0.22 x^* \quad (8)$$

The local heat-transfer parameters obtained by using equations (5) and (8), and converted back to the x^* values for

cones, are shown in the lower half of figure 13. Agreement between theory and experiment would not be expected ahead of the 47-percent length station on the experimental curve because of the increasingly poor fit of the experimental and analytical temperature curves ahead of the 10-percent length station on the converted curve ($0.10^{1/3}=0.47$, equation (7)). However, the theory of reference 5 and the experiments are in quite good agreement downstream of the 47-percent length station.

The abrupt rise in heat-transfer parameter at the rear of the heated cone as shown in figures 11, 12, and 13 ($x^*=0.93$) is not due to surface-temperature gradient effects, but is due to the beginning of transition.

In the preceding discussion it was necessary to accept a poor fit between the data and the analytical surface-temperature distribution (equation (8)) in the region $x^*=0$ to 0.1. Considerable difficulty has been experienced in obtaining sufficiently accurate analytical expressions for the surface-temperature distributions. The obvious approach to obtaining polynomial expressions is the method of least squares or some other method of curve fitting using real, positive integers as exponents. In general, the analytical curves obtained with such methods oscillated about the experimental curve and near $x^*=1.0$ abruptly diverged due to the influence of the last term in the polynomial. It was found that the alternating positive and negative surface-temperature gradients due to the oscillation of the analytical curves resulted in even greater relative oscillations in the corresponding calculated heat-transfer-parameter curves. Also, because of the effect of the values of $Y_n'(0)$ on the coefficients of the polynomial, the abrupt divergence of the heat-transfer parameter resulting from the divergence of the temperature occurred at values of x^* below 1.0. It appears that any satisfactory polynomial with integer exponents must have a large number of terms and must give a fair curve to well beyond $x^*=1.0$. The coefficients of such polynomials are tedious to determine. An alternative procedure which was used in the present investigation, consisted of using a fairly simple analytical expression to obtain a good fit between the curves over the major part of the cone length while accepting differences in the nose region. Although the results for the particular case shown in figure 13 indicate agreement between theory and experiment, downstream of the 47-percent length station, it is evident from the previous discussion that the restrictions imposed by the form of equation (4) limit the application of the theory of reference 5 to only simple variations of surface temperature with body length.

Heat transfer on a body with nonuniform pressure.—The boundary layer on a flat plate grows only in thickness with distance along the surface, while the boundary layer on a pointed body of revolution must spread circumferentially as it grows in thickness. The effect of this circumferential spreading on the boundary-layer thickness, and hence on the heat transfer, varies with the rate of change of circumference with respect to length and can be evaluated for any fair contour by relations given in reference 8. For example, it can be shown that on a cone, for which the pressure and the rate of change of circumference are constant with length, the boundary-layer thickness is less than that on a flat plate by

the factor $1/\sqrt{3}$ and the rate of heat transfer is greater by the factor $\sqrt{3}$. For a parabolic-arc body, the factors defining the boundary-layer thickness and the rate of heat transfer relative to a two-dimensional surface with an equivalent pressure distribution vary from the conical values ($1/\sqrt{3}$ and $\sqrt{3}$, respectively) at the vertex to unity at the point where the rate of change of circumference becomes zero. Therefore, the rate of growth of the boundary layer and the rate of heat transfer on such a body are the same as on a cone at the apex, but the decreasing rate of change of circumference along the body tends to increase the rate of growth of the boundary layer relative to that on a cone and tends to decrease the rate of heat transfer. Although the effect of a pressure gradient on the characteristics of a boundary layer cannot be predicted exactly, it is known that a negative pressure gradient causes the boundary layer to thicken less rapidly with length than is the case if the pressure were constant, regardless of the body shape. Thus, on a body for which both the pressure and the rate of change of circumference decrease with length, the effect of the pressure gradient tends to counteract the effect of the variation of circumference on the boundary-layer characteristics.

The heat-transfer data obtained with the parabolic-arc body and the theoretical values of heat-transfer parameter for uniform-temperature cones (0.51 at a Mach number of 1.49 and 0.48 at a Mach number of 2.18) are shown in figures 16 and 17. It is apparent in figure 16 that the data, the theory for cones, and the method of reference 7, are in fairly good agreement. After the local hot region at the nose of the body had been eliminated, the data shown in figure 16 (c) were obtained. Good agreement between the experimental data, the theory for cones, and the method of reference 7, is again apparent. Because of the good agreement between the theory for cones and the experimental data shown in figure 16 (c), and the agreement shown in figures 16 (a), 16 (b), and 17, it can be expected that when the surface temperature is uniform, good agreement would be obtained at least throughout the test range of Mach number and Reynolds number. It should be noted that the small variation of the individual values of heat-transfer parameter from a fair curve is consistent throughout and is possibly due to small deviations in thickness of the cone shell. Comparison of the data of figure 16 (c) with the data of figures 16 (a), 16 (b), and 17 reveal that there was a slight reduction in the local values of the heat-transfer parameter over most of the body when the negative temperature gradient at the nose was reduced. However, the reduction at each station on the body where it occurred is within the experimental accuracy.

Since the experimental values of the heat-transfer parameter remained essentially constant along the parabolic-arc body, the effect of the negative pressure gradient is apparently opposite and, approximately equal to the effect of the decreasing rate of change of the circumference on the growth of the boundary layer. The present investigation, however, is inconclusive as to the exact individual effects of the pressure gradient and body shape on heat transfer. It has been shown that the local heat transfer for a parabolic-arc body can be calculated with satisfactory accuracy by applying the

theory for cones. Because of the agreement between the results of the experiments, the method of reference 7, and the theory for cones, it appears reasonable to assume that this agreement would be obtained with other pointed body shapes that give rise to uniform negative-pressure gradients. On a body that is more slender than the one tested, the effects of the changing circumference and the pressure gradient would be less and would still tend to counteract each other. Conversely, on a more blunt body both effects should be larger. However, additional experiments with other body shapes covering a wider range of Mach numbers are necessary before the theory for cones can be considered to be applicable to all fair bodies of revolution with negative pressure gradients.

CONCLUSIONS

The results obtained from tests of three bodies of revolution at supersonic velocities have provided information on three important phases of the heat-transfer problem—boundary-layer transition, temperature-recovery factor, and heat-transfer parameter (for laminar boundary layers). The following conclusions are based on these results and comparisons with available theories:

1. The effect of adding heat to a laminar boundary layer is to cause premature transition and the effect of extracting heat is to delay transition although to lesser degree than could be expected from the implications of the theory of Lees (NACA Rep. 876).

2. The recovery temperature on cones and in the negative pressure-gradient region on bodies of revolution can be computed with satisfactory accuracy by use of the theoretical recovery factors for laminar and turbulent boundary layers and the local Mach number just outside the boundary layer.

3. The values of heat-transfer parameter for laminar boundary layers obtained from the theories for uniform surface temperatures (Crocco, and Hantsche and Wendt) are in good agreement with experiment for both heated and cooled bodies.

4. Values of heat-transfer parameter for laminar boundary layers obtained from the theory of Chapman and Rubesin

for nonuniform surface temperatures are in good agreement with experiment; however, the agreement is markedly dependent upon obtaining an exact analytical expression for the experimental surface-temperature distribution.

5. The values of heat-transfer parameter on the forward half of a pointed parabolic-arc body with a laminar boundary layer when based on the local Mach number just outside the boundary layer are almost identical to those of cones.

AMES AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., December 17, 1948.

REFERENCES

1. Lees, Lester: The Stability of the Laminar Boundary Layer in a Compressible Fluid. NACA Rep. 876, 1947. (Formerly NACA TN 1360.)
2. Van Driest, E. R.: Turbulent Boundary Layer in Compressible Fluids. Jour. Aero. Sci., vol. 18, no. 3, March 1951, pp. 140-161.
3. Stalder, Jackson R., Rubesin, Morris W., and Tendeland, Thorval: A Determination of the Laminar-, Transitional-, and Turbulent-Boundary-Layer Temperature-Recovery Factors on a Flat Plate in Supersonic Flow. NACA TN 2077, 1950.
4. Johnson, H. A., and Rubesin, M. W.: Aerodynamic Heating and Convective Heat Transfer—Summary of Literature Survey. Trans. ASME, vol. 71, no. 5, July 1949, pp. 447-456.
5. Chapman, Dean R., and Rubesin, Morris W.: Temperature and Velocity Profiles in the Compressible Laminar Boundary Layer with Arbitrary Distribution of Surface Temperature. Jour. of Aero. Sci., vol. 16, no. 9, Sept. 1949, pp. 457-565.
6. Kalikhman, L. E.: Heat Transmission in the Boundary Layer. NACA TM 1229, 1949.
7. Scherrer, Richard: The Effects of Aerodynamic Heating and Heat Transfer on the Surface Temperature of a Body of Revolution in Steady Supersonic Flight. NACA Rep. 917, 1948. (Formerly TN 1300, 1947.)
8. Mangler, W.: Compressible Boundary Layers on Bodies of Revolution. M. A. P. Volkenrode, VG83 (Rep. and Trans. 47T) Mar. 15, 1946.
9. Michels, Walter C.: Advanced Electrical Measurements. D. Van Nostrand Company, Inc. (New York) 2d ed., ch. 1, 1943.

REPORT 1056

THEORETICAL ANTISYMMETRIC SPAN LOADING FOR WINGS OF ARBITRARY PLAN FORM AT SUBSONIC SPEEDS¹

By JOHN DEYOUNG

SUMMARY

A simplified lifting-surface theory that includes effects of compressibility and spanwise variation of section lift-curve slope is used to provide charts with which antisymmetric loading due to arbitrary antisymmetric angle of attack can be found for wings having symmetric plan forms with a constant spanwise sweep angle of the quarter-chord line. Consideration is given to the flexible wing in roll. Aerodynamic characteristics due to rolling, deflected ailerons, and sideslip of wings with dihedral are considered. Solutions are presented for straight-tapered wings for a range of swept plan forms.

INTRODUCTION

Reference 1 has been for many years the standard reference for estimating the stability and control characteristics of wings. The lifting-line theory on which this work was based gave generally satisfactory results for straight wings having the aspect ratios considered; however, the use of wing sweep combined with low aspect ratio has made an extension of this work desirable. Lifting-line theory cannot adequately account for the increased induction effects due to sweep and low aspect ratio; consequently, it has been found necessary to turn to the more complex lifting-surface theories.

Of the many possible procedures, a simplified lifting-surface theory proposed by Weissinger and further developed and extended in reference 2 has been found especially suited to the rapid computation of characteristics of wings of arbitrary plan form. Comparisons with experiment have generally verified the theoretical predictions. In reference 2, this method has been used to compute for plain, unflapped wings, the aerodynamic characteristics dependent on symmetric loading. The same simplified lifting-surface theory can be extended to predict the span loading resulting from antisymmetric² distribution of the wing angle of attack. From such loadings the damping moment due to rolling, the rolling moment due to deflected ailerons, and the rolling moment due to dihedral angle with the wing in sideslip can be determined. A recent publication (reference 3) makes use of the simplified lifting-surface theory to find span-loading characteristics of straight-tapered swept wings in roll and loading due to dihedral angle with the wing in sideslip. Experimental checks of the theory for the damping-in-roll coefficient and rolling moment due to sideslip were very favorable. The range of plan forms considered in reference

3 is somewhat limited and aileron effectiveness was not included. The loading due to aileron deflection normally involves excessive labor when computed by means of the simplified lifting-surface theory; however, development of the theory, presented in reference 4, that deals with flap and aileron effectiveness for low-aspect-ratio wings provides a means by which the simplified lifting-surface method can be used to obtain spanwise loading due to aileron deflection.

It is the purpose of the present analysis to provide simple methods of finding antisymmetric loading and the associated aerodynamic coefficients and derivatives for wings with symmetric plan forms limited only by a straight quarter-chord line over the semispan. Means will be presented for finding quickly the aerodynamic coefficients of span loading due to rolling, of span loading due to deflected ailerons, and of span loading due to sideslip of wings with dihedral. Flexible wings, when the flexure depends principally on span loading as in loading due to rolling, can be included in the analysis.

NOTATION

A	aspect ratio $\left(\frac{b^2}{S}\right)$
b	wing span measured perpendicular to the plane of symmetry, feet
c	wing chord, feet ³
c_a	aileron chord, feet ³
c_{av}	mean wing chord $\left(\frac{S}{b}\right)$, feet ³
c_l	local lift coefficient $\left(\frac{\text{local lift}}{qc}\right)$
C_{D_i}	induced drag coefficient $\left(\frac{\text{induced drag}}{qS}\right)$
C_l	rolling-moment coefficient $\left(\frac{\text{rolling moment}}{qSb}\right)$
C_{l_r}	rolling moment due to rolling $\left[\frac{\partial C_l}{\partial (pb/2V)}\right]$, per radian
C_{l_δ}	rolling moment due to aileron deflection $\left(\frac{\partial C_l}{\partial \delta}\right)$, per radian
$\frac{c_l c}{C_l c_{av}}$	spanwise loading coefficient for unit rolling moment $\left(\frac{2AG}{C_l}\right)$
d_r	scale factor

¹ Supersedes NACA TN 2140, "Theoretical Antisymmetric Span Loading for Wings of Arbitrary Plan Form at Subsonic Speeds" by John DeYoung, 1950.

² The word "antisymmetric" is understood to indicate that a distribution of loading or angle of attack is equal in absolute magnitude on each half of the wing but of opposite sign.

³ Measured parallel to the plane of symmetry.

e_{nk}	factors of loading interpolation function	η_a	dimensionless aileron span $\left(\frac{\text{aileron span}}{b/2}\right)$
G	spanwise loading coefficient or dimensionless circulation $\left(\frac{c_l c}{2b}\right)$ or $\left(\frac{\Gamma_c}{bV}\right)$	$\eta_{c.p.}$	spanwise center of pressure on one wing panel $\left(\frac{y_{c.p.}}{b/2}\right)$
$\bar{G}$	spanwise loading coefficient due to rolling $\left(\frac{G}{pb/2V}\right)$, per radian	θ	trigonometric spanwise coordinate ϕ , indicating the edge of the aileron span, radians
$\overline{\bar{G}}$	spanwise loading coefficient due to aileron deflection $\left(\frac{G}{\delta}\right)$, per radian	κ_v	ratio of section lift-curve slope at a span station v to $\frac{2\pi}{\beta}$, both at the same Mach number
H_v	wing geometry, compressibility, and section lift-curve-slope parameter $\left[d_v \left(\frac{1}{\kappa_v}\right) \left(\frac{b}{c_v/\beta}\right)\right]$	Λ	sweep angle of the wing quarter-chord line, positive for sweepback, degrees
h_n	integration factors for spanwise loading due to ailerons	Λ_β	compressibility sweep-angle parameter $\left[\tan^{-1} \left(\frac{\tan \Lambda}{\beta}\right)\right]$, degrees
M	Mach number	λ	taper ratio $\left(\frac{\text{tip chord}}{\text{root chord}}\right)$
m	arbitrary number of span stations defined by $\eta = \cos \frac{n\pi}{m+1}$	ϕ	trigonometric spanwise coordinate ($\cos^{-1} \eta$), radians
p	rate of rolling, radians per second	SUBSCRIPTS	
$pb/2V$	wing-tip helix angle, radians	n, v	integers pertaining to specific span stations given by $\eta = \cos \frac{n\pi}{8}$ or $\eta = \cos \frac{v\pi}{8}$
p_{vn}	coefficient depending on wing geometry and indicating the influence of antisymmetric loading at span station n on the downwash angle at span station v	k	pertaining to span station k
q	free-stream dynamic pressure, pounds per square foot	$c.p.$	center of pressure
S	wing area, square feet	a	aileron
t	ratio of aileron chord to wing chord $^3 \left(\frac{c_a}{c}\right)$	t	pertaining to fraction-of-wing-chord ailerons
V	free-stream velocity, feet per second	T	wing tip
w	induced velocity, normal to the lifting surface, positive for downwash, feet per second	R	wing root
y	lateral coordinate measured from the wing root perpendicular to the plane of symmetry, feet	av	average or mean
α_v	section angle of attack at span station v , radians 3	DEVELOPMENT OF METHOD	
$\Delta\alpha_v$	angle of antisymmetric twist of the elastic wing produced by the loading due to rolling, radians 3	The simplified lifting-surface method used herein replaces a lifting surface by a lifting vortex located at the wing one-quarter-chord line. The boundary condition for determining the vortex strength distribution specifies that, along the three-quarter-chord line of the wing, there shall be no flow through the lifting surface. In effect, this specifies that, at the three-quarter-chord line, the ratio of the velocity normal to the mean camber line (induced by the bound and trailing vortices) to the velocity of the free stream shall equal the sine of the angle of attack.	
$\frac{d\alpha}{d\delta}$	rate of change of wing-section angle of attack with control-surface angle for constant section lift coefficient 3		
β	compressibility parameter $(\sqrt{1-M^2})$	Span loadings are theoretically additive. Since the symmetric angle-of-attack distribution contributes only to symmetric loading, it follows that the antisymmetric loading is independent of symmetrically distributed wing twist or camber; hence, to find antisymmetric loading, it is only necessary to consider the loading resulting from the antisymmetric distribution of the angle of attack across the wing span. In the subject case, such a distribution is experienced by the wing as induced angle due to rolling, ⁴ the effective twist due to aileron deflection, or sideslip of the wing with dihedral.	
$\bar{\beta}$	angle of sideslip, radians		
Γ	dihedral angle measured perpendicular to the plane of symmetry, radians		
Γ_c	spanwise circulation, feet squared per second		
δ	angle of deflection of full wing-chord control surface, radians 3		
$\bar{\delta}$	angle of deflection of full-wing-chord control surface, measured perpendicular to the hinge line, radians		
η	dimensionless lateral coordinate $\left(\frac{y}{b/2}\right)$		

³ Measured parallel to the plane of symmetry.

⁴ In considering the case of the angle induced by rolling as equivalent to an antisymmetric distribution of twist, it must be noted that account should be taken of the fact that a rolling wing leaves a twisted vortex trail; whereas a twisted wing does not. The difference in induction effects on the wing of the straight and twisted vortex is considered insignificant here, as has been assumed in other analyses.

For an antisymmetric angle-of-attack distribution, the loading distribution will be equal in absolute magnitude on each semispan, but of opposite sign. The loading therefore needs only to be found over the semispan, and, since the loading is zero at the wing root, only span stations outboard need be considered. The mathematical development of the simplified lifting-surface method for the case of antisymmetric loading is given in appendix A. As shown in appendix A, $(m-1)/2$ linear equations in terms of loading distribution are obtained which satisfy the wing angle-of-attack conditions⁵ at the three-quarter-chord line at m stations n , where m is an arbitrary odd integer. These equations are represented by the summations

$$\alpha_v = \sum_{n=1}^{\frac{m-1}{2}} p_{vn} G_n, \quad v=1, 2, 3, \dots, \frac{m-1}{2} \quad (1)$$

where

α_v antisymmetric angle of attack at wing station v
 p_{vn} coefficients that for a given value of m depend on wing geometry, compressibility, and section lift-curve slope
 G_n loading coefficients at span stations n

The application in appendix A of the present report is with $m=7$. Since the loading at the midspan station is known to be zero, consideration is required of only three stations: $n=1, 2, 3$, equal to wing semispan positions of $\eta = \cos(n\pi/8) = 0.924; 0.707; \text{ and } 0.383$. Equation (1) thus becomes

$$\alpha_v = \sum_{n=1}^3 p_{vn} G_n, \quad v=1, 2, 3 \quad (2)$$

where the integer v pertains to span station $\eta = \cos(v\pi/8)$

To obtain the loading coefficients $G_n = (c_l c/2b)_n$, it remains only to evaluate the coefficients p_{vn} and the spanwise variation of the antisymmetric angle of attack α_v .

EVALUATION OF COEFFICIENTS p_{vn}

Since m is chosen, p_{vn} becomes a function only of wing geometry, compressibility, and section lift-curve slope. The effects of compressibility and section lift-curve slope are equivalent to a change in wing plan form⁶ and can be accounted for by a proper adjustment of the p_{vn} values. As

shown in appendix B, p_{vn} can be conveniently presented as a function of two parameters, namely, a compressible-sweep-angle parameter defined as $\Lambda_\beta = \tan^{-1}(\tan \Lambda/\beta)$ and a parameter H_v involving the ratio of wing span to wing chord and variable section lift-curve slope, defined by

$$H_v = d_v \left(\frac{1}{\kappa_v} \right) \left(\frac{b}{c_v/\beta} \right) \quad (3)$$

where

κ_v ratio of experimental section lift-curve slope at span station v to the theoretical value of $2\pi/\beta$, both at the same Mach number

c_v wing chord at span station v

The value d_v is a scale factor given by

$$\left. \begin{aligned} d_v &= 0.061 \text{ for } v=1 \\ &= 0.234 \text{ for } v=2 \\ &= 0.381 \text{ for } v=3 \end{aligned} \right\} \quad (4)$$

Equation (3) can be written in alternative form that gives H_v in terms of wing geometry parameters that are more significant; thus

$$H_v = d_v \left(\frac{\beta A}{\kappa_{av}} \right) \left[\frac{1}{(\kappa_v/\kappa_{av})(c_v/c_{av})} \right] \quad (5)$$

where

κ_{av} ratio of average section lift-curve slope to $2\pi/\beta$ both at the same Mach number
 κ_v/κ_{av} spanwise distribution of section lift-curve slope for a given Mach number
 c_v/c_{av} spanwise distribution of the wing chord
 $(\beta A/\kappa_{av})$ compressible aspect ratio and average section lift-curve-slope parameter

The term $\frac{1}{(\kappa_v/\kappa_{av})(c_v/c_{av})}$ of equation (5) gives an effective aerodynamic taper of a wing. The distribution of κ_v/κ_{av} may vary with Mach number, particularly at transonic speeds (e. g., due to spanwise variation of airfoil section). However, since the distribution contributes to taper effect, the loading distribution and not the total loading will be appreciably affected.

With H_v determined from equations (3) or (5) and (4), the values of p_{vn} , nine in all, are presented in figure 1 where p_{vn} is given as a function of H_v for various values of Λ_β .

⁵ The reader should note that the boundary condition is given by $w_s = V \sin \alpha$, from which $(w/V)_s$ is seen to equal $\sin \alpha$. The substitution of α_v for $\sin \alpha$, has the effect of increasing the value of loading on the wing above that necessary to satisfy the boundary condition. However, the boundary condition was fixed assuming that the shed vortices moved downstream in the extended chord plane. A more realistic picture is obtained if the vortices are assumed to move downstream in a horizontal plane from the wing trailing edge. It can be seen readily that, if this occurs, the normal component of velocity induced by the trails at the three-quarter-chord line is reduced and, if the boundary condition is to continue to be satisfied, the strength of the bound vortex must increase. It follows that substitution of α_v for $\sin \alpha$, then has the effect of accounting for the bending up of the trailing vortices. It is not known how exact the correction is, but the calculations and experimental verification show it to be of the correct order.

⁶ Compressibility and section lift-curve slope are discussed in the section "Discussion" and in the appendix B.

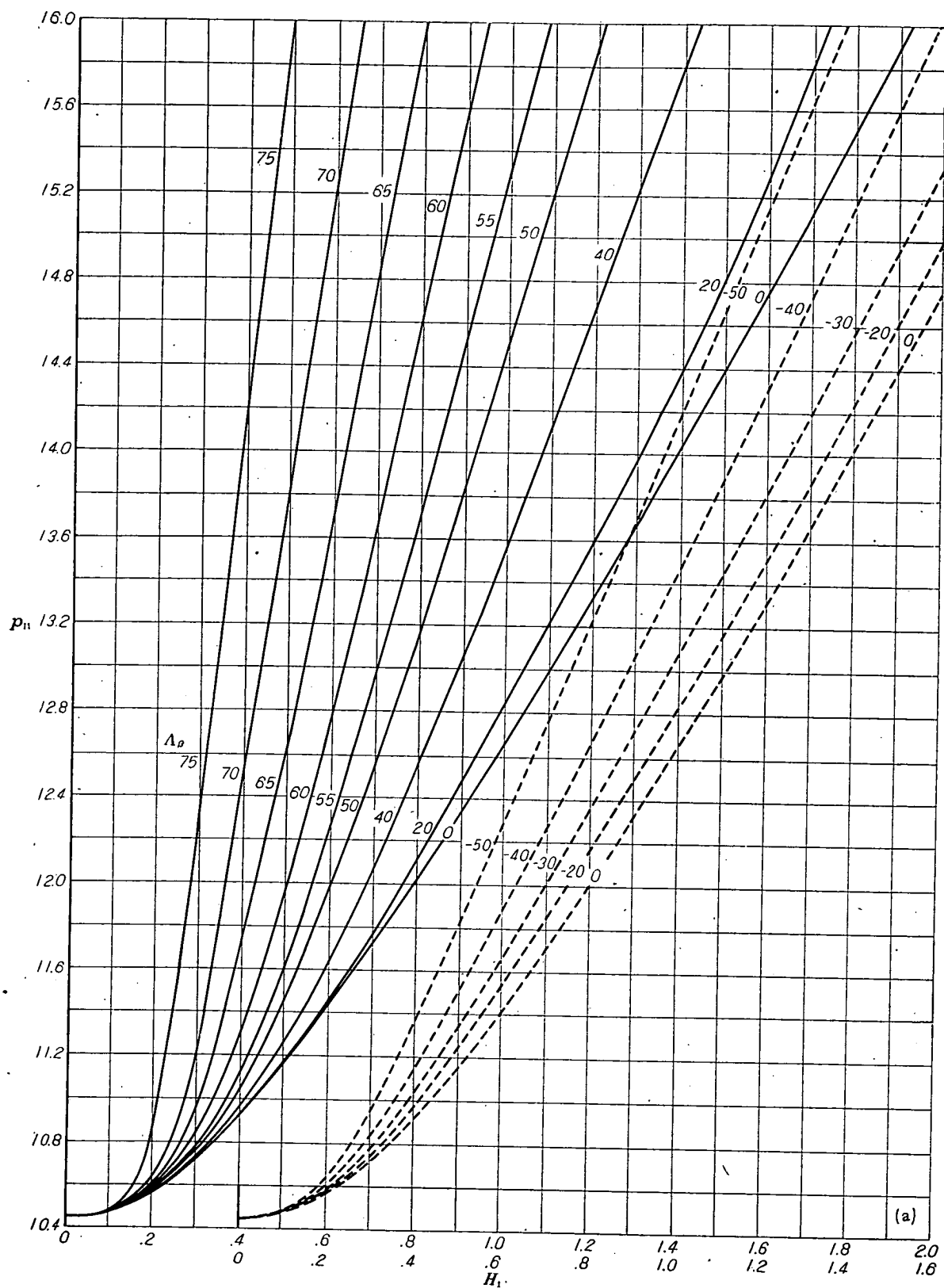
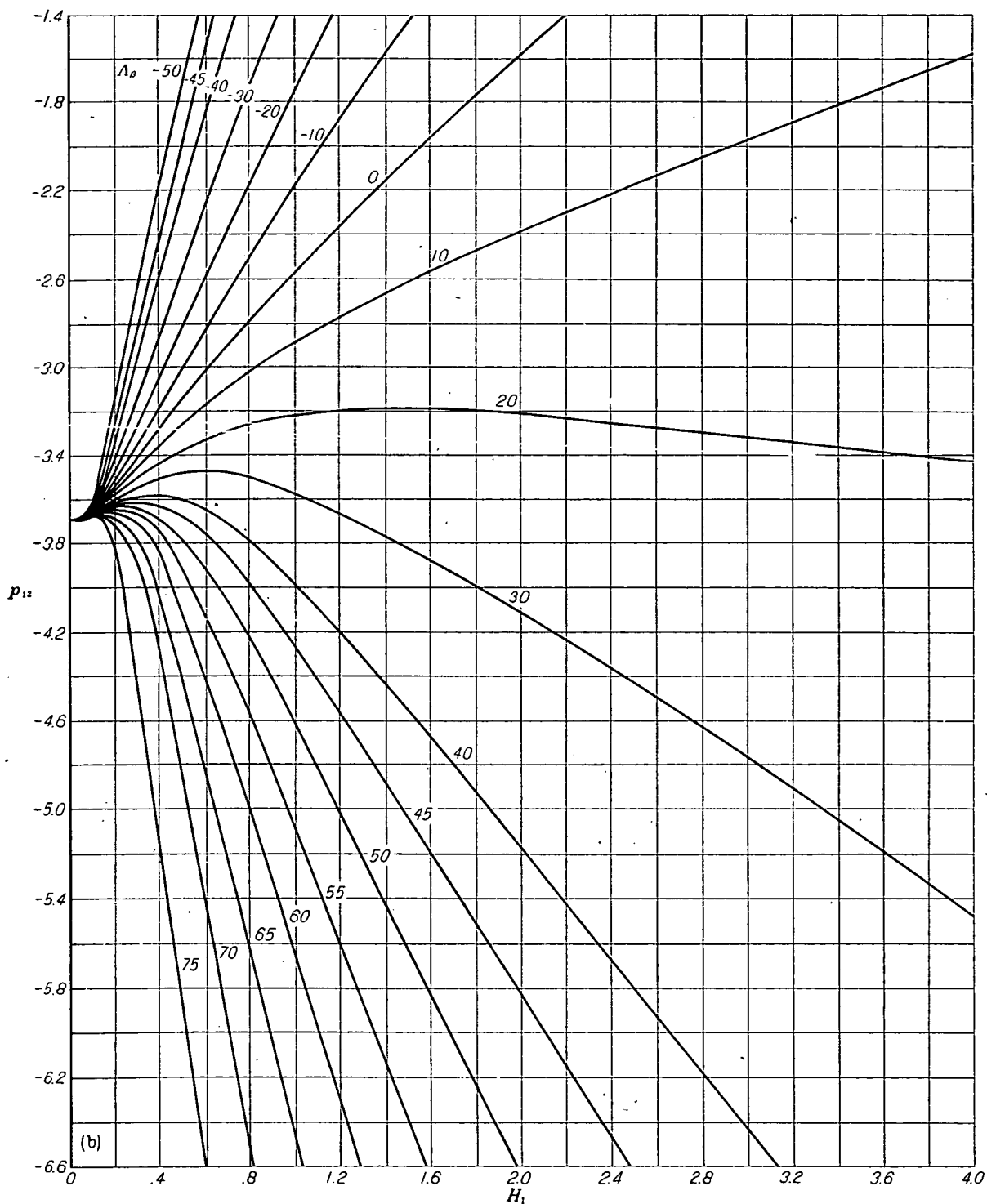
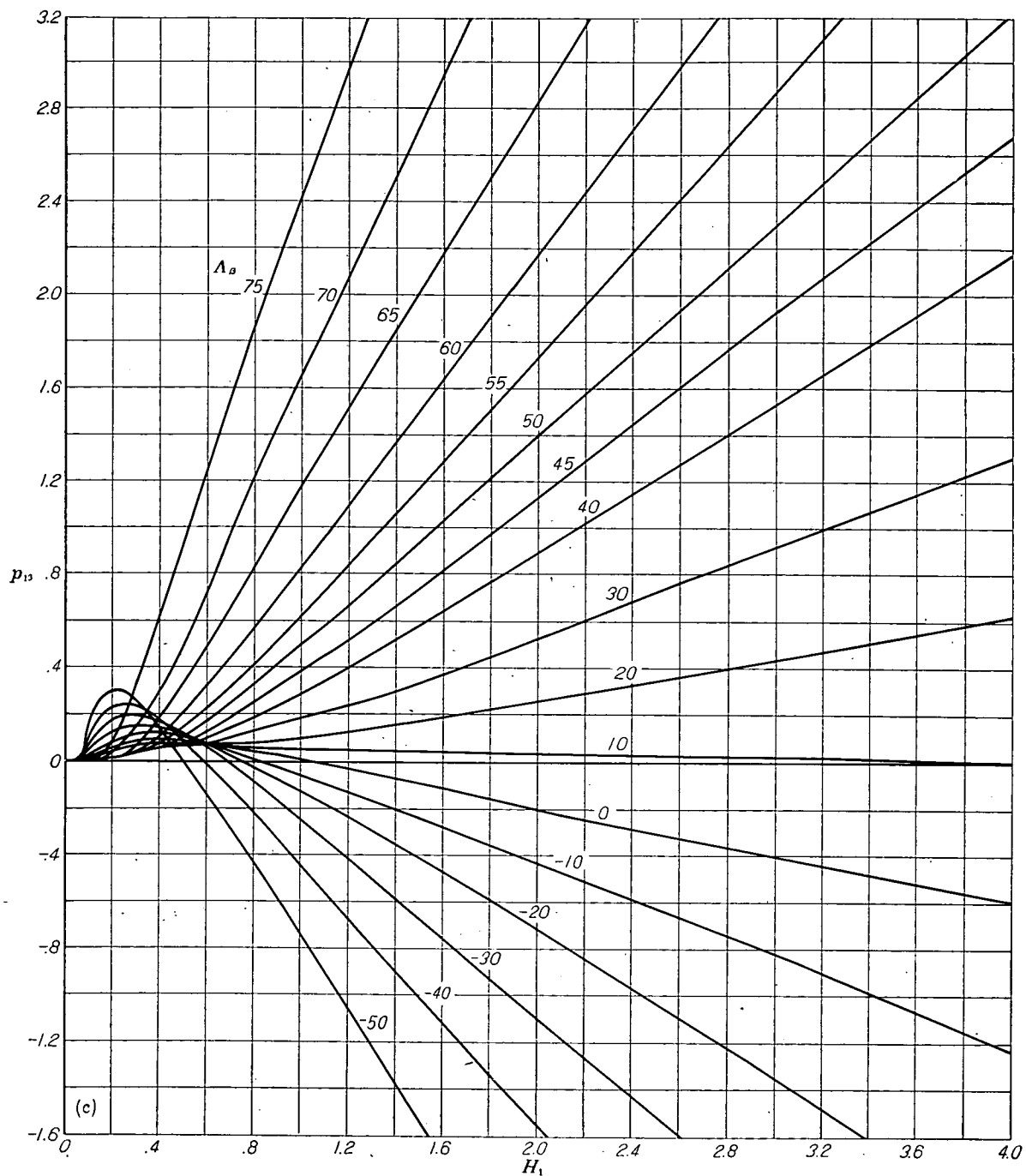
(a) $r=1, n=1, d_1=0.061$.

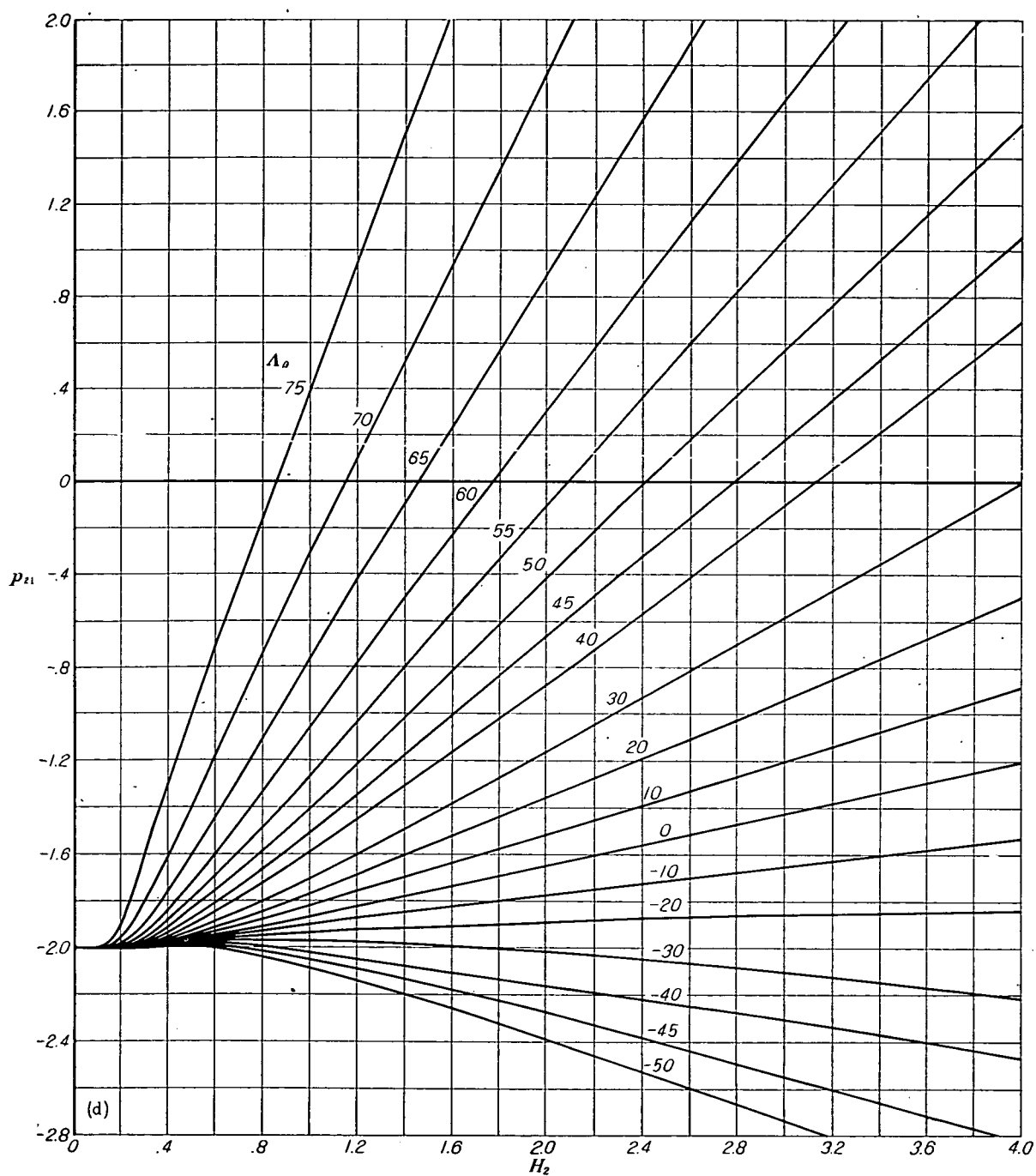
FIGURE 1.—Influence coefficients, p_n , for antisymmetric spanwise loading plotted as a function of the wing geometric parameter, H_1 , for values of the compressible sweep parameter, Λ_p degrees.



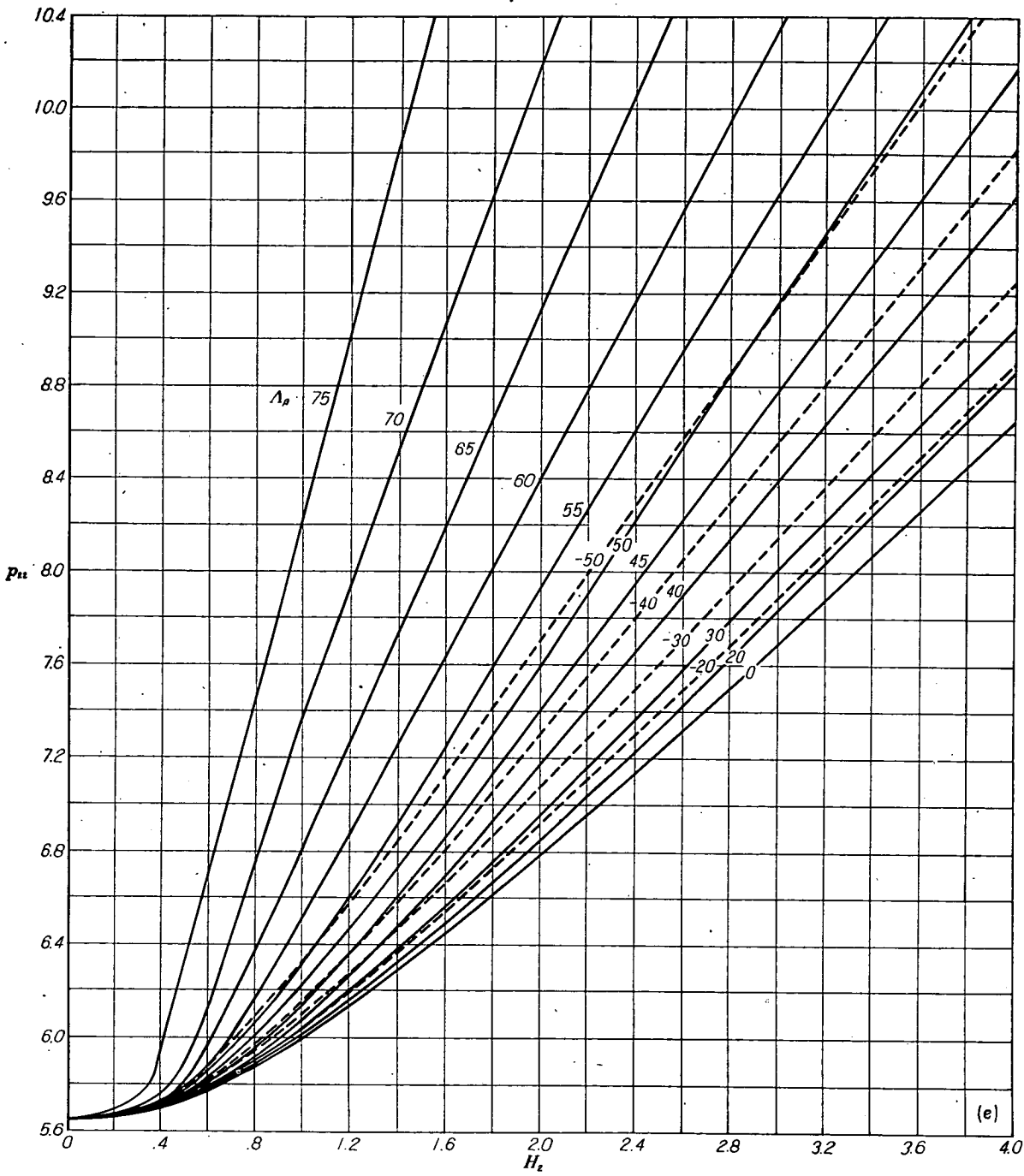
(b) $\nu=1$, $n=2$, $d_1=0.061$.
FIGURE 1.— Continued.



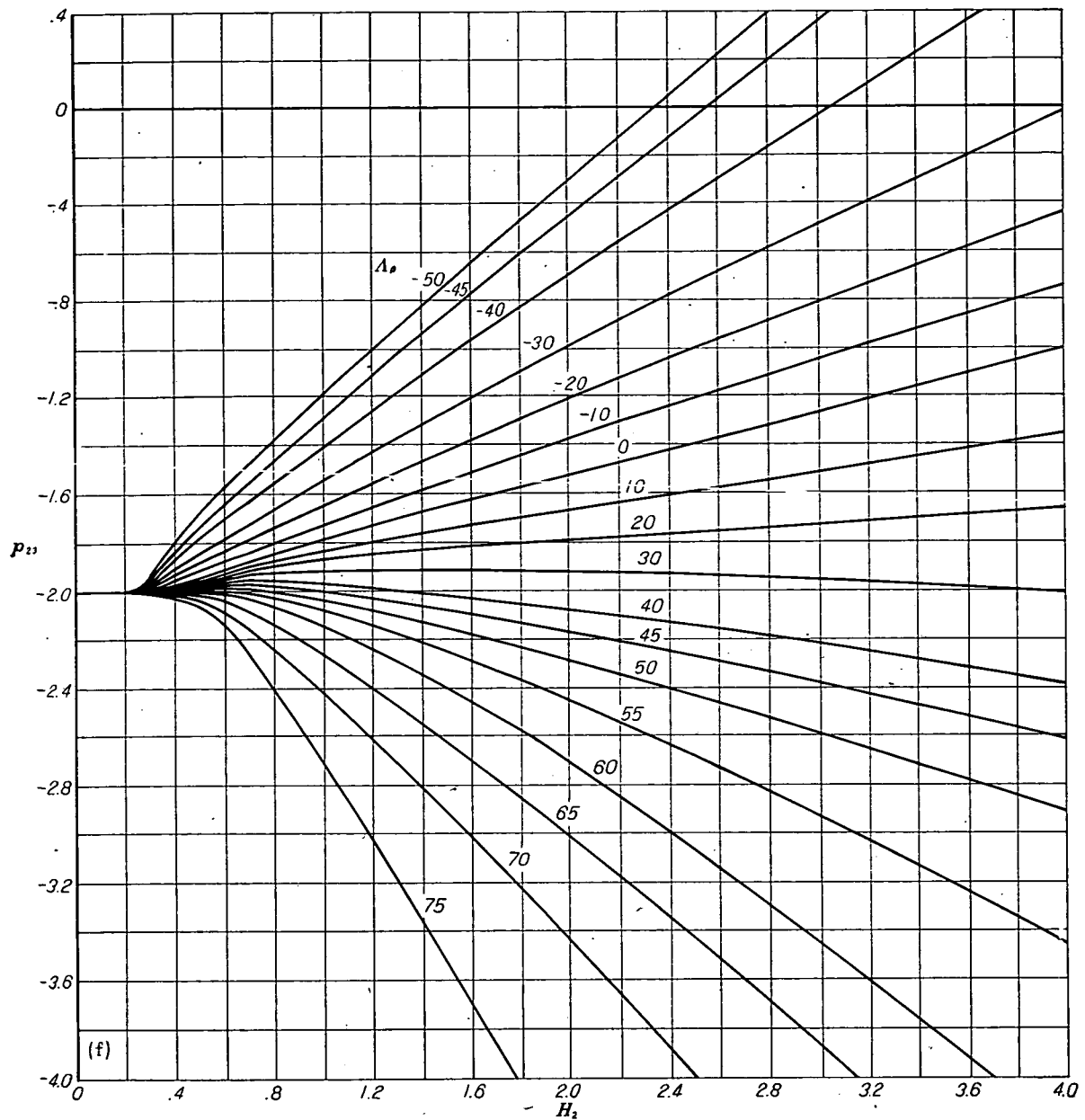
(c) $p=1$, $n=3$, $d_1=0.061$.
FIGURE 1.—Continued.



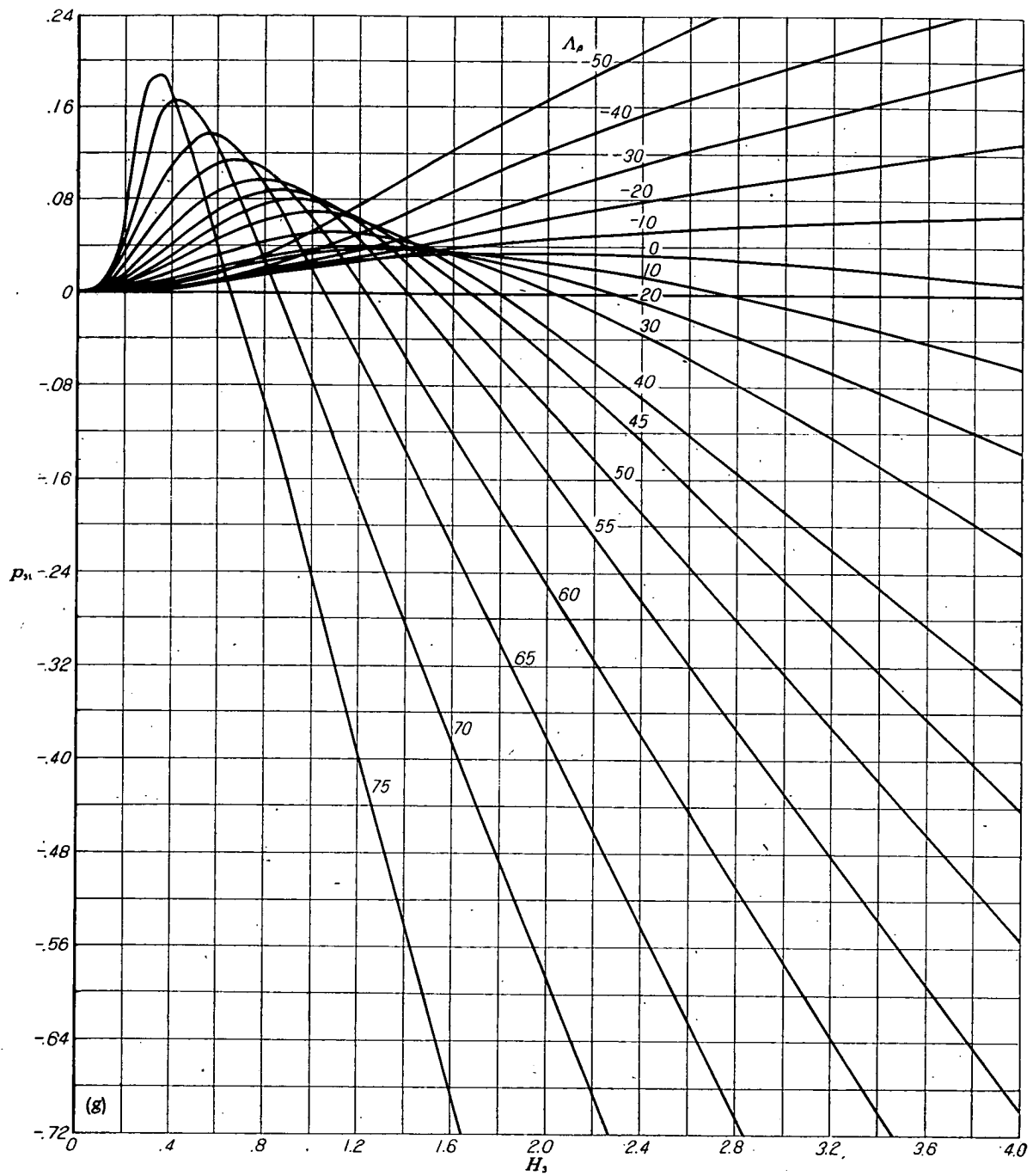
(d) $\nu=2$, $n=1$, $d_1=0.234$.
FIGURE 1.—Continued.



(e) $\nu=2, n=2, d_1=0.234$.
FIGURE 1.—Continued.



(f) $\nu=2, n=3, d_1=0.234$.
FIGURE 1.—Continued.



(g) $\nu=3, n=1, d_s=0.381$.
FIGURE 1.—Continued.

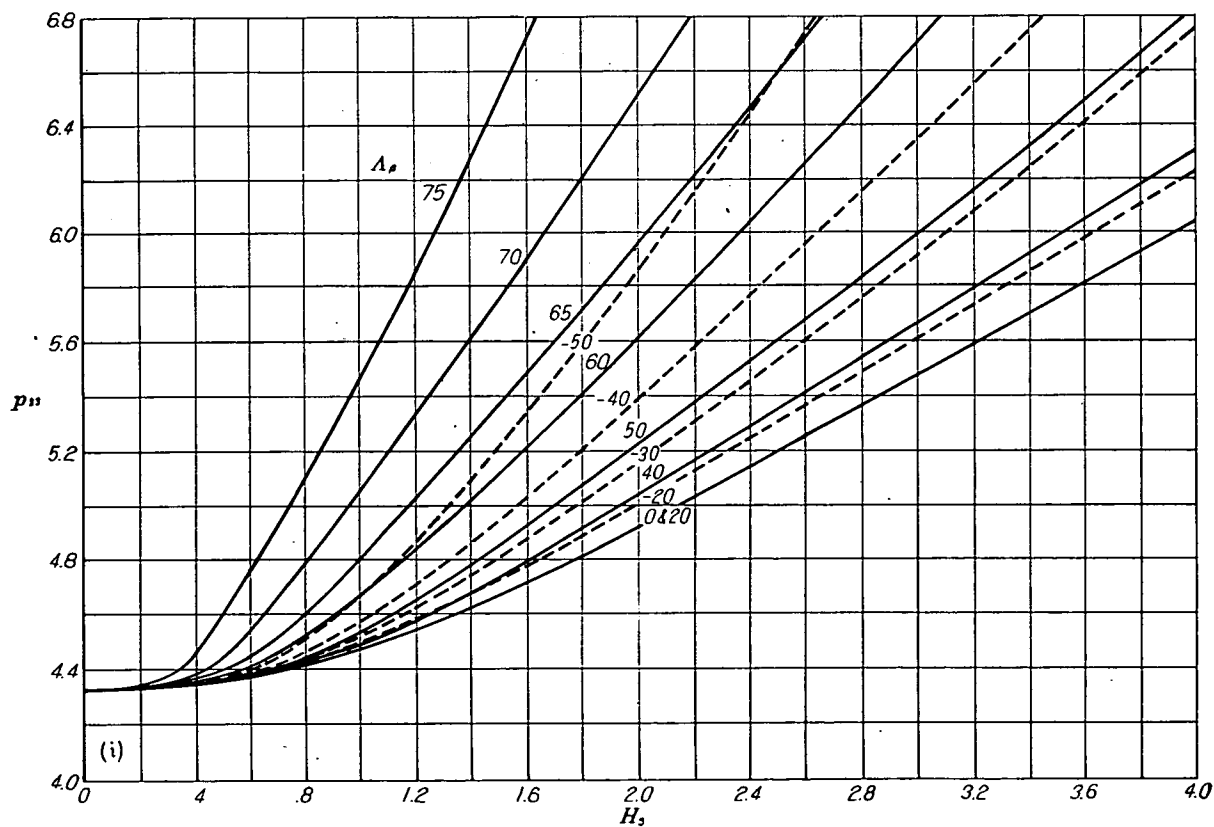
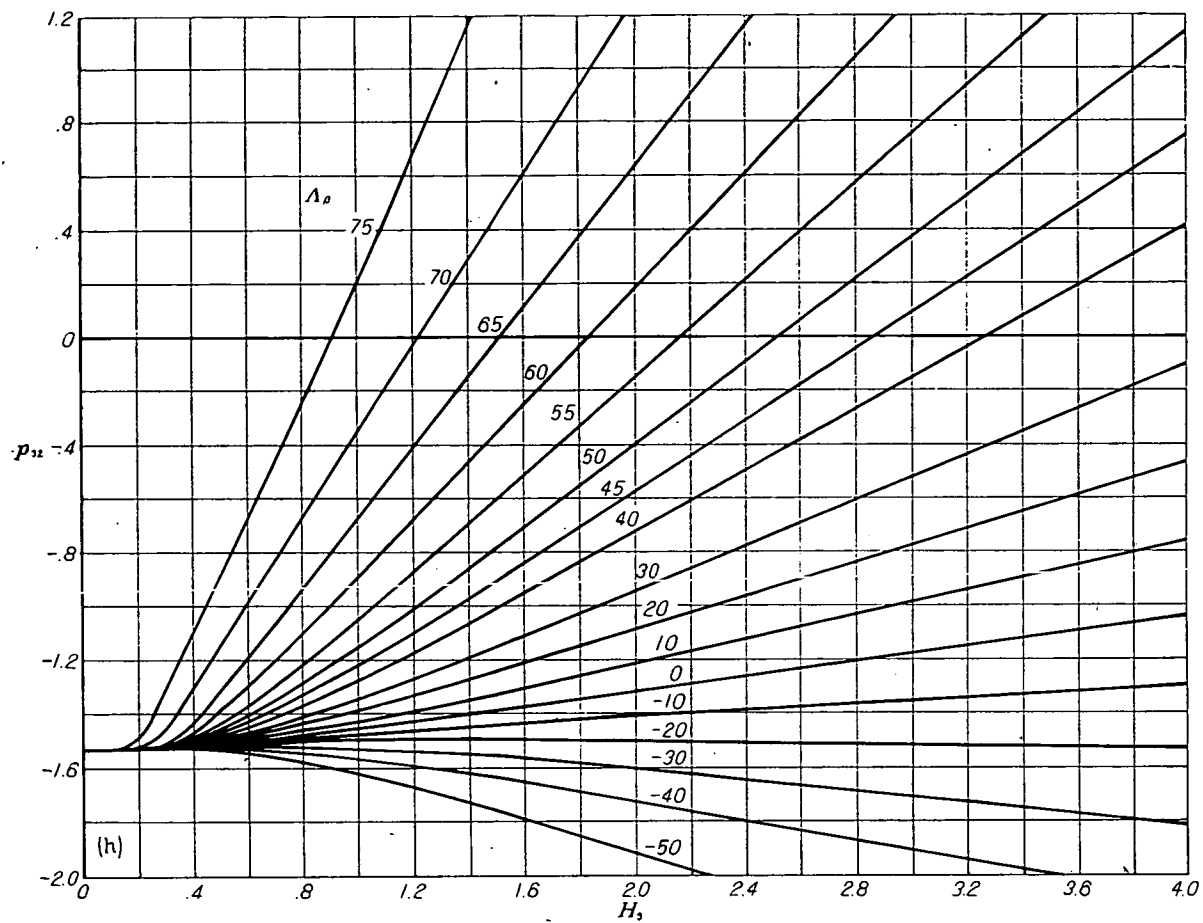
(h) $\nu=3, n=2, d_3=0.381$;(i) $\nu=3, n=3, d_3=0.381$.

FIGURE 1.—Concluded.

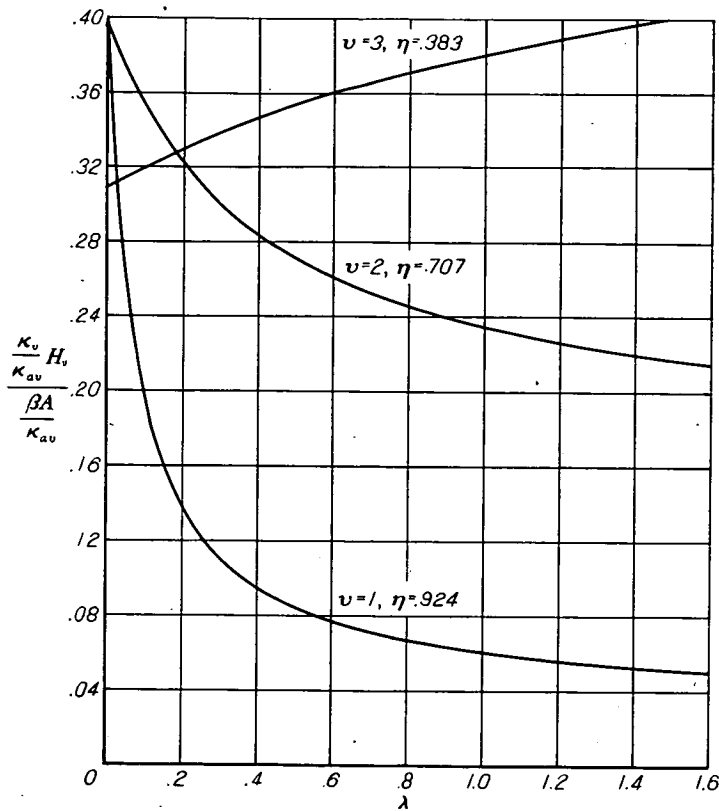


FIGURE 2.—Variation of the geometric parameter $\frac{\kappa_v}{\kappa_{av}} \frac{H_v}{\beta A}$ with taper ratio λ for straight-tapered wings.

For the case of straight-tapered wings with arbitrary section lift-curve-slope distribution for which the chord distribution is specified by taper ratio, evaluation of equation (5) is given in figure 2 where $\frac{(\kappa_v/\kappa_{av})H_v}{(\beta A/\kappa_{av})}$ for each of the three span stations is shown as a function of taper ratio.

EVALUATION OF ANTISYMMETRIC ANGLE-OF-ATTACK DISTRIBUTION α_v

The antisymmetric angle-of-attack distributions most commonly encountered are those resulting from rolling wings, aileron deflection, and sideslip of wings with dihedral. Evaluation of the angle-of-attack distributions for these various cases is outlined in the sections immediately following.

Rolling wings.—For the case of the rigid wing, the induced velocity normal to the wing surface is equal to the upwash velocity experienced by the rolling wing. Thus, at span station v

$$\alpha_v = \frac{w_v}{V} = -\left(\frac{pb}{2V}\right) \eta_v \quad (6)$$

where $pb/2V$ is the tip helix angle. It should be noted that the relation given by equation (6) assumes the wing structure to be rigid in that the distribution of α_v is completely defined by the linear distribution of helix angle. In the case of flexible wings, however, the expression for α_v must be modified to account for the streamwise angle-of-attack change which may occur due to bending or torsional deflections. In this case,

$$\alpha_v = -\left(\frac{pb}{2V}\right) \eta_v + \Delta\alpha_v \quad (7)$$

where $\Delta\alpha_v$ represents the modifying influence of flexibility. Normally, $\Delta\alpha_v$ is not considered for straight wings since only the effect of torsion (which is usually small) is involved. On swept wings, however, the effect of bending can cause $\Delta\alpha_v$ to be quite large so that the α_v distribution may be affected considerably. Due to the interaction existing between the aerodynamic and structural forces, $\Delta\alpha_v$ cannot be determined directly, but must be found through equations of equilibrium or by iteration. With the loading for the rigid wing provided, however, the iteration procedure becomes relatively easy to apply. The first approximation of α_v is found from the loading of the rigid wing and further refinements of α_v may be found utilizing the successive loadings for the flexible wing as determined.

Deflected ailerons.—Where the spanwise distribution of the angle α_v is to be considered equivalent to antisymmetric aileron deflection, it must suffer a discontinuity at the spanwise end of the control surface. The loading when such a discontinuity is present can be duplicated by a proper distribution of antisymmetric twist. In appendix C, the antisymmetric twist distribution required by the present theory to give accurate span loading distribution due to ailerons is found with the aid of zero-aspect-ratio wing theory given by reference 4. To minimize the computation involved, it is convenient to consider both the case of outboard and inboard ailerons.

1. **Outboard ailerons.**—With $m=7$, three different aileron spans can be conveniently defined for the outboard ailerons. For the aileron spans η_a , measured from the wing tip inboard, the antisymmetric twist distribution required per unit deflection of full-wing-chord ailerons, α_v/δ , is given by

Case.	I	II	III
η_a	0.169	0.444	0.805
$\frac{\alpha_1}{\delta}$	1.003	0.971	0.998
$\frac{\alpha_2}{\delta}$	.017	.996	.991
$\frac{\alpha_3}{\delta}$	.006	.014	.978

(8)

Inboard ailerons.—With $m=7$, three different aileron spans can be conveniently defined for the inboard ailerons. For the aileron spans η_a measured from the wing midspan outboard, the antisymmetric twist distribution required per unit deflection of full-wing-chord ailerons, α_v/δ , is given by

Case	IV	V	VI
η_a	0.556	0.831	1.000
$\frac{\alpha_1}{\delta}$	0.044	0.013	1.016
$\frac{\alpha_2}{\delta}$	-.017	.961	.979
$\frac{\alpha_3}{\delta}$	1.087	1.095	1.101

(9)

Sideslip of wings with dihedral.—For calculating the rolling moment caused by dihedral angle for the sideslipping wing, the effect of the skewness of the vortex field in altering the effects of the dihedral angle will be assumed to be small (as assumed in reference 3). The problem then simplifies to

finding the rolling moment due to antisymmetric angle of attack with the unskewed vortex field. The solution to this problem is the same as for the ailerons which has already been solved.

The antisymmetric distribution of angle of attack for the sideslipping wing with dihedral is given by

$$\alpha_v = \bar{\beta} \Gamma \quad (10)$$

where

- α_v effective angle-of-attack distribution
- $\bar{\beta}$ angle of sideslip measured positive in the counterclockwise direction from the plane of symmetry
- Γ dihedral angle

The wing parameter Γ is not affected by compressibility. Equation (10) is approximate for small values of $\bar{\beta}$ and Γ .

For unit $\bar{\beta} \Gamma$ over the span of the ailerons considered,

$$\bar{\beta} \Gamma = \delta \quad (11)$$

can be substituted for δ in equations (8) and (9).

APPLICATION OF METHOD

For the cases of antisymmetric angle-of-attack distributions resulting from rolling, aileron deflection, or sideslip with dihedral, it is possible to present a set of simultaneous equations which are required for the solution of the load distribution for an arbitrary plan form. With the loading known, integration formulas can be given to determine aerodynamic coefficients.

The loading-distribution coefficient G_n determined from the solutions of the simultaneous equations, are functions of p_{rn} which has been shown in a preceding section to be a function of wing geometry, compressibility, and section lift-curve slope. The aerodynamic coefficients are integrations of the load distribution and, therefore, will also be a function of wing geometry, compressibility, and section lift-curve-slope parameters. Application of the method to the general solution for arbitrary chord distribution is outlined and solutions are presented for the case of straight taper.

GENERAL SOLUTION

Aerodynamic characteristics due to rolling.—The solutions for the aerodynamic effects due to the rolling wing will be found and loading, rolling moment, spanwise center of pressure, and induced drag will be obtained.

1. *Simultaneous loading equations.*—The p_{rn} values are obtained from figure 1 and table I⁷ with values of H , given by equations (3) or (5).

The simultaneous equations (2), for the rigid and flexible wing, respectively, become:

$$\left. \begin{aligned} -0.924 &= p_{11} \bar{G}_1 + p_{12} \bar{G}_2 + p_{13} \bar{G}_3 \\ -0.707 &= p_{21} \bar{G}_1 + p_{22} \bar{G}_2 + p_{23} \bar{G}_3 \\ -0.383 &= p_{31} \bar{G}_1 + p_{32} \bar{G}_2 + p_{33} \bar{G}_3 \end{aligned} \right\} \quad (12)$$

where

$$\bar{G}_n = \frac{G_n}{pb/2V}$$

and

$$\left. \begin{aligned} -0.924 + \frac{\Delta \alpha_1}{pb/2V} &= p_{11} \bar{G}_1 + p_{12} \bar{G}_2 + p_{13} \bar{G}_3 \\ -0.707 + \frac{\Delta \alpha_2}{pb/2V} &= p_{21} \bar{G}_1 + p_{22} \bar{G}_2 + p_{23} \bar{G}_3 \\ -0.383 + \frac{\Delta \alpha_3}{pb/2V} &= p_{31} \bar{G}_1 + p_{32} \bar{G}_2 + p_{33} \bar{G}_3 \end{aligned} \right\} \quad (13)$$

where $\bar{G}_n = \frac{G_n}{pb/2V}$ and $\Delta \alpha_v$ is the incremental angle of attack due to aeroelastic effects.

2. *Loading distribution.*—The loading-distribution coefficient is given by $G = c_l c / 2b$. Other forms of the loading coefficient are given by the identities

$$G = \frac{1}{2A} \frac{c_l c}{c_{av}} = \frac{C_l}{2A} \frac{c_l c}{C_l c_{av}} \quad (14)$$

The loading is known to be zero at $\eta = 0$ and 1 and is determined at three intermediate span stations. Values of loading at other span stations can be obtained from a loading function derived in appendix B or, with equations (B23) or (B24) of appendix B, the loading can be found at span positions $\eta = 0.981, 0.831, 0.556$, and 0.195.

3. *Rolling moment.*—The damping-in-roll derivative for the solutions of equations (12) or (13) is derived in appendix B and given by

$$\frac{\beta C_{lp}}{\kappa_{av}} = \frac{\pi}{16} \left(\frac{\beta A}{\kappa_{av}} \right) [\bar{G}_2 + 0.707 (\bar{G}_1 + \bar{G}_3)] \quad (15)$$

4. *Spanwise center of pressure.*—The equation giving center of pressure on the wing semispan is shown in appendix B to be

$$\eta_{c.p.} = \frac{\beta C_l / \kappa_{av}}{\frac{\beta A}{\kappa_{av}} (0.163 \bar{G}_1 + 0.248 \bar{G}_2 + 0.430 \bar{G}_3)} = \frac{1}{0.082 \left(\frac{c_l c}{C_l c_{av}} \right)_1 + 0.124 \left(\frac{c_l c}{C_l c_{av}} \right)_2 + 0.215 \left(\frac{c_l c}{C_l c_{av}} \right)_3} \quad (16)$$

5. *Induced drag.*—The induced drag is derived in appendix B and given by

$$\frac{\beta C_{Di}}{\kappa_{av}} = \frac{\pi}{2} \left(\frac{\beta A}{\kappa_{av}} \right) \left[G_1^2 + G_2^2 + G_3^2 - \frac{\sqrt{2}}{2} G_2 (G_1 + G_3) \right] \quad (17)$$

Aerodynamic characteristics due to aileron deflection.—The solutions for the aerodynamic effects due to ailerons will be found for three different spans of outboard and inboard ailerons. Cross plots of these data provide curves for arbitrary aileron spans.

1. *Simultaneous loading equations.*—The p_{rn} values are obtained from figure 1 and table I with values of H , given by equations (3) or (5).

(a) *Deflected outboard ailerons.*—The aileron spans measured from the wing tip inboard are given by η_a . The simultaneous solution for antisymmetric spanwise loading due to

⁷ Values of p_{rn} beyond the scope of figure 1 are included in table I. For values of H larger than those included in figure 1 and table I, the p_{rn} curves can be obtained from equation (BS) which gives the linear asymptotes of the p_{rn} function.

deflection of any of the three following aileron spans can be obtained from the appropriate set of the following equations:

Case	I	II	III
η_a	0.169	0.444	0.805
$\frac{\alpha_1}{\delta}$	1.003	0.971	0.998
$\frac{\alpha_2}{\delta}$	.017	.996	.991
$\frac{\alpha_3}{\delta}$	.006	.014	.978

$$\begin{aligned}
 &= p_{11}\bar{G}_1 + p_{12}\bar{G}_2 + p_{13}\bar{G}_3 \\
 &= p_{21}\bar{G}_1 + p_{22}\bar{G}_2 + p_{23}\bar{G}_3 \quad (18) \\
 &= p_{31}\bar{G}_1 + p_{32}\bar{G}_2 + p_{33}\bar{G}_3
 \end{aligned}$$

where $\bar{G}_n = G_n/\delta$.

(b) *Deflected inboard ailerons.*—The aileron spans measured from the wing midspan outboard are given by η_a . The simultaneous solution for antisymmetric spanwise loading due to deflection of any of the three following aileron spans can be obtained from the appropriate set of the following equations:

Case	IV	V	VI
η_a	0.556	0.831	1.000
$\frac{\alpha_1}{\delta}$	0.044	0.013	1.016
$\frac{\alpha_2}{\delta}$	-.017	.961	.979
$\frac{\alpha_3}{\delta}$	1.087	1.095	1.101

$$\begin{aligned}
 &= p_{11}\bar{G}_1 + p_{12}\bar{G}_2 + p_{13}\bar{G}_3 \\
 &= p_{21}\bar{G}_1 + p_{22}\bar{G}_2 + p_{23}\bar{G}_3 \quad (19) \\
 &= p_{31}\bar{G}_1 + p_{32}\bar{G}_2 + p_{33}\bar{G}_3
 \end{aligned}$$

where $\bar{G}_n = G_n/\delta$.

2. *Loading distribution.*—The spanwise loading distributions due to various aileron configurations include:

(a) *Full-wing-chord ailerons.*—The loading is known to be zero at $\eta=0$ and 1, and is determined at three intermediate span stations. With equation (C13) and tables C6, B1, and C7, the loading can be found at span stations $\eta=0.981$, 0.831, 0.556, and 0.195 for each of the aileron spans considered. With these given points and the knowledge that the slope of the loading distribution curve is theoretically infinite at the point of angle-of-attack discontinuity (aileron spanwise end), the loading distribution can be faired.

(b) *Constant fraction of wing-chord ailerons.*—The spanwise loading of constant fraction of wing-chord ailerons is equal to the product of the loading due to full-wing-chord ailerons and the effective change of angle of attack with aileron angle,⁵ $d\alpha/d\delta$. The factor $d\alpha/d\delta$ is a function of the ratio of aileron chord to wing chord $t=c_a/c$. The change of section angle of attack with aileron angle $d\alpha/d\delta$ is presented in figure 3, which is reproduced from figure 18 of reference 5.

Although figure 3 taken from reference 5 limits the Mach number range to Mach numbers less than 0.2, this limitation is believed to be unwarranted since theory indicates that $d\alpha/d\delta$ is unaffected by compressibility for the two-dimensional wing. However, as indicated in reference 4, $d\alpha/d\delta$ is strongly affected by low aspect ratio and will change appreciably if the parameter βA becomes much less than two; hence, the values of $d\alpha/d\delta$ from figure 3 appear to be valid for $\beta A > 2$.

⁵ In using $d\alpha/d\delta$ here, it should be noted that the assumption is made that the effective airfoil section is taken as being parallel to the plane of symmetry and that the section approaches a two-dimensional section. The validity of this assumption can be questioned; however, limited checks with experiment show it to be at least approximately correct.

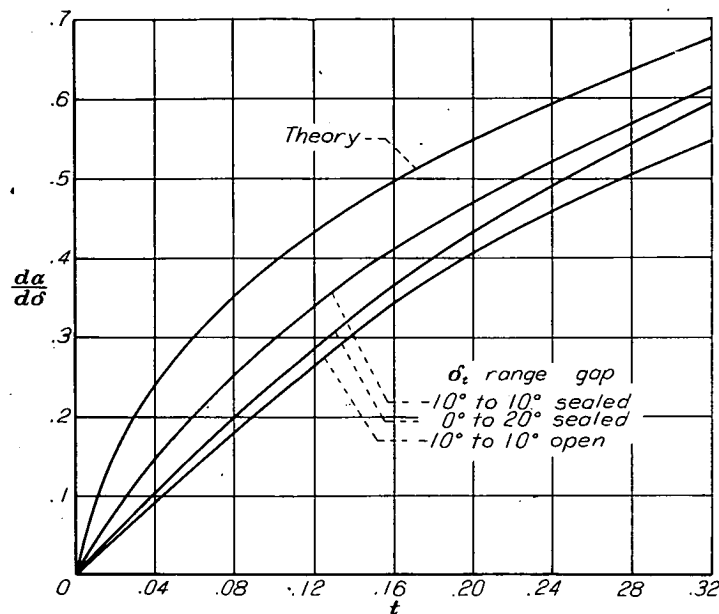


FIGURE 3.—Variation of lift-effectiveness parameter with aileron chord ratio, $t = \frac{c_a}{c}$. Average trailing-edge angle about 10° ; $M \leq 0.2$. Curves from reference 5.

(c) *Arbitrary spanwise distribution of aileron chord.*—The aileron can be divided into several spans with constant $d\alpha/d\delta$, then the total loading is the sum of the products of the full-wing-chord loading of each span and its respective $d\alpha/d\delta$.

3. *Rolling moment.*—The rolling moment can be found for the following aileron configurations:

(a) *Full-wing-chord ailerons.*—The spanwise loading due to aileron deflection cannot be integrated with sufficient accuracy with equation (15). In appendix C, a similar integration formula is developed that applies to each given aileron span. Equation (C10) and table C5 give

$$\frac{\beta C_{l_{\delta}}}{\kappa_{av}} = \left(\frac{\beta A}{\kappa_{av}} \right) (h_1 \bar{G}_1 + h_2 \bar{G}_2 + h_3 \bar{G}_3) \quad (20)$$

where for each of the cases of equations (18) and (19) the h_n values are given by

Case	I	II	III	IV	V	VI
h_1	0.140	0.139	0.138	0.146	0.141	0.140
h_2	.199	.196	.196	.200	.197	.198
h_3	.145	.139	.138	.140	.139	.140

(b) *Constant fraction of wing-chord ailerons.*—For constant fraction of wing-chord ailerons with aileron angle measured parallel to the plane of symmetry, the aileron effectiveness is given by

$$\frac{\beta C_{l_{\delta}}}{\kappa_{av}} = \frac{d\alpha}{d\delta} \left(\frac{\beta C_{l_{\delta}}}{\kappa_{av}} \right) \quad (21)$$

(c) *Arbitrary spanwise distribution of aileron chord.*—The deflection of ailerons for which t varies spanwise on the wing can be considered as an equivalent wing-twist distribution. The effective antisymmetric twist of the wing is given by

$$\alpha_r = \frac{d\alpha}{d\delta} \delta_i \quad (22)$$

wise, the rolling moment can be obtained by integration as in equation (23). The integral becomes

$$\frac{\beta C_{l_{\bar{\beta}}}}{\kappa_{av}} = \int_0^1 \Gamma(\eta) \frac{d(\beta C_{l_{\bar{\beta}}}/\kappa_{av} \Gamma)}{d\eta} d\eta \quad (29)$$

where $\frac{d(\beta C_{l_{\bar{\beta}}}/\kappa_{av} \Gamma)}{d\eta}$ is the slope of the curve described in part (b) above.

SOLUTION FOR STRAIGHT-TAPERED WINGS

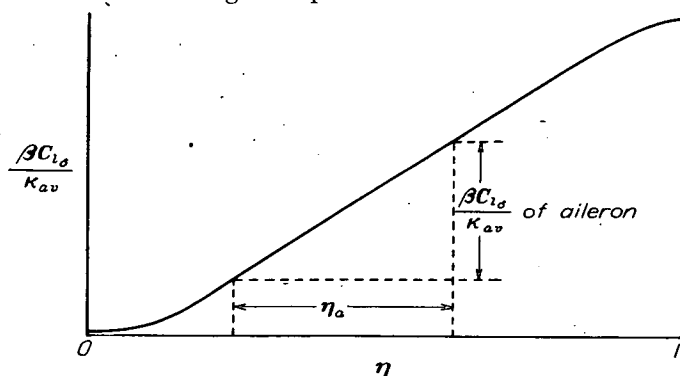
Charts of aerodynamic characteristics for straight-tapered wings can be presented in terms of geometric, compressibility, and average section lift-curve-slope parameters. These charts provide a ready means of obtaining data directly.

Aerodynamic characteristics due to rolling.—The application of equation (12) for a constant value of section lift-curve slope⁹ provides the spanwise loadings at span stations 0.383, 0.707, and 0.924 which are presented in figure 4 for a wide range of plan forms. The interpolation formula of equation (B24) will give values of loading due to rolling at span stations other than those presented. With equation (15), the damping-in-roll coefficients $\beta C_{l_{\bar{\beta}}}/\kappa_{av}$ can be obtained and are presented in figure 5 for a wide range of plan forms.

Aerodynamic characteristics due to aileron deflection.—

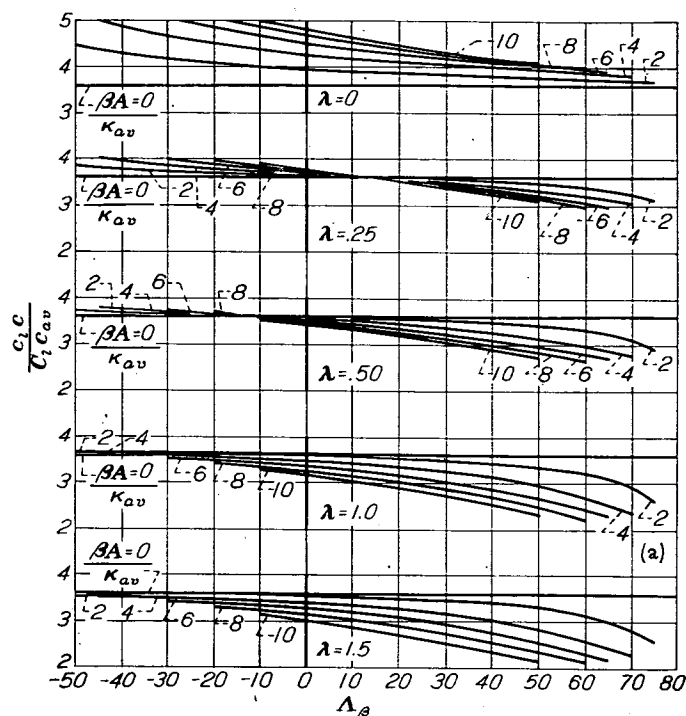
The application of equation (19), case III of equation (18), and equation (20) provide aileron effectiveness in the coefficient form $\beta C_{l_{\bar{\beta}}}/\kappa_{av}$ for several aileron spans. In figure 6, $\beta C_{l_{\bar{\beta}}}/\kappa_{av}$ is plotted against extent of unit antisymmetric angle of attack from the wing semispan root outboard for a range of wing parameters.

As presented, figure 6 gives directly the effectiveness of full-wing-chord inboard ailerons for aileron spans measured from the plane of symmetry outboard. The effectiveness of full-wing-chord outboard ailerons for aileron spans measured from the wing tip inboard is given by figure 6 directly by the relations of equation (24). For full-wing-chord ailerons located arbitrarily on the wing semispan, the aileron effectiveness can be obtained directly from figure 6 as indicated in the following example sketch.



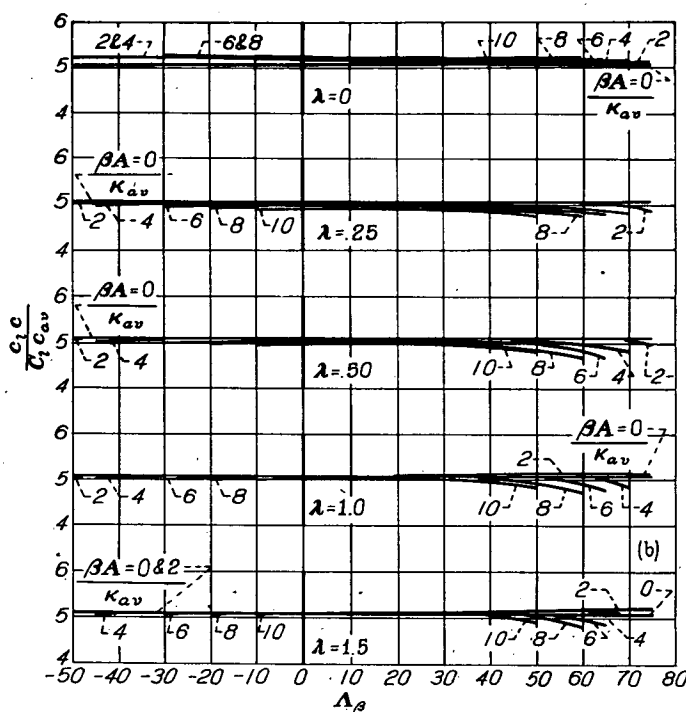
With the full-wing-chord values given above, the effectiveness of constant fraction of wing-chord ailerons or ailerons of arbitrary spanwise chord distribution can be found through use of equations (21) or (23) with the $d\alpha/d\delta$ values of figure 3.

⁹ Throughout the figures, κ_{av} is the constant spanwise section lift curve slope or the average of a small variation. For large spanwise variations of κ that follow the function given by equation (B11) developed in appendix B, the parameters $\beta A/\kappa_{av}$ and λ can be replaced by the parameters $\frac{\beta A}{(\kappa_R + \kappa_T \lambda)/(1 + \lambda)}$ and $\frac{\kappa_T}{\kappa_R} \lambda$, respectively. For large spanwise variations of κ that do not follow the curve of equation (B11), the simultaneous equations for the general solution can be solved for arbitrary distributions of κ . The H_s values can be obtained from figure 2.



(a) $\eta = 0.3827$.

FIGURE 4.—Variation of loading due to rolling coefficient $\frac{c_l c}{C_{l_{\bar{\beta}} \kappa_{av}}}$ with compressible sweep parameter Δ_β , degrees, for straight-tapered wings.



(b) $\eta = 0.7071$.

FIGURE 4.—Continued.

Aerodynamic characteristics due to sideslip of wings with dihedral.—The application of equation (19), case III of equation (18), but with $\delta = \bar{\beta} \Gamma$, and $\bar{G} = G/\bar{\beta} \Gamma$, and the use of equation (28) provides rolling moments due to dihedral angle for the wing in sideslip. These rolling moments are given in the coefficient form $\beta C_{l_{\bar{\beta}}}/\kappa_{av} \Gamma$ which is the same function of η as $\beta C_{l_{\bar{\beta}}}/\kappa_{av}$ and is presented with $\beta C_{l_{\bar{\beta}}}/\kappa_{av}$ in figure 6. Figure 6 with equation (29) will provide the rolling moment

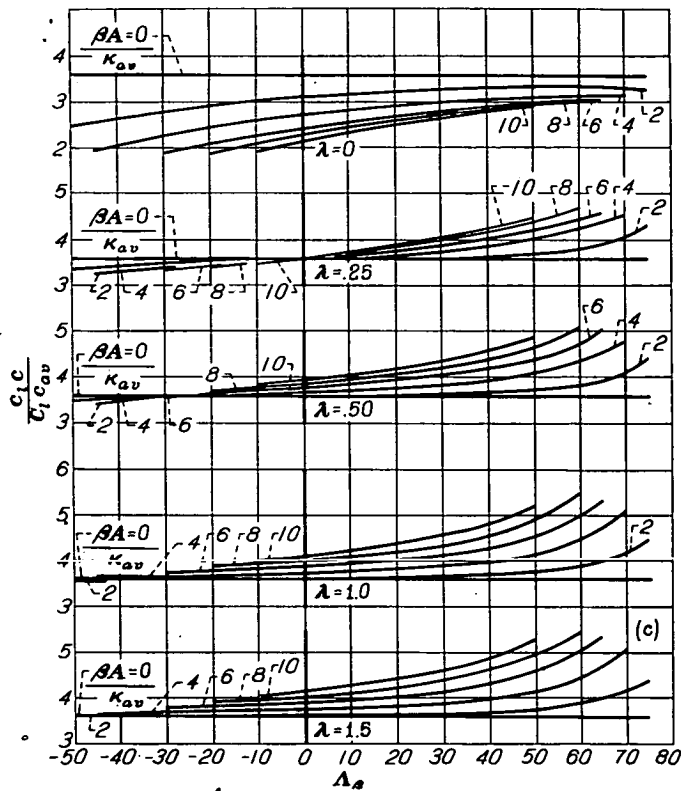
(c) $\eta = 0.9239$.

FIGURE 4.—Concluded.

due sideslip for any symmetric spanwise distribution of dihedral angle.

For dihedral angle constant spanwise, the rolling moment is given by the value at $\eta = 1$ in figure 6. These values for constant spanwise dihedral angle are presented in figure 7 as a function of aspect ratio for various values of sweep angle and taper ratio.

DISCUSSION

Effects of plan-form parameters on aerodynamic characteristics for straight-tapered wings are shown by plots against the various parameters. Compressibility is discussed and formulas given for a range of plan forms at sonic speeds. Theoretical considerations and experimental comparisons indicate the order of reliability of the present theoretical results.

STRAIGHT-TAPERED WINGS

The spanwise loading distribution due to rolling for several plan forms is presented in figure 8. These curves are the result of applying figure 4 and the loading interpolation formula of appendix B. The loading coefficient is given as $\frac{\eta_{c.p.}}{(\eta_{c.p.})_{A=0}} \left(\frac{c_l c}{C_l c_{av}} \right)$ to make the total loading on the semispan constant and thus show more clearly the changes of distribution due to sweep and taper ratio. Figure 8 shows large changes in loading distribution for the zero tapered wing. The effects of sweep are generally as expected, namely, that sweepback shifts the loading outboard.

Effects of plan form on the rolling moment due to rolling is shown from cross plots of figure 5 which are presented in figures 9 and 10. For higher aspect ratio, figures 4, 9, and 10 show the marked lowering of rolling moment due to sweep. Figure

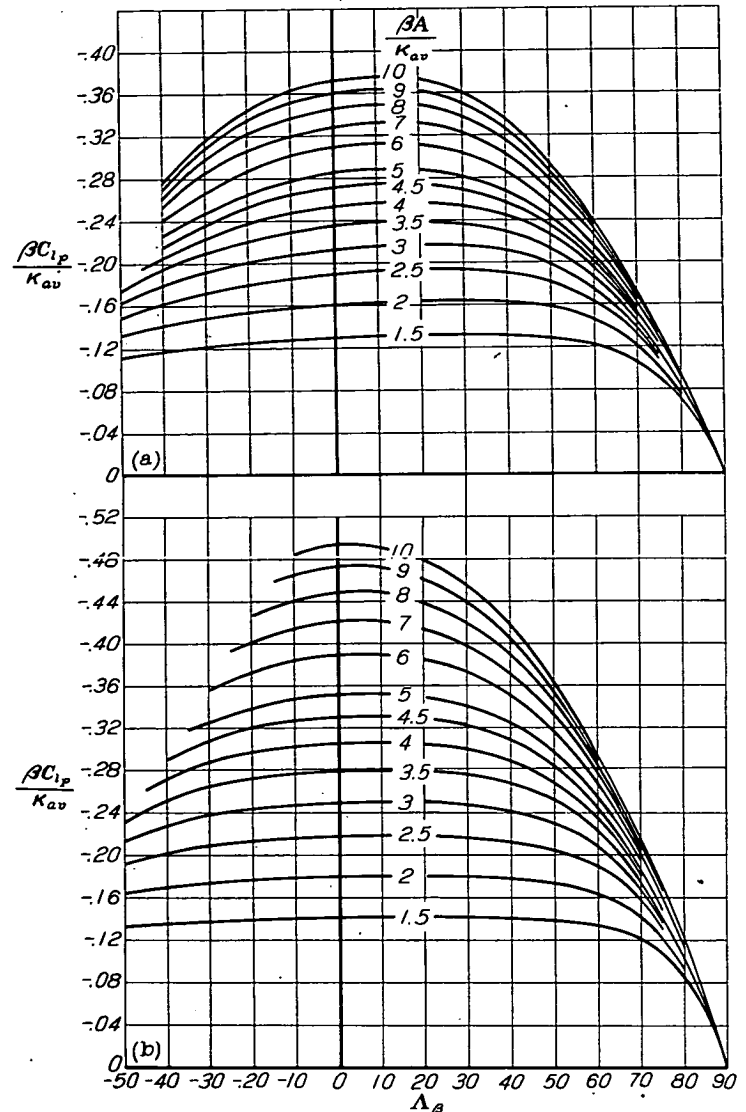
(a) $\lambda = 0$.(b) $\lambda = 0.25$.

FIGURE 5.—Variation of damping-in-roll parameter $\frac{\beta C_{l_p}}{\kappa_{av}}$ with compressible sweep parameter Δ_β , degrees, for straight-tapered wings.

9 indicates that for low aspect ratio, the rolling moment becomes essentially independent of sweep and taper. The taper effects on rolling moment as seen in figure 10 are small except for values of taper ratio less than 0.25.

Typical spanwise loading distributions due to full-wing-chord aileron deflection are shown in figure 11. These curves were faired with the aid of the loading interpolation function of appendix C and, at the aileron spanwise end, care was taken to make the slope large.

Wing geometry effects on aileron effectiveness for full-chord outboard partial-span ailerons (with aileron angle measured parallel to the plane of symmetry) are given in figure 12. The geometry effects on $\beta C_{l_p} / \kappa_{av}$ are similar to those on the damping-in-roll coefficient. Comparison of figure 12 (a) with figure 9 shows that C_{l_p} approaches the zero-aspect-ratio value in the same manner as does C_{l_p} . Figure 13 gives comparative effectiveness of inboard and outboard ailerons for swept wings. As sweep increases, the difference of effectiveness between inboard and out-

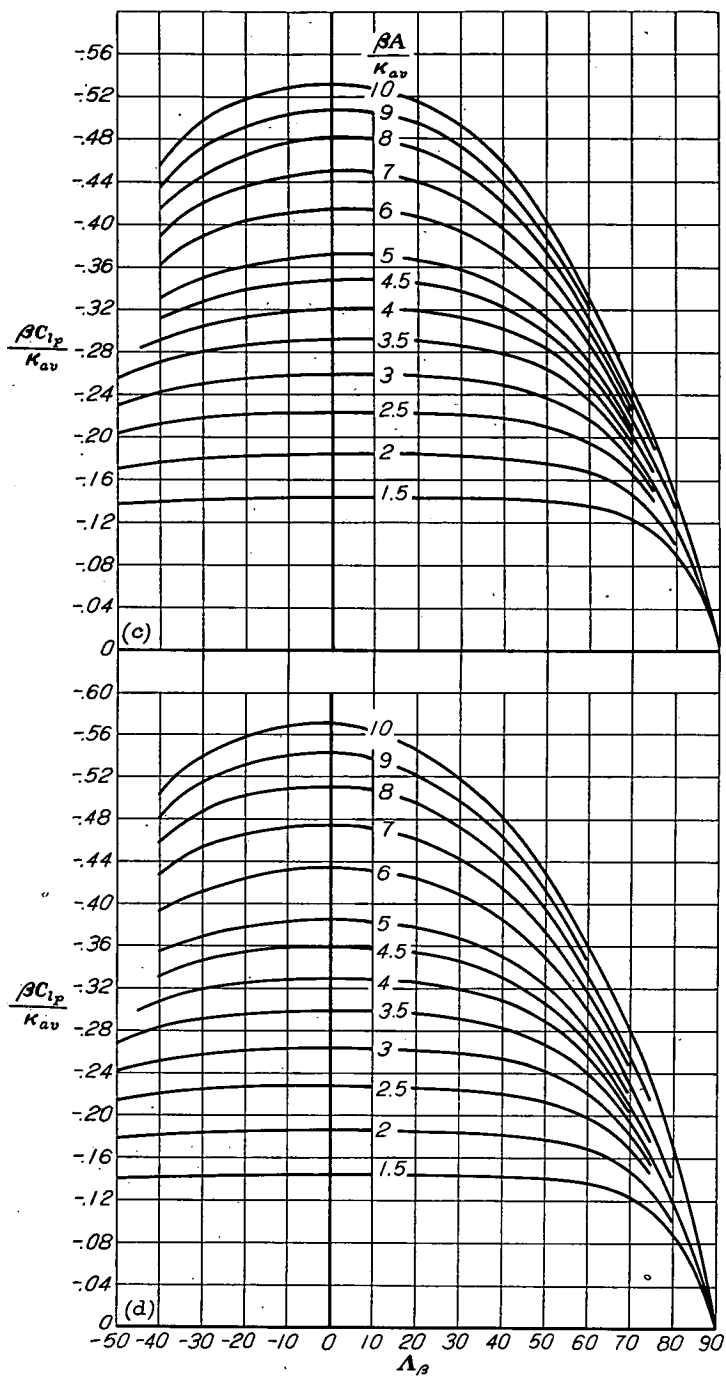
(c) $\lambda = 0.50$.(d) $\lambda = 1.0$.

FIGURE 5.—Continued.

board ailerons decreases showing that inboard ailerons for highly swept-back wings approach the effectiveness of outboard ailerons. Since $d\alpha/d\delta$ becomes large rapidly at small values of t (fig. 3), then, for a given aileron area, narrow full-span ailerons for swept-back wings may be more desirable than larger-chord outboard ailerons. The relative effects of figures 12 and 13 apply equally well for constant fraction of chord ailerons, since the data would differ only by a constant factor $d\alpha/d\delta$.

COMPRESSIBILITY

From the three-dimensional linearized-compressible-flow equation, it can be shown that the effects of compressibility will be properly taken into account if the longitudinal com-

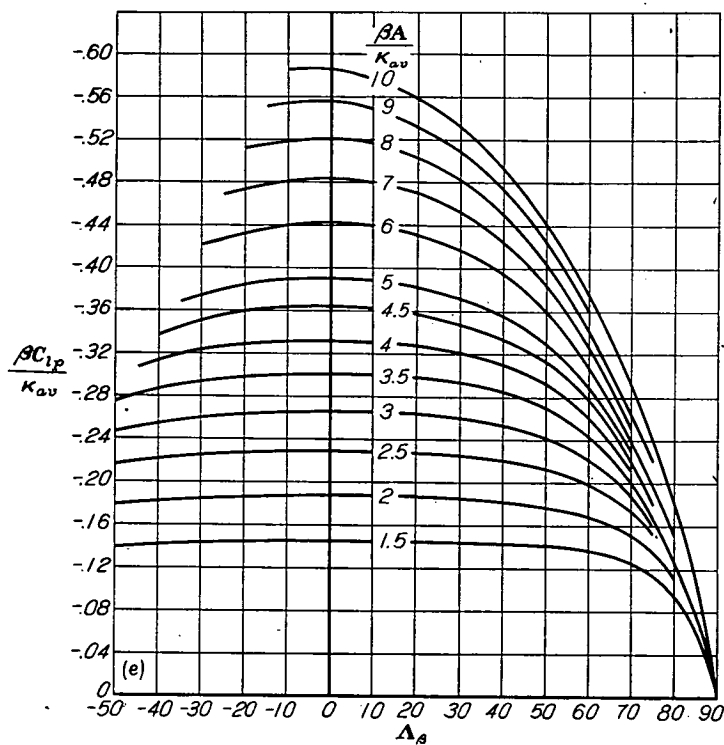
(e) $\lambda = 1.5$.

FIGURE 5.—Concluded.

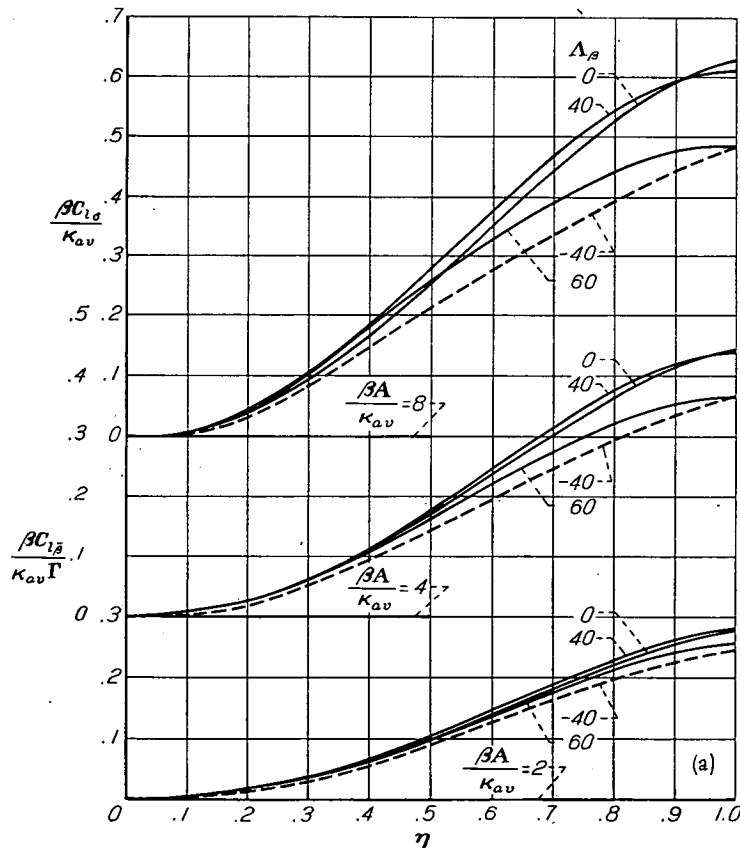
(a) $\lambda = 0$.

FIGURE 6.—Aileron rolling-moment parameter $\frac{\beta C_{l_p}}{K_{av}}$, per radian, and rolling moment due to sideslip with dihedral $\frac{\beta C_{l_\beta}}{K_{av}^2}$, per radian squared, for extent of unit antisymmetric angle of attack from the wing root outboard.

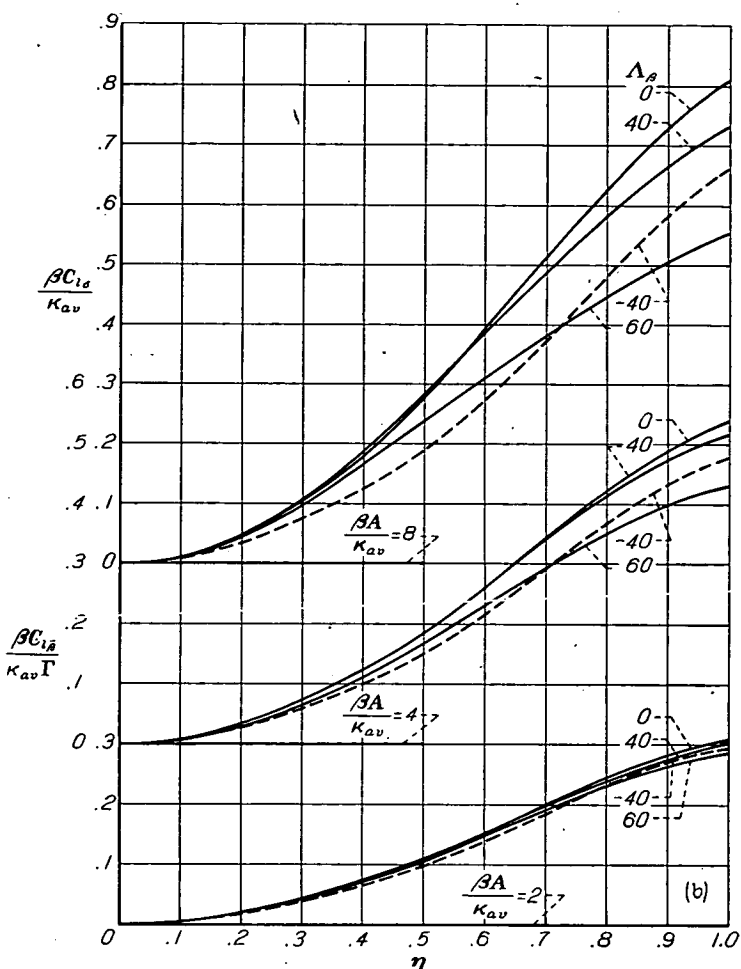
(b) $\lambda = 0.5$.

FIGURE 6.—Continued.

ponents of a wing plan form are increased by the factor $1/\beta$. Or, alternatively, if the linearized compressible flow equation be divided through by β^2 , then the lateral and vertical components of a plan form are decreased by the factor β . In both cases, the incompressible local lift is increased by the factor $1/\beta$ and the compressible local lift coefficient can be written as the parameter βc_l .

With these relations known, an incompressible theory can be made into a compressible theory subject to the limitations of the linearized compressible flow equation. The geometric parameters of a wing are simply substituted by βA , $\Lambda_{\beta} = \tan^{-1} \frac{\tan \Lambda}{\beta}$ and βb . With local lift coefficient given by βc_l , the dimensionless loading becomes $G = \frac{\beta c_l c}{2\beta b} = \frac{c_l c}{2b}$. The wing-chord distribution remains unaltered.

The sonic speed results of reference 4 can be used as a limit point in the present theory for a curve of the variation of antisymmetric aerodynamic characteristics with Mach number. The following equations apply at the speed of sound to plan forms with all points of the trailing edge at or behind the upstream line of maximum wing span:

$$C_{l_p} = \frac{-\pi A}{32}$$

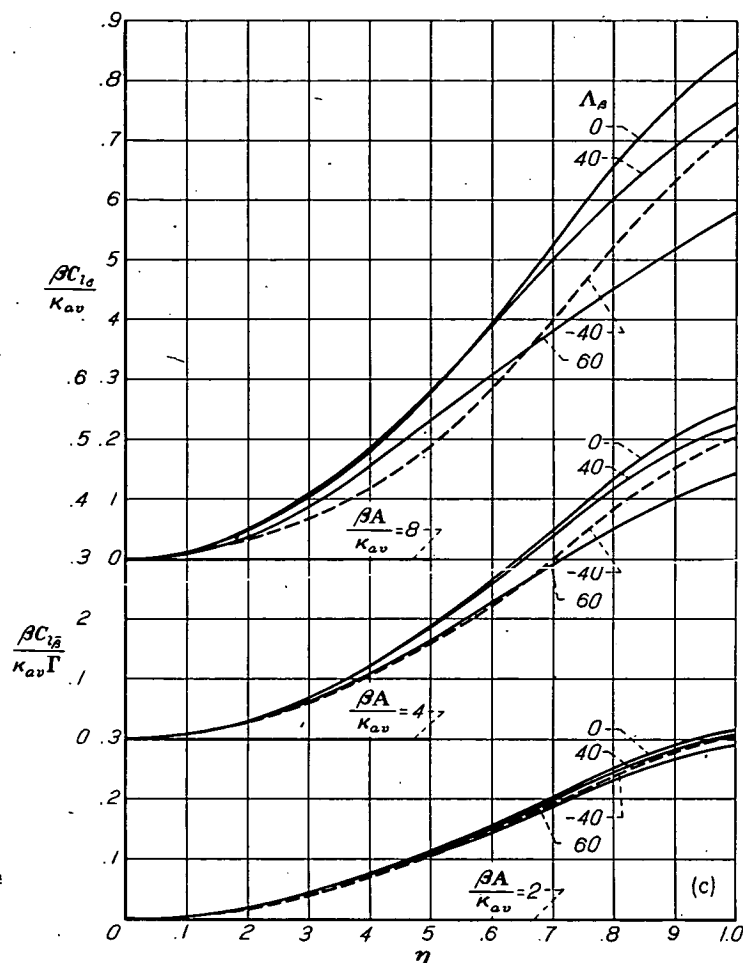
(c) $\lambda = 1.0$.

FIGURE 6.—Concluded.

For outboard ailerons,

$$C_{l_s} = \frac{A}{6} \sin^3 \theta, \text{ where } \eta_a = 1 - \cos \theta$$

For inboard ailerons,

$$C_{l_s} = \frac{A}{6} (1 - \sin^3 \theta), \text{ where } \eta_a = \cos \theta$$

Reference 4 shows that aileron effectiveness at the speed of sound is independent of the chordwise location of the aileron hinge line, provided the hinge line remains ahead of all points of the trailing edge.

ACCURACY OF THE SEVEN-POINT SOLUTION FOR AILERONS

The prediction of aileron effectiveness for given aileron spans with wing twist determined by zero-aspect-ratio theory at only seven span points to satisfy the boundary conditions has been theoretically shown to be sufficient by comparing results with the computation of a typical 3.5 aspect ratio, 45° swept wing with 15 span points satisfying the boundary conditions. The process of finding aileron spans for the 15-point method was the same as that in appendix C. The curves showing the variation of C_{l_s} with aileron span for the 7- and 15-point computations were identical.

The solution for the angle-of-attack distribution that includes a discontinuity can be compared with the solution

for the continuous angle-of-attack distribution by considering an aileron such that the angle-of-attack distribution is equivalent to that of the rolling wing. The damping-in-roll coefficient then can be found by use of equation (23) which reduces to the form

$$C_l = \int_0^1 \alpha \frac{dC_{l_s}}{d\eta} d\eta$$

for $\alpha = -\left(\frac{pb}{2V}\right)_\eta$, and integrating by parts

$$C_{l_p} = \int_0^1 C_{l_s} d\eta - C_{l_s}_{\eta=1}$$

This relation states that C_{l_p} is equal to the area between a curve of figure 6 and the line of C_{l_s} for $\eta=1$. The curves of figure 6 were found by the simplified lifting-surface theory with antisymmetric twist determined by zero-aspect-ratio theory. The values of C_{l_p} obtained in this manner from figure 6 were identical to the C_{l_p} values given by simplified lifting-surface theory for continuous linear antisymmetric-twist distribution.

As further theoretical check, the values of rolling moment due to constant spanwise dihedral angle are obtained from 15-point computations in reference 3 for taper ratio equal to one, with which the present theory for the 7-point method is in exact agreement.

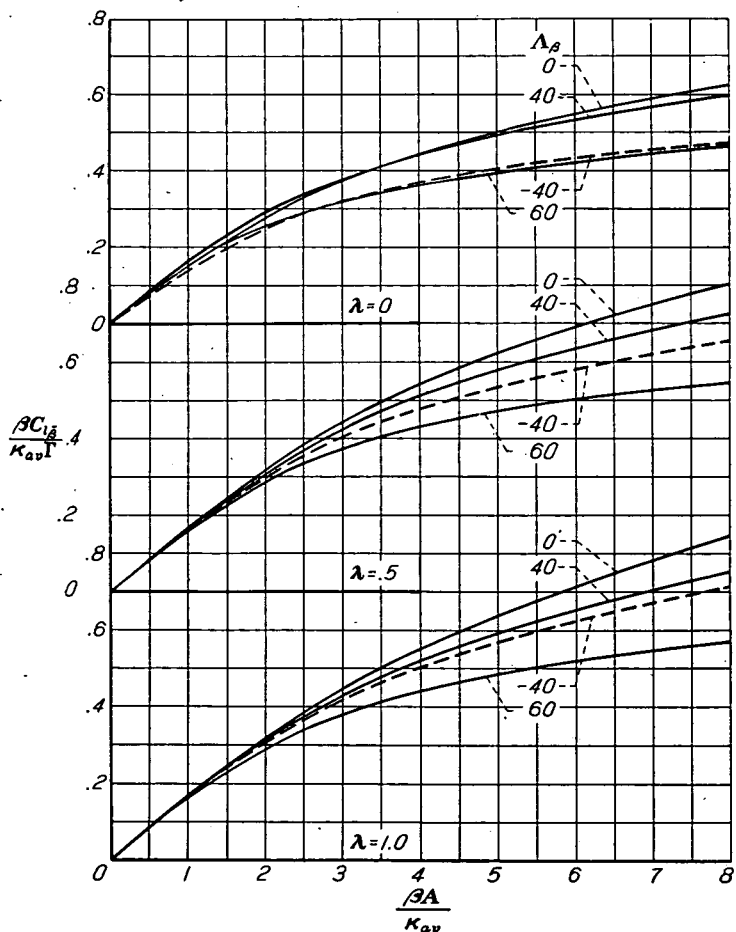


FIGURE 7.—Rolling moment due to sideslip of wing with dihedral $\frac{\beta C_{l_p}}{\kappa_{av} A}$, per radian squared, for unit constant spanwise dihedral angle for straight-tapered wings.

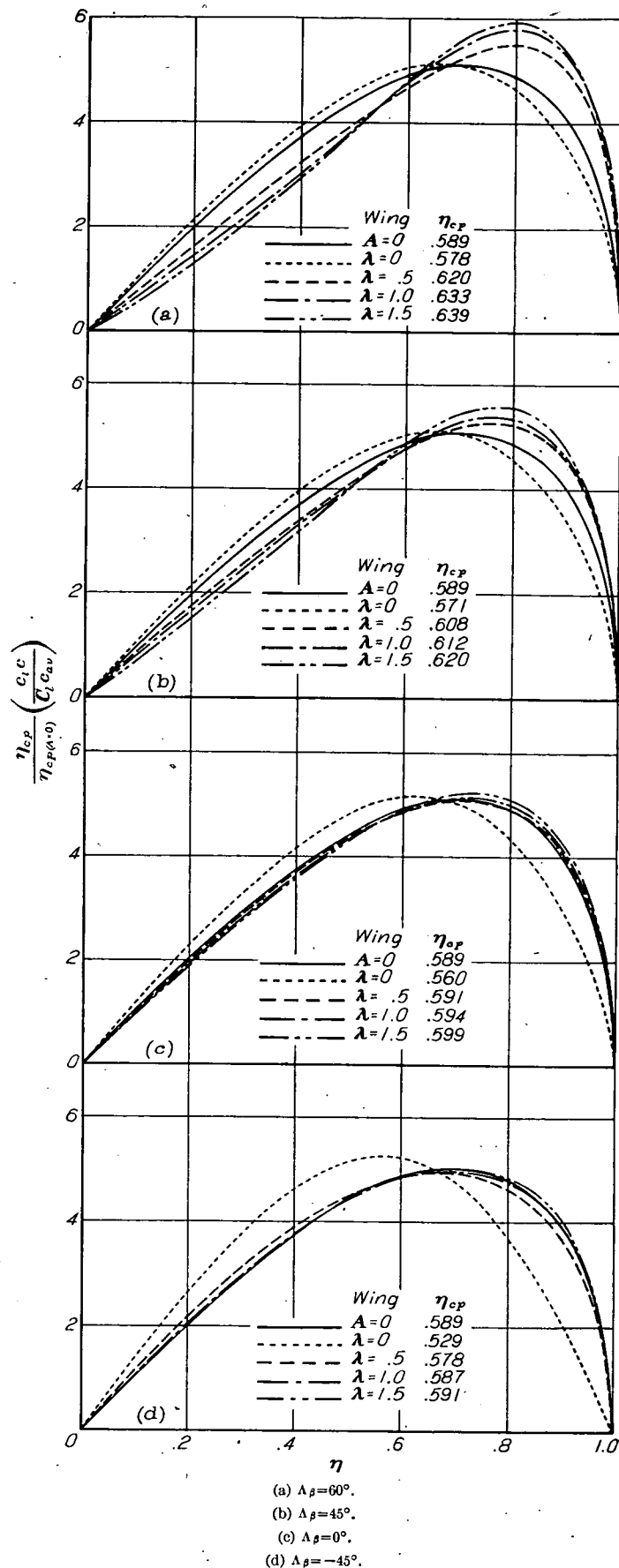


FIGURE 8.—Spanwise loading due to rolling of wings with various taper-ratio and sweep-angle parameters of aspect-ratio parameter $\frac{\beta A}{\kappa_{av}} = 4$. The curve for $A=0$ serves as a basis for comparison and the factor $\frac{\eta_{cp}}{\eta_{cp(A=0)}}$ gives the curves constant area equal to 3.454.

COMPARISON OF THEORETICAL AND EXPERIMENTAL RESULTS

The electro-magnetic analogy method of reference 6 provides damping-in-roll coefficients for an aspect-ratio range of unswept, tapered wings. The results of the present theory and those of reference 8 are compared in figure 14. Except for the taper ratio effects on C_{lp} , the comparison is good. The rounded-wing-tip values of C_{lp} given by NACA Rep. 635 (reference 1) are included in figure 14. Since rounded wing tips generally give values of C_{lp} about 6 percent lower than straight wing tips, the values of NACA Rep. 635 appear to be appreciably too high for lower-aspect-ratio wings. The present theory and the theory of refer-

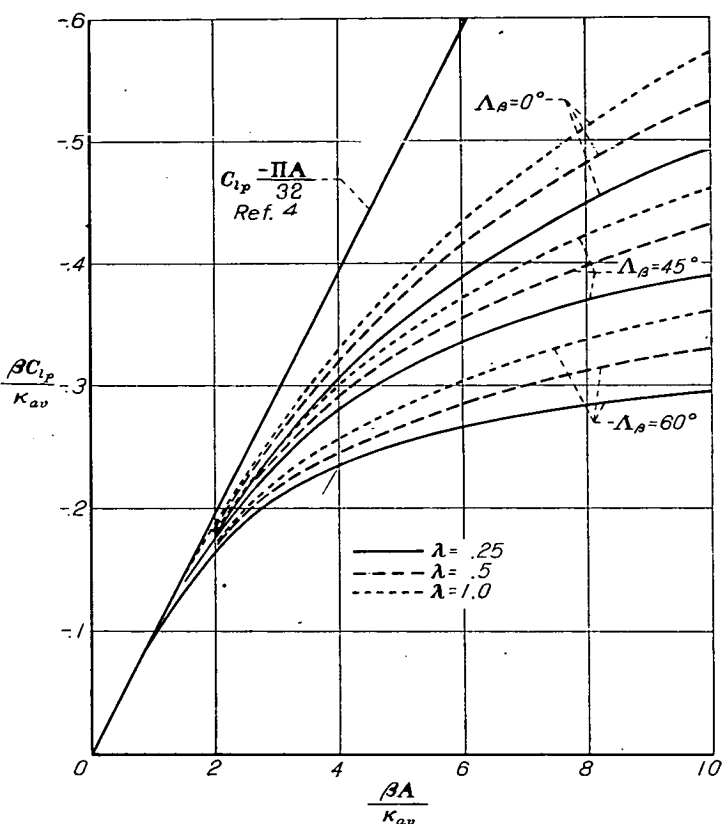


FIGURE 9.—Variation of the damping-in-roll parameter $\frac{\beta C_{lp}}{\kappa_{av}}$ with aspect-ratio parameter $\frac{\beta A}{\kappa_{av}}$ for various sweep angles and taper ratios.

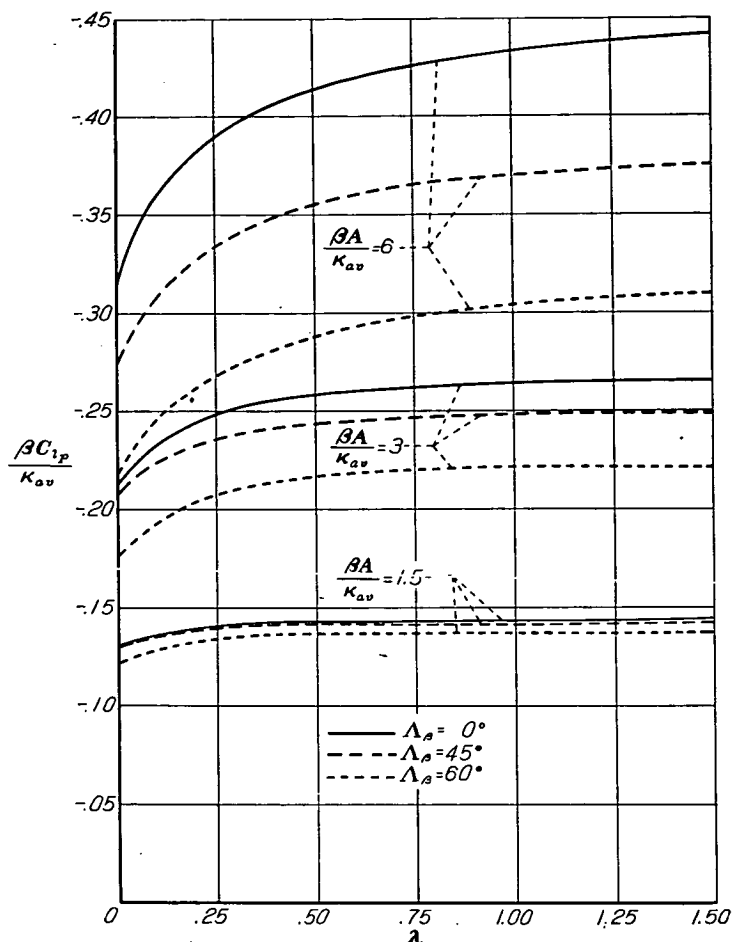


FIGURE 10.—Variation of the damping-in-roll parameter $\frac{\beta C_{lp}}{\kappa_{av}}$ with taper ratio for wings with various aspect-ratio and sweep-angle parameters.

ence 6 approach the value given by the zero-aspect-ratio theory of reference 4 quite satisfactorily. The results of the present theory may be further assessed by the comparison with the results of low-speed experiment as given in figure 15 for the range of plan forms presented. For further experimental verification of the accuracy with which C_{lp} can be determined by the present theory, the reader is referred to reference 3 which supports the theory as well or better than figure 15 of the present report.

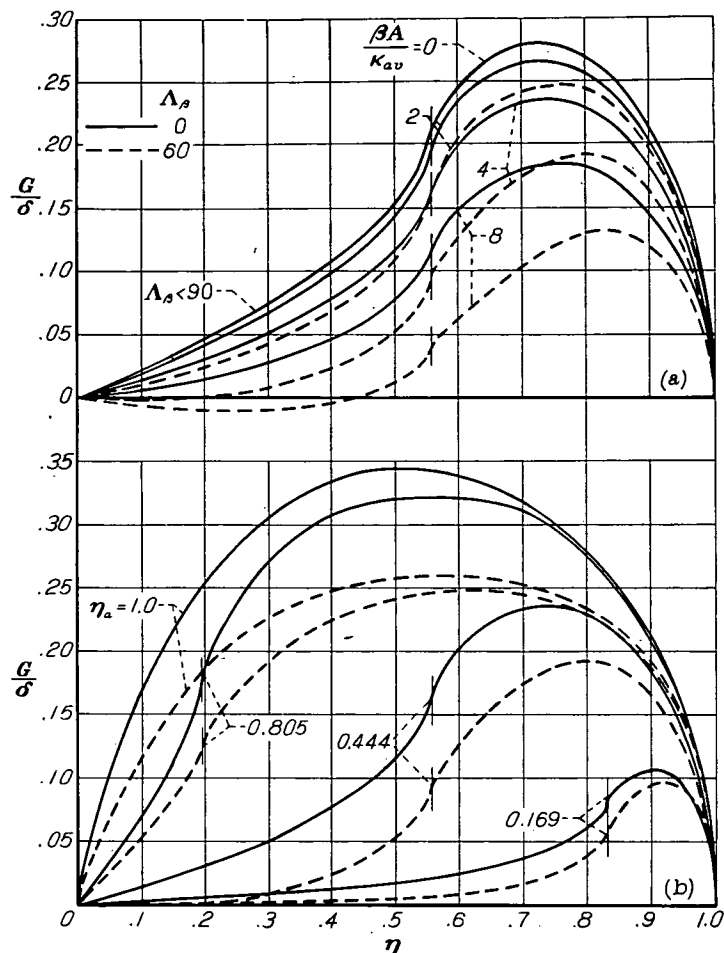


FIGURE 11.—Antisymmetric spanwise loading distribution due to deflection of outboard ailerons, for $c_a = c$, on wings having a taper ratio of 0.5. (a) $\eta_a = 0.444$. (b) $\frac{\beta A}{K_{av}} = 4.0$.

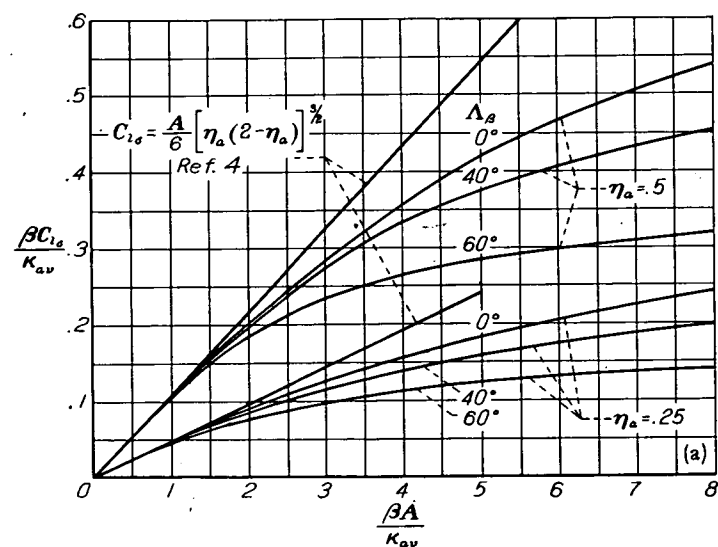
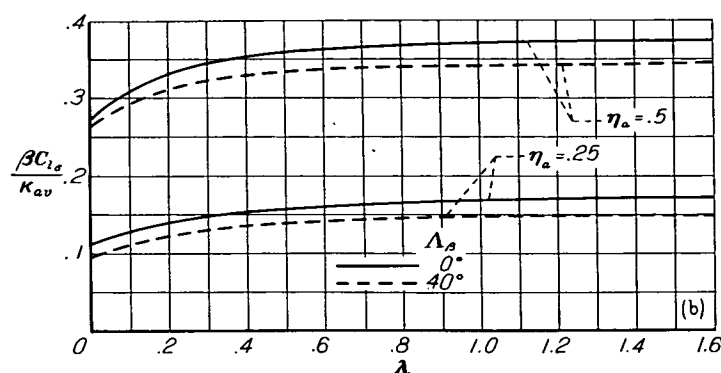
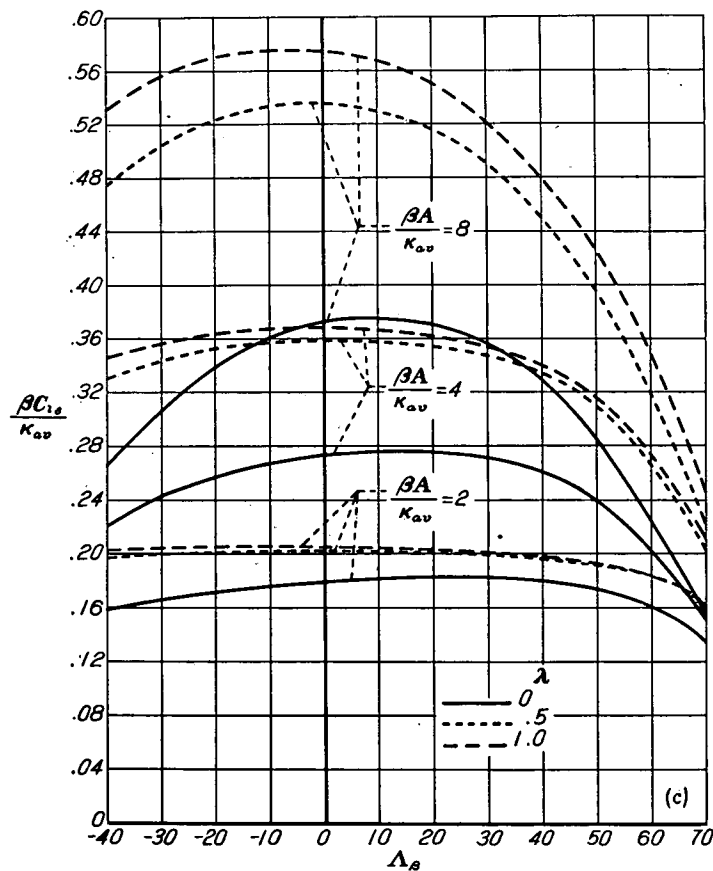


FIGURE 12.—Aileron rolling-moment-coefficient parameter $\frac{\beta C_{l_a}}{K_{av}}$, per radian, for outboard ailerons as a function of aspect-ratio parameter, taper ratio, and compressible-sweep parameter. (a) $\lambda = 0.5$.



(b) $\frac{\beta A}{K_{av}} = 4.0$.

FIGURE 12.—Continued.



(c) Effect of sweep, $\eta_a = 0.5$.

FIGURE 12.—Concluded.

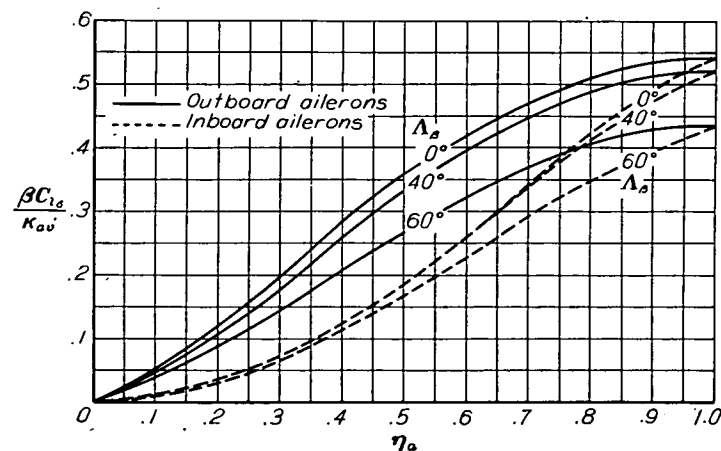


FIGURE 13.—Aileron rolling-moment parameter $\frac{\beta C_{l_a}}{K_{av}}$, per radian, for $\frac{\beta A}{K_{av}} = 4$, $\lambda = 0.5$, as a function of outboard and inboard aileron spans.

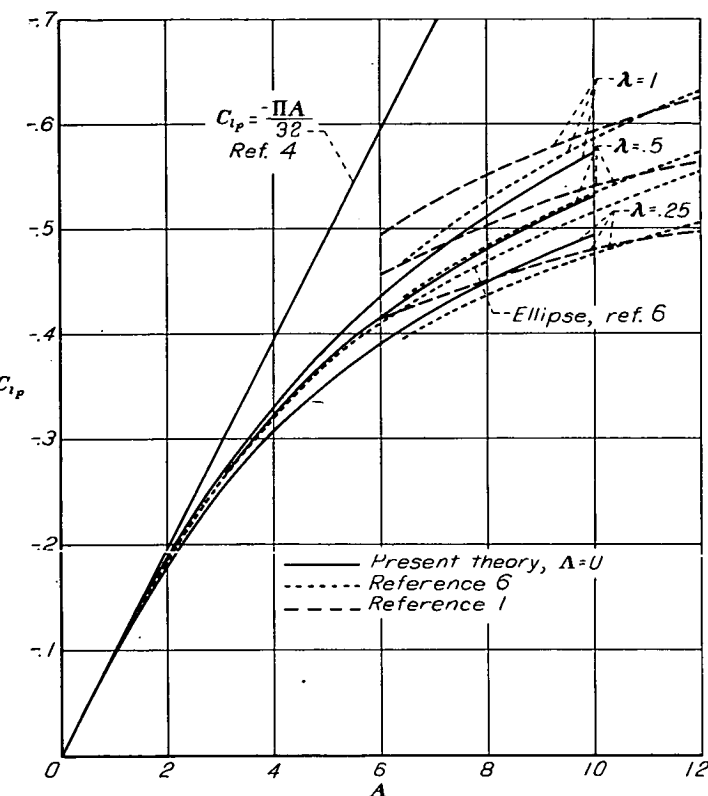


FIGURE 14.—Comparison of damping-in-roll coefficient C_{lr} of the present theory with those of the theories of references 1, 4, and 6.

The loading distributions due to rolling as given by the present theory are compared in figure 16 with low-speed experimental results for a range of swept wings. The sweep angle seems to have considerably more influence on loading distribution as given by experiment than the theory indicates. The experimental pressure data, however, were very erratic and no firm conclusion can be made.

Experimental values of rolling effectiveness due to aileron deflection are compared with theoretically predicted values in a correlation diagram given by figure 17. Included are the results of a wide range of plan forms which do not vary consistently with any geometric parameter or aileron configuration. Sketches of the plan forms and ailerons are drawn about the points of correlation. The theory makes use of the curve of figure 3 giving $d\alpha/d\delta$ for a sealed-gap aileron over a range of deflection of $\pm 10^\circ$. Experimental results for aileron deflections greater than 15° measured perpendicular to the hinge line were not included. The correlation points of figure 17 scatter appreciably; however, the mean line of the points does approximate the line of perfect correlation.

Figure 17 does not account for effective plan-form change due to $d\alpha/d\delta$. Only the effectiveness of the low-aspect-ratio triangular wing of figure 17 is exceedingly in error, which is the result of neglecting plan-form change.

The plan-form change due to $d\alpha/d\delta$ can, in part, be considered analogous to that due to section lift-curve-slope change. Thus, the total section lift of a wing, the chord of which is reduced by $d\alpha/d\delta$ and which is at an angle of attack δ , is equal to the lift of the wing-aileron section for which the aileron only is deflected at the angle δ . This change in plan form, unlike the section lift-curve-slope change for

which the chordwise loading remains constant, does not account for a large change in chordwise loading. If the lifting line is considered to be at the chordwise center of pressure, then, for partial-wing-span ailerons, the lifting line is in effect broken at the aileron spanwise end and the present theory becomes invalid. For the case of full-wing-span ailerons, the lifting line in effect remains unbroken and lies along the center of chordwise pressure. For this case the wing chord can be reduced by $d\alpha/d\delta$ to account for plan-form change; however, although in the limit of zero aspect ratio the results are the same as those of reference 4, this procedure does not with sufficient accuracy account for the chordwise loading shifting aft at intermediate aspect ratios. For control surfaces, the effective plan-form change due to $d\alpha/d\delta$ is appreciable for the low-aspect-ratio wings such that in the limit of zero aspect ratio the spanwise loading is independent of the ratio of aileron chord to wing chord (reference 4). However, for moderate aspect ratios, $d\alpha/d\delta$ can be used without accounting for plan-form changes as comparison with experiment indicates.

Experimental values of C_{lr}/Γ are not compared with the present theory since reference 3 gives ample support of the theory.

CONCLUDING REMARKS

The determination of antisymmetric loading for arbitrary wings is shown to be easily obtained by the solution of three simultaneous equations. The coefficients of the simultaneous equations are presented in charts of parameters that include wing geometry, compressibility, and section lift-curve slope as arbitrary quantities. Thus the loading for an arbitrary antisymmetric angle-of-attack distribution can be simply found once the angle-of-attack distribution is chosen.

For the important cases of antisymmetric loading, roll, and aileron deflection, the angle-of-attack distribution is given and the simultaneous equations are formed. Loading for these cases can be found by simply obtaining from charts the coefficients corresponding to the wing geometry, Mach number, and lift-curve slope, inserting in the appropriate equations and solving.

Integration formulas for the loading distributions are given which enable the aerodynamic coefficients C_{lr} and C_{ls} to be found. The rolling moment due to sideslip of a wing with dihedral is shown to be equivalent to that of aileron deflection and a procedure for determining its value is given.

For the special case of straight-tapered wings, the loading distributions and values of C_{lr} and C_{ls} are given in the chart form for a range of wing plan forms.

Experimental and theoretical verification of the theory is shown to be good. The theory is applicable for large aerodynamic angles, provided the flow remains unseparated. The compressibility considerations are reliable to the speed of sound subject to the limitations of the linearized compressible-flow equation.

AMES AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
MOFFETT FIELD, CALIF., December 22, 1949.



FIGURE 15.—Correlation of theoretical and low-speed experimental damping-in-roll coefficient C_{l_r} for various plan forms.

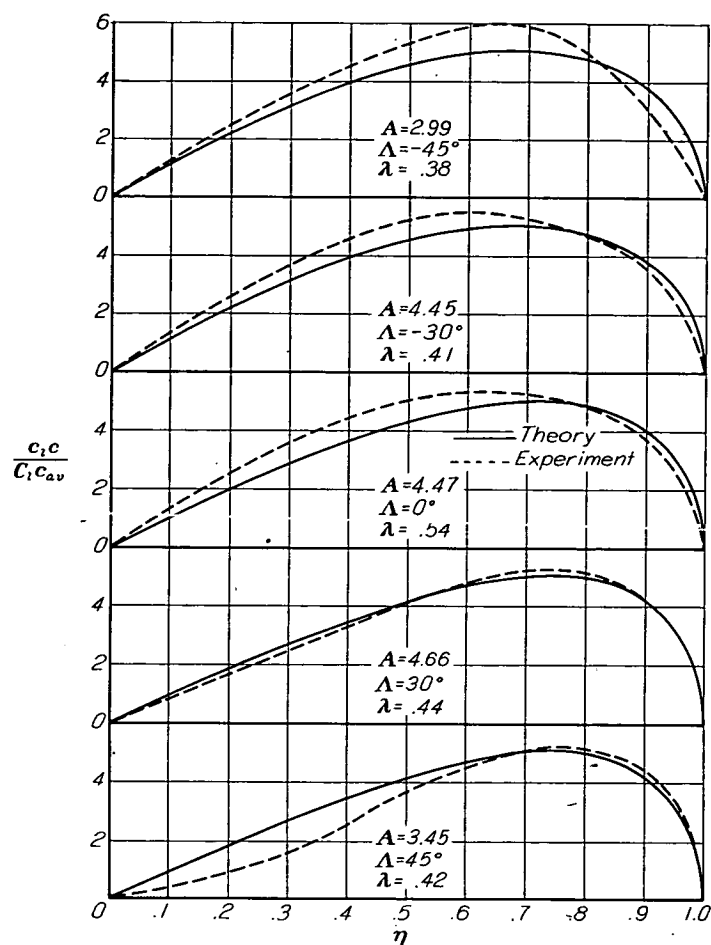


FIGURE 16.—Comparisons of theoretical and low-speed experimental spanwise loading coefficients $\frac{c_l c}{C_l c_{a_v}}$ due to rolling of various swept wings.

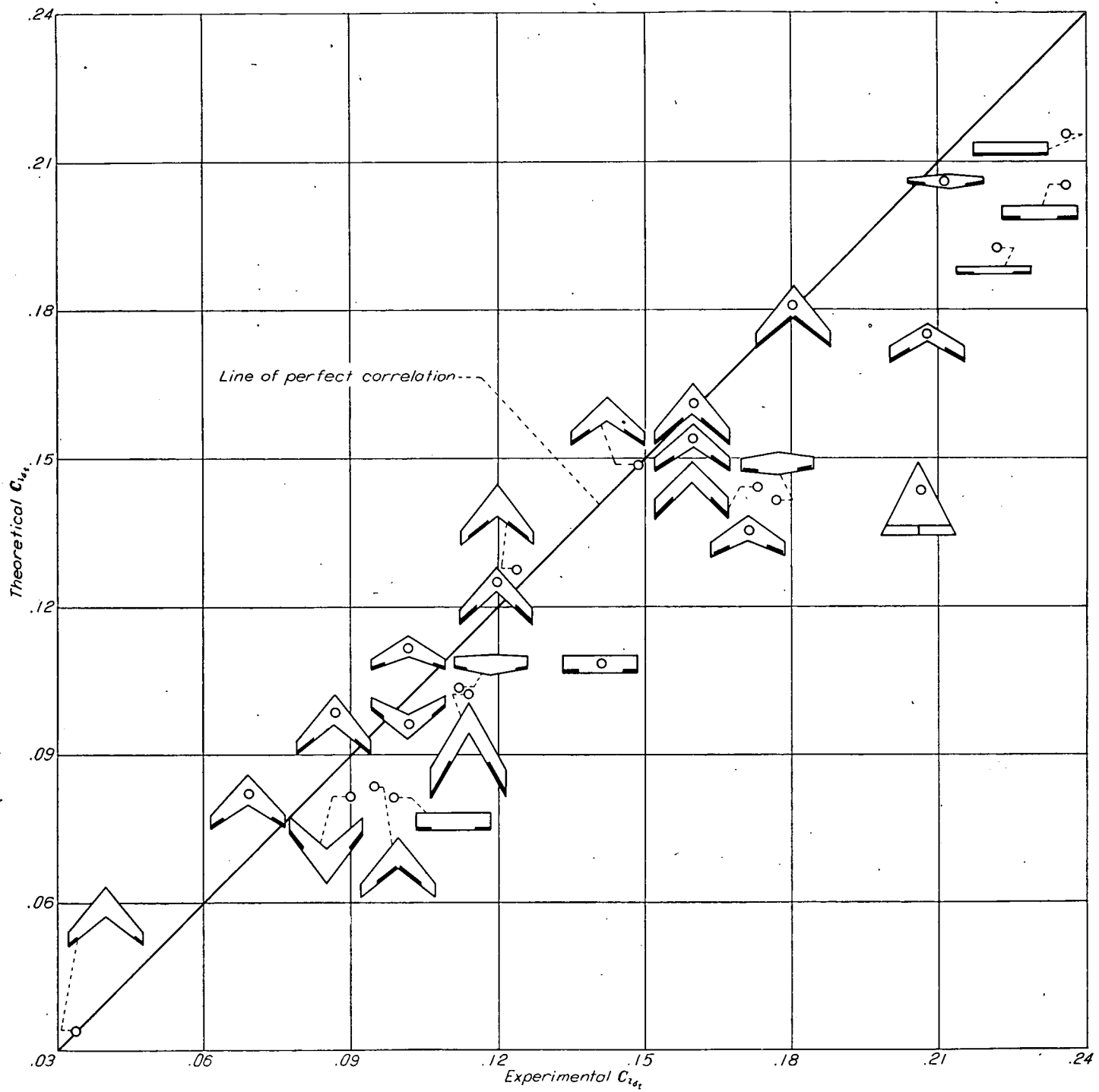


FIGURE 17.—Correlation of theoretical and low-speed experimental aileron-effectiveness $C_{l_{\delta}}$, per radian, due to two antisymmetrically deflected ailerons, for various plan forms.

APPENDIX A

EQUATIONS FOR THE DETERMINATION OF ANTISYMMETRIC LOADING

UNSYMMETRIC LOADING

From NACA Rep. 921 (reference 2), the aerodynamic loading is obtained by solution of linear simultaneous equations

$$\alpha_v = \left(\frac{w}{V} \right)_v = \sum_{n=1}^m A_{vn} G_n, \quad v = 1, 2, \dots, m \quad (A1)$$

where

$$G = \frac{\Gamma_c}{bV} \quad (A2)$$

$$\left. \begin{aligned} A_{vn} &= 2b_{vv} + \left(\frac{b}{c_v} \right) g_{vv} \text{ for } n=v \\ &= -2b_{vn} + \left(\frac{b}{c_v} \right) g_{vn} \text{ for } n \neq v \end{aligned} \right\} \quad (A3)$$

for $\bar{\eta}_\mu \geq 0$ or $\mu \leq \frac{M+1}{2}$.

$$L(v, \mu) = \left\{ \frac{\sqrt{\left[1 + \left(\frac{b}{c_v} \right) \tan \Lambda (|\eta_v| - \bar{\eta}_\mu) \right]^2 + \left(\frac{b}{c_v} \right)^2 (\eta_v - \bar{\eta}_\mu)^2}}{1 + \left(\frac{b}{c_v} \right) (|\eta_v| - \eta_v) \tan \Lambda} - 1 \right\} \frac{1}{\left(\frac{b}{c_v} \right) (\eta_v - \bar{\eta}_\mu)}$$

and

or $\bar{\eta}_\mu \leq 0$ or $\mu \geq \frac{M+1}{2}$

$$\left. \begin{aligned} L(v, \mu) &= \left\{ \frac{\sqrt{\left[1 + \left(\frac{b}{c_v} \right) \tan \Lambda (|\eta_v| + \bar{\eta}_\mu) \right]^2 + \left(\frac{b}{c_v} \right)^2 (\eta_v - \bar{\eta}_\mu)^2}}{1 + \left(\frac{b}{c_v} \right) (|\eta_v| + \eta_v) \tan \Lambda} - 1 \right\} \frac{1}{\left(\frac{b}{c_v} \right) (\eta_v - \bar{\eta}_\mu)} + \\ &\quad \frac{2 \tan \Lambda \sqrt{\left[1 + |\eta_v| \left(\frac{b}{c_v} \right) \tan \Lambda \right]^2 + \left(\frac{b}{c_v} \right)^2 \eta_v^2}}{\left[1 + \left(\frac{b}{c_v} \right) (|\eta_v| - \eta_v) \tan \Lambda \right] \left[1 + \left(\frac{b}{c_v} \right) (|\eta_v| + \eta_v) \tan \Lambda \right]} \end{aligned} \right\} \quad (A5)$$

where

$\eta_v = \cos \frac{v\pi}{m+1}$ spanwise position at which downwash is computed

$\bar{\eta}_\mu = \cos \frac{\mu\pi}{M+1}$ spanwise position of incremental loading at the one-quarter-chord line.

The above equations involve computations over the entire wing. However, if the loading is assumed to be symmetric or antisymmetric, the computations can be reduced to less than half the work. The case of symmetric loading is developed in reference 2 and the antisymmetric case is developed in the following section.

ANTISYMMETRIC LOADING

For antisymmetric loading, the loading on each side of the wing has the same magnitude and distribution but with opposite sign, or

$$\left. \begin{aligned} \alpha_v &= -\alpha_{m+1-v} \\ G_v &= -G_{m+1-v}, \text{ or } G_n = -G_{m+1-n} \end{aligned} \right\} \quad (A6)$$

b_{vv} and b_{vn} are coefficients independent of plan form.

$$g_{vn} = \frac{-1}{2(M+1)} \left[\frac{L(v, 0)f_{no} + L(v, M+1)f_{n,M+1} + \sum_{\mu=1}^M L(v, \mu)f_{n\mu}}{2} \right] \quad (A4)$$

where

$f_{n\mu}$ are coefficients independent of plan form.

The $L(v, \mu)$ functions, which can have η_v negative to find unsymmetrical loading, are given by

Equation (A1) can then be written as

$$\alpha_v = \sum_{n=1}^{\frac{m-1}{2}} (A_{vn} - A_{v, m+1-n}) G_n \quad (A7)$$

where the summation is only to $\frac{m-1}{2}$, since $G_{\frac{m+1}{2}} = 0$ for antisymmetric loading.

With equations (A3), equation (A7) becomes

$$\alpha_v = \left[2(b_{vv} - b_{v, m+1-v}) + \left(\frac{b}{c_v} \right) (g_{vv} - g_{v, m+1-v}) \right] G_v - \sum_{n=1}^{\frac{m-1}{2}} \left[2(b_{vn} - b_{v, m+1-n}) - \left(\frac{b}{c_v} \right) (g_{vn} - g_{v, m+1-n}) \right] G_n \quad (A8)$$

(The prime indicates the value for $n=\nu$ is not summed.)
Now, from equation (A4)

$$g_{\nu n} + g_{\nu, m+1-n} = \frac{-1}{2(M+1)} \left[\frac{L(\nu, 0)(f_{n0} - f_{m+1-n, 0})}{2} + \frac{L(\nu, M+1)(f_{n, M+1} - f_{m+1-n, M+1})}{2} + \sum_{\mu=1}^M L(\nu, \mu)(f_{n\mu} - f_{m+1-n, \mu}) \right] \quad (A9)$$

where

$$f_{n\mu} = \frac{2}{m+1} \sum_{\mu_1=1}^m \mu_1 \sin \mu_1 \phi_n \cos \mu_1 \phi_\mu \quad (A10)$$

and

$$\phi_n = \frac{n\pi}{m+1}, \quad \phi_\mu = \frac{\mu\pi}{M+1}$$

From equation (A9), $f_{n\mu} - f_{m+1-n, \mu}$ can be defined as

$$f_{n\mu}^* = f_{n\mu} - f_{m+1-n, \mu}$$

then, using equation (A10),

$$\begin{aligned} f_{n\mu}^* &= \frac{2}{m+1} \sum_{\mu_1=1}^m \mu_1 \cos \mu_1 \phi_\mu (\sin \mu_1 \phi_n - \sin \mu_1 \phi_{m+1-n}) \\ &= \frac{2}{m+1} \sum_{\mu_1=1}^m \mu_1 \cos \mu_1 \phi_\mu \sin \mu_1 \phi_n (1 + \cos \mu_1 \pi) \end{aligned}$$

and, since the terms of the summation for odd μ_1 vanish,

$$f_{n\mu}^* = \frac{4}{m+1} \sum_{\mu_1=2, 4, 6, \dots, \text{even}}^m \mu_1 \sin \mu_1 \phi_n \cos \mu_1 \phi_\mu \quad (A11)$$

From equation (A11),

$$f_{n\mu}^* = f_{n, M+1-\mu}^* \quad (A12)$$

Combining equation (A9) with (A12) and defining

$$g_{\nu n}^* = g_{\nu n} - g_{\nu, m+1-n}$$

then

$$g_{\nu n}^* = \frac{-1}{2(M+1)} \sum_{\mu=0}^{\frac{M+1}{2}} [L(\nu, \mu) + L(\nu, M+1-\mu)] f_{n\mu}^* \quad (A13)$$

where for $\mu=0$ and $\frac{M+1}{2}$, $f_{n\mu}^*$ is equal to half the values given by equation (A11) in order that the products can be fitted into the summation. With equation (A13), equation (A8) can be written as

$$\begin{aligned} \alpha_\nu &= \left(2C_\nu + \frac{b}{c_\nu} g_{\nu\nu}^* \right) G_\nu - \sum_{n=1}^{\frac{m-1}{2}} \left(2C_{\nu n} - \frac{b}{c_\nu} g_{\nu n}^* \right) G_n \quad (A14) \\ \nu &= 1, 2, 3, \dots, \frac{m-1}{2} \end{aligned}$$

where

$$C_\nu = b_{\nu\nu} - b_{\nu, m+1-\nu}$$

$$C_{\nu n} = b_{\nu n} - b_{\nu, m+1-n}$$

From reference 2,

$$b_m = \frac{\sin \phi_n}{(\cos \phi_n - \cos \phi_\nu)^2} \left[\frac{1 - (-1)^{n-\nu}}{2(m+1)} \right]$$

which gives zero values for $b_{\nu n}$ for even $(n-\nu)$ values. Then, since m is odd,

$$b_{\nu, m+1-\nu} = 0$$

and

$$C_\nu = b_{\nu\nu}$$

It should be noted that $L(\nu, \mu)$ simplifies somewhat for the antisymmetrically loaded wing since η now is only positive in equation (A5). If only positive values of $\bar{\eta}$ are used, then equation (A5) can be written as

$$\begin{aligned} L^*(\nu, \mu) &= L^*_{\nu\mu} = L(\eta, \bar{\eta}) + L(\eta, -\bar{\eta}) \\ &= L(\nu, \mu) + L(\nu, M+1-\mu) \end{aligned}$$

In summary, the foregoing analysis for the antisymmetrically loaded wing gives

$$\alpha_\nu = \sum_{n=1}^{\frac{m-1}{2}} p_{\nu n} G_n \quad (A15)$$

$$\nu = 1, 2, 3, \dots, \frac{m-1}{2}$$

where

$$p_{\nu n} = 2b_{\nu\nu} + \frac{b}{c_\nu} g_{\nu\nu}^* \text{ for } n=\nu \quad (A16)$$

$$= -2C_{\nu n} + \frac{b}{c_\nu} g_{\nu n}^* \text{ for } n \neq \nu$$

$$C_{\nu n} = b_{\nu n} - b_{\nu, m+1-n}$$

$$b_{\nu\nu} = \frac{m+1}{4 \sin \phi_\nu}$$

$$b_{\nu n} = \frac{\sin \phi_n}{(\cos \phi_n - \cos \phi_\nu)^2} \left[\frac{1 - (-1)^{n-\nu}}{2(m+1)} \right]$$

$$g_{\nu\nu}^* = g_{\nu\nu}^* \text{ for } n=\nu$$

$$g_{\nu n}^* = -\frac{1}{2(M+1)} \sum_{\mu=0}^{\frac{M+1}{2}} L^*_{\nu\mu} f_{n\mu}^*$$

$$f_{n\mu}^* = \frac{4}{m+1} \sum_{\mu_1=\text{even}}^m \mu_1 \sin \mu_1 \phi_n \cos \mu_1 \phi_\mu$$

$$f_{n0}^* = \frac{f_{n\mu}^*}{2} \text{ for } \mu=0$$

$$f_{n, \frac{M+1}{2}}^* = \frac{f_{n\mu}^*}{2} \text{ for } \mu = \frac{M+1}{2}$$

APPENDIX B

DERIVATION OF RELATIONS USED IN THE METHOD

APPLICATION OF APPENDIX A

With appendix A, the antisymmetrical loading on a plan form for any antisymmetrical distribution of α_v can be found. The principal work in the computations is to obtain the coefficients of the simultaneous equations (A15). These coefficients can be presented in charts for the complete range of geometric plan-form parameters into which are introduced the effects of compressibility and section lift-curve slope. With the loading due to rolling known, the coefficients and derivatives are obtained by integration formulas.

Section lift-curve-slope effect.—For a two-dimensional wing with the loaded line at the quarter-chord position, the position x aft of the loaded line where the induced downwash equals the angle of attack of the wing can be obtained by the Biot Savart Law as

$$w = \frac{\Gamma_c}{2\pi x} \text{ where } \Gamma_c = \frac{c_l c V}{2}$$

or

$$\frac{w}{V} = \frac{c_l c}{4\pi x} = \alpha$$

or

$$\frac{dc_l}{d\alpha} = \frac{4\pi x}{c}$$

then

$$x = \left(\frac{c}{4\pi} \right) \frac{dc_l}{d\alpha}$$

where $dc_l/d\alpha$ is the section lift-curve slope. Two dimensional section compressibility effects that do not follow the Prandtl-Glauert rule can be given consideration by taking the ratio of $(dc_l/d\alpha)_{\text{compressible}}$ at a Mach number to $2\pi/\beta$. Let κ be the ratio of the section lift-curve slope at a given Mach number to $2\pi/\beta$ or $(dc_l/d\alpha)_{\text{compressible}} = 2\pi\kappa/\beta$, then

$$x = \kappa(c/2)$$

Then the induced angle, $\kappa(c/2)$ aft of the loaded line, is equal to the angle of attack of the wing. For $\kappa=1$, this is at the three-quarter-chord line. For section lift-curve slope less than 2π , κ is less than one and the downwash is equal to the angle of attack at some point between the one-quarter- and three-quarter-chord line.

To take into account the section lift-curve-slope variation in the present theory, the downwash must be found at a distance $\kappa(c/2)$ aft of the loaded line. From the formulas of the summation in appendix A, b/c_v should be taken as $b/\kappa_v c_v$, where κ_v is the ratio of section lift-curve slope for a given Mach number at span station v , to $2\pi/\beta$.

Derivation of parameters for p_{vn} .—The p_{vn} coefficients, as defined by equation (A16) in appendix A, depend on plan-

form geometry in the $(b/c_v)L_{vn}^*$ functions only, or p_{vn} is a function of b/c_v and sweep angle. As previously shown, b/c_v is also a function of the spanwise variation of section lift-curve slope and is effectively equivalent to $b/\kappa_v c_v$, where κ_v is the ratio of section lift-curve slope for a given Mach number at span station v to $2\pi/\beta$. The p_{vn} coefficients can be plotted against $b/\kappa_v c_v$ with sweep angle as a parameter; however, $b/\kappa_v c_v$ will vary from zero to very large values for a range of plan-form geometry, and the plots become unwieldy. For a range of aspect ratio, the values of $b/\kappa_v c_v$ are a maximum for the zero tapered wings when $\eta_v \geq 0.5$ (provided plan-form edges are not concave) and a maximum for the inverse-tapered wings for $\eta_v \leq 0.5$. The ratio of $b/\kappa_v c_v$ for $\eta_v \geq 0.5$ for any plan form to those of the zero tapered wing or the ratio of $b/\kappa_v c_v$ for $\eta_v \leq 0.5$ for any plan form to those of the inverse-tapered wing gives a geometric parameter for any plan form that has maximum values that depend only on aspect ratio.

The chord distribution for straight-tapered wings is given by

$$\frac{b}{c_v} = \frac{A(1+\lambda)}{2[1-|\eta_v|(1-\lambda)]} \quad (B1)$$

Then, for $\lambda=0$,

$$\frac{b}{Ac_v} = \frac{1}{2(1-|\eta_v|)} \quad (B2)$$

and for $\lambda=1.5$

$$\frac{b}{Ac_v} = \frac{5}{2(2+|\eta_v|)} \quad (B3)$$

The ratio of $b/\kappa_v c_v$ to equations (B2) and (B3) gives, respectively, a geometric parameter as

$$\left. \begin{aligned} \frac{b/\kappa_v c_v}{(b/Ac_v)_{\lambda=0}} &= 2(1-\eta_v) \left(\frac{b}{\kappa_v c_v} \right) \text{ for } 0.5 \leq \eta_v < 1 \\ \frac{b/\kappa_v c_v}{(b/Ac_v)_{\lambda=1.5}} &= \frac{2(2+\eta_v)}{5} \left(\frac{b}{\kappa_v c_v} \right) \text{ for } 0 \leq \eta_v \leq 0.5 \end{aligned} \right\} \quad (B4)$$

Let H_v be defined as two-fifths times the values of equation (B4) (the fraction two-fifths is introduced to give H_v the approximate values of p_{vn} to simplify plotting procedures), then adding effects of compressibility (see Discussion section)

$$H_v = d_v \left(\frac{\beta b}{\kappa_v c_v} \right) \quad (B5)$$

where

$$\begin{aligned} d_v &= \frac{4(1-\eta_v)}{5} \text{ for } 0.5 \leq \eta_v < 1 \\ &= \frac{4(2+\eta_v)}{25} \text{ for } 0 \leq \eta_v \leq 0.5 \end{aligned}$$

For $M=m$, $f_{n\mu}^*$ simplifies to

$$f_{n\mu}^* = \left[\frac{2(-1)^{n+\mu} \sin 2\phi_n}{\cos 2\phi_n - \cos 2\phi_\mu} \right]_{\mu \neq n} = \left(\frac{-\sin 4\phi_n}{1 - \cos 4\phi_n} \right)_{n=\mu}$$

$$f_{no}^* = \frac{f_{nu}^*}{2} \text{ for } \mu=0$$

$$f_{n \frac{m+1}{2}}^* = \frac{f_{n\mu}^*}{2} \text{ for } \mu = \frac{m+1}{2}$$

$$L_{\nu\mu}^* = \frac{1}{\left(\frac{b}{c_\nu}\right)(\eta_\nu - \bar{\eta}_\mu)}$$

$$\left\{ \sqrt{\left[1 + \left(\frac{b}{c_\nu}\right)(\eta_\nu - \bar{\eta}_\mu) \tan \Lambda \right]^2 + \left(\frac{b}{c_\nu}\right)^2 (\eta_\nu - \bar{\eta}_\mu)^2} - 1 \right\} +$$

$$\frac{1}{\left(\frac{b}{c_\nu}\right)(\eta_\nu + \bar{\eta}_\mu)}$$

$$\left\{ \frac{\sqrt{\left[1 + \left(\frac{b}{c_\nu}\right)(\eta_\nu - \bar{\eta}_\mu) \tan \Lambda \right]^2 + \left(\frac{b}{c_\nu}\right)^2 (\eta_\nu + \bar{\eta}_\mu)^2}}{1 + 2 \left(\frac{b}{c_\nu}\right) \eta_\nu \tan \Lambda} - 1 \right\} +$$

$$\frac{2 \tan \Lambda \sqrt{\left[1 + \left(\frac{b}{c_\nu}\right) \eta_\nu \tan \Lambda \right]^2 + \left(\frac{b}{c_\nu}\right)^2 \eta_\nu^2}}{1 + 2 \left(\frac{b}{c_\nu}\right) \eta_\nu \tan \Lambda}$$

$$\eta_\nu = \cos \phi_\nu \text{ where } \phi_\nu = \frac{\nu \pi}{m+1}$$

$$\bar{\eta}_\mu = \cos \phi_\mu \text{ where } \phi_\mu = \frac{\mu \pi}{M+1}$$

$$\bar{\eta}_n = \cos \phi_n \text{ where } \phi_n = \frac{n \pi}{m+1}$$

For a discussion of the relative accuracies obtained for a choice of values of M and m , see reference 7. The most favorable application is with $M=m$.

For tapered wings, H_v simplifies to

$$\left. \begin{aligned} H_v &= \frac{2(1-\eta_v)(1+\lambda)}{5[1-\eta_v(1-\lambda)]} \left(\frac{\beta A}{\kappa_v} \right) \text{ for } 0.5 \leq \eta_v < 1 \\ &= \frac{2(2+\eta_v)(1+\lambda)}{25[1-\eta_v(1-\lambda)]} \left(\frac{\beta A}{\kappa_v} \right) \text{ for } 0 \leq \eta_v \leq 0.5 \end{aligned} \right\} \quad (\text{B6})$$

Plots of p_{vn} against H_v in the range of $H_v=0$ to 4 will give p_{vn} coefficients for wings of any chord distribution for aspect ratios up to 10 or 12.

Linear asymptotes of p_{vn} .—For large values of H_v , the p_{vn} functions become linearly proportional to H_v . Since this linear characteristic appears at relatively low values of H_v , the simple linear relation between p_{vn} and H_v is quite usable.

The L^*_{vn} function of appendix A is multiplied by b/c_v and the product is linearized.

$$\left. \begin{aligned} \left(\frac{b}{c_v} \right) L^*_{vn} &= \frac{2}{\cos \Lambda} \left(\frac{b}{c_v} \right) - \frac{2\eta}{\eta^2 - \bar{\eta}^2} + \left(\frac{1}{|\eta - \bar{\eta}|} + \frac{1}{\eta} \right) \sin \Lambda - \\ &\quad \frac{1}{2\eta} \tan \frac{\Lambda}{2} - \\ &\quad \frac{1}{2\eta} \left[\frac{1 - \sqrt{1 + \left(\frac{\eta - \bar{\eta}}{\eta + \bar{\eta}} \tan \Lambda \right)^2}}{\tan \Lambda} \right] \dots \text{for } \bar{\eta} < \eta \\ &= \frac{-2\eta}{\eta^2 - \bar{\eta}^2} + \left(\frac{1}{|\eta - \bar{\eta}|} + \frac{1}{\eta} \right) \sin \Lambda - \frac{1}{2\eta} \tan \frac{\Lambda}{2} - \\ &\quad \frac{1}{2\eta} \left[\frac{1 - \sqrt{1 + \left(\frac{\eta - \bar{\eta}}{\eta + \bar{\eta}} \tan \Lambda \right)^2}}{\tan \Lambda} \right] \dots \text{for } \bar{\eta} > \eta \\ &= \left(\frac{1}{\cos \Lambda} + \tan \Lambda \right) \left(\frac{b}{c_v} \right) - \frac{1}{2\eta} + \frac{1}{\eta} \sin \Lambda - \\ &\quad \frac{1}{2\eta} \tan \frac{\Lambda}{2} \dots \text{for } \bar{\eta} = \eta \\ &= \frac{2}{\cos \Lambda} \left(\frac{b}{c_v} \right) - \frac{2}{\eta} + \frac{2 \sin \Lambda}{\eta} \dots \text{for } \bar{\eta} = 0 \end{aligned} \right\} \quad (\text{B7})$$

With the values of equation (B7) substituted into equation (A16) the values of p_{vn} for arbitrary sweep angle are obtained. Thus, for $m=7$, the following equation (B8) gives values for p_{vn} as

$$\begin{aligned} p_{11} &= \left(\frac{3.928}{\cos \Lambda} + 1.026 \tan \Lambda \right) H_1 + 7.968 - 1.494 \sin \Lambda + \\ &\quad 0.014 \tan \frac{\Lambda}{2} + 0.082 \left(\frac{1 - \sqrt{1 + 0.0016 \tan^2 \Lambda}}{\tan \Lambda} \right) - \\ &\quad 0.068 \left(\frac{1 - \sqrt{1 + 0.0177 \tan^2 \Lambda}}{\tan \Lambda} \right) + \\ &\quad 0.034 \left(\frac{1 - \sqrt{1 + 0.1717 \tan^2 \Lambda}}{\tan \Lambda} \right) \\ p_{12} &= \left(\frac{0.851}{\cos \Lambda} - 2.901 \tan \Lambda \right) H_1 - 3.138 + 1.080 \sin \Lambda - \end{aligned}$$

$$\begin{aligned} &0.034 \tan \frac{\Lambda}{2} - 0.034 \left(\frac{1 - \sqrt{1 + 0.0016 \tan^2 \Lambda}}{\tan \Lambda} \right) - \\ &0.096 \left(\frac{1 - \sqrt{1 + 0.1717 \tan^2 \Lambda}}{\tan \Lambda} \right) \\ p_{13} &= \left(\frac{-0.176}{\cos \Lambda} + 1.026 \tan \Lambda \right) H_1 + 0.129 - 0.869 \sin \Lambda + \\ &0.082 \tan \frac{\Lambda}{2} + 0.014 \left(\frac{1 - \sqrt{1 + 0.0016 \tan^2 \Lambda}}{\tan \Lambda} \right) + \\ &0.068 \left(\frac{1 - \sqrt{1 + 0.0177 \tan^2 \Lambda}}{\tan \Lambda} \right) + \\ &0.034 \left(\frac{1 - \sqrt{1 + 0.1717 \tan^2 \Lambda}}{\tan \Lambda} \right) \\ p_{21} &= \left(\frac{0.221}{\cos \Lambda} + 0.534 \tan \Lambda \right) H_2 - 2.088 - 0.383 \sin \Lambda - \\ &0.018 \tan \frac{\Lambda}{2} - 0.088 \left(\frac{1 - \sqrt{1 + 0.0177 \tan^2 \Lambda}}{\tan \Lambda} \right) - \\ &0.044 \left(\frac{1 - \sqrt{1 + 0.0294 \tan^2 \Lambda}}{\tan \Lambda} \right) - \\ &0.037 \left(\frac{1 - \sqrt{1 + 0.0886 \tan^2 \Lambda}}{\tan \Lambda} \right) \\ p_{22} &= \frac{0.975}{\cos \Lambda} H_2 + 4.596 - 0.146 \sin \Lambda - 0.044 \tan \frac{\Lambda}{2} + \\ &0.125 \left(\frac{1 - \sqrt{1 + 0.0177 \tan^2 \Lambda}}{\tan \Lambda} \right) - \\ &0.044 \left(\frac{1 - \sqrt{1 + 0.0294 \tan^2 \Lambda}}{\tan \Lambda} \right) - \\ &0.125 \left(\frac{1 - \sqrt{1 + 0.0886 \tan^2 \Lambda}}{\tan \Lambda} \right) \\ p_{23} &= \left(\frac{0.221}{\cos \Lambda} - 0.534 \tan \Lambda \right) H_2 - 1.912 + 0.221 \sin \Lambda + \\ &0.107 \tan \frac{\Lambda}{2} - 0.044 \left(\frac{1 - \sqrt{1 + 0.0177 \tan^2 \Lambda}}{\tan \Lambda} \right) + \\ &0.018 \left(\frac{1 - \sqrt{1 + 0.0294 \tan^2 \Lambda}}{\tan \Lambda} \right) + \\ &0.044 \left(\frac{1 - \sqrt{1 + 0.0886 \tan^2 \Lambda}}{\tan \Lambda} \right) \\ p_{31} &= \left(\frac{-0.028}{\cos \Lambda} - 0.164 \tan \Lambda \right) H_3 + 0.149 + 0.324 \sin \Lambda + \\ &0.034 \tan \frac{\Lambda}{2} - 0.082 \left(\frac{1 - \sqrt{1 + 0.1717 \tan^2 \Lambda}}{\tan \Lambda} \right) - \\ &0.163 \left(\frac{1 - \sqrt{1 + 0.0886 \tan^2 \Lambda}}{\tan \Lambda} \right) + \\ &0.197 \left(\frac{1 - \sqrt{1 + 0.1993 \tan^2 \Lambda}}{\tan \Lambda} \right) \\ p_{32} &= \left(\frac{0.136}{\cos \Lambda} + 0.464 \tan \Lambda \right) H_3 - 1.570 - 0.389 \sin \Lambda - \end{aligned}$$

$$\begin{aligned}
& 0.082 \tan \frac{\Lambda}{2} + 0.231 \left(\frac{1 - \sqrt{1 + 0.1717 \tan^2 \Lambda}}{\tan \Lambda} \right) - \\
& 0.082 \left(\frac{1 - \sqrt{1 + 0.1993 \tan^2 \Lambda}}{\tan \Lambda} \right) \\
p_{33} = & \left(\frac{0.628}{\cos \Lambda} - 0.164 \tan \Lambda \right) H_3 + 3.417 + 0.083 \sin \Lambda + \\
& 0.197 \tan \frac{\Lambda}{2} - 0.082 \left(\frac{1 - \sqrt{1 + 0.1717 \tan^2 \Lambda}}{\tan \Lambda} \right) + \\
& 0.163 \left(\frac{1 - \sqrt{1 + 0.0886 \tan^2 \Lambda}}{\tan \Lambda} \right) + \\
& 0.034 \left(\frac{1 - \sqrt{1 + 0.1993 \tan^2 \Lambda}}{\tan \Lambda} \right) \quad (B8)
\end{aligned}$$

Linear spanwise distribution of $(\kappa c)_v/(\kappa c)_{av}$.—With the condition that the product of section lift-curve slope and wing chord varies linearly spanwise, then

$$\kappa c = \frac{2b}{A_* [1 + (\kappa_T/\kappa_R)\lambda]} \{1 - \eta_v [1 - (\kappa_T/\kappa_R)\lambda]\}$$

and equation (3) becomes

$$H_v = d_v (\beta A_*) \frac{1 + (\kappa_T/\kappa_R)\lambda}{2 \{1 - \eta_v [1 - (\kappa_T/\kappa_R)\lambda]\}} \quad (B9)$$

where A_* is the aspect ratio based on the wing chord equal to κc . In equation (B9), H_v is reduced to terms of two parameters. Expressions of A_* in terms of aspect ratio for straight-tapered wings and the distribution of section lift-curve slope can be found.

For straight-tapered wings

$$A = \frac{2b}{c_R (1 + \lambda)}$$

and since κc is linear

$$A_* = \frac{2b}{\kappa_R c_R [1 + (\kappa_T/\kappa_R)\lambda]}$$

then

$$A_* = \frac{A}{(\kappa_R + \kappa_T \lambda)/(1 + \lambda)}$$

and equation (B9) becomes

$$H_v = d_v \left[\frac{\beta A}{(\kappa_R + \kappa_T \lambda)/(1 + \lambda)} \right] \frac{1 + (\kappa_T/\kappa_R)\lambda}{2 \{1 - \eta_v [1 - (\kappa_T/\kappa_R)\lambda]\}} \quad (B10)$$

The distribution of κ for straight-tapered wings is given by

$$\kappa_v = \frac{(\kappa c)_v}{c_v} = \kappa_R \frac{1 - \eta_v [1 - (\kappa_T/\kappa_R)\lambda]}{1 - \eta_v (1 - \lambda)} \quad (B11)$$

Equation (B10) is in terms of two parameters given by $\left[\frac{\beta A}{(\kappa_R + \kappa_T \lambda)/(1 + \lambda)} \right]$ and $(\kappa_T/\kappa_R)\lambda$. Solutions for spanwise loading in terms of these two parameters and Λ_β are valid for the distribution of section lift-curve slope given by equation (B11). Equation (B11) indicates that at $\lambda=1$, κ_v is a linear function and at $\lambda=0$, κ_v is a constant. For values of λ

between 0 and 1, κ_v is a curve in the region between the linear function and a constant.

Equation (B10) is given by figure 2 for $m=7$, but with the ordinate given by the parameter $\frac{H_v}{\beta A/[(\kappa_R + \kappa_T \lambda)/(1 + \lambda)]}$ and the abscissa by $(\kappa_T/\kappa_R)\lambda$.

For the case of linear distribution of $(\kappa c)_v$ and straight-tapered wings for which the chord and section lift-curve slope can be specified in three parameters, the loading and associated aerodynamic characteristics can be presented for a range of the parameters Λ_β , $\beta A/[(\kappa_R + \kappa_T \lambda)/(1 + \lambda)]$, and $(\kappa_T/\kappa_R)\lambda$.

INTEGRATION OF ANTISYMMETRIC LOADING

Rolling-moment coefficient and derivatives.—Rolling-moment coefficient is given by

$$\beta C_l = \frac{\beta A}{2} \int_{-1}^1 G(\bar{\eta}) \bar{\eta} d\bar{\eta} \quad (B12)$$

where

$$\bar{\eta} = \cos \phi$$

which, by an integration formula,

$$\int_{-1}^1 f(\bar{\eta}) d\bar{\eta} = \frac{\pi}{m+1} \sum_{n=1}^m f(\bar{\eta}_n) \sin \phi_n \quad (B13)$$

becomes

$$\begin{aligned}
\beta C_l &= \frac{\pi \beta A}{2(m+1)} \sum_{n=1}^m G_n \cos \phi_n \sin \phi_n \\
&= \frac{\pi \beta A}{4(m+1)} \sum_{n=1}^m G_n \sin 2\phi_n
\end{aligned}$$

Since the loading is antisymmetric, $\frac{G_{m+1}}{2} = 0$, and

$$\beta C_l = \frac{\pi \beta A}{2(m+1)} \sum_{n=1}^{\frac{m-1}{2}} G_n \sin 2\phi_n \quad (B14)$$

For spanwise loading due to rolling, the loading is found as a function of $pb/2V$, then equation (B14) divided by $pb/2V$ gives

$$\beta C_{l_p} = \frac{\pi \beta A}{2(m+1)} \sum_{n=1}^{\frac{m-1}{2}} \bar{G}_n \sin 2\phi_n \quad (B15)$$

where

$$\bar{G} = G/(pb/2V)$$

The rolling moment due to ailerons will be found in appendix C.

Induced drag.—The induced drag coefficient is, with equation (B13), given by

$$\beta C_{D_i} = \beta A \int_{-1}^1 \alpha_i G d\eta = \frac{\pi \beta A}{m+1} \sum_{n=1}^m G_n \alpha_i \sin \phi_n$$

where α_i is one-half the induced angle of the wing wake given by equation (A14) for $c_v = \infty$, then for antisymmetric loading

$$\beta C_{D_i} = \frac{2\pi \beta A}{m+1} \sum_{n=1}^{\frac{m-1}{2}} \left(b_n G_n^2 - G_n \sum_{n=1}^{\frac{m-1}{2}} C_{vn} G_n \right) \sin \phi_n \quad (B16)$$

where the prime indicates the value of $n=\nu$ is not summed.

Spanwise center of pressure.—The center of pressure on the wing half panel is given by

$$\eta_{c.p.} = \frac{\int_0^1 G \bar{\eta} d\bar{\eta}}{\int_0^1 G d\bar{\eta}}$$

The numerator is equal to $\beta C_l / \beta A$. If the Fourier series for loading is assumed,

$G(\phi) = \sum_{\mu_1=\text{even}}^m a_{\mu_1} \sin \mu_1 \phi$, the denominator becomes

$$\sum_{\mu_1=\text{even}}^m a_{\mu_1} \int_0^{\pi/2} \sin \phi \sin \mu_1 \phi d\phi = \sum_{\mu_1=\text{even}}^m -a_{\mu_1} (-1)^{\frac{\mu_1}{2}} \left(\frac{\mu_1}{\mu_1^2 - 1} \right)$$

then

$$\eta_{c.p.} = \frac{\beta C_l}{-\beta A \sum_{\mu_1=\text{even}}^m a_{\mu_1} (-1)^{\frac{\mu_1}{2}} \left(\frac{\mu_1}{\mu_1^2 - 1} \right)} \quad (\text{B17})$$

where a_{μ_1} are the Fourier coefficients.

Loading-due-to-rolling function and interpolation table.—The Fourier series that approximates the antisymmetric loading with only a few terms is given by

$$G(\phi) = \sum_{\mu_1=\text{even}}^m a_{\mu_1} \sin \mu_1 \phi \quad (\text{B18})$$

The loading G_n is determined at span positions of $\bar{\eta} = \cos \phi_n$

where $\phi_n = \frac{n\pi}{m+1}$. The a_{μ_1} are given by

$$a_{\mu_1} = \frac{2}{\pi} \int_0^\pi G(\phi) \sin \mu_1 \phi d\phi \quad (\text{B19})$$

With the quadrature formula of equation (B13), equation (B19) becomes, for antisymmetric loading,

$$a_{\mu_1} = \frac{4}{m+1} \sum_{n=1}^{\frac{m-1}{2}} G_n \sin \mu_1 \phi_n \quad (\text{B20})$$

For $m=7$, the a_{μ_1} coefficients are equal to (for even μ_1)

$$\left. \begin{aligned} a_2 &= \frac{1}{2} \left(\frac{\sqrt{2}}{2} G_1 + G_2 + \frac{\sqrt{2}}{2} G_3 \right) \\ a_4 &= \frac{1}{2} (G_1 - G_3) \\ a_6 &= \frac{1}{2} \left(\frac{\sqrt{2}}{2} G_1 - G_2 + \frac{\sqrt{2}}{2} G_3 \right) \end{aligned} \right\} \quad (\text{B21})$$

Equation (B18) with (B21) can be arranged to give

$$\left. \begin{aligned} G(\phi) &= \frac{1}{2} \left(\frac{\sqrt{2}}{2} \sin 2\phi + \sin 4\phi + \frac{\sqrt{2}}{2} \sin 6\phi \right) G_1 + \\ &\quad \frac{1}{2} (\sin 2\phi - \sin 6\phi) G_2 + \\ &\quad \frac{1}{2} \left(\frac{\sqrt{2}}{2} \sin 2\phi - \sin 4\phi + \frac{\sqrt{2}}{2} \sin 6\phi \right) G_3 \end{aligned} \right\} \quad (\text{B22})$$

With equation (B22) the loading due to rolling can be determined at any span position. Letting $\phi = \phi_k = \frac{k\pi}{8}$ and tabulating the factors of G_n as e_{nk} , an interpolation table may be obtained to determine loading at span station k .

TABLE B1, e_{nk}
[$m=7$]

η_k	0.981	0.831	0.556	0.195
k				
n	1/2	3/2	5/2	7/2
1.....	0.8155	0.5449	-0.1622	0.1684
2.....	-.2706	.6533	.6533	-.2706
3.....	.1084	-.1622	.5449	.8155

$$G_k = \sum_{n=1}^3 e_{nk} G_n \quad (\text{B23})$$

Equation (B23) may be used for interpolation of any form of loading coefficient, thus

$$\left(\frac{c_l c}{c_l c_{av}} \right)_k = \sum_{n=1}^3 e_{nk} \left(\frac{c_l c}{c_l c_{av}} \right)_n \quad (\text{B24})$$

APPENDIX C

DETERMINATION OF ANTISYMMETRIC WING TWIST FOR FINDING SPANWISE LOADING DUE TO AILERON DEFLECTION

WING TWIST FOR A GIVEN AILERON SPAN

The determination of loading for an angle-of-attack distribution that contains a discontinuity by a method which satisfies the boundary conditions at a finite number of points can be made by increasing the number of points until the solutions become sufficiently accurate. For the method as given in appendix A, the number of points that satisfy the boundary conditions is given by m . For the large value of m required for accurate results, the computations become exceedingly laborious; however, a procedure using a moderate value of m can be determined by use of a low-aspect-ratio theory with which a wing twist can be found that duplicates the results of the discontinuous angle-of-attack distribution.

A theoretical but relatively simple method of finding spanwise loading due to inboard and outboard ailerons for wings of low aspect ratio is given by reference 4. In the present theory, as aspect ratio approaches zero, g^*_{vn} values of appendix A become zero and the p_{vn} coefficients given by equation (A16) become constant or independent of plan-form shape and equal to

$$\left. \begin{aligned} p_{vn} &= -2C_{vn} \\ p_{vv} &= 2b_{vv} \end{aligned} \right\} \quad (C1)$$

These coefficients are given by the relations under equation (A15) and p_{vn} can be tabulated.

TABLE C1.— p_{vn}

[For $m=7$ and $A=0$]

$n \backslash v$	1	2	3
1	10.4524	-2.0000	0
2	-3.6954	5.6568	-1.5308
3	0	-2.0060	4.3296

With equation (A15), antisymmetric loading can be found for zero-aspect-ratio wings. As a comment on the accuracy of the present theory for $m=7$, the solution of equation (A15), with the $A=0$ p_{vn} values for loading due to rolling gave the same values at the three semispan stations as does reference 4, namely, $G(\phi) = -\frac{(\dot{p}b/2V) \sin 2\phi}{4}$.

The zero-aspect-ratio theory of reference 4 shows that all span loading characteristics are independent of plan-form shape for zero aspect ratio. This independence makes that theory ideal for obtaining the boundary conditions of the present theory for zero aspect ratio, which should apply with the present theory for higher aspect ratios for which plan-form shape has an effect on spanwise loading. The boundary conditions of the present theory are given by the antisymmetric values of α_v in equation (A15). The problem is to find what antisymmetrical distribution of α_v is required for the present theory to duplicate the exact loading

distribution given by reference 4 for a given aileron span.

The aileron spans are arbitrarily chosen for the present theory as the mean value of the spanwise trigonometric coordinate of the downwash point at a section angle of attack equal to zero. For $m=7$, three aileron spans can be defined for both outboard and inboard ailerons. Let η_a be the aileron span, and θ the spanwise point of the end of the aileron, then

$$\eta_a = 1 - \cos \theta \text{ for outboard ailerons}$$

$$\eta_a = \cos \theta \text{ for inboard ailerons}$$

For the present theory, the aileron spans defined are tabulated as follows:

TABLE C2

Case	Outboard			Inboard		
	I	II	III	IV	V	VI
θ	$\frac{3\pi}{16}$	$\frac{5\pi}{16}$	$\frac{7\pi}{16}$	$\frac{5\pi}{16}$	$\frac{3\pi}{16}$	0
η_a	0.1685	0.4444	0.8049	0.5556	0.8315	1.0000

For the aileron spans listed in table C2, the exact span loading distribution can be found from reference 4. With the p_{vn} values listed in table C1 and the exact values of G_1 , G_2 , and G_3 from reference 4, equation (A15) gives the twist required for the present theory to give the loading distribution for each case listed in table C2 or

$$\left. \begin{aligned} \frac{\alpha_1}{\delta} &= 10.4524 \left(\frac{G_1}{\delta} \right) - 3.6954 \left(\frac{G_2}{\delta} \right) \\ \frac{\alpha_2}{\delta} &= -2 \left(\frac{G_1}{\delta} \right) + 5.6568 \left(\frac{G_2}{\delta} \right) - 2 \left(\frac{G_3}{\delta} \right) \\ \frac{\alpha_3}{\delta} &= -1.5308 \left(\frac{G_2}{\delta} \right) + 4.3296 \left(\frac{G_3}{\delta} \right) \end{aligned} \right\} \quad (C2)$$

The spanwise loading distribution from reference 4 for outboard ailerons is given by

$$\left[\frac{G(\phi)}{\delta} \right]_{\text{outboard}} = \frac{1}{\pi} \left[(\cos \phi - \cos \theta) \ln \left| \frac{\sin \frac{\theta + \phi}{2}}{\sin \frac{\theta - \phi}{2}} \right| - (\cos \phi + \cos \theta) \ln \left| \frac{\cos \frac{\theta + \phi}{2}}{\cos \frac{\theta - \phi}{2}} \right| \right] \quad (C3)$$

For the full-wing-span aileron, $\theta = \frac{\pi}{2}$ or $\eta_a = 1$

$$\left[\frac{G(\phi)}{\delta} \right]_{\eta_a=1} = \frac{2}{\pi} \cos \phi \ln \left| \frac{1 + \sin \phi}{\cos \phi} \right| \quad (C4)$$

For inboard ailerons, with the same value of θ

$$\left[\frac{G(\phi)}{\delta} \right]_{\text{inboard}} = \left[\frac{G(\phi)}{\delta} \right]_{\eta_a=1} - \left[\frac{G(\phi)}{\delta} \right]_{\text{outboard}} \quad (C5)$$

With equations (C3), (C4), and (C5), the spanwise loading G_1 , G_2 , and G_3 at span stations $\phi = \pi/8$, $\pi/4$, and $3\pi/8$, or $\eta = 0.9239$, 0.7071 , and 0.3827 can be tabulated for each of the cases given in table C2.

TABLE C3

$\frac{G_n}{\delta}$	Case I	II	III	IV	V	VI
$\frac{G_1}{\delta}$	0.1136	0.1919	0.2316	0.0454	0.1237	0.2373
$\frac{G_2}{\delta}$	0.0500	.2800	.3851	.1164	.3464	.3964
$\frac{G_3}{\delta}$	.0190	.1022	.3620	.2922	.3754	.3944

The twist distribution required for each case is obtained with equation (C2) and table C3, tabulating

TABLE C4

$\frac{\alpha}{\delta}$	Case I	II	III	IV	V	VI
$\frac{\alpha_1}{\delta}$	1.0029	0.9713	0.9979	0.0444	0.0128	1.0157
$\frac{\alpha_2}{\delta}$	.0174	.9957	.9913	-.0169	.9614	.9788
$\frac{\alpha_3}{\delta}$	.0056	.0139	.9777	1.0868	1.0951	1.1007

With the twist distribution given by table C4, equation (A15) can be used to solve for spanwise loading due to ailerons for any of the six cases.

ROLLING MOMENT DUE TO AILERON DEFLECTION

The rolling moment is given by

$$C_l = \frac{A}{4} \int_0^\pi G(\phi) \sin 2\phi d\phi \quad (C6)$$

For span loading due to ailerons, the loading distribution is distorted sufficiently such that the quadrature formula given by equation (B13) is not sufficiently accurate for $m=7$ to integrate equation (C6). With equation (B18)

$$C_l = \frac{\pi A}{8} a_2 \quad (C7)$$

Expanding equation (B18) for $\phi = \pi/8$, $\pi/4$, and $3\pi/8$, or obtaining G_1 , G_2 , and G_3 in series of a 's, the sum of the G 's gives

$$a = \frac{1}{2} (0.7071G_1 + G_2 + 0.7071G_3) + a_{14} - a_{18} + a_{30} - a_{34} \quad (C8)$$

The higher harmonic coefficients can be put as factors of the G_n . The rolling-moment coefficient becomes

$$C_l = A \left\{ \frac{0.7071\pi}{16} \left[1 + \frac{(a_{14} - a_{18} + a_{30} - a_{34})}{1.2071G_1} \right] G_1 + \frac{\pi}{16} \left[1 + \frac{(a_{14} - a_{18} + a_{30} - a_{34})}{1.2071G_2} \right] G_2 + \frac{0.7071\pi}{16} \left[1 + \frac{(a_{14} - a_{18} + a_{30} - a_{34})}{1.2071G_3} \right] G_3 \right\} \quad (C9)$$

When h_n is defined as the coefficients of G_n

$$C_l = A (h_1 G_1 + h_2 G_2 + h_3 G_3) \quad (C10)$$

The ratio of $(a_{14} - a_{18} + a_{30} - a_{34})$ to G_n can be evaluated by the zero-aspect-ratio theory. It is expected this ratio will not vary appreciably with aspect ratio. From reference 4, the loading series expansion gives for equation (B18)

$$\left. \begin{aligned} \left(\frac{a_{\mu_1}}{\delta} \right)_{\text{outboard}} &= \frac{4}{\pi \mu_1 (\mu_1^2 - 1)} (\cos \theta \sin \mu_1 \theta - \mu_1 \sin \theta \cos \mu_1 \theta) \\ \left(\frac{a_{\mu_1}}{\delta} \right)_{\eta_a=1} &= \frac{-(-1)^{\frac{\mu_1}{2}}}{\mu_1^2 - 1} \\ \left(\frac{a_{\mu_1}}{\delta} \right)_{\text{inboard}} &= \left(\frac{a_{\mu_1}}{\delta} \right)_{\eta_a=1} - \left(\frac{a_{\mu_1}}{\delta} \right)_{\text{outboard}} \end{aligned} \right\} \quad (C11)$$

These high harmonic coefficients are small, but are not negligible for loading due to ailerons. The h_n are tabulated for each of the cases

TABLE C5

h_n	Case I	II	III	IV	V	VI
h_1	0.1398	0.1388	0.1379	0.1462	0.1407	0.1402
h_2	.1994	.1963	.1955	.2004	.1973	.1975
h_3	.1446	.1388	.1382	.1400	.1394	.1397

SPANWISE LOADING DISTRIBUTION

The spanwise loading distributions due to the twist distributions of table C4 are found at three span stations, and, since these loadings are not completely defined by a few terms of the assumed loading series, the values of loading at other span stations cannot be found accurately by direct use of equation (B23) and table B1. For zero-aspect-ratio wings, the spanwise loading distribution due to aileron deflection is given to all span stations by equation (C3). The loading distribution for other than zero-aspect-ratio wings will fluctuate about the value given by equation (C3) in a manner similar to the manner that loading due to rolling varies about the function $\sin 2\phi$ of zero-aspect-ratio theory. Since the interpolation table of equation (B23) applies only to loadings that vary about the function $\sin 2\phi$ the loading due to aileron deflection can be divided by the ratio of equation (C3) to $\sin 2\phi$ and the resulting loading will be approximately given by $\sin 2\phi$.

The zero-aspect-ratio values of equation (C3) can be tabulated as ratios of $\frac{G(\phi)/\delta}{\sin 2\phi}$. Define

$$R_n = \frac{G(\phi_n)/\delta}{\sin 2\phi_n} \quad (C12)$$

The zero-aspect-ratio values of R_n can be tabulated for each aileron-span case considered.

TABLE C6.— R_n

Case	Outboard			Inboard		
	I	II	III	IV	V	VI
η_a	0.1685	0.4444	0.8049	0.5556	0.8315	1.0000
1.....	0.1607	0.2714	0.3275	0.0642	0.1749	0.3358
2.....	.0500	.2800	.3851	.1164	.3464	.3964
3.....	.0269	.1445	.5119	.4132	.5309	.5578

The interpolation series of equation (B23) becomes

$$\left(\frac{\bar{G}}{\bar{R}}\right)_k = \sum_{n=1}^3 e_{nk} \left(\frac{\bar{G}}{\bar{R}}\right)_n \quad (C13)$$

where $\bar{G} = \frac{G}{\delta}$ and e_{nk} are given by table B1. With R_k tabulated, values of loading at span stations $\eta_k = \cos \frac{k\pi}{8}$ are obtained.

TABLE C7.— R_k

η_k	Case k	I	II	III	IV	V	VI
0.981	1/2	0.1738	0.2663	0.3162	0.0559	0.1484	0.3222
.831	3/2	.1056	.2787	.3499	.0802	.2533	.3589
.556	5/2	.0341	.2250	.4365	.2424	.4225	.4566
.195	7/2	.0233	.1212	.5304	.6363	.7343	.7575

REFERENCES

1. Pearson, Henry A., and Jones, Robert T.: Theoretical Stability and Control Characteristics of Wings With Various Amounts of Taper and Twist. NACA Rep. 635, 1938.
2. DeYoung, John, and Harper, Charles W.: Theoretical Symmetric Span Loading at Subsonic Speeds for Wings Having Arbitrary

Plan Form. NACA Rep. 921, 1948. (Formerly NACA TN's 1476, 1491, 1772)

3. Bird, John D.: Some Theoretical Low-Speed Span Loading Characteristics of Swept Wings in Roll and Sideslip. NACA TN 1839, 1949.
4. DeYoung, John: Spanwise Loading for Wings and Control Surfaces of Low Aspect Ratio. NACA TN 2011, 1950.
5. Toll, Thomas A.: Summary of Lateral-Control Research. NACA TN 1245, 1947.
6. Swanson, Robert S., and Priddy, E. LaVerne: Lifting-Surface-Theory Values of the Damping in Roll and of the Parameter Used in Estimating Aileron Stick Forces. NACA ARR L5F23, 1945.
7. Weissinger, J.: The Lift Distribution of Swept-Back Wings. NACA TM 1120, 1947.

TABLE I.—ANTISYMMETRIC INFLUENCE COEFFICIENTS, p_{rn} , BEYOND THE SCOPE OF FIGURE 1

p_{11}										
Λ H_1	-50	-40	-20		20	40	50	60	70	75
0.4										14.22
0.6									15.23	17.99
0.8									18.05	21.84
1.2	15.15					14.69	15.93	18.36	23.86	29.34
1.6	17.05	15.68	15.01	14.78	15.23	16.91	18.77	22.17	29.81	37.09
2.0	18.89	16.91	16.48	16.25	16.96	19.25	21.62	26.03	35.88	
2.4		19.35	17.93	17.73	18.68	21.60	24.46	30.02		
2.8			19.53	19.26	20.46	23.91	27.32	34.07		
3.2			21.19	20.81	22.27	26.22	30.20	38.24		
3.6				22.42	24.08	28.53				
4.0					24.05	25.86	30.84			

p_{12}										
Λ H_1	-50	-40	-20	0	20	40	50	60	70	75
0.6	-1.17									-6.59
0.8	-1.25	-1.11							-6.51	-8.17
1.2	1.70	.26	-1.33					-6.29	-8.71	-9.67
1.6	3.64	1.68	-.52					-7.58	-11.01	-11.18
2.0	3.61	3.12	.28					-6.66	-8.90	-13.23
2.4		4.62	1.08	-1.22				-7.50	-10.24	-15.54
2.8			1.86	-.87				-8.35	-11.57	
3.2			2.61	-.51			-6.70	-9.21	-12.92	
3.6			3.37	-.16			-7.20	-10.09		
4.0				.18			-7.72	-10.95		

REPORT 1057

ANALYSIS OF THE EFFECTS OF BOUNDARY-LAYER CONTROL ON THE TAKE-OFF AND POWER-OFF LANDING PERFORMANCE CHARACTERISTICS OF A LIAISON TYPE OF AIRPLANE¹

By ELMER A. HORTON, LAURENCE K. LOFTIN, Jr., STANLEY F. RACISZ, and JOHN H. QUINN, Jr.

SUMMARY

A performance analysis has been made to determine whether boundary-layer control by suction might reduce the minimum take-off and landing distances of a four-place or five-place airplane or a liaison type of airplane below those obtainable with conventional high-lift devices. The airplane was assumed to have a cruise duration of 5 hours at 60-percent power and to be operating from airstrips having a ground friction coefficient of 0.2 or a combined ground and braking coefficient of 0.4. The payload was fixed at 1500 pounds, the wing span was varied from 25 to 100 feet, the aspect ratio was varied from 5 to 15, and the power was varied from 300 to 1300 horsepower. Maximum lift coefficients of 5.0 and 2.8 were assumed for the airplanes with and without boundary-layer control, respectively. A conservative estimate of the boundary-layer-control-equipment weight was included. The effects of the boundary-layer control on total take-off distance, total power-off landing distance, landing and take-off ground run, stalling speed, sinking speed, and gliding speed were determined.

The more important results of the analysis can be summarized as follows: The absolute minimum total take-off distance which was obtained with an airplane having a low wing loading and a moderately low aspect ratio is not reduced by the addition of boundary-layer control. The effectiveness of boundary-layer control in reducing the total take-off distance for a given maximum speed improves with increasing aspect ratio, and, for wing loadings of 10 pounds per square foot or more and an aspect ratio of 10 or more, the addition of boundary-layer control results in a decrease in the total take-off distance of as much as 14 percent. The total landing distance for a given maximum speed is reduced for all configurations by the use of boundary-layer control. The reduction varies from 25 to 40 percent depending on the wing loading.

The reduction in ground run for take-off was negligible for an aspect ratio of 5 but was of the order of 10 to 30 percent for aspect ratios of 10 and 15; whereas, the reduction in ground run for landing was from 25 to 40 percent for all configurations. The stalling speed for a given maximum speed was reduced 20 to 25 percent for all configurations by the application of boundary-layer control.

For the landing condition, boundary-layer control also reduced the gliding speed but resulted in a slightly higher sinking speed, or vertical velocity, than that for the conventional airplane having the same wing span.

INTRODUCTION

The design of a new airplane usually involves a compromise between several desired high-speed performance characteristics and the practical necessity for operating the airplane in and out of airports of reasonable size. The degree of necessary compromise has been reduced by the use of high-lift devices to increase the maximum lift coefficient. Such devices as leading- and trailing-edge flaps which are now in use on operational aircraft permit the attainment of maximum airplane lift coefficients, power-off, of the order of 2.8 (reference 1). In the belief that much higher airplane maximum lift coefficients would be desirable, numerous wind-tunnel investigations have been made of the effectiveness of boundary-layer control as a means for obtaining high maximum lift coefficients. Airfoil-section maximum lift coefficients as high as 5.5 have been obtained in wind-tunnel tests (see, for example, reference 2), and in a limited flight investigation airplane lift coefficients of 4.2 were obtained (reference 3).

There is, however, some question as to the exact benefits to be derived from the use of the high lift coefficients available with boundary-layer control. In an effort to obtain some idea of the extent to which the high lift coefficients available with boundary-layer control might be useful, an analytical investigation has been made of the effect of lift coefficient on the distance required for a four or five place or liaison type of airplane to take off and land over a 50-foot obstacle.

A liaison type of airplane was selected for the analysis since such an airplane might be expected to operate from small or makeshift airports where take-off and landing distances would be of primary importance. A 1,500-pound payload and sufficient fuel for a 5-hour flight were assumed. The power, wing span, and aspect ratio of the airplane configurations investigated were varied over a wide range.

¹ Supersedes NACA TN 1597, "Analysis of the Effects of Boundary-Layer Control on the Take-Off Performance Characteristics of a Liaison-Type Airplane" by Elmer A. Horton and John H. Quinn, Jr., 1948, and NACA TN 2143, "Analysis of the Effects of Boundary-Layer Control on the Power-Off Landing Performance Characteristics of a Liaison-Type Airplane" by Elmer A. Horton, Laurence K. Loftin, Jr., and Stanley F. Racisz, 1950.

Allowances were made for changes in the gross weight resulting from changes in the airplane configuration and for the weight of the boundary-layer-control equipment. A maximum lift coefficient of 2.8 was assumed for the airplanes without boundary-layer control and a value of 5.0 was assumed to be the highest maximum lift coefficient available with boundary-layer control.

In addition to calculations of the distance required to land and take off over a 50-foot obstacle, the ground-run distance corresponding to landing and take-off, stalling speed, gliding speed, and sinking speed were calculated for all the airplanes. The maximum speed of each airplane configuration was also calculated in order to provide some indication of the relation between high-speed performance and landing and take-off performance. The landing maneuver was assumed to be executed without the use of power.

SYMBOLS

W	airplane gross weight, pounds
w	weight of airplane components, pounds
g	acceleration due to gravity (assumed equal to 32.2), feet per second per second
T	thrust, pounds
T_0	static thrust, pounds
$T_{V_{max}}$	thrust at maximum velocity, pounds
S	wing area, square feet
θ	angle of flight path with respect to ground, degrees
V	velocity, feet per second
$\bar{V}$	average flight velocity during transition arc, feet per second $\left(\frac{V_G + V_s}{2}\right)$
D	total drag, pounds
C_D	airplane drag coefficient (D/qS)
D_0	wing profile drag, pounds
C_{D_0}	wing profile-drag coefficient (D_0/qS)
C_{Di}	induced drag coefficient ($C_L^2/\pi Ae$)
L	total lift, pounds
C_L	airplane lift coefficient (L/qS)
C_{LT}	lift coefficient that would be required for steady level flight at speed $\bar{V}$
$\Delta C_L = C_{L_{max}} - C_{LT}$	
s	horizontal distance, feet
s_i	total take-off distance over 50-foot obstacle, feet
s_L	landing distance from 50-foot obstacle, feet
R	radius of transition arc, feet
q	dynamic pressure, pounds per square foot $\left(\frac{1}{2} \rho V^2\right)$
H	total pressure, pounds per square foot
C_P	pressure coefficient $\left(\frac{H_0 - H_d}{q_0}\right)$
Q	quantity rate of flow, cubic feet per second
C_Q	quantity rate of flow coefficient (Q/SV_0)
P	brake horsepower

A	aspect ratio (b^2/S)
h	altitude at which flare is started, feet
b	span, feet
e	wing efficiency factor based on variation of spanwise loading from an elliptical loading with no ground effect (assumed equal to 0.9)
$A_P = \frac{T_0}{P}$	constants for calculating propeller thrust
$B = \frac{2(T - T_0)}{P \rho V^2}$	
$C = \frac{T_{V_{max}}}{P}$	
η	efficiency factor of blower (assumed equal to 0.9)
μ	ground or braking friction coefficient or both
ρ	mass density of air, slugs per cubic foot
γ	ratio of specific heats at constant volume and constant pressure (1.4 for air)
τ	time, seconds
Subscripts:	
c	conventional airplane
BLC	boundary-layer-control airplane
0	free-stream conditions
d	conditions in boundary-layer-control duct
L	conditions at point of ground contact on landing
max	maximum
u	pay load
G	glide
F	float
g	ground conditions for take-off
1	conditions during ground run of airplane for take-off
B	ground conditions for landing
t	conditions at take-off of airplane
T	transition
s	stalling
opt	optimum conditions

METHOD OF ANALYSIS

In calculating the take-off and landing performance characteristics for the various airplanes, a number of basic assumptions were made concerning the airplane configurations, the aerodynamic characteristics of the wing both with and without boundary-layer control, the method of estimating the weight of the airplane and the auxiliary boundary-layer-control equipment, and the method used in performing the take-off and landing maneuvers. The final comparative results should be unaffected by these assumptions inasmuch as the same assumptions were used for both the conventional and boundary-layer-control airplanes, except for the assumptions concerning the weight of the boundary-layer-control equipment which, in this instance, are believed to be conservative. In general, the assumptions were compatible with data from existing airplanes.

AIRPLANE CONFIGURATION

The airplane was assumed to have a cantilever semimonocoque wing, rectangular in plan form, with airfoil sections tapering from a thickness-chord ratio of 0.18 at the root to 0.12 at the tip. The empennage area was considered to be $0.25S$. The fuselage frontal area F for the constant payload w_u of 1,500 pounds was determined from the following equation obtained from reference 4:

$$F = 0.15w_u^{2/3}$$

The dimensions of the fuselage and landing gear remained constant.

The propeller was considered to be fully automatic in order that maximum engine speed and power could be obtained at all airspeeds. The fuel and oil supplies were assumed sufficient for 5 hours of cruising at 60 percent of maximum power with a specific fuel consumption of 0.50 pound per brake horsepower per hour.

It was assumed that an auxiliary engine and a blower were used to apply suction through the duct provided by the internal space of the semimonocoque wing to the boundary-layer-control slots. The boundary-layer-control apparatus was assumed to have a fuel supply sufficient for the duration of the flight.

AERODYNAMIC CHARACTERISTICS

The variation of wing profile-drag coefficient with lift coefficient, shown in figure 1, was determined from section data contained in references 5 to 8. The data are for the smooth-surface condition of the wings with and without boundary-layer control. The use of boundary-layer suction is seen to cause only relatively small changes in the profile drag in the range of lift coefficients from 0 to 1.6. On a wing provided with suction slots to improve the maximum lift, however, suction through these slots must be maintained in the cruising range of lift coefficients in order that the profile drag will not be increased by outflow through the slots. For

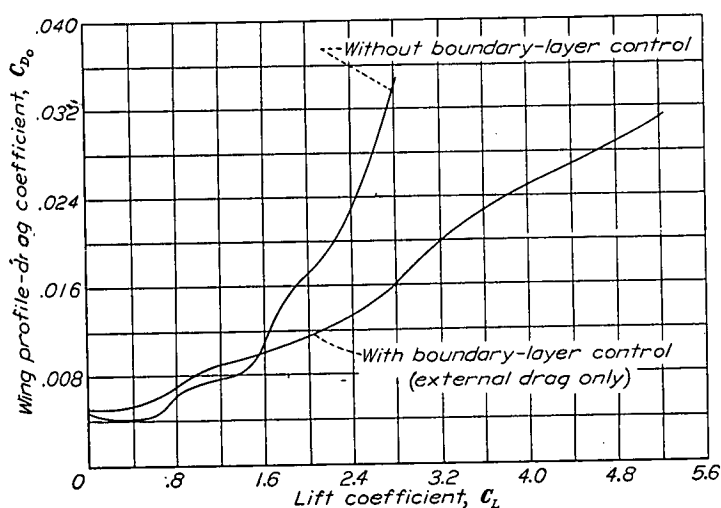


FIGURE 1.—Assumed profile-drag coefficient of the wing with and without boundary-layer control.

this reason, the previously mentioned provision of enough fuel to operate the boundary-layer-control apparatus continuously during the 5-hour flight was considered necessary. The use of a drag polar based on airfoil-section data for the rough-surface condition might represent a more realistic appraisal of the high-speed characteristics of the airplane configurations investigated. Enough data were not available, however, to permit the determination of the drag polar for the rough-surface condition. The assumed empennage drag coefficient based on the empennage area was 0.01 and the assumed fuselage and landing-gear drag coefficients were 0.20 and 0.05, respectively, based on the fuselage frontal area (reference 9). The induced drag coefficients were calculated from the equation

$$C_{D_i} = \frac{C_L^2}{\pi A e}$$

where the value of e was assumed to be 0.9. The maximum attainable lift coefficients were assumed to be 2.8 and 5.0 for the airplane without and with boundary-layer control, respectively.

WEIGHT ANALYSIS

It was found convenient to express the gross weight of the airplane in terms of the wing span, aspect ratio, and power. The relation expressing the gross weight as a function of these variables was found by determining the weights of various airplane components as functions of one or more of the variables. The airplane components are designated by the following subscripts:

m	engine
p	propeller, hub, and engine auxiliaries
g	gasoline and oil
F	fuselage
L	landing gear
E	empennage
w	wing
b	blower
bm	blower engine

The following empirical relations giving the weights of engine, engine auxiliaries, propeller, and hub were determined from an analysis of 65 airplanes and 225 engines ranging from 50 to 2,000 horsepower (references 9 and 10):

$$w_m = P \left(\frac{192}{P-30} + 1.1 \right) \quad (1)$$

$$w_p = P \left(\frac{4.58}{P^{0.68}} + 0.48 \right) \quad (2)$$

The airplane was assumed to have a cruising duration of 5 hours at 60-percent full power with a specific fuel consumption of 0.5 pound per horsepower per hour and an oil requirement of 1 gallon per 16 gallons of gasoline (reference 9). Thus, the weight of gasoline and oil is

$$w_g = 1.62P \quad (3)$$

The empirical relations giving the weight of fuselage, landing gear, empennage, and wing are from reference 9 and are as follows:

$$w_F = 0.172 W^{0.94} \quad (4)$$

$$w_L = 0.067 W^{0.98} \quad (5)$$

$$w_E = 0.25S \quad (6)$$

$$w_w = 0.046 S A^{0.47} \left(\frac{W}{b} \right)^{0.53} \left(\frac{b}{t} \right)^{0.115} \quad (7)$$

For the analysis, a value of $\frac{b}{t} = 35$, which is a representative value for the type airplane considered, was assumed in evaluating equation (7). The ratio of span to root thickness b/t enters in the wing weight equation to the 0.115 power and, since the wing weight is only approximately 15 percent of the gross weight, this ratio could vary appreciably without causing a change in the gross-weight estimate of more than 1 to 2 percent.

A summation of equations 1 to 7 plus the assumed pay load of 1,500 pounds results in the following empirical relation giving the gross weight of the conventional airplane as a function of span, aspect ratio, and horsepower:

$$W_c = P \left(\frac{192}{P-30} + \frac{4.58}{P^{0.68}} + 3.20 \right) + 1500 + 0.172 W^{0.94} + 0.067 W^{0.98} + S \left[0.25 + 0.07 A^{0.47} \left(\frac{W}{b} \right)^{0.53} \right] \quad (8)$$

The gross weight of the boundary-layer-control airplane is then the gross weight of the conventional airplane plus the gross weight of the blower engine w_{bm} and blower w_b ; that is,

$$W_{BLC} = W_c + w_{bm} + w_b \quad (9)$$

The estimate of the blower-engine power was made in terms of the compression ratio, quantity flow, absolute entrance pressure, and blower efficiency by the following expression for an adiabatic gas flow:

$$P_{bm} = \frac{\gamma}{\gamma-1} \frac{H_a Q}{550 \eta} \left[\left(\frac{H_0}{H_a} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right] \quad (10)$$

Reference 2 indicated that sufficient boundary-layer control for a maximum lift coefficient of 5.0 could be obtained with a flow coefficient $C_Q = 0.03$ and a pressure coefficient $C_P = 4.0$. However, in order to make a conservative estimate of the weight of the boundary-layer-control equipment, a flow coefficient of 0.04 and a pressure coefficient of 15.0 (reference 11) were used and, by substitution, equation (10) becomes, for $\eta = 0.9$,

$$P_{bm} = 0.00367 \left(H_0 - 3 \frac{W}{S} \right) \sqrt{WS} \left[\left(\frac{H_0}{H_0 - 3 \frac{W}{S}} \right)^{0.286} - 1 \right] \quad (11)$$

The blower-engine weight was then obtained by assuming an engine weight of 2.5 pounds per horsepower and a flight duration of 5 hours at 60-percent power with a specific fuel consumption of 0.5 pound per horsepower per hour. With these assumptions, the blower-engine weight, including fuel, is

$$w_{bm} = 4 P_{bm} = 0.0147 \left(H_0 - 3 \frac{W}{S} \right) \sqrt{WS} \left[\left(\frac{H_0}{H_0 - 3 \frac{W}{S}} \right)^{0.286} - 1 \right] \quad (12)$$

The weight of the blower was obtained by assuming an axial-flow stator-rotor type constructed of aluminum alloy having a hub-to-tip ratio of 0.6 and an axial velocity of 400 feet per second. The outer casing was assumed to be 0.125 inch thick and 48 inches long, the rotor, blades, and shaft, to be equivalent to a disk 2 inches thick with a diameter 0.8 of the tip diameter, and the stator vanes, to be equivalent to a disk 0.25 inch thick with the same diameter as the complete rotor. With these assumptions, the blower-weight equation was developed and is as follows:

$$w_b = 0.044 \sqrt{WS} + 1.13 (WS)^{0.25} \quad (13)$$

TAKE-OFF PERFORMANCE ANALYSIS

Take-off maneuver.—The take-offs were assumed to be made at full power, with no head wind, and to consist of three phases: (1) an accelerated run on the ground at the attitude for least total resistance until the speed for take-off was reached; (2) the transition arc or period of change of the flight path from ground run to steady climb; and (3) steady climb to an altitude of 50 feet where take-off is considered complete. A sketch illustrating the assumed maneuver is presented in figure 2.

Equations for total take-off distance.—The following equation for the total take-off distance was obtained from reference 12 by combining the expressions giving the distance

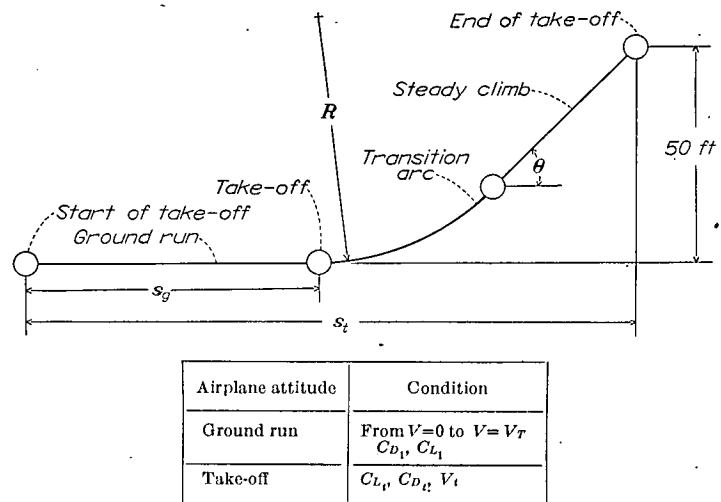


FIGURE 2.—Illustration of assumed maneuver to clear a 50-foot obstacle on take-off.

required for ground run, transition arc, and climb:

$$s_t = \frac{W/S}{\rho g} \left\{ \frac{1}{(\mu C_{L_1} - C_{D_1}) - B \frac{W/S}{W/P}} \log_e \left[1 + \frac{(\mu C_{L_1} - C_{D_1}) - B \frac{W/S}{W/P}}{\left(\frac{A_P}{W/P} - \mu \right) C_{L_1}} \right] + \frac{2 \tan \frac{\theta}{2}}{C_{L_{max}} - C_{L_1}} + \frac{\rho g h}{S \tan \theta} \right\} \quad (14)$$

where

$$\theta = \sin^{-1} \left[\frac{A_P}{W/P} - \left(B \frac{W/S}{W/P} + C_{D_1} \right) \frac{1}{C_{L_1}} \right] \quad (15)$$

and

$$C_{L_1} = 0.9 C_{L_{max}}$$

which is the usual value assumed for C_{L_1} in an analysis of this nature.

The attitude of least air and ground resistance during the ground run, as shown in reference 13, is defined by the expression:

$$C_{L_1} = \frac{1}{2} \mu \pi A e \quad (16)$$

In using equation (16) in the analysis, the profile-drag variation is neglected. The assumed ground friction coefficient $\mu=0.2$ is equivalent to that of deep grass or sand. A lower value of μ corresponding to that of concrete would reduce the take-off distance of both the conventional and

boundary-layer-control airplanes by approximately the same percentage; thus the comparative results would be equal to those given in this report.

The power constants A_P and B used in equations 14 and 15 were obtained from reference 12 and are reproduced herein as figure 3. Use of figure 3 requires determination of V_{max} as a function of span, which was done by equating thrust to airplane drag as follows:

$$T_{V_{max}} = \frac{1}{2} \rho V_{max}^2 \frac{b^2}{A} C_D \quad (17)$$

where C_D is the summation of the assumed drags of the airplane components in coefficient form. Also, from reference 12,

$$T_{V_{max}} = CP \quad (18)$$

where, from figure 3,

$$C = 3.09 - 0.005 V_{max} \quad (19)$$

equation (17) can then be expressed as

$$P(3.09 - 0.005 V_{max}) = \frac{1}{2} \rho V_{max}^2 \frac{b^2}{A} C_D \quad (20)$$

From this equation V_{max} as a function of span for various powers and aspect ratios was obtained for both the conventional and boundary-layer-control airplane, and the results are given in figures 4 and 5. Once V_{max} is known as a function of span, the power constants A_P and B are obtained for the various spans from figure 3.

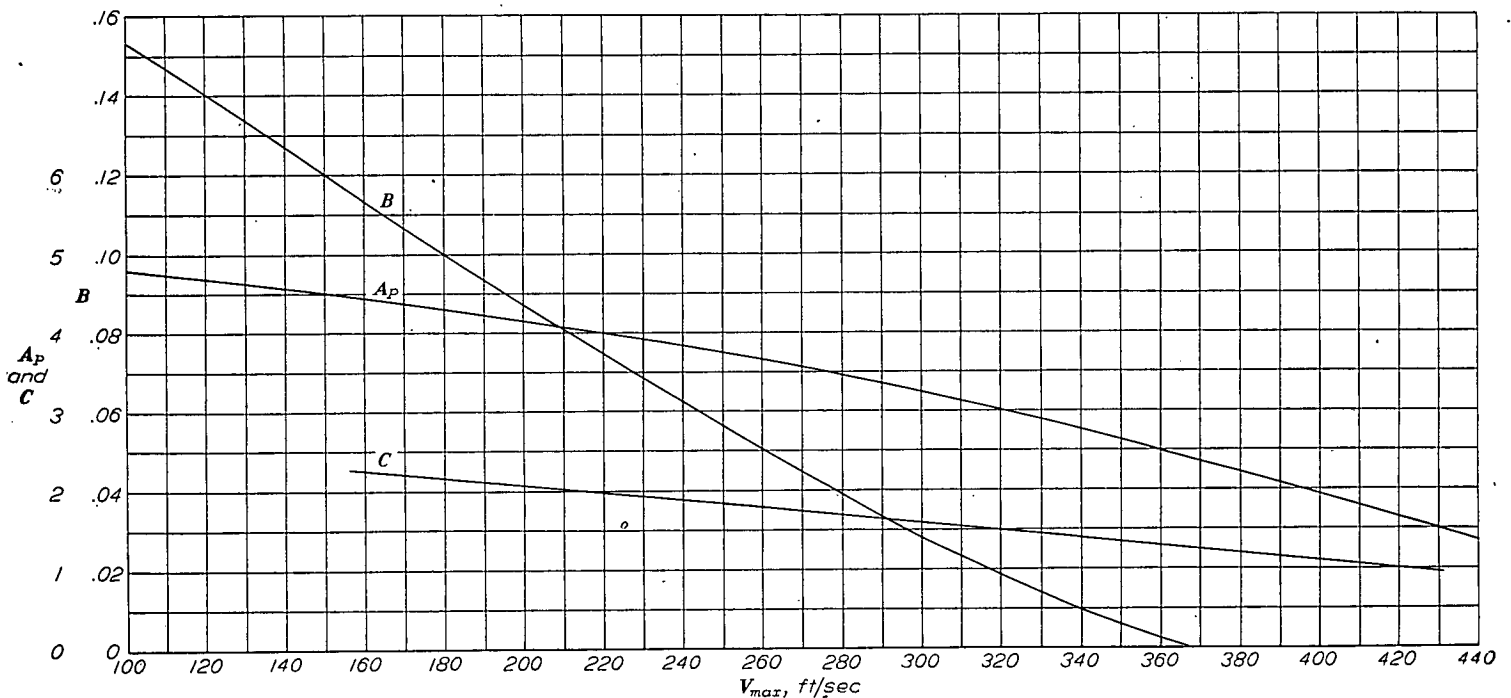


FIGURE 3.—Thrust factors as functions of maximum speed for automatic propellers (reference 12).

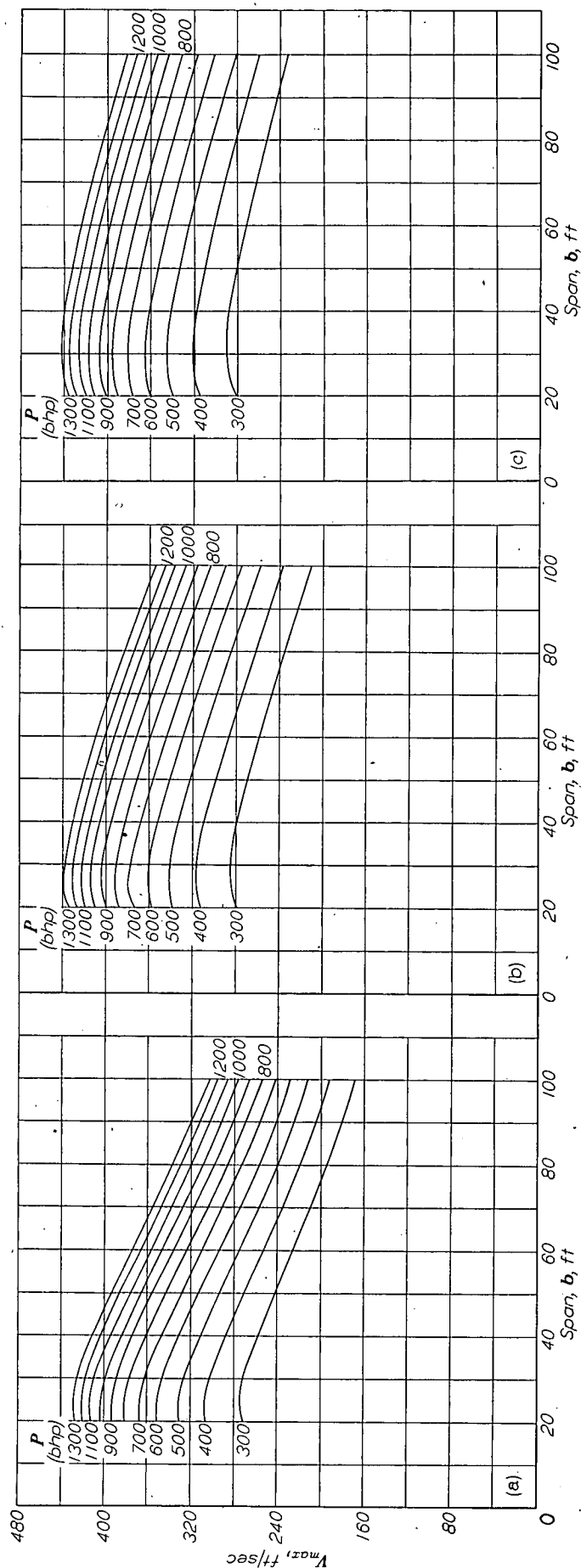
(a) $A=5$.(b) $A=10$.

FIGURE 4.—Maximum speed of assumed airplane without boundary-layer control as a function of span for various powers.

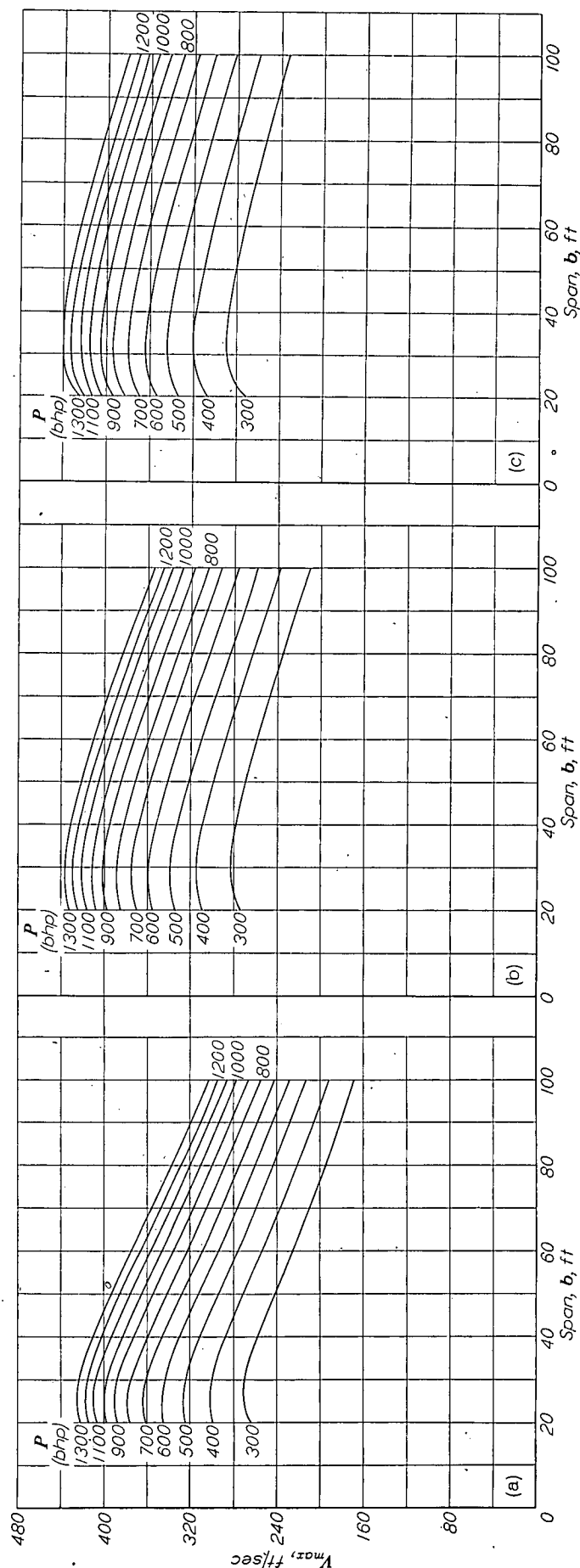
(a) $A=5$.(b) $A=10$.

FIGURE 5.—Maximum speed of assumed airplane with boundary-layer control as a function of span for various powers.

Ground-run distance and stalling speed.—The ground run required for take-off was calculated by use of the following expression from reference 12:

$$s_g = \frac{W/S}{\rho g \left[(\mu C_{L_1} - C_{D_1}) - B \frac{W/S}{W/P} \right]} \log_e \left[1 + \frac{(\mu C_{L_1} - C_{D_1}) - B \frac{W/S}{W/P}}{\left(\frac{A_P}{W/P} - \mu \right) C_{L_1}} \right] \quad (21)$$

The stalling speed V_s was found for each airplane from the relation

$$V_s = \sqrt{\frac{2W}{\rho S C_{L_{max}}}} \quad (22)$$

LANDING PERFORMANCE ANALYSIS

Landing maneuver.—The landing maneuver was considered to consist of four phases: (1) the steady glide, (2) a transition path executed at maximum lift coefficient to bring the airplane from a steady glide to level flight, (3) a floating period of 2 seconds to allow for lag in control response and for the application of brakes (see reference 14), and (4) the ground run. The beginning of the landing was considered as the point at which the altitude was 50 feet; the total landing distance was considered to be the horizontal distance from this point to the end of the ground run. The maneuver was considered to be performed without the use of power—that is, no propeller drag or thrust—and with no wind. A sketch illustrating the assumed maneuver is presented in figure 6.

Basic assumptions.—In calculating the total landing distance, certain simplifying assumptions were made in connection with the manner in which the transition from the steady-glide speed and attitude to level-flight speed and attitude was executed. These assumptions were based on the concept that the horizontal distance covered during the transition period for the type of airplane considered is a relatively small portion of the total landing distance so that a precise determina-

tion of the transition path is not required. The simplifying assumptions were:

1. The airplane was assumed to execute the transition at maximum lift coefficient and the transition path was assumed to be represented by an arc of constant radius. This assumption implies, of course, a constant speed during the transition.

2. Although a constant speed was assumed for the transition arc, it is, of course, obvious that in the actual case the speed during the transition must vary from the steady-glide speed to the landing speed. The constant speed implied by the assumption of a transition arc of constant radius was determined by assuming a linear variation in speed from the steady-glide speed to the stalling speed and taking the constant speed as the arithmetic mean of these two values. This assumption implies a constant decelerating force during the transition.

These assumptions are somewhat similar to those found in approximate methods for calculating the transition path following take-off (reference 12). Such approximate methods for calculating the take-off distance have been found to give good results and, in those cases for which experimental data were available, the method outlined for calculating the landing distance was also found to give good results.

Development of landing equations.—On the basis of assumptions 1 and 2 the following equations for the total landing distance can be derived. The horizontal distance covered during the transition arc s_T is considered first. Reference to figure 6 shows that

$$s_T = R \sin \theta_G \quad (23)$$

where θ_G is the angle of steady glide and R is the radius of the transition arc. The instantaneous radius of curvature during a pull-up at maximum lift is given by the expression

$$R = \frac{2}{\rho g} \frac{W}{S} \frac{1}{C_{L_{max}} - C_{L_T} \cos \theta} \quad (24)$$

where C_{L_T} in this equation corresponds to the lift coefficient for unaccelerated level flight at the velocity at which the pull-up is being executed and θ is the instantaneous flight-path angle. If the cosine of the glide-path angle is assumed to be 1.0, equation (24) can be written as follows:

$$R = \frac{2}{\rho g} \frac{W}{S} \frac{1}{\Delta C_L} \quad (25)$$

where ΔC_L is the difference between the maximum lift coefficient and the lift coefficient corresponding to the previously defined mean speed used during the transition. Since the stalling speed is known, the value of the steady-glide speed V_G is all that is required for the determination of R and the horizontal distance covered during the transition. The value of V_G must be chosen in such a way that the time required for the velocity to decrease from V_G to V_L is the same as the time required for the airplane to traverse the distance s_T . The tangential forces acting on the airplane during the transition arc are composed of the drag which is a decelerating force and the component of weight along the flight path which is an accelerating force. The mean decelerating drag force

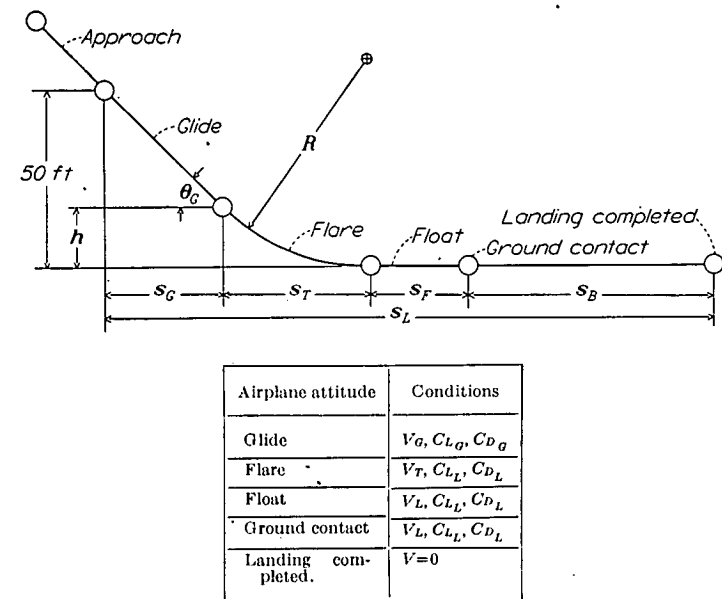


FIGURE 6.—Illustration of assumed maneuver to clear a 50-foot obstacle in landing.

D_T is determined from the drag coefficient at the maximum lift coefficient and the mean speed $\bar{V}$.

There is, however, an accelerating force which may be determined in the following manner. At the end of the steady glide the following relation holds:

$$D_G = W \sin \theta_G$$

where D_G is the drag in the steady glide. Since the glide angle θ_G is usually small, θ varies in a nearly linear manner with s_T during the transition, and since $\sin \theta$ also varies in a nearly linear manner with θ for small values of θ , the mean accelerating force during the transition may be written as

$$\frac{W \sin \theta}{2} = \frac{D_G}{2}$$

Therefore, the time required for the airplane to decelerate from the steady-glide speed V_G to the landing speed V_L is then given by the following expression:

$$\tau = \frac{W}{g} \frac{V_G - V_L}{D_T - \frac{D_G}{2}} \quad (26)$$

If the cosine of the flight-path angle is considered to be unity, the time required to traverse the distance s_T is

$$\tau = \frac{s_T}{\bar{V}} \quad (27)$$

where $\bar{V}$ is the mean speed. Since the two intervals of time expressed by equations (26) and (27) must be equal, the distance s_T may be expressed in the following form:

$$s_T = \frac{2\bar{V}(V_G - V_L)W}{g(2D_T - D_G)} \quad (28)$$

If

$$\sin \theta_G = \sin \tan^{-1} \left(\frac{C_D}{C_L} \right)_G = \left(\frac{C_D}{C_L} \right)_G$$

the distance s_T as given by equations (23) and (25) may be written

$$s_T = \frac{2}{\rho g} \frac{W}{S} \frac{1}{\Delta C_L} \left(\frac{C_D}{C_L} \right)_G \quad (29)$$

A simultaneous solution of equations (28) and (29) gives, after some algebraic manipulation, the following equation:

$$\begin{aligned} & \left[\left(\frac{C_D}{C_L} \right)_L \left(\frac{C_D}{C_L} \right)_G - 6 \right] \left(\frac{V_L}{V_G} \right)^3 + \left[3 \left(\frac{C_D}{C_L} \right)_G \left(\frac{C_D}{C_L} \right)_L + 10 \right] \left(\frac{V_L}{V_G} \right)^2 + \\ & \left[\left(3 - 2 \frac{C_{D_G}}{C_{D_L}} \right) \left(\frac{C_D}{C_L} \right)_G \left(\frac{C_D}{C_L} \right)_L - 2 \right] \left(\frac{V_L}{V_G} \right) + \\ & \left[\left(1 - 2 \frac{C_{D_G}}{C_{D_L}} \right) \left(\frac{C_D}{C_L} \right)_G \left(\frac{C_D}{C_L} \right)_L - 2 \right] = 0 \end{aligned} \quad (30)$$

An exact solution of equation (30) for $\frac{V_L}{V_G}$ requires additional analytic expressions relating $\left(\frac{C_D}{C_L} \right)_G$ and $\frac{C_{D_G}}{C_{D_L}}$ to $\frac{V_L}{V_G}$. Such relations can, of course, be found by expressing the drag polars for the various airplanes in analytic form. It was found more convenient, however, to perform a simultaneous solution of equations (28) and (29) by a trial and error process. Once the correct value of V_G is determined from equations (28) and (29), the horizontal distance covered in the transition arc is easily calculated for a particular airplane from equation (29).

The horizontal distance s_G , covered in the steady glide from a height of 50 feet to the height h at which the transition is begun, can be calculated by the following equations (see fig. 6):

$$s_G = \frac{50 - h}{\tan \theta_G} = \frac{50 - h}{\left(\frac{C_D}{C_L} \right)_G}$$

However,

$$h = R(1 - \cos \theta_G)$$

so that

$$s_G = \frac{50 - R \left[1 - \cos \tan^{-1} \left(\frac{C_D}{C_L} \right)_G \right]}{\left(\frac{C_D}{C_L} \right)_G} \quad (31)$$

The values of R and $\left(\frac{C_D}{C_L} \right)_G$ are already known from the previous calculations of the transition path so that s_G may be readily determined. The distance covered during the floating period is merely

$$s_F = 2 V_L = 2 \sqrt{\frac{2W}{\rho S C_{L_{max}}}} \quad (32)$$

The equation for determining the ground run or braking distance, obtained from reference 15, is

$$s_B = \frac{V_L^2}{2g \left(\frac{C_D}{C_L} - \mu \right)} \log_e \left(\frac{C_D/C_L}{\mu} \right) \quad (33)$$

where $\frac{C_D}{C_L}$ is the lift-drag ratio for the maximum-lift condition and V_L corresponds to the stalling speed. The combined ground and braking friction coefficient was assumed to be 0.4. This value of the friction coefficient can be obtained with cinders on ice. (See reference 14.) The ground effect on the induced drag was neglected. The total landing distance is obtained from a summation of the horizontal distances covered during the four phases of the maneuver:

$$s_L = s_G + s_T + s_F + s_B \quad (34)$$

where these four components are calculated by means of equations (31), (29), (32), and (33), respectively.

SCOPE OF CALCULATIONS

The landing and take-off performance characteristics calculated included the total landing and take-off distances, the ground run corresponding to landing and take-off, the stalling speed, the gliding speed, and the sinking speed. The airplanes for which the landing and take-off performances were calculated had wing spans varying from 25 feet to 100 feet, engine brake horsepower varying from 300 to 1,300, and aspect ratios of 5, 10, and 15. As previously stated, the wing span, aspect ratio, and power determine the weight of an airplane and the airplane configuration. The actual values of the engine horsepower for which calculations were made were somewhat different for the landing and take-off analysis; however, the range of horsepower covered in the two analyses was the same. The landing and take-off performances were calculated for each airplane with and without boundary-layer control. The highest attainable value of the maximum lift coefficient was assumed to be 2.8 for the airplanes without boundary-layer control and, 5.0 for the airplanes with boundary-layer control. The landing performance calculations were made only for lift coefficients of 2.8 and 5.0. The take-off performance calculations, however, were made for a number of lift coefficients in order to determine the optimum value for each configuration. The effect of the additional weight of the boundary-layer-control equipment on the take-off performance characteristics was isolated by calculating the take-off performance characteristics of the boundary-layer-control airplane with and without the additional weight of the boundary-layer-control equipment included in the gross-weight estimate. This calculation was made for wings of aspect ratio 10 only inasmuch as the effect would be relatively the same for other aspect ratios.

Data defining the range of airplane configurations for which the performance calculations were made are presented in figures 7 and 8 for the airplanes without and with boundary-layer control, respectively. These data were obtained by cross-plotting the data derived from equations 8, 12, and 13, examples of which are given in figures 9 and 10 for the aspect-ratio-10 configuration. From figures 7 and 8 it is seen that the wing loading of the airplanes investigated varied from about 4 pounds per square foot to 160 pounds per square foot without boundary-layer control and, from 4 pounds per square foot to 180 pounds per square foot with boundary-layer control.

The maximum velocity of the different airplane configurations without and with boundary-layer control was calculated in order to provide a basis of comparison for the high- and low-speed performances and is given in figures 4 and 5. From figures 4 and 5 it is seen that for a given wing span, aspect ratio, and brake horsepower of the main propulsive unit the maximum velocities of the airplanes with and without boundary-layer control are nearly the same. The slight variation in speed is due to the additional weight of the boundary-layer-control equipment which increases the wing loading and thus the lift and drag coefficient for any given speed and also the small extent to which the drag polars of

the airplanes with and without boundary-layer control differ in the low lift-coefficient range (fig. 1).

RESULTS AND DISCUSSION

The discussion is intended to show the effects of increasing the maximum lift coefficient by boundary-layer control upon the landing and take-off performance characteristics and upon the relation between high-speed performance and the take-off and landing performance as the airplane configuration is varied. The pertinent take-off and landing performance characteristics are presented in terms of the wing span, power, and aspect ratio for the airplanes with and without boundary-layer control. The choice of variables employed in presenting the data was arbitrary to some extent. Although other parameters could have been employed, span, aspect ratio, and power were chosen because these variables indicate the physical size and practicability of the airplane. In some cases, the performance parameters were plotted against wing loading or power loading as well as wing span because their use tended to clarify the results. The effect upon the take-off characteristics of increasing the maximum lift coefficient by boundary-layer control is discussed first.

TAKE-OFF CHARACTERISTICS

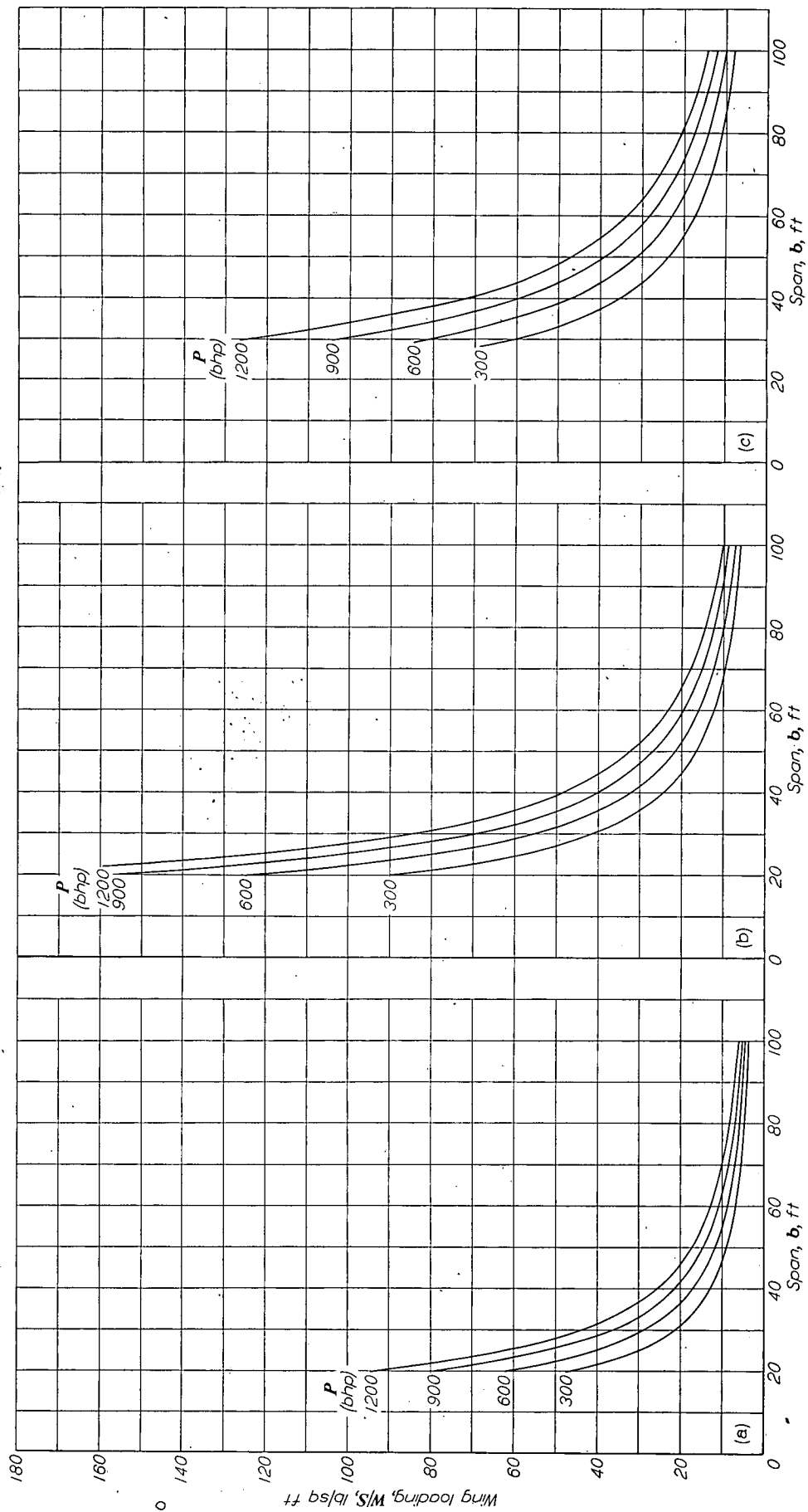
The take-off characteristics to be discussed are:

- (1) The total take-off distance
- (2) The ground run and stalling speeds

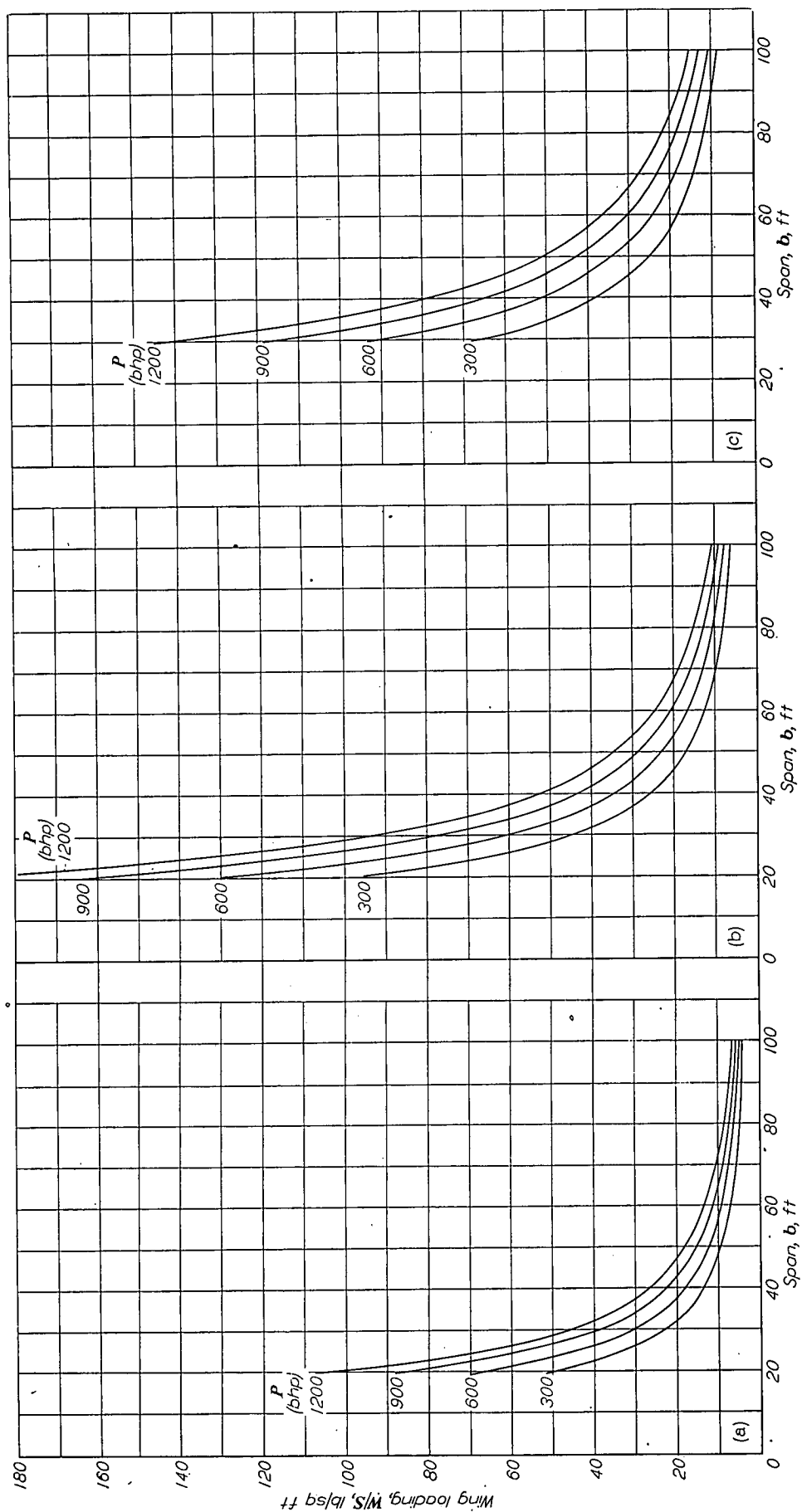
TOTAL TAKE-OFF CHARACTERISTICS

Examples of the variations of total take-off distance of the boundary-layer-control airplane with maximum lift coefficient for various spans and horsepower at an aspect ratio of 10 are presented in figure 11. For a given aspect ratio, the lift coefficient for minimum take-off distance increases as the span decreases and the wing loading increases. These results were cross-plotted in figure 12 to show the variation of optimum C_L with wing loading for the various aspect ratios and horsepower. The figure shows that at an aspect ratio of 5, regardless of wing loadings, the optimum lift coefficient is less or slightly greater than that available with conventional high-lift devices. For aspect ratios of 10 and 15 and wing loadings of less than 10 pounds per square foot, although the optimum maximum lift coefficient for take-off exceeds the maximum lift coefficient attainable without boundary-layer control, the use of lift coefficients greater than 2.8 will decrease the take-off distance very little. (See fig. 11.) For the larger wing loadings, however, the rate of change of the take-off distance with lift coefficient is large and the use of the optimum lift coefficient offers a considerable decrease in take-off distance.

Throughout the remainder of the analysis, the effects of other variables on total take-off distance are discussed for the optimum lift coefficient unless it exceeds 5.0, in which case the take-off distance was calculated for a maximum lift coefficient of 5.0.



(a) $A=5$.
 (b) $A=10$.
 (c) $A=15$.
 FIGURE 7.—Wing loading of assumed airplane without boundary-layer control as a function of span for various powers.



(a) $A=5$.
(b) $A=10$.
(c) $A=15$.
FIGURE 8.—Wing loading of assumed airplane with boundary-layer control as a function of span for various powers.

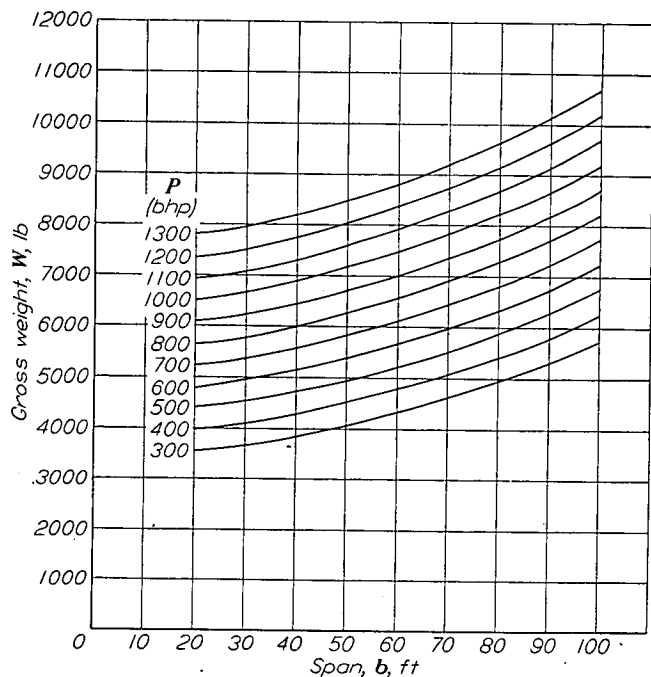


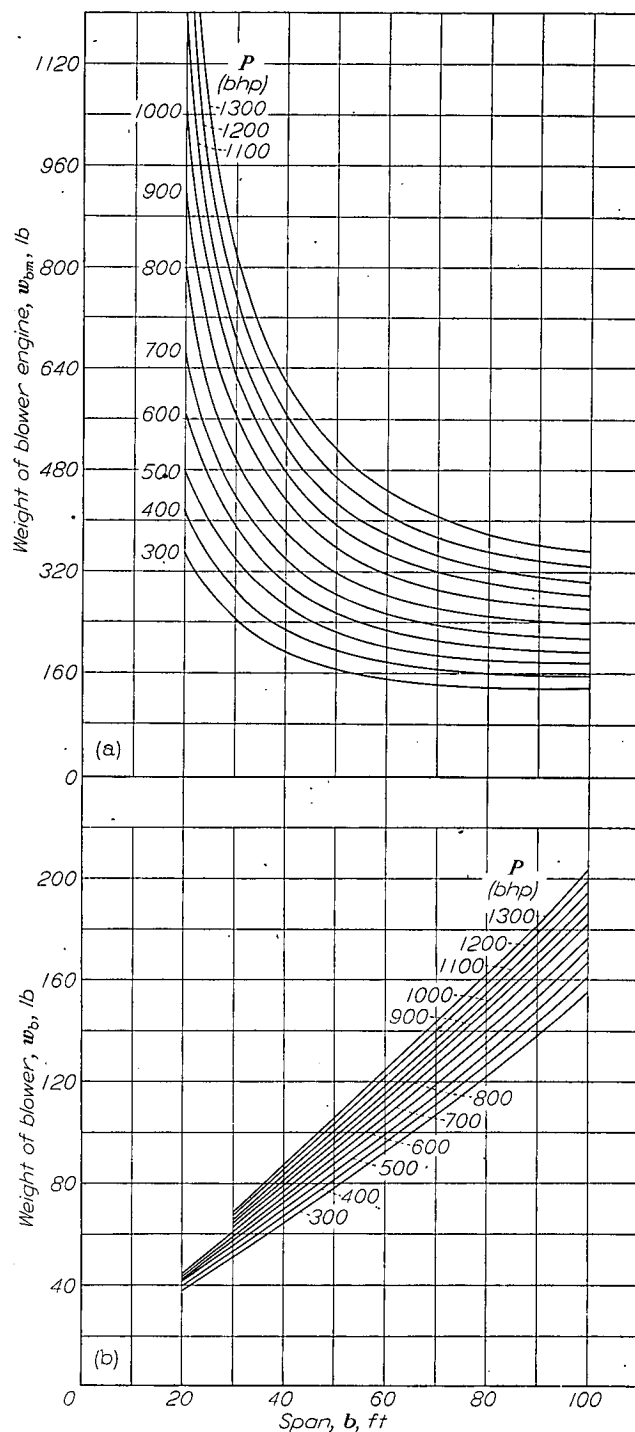
FIGURE 9.—Gross weight of conventional airplane as a function of span for various powers. $A=10$.

Effect of boundary-layer control on take-off.—The variation of take-off distance with span for various horsepowers is presented for aspect ratios of 5, 10, and 15 in figures 13 and 14 for the conventional and boundary-layer-control airplanes, respectively. The effect of the weight of the boundary-layer-control equipment on the take-off characteristics was found for an aspect ratio of 10 by assuming that no weight was added by the auxiliary blower and motor. These data are presented in figure 15.

The effect of boundary-layer control on the total take-off distance of the airplane may be seen in figure 16, which shows the total take-off distance as a function of maximum speed for both the conventional and boundary-layer-control airplanes with varying aspect ratio and horsepower. Figure 16 shows that for a given maximum speed and an aspect ratio of 5, regardless of span, the boundary-layer-control airplane generally requires more distance for take-off than the conventional airplane. As the aspect ratio increases, however, boundary-layer control becomes more effective, and for an aspect ratio of 10 or more with a wing loading of 10 pounds per square foot or more the addition of boundary-layer control decreases the total take-off distance. It follows that, for a given take-off distance, the boundary-layer-control airplane would have a greater maximum speed.

The effect of the weight of the boundary-layer-control equipment on the total take-off distance is shown in figure 16 (b) for aspect ratio of 10. This figure shows that the total take-off distance may be decreased appreciably by decreasing the weight; therefore, every effort should be made to decrease the weight of the boundary-layer-control equipment.

Figure 16 also shows that the absolute minimum total take-off distance obtained with a low wing loading and moderately low aspect ratio is not decreased by the addition of boundary-layer control.



(a) Weight of blower engine.
(b) Weight of blower.

FIGURE 10.—Weight of boundary-layer-control equipment as a function of span for various main engine powers. $A=10$.

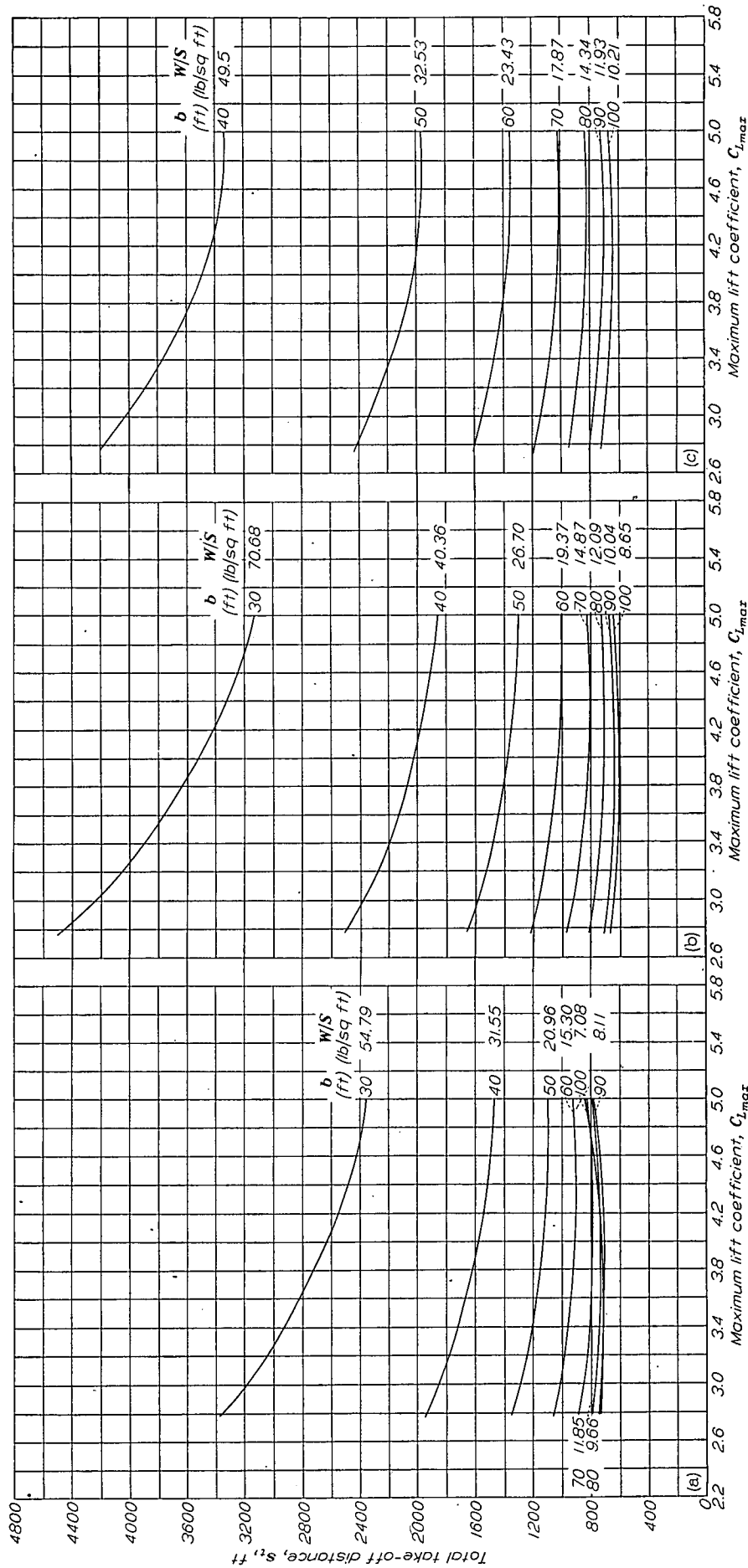


FIGURE 11.—Total take-off distance of an airplane with boundary-layer control as a function of maximum lift coefficient for various spans and wing loadings. $A=10$.

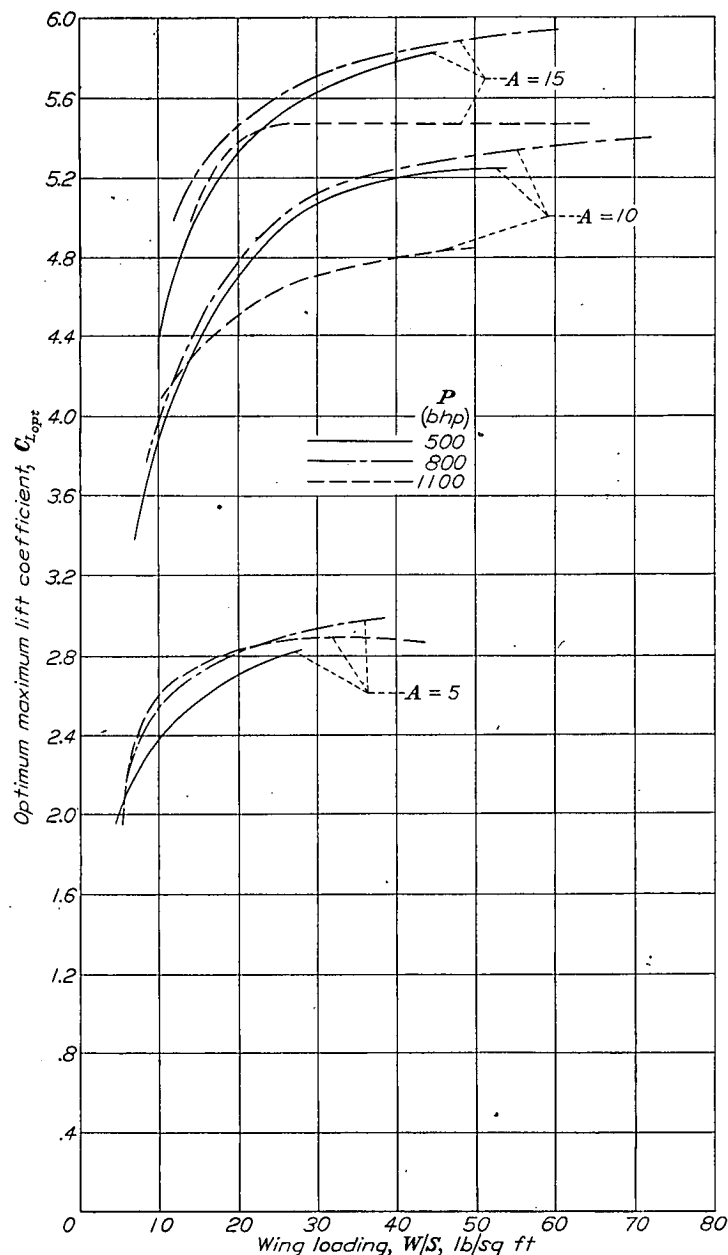


FIGURE 12.—Optimum maximum lift coefficient for minimum total take-off distance of an airplane with boundary-layer control as a function of wing loading for various aspect ratios and powers.

Effect of power loading on take-off distance.—The power loading is shown as a function of take-off distance for various wing loadings and aspect ratios in figures 17 and 18 for the conventional and boundary-layer-control airplanes, respectively. As is shown, the optimum power loading, which is nearly independent of wing loading and aspect ratio, is approximately 8.5 and 9.0 pounds per horsepower for the conventional airplane and the boundary-layer-control airplane, respectively. It should be noted that increasing the horsepower above the optimum value increases the take-off distance. This result is due to the accompanying change in engine, fuel, and structural weight.

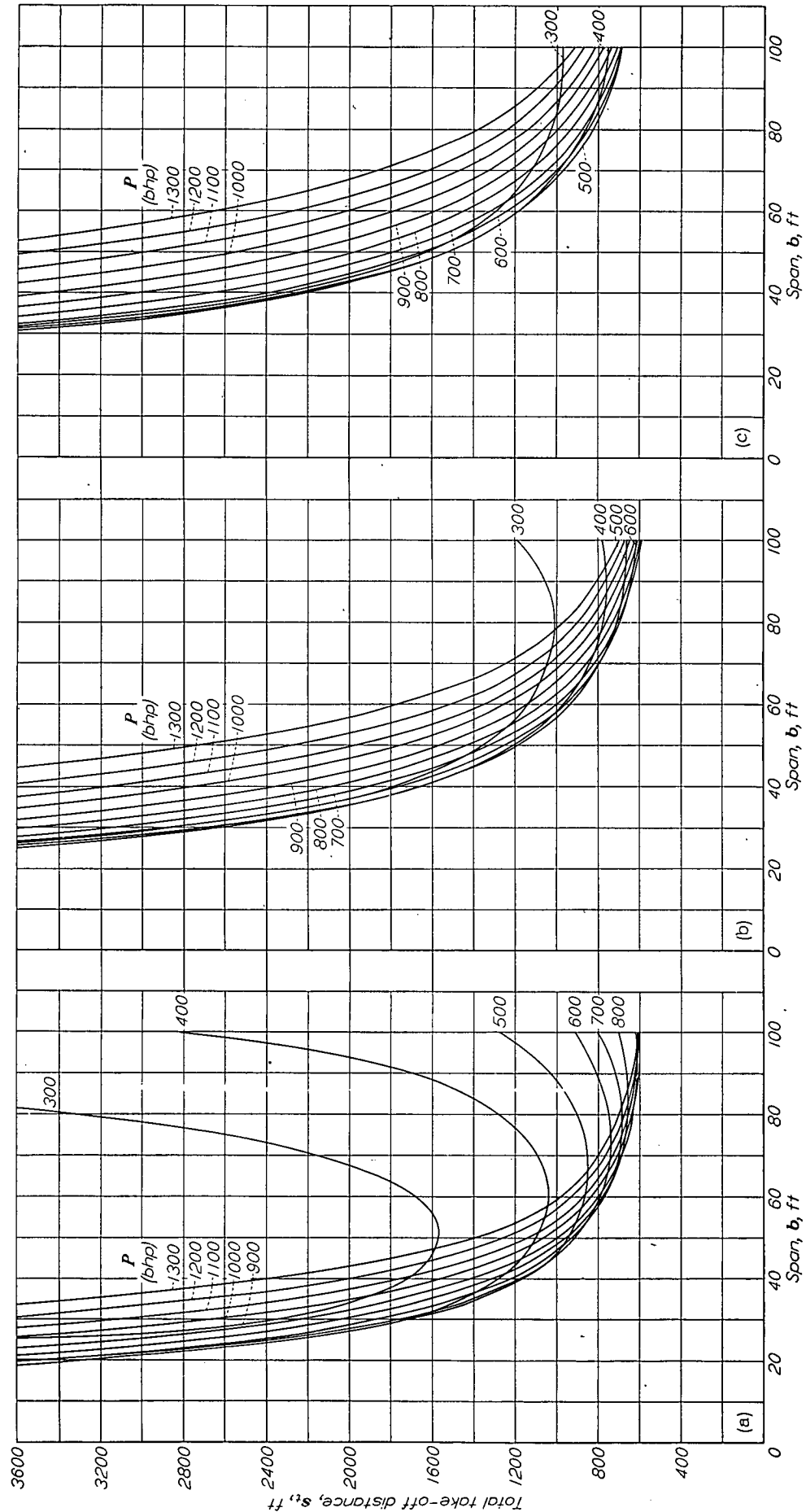
GROUND-RUN AND STALLING-SPEED CHARACTERISTICS

In order to obtain the minimum ground run, which is given in figures 19 and 20, the calculations were made by considering the ground run completed when a speed was reached corresponding to a flying speed at 0.9 of the assumed maximum lift coefficient. During the analysis, it was found that, because the induced drags were large for aspect ratio of 5 of the boundary-layer-control airplane, the power was insufficient to maintain level flight at lift coefficients greater than 3.8; therefore, the ground run for an aspect ratio of 5 was calculated for a maximum lift coefficient of 3.8. The variation of ground run with span for various horsepower and aspect ratios is shown in figures 19 and 20 for the conventional and boundary-layer-control airplanes, respectively, and in figure 21 for the airplane with boundary-layer control but with the weight of the additional equipment disregarded. These data are compared in figure 22 where the ground run has been plotted as a function of V_{max} for various horsepower and aspect ratios.

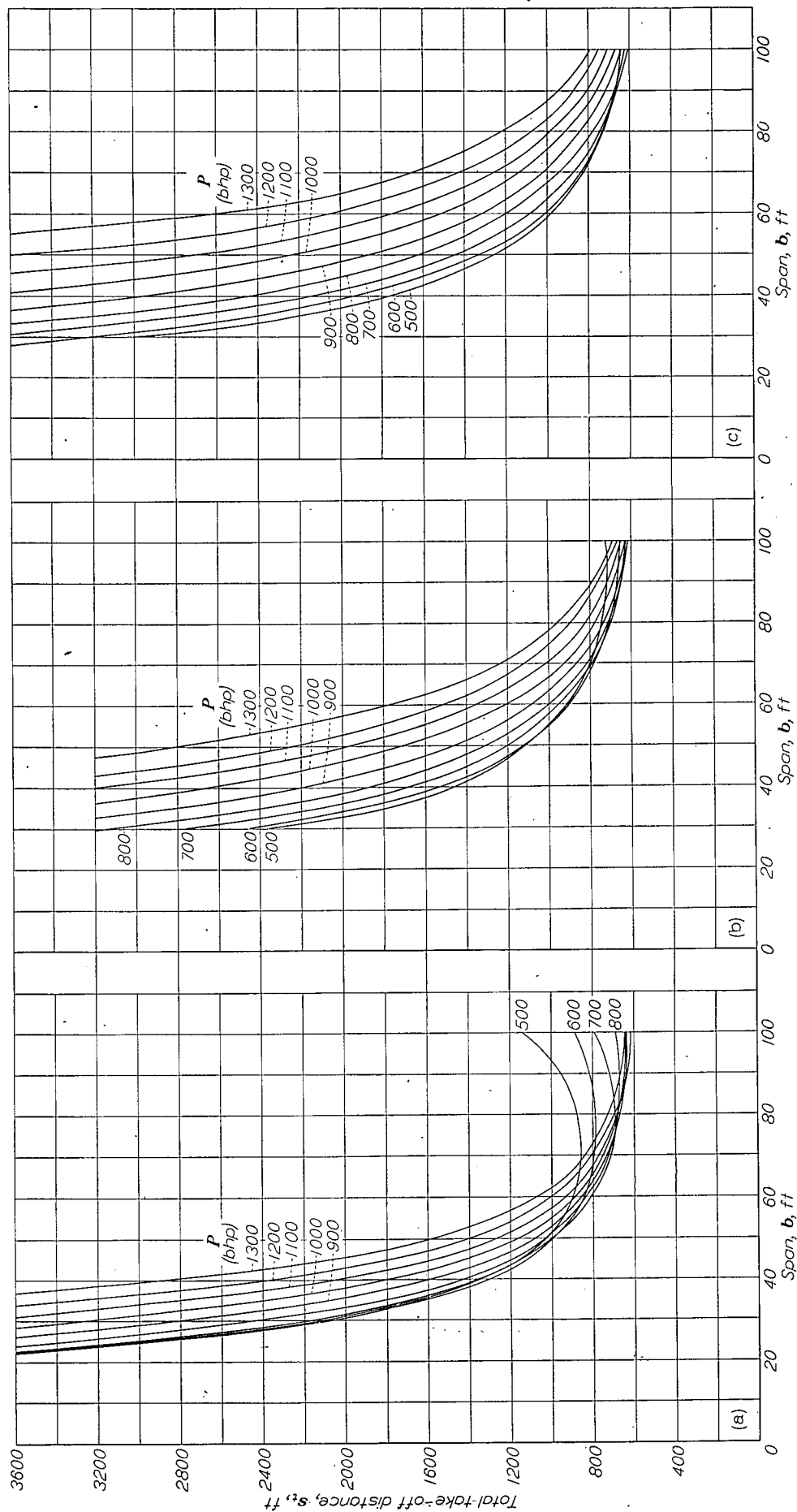
The boundary-layer-control airplane had shorter ground runs than the conventional airplane for all configurations considered. The reduction was negligible for an aspect ratio of 5 and a maximum lift coefficient of 3.8. At aspect ratios of 10 and 15 and maximum lift coefficient of 5.0, however, the ground run was decreased 10 to 30 percent by the addition of boundary-layer control. The beneficial effect of reducing the boundary-layer-control-equipment weight, as previously noted for the total take-off distance, was again observed for the case of the ground run (fig. 22 (b)).

This reduced ground run produced by use of high maximum lift coefficients associated with boundary-layer control may prove to be most advantageous for carrier-based airplanes or seaplanes.

The stalling speed V_s is presented as a function of maximum speed in figure 23 for various aspect ratios and horsepower. The stalling speed was 20 to 25 percent less for the boundary-layer-control airplane than for the conventional airplane for all configurations considered.



(a) $A=5$.
(b) $A=10$.
(c) $A=15$.
FIGURE 13.—Total take-off distance of an airplane without boundary-layer control as a function of span for various powers.



(a) $A=5$.
 (b) $A=10$.
 (c) $A=15$.
 FIGURE 14.—Total take-off distance of an airplane with boundary-layer control as a function of span for various powers.

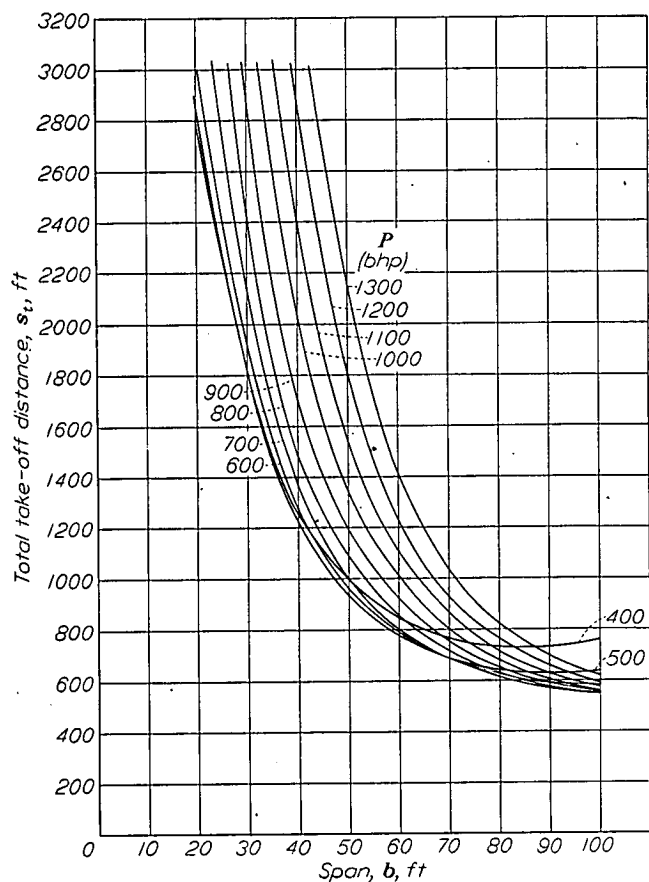
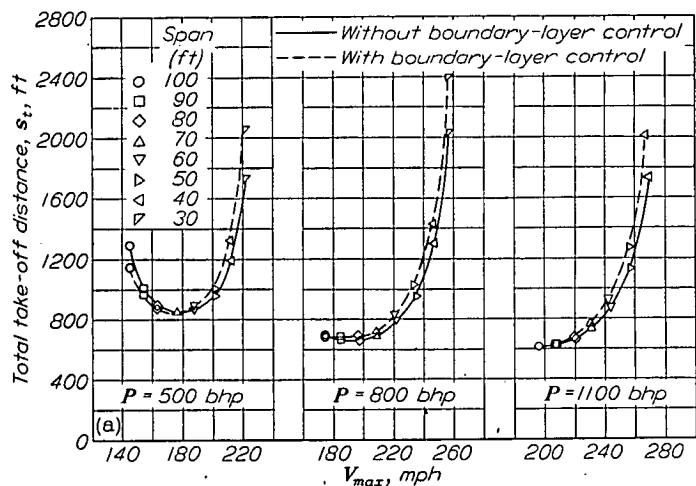
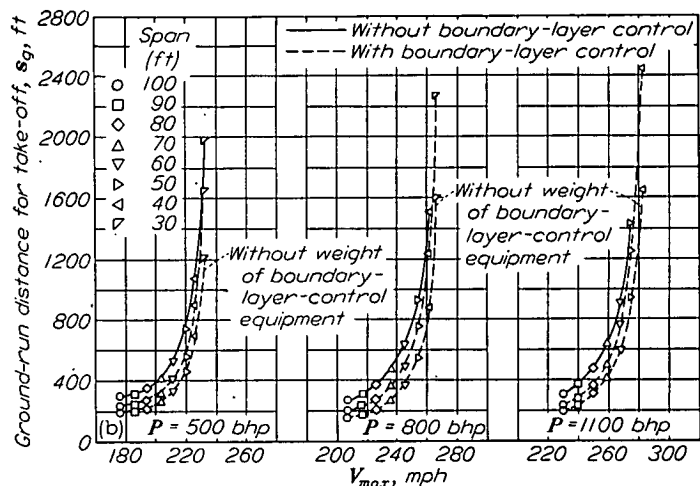


FIGURE 15.—Total take-off distance of an airplane with boundary-layer control, but with weight of boundary-layer control equipment excluded, as a function of span for various powers. $A=10$.



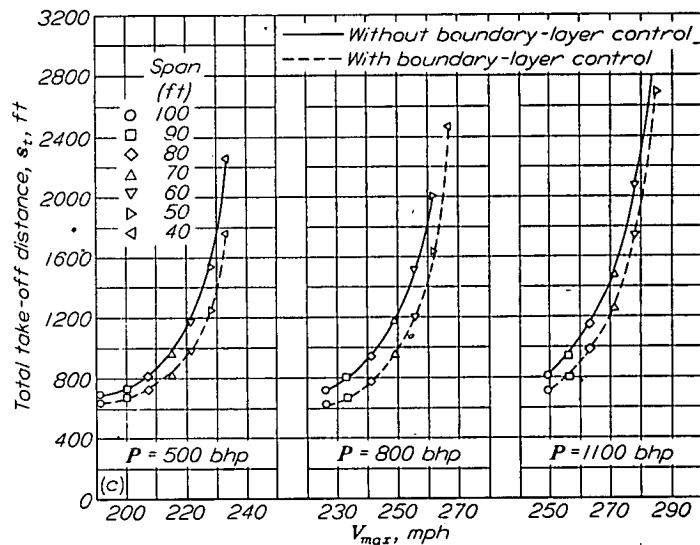
(a) $A=5$.

FIGURE 16.—Total take-off distance of an airplane with and without boundary-layer control as a function of maximum speed for various powers.



(b) $A=10$.

FIGURE 16.—Continued.



(c) $A=15$.

FIGURE 16.—Concluded.

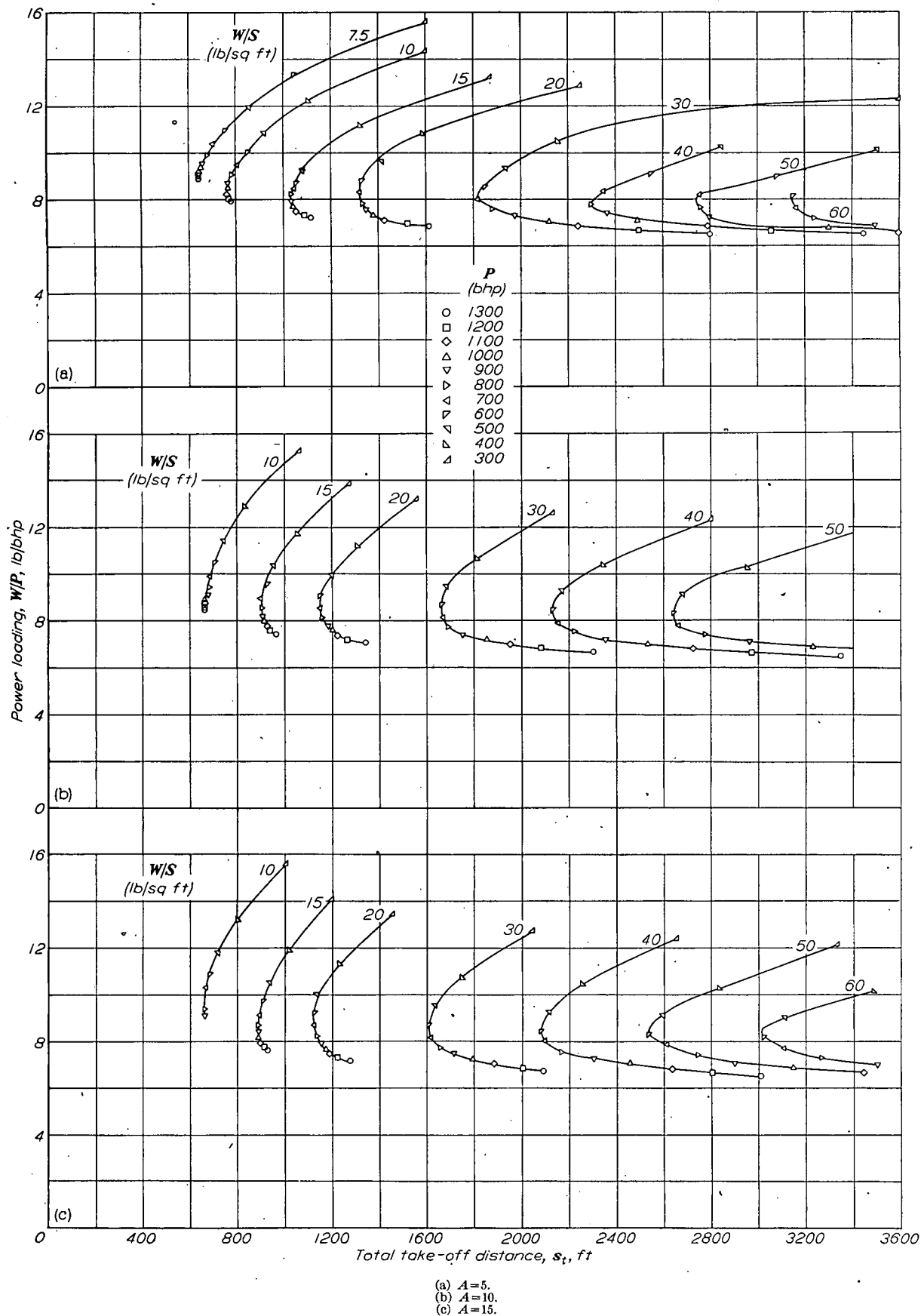


FIGURE 17.—Total take-off distance of an airplane without boundary-layer control as a function of power loading for various wing loadings.

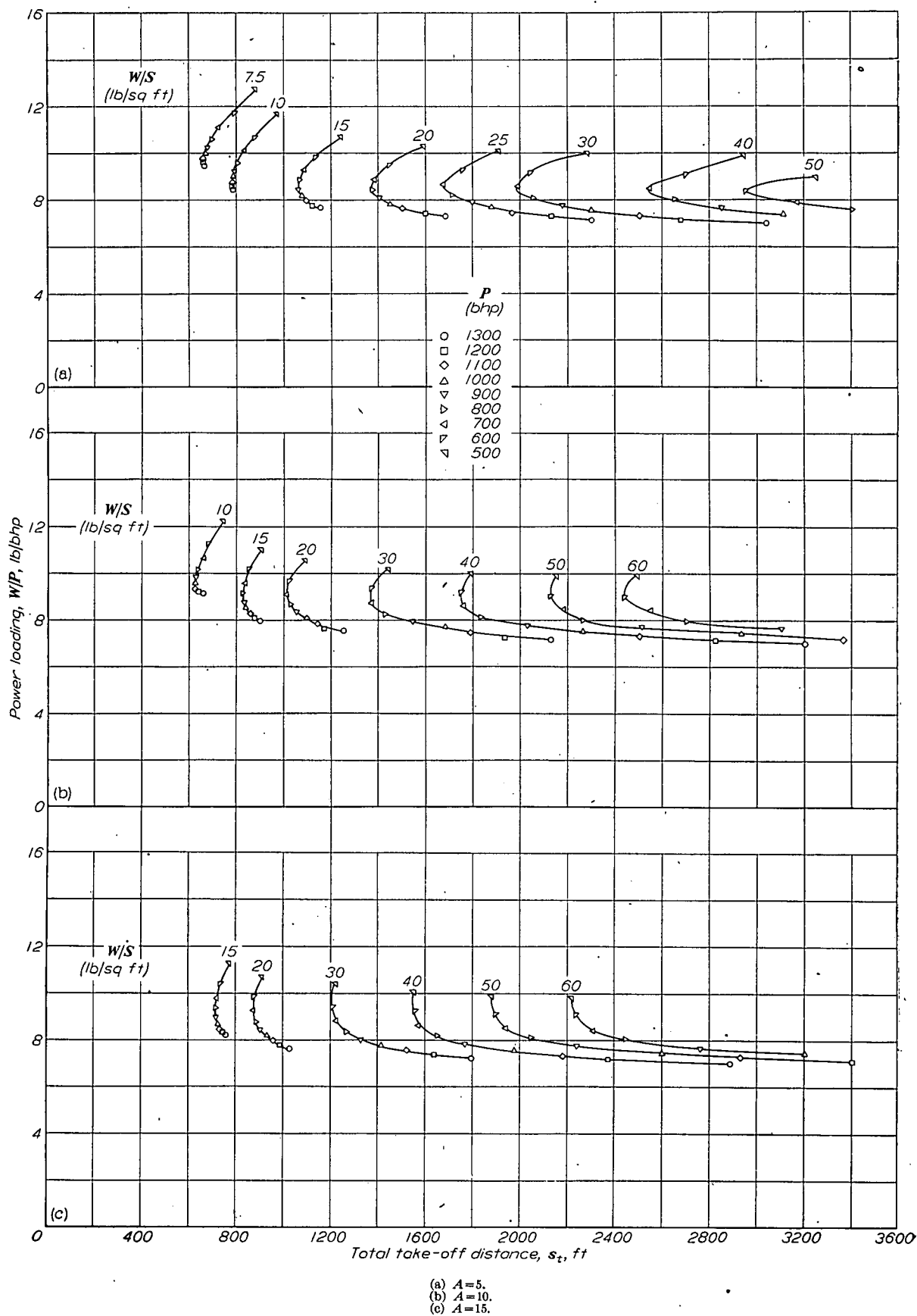


FIGURE 18.—Total take-off distance of an airplane with boundary-layer control as a function of power loading for various wing loadings.

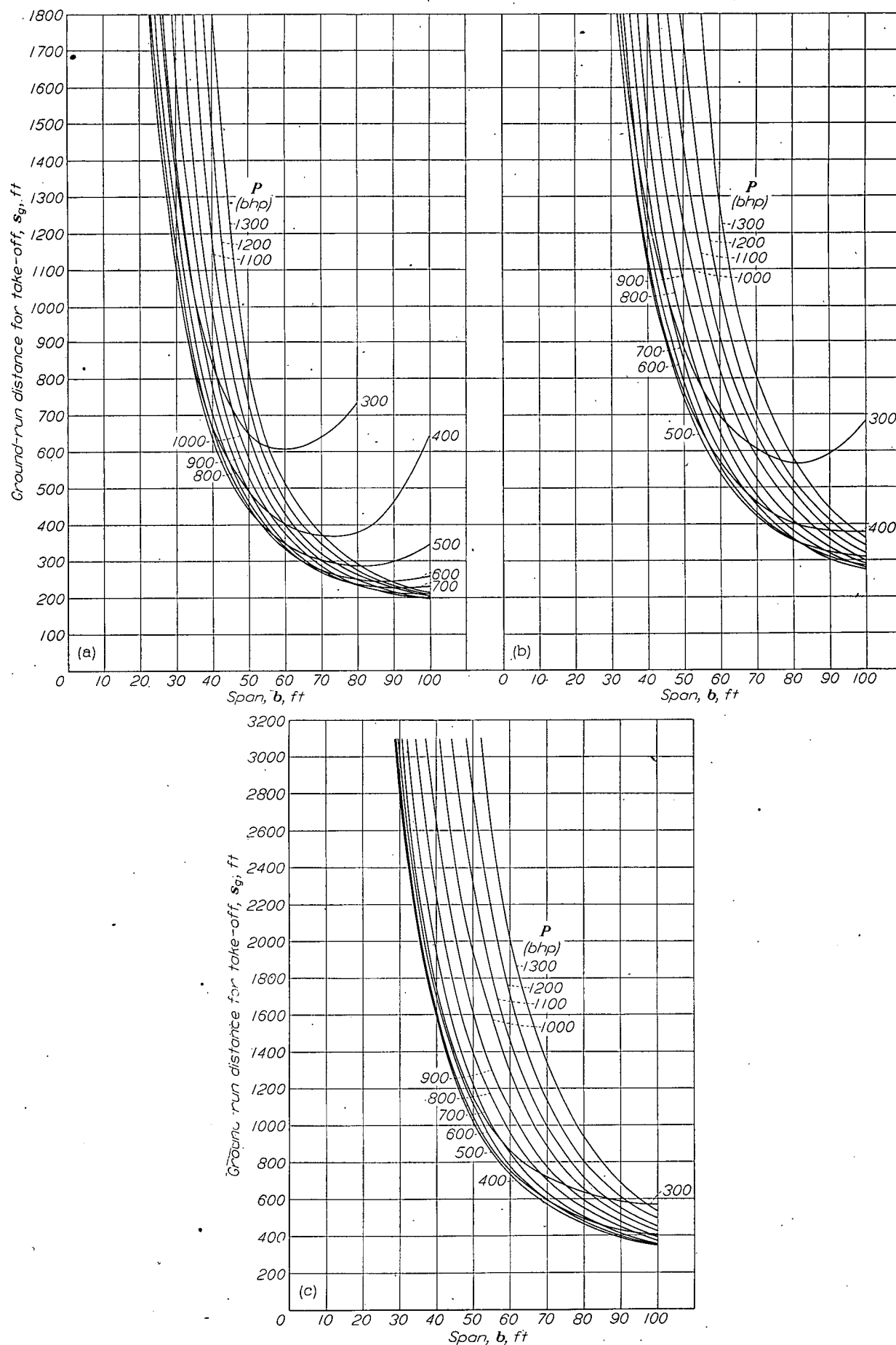


FIGURE 19.—Ground-run distance for take-off for an airplane without boundary-layer control as a function of span for various powers.

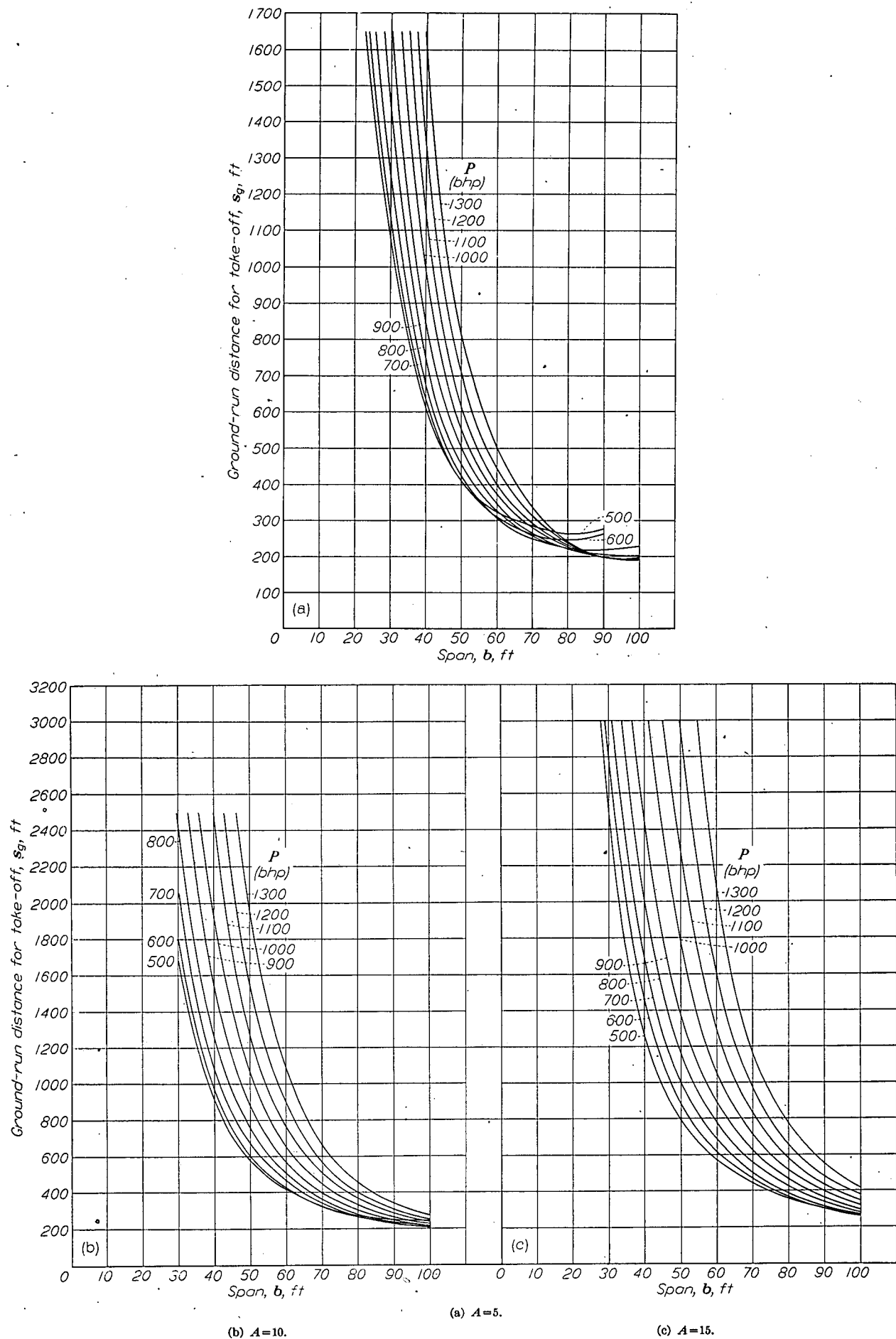


FIGURE 20.—Ground-run distance for take-off for an airplane with boundary-layer control as a function of span for various powers.

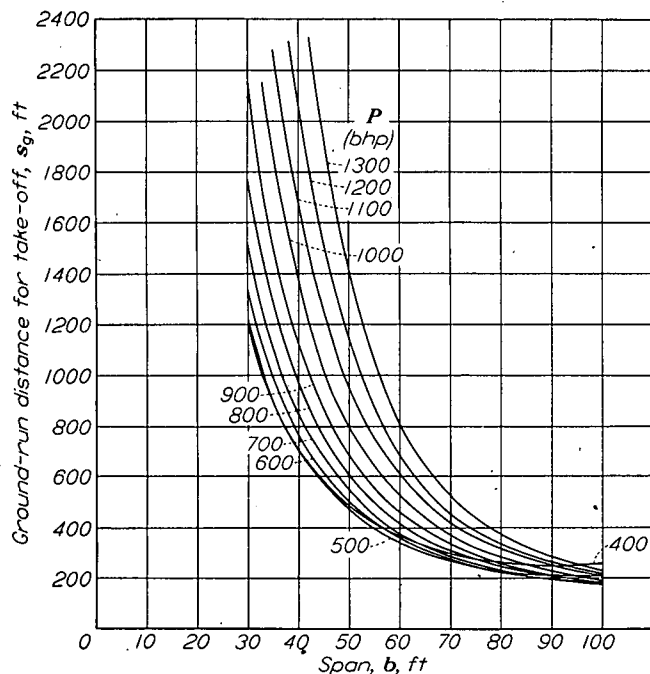
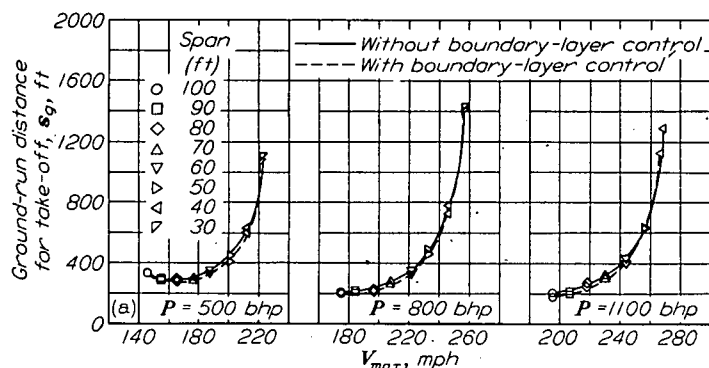
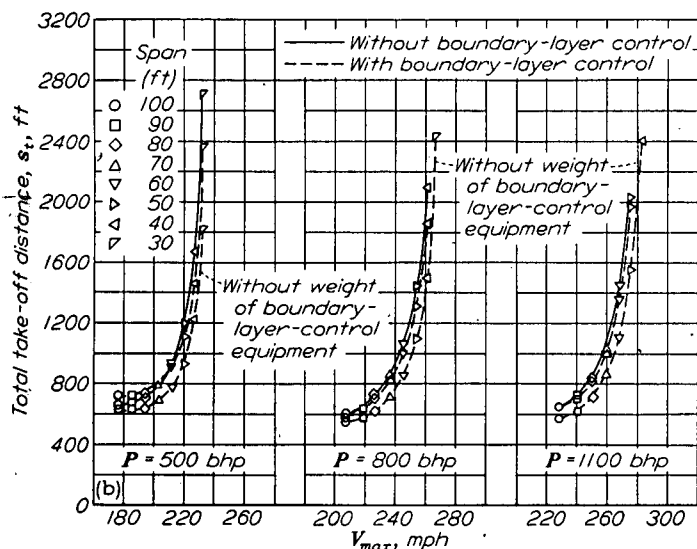


FIGURE 21.—Ground-run distance for take-off for an airplane with boundary-layer control, but with weight of boundary-layer equipment excluded, as a function of span for various powers. $A=10$.



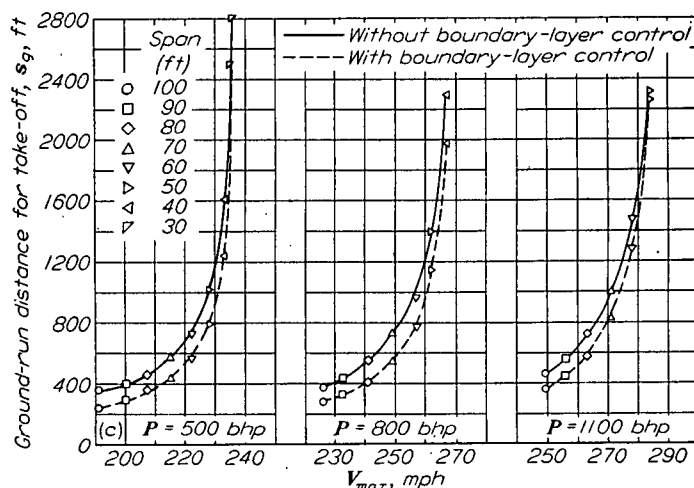
(a) $A=5$; $CL_{max}=3.8$.

FIGURE 22.—Ground-run distance for take-off for an airplane with and without boundary-layer control as a function of maximum speed for various powers.

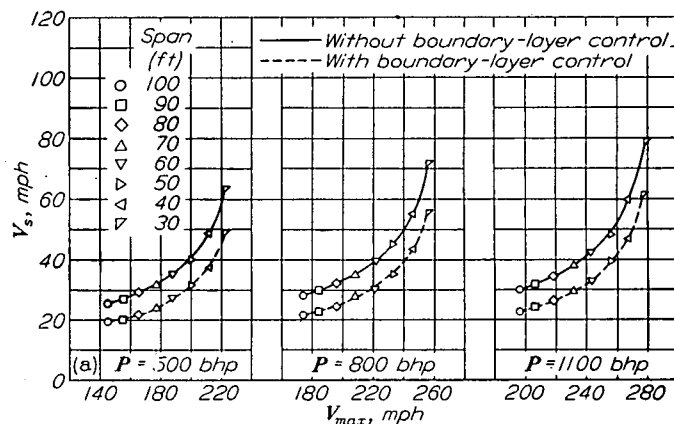


(b) $A=10$; $CL_{max}=5.0$.

FIGURE 22.—Continued.

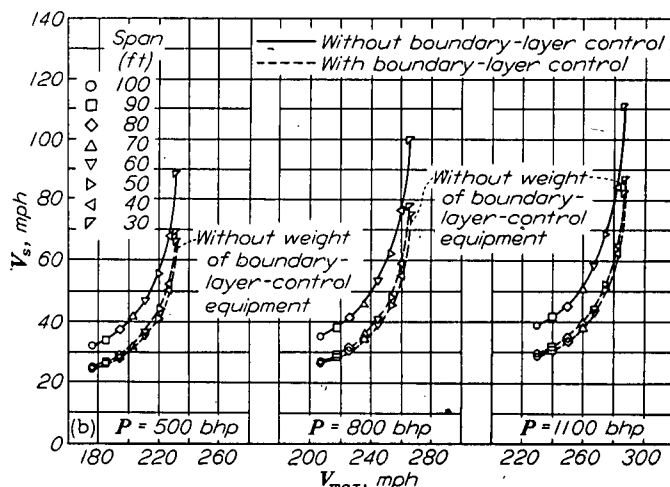


(c) $A=15$; $CL_{max}=5.0$.
FIGURE 23.—Concluded.



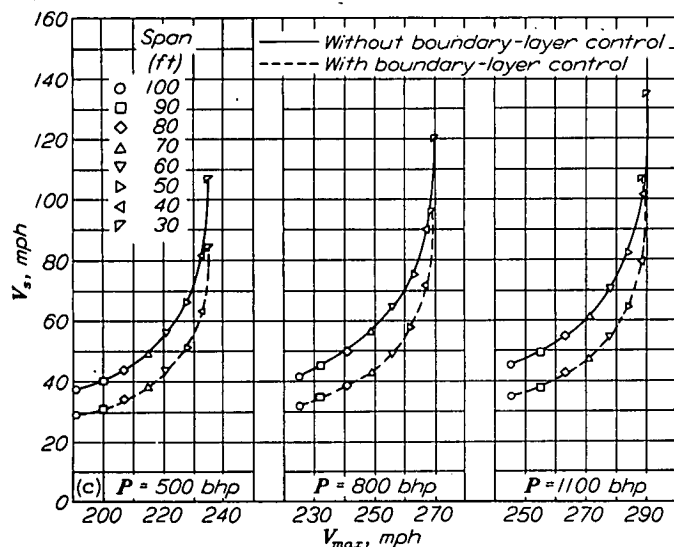
(a) $A=5$.

FIGURE 23.—Stalling speed of an airplane with and without boundary-layer control as a function of maximum speed for various powers.



(b) $A=10$.

FIGURE 23.—Continued.



(c) $A=15$.
FIGURE 23.—Concluded.

POWER-OFF LANDING CHARACTERISTICS

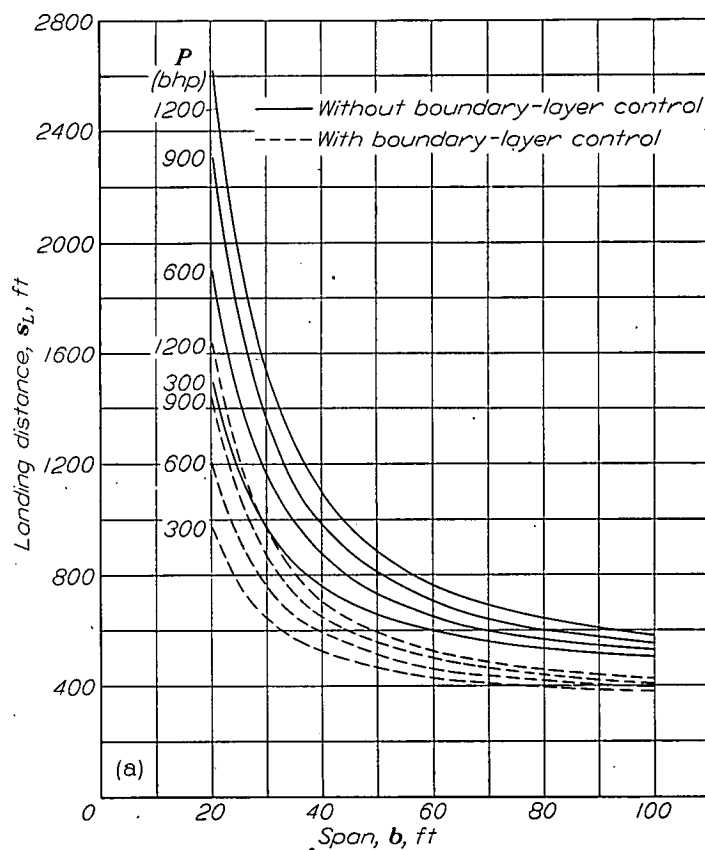
The power-off landing characteristics to be discussed are

- (1) The total landing distance
- (2) The ground-run distance
- (3) The speed at which the different phases of the landing maneuver are executed

Total landing distance.—The total landing distance is presented as a function of wing span in figure 24 with power as the parameter. The data are for aspect ratios of 5, 10, and 15 and are for the airplanes with and without boundary-layer control. An examination of the data of figure 24 indicates that, for a given engine power and aspect ratio, the landing distance decreases rapidly with increasing span over a certain range of spans, after which further increases in span have little effect. This is a result of the manner in which the wing loading varies with span. (See figs. 7 and 8.) For a given wing span, the landing distance is seen to increase with increasing engine power. In all cases, increasing the aspect ratio for a fixed span and power increases the total landing distance. For any given aspect ratio, the shortest landing distance is obtained for the airplane with largest span and lowest power. These trends are evident in the data for all configurations investigated. The effect of boundary-layer control on the total landing distance can best be seen in figure 25. In this figure the ratio of the total landing distance with boundary-layer control to the total distance without boundary-layer control is plotted as a function of span. The data clearly indicate that, regardless of engine power or aspect ratio, the use of maximum lift coefficients of the order of 5.0 which can be obtained with boundary-layer control as compared with lift coefficients of 2.8 which can be obtained without boundary-layer control results in decreases in the total landing distance which vary between 25 and 40 percent.

The data of figure 26 show that, for a constant wing loading, the use of boundary-layer control results in reductions

of the total landing distance which vary from about 27 to 43 percent. The slightly more favorable effect of boundary-layer control when the comparison is based on a constant wing loading rather than on a constant span is explained by the fact that the addition of boundary-layer control to the airplane of constant span increases the wing loading by a small amount which has an adverse effect on the landing distance. For a constant wing loading, variations in the engine power have a negligible effect upon the landing distance (fig. 26); hence, the relatively large adverse effect of increasing the power upon the landing distance of an airplane of constant span, shown by the data of figure 24, results from the effect of engine power on wing loading. It might also be thought that the adverse effect upon the landing distance of increasing the aspect ratio for a given span and power (fig. 24) could be attributed entirely to an increase in wing loading. The data of figure 26, however, show that for a given wing loading, increasing the aspect ratio also causes some increase in the landing distance. This unfavorable effect of increasing aspect ratio on the landing distance results from the fact that as the aspect ratio is increased the airplane lift-drag ratio is also increased so that there results a flatter glide and, hence, a greater horizontal distance from the 50-foot obstacle to the point of ground contact. The proper application of a spoiler or air brake might, therefore, reduce or eliminate the unfavorable effect of increasing aspect ratio on the total landing distance.



(a) $A=5$.
FIGURE 24.—Landing distance of an airplane with and without boundary-layer control as a function of span for various powers.

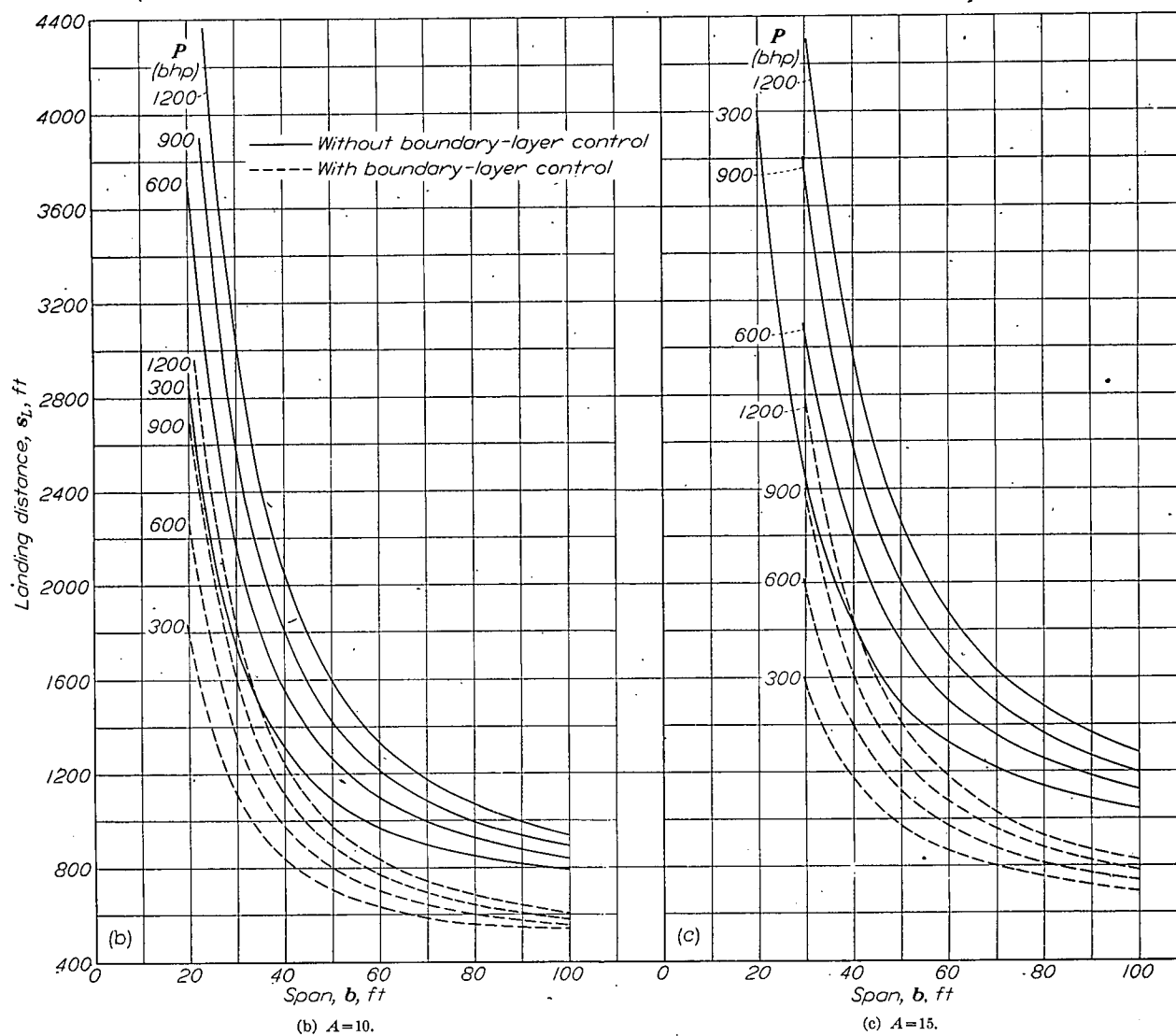


FIGURE 24.—Concluded.

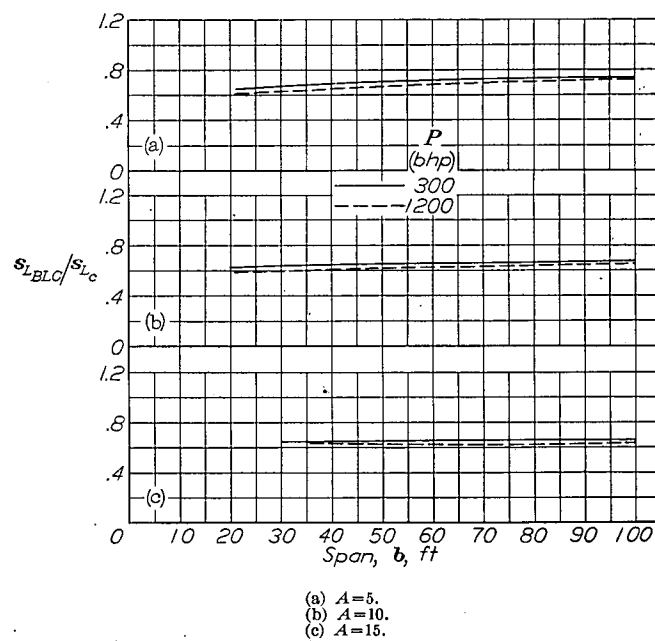


FIGURE 25.—Landing distance of airplane with boundary-layer control as a fraction of landing distance of same airplane without boundary-layer control.

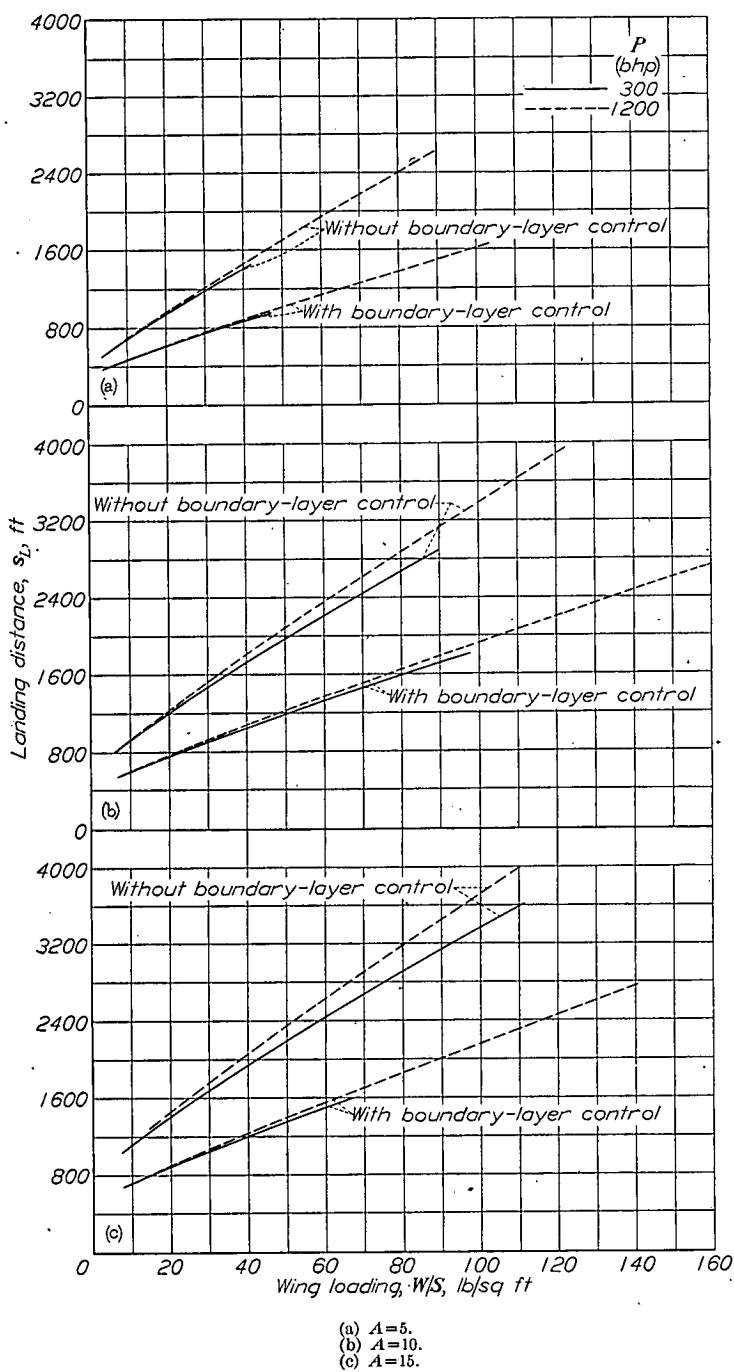


FIGURE 26.—Landing distance of an airplane with and without boundary-layer control as a function of wing loading.

The over-all conclusion to be drawn from the data of figures 24 to 26 is that boundary-layer control causes a substantial reduction in the total landing distance of all the airplane configurations investigated. The minimum landing distance for the configurations investigated was obtained for the airplane configuration having boundary-layer control and the lowest wing loading and aspect ratio—that is, a wing loading of 4 pounds per square foot and an aspect ratio of 5.

As previously pointed out, the application of boundary-layer control does not have any appreciable effect upon the maximum speed. Consequently, the reductions in landing

distance resulting from boundary-layer control (figs. 24 to 26) can be obtained without any sacrifice in maximum speed in most cases. In order to show this effect more clearly, the total landing distance has been plotted against maximum speed in figure 27 for the airplanes with and without boundary-layer control. Figure 27 shows that for a given maximum speed the use of boundary-layer control results in a 25 to 40 percent decrease in the landing distance. The wing spans of the different airplanes are indicated by symbols on these curves. It is interesting to note that for most cases large increases in the maximum speed can be obtained with no increase in the landing distance by the use of boundary-layer control along with reduction in span. The unfavorable effect of increasing aspect ratio on the landing distance for a given maximum speed is, as previously pointed out, a result of the higher lift-drag ratio of the airplanes of high aspect ratio. The fact that boundary-layer control does not have a favorable effect upon the landing distance for the highest maximum speeds obtainable with a given power is explained by the data of figures 4 and 5 which show that the highest possible speed for a given power is slightly higher for the airplane without boundary-layer control than for the airplane with boundary-layer control. This is due to the increased wing loading of the boundary-layer-control airplane.

The data presented in figures 24 to 27 lead to the conclusion that the high lift coefficients available with boundary-layer control are very effective in reducing the landing distance of the type of airplane considered in this investigation. A somewhat different conclusion was reached with respect to the effect on the total take-off distance of the increased lift coefficients available with boundary-layer control. The data of figure 16 showed that there was no appreciable decrease in the total take-off distance due to boundary-layer control for a given maximum speed unless the aspect ratio was of the order of 15. Even for the higher aspect ratios, the relative effect of boundary-layer control on the total take-off distance is small as compared with its effect on the landing distance.

Ground-run distance.—The ground-run distance is plotted against wing span for different aspect ratios and engine horse-powers in figure 28 and against maximum speed in figure 29. The data of figure 28 indicate that the use of boundary-layer control results in reductions of the ground-run distance which vary from 30 to 40 percent depending upon the configuration. The use of the lowest possible wing loading—that is, low aspect ratio and engine of low power—gives the the shortest ground-run distance for a given span.

The data of figure 29 indicate that, for nearly all configurations, reductions in the ground-run distance of 35 to 40 percent can be obtained by the use of boundary-layer control without compromising the maximum speed. In comparison with the trends of figure 29, the data of figure 22 indicated that boundary-layer control has an important effect upon the ground run to take-off for a given maximum speed only if the aspect ratio is of the order of 10 to 15 and that the ground run for take-off is generally longer than that for landing.

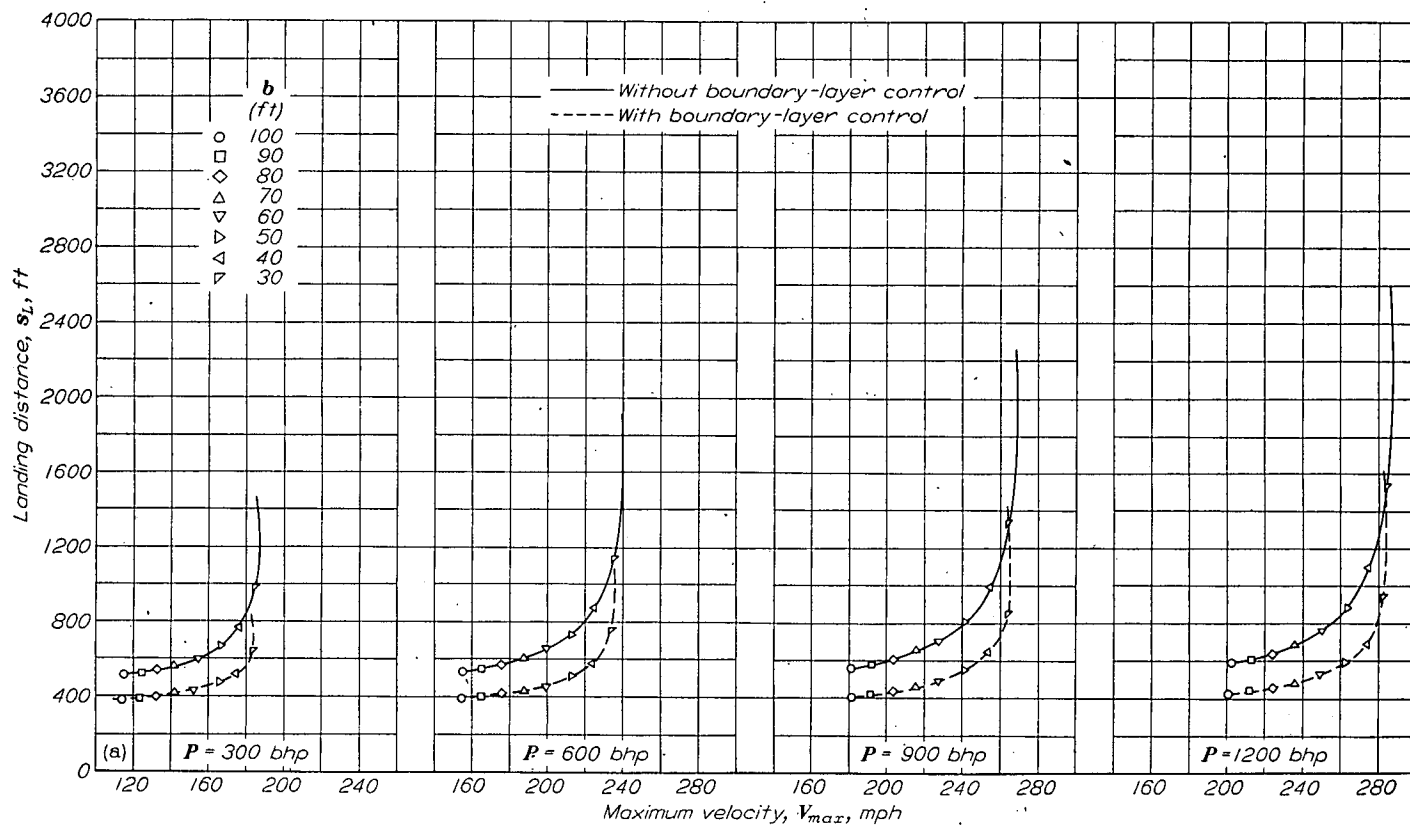
(a) $A=5$.

FIGURE 27.—Landing distance of an airplane with and without boundary-layer control as a function of velocity for various powers.

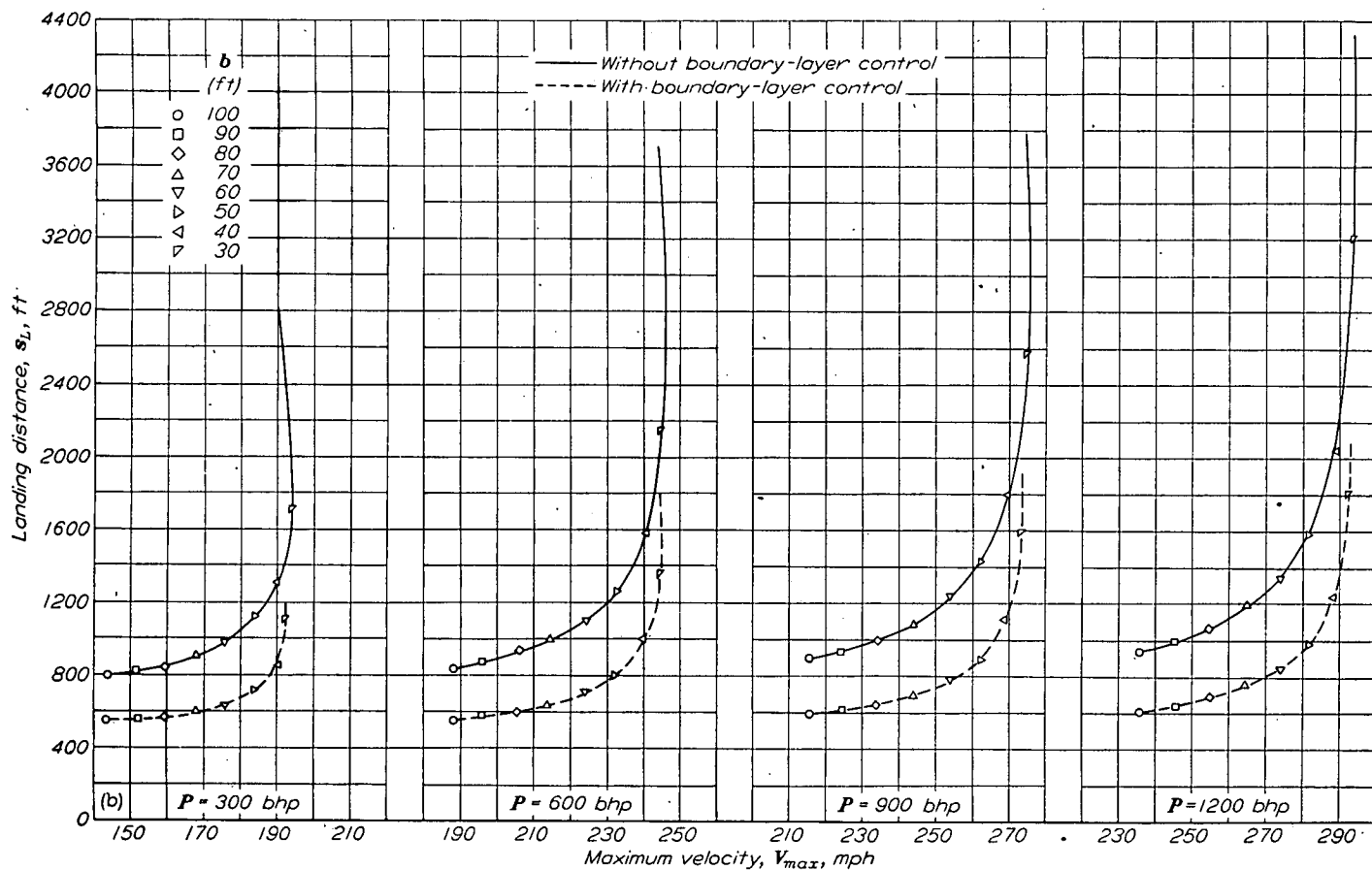
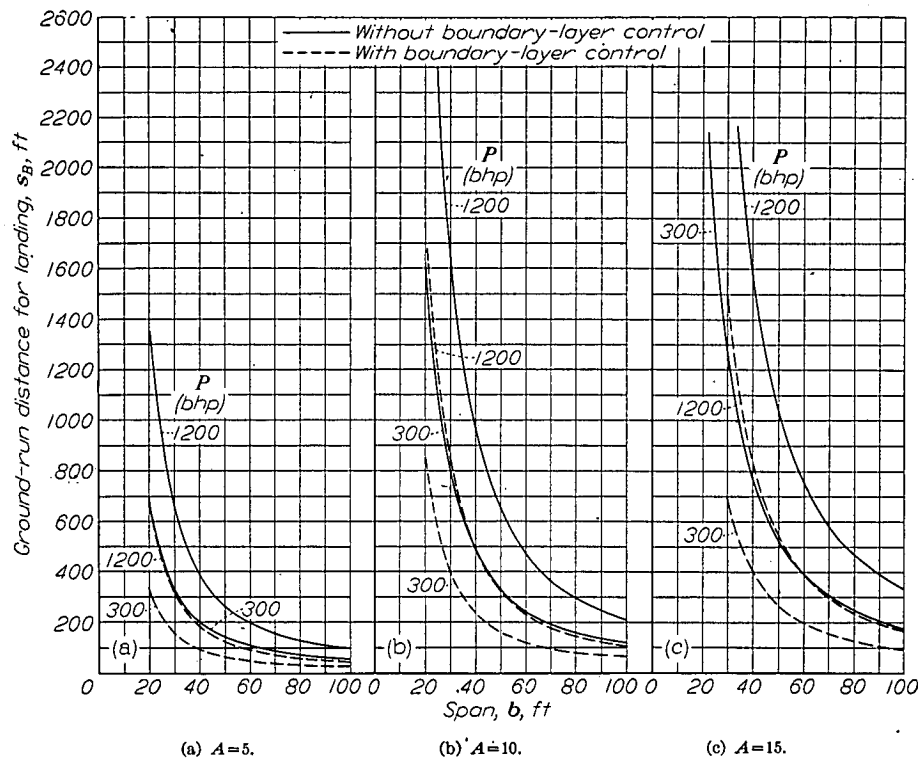
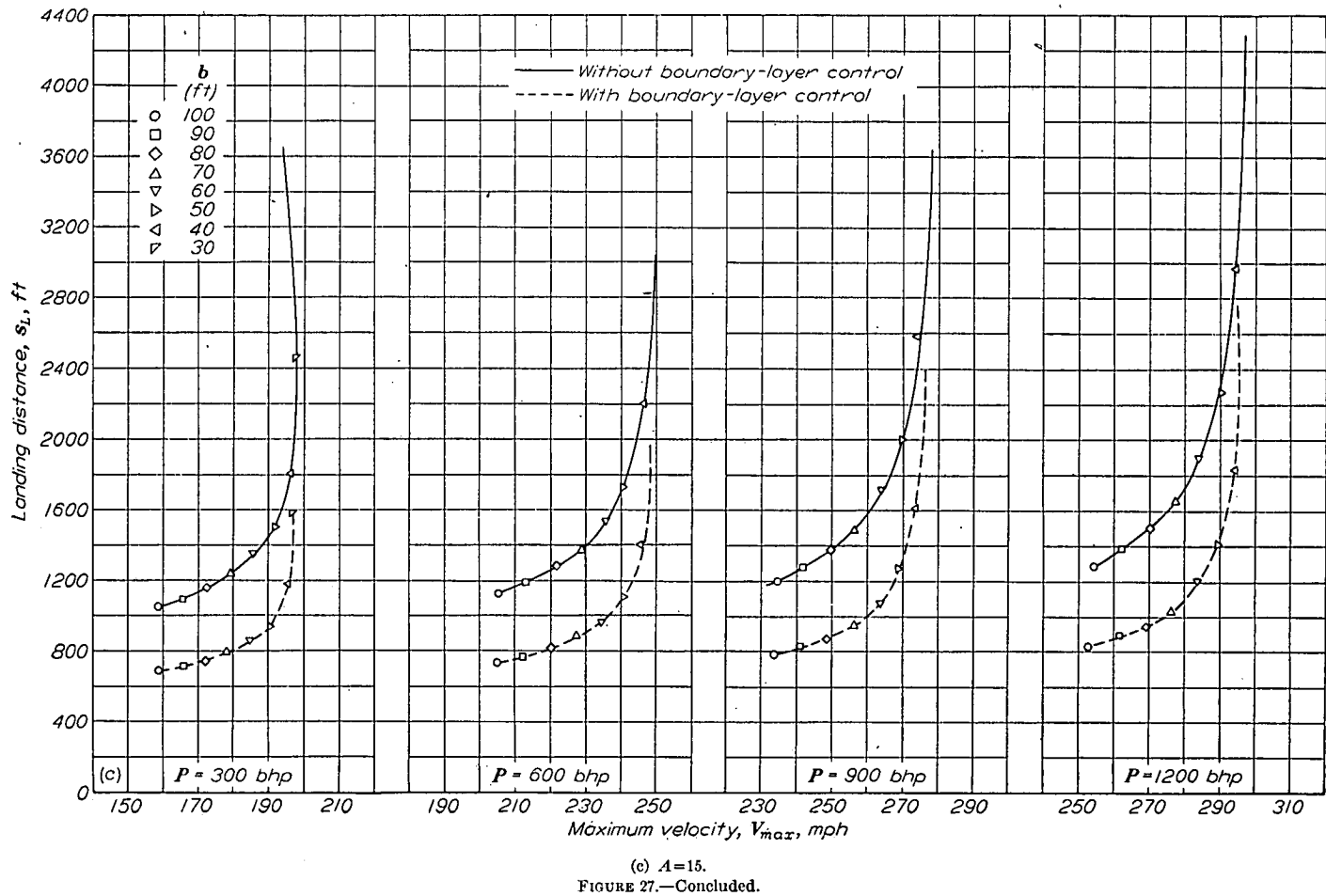
(b) $A=10$.

FIGURE 27.—Continued.



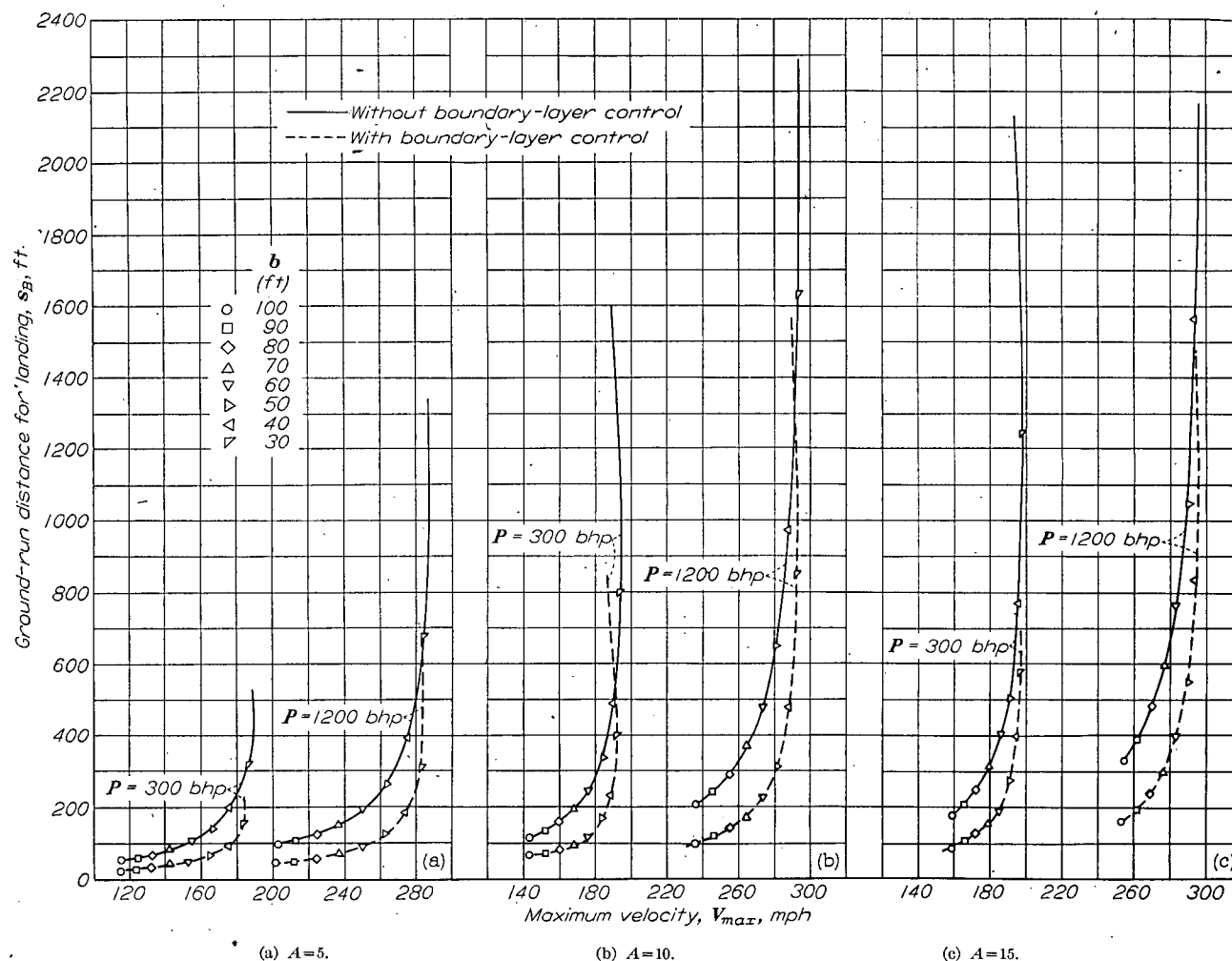


FIGURE 29.—Ground-run distance of an airplane with and without boundary-layer control as a function of maximum velocity.

Landing speeds.—The speeds with which the various phases of the landing maneuver are executed are of some importance as an indication of the piloting skill required to land a particular airplane. For this reason, data are given in figures 30 and 31 pertaining to the effect of boundary-layer control on the vertical or sinking speed in the steady glide and steady-glide speed.

The effect of boundary-layer control on the sinking speed is shown in figure 30 where the vertical velocity is plotted against wing span for various horsepower and aspect ratios for the airplanes with and without boundary-layer control. The data show that boundary-layer control has only a relatively small effect on the sinking speed in all cases. For all the airplanes both with and without boundary-layer control, reducing the span for a given aspect ratio and engine power is seen to increase the sinking speed.

In figure 31 the velocity in the steady glide is plotted against wing span for the airplanes of different aspect ratio and power both with and without boundary-layer control. In all cases, the use of boundary-layer control is seen to reduce the speed in the steady glide by 20 to 25 percent. As would be expected, the steady-glide speed increases with

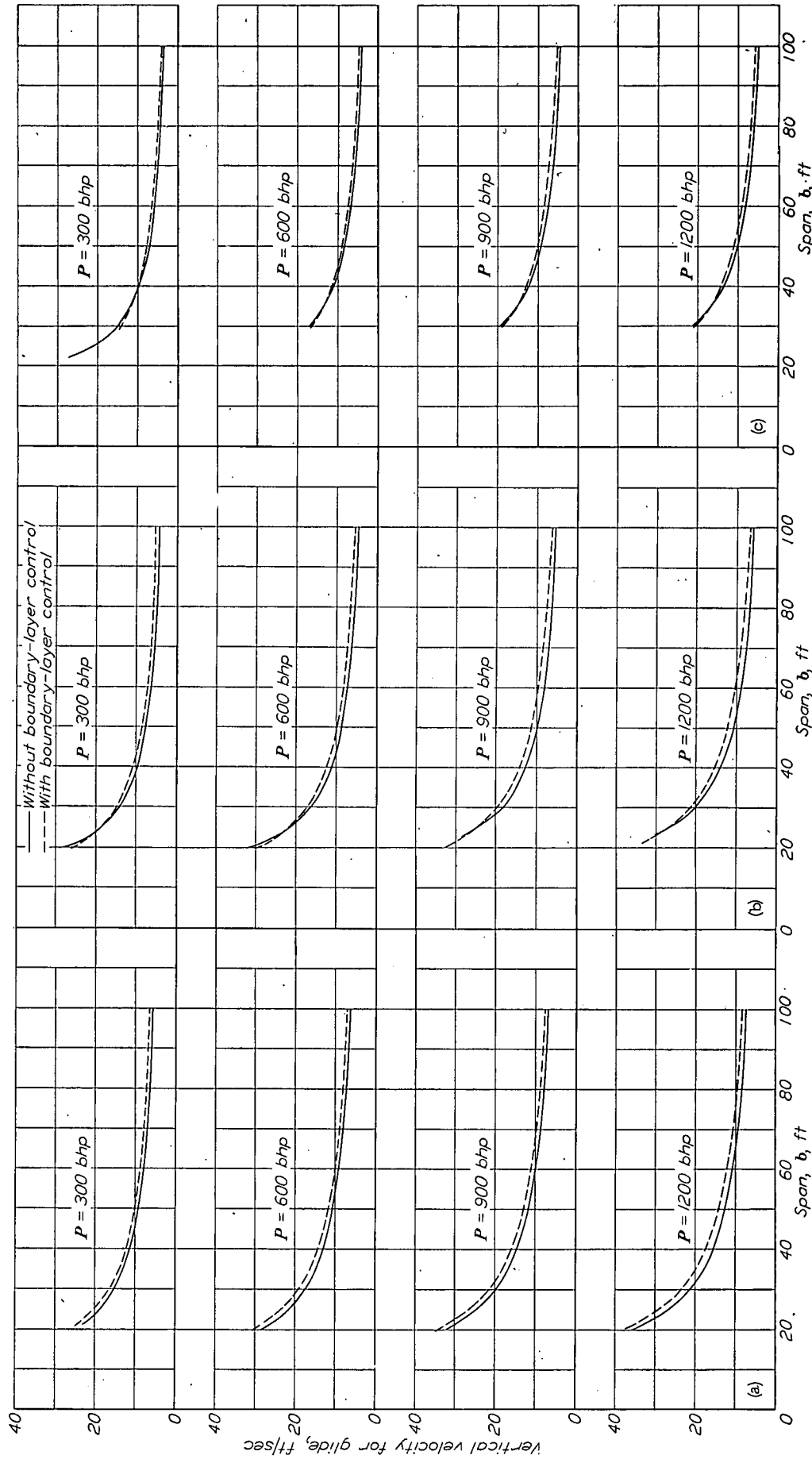
decreasing span for a fixed power and aspect ratio in all cases. Increasing the aspect ratio for a given span and power also increases the gliding speed because of the associated increase in wing loading and wing lift-drag ratio.

EFFECT OF ASSUMPTIONS ON RESULTS

As was previously pointed out, most of the many assumptions employed in the analysis would not be expected to have any large effect on the landing and take-off performances of the boundary-layer-control airplane as compared with the airplane without boundary-layer control. Three assumptions were made, however, which should be considered in comparing the performance characteristics of the conventional and boundary-layer-control airplanes. These assumptions are

- (1) No head wind
- (2) No ground effect
- (3) A ratio of span to root thickness of 35 and a thickness to chord ratio of 0.18 at the root and 0.12 at the tip

These three assumptions would probably have a greater effect on the boundary-layer-control airplane than on the conventional airplane for the following reasons.



(a) $A=5$.
(b) $A=10$.
(c) $A=15$.
FIGURE 30.—Vertical velocity as a function of span for an airplane with and without boundary-layer control.

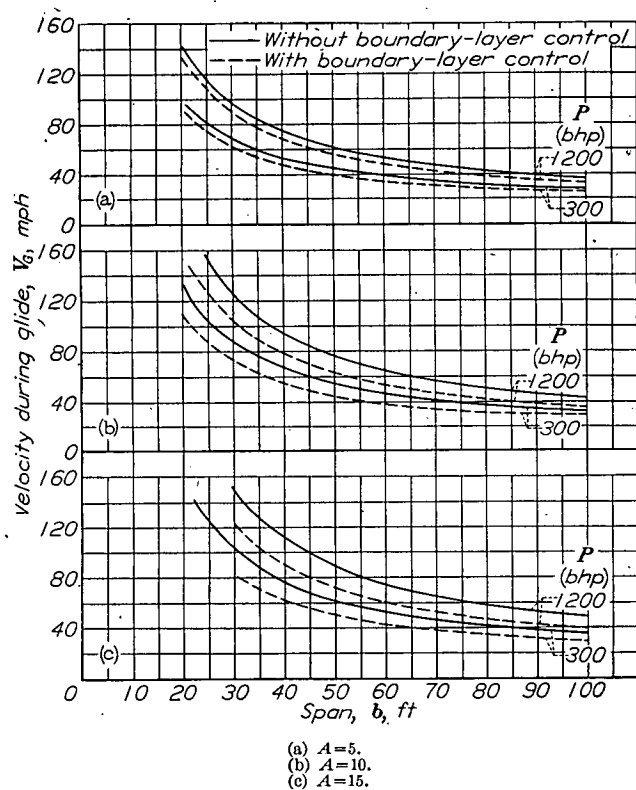


FIGURE 31.—Velocity during glide for an airplane with and without boundary-layer control as a function of span.

Head wind.—Because the maximum lift coefficients of the boundary-layer-control airplanes were greater than those of the conventional airplanes, the horizontal speed during the landing or take-off maneuver was less for the boundary-layer-control airplane than for the conventional airplane. Given a uniform head wind, the airspeeds of the two airplanes would remain unchanged, but the horizontal speed with respect to the ground of the slower airplane would be reduced by a greater percentage than that of the faster airplane. Therefore, the horizontal distance required to land from or take off and climb to a given altitude would be decreased in a head wind by a greater percentage for the boundary-layer-control airplane than for the conventional airplane.

Ground effect.—The effect of proximity to the ground is mainly that of increasing the effective aspect ratio. The greater aspect ratio would result in proportionately greater decreases in induced drag for the boundary-layer-control airplane with its high maximum lift coefficient than for the conventional airplane; therefore, the take-off distance for the boundary-layer-control airplane would be decreased by a greater percentage than that for the conventional airplane. For a more thorough treatment of this subject, see reference 16.

Wing thickness-chord ratios.—If the ratio of wing span to root thickness were maintained at 35, the root thickness-chord ratios of the wing would greatly exceed 0.18 for the larger spans and aspect ratios. The wing profile drag of the

conventional airplane would, therefore, be considerably greater than the values used because of the large profile drags associated with airfoil sections having thickness ratios greater than 0.21 (reference 17). With boundary-layer control, however, it is possible to use the thicker airfoil sections without greatly increasing the profile drag as experimental results have indicated that, when separated flow exists, the drag of an airfoil section, including the boundary-layer-control power, may be less than the drag without boundary-layer control (references 2, 7, and 8).

CONCLUSIONS

An analysis was made to determine the effect of boundary-layer control on the take-off and power-off landing performance characteristics of a liaison type of airplane having aspect ratios ranging from 5 to 15, wing spans ranging from 25 to 100 feet, and engine brake horsepower ranging from 300 to 1,300. The airplanes were assumed to have a 1,500-pound pay load and a cruising duration at 60-percent power of 5 hours. The results of the analysis indicate the following conclusions:

1. The addition of boundary-layer control does not reduce the absolute minimum total take-off distance which is obtained with a low wing loading and a moderately low aspect ratio.
2. The effectiveness of boundary-layer control in reducing the total take-off distance for a given maximum speed improves with increasing aspect ratio and, for wing loadings of 10 pounds per square foot or more and an aspect ratio of 10 or more, the addition of boundary-layer control results in a decrease in the total take-off distance of as much as 14 percent.
3. For a given maximum speed the ground-run distance for take-off was reduced for all configurations by the use of boundary-layer control. This reduction was negligible for an aspect ratio of 5 but was from 10 to 30 percent for aspect ratios of 10 and 15.
4. For a given maximum speed, the use of boundary-layer control resulted in a reduction in stalling speed of 20 to 25 percent for all configurations.
5. A reduction in the weight of the boundary-layer-control equipment would result in an appreciable decrease in the total take-off distance and ground-run distance for take-off, but its effect on the stalling speed would be negligible.
6. The optimum horsepower loading for minimum take-off distance was found to be approximately 8.5 and 9.0 pounds per horsepower for the conventional and boundary-layer-control airplanes, respectively.
7. For a specified airplane maximum speed, the total landing distance was reduced from 25 to 40 percent and the landing ground-run distance was reduced 30 to 40 percent by the use of boundary-layer control.

8. The gliding speeds were 20 to 25 percent lower for most of the airplanes with boundary-layer control than those for the airplanes without boundary-layer control.

9. For a fixed wing span, the sinking speed, or vertical velocity for the landing condition was slightly higher for the airplane with boundary-layer control than that for the conventional airplane.

LANGLEY AERONAUTICAL LABORATORY,
NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LANGLEY FIELD, VA., *October 4, 1951.*

REFERENCES

1. Hagerman, John R.: Wind-Tunnel Investigation of the Effect of Power and Flaps on the Static Lateral Stability and Control Characteristics of a Single-Engine High-Wing Airplane Model. NACA TN 1379, 1947.
2. Regenscheit, B.: Hochauftriebsversuche mit Absaugeklappenflügeln. Bericht A64 der LGL, 1938, pp. 27-34.
3. Stüper: Flight Experiences and Tests on Two Airplanes with Suction Slots. NACA TM 1232, 1950.
4. Ivey, H. Reese, Fitch, G. M., and Schultz, Wayne F.: Performance Selection Charts for Gliders and Twin-Engine Tow Planes. NACA MR L5E04, 1945.
5. Abbott, Ira H., Von Doenhoff, Albert E., and Stivers, Louis S., Jr.: Summary of Airfoil Data. NACA Rep. 824, 1945. (Formerly NACA ACR L5C05.)
6. Schrenk, Oskar: Experiments with a Wing from Which the Boundary Layer Is Removed by Suction. NACA TM 634, 1931.
7. Quinn, John H., Jr.: Tests of the NACA 65₃-018 Airfoil Section with Boundary-Layer Control by Suction. NACA CB L4H10, 1944.
8. Quinn, John H., Jr.: Wind-Tunnel Investigation of Boundary-Layer Control by Suction on the NACA 65₃-418, $\alpha=1.0$ Airfoil Section with a 0.29-Airfoil-Chord Double Slotted Flap. NACA TN 1071, 1946.
9. Wood, Karl D.: Airplane Design. Fourth ed., Cornell Co-Op Soc., Ithaca, N. Y., June 1939.
10. Bridgman, Leonard, ed.: Janes's All the World's Aircraft. The Macmillan Co., 1942.
11. Krüger, W.: Calculations and Experimental Investigations on the Feed-Power Requirement of Airplanes with Boundary-Layer Control. NACA TM 1167, 1947.
12. Wetmore, J. W.: Calculated Effect of Various Types of Flap on Take-Off over Obstacles. NACA TN 568, 1936.
13. Hartman, Edwin P.: Considerations of the Take-Off Problem. NACA TN 557, 1936.
14. Gustafson, F. B.: Tire Friction Coefficients and Their Relation to Ground-Run Distance in Landing. NACA ARR, June 1942.
15. Diehl, Walter Stuart: Engineering Aerodynamics. The Ronald Press Co., rev. ed., 1936, p. 447.
16. Reid, Elliott G.: A Full-Scale Investigation of Ground Effect. NACA Rep. 265, 1927.
17. Neely, Robert H., Bollech, Thomas V., Westrick, Gertrude C., and Graham, Robert R.: Experimental and Calculated Characteristics of Several NACA 44-Series Wings with Aspect Ratios of 8, 10, and 12 and Taper Ratios of 2.5 and 3.5. NACA TN 1270, 1947.

REPORT 1058

INFLUENCE OF CHEMICAL COMPOSITION ON RUPTURE PROPERTIES AT 1200° F OF FORGED CHROMIUM-COBALT-NICKEL-IRON BASE ALLOYS IN SOLUTION-TREATED AND AGED CONDITION¹

By E. E. REYNOLDS, J. W. FREEMAN, and A. E. WHITE

SUMMARY

The influence of systematic variations of chemical composition on rupture properties at 1200° F was determined for 62 modifications of a basic alloy containing 20 percent chromium, 20 percent nickel, 20 percent cobalt, 3 percent molybdenum, 2 percent tungsten, 1 percent columbium, 0.15 percent carbon, 1.7 percent manganese, 0.5 percent silicon, 0.12 percent nitrogen, and the balance iron. These modifications included individual variations of each of 10 elements present and simultaneous variations of molybdenum, tungsten, and columbium. Laboratory induction furnace heats were hot-forged to round bar stock, solution-treated at 2200° F, and aged at 1400° F. The melting and fabrication conditions were carefully controlled in order to minimize all variable effects on properties except chemical composition.

For the limited number of composition variables studied the range in 100-hour rupture strengths was from 26,000 to 52,000 psi. Major strengthening effects resulted from additions of molybdenum, tungsten, and columbium, individually or simultaneously. The lowest-strength alloy contained none of these elements. Chromium also had a major strengthening influence. However, no alloy was obtained which had properties which were outstanding compared with those of the basic analysis.

Carbon (varied from 0.08 to 0.60 percent), nitrogen (0.08 to 0.18 percent), manganese (0.30 to 2.5 percent), nickel (10 to 20 percent), cobalt (20 to 32 percent), and columbium (2 to 4 percent) had little influence on rupture properties. Nitrogen (0.004 to 0.08 percent), chromium (10 to 30 percent), nickel (0 to 10 percent), cobalt (0 to 20 percent), molybdenum (0 to 4 percent), tungsten (0 to 4 percent), and columbium (0 to 1 percent) increased strength appreciably. Silicon (0.5 to 1.6 percent) and nickel (20 to 30 percent) had a weakening influence. Columbium had a marked influence on increasing total elongation to fracture.

Rupture strengths varied directly with a measure of the resistance to creep of the alloys with total elongation to fracture as a parameter. This indicated that rupture strengths were a function of the effect of the composition modifications on both the inherent creep resistance and the amount of deformation the alloys would tolerate before fracture.

Interpretation of the results on the basis of microstructural studies indicated that molybdenum and tungsten improved creep resistance by entering solid solution. Nickel and cobalt, two elements forming part of the matrix solid solution, appeared to improve strength by increasing the solubility of molybdenum

and tungsten. Chromium improved creep resistance, at least in part, as a result of an aging reaction. Columbium did not appreciably affect creep resistance but improved rupture strength by increasing the elongation to fracture.

Information is presented which indicates that melting and hot-working conditions play an important role in high-temperature properties of alloys of the type investigated.

INTRODUCTION

This investigation had as its object a study of the influence of systematic variations of chemical composition on the 1200° F rupture properties of forged alloys in the solution-treated and aged condition in which the composition was varied from the following basic analysis:

Chemical composition (percent)

Carbon, C	0.15	Cobalt, Co	20
Manganese, Mn	1.7	Molybdenum, Mo	3
Silicon, Si	0.5	Tungsten, W	2
Chromium, Cr	20	Columbium, Cb	1
Nickel, Ni	20	Nitrogen, N	0.12
		Iron, Fe	32

Sixty-two modifications of this alloy were studied in which each of 10 elements was systematically varied individually and Mo, W, and Cb were varied simultaneously.

The investigation was prompted by the fact that at present there are no published data which make possible a correlation between systematic variations in chemical composition and the high-temperature properties of forged Cr-Ni-Co-Fe-Mo-W-Cb alloys of the type investigated. Past research has shown that close control over processing of alloys is necessary to reproduce high-temperature properties between different heats of the same analysis. Lack of such control of processing variables has resulted in the failure of attempts to correlate the published high-temperature data on these types of alloys. Such correlations are needed to provide a basis for establishing optimum chemical compositions for heat-resistant alloys, for reducing the required alloy content while retaining worth-while properties, and for determining the fundamental mechanisms by which alloying elements influence properties at high temperatures.

As an initial approach to the solution of a large problem, the investigation was necessarily limited in scope. The

¹ Supersedes NACA TN 2449, "Investigation of Influence of Chemical Composition on Forged Modified Low-Carbon N-155 Alloys in Solution-Treated and Aged Condition as Related to Rupture Properties at 1200° F" by E. E. Reynolds, J. W. Freeman, and A. E. White, 1951.

number of compositions studied, the use of only one condition of processing, the evaluation of high-temperature characteristics only by rupture properties for 100 hours at 1200° F, and the use of only microstructural and hardness data to provide interpretation of results were the major limitations placed on the investigation.

This work was conducted at the University of Michigan under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

EXPERIMENTAL PROCEDURE

The basic analysis selected for study of the influence of systematic variations in chemical composition on rupture properties at 1200° F was that of a commercial alloy which has the following nominal composition:

Chemical composition (percent)			
Carbon.....	0.15	Cobalt.....	20
Manganese.....	1.7	Molybdenum.....	3
Silicon.....	0.5	Tungsten.....	2
Chromium.....	20	Columbium.....	1
Nickel.....	20	Nitrogen.....	0.12
		Iron.....	32

The influence of composition was evaluated on 62 modifications of this alloy with the following composition variables:

(1) Individual variations of the elements in the basic analysis:

C, percent.....	0.08, 0.40, 0.60
Mn, percent.....	0, 0.30, 0.50, 1.0, 2.5
Si, percent.....	1.2, 1.6
Cr, percent.....	10, 30
Ni, percent.....	0, 10, 30
Co, percent.....	0, 10, 32
Mo, percent.....	0, 1, 2, 5, 7
W, percent.....	0, 1, 5, 7
Cb, percent.....	0, 2, 4, 6
N, percent.....	0.004, 0.08, 0.18

(2) Simultaneous variations, in steps of 2 percent, of Mo, W, and Cb from 0 to 4 percent

In all cases Fe variations compensated for the variations of total alloy content.

The first step in the investigation was the development of melting, forging, and heat-treatment procedures which would minimize all variable effects on properties except chemical composition. This work was done on the basic alloy with the reproducibility of 1200° F rupture properties from heat to heat being the main criterion for final adoption of preparation procedures. Six heats were prepared to develop melting practice. Three additional heats were prepared using varying deoxidation practices. The forging procedure was developed on six additional heats of the basic alloy.

The work on procedure development and detailed descriptions of the final procedures adopted are summarized in the subsequent section "Preparation of Experimental Alloys." Briefly these consisted of melting the alloys as 9-pound induction-furnace heats, hot-forging the ingots between 2200° and 1800° F to bar stock, heat-treating at 2200° F for 1 hour, and water-quenching followed by aging at 1400° F for 24 hours.

All alloys were chemically analyzed for the modified element while complete analyses were made only on spot heats

within a group after it was established that melting practice was yielding the desired compositions.

Metallographic examinations were made of all the alloys in the as-cast, hot-forged, solution-treated, aged, and rupture-tested conditions. Vickers hardness tests were made on bar stock in the solution-treated and the aged conditions.

The high-temperature load-carrying ability of the alloys was evaluated by means of stress-rupture properties at 1200° F. Rupture test specimens, machined from the heat-treated bar stock, were 0.250 inch in diameter with a 1-inch gage length. The stress-rupture tests were made in individual stationary units with the load applied by a simple beam acting through a system of knife edges. At least two and usually three or four tests at various stresses were made on each alloy and were of sufficient duration to establish the 100-hour rupture strengths and to permit at least an estimate of the 1000-hour strengths. Time-elongation data were taken during the rupture tests by the drop-of-the-beam method and also, in the case of many of the longer time tests, by means of modified Martens type extensometers with a sensitivity of 0.00005 inch per inch. There was good agreement between curves from the two types of deformation measurements in all cases where both types were used on the same tests.

PREPARATION OF EXPERIMENTAL ALLOYS

This section discusses the development of techniques for melting, forging, and heat-treating, describes the procedures finally adopted, presents the observed behavior of the alloys during processing, and shows the reproducibility of rupture properties of the basic alloy resulting from the processing procedures used.

MELTING

Preliminary melting experiments.—Six heats were prepared originally to develop and standardize melting practice, particularly to obtain control over final composition from charge calculations and reproducibility from heat to heat. Actual analyses of these heats are given in table I. It was found necessary to make certain minor corrections in the charged materials and the next five heats of the basic alloy had reasonably consistent compositions.

Melting procedure.—The alloys were melted in a 12-pound-capacity induction furnace as 9-pound heats. The heats were poured into 9-inch-long tapered (1½- to 1¼-in.-sq.) cast-iron ingot molds with 2½-inch-square hot tops. The life of the magnesia melting crucible was from 10 to 12 heats.

A typical melting charge and schedule are given in table II. This table shows the various ferroalloys used to make up the charges. The Fe, Cr, Ni, and Co charge was first melted down; Mn and Si were added, followed by Mo, W, and Cb. The heat was then deoxidized with 15 grams of calcium-silicon alloy, power turned off, bath temperature taken with both a Leeds and Northrup optical pyrometer and a platinum, platinum-rhodium immersion thermocouple, and the metal poured. The calibration of the immersion thermocouple was checked periodically.

Immediately after pouring, a Chromel-Alumel thermocouple was immersed into the molten metal in the hot top and the cooling curve determined by readings at 5-second intervals on a Leeds and Northrup portable potentiometer.

In a subsequent study of the effect of deoxidation on properties, three heats of the basic alloy were prepared using deoxidation practices varying from the normal use of 15 grams of calcium-silicon alloy. The three variations were: No deoxidation (alloy 74); 15 grams of zirconium-silicon-iron alloy (75); and melting under a lime-fluorspar slag (76).

Melting characteristics.—Table I gives the chemical compositions of the experimental alloys. The intended modifications from the basic alloy are given along with actual values where chemical analyses were made. In general, the actual analyses were consistently close to the desired composition.

Table III gives bath temperatures just prior to pouring the alloys. There was a fair agreement and no consistent difference between the optical pyrometer and the immersion thermocouple readings. Cleanness of the bath surface was found to be of importance in the pyrometer readings. The agreement between the two methods in so many cases indicates that the temperatures given are reliable.

A macrographic section of the ingot cross section was taken from each heat from just below the hot top to determine if pipe or porosity existed. All of the heats actually tested were sound. The ingot grain structure was well-defined on these sections, a typical example of which is shown in figure 1 for several heats. In general the various ingot macrostructures reflected the relative pouring temperatures rather than variations in chemical composition. The higher the pouring temperature, the greater was the total area covered by the columnar grains and, the greater the size, the less the total area covered by the equiaxed grains at the center.

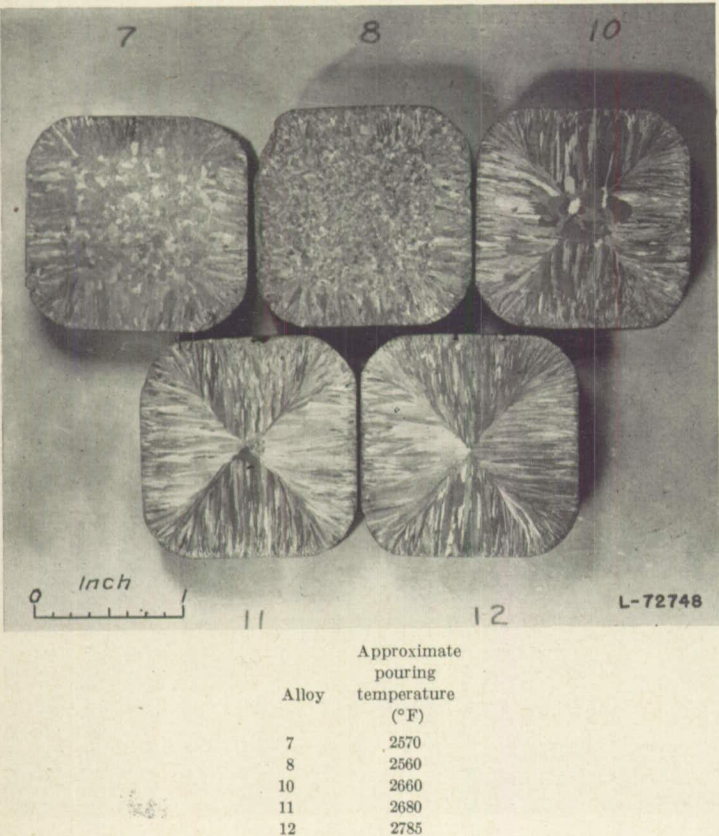


FIGURE 1.—Typical macrostructures of ingots. Etchant, Marble's reagent.

The cooling curves obtained on the metal in the ingot hot tops were similar in that, in all cases when readings were begun early enough to determine it, they had a definite halt in temperature within what seemed to be the solidification range for the alloys. Usually these halts were preceded on the curve by what appeared to be an undercooling effect. The time of the constant temperature halt varied for the alloys from 10 to 60 seconds. After these halts the cooling curves were smooth with a gradual decrease in cooling rate as the temperature decreased. The last column in table III contains the temperatures of halts in the cooling curves during solidification. In cases where temperature readings were not started early enough the halt temperature is indicated as being greater than that at which the cooling curve was started.

The over-all temperature range of the halts was from about 2420° to 2610° F and they varied consistently with systematic chemical-composition variations. Alloying additions tended to lower the temperature of the halt, with Cb having the greatest effect. The exact physical significance of the temperature halts, beyond an indication of the temperatures at which the major portion of the alloy solidified, is uncertain.

FORGING

Preliminary forging experiments.—There was available for the forging work a Nazel air forging hammer (size 3, type B) which is approximately equivalent to a 400-pound steam hammer. This hammer was fitted with 3½- by 8-inch flat dies.

Preliminary forging to determine the forging temperature range for the alloys was done on ¾-inch-square bar stock of the basic alloy, reducing it to 0.43-inch squares from 2100°, 2200°, 2300°, and 2400° F. The plasticity increased with temperature, but burning, which resulted in forging cracks, occurred at 2400° F. Further experiments on a 0.55-percent-C heat (alloy 5) indicated that burning could occur at 2300° F in this alloy. Since the alloy was quite difficult to forge with an initial temperature of 2100° F it was decided to use 2200° F as the initial forging temperature. Subsequent work on the heats with variable composition showed that all the alloys were forgeable from this temperature, the forgeability, however, varying with the composition.

Finishing forging temperatures were kept at 1800° F or above as judged by color, in order to minimize hot-cold-work, which is known to influence the properties of these alloys.

Several ingots were forged to ½-inch-square bars between the flat dies, a reduction of approximately 90 percent. Microstructural examination of these forged bars showed a nonuniformity of grain size in the cross section. A diagonal "X," distinctly visible on visual examination of the etched cross section, was present in all the bars. The grains were appreciably finer along these diagonal planes than in the triangles, under the flat surfaces of the bar, formed between these planes.

A number of different methods of forging were tried in order to determine the cause of and in an effort to eliminate this grain size nonuniformity. The methods used and their results were:

(1) Forging at a 45° angle to the original square of the ingot resulted in the same diagonal "X" from the corners of the final square indicating that the ingot structure bore no relation to this characteristic.

(2) Upset forging prior to final reduction to a square showed no advantage of the added working on the structural uniformity.

(3) Finishing as an octagon between flat dies yielded areas of fine grains originating at all eight corners of the bar.

(4) Forging wholly between a flat top die and bottom "V" dies to an octagon appeared to abuse the metal more than any other method used. It was difficult to draw the metal out, the cross section merely being changed back and forth. This tended to open up the center of the bar.

(5) Forging to approximately 1 inch square and finishing to ½ inch round between hand swages in four steps gave the best structural uniformity. A narrow band of fine grains was sometimes present near the surfaces of the round bars but the over-all grain size was quite uniform.

The above experiments showed that the forging "X" in the square bars was the result of the relatively greater amount of hot-work along the diagonal planes. This was further verified by the fact that occasionally in forging the square bars diagonal cracks would form completely through the bars.

As a result of the forging experiments it was decided to fit the hammer with a set of dies with swaging impressions. Subsequent experiments using this method proved satisfactory so it was adopted as the standard forging procedure for the experimental alloys.

Forging procedure.—The general arrangement of the 400-pound-capacity air forging hammer and the heating furnace is shown in figure 2. A picture of the forging operation in swaging dies is shown in figure 3. Figure 4 shows a bar which was forged to indicate the steps in reduction from the ingot to the final round bar. A typical forging record is presented in table IV.

The temperature of the furnace was controlled by an automatic temperature controller through a platinum, platinum-rhodium thermocouple. The temperature of the forg-

ing stock was measured by a Chromel-Alumel thermocouple attached to a small bar placed next to the stock.

The cast ingots were ground to remove all surface defects prior to forging. The ingots were preheated to approximately 1400° F, then placed just inside the door in the coolest region of the furnace, and finally moved to the center of the furnace at 2200° F.

Generally half of an ingot was forged at a time. After holding at least 30 minutes at 2200° F the ingot was forged from approximately 1.4 inches square to slightly over 1 inch square between flat dies with from two to four reheats. A number of the blows in the flat dies were on the corners of the bars to prevent corner cracks and to prepare the piece for the first swage. The bar was reheated and forged to approximately 0.95 inch round in the first swage with from one to three reheats. The second swage reduced the bar to approximately 0.75 inch round with two to four reheats. The forged bar was then cut off the unforged half of the ingot and recharged to the furnace. The third swage reduced the bar to approximately 0.58 inch round with three to six reheats. The bar was then cut in two equal pieces for the final swaging. The last swage finished these bars to 0.40 to 0.50 inch round with from 5 to 10 reheats. The number of blows from the hammer were counted and tabulated. The number of reheats and blows varied depending on the ease of forging of the alloy and the size of the stock. Total reduction of area during forging was approximately 90 percent

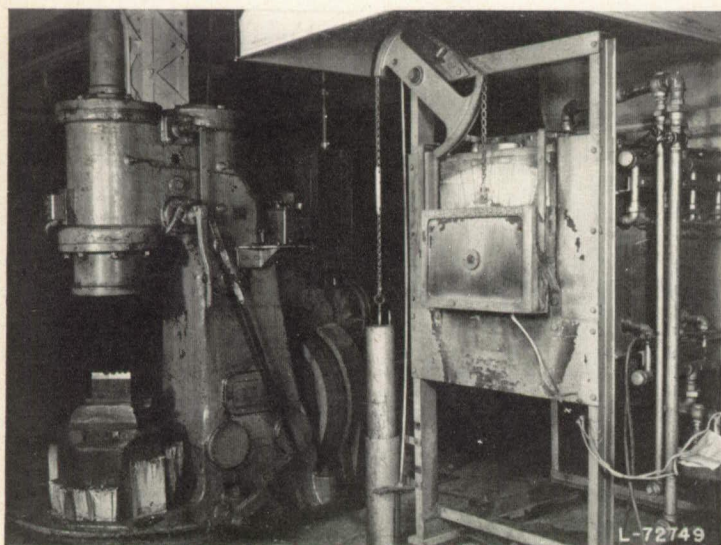


FIGURE 2.—Arrangement of forging hammer and furnace.



FIGURE 3.—Forging experimental alloy in swaging dies.

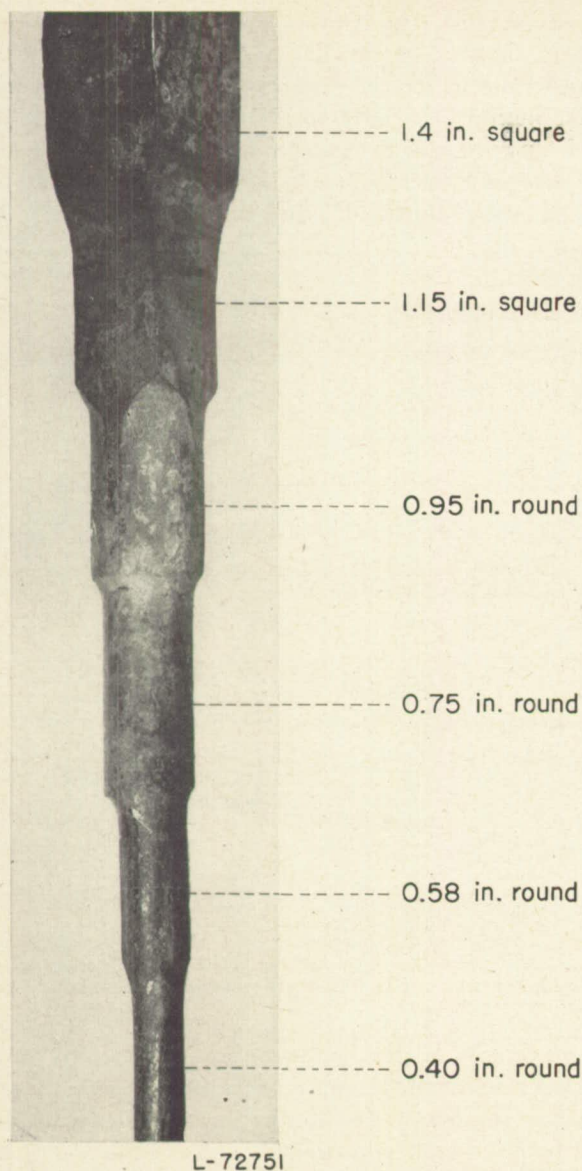


FIGURE 4.—Forged bar showing reduction steps used in swaging experimental alloys from original ingot to final round bar stock.

Reduction in the individual swages was done in steps; that is, after the first reheat only a portion of the bar was reduced to the swage diameter, after the second reheat another portion of the bar was reduced, but in the last reheat or two the whole length of the bar was worked to the final diameter of that swage. In particular, in the finishing swage, the complete length of the bar was given some work during the final two or three operations.

Finishing temperature of forging was judged by the color of the stock by an experienced forger. An attempt was made to hold this temperature above 1800° F.

Forging characteristics.—During forging an effort was made to evaluate the relative forgeability of the experimental alloys. This evaluation consisted of observing the forging characteristics of the alloys—ease of reduction and tendency to form cracks—and of keeping complete records of the number of reheats and blows required to forge each alloy. These methods were at best only very approximate.

Since the small hammer used was taxed to capacity during the initial stages of the breakdown by the rigidity of most of the alloys and since the force of the blows was varied some-

what, depending on the forgeability of the alloys, the method of comparing the required blows was usable only for distinguishing between alloys with wide differences in plasticity. Judging the relative response of the alloys to the blows from the hammer seemed to be the best method to evaluate forgeability. Relative forgeability could be only qualitatively estimated, however, by this method, because only two ingots were forged together and only a few forged in 1 day.

In table V a summary is given of forging data and the estimated forgeability of all the alloys compared with that of the basic alloy. This forgeability is listed merely as "better," "poorer," or the "same" as that of the basic alloy. In cases where the alloys are either poorer or better, remarks are given to explain this rating.

On the basis of these forgeability ratings the following effects of composition on forgeability were evident:

(1) Additions of Si (0.5 to 1.6 percent), Mo (0 to 7 percent), and N (0.004 to 0.18 percent) had no observable effect on forgeability.

(2) It appeared that Mn (0 to 2.5 percent) increased forgeability by lowering the tendency of cracking.

(3) Additions of C (0.08 to 0.60 percent) decreased forgeability by lowering the plasticity and increasing the tendency of cracking.

(4) The elements Ni (0 to 30 percent), Co (0 to 32 percent), and Cr (10 to 30 percent) lowered plasticity and thus forgeability, Cr seemingly having the greatest effect.

(5) Additions of W (0 to 7 percent) lowered plasticity and forgeability.

(6) The effect of Cb (0 to 6 percent) was unique among the elements. Alloys without Cb were subject to severe cracking during the initial breakdown of the ingot between flat dies, the forgeability thus being poorer than that for the basic alloy. Increasing Cb to the 1 percent of the basic alloy seemed completely to alleviate this situation. When it was increased to 4 and 6 percent the material was more difficult to forge as a result of the decreased plasticity.

HEAT TREATMENT

Originally a solution treatment at 2100° F was considered to be a satisfactory condition for testing. This was on the basis that for one heat of the basic alloy previously studied (reference 1), for which the solution-treatment temperature was varied from 1800° to 2300° F, a treatment at 2100° F gave slightly higher rupture strengths than the other conditions. Further work, however, indicated that there was poor agreement between properties of two heats in this condition and the best agreement between heats resulted from a 2200° F solution treatment followed by a 1400° F aging treatment. The variability in properties between heats was attributed to the relative residual effects of the hot-working which had not been removed by the 2100° F solution treatment. It appeared that the 2200° F treatment tended to minimize, but did not completely remove, the residual hot-working effects without excessively coarsening the grain size. It was therefore selected as the solution temperature for the alloys in this investigation.

Previous investigation of the basic alloy had shown that at temperatures of solution treatment above 2050° F the 100-hour rupture test elongation was quite low and decreased

with increasing solution temperatures. Accompanying this low elongation was a tendency for the alloy to be sensitive to stress concentrations at the specimen fillets, fractures occurring in these regions. However, proper aging of the solution-treated alloy eliminated this stress-concentration sensitivity and increased the 100-hour rupture strength slightly while not appreciably affecting 1000-hour strengths. Aging between 1350° and 1500° F did not produce significant changes in properties. A temperature of 1400° F for 24 hours was selected for aging the experimental alloys in this investigation so that variable effects resulting from stress-concentration sensitivity would be minimized and the materials would be in a more stable condition than when only solution-treated.

It is known that the treatment used does not give the highest strengths at high temperatures for the basic alloy and it is not believed that this treatment necessarily would give the best properties for any of the composition modifications studied. But it is believed that use of this treatment, plus the careful control to hold melting and fabrication conditions as constant as possible, has minimized all variable effects on properties except chemical composition in this investigation.

Heat-treatment procedure.—All of the alloys were solution-treated for 1 hour at 2200° F, water-quenched, aged 24 hours at 1400° F, and air-cooled prior to rupture testing. Solution treatments were made in a gas-fired furnace and aging was in an electric resistance furnace.

Heat-treatment observations.—An observation made during heat treatment of the effect of Cb on scaling characteristics of the alloys appeared to be significant. In alloys containing four or more percent of Cb relatively heavy scaling occurred during the 1-hour solution treatment at 2200° F. This scaling effect was greater on these high-Cb alloys than on the alloy containing 10 percent Cr which was expected to have the poorest corrosion resistance of the alloys studied.

REPRODUCIBILITY OF RUPTURE PROPERTIES AT 1200° F OF BASIC ALLOY

The final objective of the work on techniques for preparing the experimental alloys was the development of a condition in which the rupture properties at 1200° F would be reproducible from heat to heat of a given composition. The interpretation of significant property variations with chemical composition depends on the range of this reproducibility.

The rupture test characteristics at 1200° F for six heats of the basic alloy, the alloy used to evaluate the preparation procedures, are given in table VI, for both square and round bar stock. The range of reproducibility of rupture properties indicated for the round stock forged by the finally adopted swaging procedure was as follows:

100-hour rupture strength, psi.....	48,000 to 50,000
1000-hour rupture strength, psi.....	37,000 to 38,000
100-hour rupture elongation, percent.....	19 to 25

These ranges are used in interpreting the significant influences of composition variables for the materials forged as rounds. Properties of alloys with variable C and Mn, forged early in the program as square stock and tested as such, are compared with the wider property range indicated in table VI for square stock of the basic alloy.

The stress and rupture-time data for the six heats are plotted in figure 5 with curves drawn which indicate the range in strengths for the round stock. Figure 6 presents in a similar manner the data for stress against creep rate obtained from curves of elongation against time for the rupture tests. The narrower property ranges for the round stock compared with those of the square stock are noted.

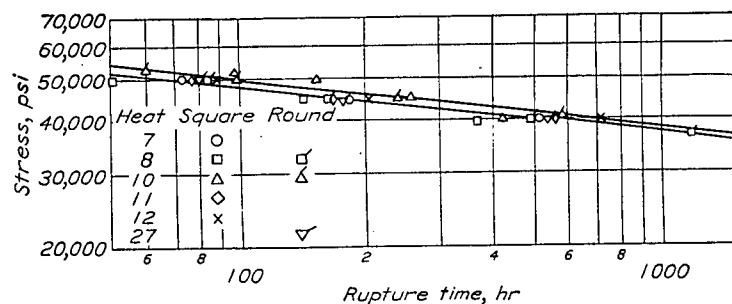


FIGURE 5.—Curves of stress against time for rupture for six heats of basic alloy. Range indicated for round bar stock. Treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours.

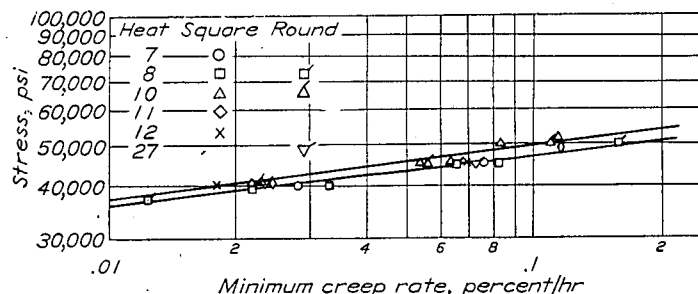


FIGURE 6.—Curves of stress against minimum creep rates at 1200° F for six heats of basic alloy. Range indicated for round bar stock. Treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours.

As an example of the narrowing of the range by using the more easily controlled and duplicated swaging procedure, alloy 8 had the lowest (46,500 psi) and alloy 10 the highest (52,000 psi) 100-hour rupture strengths of the 5500-psi over-all range for square bar stock. After forging as rounds these same two heats had 100-hour rupture strengths of 48,500 and 50,000 psi, thus narrowing the over-all range of 100-hour rupture strengths to 1500 psi. The 1000-hour rupture-strength range was narrowed from 2500 to 1000 psi by changing the forging procedure. Similar changes in ranges for squares and rounds are indicated in table VI for stresses at constant creep rates and for the minimum creep rate at constant stress.

Table VI also gives properties of alloys 74 and 75 which indicate that variation in melting deoxidation practice can have a pronounced effect on high-temperature characteristics of alloys of this type.

The influence on rupture test characteristics of changing hot-working and deoxidation procedures indicates that the solution treatment at 2200° F did not completely eliminate variables in prior processing. However, the use of the same deoxidation practice for all the alloys and the control of hot-working resulting from the adoption of the procedure producing rounds rather than squares plus the use of the 2200° F solution treatment are believed to have minimized the effects of variables in processing on the high-temperature properties.

Data for two commercial heats of the basic alloy with the same heat treatment as the experimental alloys have been included in table VI to show the relationship between the experimental alloys and normal commercial properties.

INFLUENCE OF CHEMICAL-COMPOSITION VARIABLES ON ROOM-TEMPERATURE METALLURGICAL CHARACTERISTICS OF MODIFIED ALLOYS

Microstructural studies and hardness tests were made on all the experimental alloys to obtain metallurgical information to aid in interpretation of the influence of chemical composition on rupture properties.

MICROSTRUCTURE

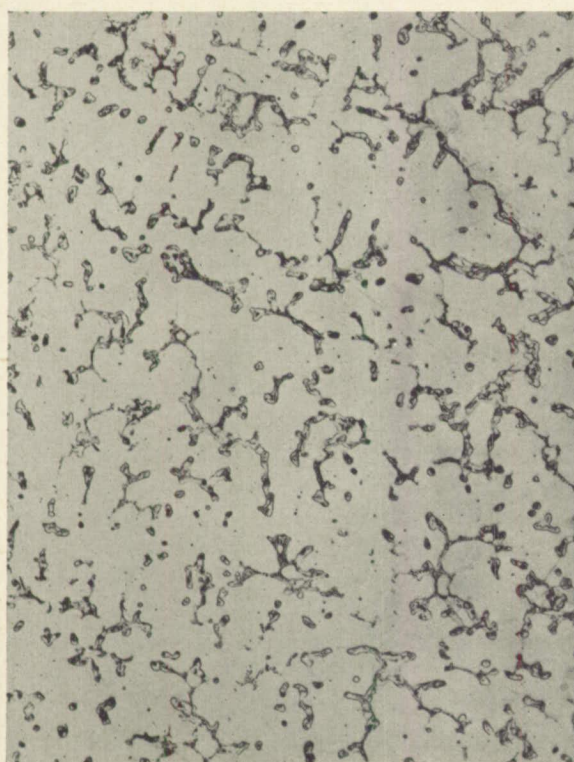
Microstructural studies were made of each of the alloys in the following conditions: As-cast; hot-forged; solution-treated 1 hour at 2200° F and water-quenched; aged 24 hours at 1400° F after solution treatment; and rupture-tested the maximum time at 1200° F. An electrolytic solution of 10 percent chromic acid was used as an etchant.

Effect of treatment on microstructures.—The as-cast structure revealed the amount and distribution of the excess constituents in the cast dendritic pattern. The hot-forged structures were examined for flaws, uniformity of grain size, and distribution of excess constituents. Only limited information is included on the cast and forged structures, the main emphasis being placed on structures of the alloys prior to testing in the heat-treated condition. The solution-treated structure showed the amount of insoluble excess constituents after light etching and the grain size after deep etching. The grain size was much easier to determine in the

aged structures because the grain boundaries were revealed by light etching. The aged structure also revealed the amount and mode of matrix precipitation. From the structure of the maximum-time rupture test any change occurring during testing and the mode of fracture could be observed.

An example of microstructural variation with treatment for the basic alloy is shown in figure 7. The as-cast condition contained a considerable amount of excess constituent. After hot-forging the alloy was uniformly fine-grained and contained excess constituent at the grain boundaries and in the matrix. Solution-treating at 2200° F dissolved the grain boundary constituent but not the matrix constituent. Solution-treating also coarsened the grains considerably as is best shown by the aged structure. Aging caused considerable grain boundary precipitate and some random matrix precipitation. Rupture testing at 1200° F increased the matrix precipitate slightly.

As an example of one extreme to which microstructures varied with composition, figure 8 shows microstructures for the 0Mo-0W-0Cb alloy in the as-cast, solution-treated, aged, and rupture-tested conditions. The structures differed from those of the basic alloy in that: There was much less excess constituent in the as-cast and the solution-treated conditions; grain size after solution treatment was much larger; and aging produced more precipitate near the grain boundaries and a preferred type of matrix precipitate which tended to follow definite crystallographic planes. The two alloys also differed in that the 0Mo-0W-0Cb alloy had a completely intergranular fracture while that of the basic alloy was approximately half intergranular and half transgranular.



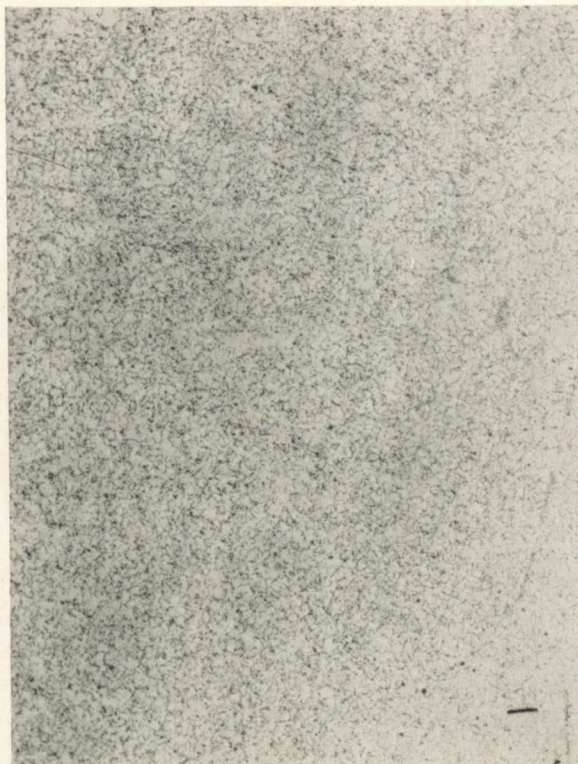
100X

(a) Ingot.



1000X

FIGURE 7.—Microstructures of basic alloy (heat 8).

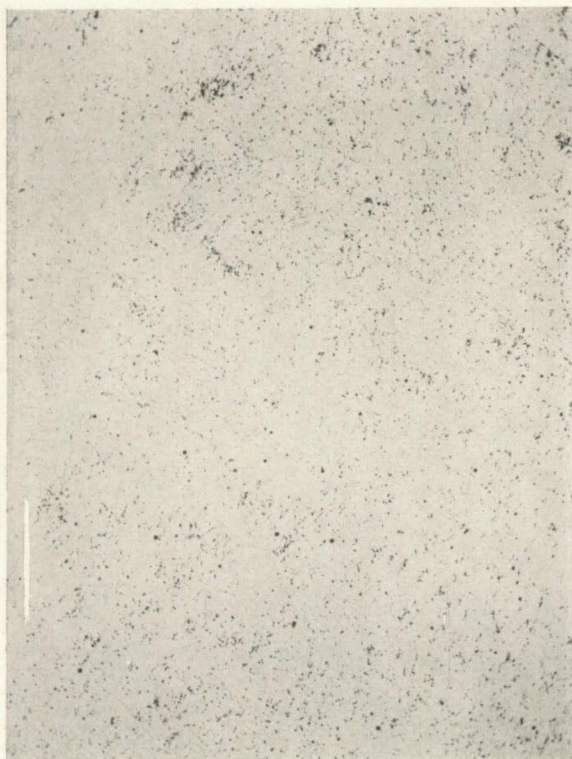


100X



1000X

(b) As-forged 0.4-inch-diameter bar stock.



100X



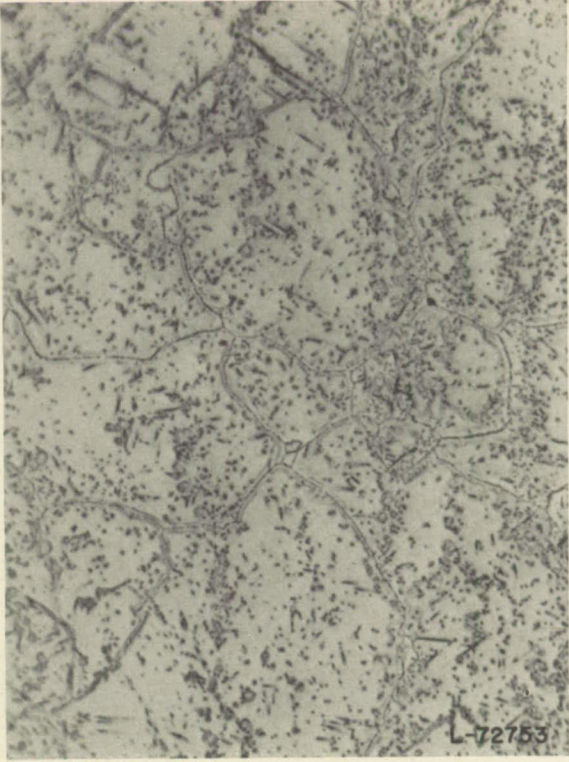
1000X

(c) Solution-treated; 1 hour at 2200° F, water-quenched.

FIGURE 7.—Continued.

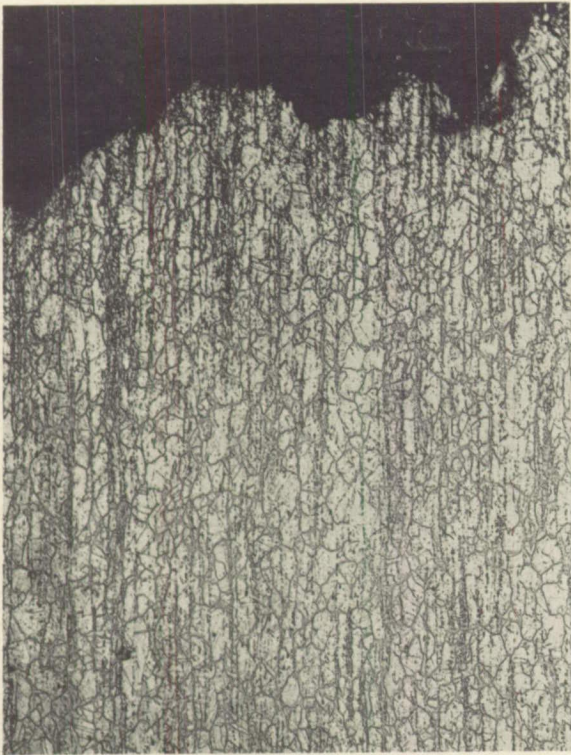


100X

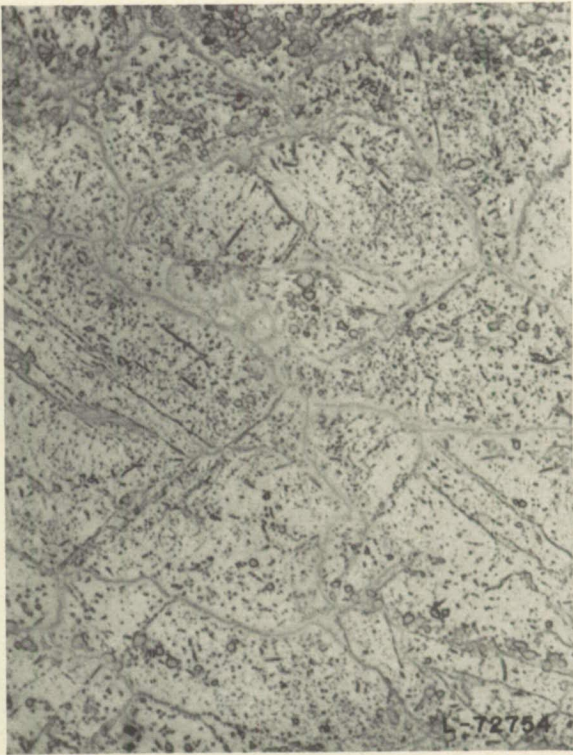


1000X

(d) Solution-treated and aged 24 hours at 1400° F.



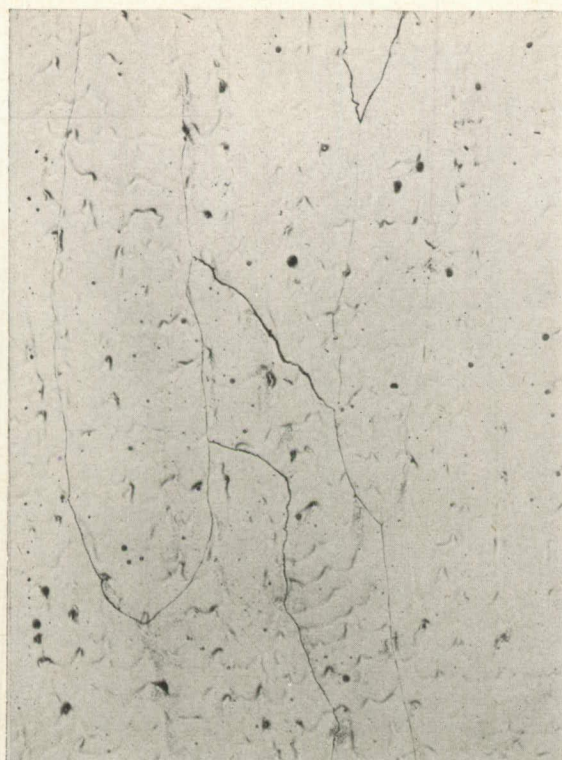
Fracture 100X



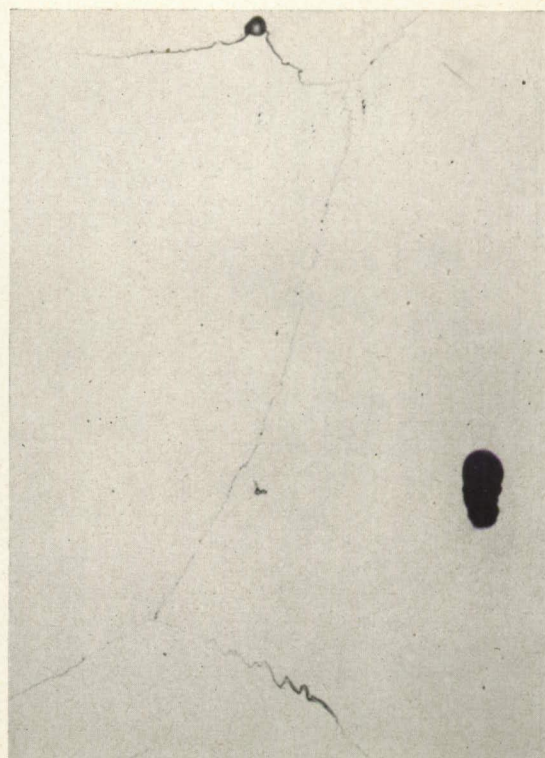
1000X

(e) Rupture specimen; 499 hours for rupture at 1200° F under 40,000 psi.

FIGURE 7.—Concluded.

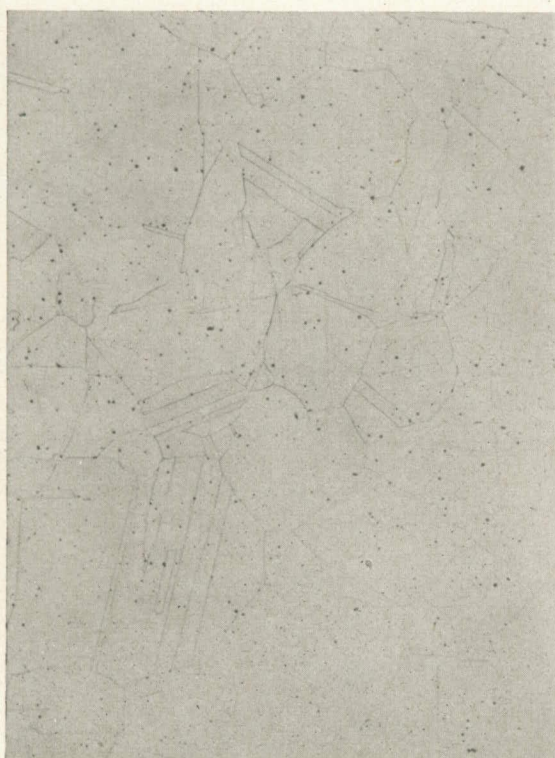


100X

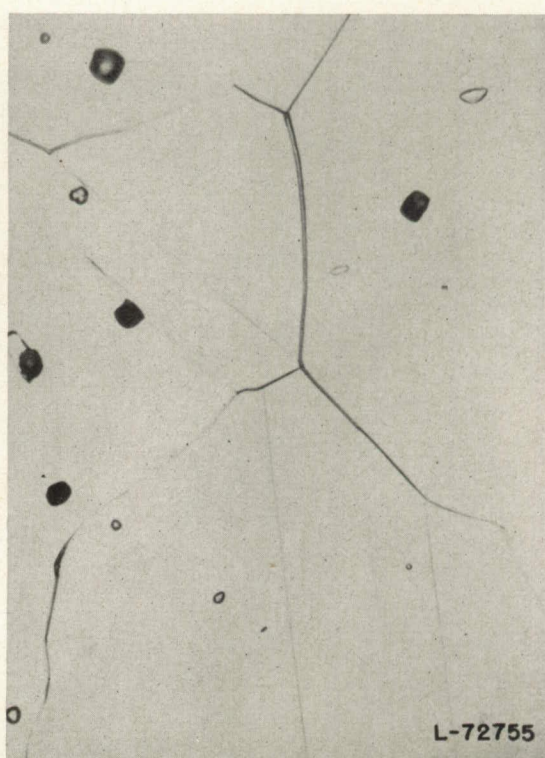


1000X

(a) Ingot.



100X



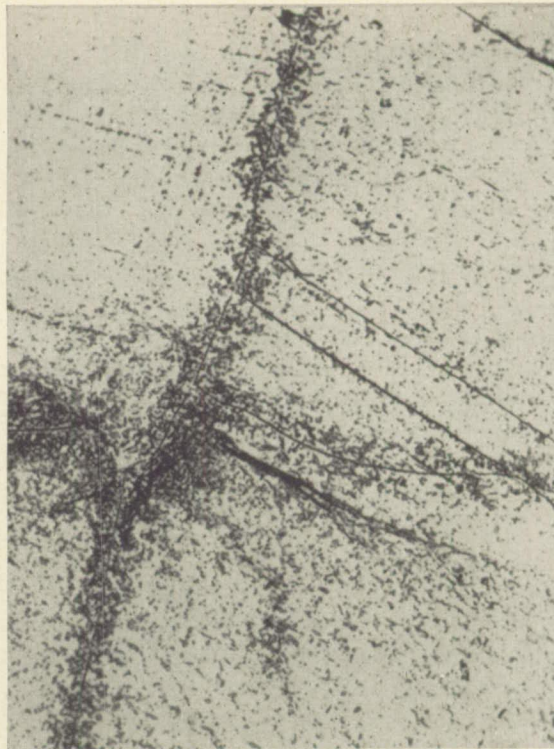
1000X

(b) Solution-treated; 1 hour at 2200° F, water-quenched.

FIGURE 8.—Effect on microstructure of omitting molybdenum, tungsten, and columbium from basic analysis (heat 43).



100X



1000X

(c) Solution-treated and aged 24 hours at 1400° F.



100X



1000X

(d) Rupture specimen; 819 hours for rupture at 1200° F under 20,000 psi.

FIGURE 8.—Concluded.

The above comparison has pointed out the major similarities and differences in structures which were found for all the alloys. Similarities were:

(1) The amount of excess constituent in the solution-treated condition was proportionate, for any given alloy, to the amount of excess constituent in the as-cast condition. The structures indicate that the rapid cooling of the ingot after solidification effectively solution-treated the material.

(2) All grain boundary constituents were dissolved during solution treatment.

(3) Precipitation occurred during aging at 1400° F.

(4) Very little additional precipitation occurred during rupture testing at 1200° F for any of the alloys with the exception of low-Co modifications.

Major structural differences between the alloys were:

(1) Amount of excess constituent after solution treatment.

(2) Grain size.

(3) Location, type, and amount of aging precipitate.

Classification of microstructures.—In order to condense the description of and to make possible a quick comparison of the changes in microstructure occurring with variation in chemical composition the following classification of microstructures of the solution-treated and the aged materials was devised. Symbols were used to indicate the differences in the basic structural characteristics.

(1) Solution-treated structures

Amount of insoluble constituent:

I—small

II—medium

III—large

IV—very large

(2) Aged structures

A. S. T. M. grain size number:

1 (up to $1\frac{1}{2}$ grains/sq in. at 100X magnification) to 8 (96 grains or more/sq in. at 100X magnification)

Amount of grain boundary precipitate:

x—small

y—medium

z—large

Type of matrix precipitate:

C—precipitate tends to follow crystallographic planes

R—random matrix precipitate

Amount of matrix precipitate:

a—small

b—medium

c—large

d—very large

As an example of how this classification works the structures of the basic alloy (fig. 7) and the 0Mo-0W-0Cb modification (fig. 8) are classified below:

(1) Basic alloy:

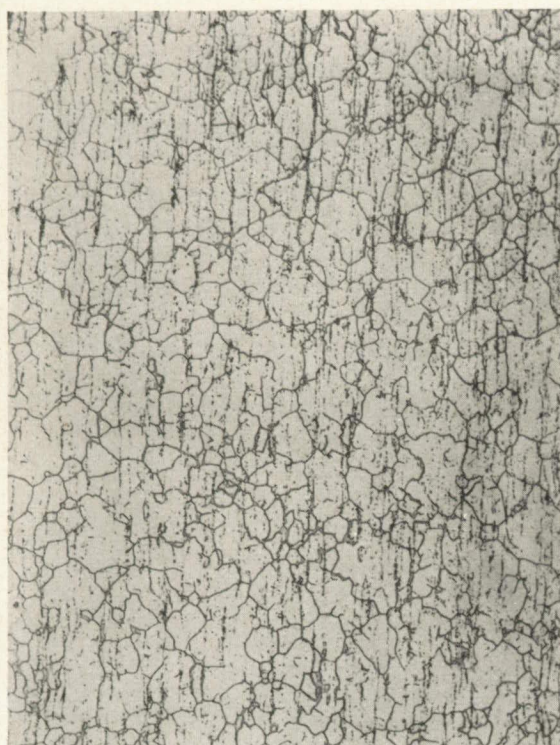
Classification—II 5 y Rb

II—medium amount of insoluble constituent

5—number 5 A. S. T. M. average grain size

y—medium amount of precipitate near the grain boundaries

Rb—medium (b) amount of random (R) matrix precipitate



100X



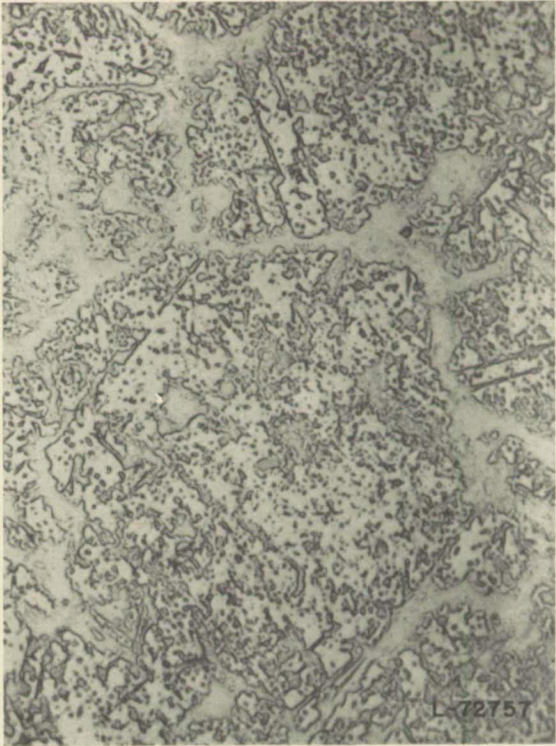
1000X

(a) Carbon, 0.08 percent (heat 13).

FIGURE 9.—Influence of carbon content on microstructure of solution-treated and aged basic alloy. Treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours.



100X

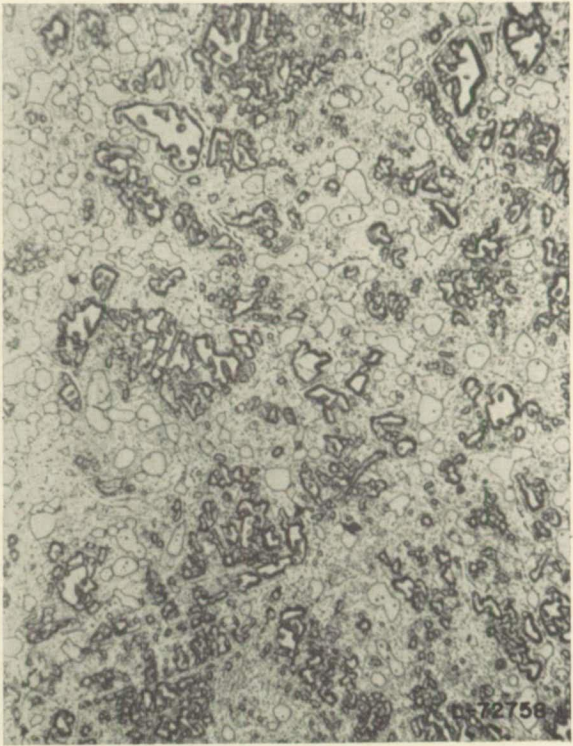


1000X

(b) Carbon, 0.40 percent (heat 15).



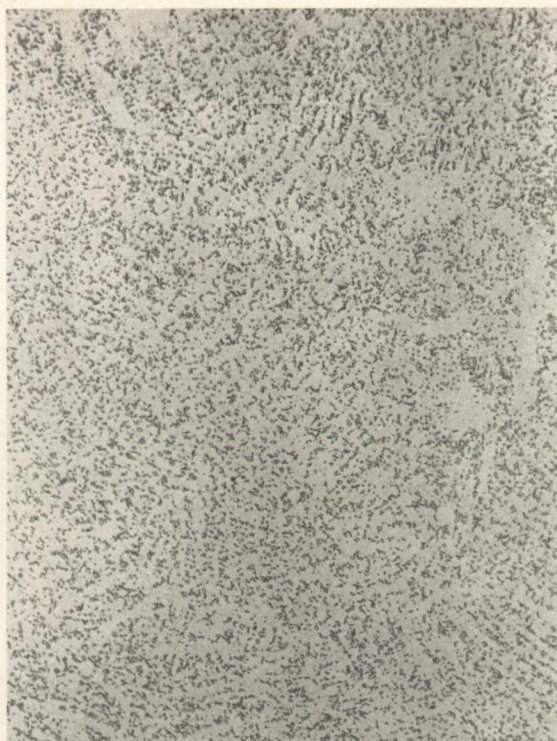
100X



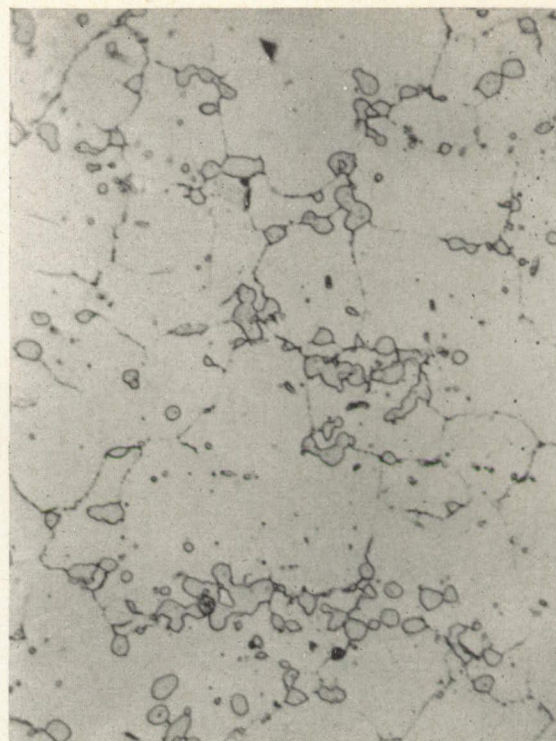
1000X

(c) Carbon, 0.60 percent (heat 16).

FIGURE 9.—Concluded.

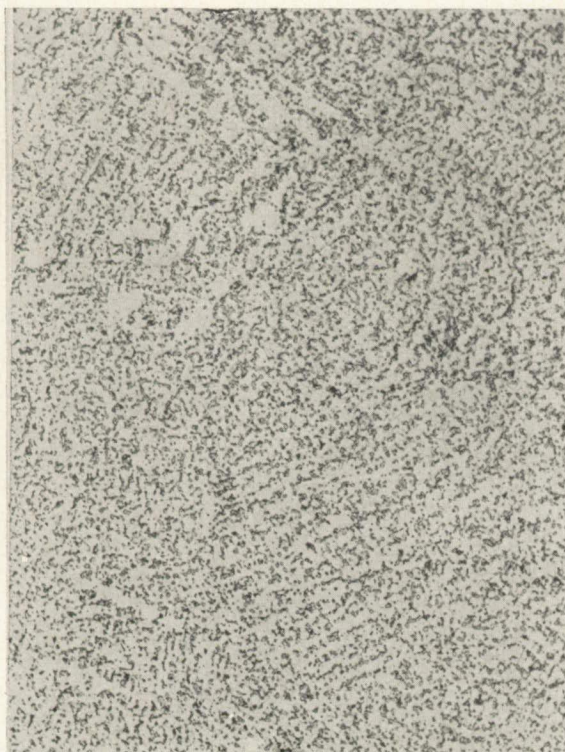


100X

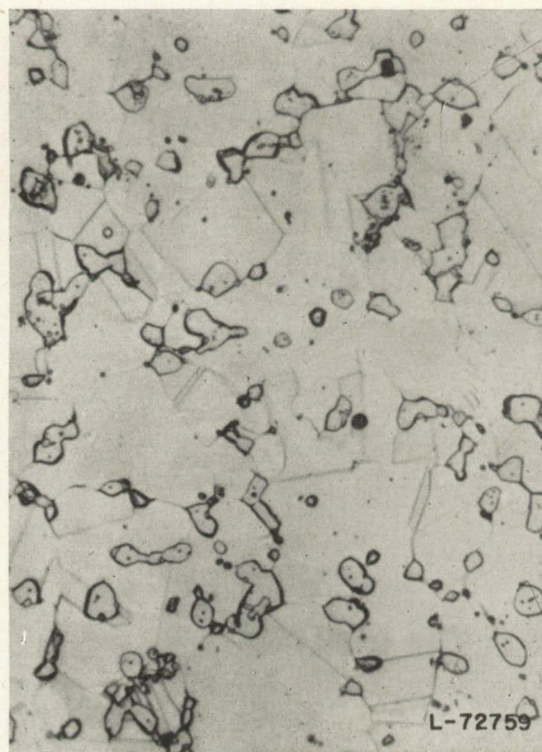


1000X

(a) 1 hour at 2200° F, water-quenched.



100X



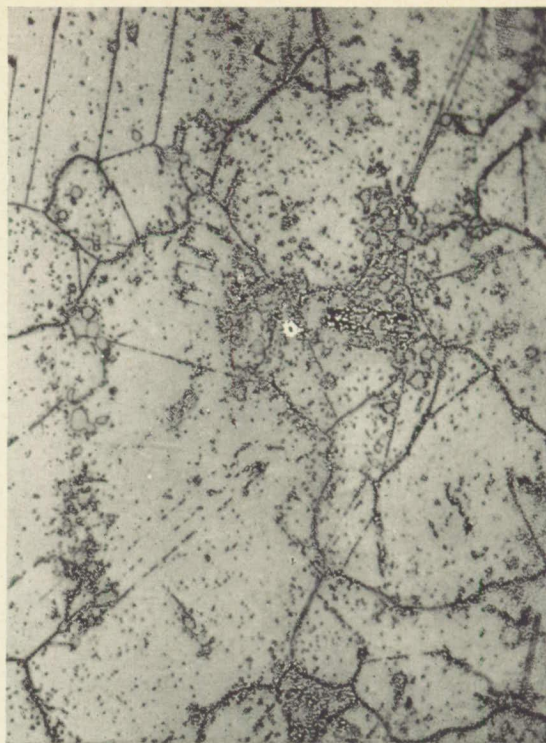
1000X

(b) 10 hours at 2200° F, water-quenched.

FIGURE 10.—Effect of increasing solution time on microstructure of 0.60-percent-carbon basic alloy (heat 16).



100X

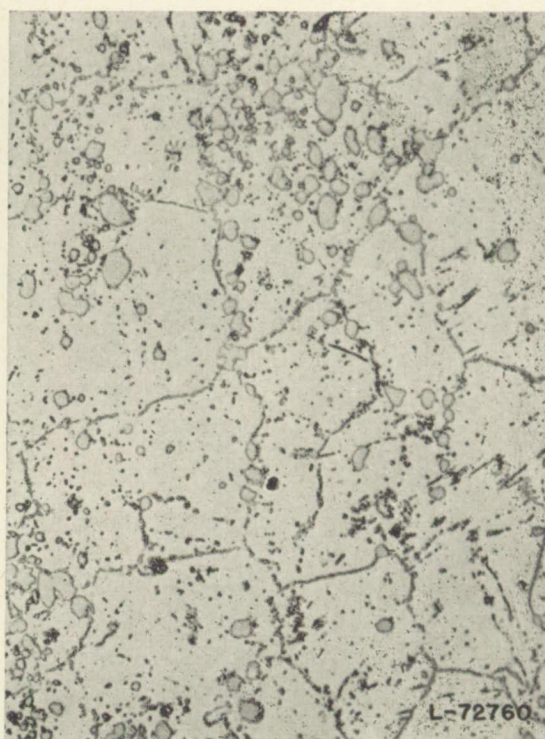


1000X

(a) Silicon, 1.2 percent (heat 28).



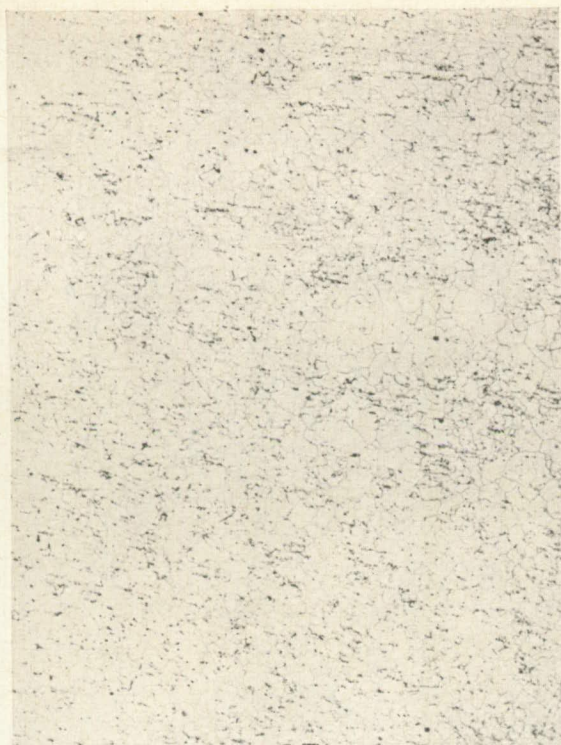
100X



1000X

(b) Silicon, 1.6 percent (heat 80).

FIGURE 11.—Influence of silicon content on microstructure of solution-treated and aged basic alloy. Treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours.



100X

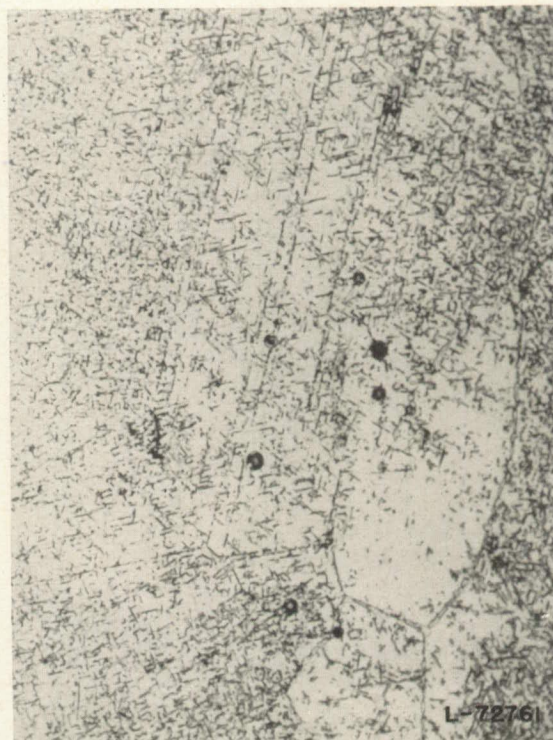


1000X

(a) Chromium, 10 percent (heat 51).



100X



1000X

(b) Chromium, 30 percent (heat 52).

FIGURE 12.—Influence of chromium content on microstructure of solution-treated and aged basic alloy. Treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours.

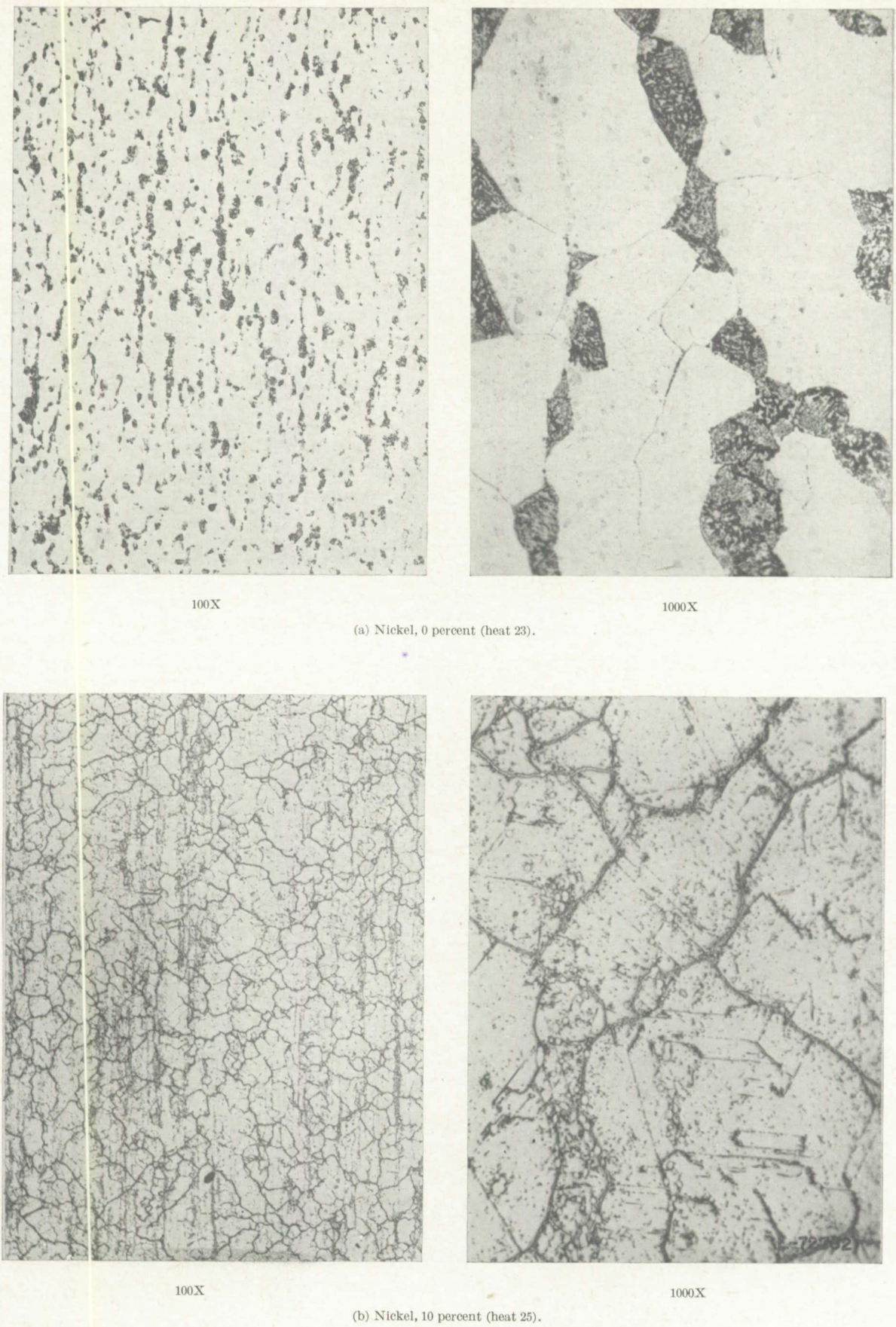
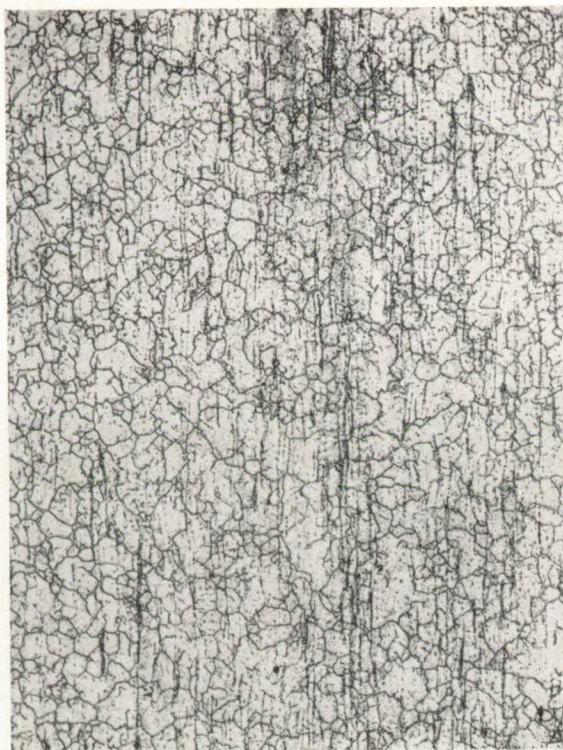
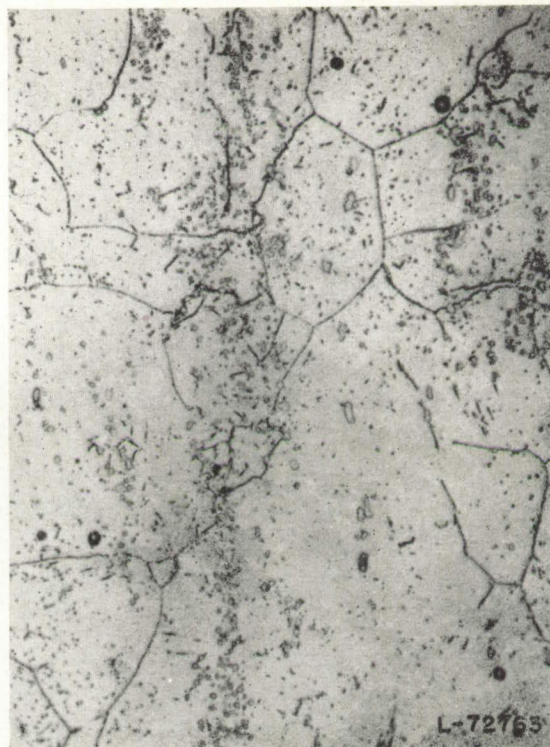


FIGURE 13.—Influence of nickel content on microstructure of solution-treated and aged basic alloy. Treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours.



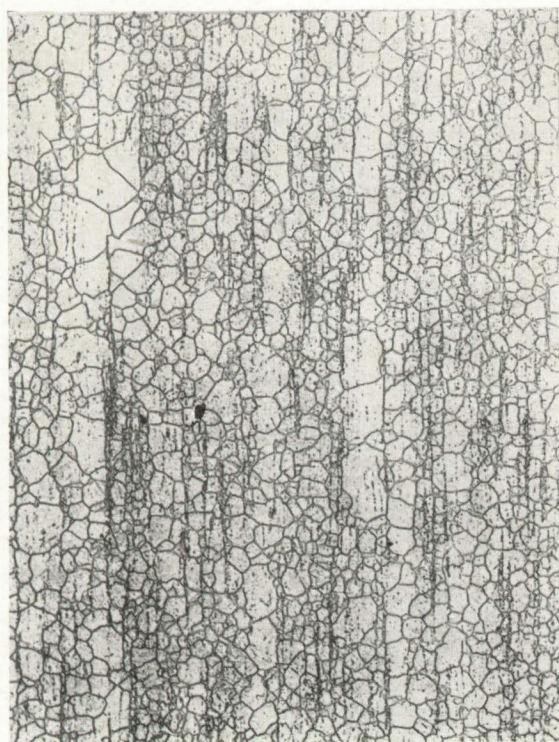
100X



1000X

(c) Nickel, 30 percent (heat 26).

FIGURE 13.—Concluded.



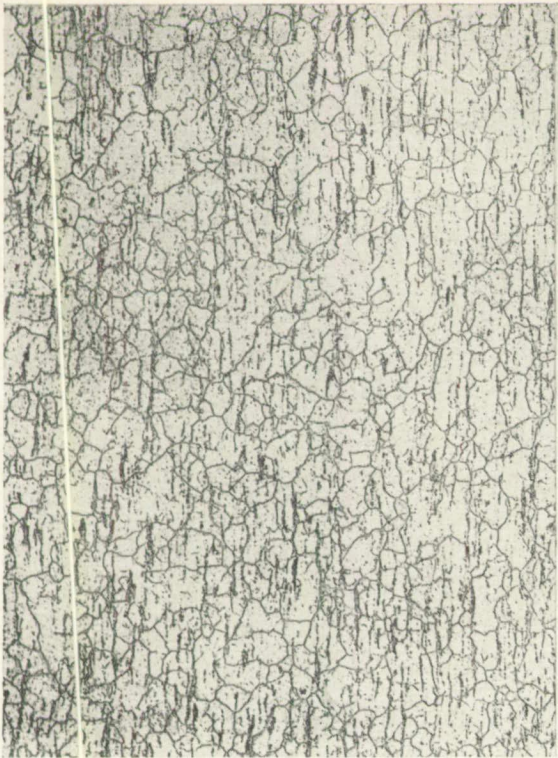
100X



1000X

(a) Cobalt, 0 percent (heat 29).

FIGURE 14.—Influence of cobalt content on microstructure of solution-treated and aged basic alloy. Treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours

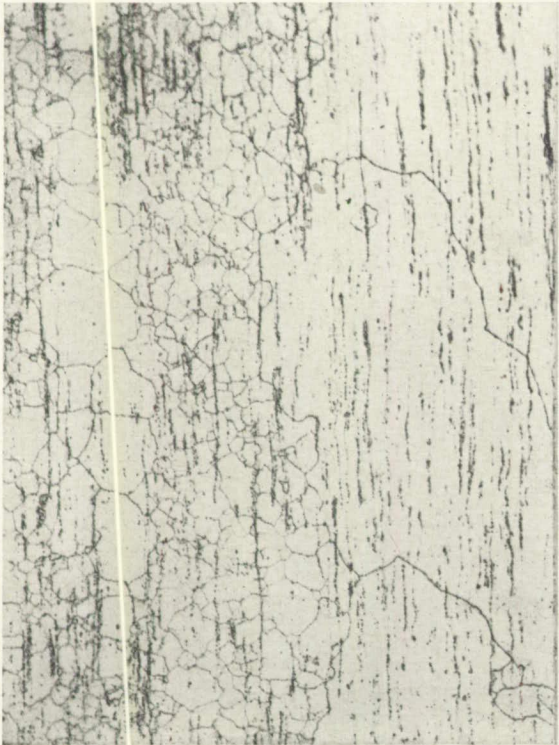


100X

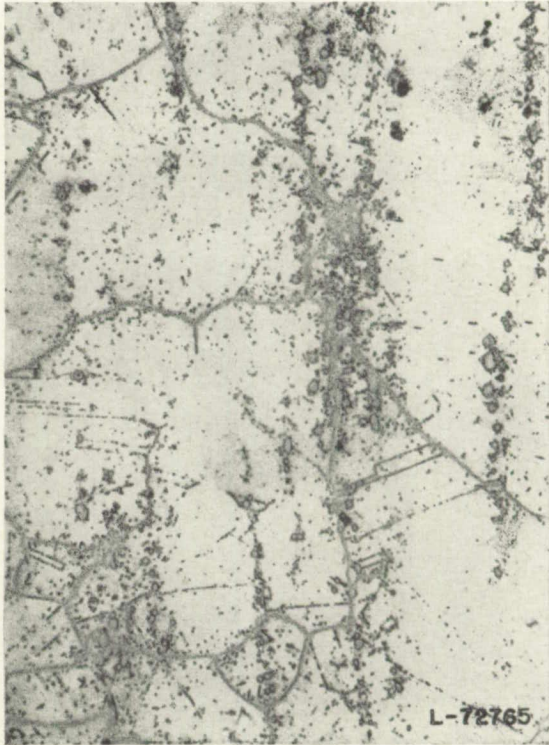


1000X

(b) Cobalt, 10 percent (heat 30).



100X



1000X

(c) Cobalt, 32 percent (heat 31).

FIGURE 14.—Concluded.

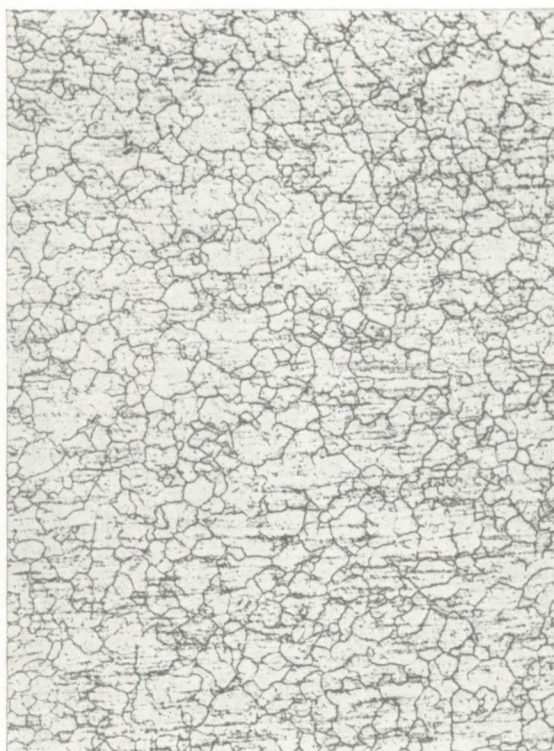


100X



1000X

(a) Columbium, 0 percent (heat 47).



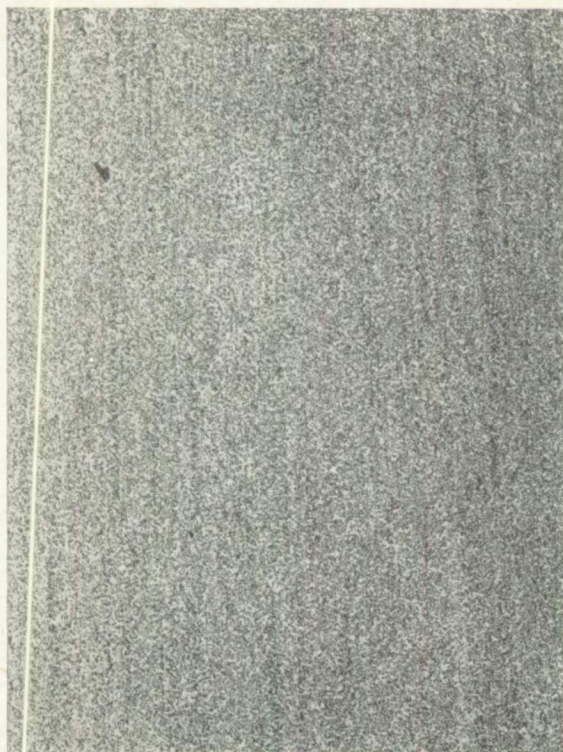
100X



1000X

(b) Columbium, 2 percent (heat 48).

FIGURE 15.—Influence of columbium content on microstructure of solution-treated and aged basic alloy. Treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours.

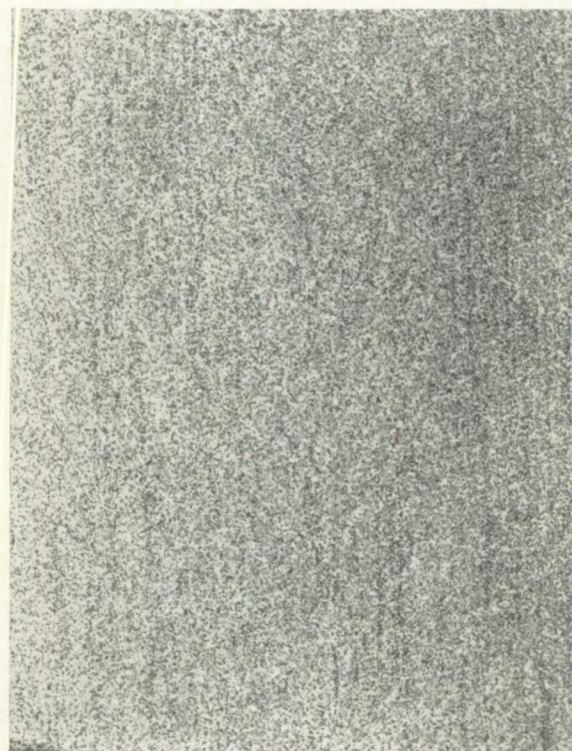


100X

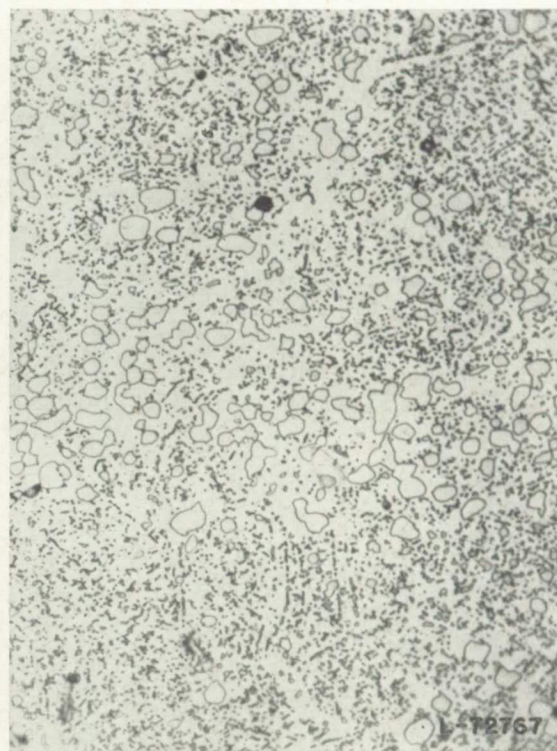


1000X

(c) Columbium, 4 percent (heat 49).



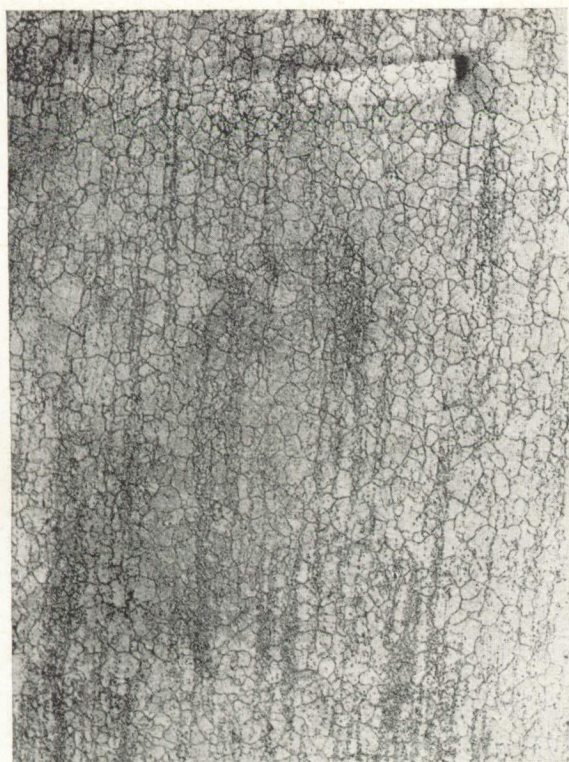
100X



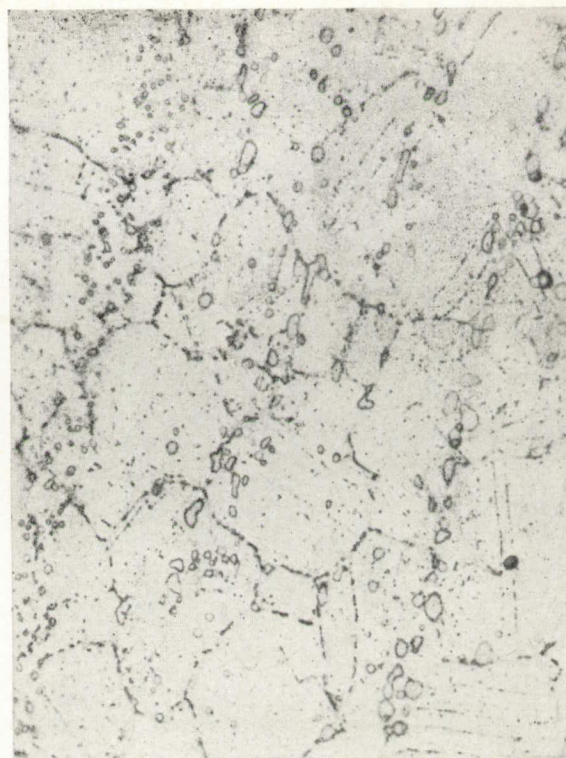
1000X

(d) Columbium, 6 percent (heat 50).

FIGURE 15.—Concluded.

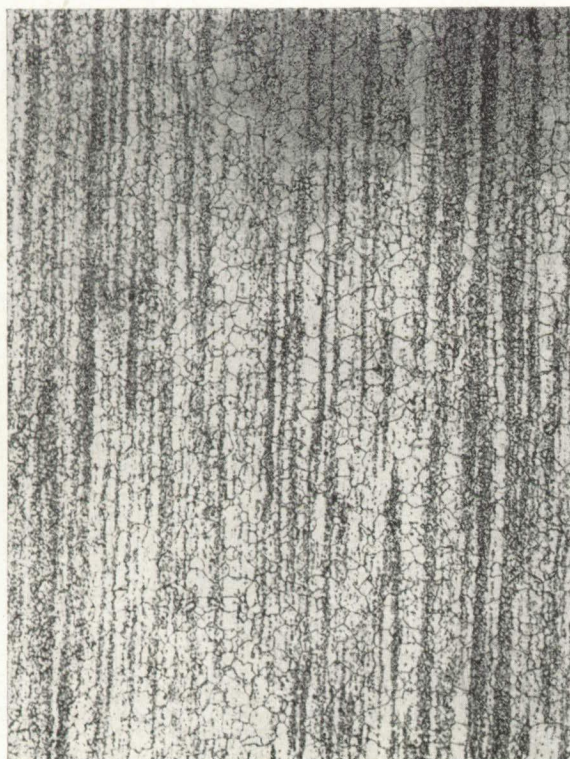


100X

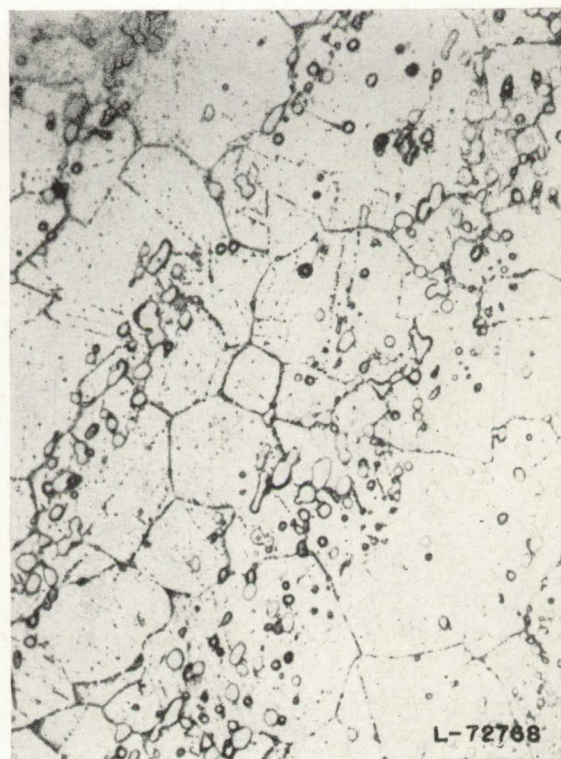


1000X

(a) Molybdenum, 0 percent; tungsten, 0 percent; columbium, 4 percent (heat 69).



100X



1000X

(b) Molybdenum, 2 percent; tungsten, 2 percent; columbium, 4 percent (heat 62).

FIGURE 16.—Influence of varying molybdenum and tungsten additions on microstructure of modified alloys containing 4 percent columbium. Treatment: 2200° F for 1 hour, water-quenched 1400° F for 24 hours.



100X

(c) Molybdenum, 4 percent; tungsten, 4 percent; columbium, 4 percent (heat 67).



1000X

FIGURE 16.—Concluded.

(2) 0Mo-0W-0Cb alloy:

Classification—I 1 : Cb

I—small amount of insoluble constituent

1—number 1 A. S. T. M. average grain size

z—large amount of precipitate near the grain boundaries

Cb—medium (b) amount of matrix precipitate on crystallographic planes (C)

The microstructural classifications are given in table VII for all the experimental alloys arranged for comparative purposes on the basis of the alloying element varied. For the alloys in which simultaneous variations were made in Mo, W, and Cb the classifications are repeated to show the influence of individual variations of these elements.

Influence of alloying elements on microstructure.—The effects on microstructure of variations of the elements will be given on the basis of the classification of table VII, supplemented by photomicrographs showing only the more pronounced effects observed.

Carbon: It is indicated in table VII and by the microstructures in figure 9 that as C was increased from 0.08 to 0.60 percent in the basic alloy the amount of insoluble constituent and matrix precipitate increased considerably and the grain size decreased, while the amount of grain boundary precipitate and the randomness of the matrix precipitate did not change. Figure 10 also shows that the amount of insoluble constituent was not reduced by heating the 0.60-percent-C alloy 10 hours at 2200° F over what it was after 1 hour.

The increase in the amount of soluble and insoluble constituents indicates that C is an important component in both

the soluble and precipitating phases found in the basic alloy and its modifications.

Manganese: There was no apparent effect of Mn on microstructure.

Silicon: As will be shown later, Si was the only element which consistently lowered rupture strength over the complete composition range studied (from 0.5 percent in the basic alloy to 1.6 percent). The microstructures in figure 11 indicate that variation of Si did not change microstructural characteristics except for a smaller grain size for the high-Si alloy.

Chromium: Increasing Cr from 10 percent to the 20 percent of the basic alloy and to 30 percent drastically changed the microstructure as shown by the classification in table VII and the microstructures in figure 12. Increasing Cr caused the following changes: (1) The grain size became larger; (2) the amount of grain boundary precipitate increased between 10 and 20 percent Cr; (3) the matrix precipitate became heavier, especially between 20 and 30 percent Cr; and (4) for the 30-percent-Cr alloy the matrix precipitation occurred on the crystallographic planes rather than having a random pattern as for the lower-Cr alloys.

Nickel: The only pronounced structural variation for the series of Ni alloys was the ferrite-sigma-type phase which was present in the 0-percent-Ni alloy (see fig. 13). In the solution-treated alloy this phase was free from precipitate and the alloy was magnetic. In the aged condition a large amount of precipitation occurred in this phase and the alloy was nonmagnetic, signifying the transformation from the magnetic ferrite-type phase to the nonmagnetic sigma phase.

Upon rupture testing at 1200° F the magnetism increased with testing time, indicating some reversion to the ferrite-type phase. The 0-percent-Ni alloy also had less grain boundary precipitate and less precipitate in the austenite matrix than the higher-Ni alloys. None of the ferrite-type phase was observed in the alloy containing 10 percent Ni or in any of the other alloys studied in this investigation. Alloys containing 10 and 30 percent Ni were very similar in structure to the 20-percent-Ni basic alloy.

Cobalt: Varying Co from 0 to 32 percent produced two marked effects (see table VII and fig. 14). The 0-percent-Co alloy had a large amount of random matrix precipitation in the aged condition. The 10-percent-Co alloy resembled the basic 20-percent-Co alloy having only a medium amount of matrix precipitate. Increasing Co from 20 to 32 percent produced an alloy in which marked germination occurred at the 2200° F solution-treating temperature, resulting in several very massive grains surrounded by grains of normal size across the bar.

Another effect of Co on the structure occurred during rupture testing. Much additional matrix precipitation occurred in the 0-percent-Co alloy during testing; some also occurred in the 10-percent-Co alloy, and there was very little additional precipitate in the 20- and 32-percent alloys.

There was no evidence that Co changed the amount of the insoluble constituent or the grain boundary precipitate in the alloys.

Nitrogen: There was no appreciable influence from N varied from 0.004 to 0.18 percent.

Molybdenum: There were 10 modifications of alloys in which Mo was varied between 0 and 4 percent with 10 different constant ratios of W and Cb.

In the basic composition Mo was varied from 0 to 7 percent (alloys 32 to 36 in table VII). The only apparent effect of Mo was the larger grains at both the lower and the higher percentages.

For the nine other alloy modifications (alloys 43, 45, 46, 53 to 73, and 82 to 84) increases in Mo to 2 and 4 from

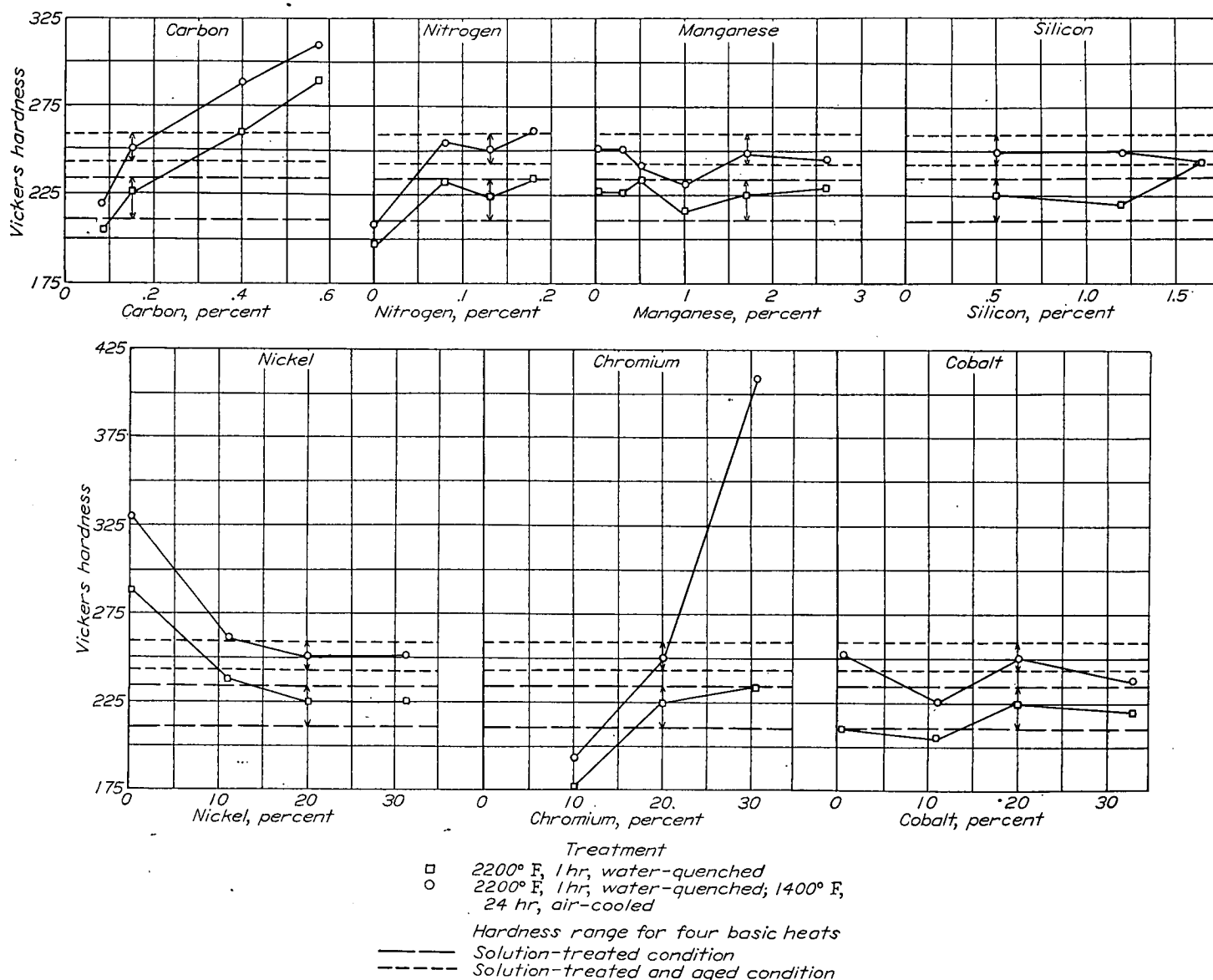


FIGURE 17.—Curves of hardness against alloy content for individual element modifications.

0 percent produced only very slight changes in structure except in the alloys containing Cb (particularly 4 percent Cb) in which case the grain size was refined and the amount of matrix precipitation during aging was increased.

Major structural changes between these alloy modifications were the result of variation in Cb.

Tungsten: In the 10 alloy modifications in which W was varied systematically (alloys 37 to 40, 43, 45, 46, 53 to 73, and 82 to 84 in table VII) similar observations to those for the influence of Mo were made. Increases in W slightly refined the grains and increased the matrix precipitation to a small degree in some cases.

Columbium: As shown in table VII for the 10 alloy modifications with variable Cb, this element had a marked influence on all the structural classification variables. A typical example of microstructural changes occurring with Cb additions from 0 to 6 percent to the basic composition is shown in figure 15.

Increasing Cb from 0 to the 1 percent of the basic alloy produced the following structural changes: (1) Increased the insoluble constituent from very little to an appreciable amount; (2) drastically reduced grain size; (3) decreased the concentration of grain boundary precipitation during aging; (4) changed the mode of matrix precipitation from internal

crystallographic planes to a random pattern; and (5) decreased the amount of matrix precipitation during aging.

Increasing Cb from the 1 percent of the basic alloy to 4 percent continued to increase markedly the amount of insoluble constituent, to further gradually reduce grain size, and not to alter appreciably the grain boundary concentration effects or the randomness of precipitation while increasing the amount of matrix precipitation during aging in alloys containing four or more percent of Mo or W.

While Cb did produce major changes in microstructure, it is noted that for the alloys arranged in table VII with Cb as the systematic variable the presence of either or both Mo or W in the alloys increased the tendencies for these changes. This is shown in figure 16 for 4-percent-Cb alloys containing increasing Mo and W contents.

Summary of influence of alloying elements on microstructure.—A summary of the effects of alloying elements on tendencies for microstructural changes is given in table VIII.

Two elements producing major microstructural changes were Cb and C. Both elements increased the amounts of insoluble constituent and aging precipitate and refined the grain size of material solution-treated at 2200° F. In addition, Cb changed the mode of matrix precipitation from

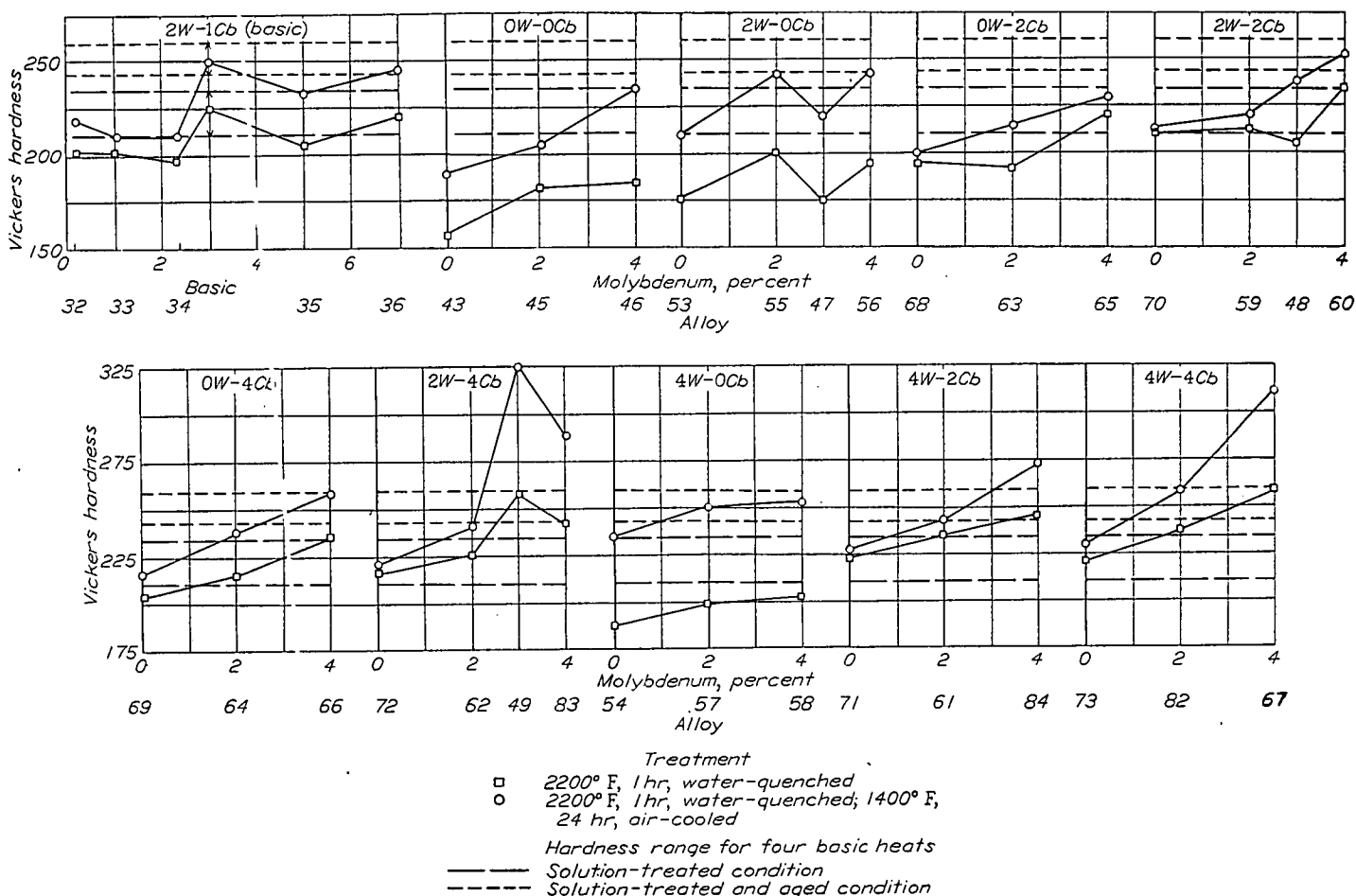


FIGURE 18.—Curves of hardness against molybdenum content for 10 tungsten-columbium modifications.

preferred, when no Cb was present, to random, when the 1 percent of the basic alloy or more was present.

Additions of Mo and W produced little change in microstructure except in alloys containing 4 percent Cb in which case these elements decreased the grain size and increased matrix precipitation.

Between 20 and 30 percent of Cr a pronounced increase occurred in the amount of aging precipitate and the mode of precipitation changed from random to preferred.

Grain size was mildly increased and the amount of aging precipitate decreased by Co.

Additions of Ni (between 10 and 30 percent) produced only minor changes in structure. The 0-percent-Ni alloy contained a ferrite-sigma type of phase which disappeared when Ni was raised to 10 percent.

The only effect of Si, within the limits studied, was to refine grain size slightly. No discernible change in structure was produced by Mn and N.

HARDNESS

Vickers hardness tests at room temperature were made on metallographic specimens of all the experimental alloys in the solution-treated (2200°F , 1 hr, water-quenched) and the aged (1400°F , 24 hr, air-cooled) conditions. Preliminary hardness surveys on several alloys indicated that the forged

bars had uniform hardnesses both in the lengthwise and the crosswise directions.

Hardness test results, representing the average of at least four tests on each sample, are given in table IX and are plotted against composition variables in figures 17 to 20.

There was fair agreement between the hardnesses of five heats of the basic alloy. Additions of C, N, Cr, Mo, W, and Cb tended to increase hardness in both conditions; Ni, particularly from 0 to 10 percent, lowered hardness; and Mn, Si, and Co did not appreciably affect hardness. Perhaps an exception to this was the effect observed for the 1.6-percent-Si alloy, which was the only one of all the alloys studied which did not show hardening as a result of aging. The addition of Cr from 20 to 30 percent markedly increased the age-hardening characteristics of the alloy.

Figure 18 shows the influence of Mo for 10 W-Cb modifications. Additions of Mo, while tending to increase hardness, did not appreciably increase the age-hardening tendency except for the modifications containing more than 4 percent total of W plus Cb. For the 10 Mo-Cb modifications (fig. 19) the effect of W additions was similar to that observed for Mo, although there was less tendency for increasing age hardening with higher alloy contents. In both figures 18 and 19 it is seen that there was a greater difference between the hardness in the solution-treated and the aged conditions

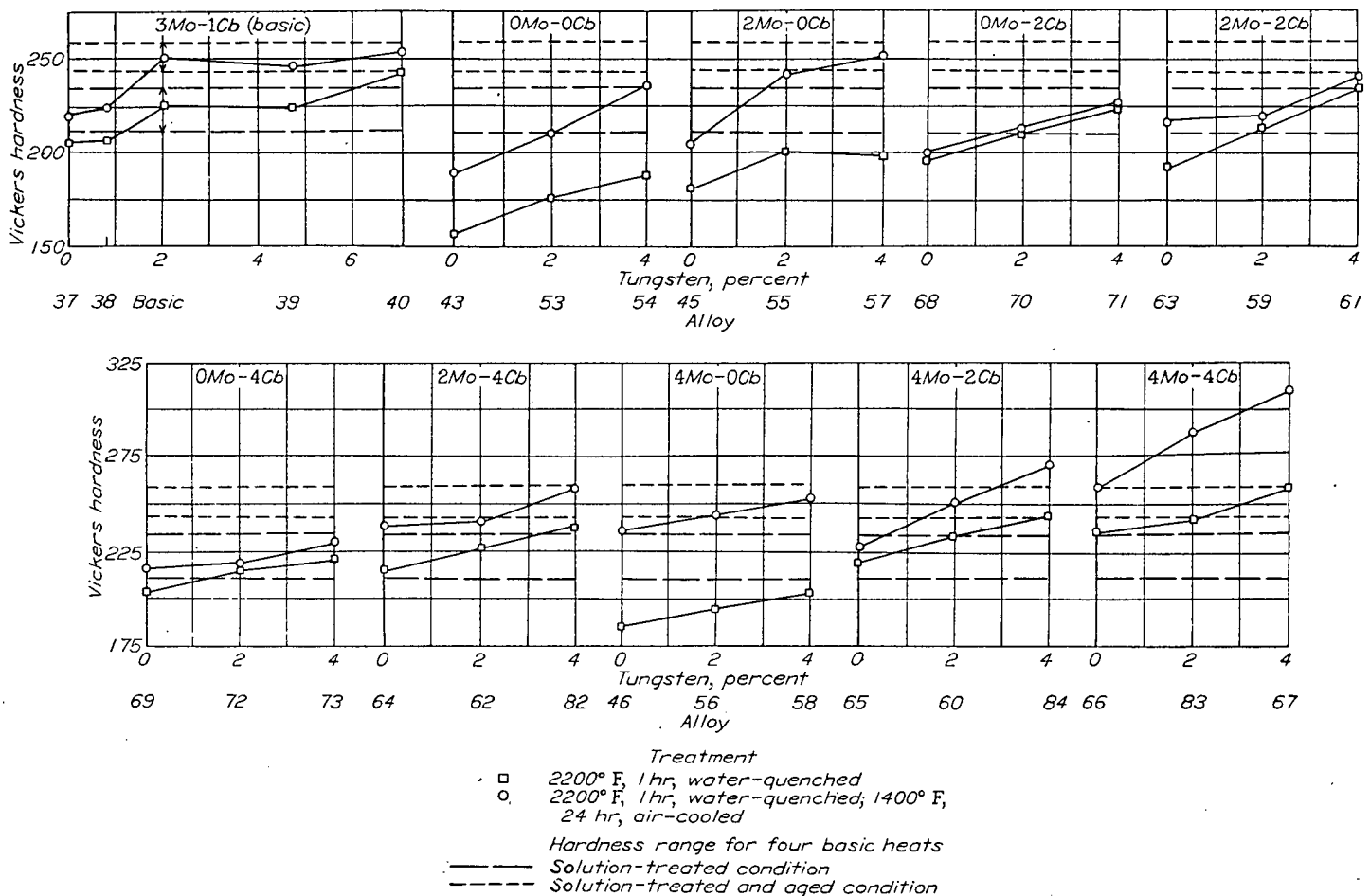


FIGURE 19.—Curves of hardness against tungsten content for 10 molybdenum-columbium modifications.

for the modifications not containing Cb than for the modifications containing Cb. This effect is shown again in figure 20 in that Cb additions from 0 to 2 percent to the 10 Mo-W modifications narrowed the age-hardening range. Additions of Cb from 2 to 4 percent tended to broaden the age-hardening range for modifications containing the larger amounts of Mo and W.

It is recognized that these hardness results are not a true evaluation of the comparative aging effects caused by composition variables. Such an evaluation would necessarily require aging-time-temperature data.

INFLUENCE OF CHEMICAL-COMPOSITION VARIABLES ON RUPTURE TEST CHARACTERISTICS AT 1200° F

The results of rupture tests at 1200° F for all the alloys studied are given in table X. Included in this table are the times for rupture, elongations, reductions of area, and minimum creep rates obtained from the curves of elongation against time for the individual tests. Also included are 100- and 1000-hour rupture strengths obtained from double-logarithmic plots of stress against rupture time (an example

is shown in fig. 5); estimated elongation to rupture in 100 hours; stresses for creep rates of 0.1 and 0.01 percent per hour obtained from double-logarithmic plots of stress against minimum creep rates from the rupture tests (an example is shown in fig. 6); and minimum creep rates at 40,000 psi. The alloys are listed in table X in the approximate order of the element varied in the basic analysis.

Rupture tests of sufficient number and duration were conducted to establish the 100-hour rupture strengths. The 1000-hour rupture strengths were obtained by extrapolation of the double-logarithmic curves of stress against rupture time. Unless the maximum-time rupture test was longer than 500 hours, however, the 1000-hour strengths are listed in table X as estimated and are indicated as such on the curves of strength against chemical composition.

In this evaluation of the influence of variations in chemical composition on the properties most emphasis has been placed on rupture properties at 1200° F. Because the creep data were obtained from the rupture tests, it is emphasized that the rates of deformation were much higher than those usually associated with reported "creep strength."

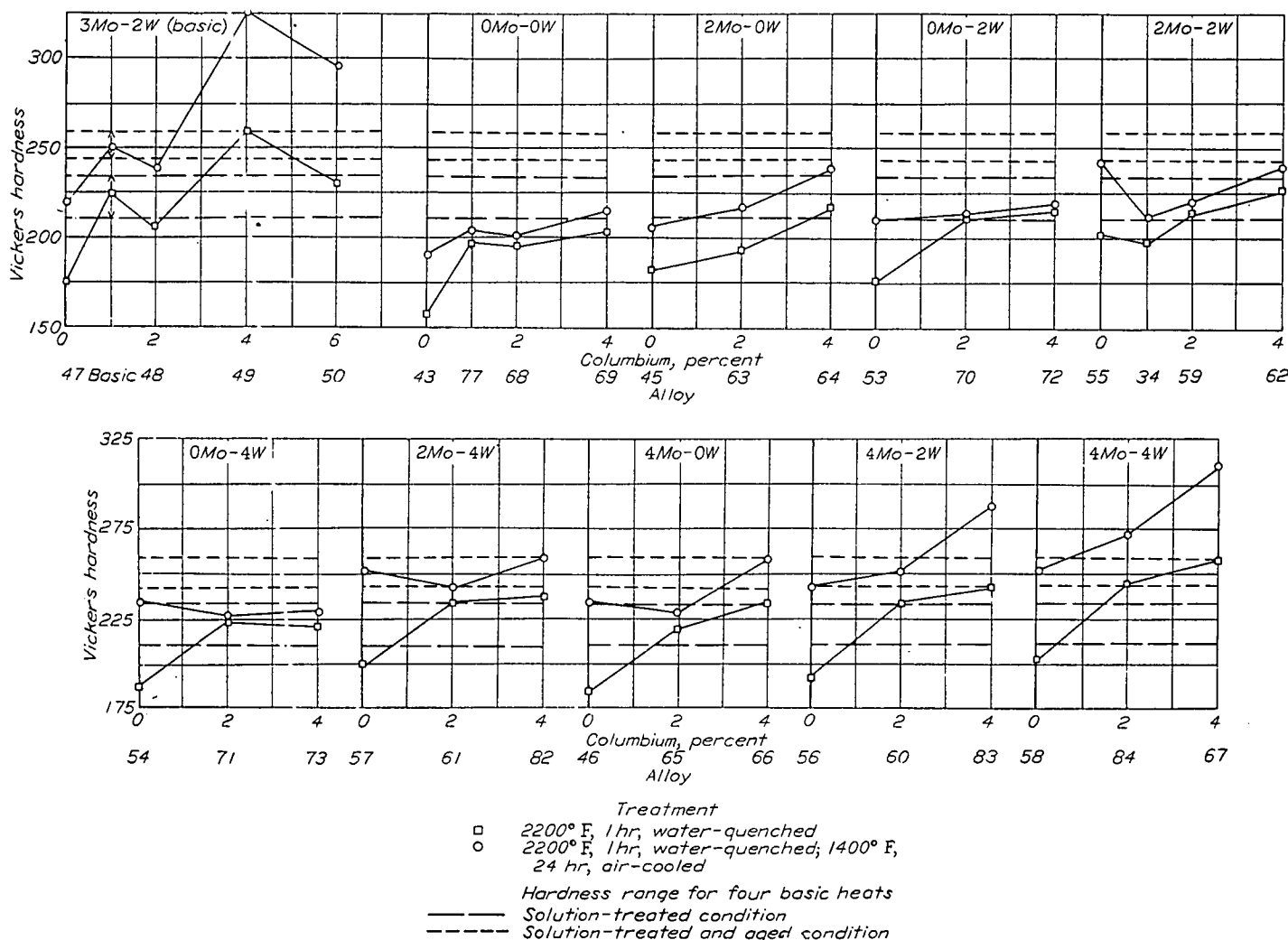


FIGURE 20.—Curves of hardness against columbium content for 10 molybdenum-tungsten modifications.

100-HOUR RUPTURE PROPERTIES AT 1200° F OF MODIFIED ALLOYS

The 100-hour rupture strengths and elongations for all of the alloys are arranged in order of increasing 100-hour rupture strengths in figure 21. The ranges in properties for the basic alloy heats, forged both as squares and as rounds, are indicated to show the significant variations in properties. The following observations are made:

(1) The over-all 100-hour rupture strength range was from 26,000 to 52,000 psi. The lowest-strength alloy was the one which did not contain Mo, W, and Cb. The two highest-strength alloys were modifications of the basic alloy, one containing 30 percent Cr, the other 7 percent W.

(2) Six alloys had rupture strengths above the range of the basic alloy forged to round bars. None of the alloys, however, were above the range for the basic alloy forged to squares where the practice was not so well controlled.

(3) It appears that all of the elements can be varied individually over relatively wide ranges without appreciably altering rupture strengths. It will be shown, however, that in most cases rupture strengths varied consistently with systematic variations in composition.

(4) Alloys having strengths near the lower end of the range were those containing the smallest amounts of Mo,

W, or Cb added separately or two at a time. Additions of greater amounts of these elements generally yielded rupture strengths which were closer to the strength range of the basic alloy.

(5) Adding 2 percent of either Mo, W, or Cb to the lowest-strength alloy increased the 100-hour rupture strength in the order of 10,000 psi or more.

(6) Strengths in the order of those of the basic alloy were obtained without the presence of Cb. Strengths almost as high were obtained without the presence of Mo or of W.

(7) Low Cr, Co, or Ni and high Si resulted in lower strengths than the range for the basic heats.

(8) Elongation at rupture in 100 hours ranged from 5 to 40 percent.

(9) There was no relationship apparent between rupture elongation and strength. The omission of Cb resulted in consistently low elongation. More Cr or Co than the 20 percent of the basic alloy also lowered elongation.

INDIVIDUAL VARIATIONS OF ELEMENTS

The influence on rupture test characteristics of systematic variations of the individual elements, shown in figures 22 to 31, was as follows:

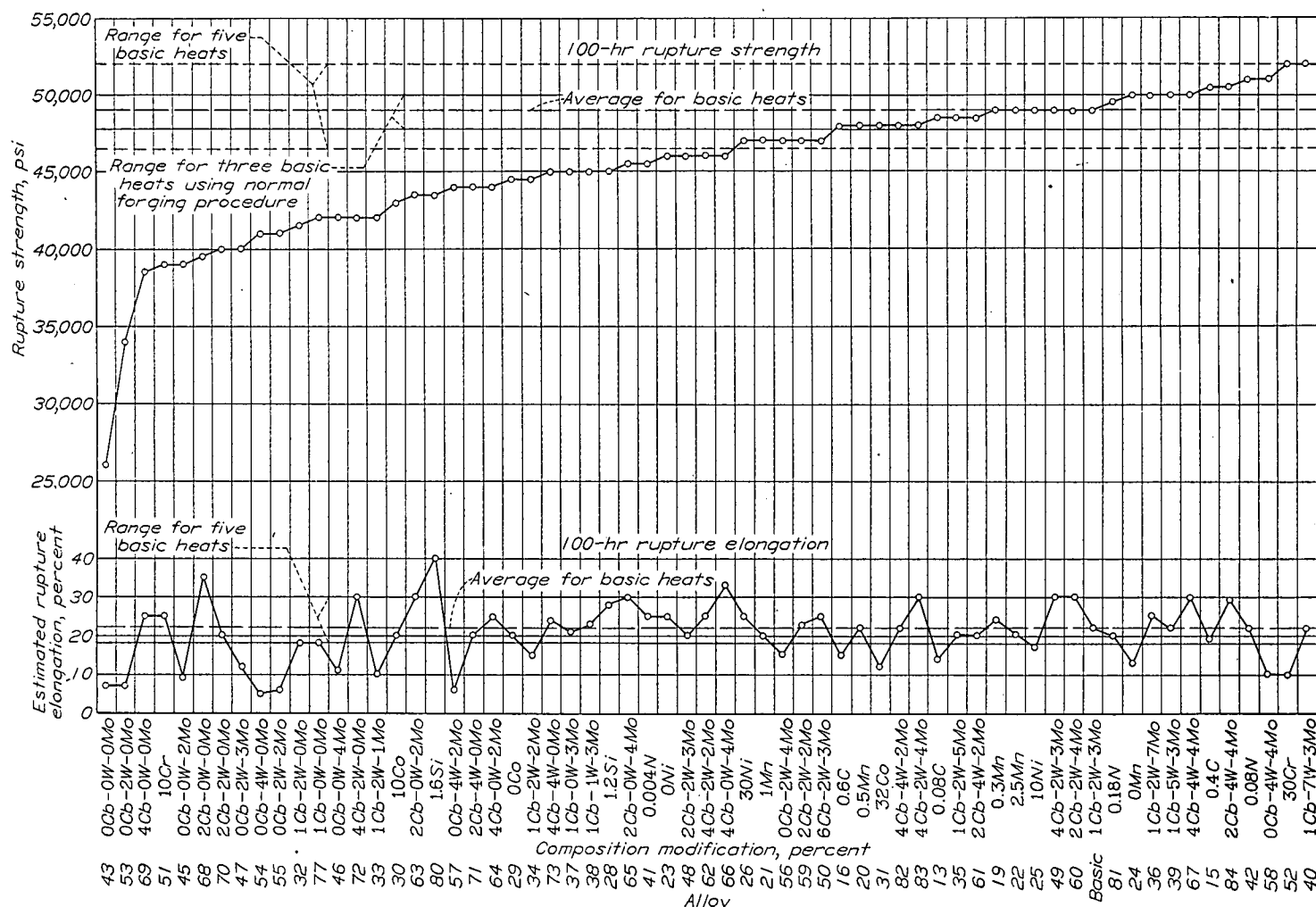


FIGURE 21.—Influence of chemical-composition modifications on 100-hour rupture properties at 1200° F. Heat treatment: 2200° F for 1 hour, water-quenched; 1400° F for 24 hours.

Carbon.—The effect of C, varied from 0.08 to 0.60 per cent, was only slight. All the rupture strengths fell within close proximity to those of the basic alloy (see fig. 22). The variable C heats were forged as squares and are compared with the property ranges found for similarly forged basic heats. The lowest and the highest C heats had rupture elongations below normal.

Nitrogen.—Variations of N, from 0.004 to 0.18 per cent, tended to increase strength somewhat (see fig. 23). The alloy containing 0.004 per cent N had strengths which were slightly below those of the basic 0.12-percent-N alloy while the 0.08-percent-N alloy was slightly stronger.

Manganese.—Figure 24 shows that Mn variations from 0 to 2.5 per cent had no significant influence on rupture

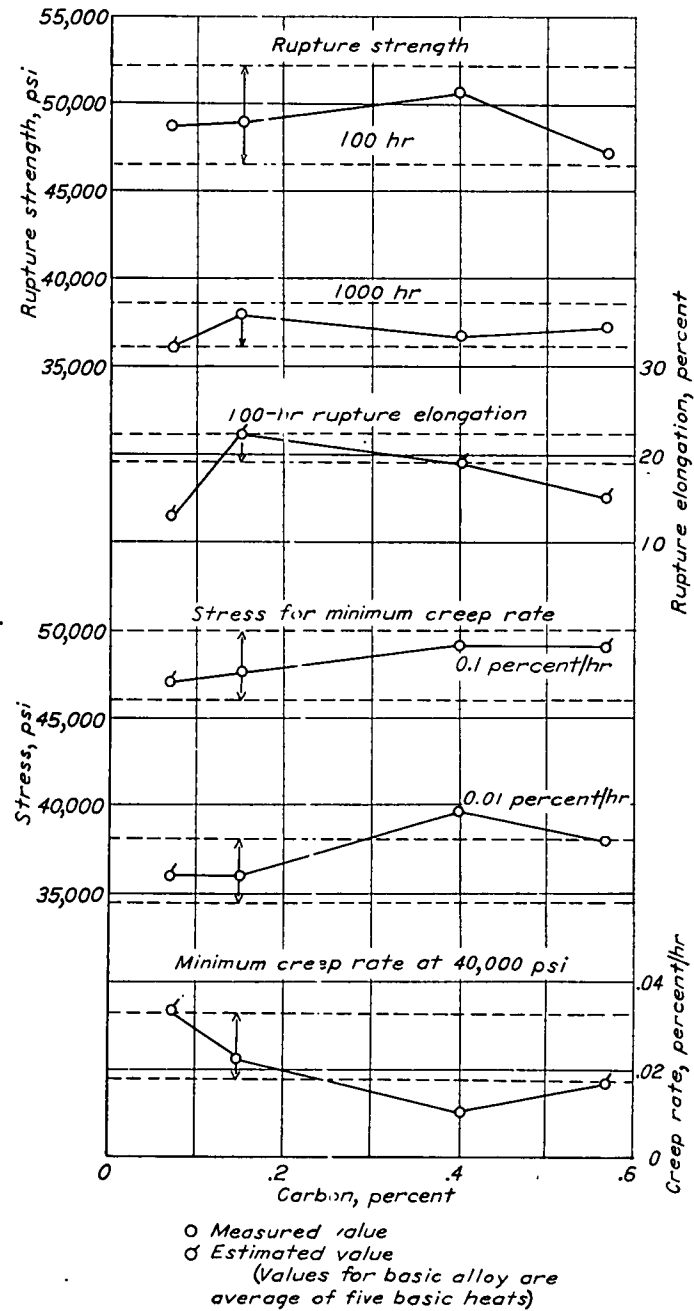


FIGURE 22.—Curves of rupture test data at 1200° F against carbon content of basic alloy. Horizontal dashed lines indicate property range for five basic heats.

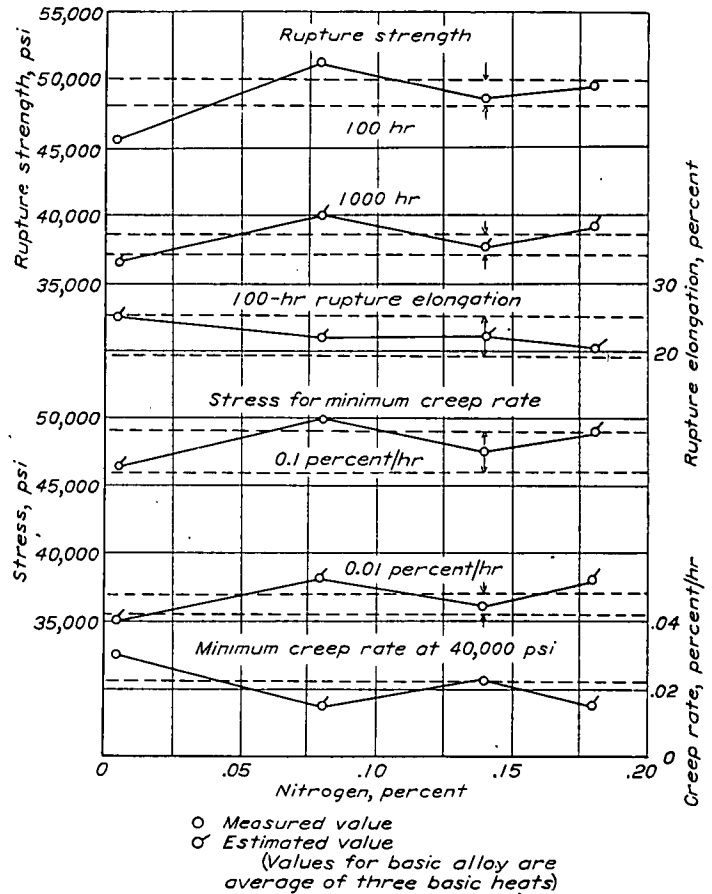


FIGURE 23.—Curves of rupture test data at 1200° F against nitrogen content of basic alloy. Horizontal dashed lines indicate property range for three basic heats.

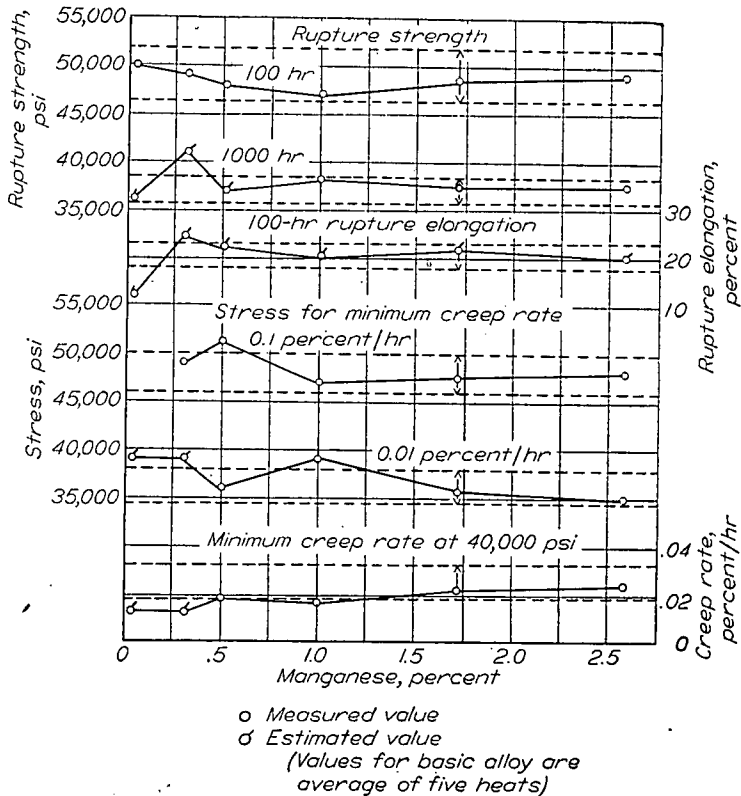


FIGURE 24.—Curves of rupture test data at 1200° F against manganese content of basic alloy. Horizontal dashed lines indicate property range for five basic heats.

properties except a somewhat lower elongation for the low-Mn heat.

Silicon.—Figure 25 shows that increasing Si from the 0.5 percent of the basic alloy to 1.6 percent produced a marked lowering of rupture strength and an increase in elongation. The stresses at constant creep rates were lowered and creep rates at 40,000 psi were noticeably raised in agreement with trends in rupture properties. This was the only case in which the addition of an element consistently lowered strength over the complete composition range studied.

Chromium.—The influence of Cr variations from 10 to 30 percent is shown in figure 26. The 100-hour rupture strengths were increased from 39,000 to 52,000 psi while rupture elongation decreased from 25 to 10 percent. Similar marked

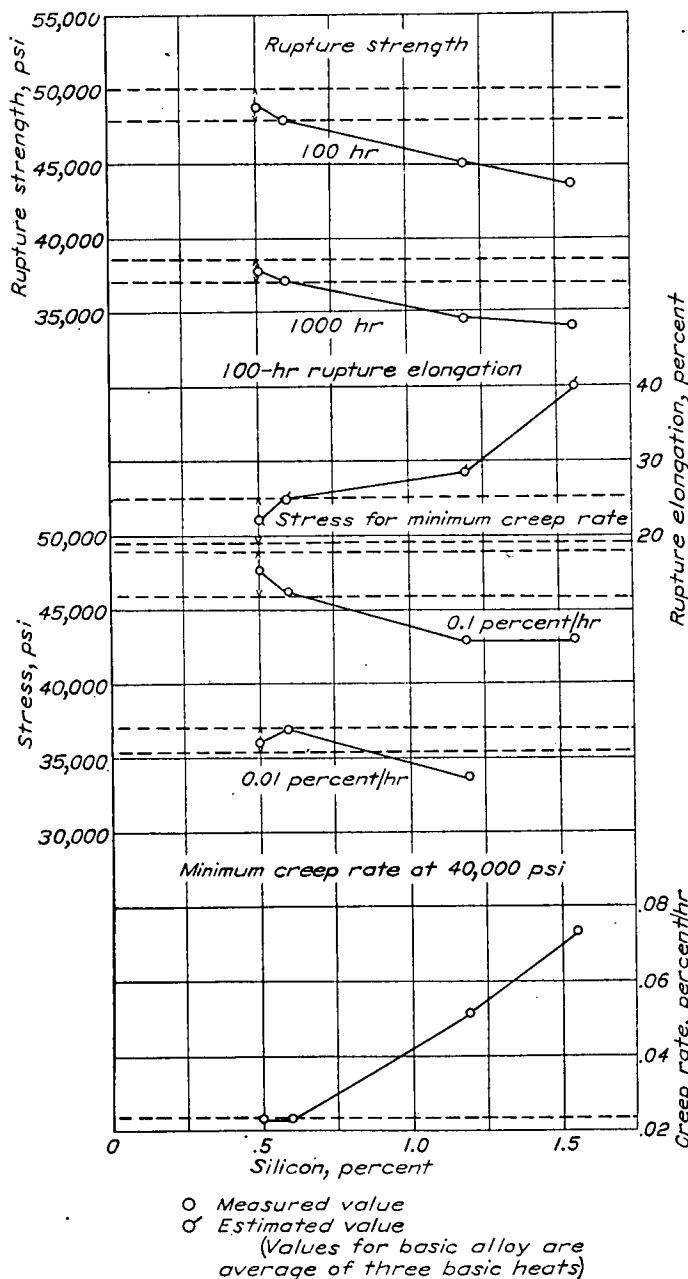


FIGURE 25.—Curves of rupture test data at 1200° F against silicon content of basic alloy. Horizontal dashed lines indicate property range for three basic heats.

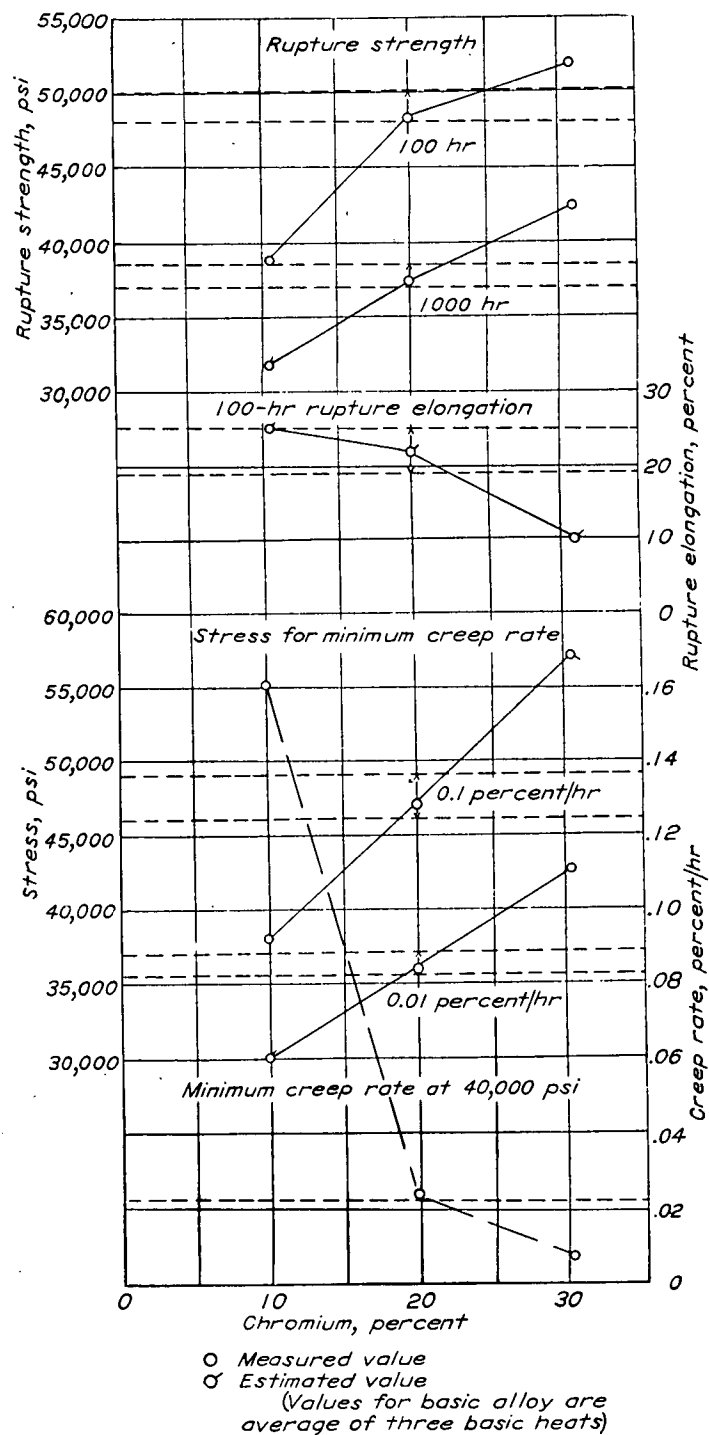


FIGURE 26.—Curves of rupture test data at 1200° F against chromium content of basic alloy. Horizontal dashed lines indicate property range for three basic heats.

increases occurred in 1000-hour rupture strengths and stresses at constant creep rates while the creep rate at 40,000 psi was drastically reduced, particularly between 10 and 20 percent Cr.

Nickel.—Additions of Ni from 0 to 30 percent had little influence on properties as shown in figure 27. The 0- and 30-percent-Ni alloys had slightly lower rupture strengths than the range for the basic 20-percent-Ni heats, producing an apparent maximum in strength between 10 and 20 per-

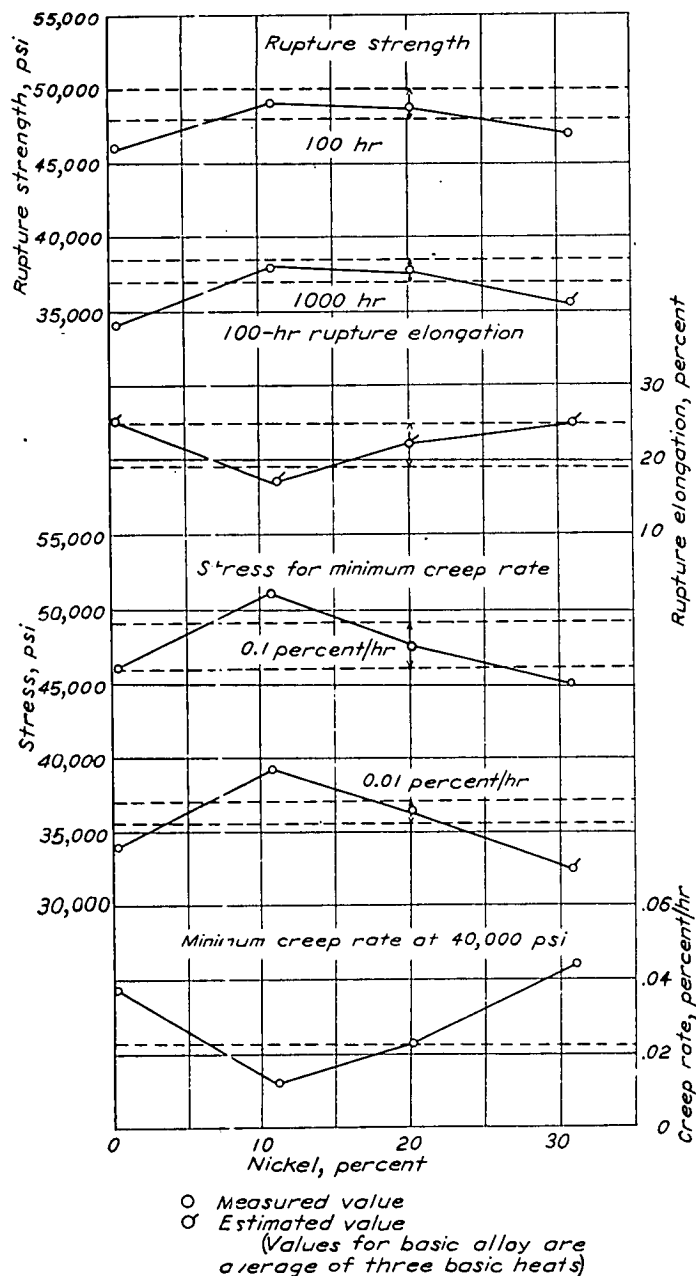


FIGURE 27.—Curves of rupture test data at 1200° F against nickel content of basic alloy. Horizontal dashed lines indicate property range for three basic heats.

cent Ni. A similar slight trend was noted in stresses at constant creep rates while the creep rate at 40,000 psi decreased and then increased with increasing Ni, following the trend of the change in elongation at fracture in 100 hours.

Cobalt.—Figure 28 shows the influence of Co variations from 0 to 32 percent. The 0- and 10-percent-Co alloys had lower rupture strengths than normal with little effect on elongation. The 20- and 32-percent-Co alloys had equal 100-hour rupture strengths at a higher level than those with lower Co. The 1000-hour rupture strengths and stresses at constant creep rates were improved, while the elongation and creep rate were lowered, when Co was increased from the 20 percent of the basic alloy to 32 percent.

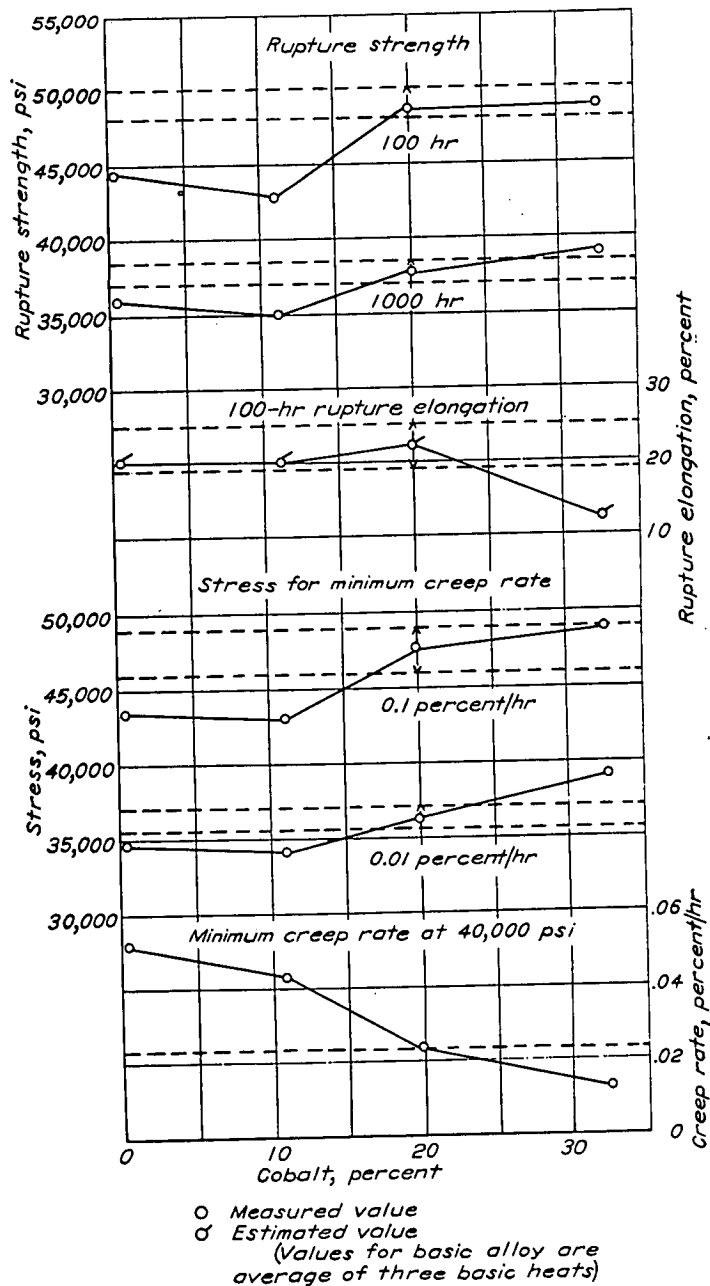


FIGURE 28.—Curves of rupture test data at 1200° F against cobalt content of basic alloy. Horizontal dashed lines indicate property range for three basic heats.

Molybdenum.—Additions of Mo from 0 to 7 percent improved rupture and creep properties as shown by figure 29. The most marked improvement in rupture strength from Mo was from additions of 1 to 3 percent.

Tungsten.—Figure 30 shows that the improvement in rupture strength with additions of W from 0 to 7 percent was gradual over the entire range, this addition increasing 100-hour strength from 45,000 to 52,000 psi. The stresses at constant creep rates showed a similar trend to rupture strengths while rupture elongation was not at all affected by W.

Columbium.—Figure 31 shows that increasing Cb from 0 to the 1 percent of the basic alloy increased the 100-hour rupture strength from 40,000 to 48,000 psi. Additions of Cb

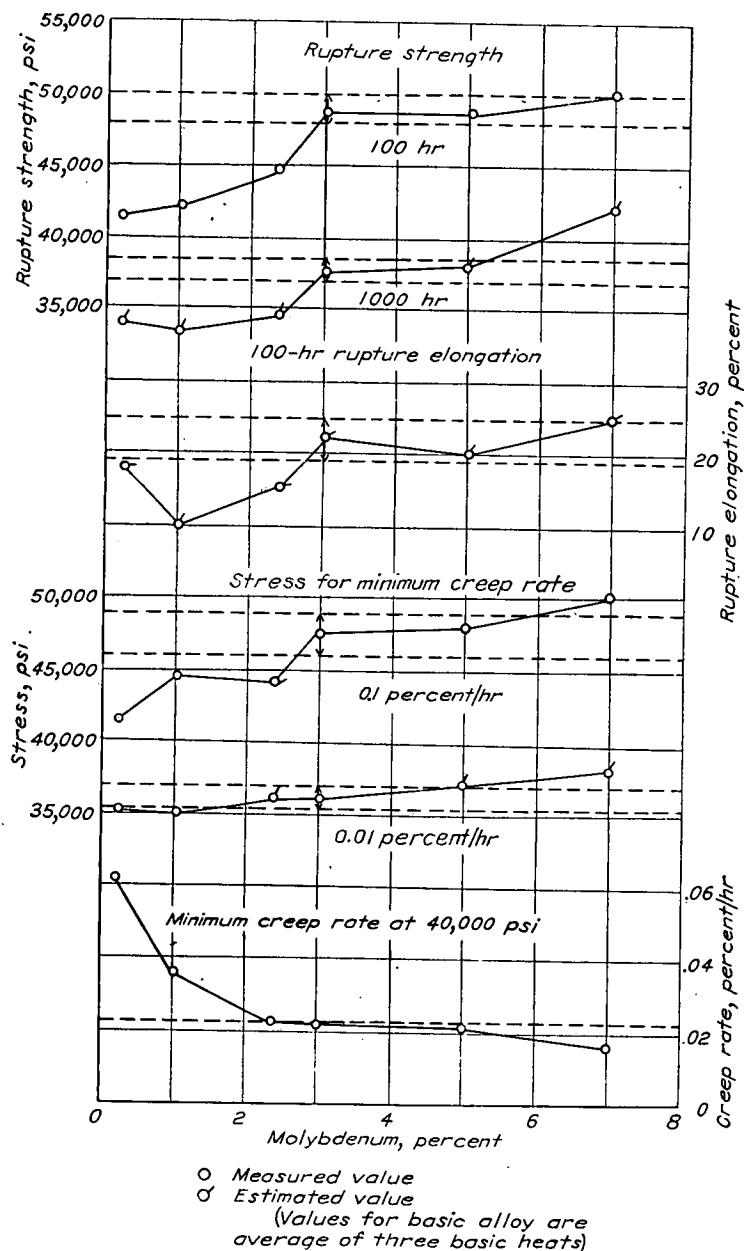


FIGURE 29.—Curves of rupture test data at 1200° F against molybdenum content of basic alloy. Horizontal dashed lines indicate property range for three basic heats.

from 1 to 6 percent produced no further improvement in strength. A similar trend was shown by creep properties. However, the increase of 44,500 to 47,500 psi in the stress for a minimum creep rate of 0.1 percent per hour between 0 and 1 percent Cb was not of the same magnitude as the rupture strength increase. Rupture test elongation was raised from 12 to 22 percent by the addition of 1 percent Cb and tended to increase further with higher Cb.

SIMULTANEOUS VARIATIONS OF MOLYBDENUM, TUNGSTEN, AND COLUMBIUM

In addition to the individual variations of all the elements the effects of simultaneous variations of Mo, W, and Cb in steps of 2 percent from 0 to 4 percent were evaluated. This made possible curves showing the influence on properties of systematic variations of one of these elements for 10 different constant ratios of the other two elements.

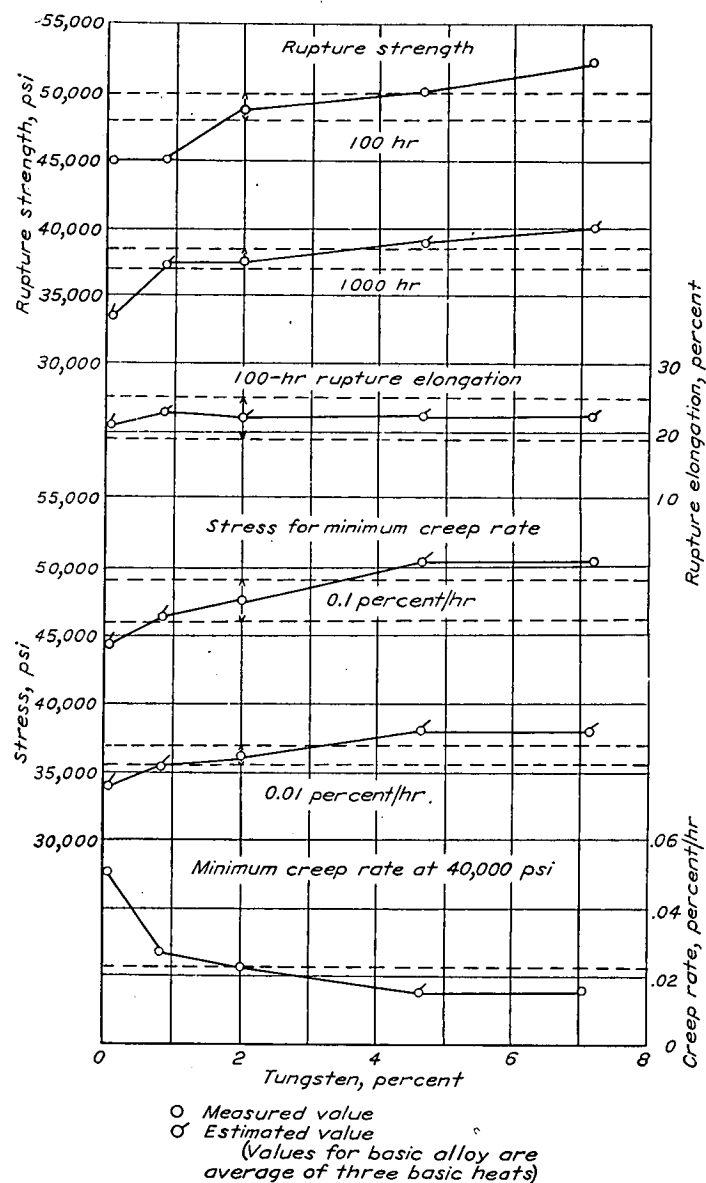


FIGURE 30.—Curves of rupture test data at 1200° F against tungsten content of basic alloy. Horizontal dashed lines indicate property range for three basic heats.

Molybdenum.—The influence of the Mo variations from 0 to 4 percent on rupture test characteristics of alloy modifications, including the basic alloy, with 10 different constant ratios of W and Cb is shown in figure 32. Curves comparing the relative effect of Mo on the 100-hour rupture strength of the 10 W-Cb modifications are shown in figure 33.

Increasing Mo from 0 to 4 percent tended continuously to increase the 100-hour rupture strength for all the ratios of W and Cb. The relative effect was greatest for the modifications containing neither W nor Cb, the strength increase being from 26,000 to 42,000 psi. In alloys containing W or Cb the strengthening effect of Mo was greatest for the alloys which contained 2 and 4 percent of W and no Cb. This strengthening effect was less for alloys containing 2 and 4 percent Cb and no W and became even less for those containing both Cb and W which had a higher level of initial strength.

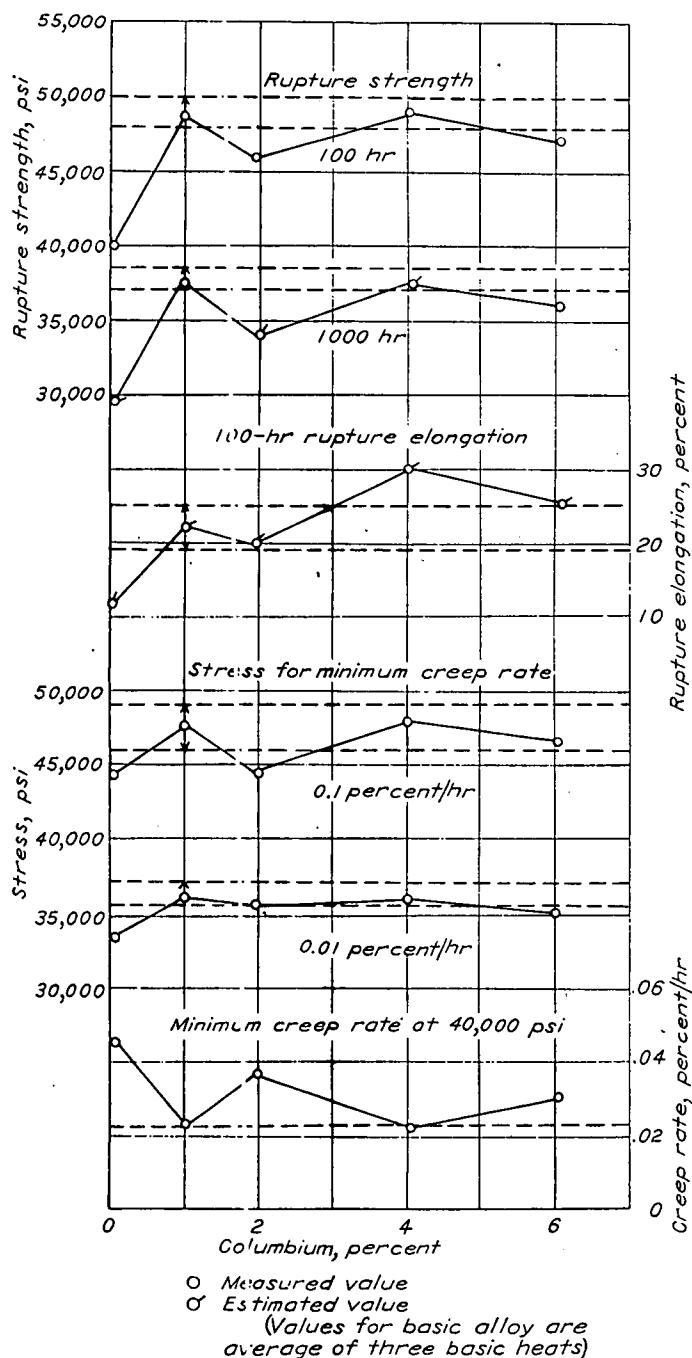


FIGURE 31.—Curves of rupture test data at 1200° F against columbium content of basic alloy. Horizontal dashed lines indicate property range for three basic heats.

The 1000-hour-strength trends followed those of the 100-hour strengths. There was a slight tendency for increasing Mo to widen the difference between the 100- and 1000-hour rupture strengths in the modifications containing more than 6 percent total of W plus Cb.

In general Mo tended to improve the 100-hour rupture elongation. It is noted that for the alloys in which Cb was absent the elongation was quite low (5 to 15 percent) as compared with the elongations above 20 percent of the alloys containing 2 and 4 percent Cb.

The stresses causing a minimum creep rate of 0.1 percent per hour followed the trends and were of the same order of magnitude as the 100-hour rupture strengths. The trends

and magnitudes of stresses causing a creep rate of 0.01 percent per hour were somewhat similar to those of the 1000-hour rupture strengths. Creep stresses were consistently higher than rupture strength, however, for the 0-percent-Cb alloys having low rupture elongation and were somewhat lower than rupture strength for the alloys with higher elongation.

Tungsten.—By rearranging the data for alloys involving simultaneous variations of Mo, W, and Cb figures 34 and 35 were obtained. These show the influence of W variations from 0 to 4 percent on rupture test characteristics of alloy modifications with 10 different constant ratios of Mo and Cb.

Additions of W produced an almost linear increase in 100-hour rupture strength. The greatest improvement, 26,000 to 41,000 psi, was for the 0Mo-0Cb modifications, with the relative improvements tapering off for the alloys with higher Mo and Cb to about 4000 psi in the range 46,000 to 50,000 psi for the 4Mo-4Cb modifications. The 1000-hour rupture strengths, although only estimated in many cases, followed a similar, but not quite so pronounced, trend. The trends in creep properties were also, in general, the same as those for the rupture strengths with the additional effect of the level of elongation on the relation between rupture strength and stresses at constant creep rates.

There was a general tendency for W to lower the 100-hour rupture elongation very slightly. The low elongation of the alloys which did not contain Cb was again noted.

Columbium.—The data arranged to define the influence of Cb are shown in figures 36 and 37. The influence of Cb on rupture strengths was significantly different from the influences of Mo and W. Additions of Cb from 0 to 2 percent, in general, increased the 100-hour rupture strength, but there was no significant strength increase with greater additions. This finding, in addition to the observation of the influence of 1 percent as compared with that of 0 percent Cb on the strength of the basic alloy, led to the preparation of an additional alloy (77) containing 0Mo-0W-1Cb which fits into the 0Mo-0W alloy modifications as shown in figure 36. This alloy had as good or better strength properties than the 0Mo-0W-2Cb modification. The addition of Cb to the 4Mo-2W and 4Mo-4W modifications evidently had no effect or a slightly detrimental effect on rupture strength.

One or two percent of Cb markedly improved the rupture elongation. An additional slight improvement was realized with even higher Cb.

The stresses causing a creep rate of 0.1 percent per hour were higher than the 100-hour rupture strengths for 0-percent-Cb alloys but lower than the rupture strengths for alloys containing Cb. Thus the creep stress did not follow the increase in rupture strength with the additions of 1 or 2 percent Cb but either remained constant or decreased. It appeared that the marked increase in elongation with small Cb additions was the factor responsible for the improvement in rupture strength.

It can be concluded that, for the alloy modifications studied, additions of Cb are necessary to produce substantial ductility in the rupture test, but additions of more than 2, or probably 1, percent of Cb add nothing to the rupture properties of the alloys.

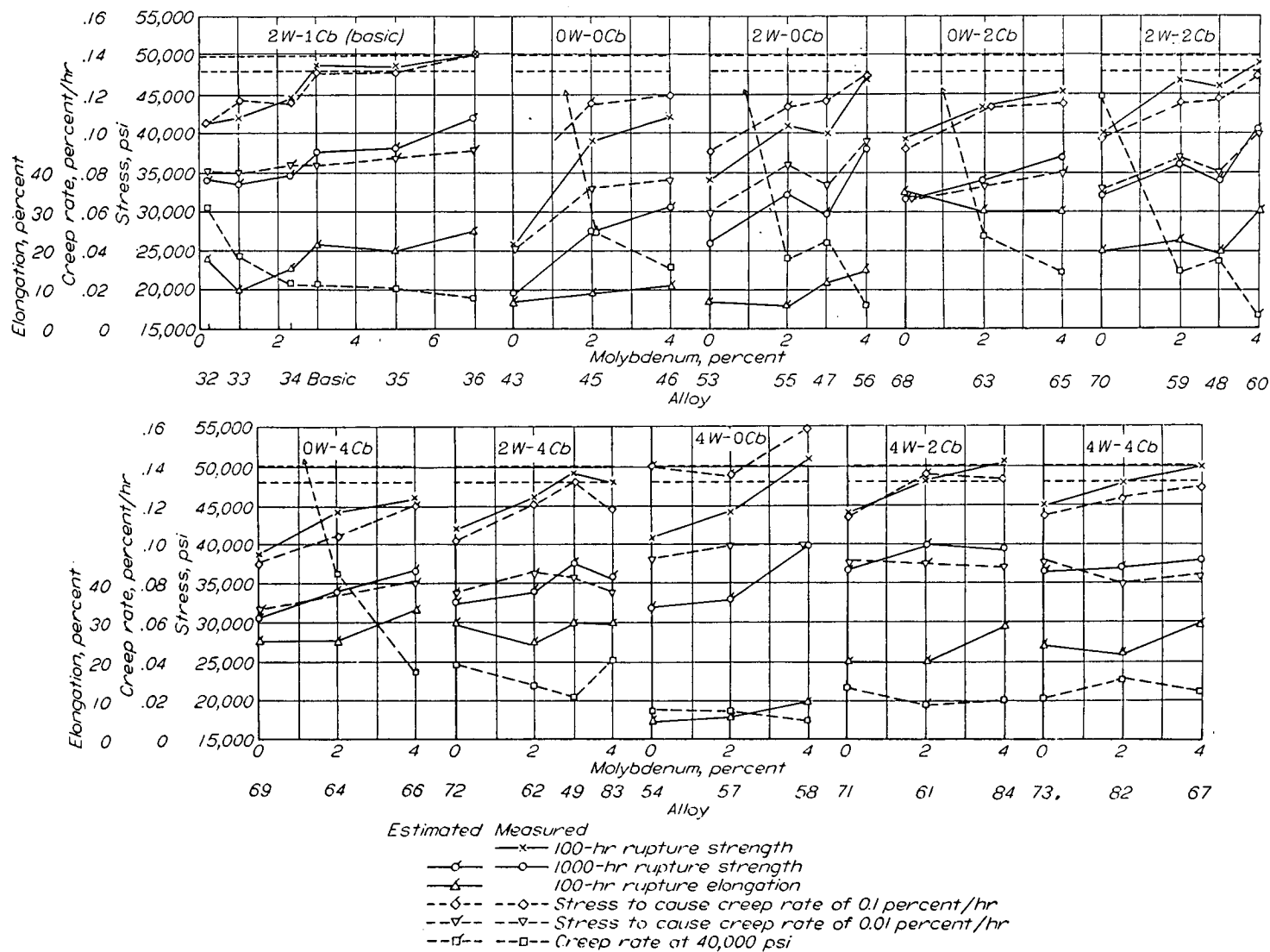


FIGURE 32.—Curves of rupture test data at 1200° F against molybdenum content for 10 tungsten-columbium modifications. Horizontal dashed lines indicate range of 100-hour rupture strength for three basic heats.

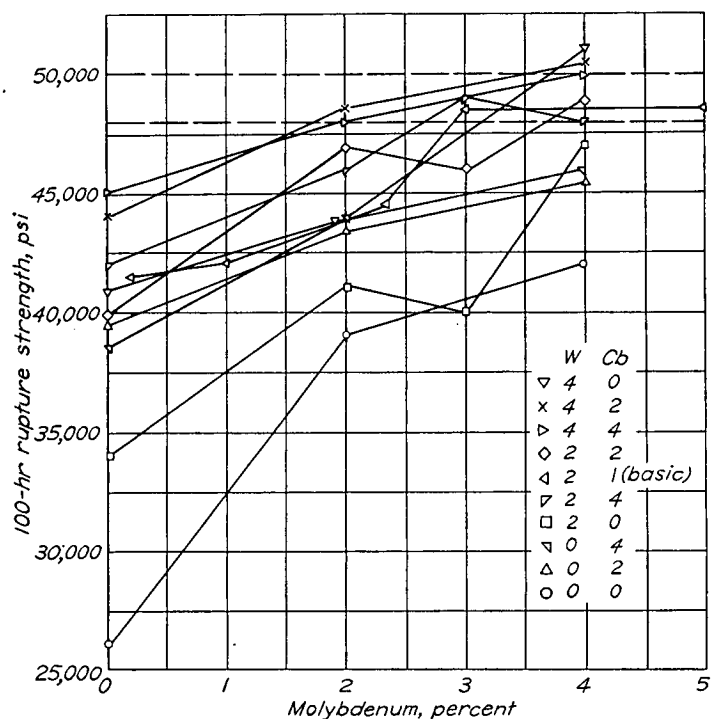


FIGURE 33.—Summary of influence of molybdenum on 100-hour rupture strength at 1200° F of 10 tungsten-columbium modifications. Horizontal dashed lines indicate property range for three basic heats.

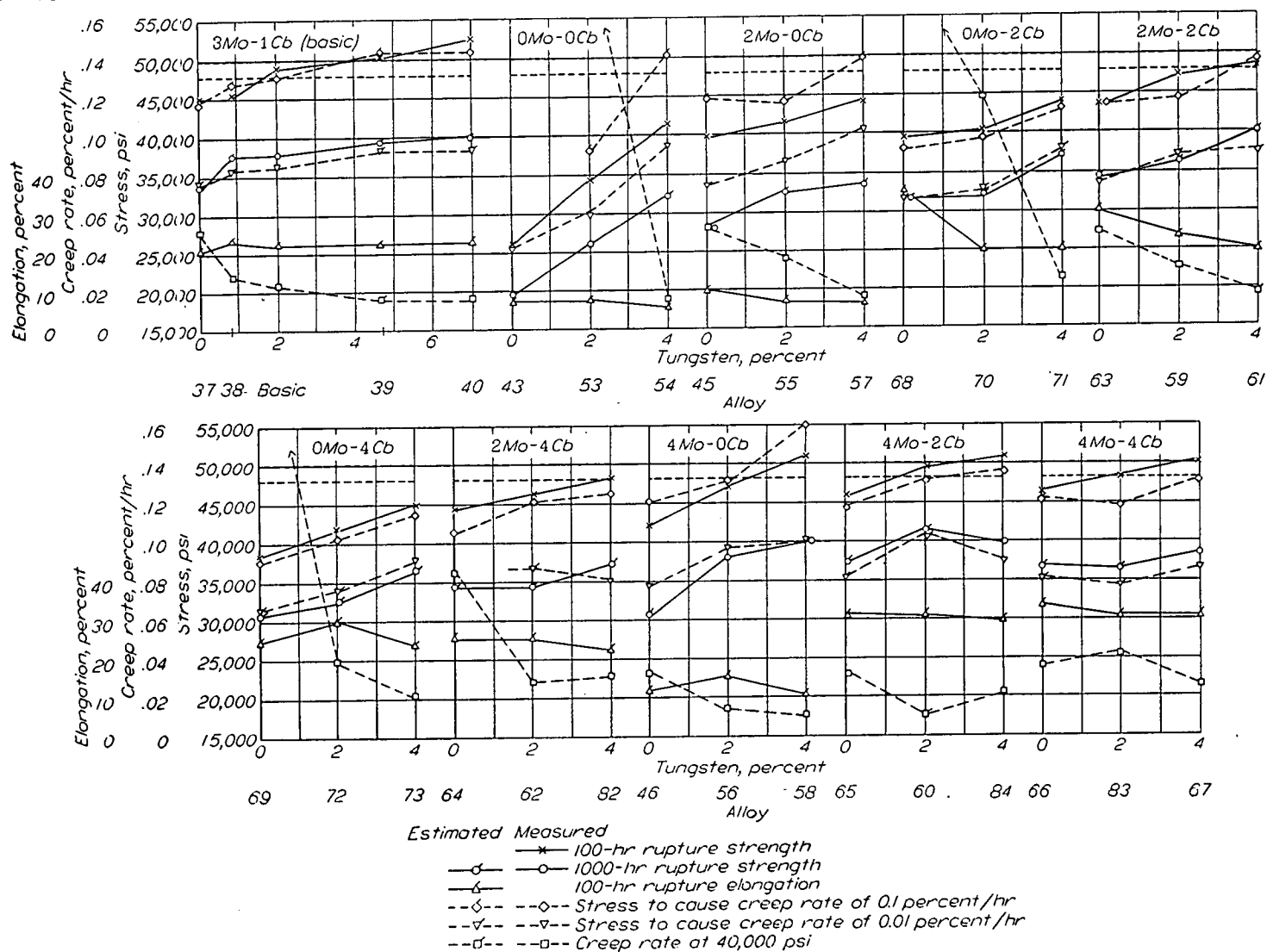


FIGURE 34.—Curves of rupture test data at 1200° F against tungsten content for 10 molybdenum-columbium modifications. Horizontal dashed lines indicate range of 100-hour rupture strength for three basic heats.

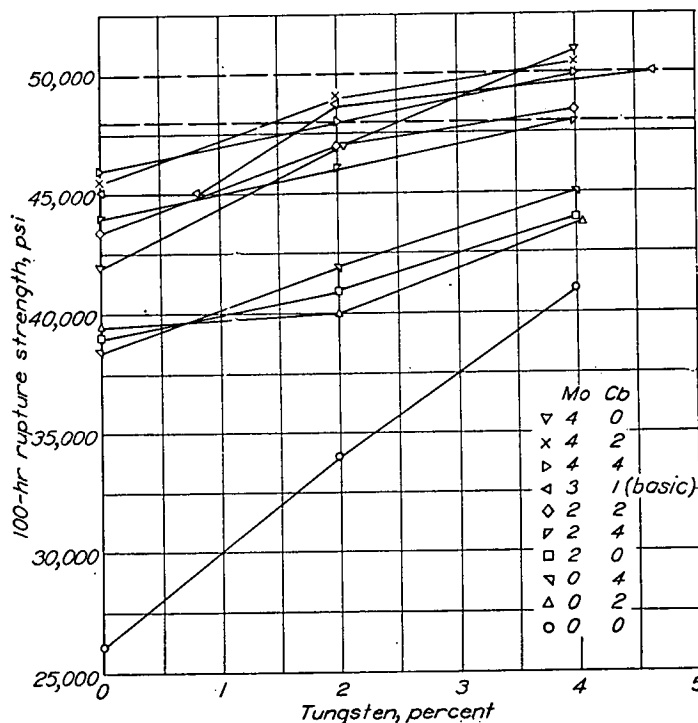


FIGURE 35.—Summary of influence of tungsten on 100-hour rupture strength at 1200° F of 10 molybdenum-columbium modifications. Horizontal dashed lines indicate property range for three basic heats.

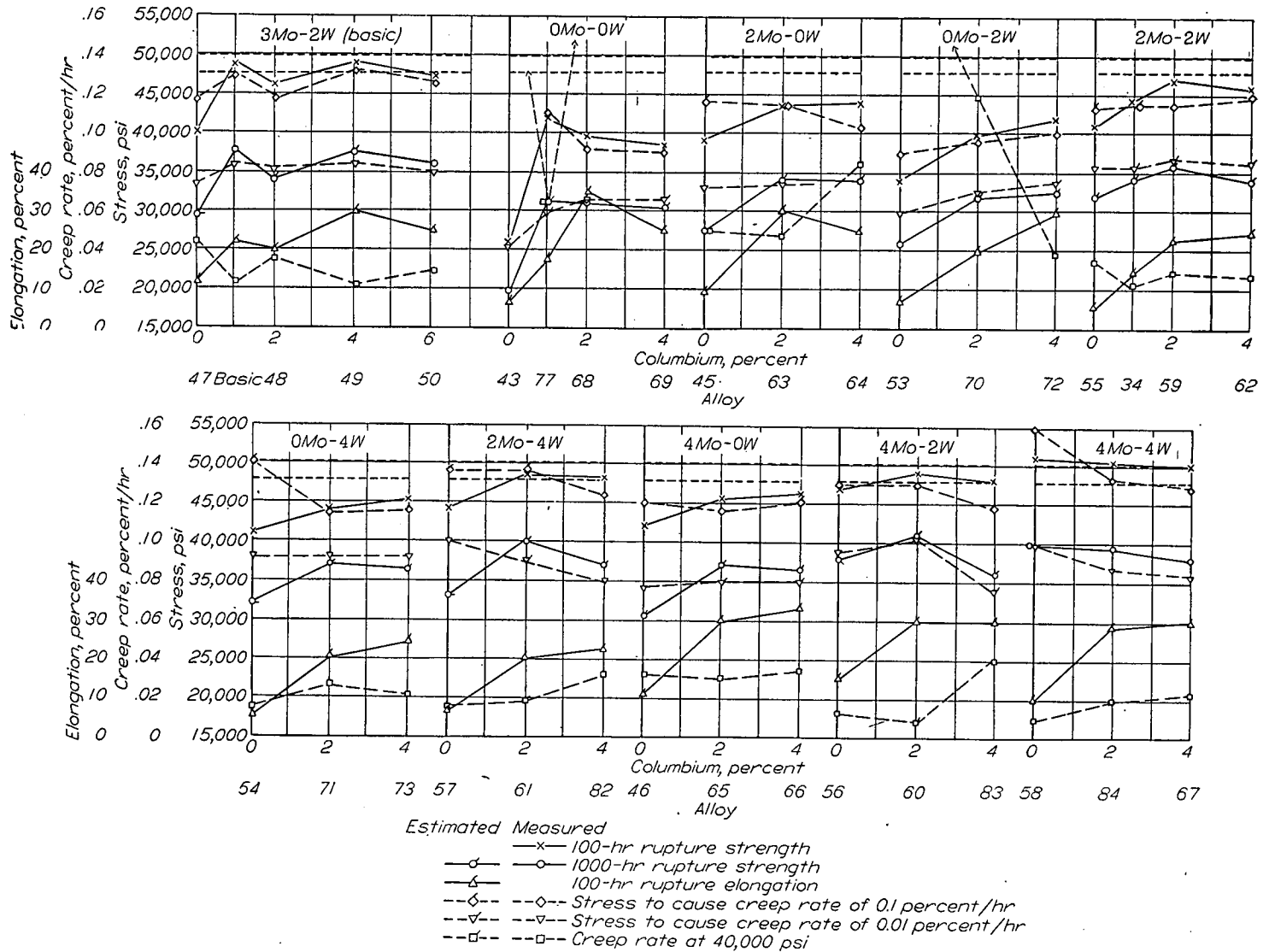


FIGURE 36.—Curves of rupture test data at 1200° F against columbium content for 10 molybdenum-tungsten modifications. Horizontal dashed lines indicate range of 100-hour rupture strength for three basic heats.

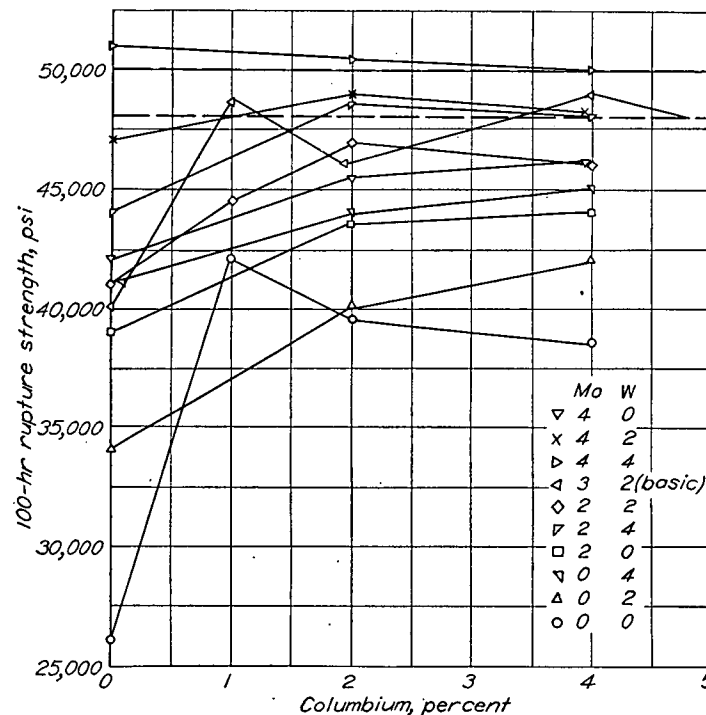


FIGURE 37.—Summary of influence of columbium on 100-hour rupture strength at 1200° F of 10 molybdenum-tungsten modifications. Horizontal dashed lines indicate property range for three basic heats.

Comparative influences of molybdenum, tungsten, and columbium.—Figure 38 shows comparative influences of Mo, W, and Cb on the 100-hour rupture properties at 1200° F for three of the alloy modifications in which these elements were varied simultaneously: 0Mo-0W-0Cb, 2Mo-2W-2Cb, and 4Mo-4W-4Cb. Two of the three elements are constant while the third is varied from 0 to 4 percent giving a family of three curves with one common composition. The 100-hour rupture strengths varied from 26,000 to 51,000 psi. Comparative influences of elements on rupture strengths were as follows:

(1) For the family of curves in which there is a common alloy with a composition of 0Mo-0W-0Cb, a Cb addition of 1 percent gave the highest strength; higher Cb lowered the strength. Increasing Mo and W increased strength continuously with Mo having the greatest influence. The strengths of the alloys on this family of curves were at a lower level, 26,000 to 42,000 psi, than those for the other two alloy modifications.

(2) For the 2Mo-2W-2Cb modifications, having strengths between 40,000 and 49,000 psi, Mo had the greatest strengthening influence; Cb was next, but strength dropped off above 2 percent Cb; and W produced a consistent, but the least, over-all strength increase.

(3) The 4Mo-4W-4Cb modifications were at the highest strength level of the three, 45,000 to 51,000 psi. Additions of Mo and W had about the same strengthening influence while Cb tended to lower strength.

The effects of the elements on 100-hour rupture elongations, also shown in figure 38, were the same for all three levels of composition: W tended to lower elongation slightly; Mo raised the elongation slightly; and Cb additions from 0 to 1 or 2 percent markedly raised the elongation from values below 10-percent elongation to 20 percent or higher.

The combined influence on 100-hour rupture strength of Mo, W, and Cb, on a total weight percent basis is shown in figures 39, 40, and 41. Each of these graphs contains the same points plotted to the same coordinates, these points representing all of the alloys in the testing program in which Mo, W, and Cb were varied, the remainder of the basic analysis being constant. The difference between the graphs is in the method of joining the points. In figure 39 the points are joined to give curves showing the effect of Mo varied from 0 to 4 percent for 10 constant ratios of W and Cb. In figure 40 the points are joined to show the effect of W and in figure 41 the curves show the effect of Cb.

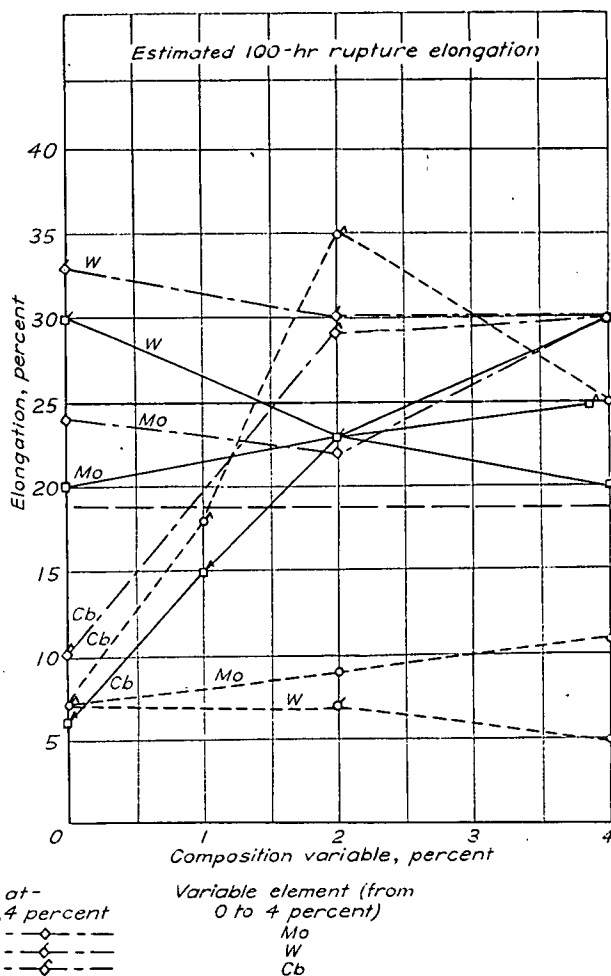
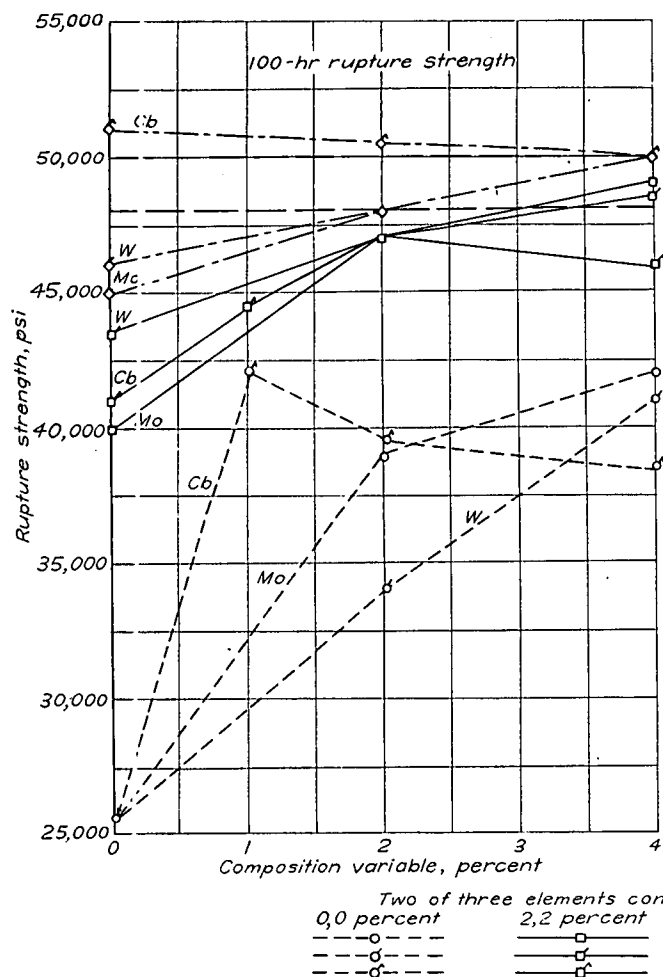


FIGURE 38.—Comparative influences of molybdenum, tungsten, and columbium on 100-hour rupture properties at 1200° F of modified alloys. Horizontal dashed lines indicate property range for three basic heats.

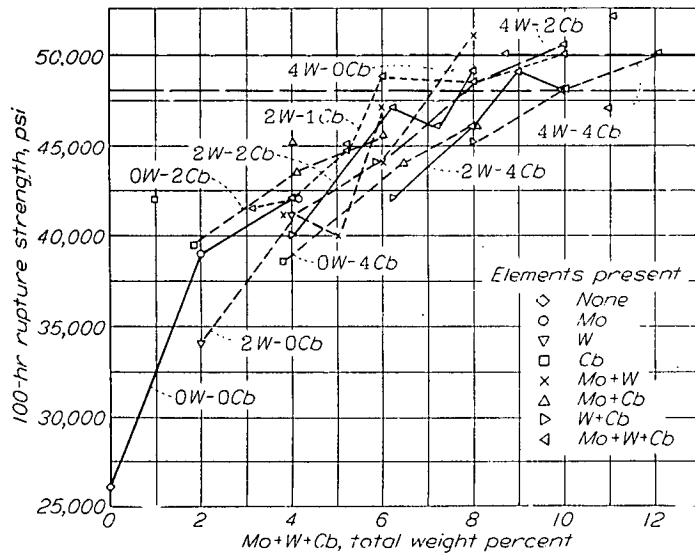


FIGURE 39.—Influence of molybdenum additions (0, 2, and 4 percent) on a total weight percent basis of molybdenum, tungsten, and columbium on 100-hour rupture strength at 1200° F of modified alloys. Horizontal dashed lines indicate property range for three basic heats.

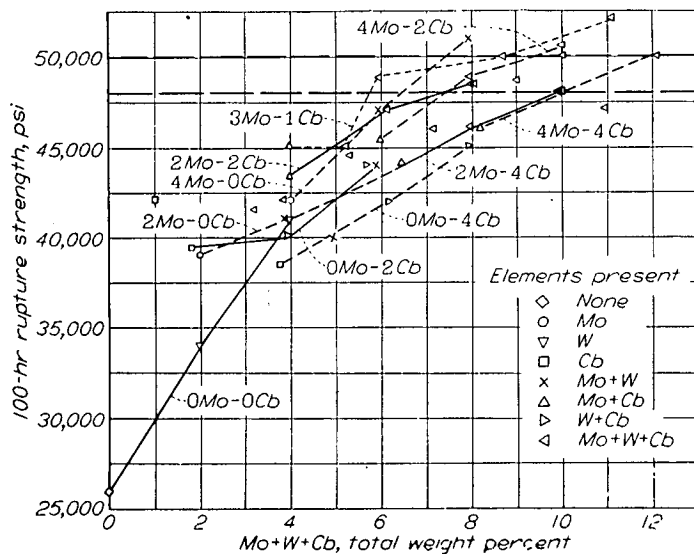


FIGURE 40.—Influence of tungsten additions (0, 2, and 4 percent) on a total weight percent basis of molybdenum, tungsten, and columbium on 100-hour rupture strength at 1200° F of modified alloys. Horizontal dashed lines indicate property range for three basic heats.

The array of points in these graphs indicates the general strength increase resulting from increasing the total alloy content of Mo plus W plus Cb from 0 to 12 percent. The main reason for the scatter of the points is evident from comparison of the curves in the three graphs. Both Mo (fig. 39) and W (fig. 40) produce strength increases over the entire composition range, while the Cb curves (fig. 41) above 1 and 2 percent Cb flatten out and cut horizontally across the property range. This emphasizes the unique influence, shown in previous graphs, of Cb on rupture strength.

The comparative influence on 100-hour rupture strength of Cb in relation to the influence of Mo and W is broken down further in the graphs of figure 42. Curves at constant Cb contents of 0, 1, 2, and 4 percent indicate the combined strengthening influence of Mo plus W. Curves represent-

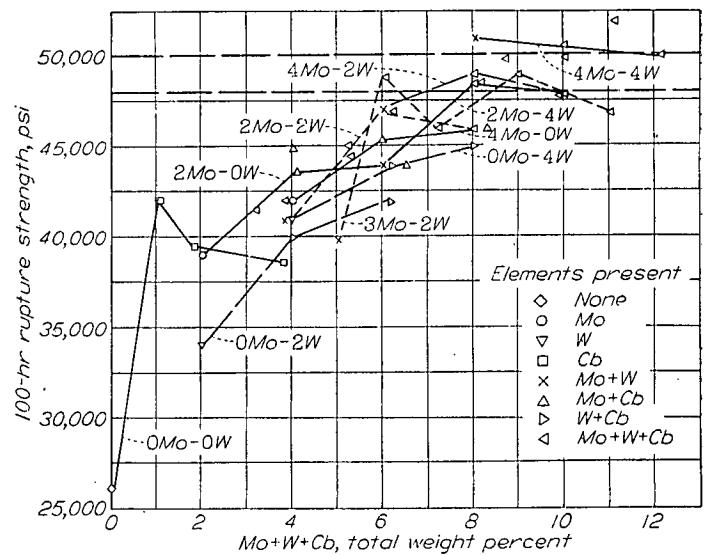


FIGURE 41.—Influence of columbium additions (0, 2, and 4 percent) on a total weight percent basis of molybdenum, tungsten, and columbium on 100-hour rupture strength at 1200° F of modified alloys. Horizontal dashed lines indicate property range for three basic heats.

ing this influence have been drawn and are summarized in the bottom graph in which Cb content is the parameter. The short-dashed curves on the upper four graphs represent the influence of Mo and the long-dashed curves, the influence of W at constant amounts of the other element.

The spread in data in the graphs of figure 42 was much less than that in figures 39, 40, and 41 in which the influence of Cb was not separated. When Mo is added in lower percentages it has the greater strengthening influence and W has the greater influence at higher percentages. These trends cause the curves representing the Mo and W influences to turn toward the center of the range representing the data spread at the higher percentages of these two elements and thus to strengthen the reliability of the single curves representing the data at constant Cb contents.

The summary curves in figure 42 indicate that additions of more than 1 percent of Cb were of no benefit to the alloys. When approximately 8 percent total of Mo plus W was present any addition of Cb was of no benefit to the strength. As has been noted previously, however, Cb additions of 1 percent markedly improved the rupture ductility over that of the 0-percent-Cb alloys, regardless of the influence of Cb on strength.

From the summary curves in figure 42 it is possible to predict the 100-hour rupture strength of alloys with any combination of Mo and W, at the four Cb levels, within the composition range in which these elements were investigated. Such predictions for alloys falling within the range of 0 to 1 percent Cb are not possible because data were not obtained. Based on the data, the largest error in these predictions would be in the vicinity of 2 percent total of Mo plus W. This error could be as much as 2500 psi, which is not large compared with the 2000-psi range in properties found from heat to heat of the basic alloy. Such predictions of course are tempered by the limitations of the investigation, particularly by the limitation of only one condition of preparation and heat treatment.

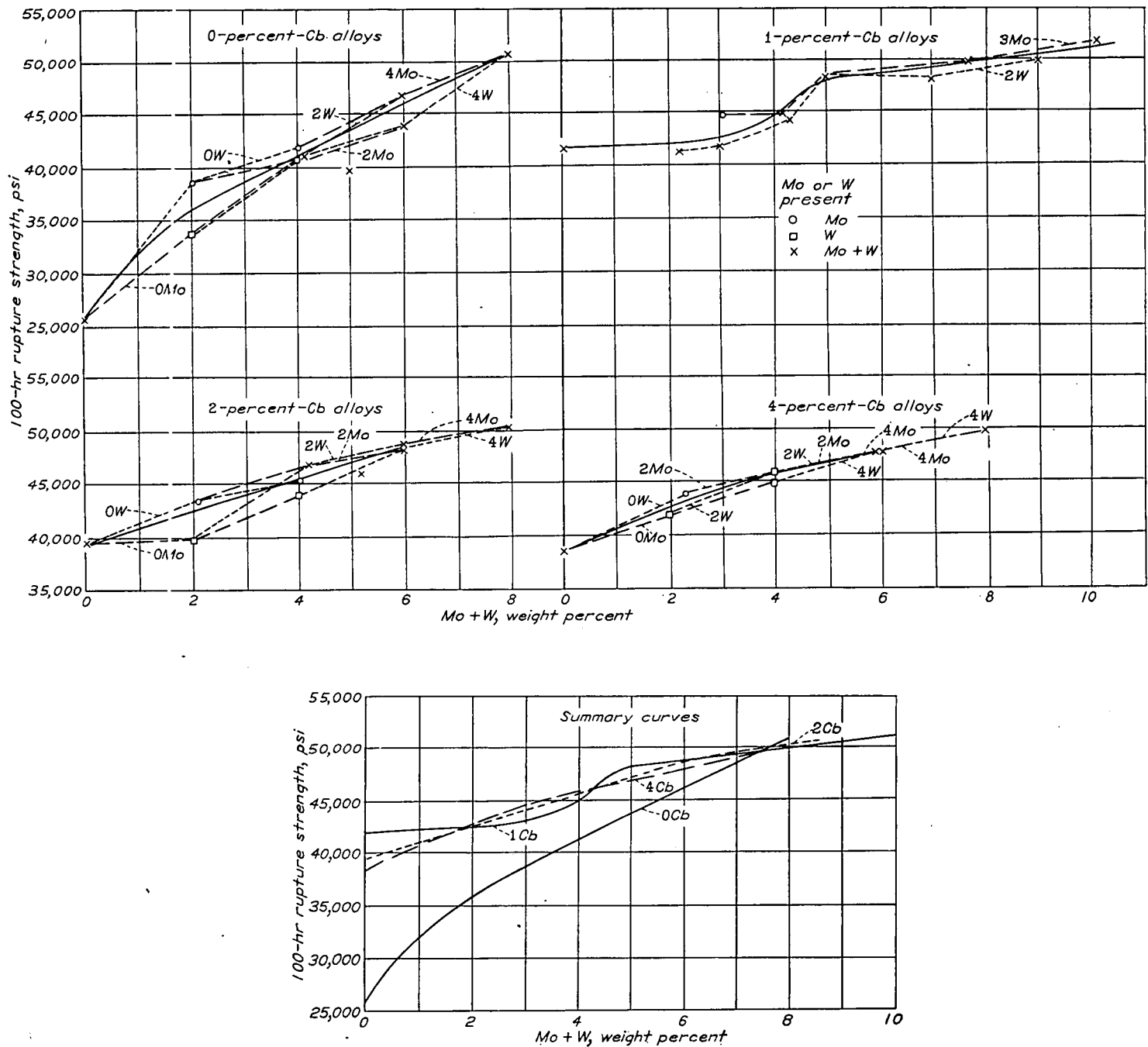


FIGURE 42.—Influence of molybdenum and tungsten additions on a total weight percent basis on 100-hour rupture strength at 1200° F of columbium modifications.

SUMMARY OF INFLUENCE OF CHEMICAL COMPOSITION ON RUPTURE TEST CHARACTERISTICS

Individual variations of elements.—Figure 43, which summarizes the rupture properties at 1200° F for variations of one element at a time in the basic analysis, indicates the following influences of chemical composition for alloys melted, forged, and heat-treated under the conditions of this investigation:

(1) Variation of C and Mn had no appreciable influence over the ranges examined.

(2) Very low N caused somewhat low rupture strength and intermediate N produced strengths slightly above normal.

(3) The effect of Si was unique in that increasing amounts lowered rupture strength and increased elongation.

(4) Increasing amounts of all other elements resulted in increased rupture strength over the complete range of variation except Ni, Co, and Cb which apparently reached a saturation content for rupture strength at 10, 20, and 1 percent, respectively. Additions of Cb caused a pronounced increase in rupture elongation.

(5) A saturation point for Mo and W was approached at the 3- and 2-percent levels of the basic alloy although improvements were obtained by further additions of these elements. The element Mo increased and W had no effect on elongation.

(6) The rupture strength was markedly increased by additions of Cr from 10 to 30 percent, and rupture elongation was decreased.

(7) Relative magnitudes of 100-hour rupture-strength improvements were: 20 percent Cr increased strength 13,000 psi; 7 percent Mo, 8500 psi; 7 percent W, 7000 psi; and 1 percent Cb, 8500 psi. The addition of 1 percent of Si decreased strength 5000 psi.

The following modifications had strengths appreciably below that of the basic alloy: 1.2 percent Si, 1.6 percent Si, 10 percent Cr, 0 percent Co, 10 percent Co, 0 percent Mo, 1 percent Mo, 2 percent Mo, 0 percent W, 1 percent W, and 0 percent Cb. The only alloys which had appreciably higher strengths than the basic alloy were those with 30 percent Cr and 7 percent W. From this it appears little was done to improve the 1200° F rupture strength of the basic alloy in the single condition of treatment studied, that rather wide individual variations of the elements can be permitted, particularly to higher values than those in the basic analysis, without appreciably altering properties, and that Cr, Co, Mo, W, and Cb are necessary for high strength.

Simultaneous variations of molybdenum, tungsten, and columbium.—The summarized influences on rupture properties at 1200° F of simultaneous variations of Mo, W, and Cb in the basic analysis, shown in figure 42 for the 100-hour rupture strength, serve to emphasize the general findings when these elements were varied individually. These influences are as follows:

(1) The absence of Mo, W, and Cb yielded an alloy with very low rupture strength, 26,000 psi for rupture in 100 hours.

(2) Separate additions of Mo, W, or Cb in amounts up to 4 percent to the 0Mo-0W-0Cb analysis raised the 100-hour rupture strength up to at least 40,000 psi as compared with 49,000 psi for the basic alloy.

(3) Simultaneous additions of the three elements in amounts up to 4 percent at least doubled the strength of the 0Mo-0W-0Cb analysis. Such additions did not, however, yield alloys with properties which were outstandingly better than those of the basic alloy.

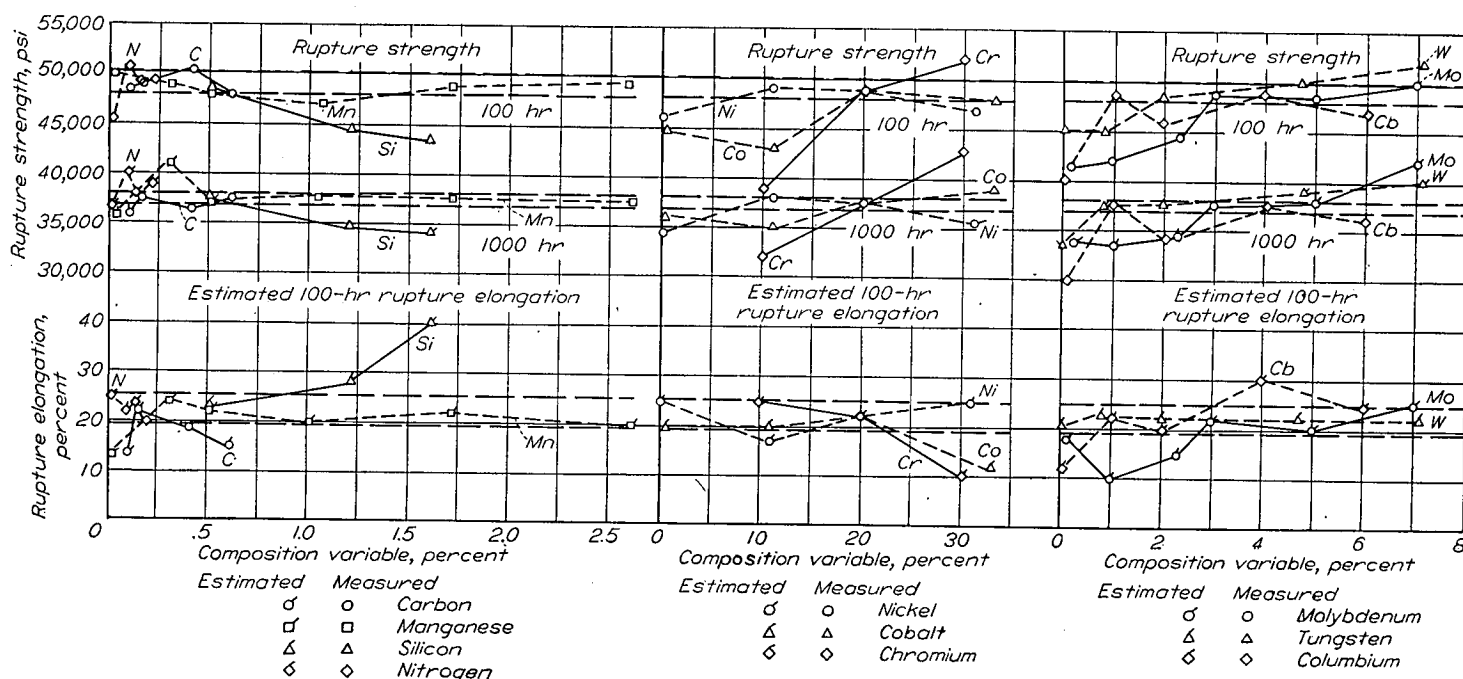


FIGURE 43.—Summary of influence of variation of individual alloying elements in basic alloy on 100-hour and 1000-hour rupture strengths and 100-hour rupture elongation at 1200° F. Horizontal dashed lines indicate property range for three basic heats.

(4) Additions of Mo and W to 4 percent raised strength progressively. Additions of more than 1 percent of Cb, however, were not beneficial to strength.

(5) Additions of Mo and W had little effect on rupture test elongation. Alloys containing no Cb had consistently low elongation and additions of 1 to 2 percent Cb markedly increased the elongation.

Relation of rupture and creep properties.—It was noted in table X and in the graphs of properties against composition that the stresses causing a minimum creep rate of 0.1 percent per hour were similar in magnitude to the 100-hour rupture strength. In general it was also noted that changes in rupture strength were accompanied by similar changes in creep properties except for alloys with relatively low rupture elongation, particularly alloys not containing Cb, where total elongation appeared to have relatively greater influence on fracture time.

It appears, therefore, that the rupture strengths of the alloys were controlled by their inherent resistance to creep, as measured by the stresses based on minimum creep rates obtained during the rupture test and their total elongation to fracture.

DISCUSSION OF RESULTS

The results indicate that by careful control of processing conditions the high-temperature characteristics of forged Cr-Ni-Co-Fe-Mo-W-Cb alloys can be related to systematic variations of chemical composition and that major changes in rupture characteristics at 1200° F accompany certain variations in composition. A summary of the influence of alloy modification on the 100-hour rupture strength at 1200° F is shown in figure 44 for the elements (Si, Cr, Ni, Co, Mo, W, and Cb) producing significant changes. There were also marked changes with alloy modification in such other metallurgical properties as microstructure, hardness, and melting and forging characteristics which indicate reasons for the observed influence of composition on rupture characteristics.

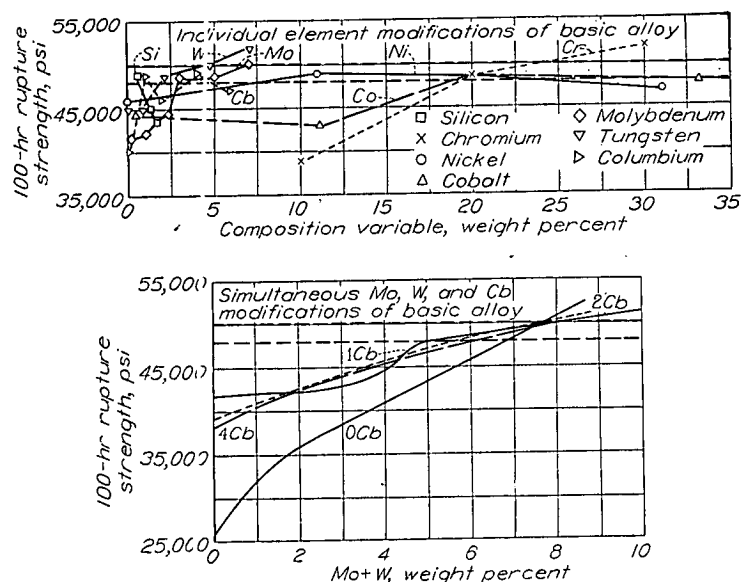


FIGURE 44.—Summary of influences of alloy modification on 100-hour rupture strength at 1200° F. Horizontal dashed lines indicate property range for three basic heats.

The range in 100-hour rupture properties associated with the composition variables studied was 26,000 to 52,000 psi for the rupture strength and 5 to 40 percent for the rupture elongation. These ranges compare with properties of other heat-resisting alloys as shown in table XI. It is seen that the rupture properties of the basic alloy can be varied over just as large a strength range, but at a higher level, as a result of variations in prior processing as was obtained by composition modifications for a single processing condition. The strength of the modification (0Mo-0W-0Cb) at the low end of the composition strength range is similar in magnitude to strengths of the standard heat-resisting alloys of the 18Cr-8Ni type. Wide property ranges are also shown for seven superalloys studied in a research program on a large number of alloys in which it was found that no accurate comparison could be made between alloys on the basis of chemical composition because prior processing conditions were not controlled. The present investigation has shown that certain variables in melting and hot-working procedures definitely result in major variations in high-temperature properties and that these variations probably cannot be completely removed by subsequent heat treatment.

LIMITATIONS OF DATA

Interpretation of the results can be made only subject to the limitations initially placed on the investigation and those which developed as a result of the investigation. The major limitations which appear to be of significance are listed below:

(1) A limited number of composition variables were studied. Simultaneous variations of elements were made only for Mo, W, and Cb. While certain of the composition variables studied showed significant influences on properties there was only limited indication of what to expect from simultaneous variation of other combinations of the elements in the alloy.

(2) Comparisons between alloys were limited to properties in one condition of prior processing. It was shown that the control exercised over processing was sufficient to obtain reliable reproducibility of properties between heats of the basic alloy. But it was also shown that, unless all the processing procedures, including both melting and hot-working conditions, are controlled, variability in properties may result. Reasons for this variability were not evident from this investigation. The relative influences of processing on alloys other than the basic are not known. Because only one processing condition and heat treatment were used for all the alloys no indication could be obtained of the optimum properties which would be expected for the alloys by variation of treatment.

(3) Limitations are also imposed in the interpretation of the influence of composition on high-temperature characteristics in general by the fact that only one type of test was used to evaluate properties and this at only one temperature. Testing time was also usually limited to that necessary to establish the 100-hour rupture strength.

In the discussion of the results of this investigation cognizance is made of these limitations.

COMPARATIVE RUPTURE AND CREEP CHARACTERISTICS

The main reason for consideration of the creep data from time-elongation curves of the rupture tests was to determine if creep resistance or elongation best correlated with the rupture strengths. It was observed that in general the stresses to cause a creep rate of 0.1 percent per hour were of the same order of magnitude and followed the same trends with composition variation as the 100-hour rupture strengths. The main exception to this observation was in the alloys in which Cb was systematically varied from 0 to 4 percent in which case the stresses for constant creep rates remained constant or decreased with increasing rupture strength (see fig. 36). It seemed apparent, therefore, that the marked influence of Cb additions from 0 to 1 or 2 percent on increasing inherent ductility was responsible for the unusual relation between creep and rupture properties. While stresses at a constant creep rate of 0.1 percent per hour were of the same order of magnitude as the 100-hour rupture strength, these stresses were higher for the 0-percent-Cb alloys (low elongation) and equal to or less than the rupture strength for the 1- and 2-percent-Cb alloys (high elongation). Apparently, therefore, the influence of Cb on changing the rupture strength was less dependent on the resistance to creep, as measured by stress at constant creep rate, than on the relative ability of the alloys to deform before fracture.

Figure 45 shows, for all the alloys studied, the variation of the 100-hour rupture strength with stress to cause a creep rate of 0.1 percent per hour. Average curves are drawn representing the points indicating the low (5 to 10 percent), intermediate (11 to 19 percent), and high (20 to 40 percent) 100-hour rupture-elongation levels encountered. While there are ranges for the rupture-creep property relations it is noted that the range representing high elongations is definitely at a higher level than that representing low elongations. The range for intermediate elongations overlaps both those for high and low elongations. These ranges, while partially the result of variation in actual elongation within the range, particularly at the lower elongations, are believed to be mainly caused by the limitation of the reproducibility of properties from heat to heat of a given analysis and by the fact that the creep properties in many cases were estimated from only a small amount of time-elongation data. Another possible cause of the ranges could be that for certain alloys the stress at constant creep rate is not an accurate representation of the creep resistance.

In general it appears, however, that the stress to cause a creep rate of 0.1 percent per hour is a measure of creep resistance which does control the 100-hour rupture strength, that the relation is linear and represented by a line with a slope of approximately 45°, but that the level of rupture properties at a given creep resistance is dependent upon inherent ability to deform before fracture. Specific examples of the relative dependence of rupture strength on stress at constant creep rate are as follows (see fig. 45):

(1) At a constant 0.1-percent-per-hour creep strength of 45,000 psi (which means all alloys on this vertical 45,000-psi line have constant creep resistance) alloys having high elongation

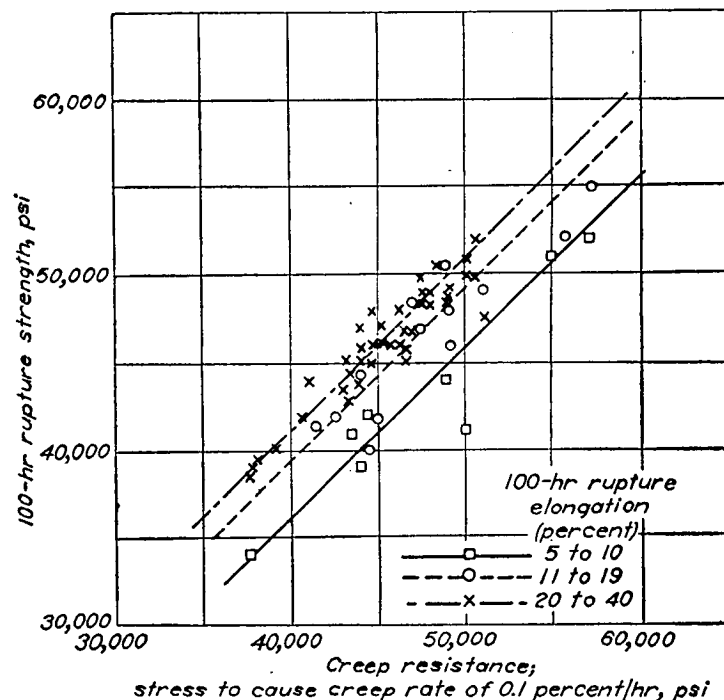


FIGURE 45.—Influence of creep resistance and 100-hour rupture elongation on 100-hour rupture strength at 1200° F of modified alloys.

will take longer to fracture than those having low elongation. The 100-hour rupture strength will be approximately 5000 psi higher for an alloy with 20-percent or more total elongation than for an alloy with 5- to 10-percent elongation.

(2) The 100-hour rupture strength must remain the same, for example at 45,000 psi, even though the stress at constant creep rate, or creep resistance, decreases by approximately 5000 psi because the ability to deform prior to fracture increases at the same time.

It is noted in figure 45 that the level of the relation between rupture strength and creep resistance is dependent on total elongation only at the lower elongations. The reason for this is the difference between the time-elongation curves for ductile and brittle materials. For ductile materials toward the end of the period of increasing creep rate (third-stage creep) there is a large amount of deformation in a very short time period. In this case the total elongation has a relatively small time dependence, the elongation having little influence on the rupture time. For brittle materials, on the other hand, the elongation change with time is more gradual near the end of the test. Thus with the same creep resistance the rupture time is more dependent on the relative inherent ability of the materials to deform before fracture.

The points in figure 45 that indicate elongations between 5 and 10 percent represent the alloys which did not contain Cb. One point at the highest stress represents the high-Cr alloy which had low ductility. It is evident that the unusual relation between creep and rupture properties observed for alloys in which Cb was varied systematically was the result of the added influence of total elongation on rupture strength.

It can be concluded that the rupture-strength variations with chemical composition observed in this investigation were the result of the changes in inherent creep resistance caused by alloy additions except for composition variations for which total elongation to fracture changed from low (5 to 10 percent) to high (above 20 percent) in which case the greater ability to deform before fracture also improved the rupture strength.

INTERPRETATION OF INFLUENCE OF CHEMICAL-COMPOSITION VARIABLES ON RUPTURE PROPERTIES AT 1200° F OF MODIFIED ALLOYS

Interpretation of the manner in which the composition variations influence the high-temperature properties of these alloys by controlling the inherent creep resistance and the ability to deform before fracture can be made in view of simultaneous effects noted in other metallurgical characteristics. The following observations of such effects were made:

(1) Only the following nine alloys of the 63 different compositions studied developed a pronounced amount of visible microstructural precipitate during aging:

Alloy	Composition variable (percent)
15	0.40C
16	0.60C
52	30Cr
29	0Co
49	3Mo-2W-4Cb
50	3Mo-2W-6Cb
82	2Mo-4W-4Cb
83	4Mo-2W-4Cb
67	4Mo-4W-4Cb

Except for the 30-percent-Cr alloy, the additional precipitate, resulting from either increased C or Cb or decreased Co, did not have any apparent beneficial effect on strength. The matrix precipitate which occurred during aging of the 30-percent-Cr alloy was of a different type from that of other alloys with heavy precipitation, being preferred rather than random. The hardness increase during aging of the 30-percent-Cr alloy was exceptionally large. Thus the substantial strength increase with the addition of Cr from 20 to 30 percent appeared to result, at least in part, from the aging characteristics caused by higher Cr.

(2) Neither Mo nor W, when varied from 0 to 7 percent in the basic alloy containing 1 percent Cb, showed the least effect upon the relative amounts of visible precipitate occurring during aging. However, accompanying the increases in these elements were improvements in creep resistance and rupture strength over the entire composition range. Likewise Mo and W, when varied simultaneously from 0 to 4 percent, produced no increase in amount of visible aging precipitate in alloys containing 2 percent or less of Cb. Creep-resistance and rupture-strength improvements were considerable and continuous in the 20 series of alloys in which either Mo or W was increased systematically. In 16 series of alloys, there was no appreciable change in the effect of these elements on aging precipitate present. The remaining four series were those containing 4 percent Cb, and in these the presence of at least 6 percent total of Mo plus W was necessary to increase the amount of visible aging precipitate. The

elements Mo and W were similar in effect on properties in these systems as in the systems containing lower Cb. However, these high-Cb alloys did not develop an appreciable relative increase in hardness during aging and had a random rather than the preferred matrix precipitate of the high-Cr alloy which apparently derived its superior strength from the aging reaction.

(3) The Cb produced its major improvement in rupture strength in additions of 1 percent to the 0-percent-Cb alloys. This increase in rupture strength appeared to be the result of the higher ductility caused by Cb rather than an improvement in creep resistance, which Cb additions either did not change or reduced. Additions of more than 2 percent Cb did not add anything to rupture strength because creep resistance was not increased and the rupture elongations were all at a high level where changes in total elongation did not appreciably affect fracture time. Microstructural examinations indicated that Cb, or the compound it caused to form, was at most only partially soluble at temperatures up to 2200° F. Limited solubility appeared to be the reason that there was no improvement in creep resistance when Cb was added. The major changes in microstructure caused by Cb—increasing the amount of insoluble constituent, changing the mode of aging precipitation from preferred to random, and refining the grain size—appeared to be associated with the marked increase in rupture elongation resulting from Cb additions. It is noted that low rupture elongation was also associated with the preferred type of aging precipitate in the 30-percent-Cr alloy. Rupture-strength improvements with Cr additions, however, were the result of increased creep resistance which was not the case for the Cb-modified alloys.

Additions of 4 percent or more of Cb in the presence of 6 percent or more of Mo plus W caused a marked increase in the amount of random matrix precipitation during aging. This additional precipitate added nothing to strength and did not appreciably increase the relative hardening during aging. However, the strength increases accompanying additions of Mo and W in this region continued to be consistent with those which appeared to be the result of solid-solution effects for the lower-Cb alloys.

(4) There were certain similarities between the effects of C and Cb on the amount of excess constituents present in the microstructures. While the excess constituents resulting from C and Cb additions were not necessarily the same phases, the appearance of large amounts of excess constituents, both in the solution-treated condition and during aging of the higher-C modifications, had little effect on properties. Apparently C combines with elements normally present in the solid solution to form certain of the observed excess constituents. In view of this the inherent matrix strength of the material should be reduced. This effect appeared to be the case for the 0.60-percent-C alloy which did show a tendency toward lower strength. However, balancing factors, which could hold strength up to a certain extent, could be that some strengthening resulted from the very heavy aging precipitation, or that C did not combine with the major elements causing the strength.

(5) Additions of Co from 0 to 20 percent which tended to raise creep resistance, and thus rupture strength of the alloy, decreased the amount of visible aging precipitate, apparently increasing the solubility of the strengthening elements. Further addition to 32 percent Co increased creep resistance but did not appreciably improve rupture strength because of the lower rupture elongation.

(6) The major strengthening effect of Ni was between 0 and 10 percent. The addition of 10 percent Ni changed the alloy from an unbalanced ferrite-austenite composition of the 0-percent-Ni alloy to the austenitic-matrix-type alloy typical of all the other alloys studied. By preventing the formation of the weak ferrite-sigma-type phase 10 percent Ni caused the elements in this phase to enter the solid solution of the matrix, thus increasing the creep resistance and rupture strength.

Each of the above observations of relations of metallurgical characteristics and high-temperature properties points to the conclusion that the major strengthening effects produced by alloy variations in this investigation were the result of the elements entering into solid solution in the alloys. The two outstanding examples of solid-solution strengthening were for Mo and W. These elements would enter the solid solution by substitution, as also would Cr, Ni, and Co, for atoms of Fe, which they replaced in the alloys. The atoms of Cr, Ni, and Co are all about the same size as Fe atoms and could substitutionally enter the crystal lattice of the solid solution with ease. Substitution of the larger or incongruous atoms of Mo and W into the lattice, however, would necessarily set up a strained condition. Such strains in the lattice would interfere with the flow conditions during creep, thus giving the material higher creep resistance. The role of Cb, which also has relatively large atoms, is complicated by the high affinity of Cb for C and by the apparent low limit of solubility of Cb, or the phase which it forms, in the basic alloy system studied. The substitution of the similarly sized elements, Cr, Ni, or Co, into the lattice would not set up particularly strained conditions and thus strengthening from solid-solution effects would not result. This has also been demonstrated by a recent investigation of Fe-Cr-Ni alloys (reference 2). The effects of Ni and Co on properties and microstructure appeared to be the result of these elements influencing the matrix solubility of other elements, the two most probable other elements being Mo and W. As noted, Cr apparently increased strength, at least in part, by aging effects.

It would appear therefore that as the number of incongruous atoms entering solid solution increased the strength should increase proportionately. The composition modifications studied most thoroughly in this investigation were those involving the simultaneous variation of Mo, W, and Cb. The modifications in which Mo and W additions caused strength increases were pointed out as examples of strengthening as a result of solution effects. Figure 46 shows the influence on the 100-hour rupture strength of total atomic additions of Mo and W to alloys containing four levels of Cb. This figure is similar to the presentation in figure 42 of the

same data on a weight percent basis. The solid-line curves drawn represent the data even better, however, when presented on an atomic percent basis. Trends in the summary curves, with Cb content as a parameter, are the same in both presentations, Cb above 1 percent being of no benefit to strength. In general, the curves in figure 46 show a continuous and approximately proportionate increase in strength with atomic additions of Mo and W. This is consistent with what would be expected if solid-solution effects are responsible for the strength increase.

The points in the graphs of figure 46 have been connected so as to show the comparative effects of Mo and W, the short-dashed lines showing the Mo effects and the long-dashed lines the W effects. There is a general trend indicated by these curves for W to cause a relatively greater strengthening than does Mo. This means that, although the average solid-line curves drawn do represent quite well the data in the graphs for the composition ranges studied, additions of atoms of W to the solid solution had a greater strengthening effect than equivalent atomic additions of Mo.

Exceptions to the solid-solution strengthening effect, particularly that of the 30-percent-Cr alloy, were noted previously. One other major exception, which does not agree with the conclusion that strengthening resulted from solid-solution effects, was that encountered in the influence of Si variations. The effect of Si, varied from 0.5 to 1.6 percent, was to lower strength properties markedly and to increase total deformation to fracture. There was no apparent change in microstructure produced by Si. It was also observed that the high-Si alloy was the only alloy of all those studied which did not show any increase in hardness during aging.

A clue to the apparently anomalous effect of Si is to be found in the results of the effect of deoxidation practice during melting on the properties of the basic alloy. Table VI shows a 100-hour rupture strength of 52,000 psi for alloy 74 which was not deoxidized before pouring. This strength of 52,000 psi is to be compared with the strength range of 48,000 to 50,000 psi for the basic alloy deoxidized in the normal manner with calcium-silicon deoxidant and with the 55,000-psi rupture strength of alloy 75 which was deoxidized with a zirconium-silicon-iron deoxidant. These results indicate that a marked influence may occur on properties of a given alloy as a result of variation of melting procedure. The element Si definitely plays an important role in melting, particularly in deoxidation practice. One heat (alloy 27) in which the charge was aimed at yielding 0.25 percent Si in the basic analysis and which was deoxidized in the normal way with 15 grams of calcium-silicon alloy resulted in 0.58 percent Si and gave properties typical of the basic alloy. Another heat (alloy 79) aimed at 0.25 percent Si, but in which only 5 grams of calcium-silicon deoxidant were used, resulted in a blow hole at the ingot center. It thus appears that inherent characteristics, imparted to the material during melting, not removed by subsequent processing, and not evident in microstructure, could be responsible for the unusually low strengths of the high-Si alloys.

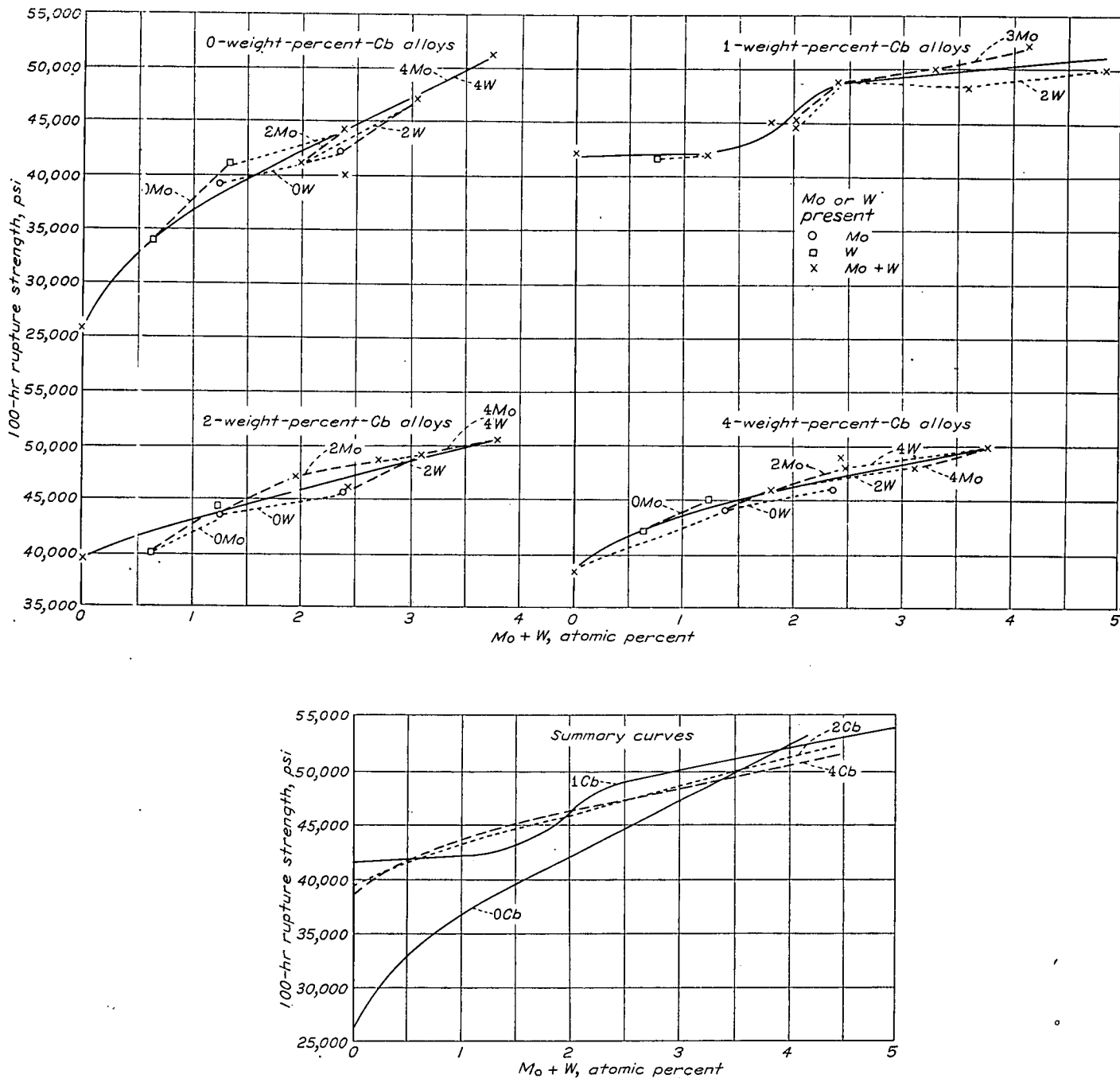


FIGURE 46 —Influence of molybdenum and tungsten additions on a total atomic percent basis on 100-hour rupture strength at 1200° F of columbium modifications.

The basic alloy has been shown by previous investigations to be an alloy which does not develop appreciable strengthening at high temperatures as a result of visible precipitation reactions during aging after solution treatment (references 1 and 3). Evidence presented in this discussion indicates that the major strengthening effects accounting for the good strength of the basic alloy result from additions of incongruous atoms of elements to the solid-solution lattice of the austenite-type matrix.

Another characteristic evaluated for the alloys was the relative forgeability. Although this evaluation was only qualitative the indications were that high-temperature characteristics as measured by rupture tests at 1200° F were in general reflected at temperatures of 1800° to 2200° F in the relative forgeability of the alloys. Alloys which had higher rupture strengths as a result of greater resistance to creep also tended to be less plastic in the forging range. The main exception to this was for Mo additions which apparently did not affect forgeability. The alloys not containing Cb, which had quite low rupture elongation, tended to crack during forging.

OPTIMUM ALLOY CONTENT AND COMPOSITION RANGES

In addition to the main objective of determining the influence of chemical-composition variations on rupture properties at 1200° F, the present investigation was also intended as an initial step toward establishment of optimum compositions for alloys of the basic-alloy type. Interpretation of the results for this purpose, however, is subject to the limitations of the number of compositions studied and the single preparation condition placed on the investigation.

While the over-all property range obtained was quite wide and information was obtained concerning the influence of each element on properties, no alloy was obtained which had properties which were outstanding compared with those of the basic analysis.

It was shown that most of the elements present in this analysis are necessary to yield the best properties, but that variations over relatively wide ranges can be made of one element at a time in the basic analysis without appreciably altering properties. Indications are that Ni can be varied from 10 to 20 percent, C from 0.08 to 0.60 percent, Co from 20 to 32 percent, Cb from 1 to 4 percent, Mn from 0 to 2.5 percent, and N from 0.08 to 0.18 percent in the basic analysis without appreciably affecting properties. It was also shown that Cr, Mo, and W are necessary for maintenance of properties and that Cb in additions not beyond 1 percent is necessary for high ductility characteristics. It is possible, however, to obtain strengths as high or higher than that of the basic alloy without the use of Cb but this involves increasing Mo and W.

Seemingly the only major reduction in required alloy content in the basic alloy, which could be made without appreciably lowering properties, is that Ni could be lowered to 10

percent. Possibly Cb could be decreased a few tenths of a percent below 1.

NATURE OF PHASES PRESENT IN EXPERIMENTAL ALLOYS

As yet little is known concerning the nature of the phases present in alloys of the type investigated. It is generally accepted that the austenite-type matrix is a solid solution which is saturated with respect to certain unidentified constituents at both the solution and aging temperatures. It is also known that Cb, Cr, Mo, and W are prone to form carbides or nitrides under certain conditions and that intermetallic compounds, such as the Fe-Cr sigma phase, and also a ferrite-type phase, are possibilities of microconstituents.

There was a limited amount of evidence gained from the microstructural studies of this investigation concerning the nature of the phases encountered. There is evidence that C entered into the reaction forming the phase which was at least partially insoluble at the solution-treatment temperature and that Cb was the other significant element in the formation of this phase. It also appeared that C was an important constituent of the precipitate which occurred during aging.

Evidently Cr contributed to the formation of the phase which precipitated during aging of the 0Mo-0W-0Cb alloy. It appeared that Cr was also quite important in the aging reaction in the 0-percent-Cb alloys since additions of Mo and W to these alloys did not change the appearance or appreciably affect the amount of aging precipitate.

Additions of Cb from 2 to 4 percent in alloys containing high Mo plus W appreciably increased the amount of aging precipitate. Evidently this occurred as a result of exceeding the solubility limit of the solution which caused the rejection during aging of the excess phase from supersaturated solution. The constituents of this phase were not evident from the data, although there was some indication that Mo and W, which apparently increased strength by substitutional entrance into solid solution, probably were not present in the precipitate. The two examples of this were:

(1) Additions of C to the basic alloy increased aging precipitate but did not appreciably lower strength which would be expected if Mo or W were forced out of solution.

(2) Additions of Cb from 2 to 4 percent in alloys also containing at least 6 percent total of Mo plus W did not affect strength appreciably while increasing the amount of aging precipitate. On the other hand, when Mo or W was raised from 2 to 4 percent in alloys containing 4 percent Cb, producing alloys with more aging precipitate, the strength was improved, indicating that Mo and W went into solution.

In the event Mo and W did maintain strength by remaining in or entering into solid solution, it is probable that Cr or some of the other elements with atoms of similar size to Cr were forced out of solution as carbides or intermetallic compounds.

On the basis of the composition range studied for N, it did not appear that this element appreciably affected the microconstituents.

CONCLUSIONS

By the use of careful control over processing conditions this investigation has shown that for forged alloys containing chromium, nickel, cobalt, iron, molybdenum, tungsten, and columbium it is possible to correlate the stress-rupture properties at 1200° F with systematic variations in chemical composition and that a wide range in properties can be obtained by such variations. However, no alloy was obtained which had properties which were outstanding compared with those of the basic analysis.

Subject to the limitations placed on this investigation of the limited number of composition variables studied, the use of only one condition of processing, the evaluation of high-temperature characteristics by only relatively short-time rupture tests at 1200° F, and the use of only microstructural and hardness data to provide interpretation of results, the findings lead to the following conclusions:

1. Carbon (varied from 0.08 to 0.60 percent), manganese (0 to 2.5 percent), nitrogen (0.08 to 0.18 percent), nickel (10 to 20 percent), cobalt (20 to 32 percent), and columbium (2 to 4 percent) do not appreciably influence 1200° F rupture properties. Nitrogen (0.004 to 0.08 percent), chromium (10 to 30 percent), nickel (0 to 10 percent), cobalt (0 to 20 percent), molybdenum (0 to 4 percent), tungsten (0 to 4 percent), and columbium (0 to 1 percent) improve the rupture strength. Silicon (0.5 to 1.6 percent) and nickel (20 to 30 percent) lower rupture strength.

2. The rupture-strength variations with chemical composition observed in this investigation were the result of changes in inherent creep resistance, caused by alloy additions, except for composition variations for which total elongation to fracture changed from low (5 to 10 percent) to high (above 20 percent), in which case the greater ability to deform before fracture also improved the rupture strength.

3. The rupture-strength improvements accompanying the increased creep resistance with additions of molybdenum and tungsten apparently are the result of the strengthening influence of these incongruous atoms entering substitutionally into the matrix solid solution. Increased creep resist-

ance with nickel and cobalt additions apparently results from the manner in which these elements improve the solubility of molybdenum and tungsten. The increased creep resistance produced by chromium additions apparently results, at least, in part, from an aging reaction. The creep resistance is not influenced by columbium which enters solid solution to only a very limited extent. Improvements in rupture strength with small additions of columbium result from the greater ability to deform before fracture of alloys containing columbium. The detrimental effect of silicon on strength properties is possibly connected with melting phenomena which are not yet understood.

4. Columbium, chromium, and carbon produce major changes in microstructure. There is limited evidence to indicate that these are the major elements in the present excess constituents in the structure of the alloys.

5. Melting and hot-working conditions have an important influence on the inherent high-temperature properties of alloys of the type studied.

UNIVERSITY OF MICHIGAN,

ANN ARBOR, MICH., *October 26, 1949.*

REFERENCES

1. Freeman, J. W., Reynolds, E. E., Frey, D. N., and White, A. E.: A Study of Effects of Heat Treatment and Hot-Cold-Work on Properties of Low-Carbon N-155 Alloy. NACA TN 1867, 1949.
2. Borzdyka, A. M.: The Influence of Alloying Elements on the High-Temperature Strength of Chromium-Nickel Austenite. Doklady Akad. Nauk SSSR, vol. 63, 1949.
3. Frey, D. N., Freeman, J. W., and White, A. E.: Fundamental Effects of Aging on Creep Properties of Solution-Treated Low-Carbon N-155 Alloy. NACA Rep. 1001, 1950. (Formerly NACA TN 1940.)
4. Anon.: Digest of Steels for High Temperature Service. Fifth ed., The Timken Roller Bearing Co., Steel and Tube Div. (Canton, Ohio), 1946.
5. Freeman, J. W., Reynolds, E. E., and White, A. E.: High-Temperature Alloys Developed for Aircraft Turbosuperchargers and Gas Turbines. Symposium on Materials for Gas Turbines, A. S. T. M. (Philadelphia), 1946.

TABLE I.—CHEMICAL COMPOSITION OF EXPERIMENTAL ALLOYS

Alloy	Aim modification from basic alloy (percent)	Chemical composition (percent) ^a									
		C	Mn	Si	Cr	Ni	Co	Mo	W	Cb	N
1 ^b	Basic analysis	0.15	1.7	0.5	20	20	20	3	2	1	0.12
2 ^b	Basic	.25	1.54	.40	18.45	19.45	20.35	3.06	2.30	1.05	.11
3 ^b	Basic	.24	1.62	.53	18.88	17.20	19.74	2.95	2.26	1.11	.12
4 ^b	Basic	.25	1.68	.60	18.99	18.60	20.64	2.77	2.62	1.12	.15
5 ^b	Basic	.30	1.87	.74	19.37	19.04	17.95	2.89	2.34	.96	.12
6 ^b	0.4 C	.55	1.58	.67	17.53	18.24	20.54	2.83	1.75	1.00	.12
7 ^c	0.4 C	.53	1.78	.84	16.27	18.77	18.04	3.32	2.50	.82	.10
8 ^c	Basic	0.16	1.40	0.66	20.64	17.34	20.52	3.01	2.34	1.11	0.15
9 ^c	Basic	.14	1.46	.41	21.17	20.16	19.81	3.03	2.08	.96	.14
10 ^c	Basic	.15	1.40	.52	20.89	20.64	19.30	3.05	1.91	1.03	.17
11 ^c	Basic	.16	1.63	.77	21.00	17.13	22.46	3.04	1.97	1.08	.14
12 ^c	Basic	.16	1.61	.67	21.04	18.39	22.00	3.03	2.01	1.15	.14
27 ^d	Basic	.15		.58							
74 ^e	Basic										
75 ^e	Basic										
13	0.07 C	0.08	1.80	0.37	20.03	20.84	19.42	3.12	2.10	1.09	
14	0.40 C	.36									
15	0.40 C	.40									
16	0.60 C	.57									
17	0.60 C	.60	1.83		18.59	20.25	18.92				
24	0.03 Mn	0.14	0.03								
19	0.25 Mn	.15	.30	0.38	20.07	20.70	19.82	2.99	2.02	1.13	
20	0.50 Mn	.14	.50								
21	1.00 Mn	.12	1.04								
22	2.50 Mn	.14	2.58								
28	1.0 Si	0.13		1.19							
80	1.5 Si	.14		1.56							
51	10 Cr	0.14			10.18						
52	30 Cr	.15			30.51						
23	0 Ni	0.15				0.01					
25	10 Ni	.16				10.70					
26	30 Ni	.14				30.64					
29	0 Co	0.19					0.31				
30	10 Co	.14					11.09				
31	30 Co	.14	1.76	0.79	20.50	20.18	32.60	3.41	2.05	1.11	
32	0 Mo	0.16						0.20			
33	1 Mo	.16	1.76	0.75	20.46	20.52	21.59	1.00	2.02	1.06	0.15
34	2 Mo	.16						2.34			
35	4 Mo	.13						4.99			
36	6 Mo	.12						6.99			
37	0 W	0.14							0.04		
38	1 W	.15	1.74	0.75	20.75	20.50	21.68	3.42	.82	1.02	0.15
39	4 W	.14							4.66		
40	6 W	.16							7.13		
47	0 Cb	0.14								0.03	
48	2 Cb	.14	1.74	0.70	20.65	20.59	20.08	3.20	1.94	1.97	0.14
49	4 Cb	.14								4.07	
50	6 Cb	.16								6.09	
41	Low N	0.14									0.004
42	0.07 N	.15									.08
81	0.22 N	.15									.18
		Mo	W	Cb							
43	0	0	0	0	0.12				0.09	0.00	0.00
44	0	0	0	0	.18	1.72	0.74	20.22	20.62	20.09	0.00
45	2	0	0	0	.14				2.09	.00	.00
46	4	0	0	0	.13				4.01	.00	.00
53	0	2	0	0	.13				.01	2.05	.00
54	0	4	0	0	.14				.02	3.97	.00
55	2	2	0	0	0.15	1.71	0.72	20.28	20.41	20.20	2.33
56	4	2	0	0	.13				4.05	2.04	.00
57	2	4	0	0	.14				1.98	4.16	.00
58	4	4	0	0	.13				4.04	4.11	.00
59	2	2	2	2	0.16				2.19	2.04	1.98
60	4	2	2	2	.13				4.12	2.07	1.98
61	2	4	2	2	.15	1.69	0.80	19.99	20.50	20.54	2.32
62	2	2	4	4	.15				2.02	1.93	4.09
63	2	0	2	2	0.11				2.10	0.00	1.93
64	2	0	4	4	.15				2.32	.03	4.23
65	4	0	4	4	.13				4.02	.00	2.10
66	4	0	4	4	.13				3.96	.00	4.08
67	4	4	4	4	.13	1.70	0.71	20.18	20.36	20.12	4.12
77	0	0	0	1	0.15						0.96
68	0	0	0	2	.18				0.00	0.00	1.81
69	0	0	4	4	.14				.01	.00	3.76
70	0	2	2	2	.16				.01	2.00	1.92
71	0	4	2	4	.17				.02	3.93	1.91
72	0	2	4	4	.14	1.80	0.66	20.21	20.39	20.51	.01
73	0	4	4	4	.15				.02	3.92	4.05
82	2	4	4	4	0.16						
83	4	2	4	4	.13						
84	4	4	2	2	.14	1.64	0.74	20.18	20.51	19.95	3.85
									1.85	4.38	0.17
									2.25		

^a Values given only where actual chemical analyses were made; blank spaces indicate same aim value as that of basic alloy.^b Alloys 1 to 6 were prepared to develop and standardize melting practice.^c Alloys 7, 8, 10, 11, and 12 were prepared to determine control of properties with processing from heat to heat of basic alloy.^d Alloy 27, aimed at low Si, resulted in basic value.^e Deoxidation practice was varied on alloys 74 and 75 of basic analysis.

TABLE II.—TYPICAL MELTING RECORD

[Alloy 72. Aim analysis, percent: 0.15 C, 1.7 Mn, 0.5 Si, 20 Cr, 20 Ni, 20 Co, 0 Mo, 2 W, 4 Cb, 0.12 N]

(a) Charge

Melting stock	Weight (grams)	Composition contributed by melting stock (percent)										
		Fe	C	Mn	Si	Cr	Ni	Co	Mo	W	Cb	N
Electrolytic nickel.....	820	-----	-----	-----	-----	-----	20.5	-----	-----	-----	-----	-----
Cobalt rondles.....	800	-----	-----	-----	-----	-----	-----	20.0	-----	-----	-----	-----
Armco iron.....	721	17.99	0.004	0.005	-----	-----	-----	-----	-----	-----	-----	-----
Ferrochromium (low C, high N).....	544	4.27	.011	-----	0.053	9.15	-----	-----	-----	-----	-----	0.110
Ferrochromium (high C, low N).....	72	.45	.088	-----	.035	1.21	-----	-----	-----	-----	-----	.001
Ferrochromium (low C, low N).....	599	4.41	.008	-----	.066	10.64	-----	-----	-----	-----	-----	.006
Electrolytic manganese.....	68	-----	-----	1.69	-----	-----	-----	-----	-----	-----	-----	-----
Ferrosilicon (50 percent Si).....	0	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
Ferromolybdenum (60 percent Mo).....	0	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----	-----
Ferrotungsten (80 percent W).....	100	0.43	0.012	0.005	0.014	-----	-----	-----	-----	2.00	-----	-----
Ferrocolumbium (60 percent Cb).....	276	2.38	.023	-----	.404	-----	-----	-----	-----	-----	4.00	-----
Totals.....	4000	29.93	0.146	1.70	0.63	21.0	20.5	20.0	0	2.00	4.00	0.117
Actual analysis.....	-----	-----	0.14	1.80	0.66	20.21	20.39	20.51	0	2.01	4.18	0.12

(b) Melt data

Time (min)	Operation
0.....	Fe, Cr, Ni, and Co melting stock placed in hot crucible (fifth heat in crucible).
0.....	Power on, 400 volts, 10 kw.
15.....	Charge melted.
17.....	Slag skimmed.
18.....	Mn added.
20.....	Ferrotungsten and ferrocolumbium added.
22.....	Power down to 300 volts, 6 kw.
23.....	15 grams of 30-percent-Ca and 65-percent-Si alloy added to deoxidize.
24.....	Slag skimmed.
25.....	Power off. Bath temperature: Leeds and Northrup optical pyrometer—2860° F. Pt, Pt-Rh immersion thermocouple—2840° F.
27.....	Heat poured.
Cooling-curve data taken on hot-top metal. Temperature halt in curve at 2495° F.	

TABLE III.—MELTING DATA FOR EXPERIMENTAL ALLOYS

Alloy	Modification from basic alloy (percent)	Bath temperature before pouring (°F)		Cooling-curve temperature halt (°F)		
		Leeds and Northrup optical pyrom- eter	Pt, Pt-Rh immersion thermocouple			
7.....	Basic.....		2570	-----		
8.....	Basic.....		2560	-----		
10.....	Basic.....		2665	-----		
11.....	Basic.....		2680	-----		
12.....	Basic.....		2785	-----		
27.....	Basic.....	2625	2600	2466		
74.....	Basic, no deoxidation.....	2780	2840	-----		
75.....	Basic, Zr-Si-Fe deoxidant.....	2730	2745	-----		
76.....	Basic, melted under lime-fluorspar slag.....	2525	2520	-----		
13.....	0.08 C.....	-----	-----	2560		
14.....	0.40 C.....	-----	2535	-----		
15.....	0.40 C.....	-----	2535	2500		
16.....	0.60 C.....	-----	2585	2518		
17.....	0.60 C.....	-----	2585	>2470		
18 °.....	0 Mn.....	2550	-----	-----		
24.....	0 Mn.....	2630	2617	>2490		
19.....	0.30 Mn.....	-----	2640	>2530		
20.....	0.50 Mn.....	2640	2587	2561		
21.....	1.0 Mn.....	-----	2560	2555		
22.....	2.5 Mn.....	-----	2585	2514		
79 °.....	Low Si.....	2675	2682	-----		
28.....	1.2 Si.....	2640	2722	>2414		
80.....	1.6 Si.....	2580	2610	2450		
51.....	10 Cr.....	-----	2758	2542		
52.....	30 Cr.....	2722	2754	>2430		
23.....	0 Ni.....	2620	-----	-----		
25.....	10 Ni.....	2600	2622	2487		
26.....	30 Ni.....	2690	2634	2456		
29.....	0 Co.....	2660	2655	2471		
30.....	10 Co.....	2690	-----	2479		
31.....	32 Co.....	2670	2657	2459		
32.....	0 Mo.....	2700	2792	2529		
33.....	1 Mo.....	2670	2660	2495		
34.....	2 Mo.....	2660	2643	>2440		
35.....	5 Mo.....	2710	2684	2471		
36.....	7 Mo.....	2710	2688	2474		
37.....	0 W.....	2710	2698	2505		
38.....	1 W.....	2705	2692	2487		
39.....	5 W.....	2685	2690	>2490		
40.....	7 W.....	2700	2693	2484		
47.....	0 Cb.....	2770	2777	2553		
48.....	2 Cb.....	-----	2780	2508		
49.....	4 Cb.....	2650	2703	2464		
50.....	6 Cb.....	2690	-----	>2390		
41.....	0.004 N.....	2709	2707	2508		
42.....	.08 N.....	2720	-----	>2480		
81.....	.18 N.....	2632	2648	2479		
	Mo	W	Cb			
43.....	0	0	0	2740	2760	2592
44.....	0	0	0	2780	2790	>2520
45.....	2	0	0	2750	2768	>2530
46.....	4	0	0	2750	2770	2537
53.....	0	2	0	2870	2805	>2520
54.....	0	4	0	2785	2783	2542
55.....	2	2	0	2752	2736	2534
56.....	4	2	0	2742	2762	2529
57.....	2	4	0	2732	2670	2518
58.....	4	4	0	2832	2835	2482
59.....	2	2	2	-----	2790	2484
60.....	4	2	2	2685	2680	2466
61.....	2	4	2	2725	2726	2474
62.....	2	2	4	2710	2703	>2400
63.....	2	0	2	2802	2847	>2500
64.....	2	0	4	2740	2792	2460
65.....	4	0	2	2750	2766	2468
66.....	4	0	4	2670	2720	2442
67.....	4	4	4	2650	2678	2424
77.....	0	0	1	-----	-----	2613
68.....	0	0	2	2810	2815	2521
69.....	0	0	4	2710	2740	2476
70.....	0	2	2	2750	2782	2532
71.....	0	4	2	2740	2757	>2510
72.....	0	2	4	2860	2840	2495
73.....	0	4	4	2730	2773	2476
82.....	2	4	4	2590	2602	>2400
83.....	4	2	4	2625	2680	2419
84.....	4	4	2	2602	2585	2442

° Blow hole in ingot of low-Mn (18) and low-Si (79) heats.

TABLE IV.—TYPICAL FORGING RECORD

[Alloy 65, 4Mo-0W-2Cb]

(a) Forging operations

Time	Operation
8:25 a. m.	Ingot moved to center of furnace at 2200° F after a preheat near door of furnace.
8:30	Ingot on temperature, 2200° F.
9:05	60 hammer blows between flat dies on square faces of ingot; forging only top half of ingot.
9:20	55 blows between flat dies on square faces; 10 blows between flat dies on corners.
9:32	60 blows between flat dies on square faces; 10 blows between flat dies on corners.
9:45	55 blows between flat dies on square faces; 10 blows between flat dies on corners to approximately 1.10-in.-square bar.
9:54	70 blows in first swage.
10:04	21 blows in first swage to approximately 0.95 in. round; 46 blows in second swage.
10:12	64 blows in second swage.
10:18	65 blows in second swage.
10:27	40 blows in second swage to approximately 0.75 in. round; forged piece cut off from unformed half of ingot; forged piece recharged to furnace.
10:40	65 blows in third swage.
10:45	47 blows in third swage.
10:54	55 blows in third swage.
11:02	55 blows in third swage.
11:10	48 blows in third swage.
11:16	55 blows in third swage to approximately 0.58 in. round; this piece cut in two, both pieces recharged to furnace for forging in the last swage.
11:24	54 blows on bar C, 59 blows on bar D in last swage.
11:30	53 blows on bar C, 55 blows on bar D in last swage.
11:36	45 blows on bar C, 52 blows on bar D in last swage.
11:41	45 blows on bar C, 48 blows on bar D in last swage.
11:47	55 blows on bar C, 56 blows on bar D in last swage.
11:52	60 blows on bar C, 62 blows on bar D in last swage.
11:57	45 blows on bar C, 46 blows on bar D in last swage.
12:02	40 blows on bar C, 35 blows on bar D in last swage, finishing to 0.40 in. round. Finishing temperatures for all forging operations were 1800° F or above as judged by color.

(b) Forging summary

Die	Heatings	Blows
Flat.....	4	260
First swage.....	2	91
Second swage.....	4	215
Third swage.....	6	325
Last swage.....	8	Bar C—397 Bar D—413
Total heatings: 16.		
Total blows:		
Bar C—797.		
Bar D—813.		
Forging range: 2200° to 1800° F.		
Initial size of ingot:		
1.42 in. square tapered to 1.04 in. square.		
8.60 in. long.		
Final size of forged bars:		
Bar C: 0.40 in. round by 19.3 in.		
Bar D: 0.40 in. round by 21.0 in.		
Reduction during forging: 63 percent.		

TABLE V.—FORGING DATA FOR EXPERIMENTAL ALLOYS

Alloy	Bar ^a	Modification from basic alloy (percent)	Total heatings	Approximate total blows	Bar size		Estimated forgeability compared with that of basic alloy		
					Length (in.)	Diameter (in.) ^b			
7.....	A	Basic.....	8.....	361	25.5	0.42 sq.....			
7.....	B		7.....	267	14.....	0.50.....			
8.....	A	Basic.....	8.....	355	25.....	0.40 sq.....			
8.....	D		18.....	875	26.....	0.41.....			
10.....	A	Basic.....	9.....	428	25.5	0.42 sq.....			
10.....	D		20.....	939	29.....	0.41.....			
11.....	B	Basic.....	3.....	384	30.....	0.50 sq.....			
12.....	A	Basic.....	8.....	434	28.....	0.41 sq.....			
27.....	D	Basic.....	14.....	824	25.2	0.43.....			
74.....	D	Basic, no deoxidation.....	18.....	799	22.....	0.40.....	Same.		
75.....	D	Basic, Zr-Si-Fe deoxidant.....	14.....	723	21.....	0.42.....	Same.		
76.....	D	Basic, melted under slag.....	-----	-----	-----	-----	Poorer—many cracks during initial flat die work; forging discontinued.		
13.....	A	0.08 C.....	5.....	292	24.5	0.40 sq.....	Better—more plastic.		
15.....	A	0.40 C.....	8.....	350	21.5	0.42 sq.....	Poorer—corner and end cracks; less plastic.		
16.....	A	0.60 C.....	5.....	169	9.5	0.51.....	Poorer—many center and corner cracks; less plastic.		
16.....	B		5.....	271	13.....	0.42 sq.....			
24.....	D	0 Mn.....	15.....	909	20.5	0.41.....	Poorer—many bad corner cracks developed during forging on flat dies.		
19.....	A	0.30 Mn.....	5.....	345	23.5	0.44 sq.....	Same—some corner cracks.		
20.....	A	0.50 Mn.....	6.....	460	27.5	0.40 sq.....	Same—some end cracks.		
21.....	A	1.0 Mn.....	6.....	444	29.....	0.41 sq.....	Same.		
22.....	A	2.5 Mn.....	4.....	344	25.....	0.41 sq.....	Slightly better.		
28.....	D	1.2 Si.....	15.....	895	26.4	0.44.....	Same.		
80.....	D	1.6 Si.....	15.....	863	23.....	0.41.....	Same.		
51.....	D	10 Cr.....	15.....	706	21.....	0.40.....	Better—much more plastic.		
52.....	D	30 Cr.....	20.....	959	21.5	0.40.....	Poorer—much less plastic.		
23.....	D	0 Ni.....	12.....	613	15.4	0.47.....	Better—more plastic.		
25.....	D	10 Ni.....	11.....	597	17.....	0.48.....	Better—more plastic.		
26.....	D	30 Ni.....	16.....	846	22.....	0.45.....	Poorer—less plastic.		
29.....	D	0 Co.....	12.....	740	25.2	0.42.....	Better—more plastic.		
30.....	D	10 Co.....	15.....	813	27.2	0.41.....	Same.		
31.....	D	32 Co.....	13.....	724	20.5	0.41.....	Poorer—less plastic.		
32.....	D	0 Mo.....	14.....	771	22.....	0.41.....	Same.		
33.....	D	1 Mo.....	18.....	660	22.....	0.42.....	Same.		
34.....	D	2 Mo.....	19.....	928	27.2	0.42.....	Same.		
35.....	D	5 Mo.....	16.....	804	22.5	0.40.....	Same.		
36.....	D	7 Mo.....	16.....	818	23.....	0.41.....	Same.		
37.....	D	0 W.....	11.....	603	17.....	0.41.....	Better—more plastic.		
38.....	D	1 W.....	14.....	674	21.5	0.41.....	Better—more plastic.		
39.....	D	5 W.....	14.....	750	21.5	0.41.....	Poorer—less plastic.		
40.....	D	7 W.....	17.....	874	22.....	0.42.....	Poorer—less plastic than 39.		
47.....	D	0 Cb.....	16.....	629	17.5	0.41.....	Poorer—many cracks developed during forging on flat dies.		
48.....	D	2 Cb.....	16.....	736	20.3	0.40.....	Same.		
49.....	D	4 Cb.....	19.....	815	21.....	0.41.....	Poorer—less plastic.		
50.....	D	6 Cb.....	18.....	889	21.....	0.40.....	Poorer—less plastic; no forging cracks.		
41.....	D	0.004 N.....	14.....	684	20.5	0.41.....	Same.		
42.....	D	0.08 N.....	16.....	770	19.5	0.41.....	Same.		
81.....	D	0.18 N.....	15.....	812	21.8	0.41.....	Same.		
		Mo	W	Cb					
43.....	D	0	0	0	16.....	854	26.....	0.45.....	Better—much more plastic.
45.....	D	2	0	0	14.....	646	22.5	0.41.....	Poorer—more plastic but more subject to cracking.
46.....	D	4	0	0	17.....	700	22.....	0.41.....	Poorer—similar to 45.
53.....	D	0	2	0	15.....	683	22.5	0.40.....	Poorer—more plastic but much more subject to cracking.
54.....	D	0	4	0	18.....	821	23.....	0.41.....	Poorer—same as 53.
55.....	D	2	2	0	19.....	1050	28.8	0.40.....	Poorer—much more tendency to crack.
56.....	D	4	2	0	19.....	1034	28.5	0.41.....	Poorer—same as 55.
57.....	D	2	4	0	20.....	881	21.....	0.41.....	Poorer—less plastic; tendency to crack.
58.....	D	4	4	0	22.....	998	21.5	0.40.....	Poorer—same as 57.
59.....	D	2	2	2	19.....	904	24.....	0.41.....	Same.
60.....	D	4	2	2	18.....	894	22.5	0.41.....	Same.
61.....	D	2	4	2	18.....	826	21.5	0.40.....	Poorer—less plastic.
62.....	D	2	2	4	19.....	917	23.....	0.40.....	Poorer—same as 61.
63.....	D	2	0	2	14.....	728	20.5	0.40.....	Better—more plastic.
64.....	D	2	0	4	14.....	754	19.....	0.40.....	Same.
65.....	D	4	0	2	16.....	813	21.....	0.40.....	Same.
66.....	D	4	0	4	16.....	859	21.5	0.40.....	Poorer—less plastic.
67.....	D	4	4	4	12.....	674	18.3	0.40.....	Poorer—less plastic.
77.....	D	0	0	1	15.....	841	26.5	0.40.....	Better—more plastic.
68.....	D	0	0	2	13.....	687	21.....	0.40.....	Better—more plastic.
69.....	D	0	0	4	10.....	521	17.3	0.41.....	Better—more plastic.
70.....	D	0	2	2	17.....	782	21.....	0.40.....	Same.
71.....	D	0	4	2	17.....	850	24.....	0.40.....	Same.
72.....	D	0	2	4	16.....	771	21.5	0.40.....	Same.
73.....	D	0	4	4	16.....	822	22.5	0.40.....	Same.
82.....	D	2	4	4	16.....	833	24.....	0.41.....	Poorer—less plastic.
83.....	D	4	2	4	16.....	850	22.....	0.41.....	Poorer—same as 82.
84.....	D	4	4	2	14.....	851	20.8	0.41.....	Poorer—same as 82.

^a Letter designates location of bar in ingot as follows:

Bar A was from bottom quarter of ingot in all cases except 16A, which was from lower middle quarter of ingot

Bar B was from lower middle quarter except 16B, which was from bottom of ingot

Bar D was from upper quarter of ingot

^b All bars were rounds except where squares (sq) are indicated.

TABLE VI.—INFLUENCE OF FORGING AND MELTING VARIABLES ON RUPTURE TEST CHARACTERISTICS AT 1200° F OF BASIC ALLOY ^a

[Heat treatment: 2200° F, 1 hr, water-quenched; 1400° F, 24 hr, air-cooled]

Alloy	Rupture strength (psi)		Estimated 100-hr rupture elongation (percent in 1 in.)	Stress (psi) to cause creep rates of—		Minimum creep rate at 40,000 psi (percent/hr)
	100 hr	1000 hr		0.01 percent/hr	0.1 percent/hr	
Final adopted forging procedure—round bar stock						
8.....	48,500	37,000	22	35,800	47,500	0.022
10.....	50,000	38,000	19	35,400	49,000	.023
27.....	48,000	37,000	25	37,000	46,200	.023
Average.....	48,800	37,300	22	36,000	47,600	0.023
Range.....	48,000 to 50,000	37,000 to 38,000	19 to 25	35,400 to 37,000	46,200 to 49,000	0.022 to .023
Preliminary forging procedure—square bar stock						
7.....	48,500	37,000	20	35,800	46,300	0.028
8.....	46,500	^b 36,000	18	34,500	46,000	.033
10.....	52,000	36,500	20	35,400	50,100	.022
11.....	48,000	38,000	22	36,000	47,500	.024
12.....	49,000	38,500	22	38,200	46,000	.018
Average.....	48,800	37,200	20	36,000	47,200	0.025
Range.....	46,500 to 52,000	36,000 to 38,500	18 to 22	34,500 to 38,200	46,000 to 50,100	0.018 to .033
Variable deoxidation practice ^a —round bar stock						
74 (none).....	52,000	^b 41,000	16	^b 40,700	^b 55,500	^b 0.0090
75 (Zr-Si-Fe).....	55,000	40,500	18	42,300	57,000	^b .0065
Comparative properties of commercial bar stock of basic alloy						
Heat 30276.....	50,000	^b 42,000	14	-----	-----	-----
A-1726.....	47,000	^b 42,000	16	-----	-----	-----

^a Results based on detailed rupture test data given in table X.^b Estimated value.^c Deoxidation practice:

None—no deoxidation

Zr-Si-Fe—zirconium-silicon-iron deoxidant

TABLE VII.—INFLUENCE OF CHEMICAL COMPOSITION ON MICROSTRUCTURAL CLASSIFICATIONS OF MODIFIED ALLOYS

Simultaneous variation of Mo, W, and Cb																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																									
Individual variation of elements in basic alloy				Variable Mo				Variable W				Variable Ob																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																													
Alloy	Alloy modification (percent)	Microstructural classification *				Alloy	Alloy modification (percent)				Alloy	Microstructural classification *				Alloy	Alloy modification (percent)				Microstructural classification *																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																				
		IC	GS	GB	MP		Mo	W	Cb	IC		GS	GB	MP	Mo		W	Cb	IC	GS		GB	MP																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																																		
Basic	0.08 C	II	5	Y	Rb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	0	I	1	z	Cb	43	0	0	

* Classification key:

Solution-treated structure	Aged structure			
	A. S. T. M. grain size number (GS)	Amount of grain boundary precipitate (GB)	Aging precipitate in matrix (MP): Type *	Amount
Amount of insoluble constituent (IC)				
I—small. II—medium. III—large. IV—very large.	1 (up to 1½ grains/sq in.) 2 to 8 (86 grains or more/sq in.)	x—small. y—medium. z—large.	C—precipitate tends to follow crystallographic planes. R—random precipitate.	a—small. b—medium. c—large. d—very large.

* Ferrite-sigma-type phase in 0-percent-Ni alloy 23. A large amount of precipitate occurred in sigma phase during aging but only a small amount in austenite matrix.

TABLE VIII.—SUMMARY OF INFLUENCE OF ALLOYING ELEMENTS ON INCREASING TENDENCY FOR MICROSTRUCTURAL CHANGES

Alloying element varied	Composition range (percent)	In soluble constituent increase	Grain size decrease	Grain boundary precipitation increase	Mode of matrix precipitation	Amount of matrix precipitation increase
C.....	0.08-0.60	Strong.....	Mild.....	None.....	Random.....	Very strong.
Mn.....	0-2.5	None.....	None.....	None.....	Random.....	None.
Si.....	0.5-1.6	None.....	Mild.....	None.....	Random.....	None.
Cr.....	10-20	None.....	Negative mild.....	Mild.....	Random.....	Mild.
Cr.....	20-30	None.....	Negative mild.....	None.....	Random to preferred.....	Strong.
Ni.....	0-10	Negative strong ^a	Negative mild.....	Negative weak.....	Random.....	None. ^a
Ni.....	10-30	None.....	Weak.....	None.....	Random.....	None.
Co.....	0-32	None.....	Negative mild.....	None.....	Random.....	Negative mild.
Mo.....	0-4	None.....	Weak.....	None.....	Random ^b	Weak.
W.....	0-4	None.....	Weak.....	None.....	Random ^b	Weak.
Cb.....	0-1	Strong.....	Strong.....	Negative mild.....	Preferred to random.....	Negative mild.
Cb.....	1-4	Strong.....	Mild.....	None.....	Random.....	Mild.
N.....	0.004-0.18	None.....	None.....	None.....	Random.....	None.

^a Ferrite-sigma phase in 0-percent-Ni alloy in air conditions. A large amount of aging precipitate occurred in sigma phase, but only a small amount in austenite matrix.

^b Mode of precipitation not affected by Mo and W. In alloys containing Cb precipitate was random; without Cb precipitate followed a preferred orientation.

TABLE IX.—HARDNESS DATA ON EXPERIMENTAL ALLOYS

Alloy	Modification from basic alloy (percent) ^a	Vickers hardness number ^b			
		Solution- treated; 2200° F, 1 hr, water- quenched	Solution- treated and aged; 1400° F, 24 hr, air-cooled		
7.....	Basic (sq).....	229	245		
8.....	Basic (sq).....	229	243		
8.....	Basic.....	229	247		
10.....	Basic (sq).....	237	272		
10.....	Basic.....	234	259		
11.....	Basic (sq).....	235	247		
12.....	Basic (sq).....	229	252		
27.....	Basic.....	211	221		
74.....	Basic, no deoxidation.....	214	233		
75.....	Basic, Zr-Si-Fe deoxidant.....	229	247		
13.....	0.08 C (sq).....	206	219		
15.....	0.40 C (sq).....	261	288		
16.....	0.60 C (sq).....	290	309		
24.....	0 Mn.....	227	251		
19.....	0.30 Mn (sq).....	227	251		
20.....	0.50 Mn (sq).....	223	241		
21.....	1.0 Mn (sq).....	216	231		
22.....	2.5 Mn (sq).....	229	245		
28.....	1.2 Si.....	221	249		
80.....	1.6 Si.....	243	242		
51.....	10 Cr.....	177	193		
52.....	30 Cr.....	233	408		
23.....	0 Ni.....	288	330		
25.....	10 Ni.....	238	261		
26.....	30 Ni.....	226	252		
29.....	0 Co.....	210	252		
30.....	10 Co.....	205	226		
31.....	32 Co.....	221	235		
32.....	0 Mo.....	203	219		
33.....	1 Mo.....	202	210		
34.....	2 Mo.....	197	211		
35.....	5 Mo.....	205	233		
36.....	7 Mo.....	221	245		
37.....	0 W.....	205	218		
38.....	1 W.....	206	223		
39.....	5 W.....	223	245		
40.....	7 W.....	243	254		
47.....	0 Cb.....	176	219		
48.....	2 Cb.....	206	238		
49.....	4 Cb.....	258	325		
50.....	6 Cb.....	229	296		
41.....	0.004 N.....	197	208		
42.....	0.08 N.....	233	254		
81.....	0.18 N.....	234	261		
	Mo	W	Cb		
43.....	0	0	0	157	190
45.....	2	0	0	182	205
46.....	4	0	0	185	235
53.....	0	2	0	176	210
54.....	0	4	0	187	235
55.....	2	2	0	201	242
56.....	4	2	0	194	243
57.....	2	4	0	198	251
58.....	4	4	0	202	252
59.....	2	2	2	213	220
60.....	4	2	2	234	252
61.....	2	4	2	234	242
62.....	2	2	4	226	240
63.....	2	0	2	192	215
64.....	2	0	4	215	238
65.....	4	0	2	221	229
66.....	4	0	4	235	258
67.....	4	4	4	258	311
77.....	0	0	1	197	204
68.....	0	0	2	196	201
69.....	0	0	4	204	216
70.....	0	2	2	211	213
71.....	0	4	2	223	227
72.....	0	2	4	215	219
73.....	0	4	4	221	229
82.....	2	4	4	237	258
83.....	4	2	4	242	288
84.....	4	4	2	245	272

^a All tests were made on round bar stock except where square bar stock is indicated (sq).^b Vickers hardness tests were made with a 50-kg load on bar-stock cross sections of metallographic specimens.

TABLE X.—RUPTURE TEST CHARACTERISTICS AT 1200° F FOR MODIFIED ALLOYS

[Heat treatment: 2200° F, 1 hr, water-quenched; 1400° F, 24 hr, air-cooled]

Alloy	Alloy modification (percent) ^a	Stress (psi)	Time (hr)	Elongation in 1 in. (percent)	Reduction of area (percent)	Minimum creep rate (percent/hr)	Rupture strength (psi)		Estimated 100-hr rupture elongation (percent in 1 in.)	Stress (psi) to cause creep rates of—		Minimum creep rate at 40,000 psi (percent/hr)
							100 hr	1000 hr		0.01 per-cent/hr	0.1 per-cent/hr	
7.....	Basic (square).....	50,000	74	20	26.8	-----	48,500	37,000	20	35,800	46,300	0.028
		45,000	184	34	39.2	0.076						
		40,000	516	27	37.4	.028						
8.....	(Round).....	50,000	82	38	34.8	-----	49,000	^b 37,000	30	-----	-----	-----
		45,000	209	31	31.5	.070						
		40,000	51	19	17.8	-----	46,500	^b 36,000	18	34,500	46,000	.033
10.....	Basic (square).....	50,000	142	17	20.5	.083						
		45,000	368	21	15.4	.033						
		40,000	87	22	18.3	.160	48,500	37,000	22	35,800	47,500	.022
11.....	(Round).....	50,000	162	23	22.6	.066						
		45,000	499	25	23.4	.022						
		40,000	1175	37	33.4	.0124						
12.....	Basic (square).....	50,000	98	20	26.1	.115	52,000	36,500	20	35,400	50,100	.022
		45,000	155	21	21.1	.083						
		40,000	259	27	30.1	.054						
13.....	(Round).....	50,000	422	21	29.5	.022						
		45,000	61	15	13.8	-----	50,000	38,000	19	35,400	49,000	.023
		40,000	100	19	24.0	.110						
14.....	Basic (square).....	50,000	240	36	37.0	.057						
		45,000	584	24	27.5	.023						
		40,000	77	26	24.0	.112	48,000	38,000	22	36,000	47,500	.024
15.....	Basic (square).....	50,000	170	19	22.6	.064						
		45,000	575	28	26.2	.024						
		40,000	88	22	23.4	-----	49,000	38,500		38,200	46,000	.018
16.....	Basic (square).....	50,000	209	35	36.6	.070						
		45,000	719	22	30.9	.018						
		40,000	82	26	25.5	-----	48,000	37,000	25	37,000	46,200	.023
17.....	Basic (round).....	50,000	176	18	16.8	.075						
		45,000	525	23	26.5	.023						
18.....	Basic, no deoxidation.....	50,000	33	22	21.3	-----	52,000	^b 41,000	16	^b 40,700	^b 55,500	^b 0.0090
		45,000	98	16	15.5	-----						
		40,000	184	15	18.4	0.039						
19.....	Basic, Zr-Si-Fe deoxidant.....	50,000	417	14	14.4	.021						
		45,000	64	16	15.5	-----	55,000	40,500	18	42,300	57,000	^b .0065
		40,000	134	18	21.3	.075						
20.....	Basic, Zr-Si-Fe deoxidant.....	50,000	263	18	19.1	.025						
		45,000	502	20	19.9	.019						
21.....	Basic, Zr-Si-Fe deoxidant.....	50,000	502	20	19.9	.019						
		45,000	502	20	19.9	.019						
		40,000	502	20	19.9	.019						
22.....	0.08 C (square).....	50,000	79	14	21.3	-----	48,500	^b 36,000	14	^b 36,000	^b 47,000	^b 0.033
		45,000	222	14	10.9	0.040						
		40,000	325	26	21.8	.033						
23.....	0.40 C (square).....	50,000	106	19	21.1	-----	50,500	36,500	19	39,500	49,000	.0102
		45,000	193	19	16.2	.040						
		40,000	518	9	12.4	.0102						
24.....	0.60 C (square).....	50,000	39	18	18.4	-----	46,000	37,500	15	38,000	^b 49,000	^b .016
		45,000	111	13	12.4	.090						
		40,000	486	12	16.2	.018						
25.....	(Round).....	50,000	78	17	16.1	-----	48,000	37,000	15	-----	-----	-----
		45,000	178	14	16.2	.036						
		40,000	528	13	15.4	.0145						
26.....	0 Mn (round).....	50,000	64	8	28.8	-----	50,000	^b 36,000	13	^b 39,000	-----	^b 0.013
		45,000	125	13	10.9	-----						
		40,000	195	10	8.7	0.036						
27.....	0.30 Mn (square).....	50,000	43	22	23.4	-----	49,000	^b 41,000	24	^b 39,000	49,000	^b .013
		45,000	83	24	22.6	.140						
		40,000	352	19	21.8	.032						
28.....	0.50 Mn (square).....	50,000	67	22	21.9	-----	48,000	^b 37,000	22	36,000	51,000	.019
		45,000	246	21	22.6	.042						
		40,000	452	23	20.5	.019						
29.....	1.0 Mn (square).....	50,000	53	21	19.9	-----	47,000	38,000	20	39,000	47,000	.015
		45,000	199	14	19.7	.059						
		40,000	617	16	17.6	.0145						
30.....	2.5 Mn (square).....	50,000	85	20	18.4	.147	49,000	37,500	20	35,000	48,000	.024
		45,000	208	19	23.4	.055						
		40,000	563	26	24.0	.024						
31.....	1.2 Si.....	50,000	41	26	20.5	-----	45,000	34,500	28	33,700	43,000	0.051
		45,000	115	28	30.9	0.145						
		40,000	224	31	26.8	.051						
32.....	1.6 Si.....	50,000	672	26	35.5	.0188						
		45,000	28	41	37.3	-----	43,500	^b 34,000	40	-----	43,000	.071
		40,000	53	42	36.0	-----						
33.....	10 Cr.....	50,000	114	38	34.2	.102						
		45,000	208	40	42.3	.071						
		40,000	208	40	42.3	.071						
34.....	30 Cr.....	45,180	16	25	24.8	-----	39,000	^b 32,000	25	^b 30,000	38,000	0.160
		42,500	36	25	27.5	-----						
		40,000	109	25	27.5	0.16						
35.....	30 Cr.....	35,000	339	21	21.8	.046						
		55,000	52	10	11.0	-----	52,000	42,500	10	42,800	^b 57,000	.006
		50,000	152	8	10.2	.035						
36.....	0 Ni.....	45,000	204	6	12.5	.0160						
		40,000	518	17	17.6	.0134						
		40,000	518	17	17.6	.0134						
37.....	10 Ni.....	50,000	86	17	14.7	-----	49,000	38,000	17	39,200	51,000	.012
		45,000	217	15	14.6	.034						
		40,000	642	14	15.4	.012						
38.....	30 Ni.....	50,000	56	24	22.6	-----	47,000	^b 35,500	25	^b 32,500	45,000	.044
		45,000	152	27	24.8	.100						
		40,000	369	29	26.2	.044						

^a Square, forged as square bar; round, forged as round bar; all heats higher than 22 were forged as round bars only.

^b Estimated.

[Heat treatment: 2200° F, 1 hr, water-quenched; 1400° F, 24 hr, air-cooled]

^a Square, forged as square bar; round, forged as round bar; all heats higher than 22 were forged as round bars only.
^b Estimated.
^c Overheated at 300 hr.

TABLE X.—RUPTURE TEST CHARACTERISTICS AT 1200° F FOR MODIFIED ALLOYS—Concluded

[Heat treatment: 2200° F, 1 hr, water-quenched; 1400° F, 24 hr, air-cooled]

Alloy	Alloy modification (percent) °			Stress (psi)	Time (hr)	Elongation in 1 in. (percent)	Reduction of area (percent)	Minimum creep rate (percent/hr)	Rupture strength (psi)		Estimated 100-hr rupture elongation (percent in 1 in.)	Stress (psi) to cause creep rates of—		Minimum creep rate at 40,000 psi (percent/hr)
									100 hr	1000 hr		0.01 per cent/hr	0.1 per cent/hr	
55-----	2	2	0	45,000	57	7	6.4	-----	41,000	32,000	6	36,000	43,500	0.036
				40,000	97	6	8.0	0.036						
				35,000	449	5	8.0	.0070						
56-----	4	2	0	50,000	43	13	16.8	-----	47,000	38,000	15	39,000	47,500	.013
				45,000	154	16	13.8	.056						
				40,000	603	12	16.8	.013						
57-----	2	4	0	45,000	87	6	10.2	.033	44,000	33,000	6	40,000	49,000	.0145
				40,000	213	6	7.1	.0145						
				37,000	418	7	5.6	.0042						
58-----	4	4	0	54,000	62	10	13.1	-----	51,000	40,000	10	40,000	55,000	.010
				50,000	131	9	15.5	.048						
				45,000	232	12	13.1	.024						
				42,000	677	17	20.5	.0137						
59-----	2	2	2	50,000	63	24	18.3	-----	47,000	36,000	23	37,000	44,000	0.030
				48,000	85	26	24.0	-----						
				45,000	105	23	24.0	0.135						
60-----	4	2	2	40,000	392	19	18.3	.030	49,000	41,000	30	40,500	47,500	.0090
				50,000	69	27	31.5	-----						
				48,000	177	30	27.5	.117						
61-----	2	4	2	45,000	241	39	30.8	.047	48,500	40,000	20	37,500	49,000	.018
				50,000	69	19	21.3	-----						
				45,000	277	21	25.5	.048						
62-----	2	2	4	50,000	51	27	20.5	-----	46,000	34,000	25	36,500	45,000	.028
				45,000	122	25	25.5	.100						
				40,950	247	31	25.5	.034						
63-----	2	0	2	45,000	70	34	35.6	-----	43,500	34,000	30	33,600	43,500	0.048
				40,000	200	25	30.1	0.048						
64-----	2	0	4	37,000	462	23	27.5	.024	44,000	34,000	25	-----	41,000	.085
				50,000	26	29	28.8	-----						
				45,000	75	21	24.0	-----						
65-----	4	0	2	40,000	236	25	28.2	.085	45,500	37,000	30	35,000	44,000	.030
				50,000	43.5	27	26.8	-----						
				45,000	101	30	30.1	.160						
66-----	4	0	4	40,000	444	38	34.8	.030	46,000	36,500	33	35,000	45,000	.034
				50,000	46	33	33.0	-----						
				45,600	114	33	35.6	-----						
67-----	4	4	4	40,000	446	43	42.3	.034	50,000	38,000	30	36,000	47,500	.025
				50,000	64	37	33.5	.200						
				54,000	84	28	30.9	.052						
				45,000	105	32	39.8	.029						
				41,000	536	31	32.2	-----						
77-----	0	0	1	45,000	66	18	16.8	0.160	42,000	31,000	18	30,000	42,500	0.065
				40,000	181	15	16.8	.065						
68-----	0	0	2	35,000	357	18	16.8	.030	39,500	31,500	35	31,500	38,000	.19
				43,000	48	41	43.7	-----						
				40,000	91	35	34.2	.19						
69-----	0	0	4	35,000	360	27	27.5	.037	38,500	30,500	25	31,500	37,500	.23
				42,000	47	31	30.9	-----						
				40,000	76	31	27.5	.23						
				38,000	95	23	24.8	.140						
				35,000	263	24	32.2	.040						
70-----	0	2	2	44,000	42	24	21.8	-----	40,000	32,000	20	32,800	39,200	0.120
				40,000	102	20	21.9	0.120						
71-----	0	4	2	35,000	424	15	20.5	.023	44,000	37,000	20	38,000	43,800	.026
				48,000	47	19	19.0	-----						
				45,000	63	17	18.4	.160						
72-----	0	2	4	40,000	368	21	19.7	.026	42,000	32,500	30	34,000	40,500	.039
				45,000	60	32	31.5	-----						
				40,000	181	27	25.4	.069						
73-----	0	4	4	37,000	302	28	30.1	.039	45,000	36,500	24	38,000	43,800	.021
				48,000	46	22	21.8	-----						
				45,000	99	24	24.8	.155						
				40,000	356	17	22.6	.021						
82-----	2	4	4	50,000	73	20	21.9	0.200	48,000	37,000	22	35,000	46,000	0.031
				45,000	196	25	25.4	.083						
83-----	2	4	4	40,000	495	32	34.8	.031	48,000	36,000	30	34,000	44,500	.041
				50,000	81	32	33.5	-----						
				45,000	160	37	38.6	.110						
84-----	4	4	2	40,000	439	42	39.8	.041	50,500	39,500	29	37,000	48,500	.020
				53,000	79	29	25.4	-----						
				50,000	109	29	27.5	.136						
				45,000	193	21	22.6	.057						
				40,000	860	28	28.2	.020						

° Square, forged as square bar; round, forged as round bar; all heats higher than 22 were forged as round bars only.

b Estimated.

TABLE XI.—COMPARISON OF 1200° F RUPTURE PROPERTIES OF WROUGHT HEAT-RESISTING ALLOYS

Alloys	Variable	Remarks	100-hr rupture strength (psi)	100-hr rupture elongation (percent)	Reference ^a
Modified alloys.	Chemical composition.	Prior processing and heat treatment controlled and held constant.	26,000 to 52,000	5 to 40	(^b)
Basic alloy.	Heat treatment and hot-cold-work.	Single commercial heat.	40,000 to 66,000	1 to 50	1
Six standard heat-resisting alloys: 18Cr-8Ni. 18Cr-8Ni-1Cb. 16Cr-13Ni-3Mo. 25Cr-12Ni. 25Cr-20Ni. 15Cr-35Ni.	Chemical composition and prior processing.	Treatment: 2000° F, water-quenched. 2250° F, water-quenched. 2000° F, water-quenched. 2200° F, water-quenched. 2150° F, water-quenched. 2000° F, water-quenched.	20,000 30,000 31,000 24,500 24,000 22,000	20 8 15 7 7 3	4
Seven heat-resisting super alloys: 19-9DL. 17W. CSA. Timken. Low-carbon N-155. High-carbon N-155. S590.	Chemical composition and prior processing.	Treatment variable: ^c Ann, HW, CW. Ann, HW, CW. Ann, HW, CW. Ann, HW, CW. Ann, HW, CW. Ann, Ann+A, HW, CW. Ann, HW, CW. Ann, Ann+A.	42,000 to 62,000 30,000 to 45,000 47,500 to 53,000 49,000 to 57,000 42,500 to 61,000 52,000 to 63,000 51,000 to 53,000	2 to 20 2 to 24 5 to 25 9 to 28 5 to 17 4 to 11 6 to 20	5

^a Numbers refer to references listed at end of text.^b Data from this investigation.^c Treatment variables: Ann—annealed (solution-treated).
Ann+A—solution-treated plus aged.
HW—hot-worked.
CW—hot-cold-worked.